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ON THE LIMIT CYCLES OF FAMILLY OF DIFFERENTIAL SYSTEM DEGREE 5

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Dedication

Thanks to **Allah** for giving us the capacity to overcome all the obstacles and achieve our goal in contributing the academic carrier.

I didicate this work to may beloved parent

"**HOUARI Brahim**" and " **HOUAMED Farida**"

who have always beging source of motivation, insperation, encouragement to me.

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Didication

Every challenging work needs self efforts as well as quidance of elders speatly those who were very close to our heart. My humble effort I dedicate my sweet and loving

Mother & Father

Ben hammada Fadila and **Ben hammada Arrezki**.

Whose effection, love, encouragement and prays of day and night make me able to gut such success and honor, I can't forgat my dear friend **Asma** , who help me a lot. All my brothers **Djallel**, **Abd alatif** and **Youcef**, and all my sisters **Donia** and **Wiam**. And finally my dear husband **Belmoumen Mohamad**.

« **BEN HAMMADA RANIA** »

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INTRODUCTION

Differential equation have important application and are powerful tool in the study of many problems in the natural sciences and in technology; they are extensively employed in mechanics, astronomy, physics, and in many problems of chemisty and biology. Direct resolution of a differential equation is usually difficult or impossible.
However, another way out it possible. This is the qualitative study of differential equations. This study makes it possible to provide information on the behavior of the solutions of a differential equation without the need to solve it explicit, and it consists in examining the properties and the characteristics of the solutions of this equation, and to justify among these solution, the existence or non existence of an isolated closed curve form called limit cycle.

An important problem of the qualitative theory of differential equations is to determine the limit cycles of a systems of a differential equations.
Usually, we ask for the number of such limit cycles as orbits, and an even more difficult problem is to give an explicit expression of them.

The limit cycles introduced for the first time by **Henri Poincaré** in 1881 in his "Dissertation on the curves defined by a differential equation" [6]. **Poincaré** was interested in the qualitative study of the solutions of the differential equations, i.e. points equilibrium, limit cycles and their stability.
This makes it possible to have an overall idea of the other orbits of the studied systems.
The mathematician **David Hilbert** presented at the second international congress of mathematics ([3], 1900), 23 problems whose future awaits resolution through new methods that will be discovered in the century that begins. The problem number 16 is to know the maximum number and relative position of the limit cycles of a planar polynomial differential systems of degree **n**. We denote **Hn** this maximum number. **Dulac** [2] in 1923, offered a proof that **Hn** is finite. In recent years, several papers have studied the limit cycles of planar polynomial differential systems. The main reason for this study is **Hilbert** 16-th unsolved problem. Later on **Van der Pol** [7] in 1926, **Liénard**[4] in 1928 and **Andronov** [1] in 1929 shown that the periodic solution of self-sustained oscillation of a circuit in a vacuum tube was a limit cycle.

The objective of this work is to give a quintic polynomial differential system of the form:

$$\begin{cases} \dot{x} = x - (\gamma(2y - ax) + \alpha(x^2 + y^2)(ax - 4y))Q(x, y), \\ \dot{y} = y - (-\gamma(2x + ay) + \alpha(x^2 + y^2)(4x + ay))Q(x, y). \end{cases} \tag{1}$$

Where $Q(x, y)$ is homogeneous polynomial of degrees 2 where α, γ and a are real constants. The main motivation of this dissertation is to prove that these systems are integrable. Moreover, we determine sufficient conditions for a polynomial differential systems to possess at most two limit cycles, one of them algebraic and the other one is non-algebraic, counted two explicit limit cycles. Concrete examples exhibiting the applicability of our result are introduced.

This dissertation is structured in three chapters. The first chapter is dedicated to reminders of some preliminary concepts on the planar differential system. In the second chapter we put $\alpha = 0$ and we get a system of degree 3 as follows:

$$\begin{cases} \dot{x} = x - \gamma(2y - ax)Q(x, y), \\ \dot{y} = y - \gamma(-2x - ay)Q(x, y). \end{cases} \tag{2}$$

We solve this system and we use the available conditions in the theories to prove that this system possesses at most one limit cycle. In the last chapter, we solve the system in the case $\alpha \neq 0$ and apply the theorems conditions in order to prove that the system possesses at most two limit cycles, and we prove as well the limit cycles algebraic or not.

Chapter 1

BASIC NOTION AND REMINDERS

1.1 Introduction

This chapter contains basic and main concepts for the qualitative study of dynamic systems. To understand this chapter, we start by definition of polynomial differential systems, we will discuss the notions of: vector field, phase portrait, solution and periodic solution, limit cycle, nature of the critical points. We also quote some theorems used as tools in our work. Most part of the results are given without proof, however references where they can found, are included.

1.2 Polynomial Differential Systems

Definition 1.1 *A polynomial differential system is a system of the form:*

$$\begin{cases} \dot{x} = P(x(t), y(t)), \\ \dot{y} = Q(x(t), y(t)), \end{cases} \tag{1.1}$$

where $P(x, y)$ and $Q(x, y)$ are real polynomials in the variables x and y.
The degree n of the system is the maximum of the degrees of the polynomial P and Q.
As usual the dot denotes derivative with respect to the independent variable t.

Definition 1.2 *A linear differential system consists of linear differential equations (the linearity relates to the unknown functions and their derivatives).*

Definition 1.3 *A non linear system consists of non linear differential equation.*

Definition 1.4 *A differential system of the form $\dfrac{dx}{dt} = f(t, x)$ is said to be autonomous if the function f depends only on the vector variable x. Otherwise, it is not autonomous, an autonomous system is written in the from $\dfrac{dx}{dt} = f(x)$.*

Remark 1.1 *If the polynomial P and Q are written in the form:*

$$\begin{cases} P(x,y) = \sum\limits_{i+j=0}^{i+j=n} a_{ij}x^i y^{n-j}, \\ Q(x,y) = \sum\limits_{i+j=0}^{i+j=n} b_{ij}x^i y^{n-j}. \end{cases} \tag{1.2}$$

We say that P and Q are homogeneous, in this case the system (1.1) is called homogeneous polynomial differential system.

1.3 Vector field

Definition 1.5 *We call vector field a region of the plane in which exists in any point a vector $\vec{V}(M,t)$. Suppose that we have a C^1 vector field in $\Omega \subset \mathbb{R}^2$, that is to say the application:*

$$M\colon \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \vec{V}(M) = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix},$$

where F_1, F_2 are C^1 in Ω.

We consider the vector field χ associated to the system (1.1)

$$\frac{d\vec{M}}{dt} = \vec{V} \Leftrightarrow \begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

which means that system (1.1) is equivalent to the vector field $\chi(P,Q)$, we can also write:

$$\chi = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

1.4 Flow

Definition 1.6 *Suppose that $f \in C^1(\Omega)$, then for all $x_0 \in \Omega$ there exists a unique solution $\phi_t(x)$ defines on an open interval $I \subset \mathbb{R}$. Given a point x belonging to Ω we note $\phi_t(x)$ the position of x after a displacement of a duration t, $(t \in I)$.*

The $\phi : \Omega \times I \longrightarrow \Omega$ application is called the flow of nonlinear differential system, satisfies the following properties:

i) $\phi_0(x) = x$.

ii) $\phi_s(\phi_t(x)) = \phi_{s+t}(x); \forall t \in \mathbb{R}$.

iii) $\phi_{-t}(\phi_t(x)) = \phi_t(\phi_{-t}(x)); \forall t \in \mathbb{R}$.

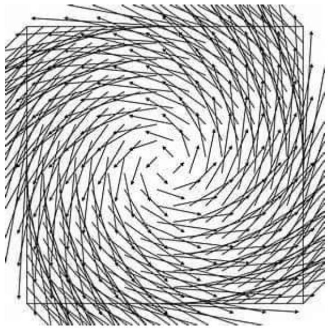

Figure 1.1: Vector Field

Example 1.1 *Let the vector field be:*

$$\dot{x} = f(x) = \begin{pmatrix} -3x \\ y + 2x^2 \end{pmatrix}.$$

The flow of the problem to the initial values:

$$\begin{cases} \dot{x} = f(x), \\ x(0) = c, \end{cases}$$

is given by

$$\Phi(c_1, c_2) = \begin{pmatrix} c_1 e^{-3t} \\ \frac{-2}{7} c_1^2 e^{-6t} + c_2 e^t + \frac{2}{7} c_1^2 e^t \end{pmatrix}.$$

1.5 Phase portrait

The plane $\mathbb{R}^2$ is called phase plane and the solutions of a vector field χ, represent in the phase plane of the orbits or the trajectories, the phase portrait of a vector field χ is the set solutions in the phase plane.

Definition 1.7 *A phase portrait is a geometric representation of the trajectories of a dynamic system in the phase space, at each set of initial conditions corresponds a curve or a point.*

1.6 Equilibrium point

The fixed points or equilibrium points play a vital role in the study of dynamic systems, Henri Poincaré (1854-1912) showed that to characterize a dynamic system with multiple variables it is not necessary to calculate the detailed solutions, it is enough to know equilibriums points and their stabilities.

Definition 1.8 *We call critical point or equilibrium point of the system* $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$*, any point* $\boldsymbol{x_0} \in \mathbb{R}^n$ *such that:*

$$\boldsymbol{f}(\boldsymbol{x_0}) = \boldsymbol{0}.$$

Definition 1.9 *Consider the system* (1.1)*, then the system:* $\dot{\boldsymbol{x}} = \boldsymbol{Ax}$ *with*

$$\boldsymbol{A} = \left(\frac{\partial \boldsymbol{f_i}}{\partial \boldsymbol{x_j}}(\boldsymbol{x_0})\right) = \boldsymbol{Df}(\boldsymbol{x_0}), \boldsymbol{1} \leqslant \boldsymbol{i}, \boldsymbol{j} \leqslant \boldsymbol{n}.$$

And since $\boldsymbol{f}(\boldsymbol{x_0}) = \boldsymbol{0}$*, is called the linearized of* (1.1) *in* $\boldsymbol{x_0}$*.*

1.6.1 Stability Of Equilibrium Points

Any non-linear system may have several equilibrium positions that may be stable or unstable. Let $(\boldsymbol{x_0}, \boldsymbol{y_0})$ be an equilibrium point of system (1.1). Note by $\boldsymbol{X} = (\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{y}))$ and
$\boldsymbol{X}(\boldsymbol{t}) = (\boldsymbol{P}(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{y}(\boldsymbol{t})), \boldsymbol{Q}(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{y}(\boldsymbol{t})))$, $\boldsymbol{X_0} = (\boldsymbol{P}(\boldsymbol{x_0}, \boldsymbol{y_0}), \boldsymbol{Q}(\boldsymbol{x_0}, \boldsymbol{y_0}))$.

Definition 1.10 *We say that:*
$(\boldsymbol{x_0}, \boldsymbol{y_0})$ *is stable if and only if*

$$\forall \varepsilon > \boldsymbol{0}, \exists \eta > \boldsymbol{0} : \|(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x_0}, \boldsymbol{y_0}\| < \eta \Rightarrow \|\boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{X_0}\| < \varepsilon, \forall \boldsymbol{t} > \boldsymbol{0}.$$

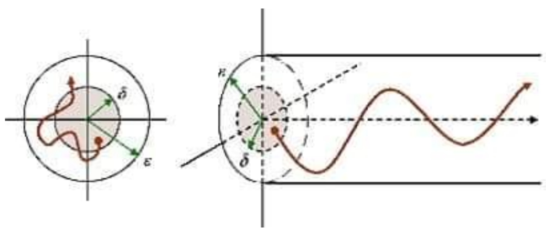

Figure 1.2: Stability Of An Equilibrium Point

$(\boldsymbol{x_0}, \boldsymbol{y_0})$ *is asymptotically stable if and only if*

$$\lim_{t \longrightarrow +\infty} \|\boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{X_0}\| = \boldsymbol{0}.$$

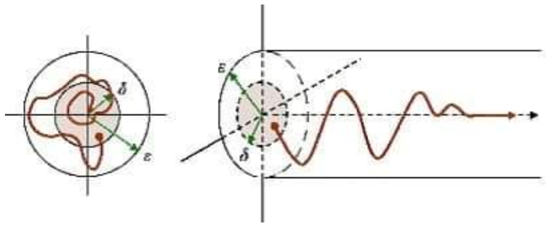

Figure 1.3: Asymptotic Stability

1.7 Invariant curve

Invariant algebraic curves play an important role in the integrability of differential planar polynomial systems, and are also used in the study of the existence and non-existence of periodic solutions and consequently the existence and non-existence of limit cycle.

Definition 1.11 *Let $f \in \mathbb{C}[x, y]$ not identically zero. The algebraic curve $f(x, y) = 0$ is an invariant algebriac curve the polynomial system (1.1) if for some polynomial $K \in \mathbb{C}[x, y]$ we have:*

$$\chi f = P(x, y)\frac{\partial f}{\partial x}(x, y) + Q(x, y)\frac{\partial f}{\partial y}(x, y) = K(x, y)f(x, y), \tag{1.3}$$

for all $(x, y) \in f$.
The polynomial K is called the cofactor of the invariant algrbraic curve $f = 0$. We note that since the polynomial system has degree m, any cofactor has degree at most $m - 1$.

Example 1.2 *The curve defined by equation $ay+b$ is an invariant curve for the following system*

$$\begin{cases} \dot{x} = -y(ay + b) - (x^2 + y^2 - 1), \\ \dot{y} = x(ay + b). \end{cases} \tag{1.4}$$

Let $f(x, y) = ay + b$, then

$$\begin{aligned} \dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y} &= 0(-y(ay + b) - (x^2 + y^2 - 1) + a(x(ay + b)) \\ &= ax(ay + b). \end{aligned}$$

Thus, $f(x, y) = 0$ is invariant curve with cofactor $k(x, y) = ax$.

1.8 First Integral

The notion of integrability for differential system is a based on the existence of first integrals, so the question that arises: If we have a differential system, how can we know if it has a first integral?

Definition 1.12 *A function $H : f \longrightarrow \mathbb{R}$ of class C^j and which is constant on each trajectory of* (1.1) *and not locally constant is called the first integral of the system* (1.1) *of class j on $U \in \mathbb{R}^2$.*
The equation $H(x, y) = c$ fixed for $c \in \mathbb{R}$, gives a set trajectories of the system in an implicit way.
When $j = 1$, these condition are equivalent to

$$P(x, y)\frac{\partial H}{\partial x} + Q(x, y)\frac{\partial H}{\partial 0} \equiv 0$$

and H not locally constant.
The search for an explicit expression of a first integral and the determination of its functional class is called the integrability problem.

Remark 1.2 *- We say that the differential system* (1.1) *is integrable on an open subset Ω if it admits a first integral on Ω of $\mathbb{R}^2$.*
- It is well know that for differential systems defined on the plan $\mathbb{R}^2$ the existence of a first integral determines their phase portrait.

1.9 Solution and Periodic Solution

Definition 1.13 *We say that $(x(t), y(t))_{t\in\mathbb{R}}$ is a solution of system* (1.1) *if the vector filed $X = (P, Q)$ is always tangent to the trajectory representing this solution in the phase plane, in other words*

$$\forall t \in \mathbb{R}, P(x(t), y(t))\dot{x} + Q(x(t), y(t))\dot{y} = 0.$$

Definition 1.14 *called periodic solution of system* (1.1)*, all solution $(x(t), y(t))$ for which there exists a real $T > 0$ such that:*

$$\forall t \in \mathbb{R}, x(t + T) = x(t) \text{ and } y(t + T) = y(t).$$

The smallest number $T > 0$ is called the period of this solution.

1.10 Limit Cycle

We have seen that the solution tend towards a singular point, another possible behavior for a trajectory is to tend towards a periodic movement in the case of a planar system, that means that the trajectories tend towards what is called a limits cycles.

Definition 1.15 *A limit cycle is an isolated closed orbit of* (1.1)*, i.e., we can not find another closed orbit in its neighborhood.*

A periodic orbit Γ *is called stable if for each* $\varepsilon > 0$ *there is a neighborhood* U *of* Γ *such that for all* $x \in U$ *and* $t > 0$*:*

$$d(\Phi(t,x),\Gamma) > \varepsilon.$$

A periodic orbit Γ *is called unstable if it is not stable, and* Γ *is called asymptotically stable if it is stable and if for all points* x *in some neighborhood* U *of* Γ

$$\lim_{x \longrightarrow \infty} d(\Phi(t,x),\Gamma) = 0.$$

Example 1.3 *The system*

$$\begin{cases} \dot{x} = -4y + x(1 - \dfrac{x^2}{4} - y^2), \\ \dot{y} = x + y(1 - \dfrac{x^2}{4} - y^2), \end{cases} \tag{1.5}$$

has a limit cycle $\Gamma(t)$ *represent by*

$$\Gamma(t) = (2\cos(2t), \sin(2t))$$

and

$$div(P,Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2 - x^2 - 4y^2,$$

let's calculates now $\int_0^{\pi} div(\Gamma(t))dt$

$$\begin{aligned} \int_0^{\pi} (2\cos(2t), \sin(2t))dt &= (2 - (2\cos(2t))^2 - 4(\sin(2t))^2) \\ &= \int_0^{2\pi} -2dt \\ &= -2\pi < 0. \end{aligned}$$

So the cycle $\Gamma(t) = (2\cos(2t), \sin(2t))$ *is a an stable limit cycle.*

Definition 1.16 *If a limit cycle is contained in an algebraic curve of the plan, then we say that it is algebraic, otherwise it is called non algebraic.*

Remark 1.3 *The limit cycle appear only in non-linear differential systems.*

1.11 The first return map

Probably the most basic tool for studying the stability of periodic orbits is the Poincaré map or first return map, defined by Henri Poincaré in 1881. The idea of Poincaré map is quite simple: If Γ is a periodic orbit of system (1.1), through the point (x_0, y_0) and Σ is a hyperplane perpendicular to Γ at (x_0, y_0), then for any point $(x, y) \in \Sigma$ sufficiently near (x_0, y_0), the solution of(1.1) through (x, y) at $t = 0$, $\Phi_t(x, y)$ will cross Σ again at a point $\Pi(x, y)$ near (x_0, y_0), the mapping $(x, y) \to \Pi(x, y)$ is called the Poincaré map.
The next theorem establishes the existence and continuity of the Poincaré map $\Pi(x, y)$ and of its first derivative $D\Pi(x, y)$.

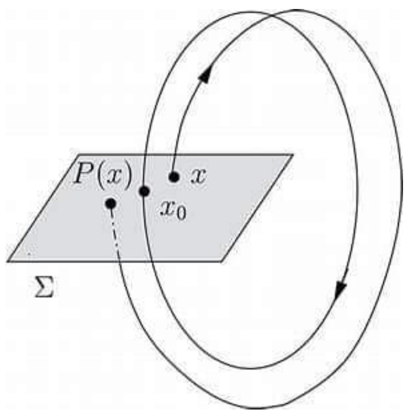

Figure 1.4: The Poincaré Map

Theorem 1.1 *[5] Let E be an open subset of $\mathbb{R}^2$ and let $(P(x, y), Q(x, y)) \in C^1(E)$. Suppose that $\Phi_t(x_0, y_0)$ is a periodic solution of (1.1) of period T and that the cycle*

$$\Gamma = \{(x, y) \in \mathbb{R}^2 | (x, y) = \Phi_t(x_0, y_0), 0 \leqslant t \leqslant T\}$$

is contained in E. Let Σ be the hyperplane orthogonal to Γ at (x_0, y_0), i.e., let

$$\Sigma = \{(x, y) \in \mathbb{R}^2 | (x - x_0, y - y_0), (P(x_0, y_0), Q(x_0, y_0)) = 0.$$

Then if a $\delta > 0$ and a unique function $\tau(x, y)$, defined and continuously differentiable for $(x, y) \in N_\delta(x_0, y_0)$, such that

$$\tau(x_0, y_0) = T$$

and

$$\Phi_{\tau(x,y)}(x, y) \in \Sigma$$

for all $(x, y) \in N_\delta(x_0, y_0)$.

Definition 1.17 *Let Γ, Σ, δ and $\tau(x, y)$ be defined as in Theorem 1.1. Then for $(x, y) \in N_\delta(x_0, y_0) \bigcap \Sigma$, the function*

$$\Pi(x, y) = \Phi_{\tau(x,y)}(x, y),$$

is called the Poincaré map for Γ at (x_0, y_0).

The following theorem gives the formula of $\Pi'(0, 0)$.

Theorem 1.2 *[5] Let $\gamma(t)$ be a periodic solution of (1.1) of period T. Then the derivative of the Poincaré map $\Pi(s)$ along a straight line Σ normal to $\Gamma = \{(x, y) \in \mathbb{R}^2 | (x, y) = \gamma(t) - \gamma(0), 0 \leqslant t \leqslant T\}$ at $(x, y) = (0, 0)$ is given by*

$$\Pi'(0) = \exp \int_0^T \nabla \cdot (P(\gamma(t)), Q(\gamma(t)))dt.$$

Corollary 1.1 *[5] Under the hypotheses of 1.4, the periodic solution $\gamma(t)$ is a stable limit cycle if*

$$\int_0^T \nabla(P(\gamma(t)), Q(\gamma(t)))dt < 0$$

and it is an unstable limit cycle if

$$\int_0^T \nabla(P(\gamma(t)), Q(\gamma(t)))dt > 0.$$

It may be a stable, unstable or semi-stable limit cycle or it may belong to a continuous band of cycles of this quantity is zero.

Chapter 2

A cubic polynomial differential systems with one explicit limit cycle

2.1 Introduction

We consider the family of the polynomial differential system of the form

$$\begin{cases} \dot{x} = x - \gamma(2y - ax)Q(x, y), \\ \dot{y} = y - \gamma(-2x - ay)Q(x, y), \end{cases} \tag{2.1}$$

where $Q(x, y)$ is homogeneous polynomial of degrees 2 and γ,a are real constants.
We prove that these systems are integrable. Moreover, we determine sufficient conditions for a polynomial differential system to possess at most one limit cycles.
We defined the trigonometric polynomial

$$Q(\theta) = Q(\cos\theta, \sin\theta).$$

Lemma 2.1 *If $Q(\theta) \neq 0$ for all $\theta \in [0, 2\pi]$, then the origin $(0, 0)$ is the unique equilibrium point of the polynomial differential systems (2.1).*

Proof. We have

$$x\dot{y} - y\dot{x} = 2(x^2 + y^2)Q(x, y)\gamma$$

then, the equilibrium points of the polynomial differential system (2.1) if there exist, are located on the curve

$$\gamma(x^2 + y^2)Q(x, y) = 0.$$

In polar coordinate this equation become $\gamma Q(\theta)r^4 = 0$. Since $Q(\theta) \neq 0$ then $r^4 = 0$, hence the origin $(0,0)$ is the unique equilibrium point of the polynomial differential system (2.1).
❒

Proposition 2.1 *The curve* $U(x,y) = -2(x^2+y^2)Q(x,y)\gamma = 0$ *is an invariant algebraic of system* (2.1) *withe cofactor*

$$K(x,y) = 4 + 2\gamma Q(x,y) - \gamma\left(2y\frac{\partial Q(x,y)}{\partial x} - 2x\frac{\partial Q(x,y)}{\partial y} - 2aQ(x,y)\right).$$

Proof. We have

$$\begin{aligned}
\dot{x}\frac{\partial U}{\partial x} + \dot{y}\frac{\partial U}{\partial y} &= (x - \gamma(2y - ax)Q)\frac{\partial U}{\partial x} + (y - \gamma(-2x - ay)Q)\frac{\partial U}{\partial y} \\
&= (x - \gamma(2y - ax)Q)(-4x\gamma Q - 2\gamma(x^2+y^2)\frac{\partial Q}{\partial x}) \\
&+(y - \gamma(-2x - ay)Q)(-4y\gamma Q - 2\gamma(x^2+y^2)\frac{\partial Q}{\partial y}) \\
&= -4\gamma(x^2+y^2)Q + 4\gamma^2\left(x(2y - ax) + y(-2x - ay)\right)Q^2 \\
&+2\gamma^2(x^2+y^2)Q\left((2y - ax)\frac{\partial Q}{\partial x} + (-2x - ay)\frac{\partial Q}{\partial y}\right) \\
&-2\gamma(x^2+y^2)\left(x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y}\right) \\
&= 2\gamma^2(x^2+y^2)Q\left((2y - ax)\frac{\partial Q}{\partial x} - (2x + ay)\frac{\partial Q}{\partial y}\right) \\
&-2\gamma(x^2+y^2)\left(x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y}\right) \\
&-(4\gamma(x^2+y^2)Q)(1 + a\gamma Q).
\end{aligned}$$

Due to the Euler's theorem for homogeneous function $Q(x,y)$, we have

$$x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} = 2Q.$$

Then,

$$\begin{aligned}
\dot{x}\frac{\partial U}{\partial x} + \dot{y}\frac{\partial U}{\partial y} &= 2\gamma^2(x^2+y^2)Q\left((2y - ax)\frac{\partial Q}{\partial x} - (2x + ay)\frac{\partial Q}{\partial y}\right) \\
&-2\gamma(x^2+y^2)\left(x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y}\right) \\
&-(4\gamma(x^2+y^2)Q)(1 + a\gamma Q) \\
&= -2\gamma(x^2+y^2)Q\left(4 + 2\gamma Q - \gamma\left(2y\frac{\partial Q}{\partial x} - 2x\frac{\partial Q}{\partial y} - 2aQ\right)\right).
\end{aligned}$$

Therefore, $U = 0$ is an invariant algebraic curve of the polynomial differential system (2.1) with the cofactor

$$K(x,y) = 4 + 2\gamma Q - \gamma\left(2y\frac{\partial Q}{\partial x} - 2x\frac{\partial Q}{\partial y} - 2aQ\right).$$

This completes the proof of proposition 2.1. ❐

2.2 Integrability

The following theorem prove the integrability of system (2.1).

Theorem 2.1 *Consider a polynomial differential system (2.1). Then the following statements hold.*
1) If $\gamma \neq 0$ and $Q(\theta) \neq 0$ for all $\theta \in [0, 2\pi]$, then system (2.1) has the first integral

$$H(x,y) = -\gamma(x^2 + y^2)e^{-a\arctan\frac{y}{x}} + \int_0^{\arctan\frac{y}{x}} \frac{e^{-as}}{Q(s)}ds.$$

2) If $Q(\theta) \equiv 0$, then system (2.1) has the first integral

$$H(x,y) = \frac{y}{x}.$$

***Proof*. Proof of statement (1)** In order to prove our results we write the polynomial differential system (2.1) in polar coordinates (r,θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then the system (2.1) becomes

$$\begin{cases} \dot{r} = r + a\gamma Q(\theta)r^3, \\ \dot{\theta} = 2\gamma Q(\theta)r^2, \end{cases} \tag{2.2}$$

where $\dot{\theta} = \frac{d\theta}{dt}$, $\dot{r} = \frac{dr}{dt}$.
Taking as new independent variable the coordinate θ. The differential system (2.2) where $Q(\theta) \neq 0$ can be written as the equivalent differential equation

$$\frac{dr}{d\theta} = \frac{1}{2\gamma Q(\theta)r} + \frac{ar}{2}, \tag{2.3}$$

we not that the differential equation (2.3) is a Bernoulli equation.
Via the change of variable $\rho = r^2$, then the equation (2.3) is transformed into the linear equation

$$\frac{d\rho}{d\theta} = \frac{1}{\gamma Q(\theta)} + a\rho, \tag{2.4}$$

solving it we find the first integral.

$$H(\rho,\theta) = -\gamma\rho e^{-a\theta} + \int_0^{\theta} \frac{e^{-as}}{Q(s)}ds.$$

Then, the first integral of system (2.2) is

$$H(r,\theta) = -\gamma r^2 e^{-a\theta} + \int_0^{\theta} \frac{e^{-as}}{Q(s)} ds.$$

Going bake through the changes of variables $r^2 = x^2 + y^2$ and $\theta = \arctan \frac{y}{x}$, we obtain

$$H(x,y) = \gamma(x^2 + y^2)e^{-a \arctan \frac{y}{x}} - \int_0^{\arctan \frac{y}{x}} \frac{e^{-as}}{Q(s)} ds.$$

Hence statement 1 of Theorem 2.1 is proved.

Proof of statement (2) If $Q(\theta) \equiv 0$, for all $\theta \in \mathbb{R}$ then $Q(x,y) \equiv 0$, for all $x, y \in \mathbb{R}$, and

$$\begin{cases} \dot{x} = x, \\ \dot{y} = y, \end{cases} \tag{2.5}$$

where $\dot{x} = \dfrac{dx}{dt}$ and $\dot{y} = \dfrac{dy}{dt}$.
This system equivalent the differential equation

$$\frac{dx}{dy} = \frac{x}{y}. \tag{2.6}$$

The equation (2.6) is separable differential equation, then the solution of this equation is

$$y = Kx$$

where k is real constant, then the first integral in the variables (x, y) of the system (2.1) is

$$H(x,y) = \frac{y}{x}.$$

Hence statement (2) of Theorem 2.1 is proved.
❒

Remark 2.1 *The curve $H = h$ with $h \in \mathbb{R}$, which is formed by trajectories of the differential system* (2.1)*, in cartesian coordinates are written as*

$$\gamma(x^2 + y^2) = \gamma(x^2 + y^2)e^{-a \arctan \frac{y}{x}} - \int_0^{\arctan \frac{y}{x}} \frac{e^{-as}}{Q(s)} ds.$$

2.3 Periodic solution

Theorem 2.2 *If $Q(\theta)$ vanishes for some $\theta \in [0, 2\pi]$, then system* (2.1) *has no periodic solutions surrounding the origin.*

***Proof*.** If $\theta^* \in [0, 2\pi]$ is a zero of $Q(\theta) = 0$, then $(\sin\theta^* x - \cos\theta^* y)$ is a factor of $Q(x, y)$, and consequently the straight line

$$\sin\theta^* x - \cos\theta^* y = 0,$$

is invariant. It is well known that if $Q = 0$ is an invariant algebraic curve, then any factor of Q is also an invariant algebraic curve. So the straight line $\sin\theta^* x - \cos\theta^* y = 0$ through the origin of coordinates is invariant, i.e., formed by solutions of systems (2.1). Therefore, it can not be periodic solutions surrounding the origin. This completes the proof of Theorem 2.2. ❐

2.4 Existence and non existence of limit cycles

For the system (2.1), we will prove the existence or non existence of one limit cycle whose explicit expression will be given.

Theorem 2.3 *Consider a polynomial differential system* (2.1)*. Then*
1) If one of the following statements hold.
i) $\gamma Q(\theta) > 0$ *and* $a < 0$ *for all* $\theta \in [0, 2\pi]$,
ii) $\gamma Q(\theta) < 0$, *and* $a > 0$,
system (2.1) *has an explicit hyperbolic limit cycle given in polar coordinates* (r, θ) *by*

$$r(\theta, r_*) = \sqrt{e^{a\theta}(r_*^2 - f(\theta))},$$

where $r_* = \sqrt{\dfrac{e^{a2\pi}}{1 - e^{a2\pi}} f(2\pi)}$, *and* $f(\theta) = \displaystyle\int_0^\theta \frac{1}{\gamma Q(s)} e^{-as} ds.$
2) if one of the following statements hold.
a) $\gamma Q(\theta) > 0$, *and* $a > 0$ *for all* $\theta \in [0, 2\pi]$,
b) $\gamma Q(\theta) < 0$, *and* $a < 0$ *for all* $\theta \in [0, 2\pi]$,
system (2.1) *has no limit cycle.*

***Proof*.** **proof of statement (1)**: Use the notation and the expressions of the proof of Theorem 2.1 we have the general solutions of linear equation (2.4) is

$$\rho(\theta, h) = e^{a\theta}\left(h + \int_0^\theta \frac{1}{\gamma Q(s)} e^{-as} ds\right).$$

And the general solution of linear equation (2.3) is

$$r^2(\theta, h) = e^{a\theta}\left(h + \int_0^\theta \frac{1}{\gamma Q(s)} e^{-as} ds\right), \tag{2.7}$$

where $h \in \mathbb{R}$.
We remark that the solution $r(\theta, r_0)$ such as $r(0, r_0) = r_0 > 0$, corresponds to the value $h = r_0^2$ provided a rewriting of the general solution $r(\theta, r_0)$ of the differential equation (2.3) as

$$r^2(\theta, r_0) = e^{a\theta}\left(r_0^2 + \int_0^\theta \frac{1}{\gamma Q(s)} e^{-as} ds\right).$$

where $r_0 = r(0)$. A periodic solution of system (2.1) must satisfy the condition $r^2(2\pi, r_0) = r^2(0, r_0)$, where

$$r^2(2\pi, r_0) = e^{a2\pi}\left(r_0^2 + \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds\right)$$

and

$$r^2(0, r_0) = r_0^2.$$

Then, the condition $r^2(2\pi, r_0) = r^2(0, r_0)$ equivalent

$$e^{a2\pi} r_0^2 + e^{a2\pi} \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds = r_0^2,$$

this imply that

$$r_0^2(1 - e^{a2\pi}) = e^{a2\pi} \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds,$$

then

$$r_0^2 = \frac{e^{a2\pi}}{1 - e^{a2\pi}} \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds$$

and there are two differences values with the property $r(2\pi, r_0) = r_0$, so one of them is equal to

$$r_0 = -\sqrt{\frac{e^{a2\pi}}{1 - e^{a2\pi}} \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds}$$

and we do not consider this case because $r_0 < 0$, we only take into consideration the following value $r_0 = r_*$ which satisfies $r(2\pi, r_*) = r_* > 0$

$$r_* = \sqrt{\frac{e^{a2\pi}}{1 - e^{a2\pi}} f(2\pi)},$$

where

$$f(2\pi) = \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds.$$

After the substitution of this value r_* into $r(\theta, r_0)$ we obtain

$$r(\theta, r_*) = \sqrt{e^{a\theta}\left(\frac{e^{a2\pi}}{1-e^{a2\pi}}\int_0^{2\pi}\frac{1}{\gamma Q(s)}e^{-as}ds + \int_0^{\theta}\frac{1}{\gamma Q(s)}e^{-as}ds\right)}. \quad (2.8)$$

Since $\gamma Q(\theta) > 0$, then

$$f(\theta) = \int_0^{\theta}\frac{1}{\gamma Q(s)}e^{-as}ds > 0,$$

for all $\theta \in [0, 2\pi]$, in particular $f(2\pi) = \displaystyle\int_0^{2\pi}\frac{1}{\gamma Q(s)}e^{-as}ds > 0,$ and $a < 0$, then $1 - e^{2a\pi} > 0$. So it follows that

$$r_* = \sqrt{\frac{e^{a2\pi}}{1-e^{a2\pi}}f(2\pi)} > 0,$$

then system (2.1) can have one limit cycle.

Strict positivity: We have

$$r^2(\theta, r_*) = e^{a\theta}\left(\frac{e^{a2\pi}}{1-e^{a2\pi}}\int_0^{2\pi}\frac{1}{\gamma Q(s)}e^{-as}ds + \int_0^{\theta}\frac{1}{\gamma Q(s)}e^{-as}ds\right).$$

Since $\dfrac{e^{a2\pi}}{1-e^{a2\pi}} > 0$, and $\gamma Q(\theta) > 0$, then

$$\begin{aligned} r^2(\theta, r_*) &\geqslant e^{a\theta}\left(\begin{array}{c}\dfrac{e^{a2\pi}}{1-e^{a2\pi}}\displaystyle\int_0^{\theta}\frac{1}{\gamma Q(s)}e^{-as}ds \\ +\displaystyle\int_0^{\theta}\frac{1}{\gamma Q(s)}e^{-as}ds\end{array}\right). \\ &= e^{a\theta}\left(\frac{1}{1-e^{a2\pi}}\int_{\theta}^{\theta}\frac{1}{\gamma Q(s)}e^{-as}ds\right) > 0, \end{aligned}$$

because $\gamma Q(\theta) > 0$, and $a < 0$.

Periodicity: Let

$$g(\theta) = e^{a\theta}\left(r_*^2 + \int_0^{\theta}\frac{1}{\gamma Q(s)}e^{-as}ds\right).$$

$\theta \in [0, 2\pi]$, then

$$g(\theta + 2\pi) = e^{a(\theta+2\pi)}\left(r_*^2 + \int_0^{\theta+2\pi}\frac{1}{\gamma Q(s)}e^{-as}ds\right).$$

Since $r_*^2 = \dfrac{e^{2a\pi}}{1-e^{2a\pi}} f(2\pi)$, then

$$g(\theta+2\pi) = e^{a\theta} e^{a2\pi} \left(\left(\frac{e^{a2\pi}}{1-e^{a2\pi}} + 1 \right) \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds + \int_{2\pi}^{\theta+2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds \right).$$

In the integral $\displaystyle\int_{2\pi}^{\theta+2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds$, we make the change of variable $u = s - 2\pi$, we obtain

$$\begin{aligned} \int_{2\pi}^{\theta+2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds &= \int_0^{\theta} \frac{1}{\gamma Q(u+2\pi)} e^{-a(u+2\pi)} du \\ &= e^{-a2\pi} \int_0^{\theta} \frac{e^{-a(u+2\pi)}}{\gamma Q(u+2\pi)} du. \end{aligned}$$

We have $Q(s+2\pi) = Q(s)$ because $Q(\cos\theta, \sin\theta)$ is homogeneous function of degrees 2 of two variables $\cos\theta$ and $\sin\theta$, where $\cos\theta$ and $\sin\theta$ are continuous 2π-periodic functions, then $\cos(\theta+2\pi) = \cos\theta$. and $\sin(\theta+2\pi) = \sin\theta.$ Then,

$$\begin{aligned} g(\theta+2\pi) &= e^{a\theta} e^{a2\pi} \left(\begin{array}{c} \left(\dfrac{e^{a2\pi}}{1-e^{a2\pi}} + 1 \right) \displaystyle\int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds \\ + e^{-a2\pi} \displaystyle\int_0^{\theta} \frac{1}{\gamma Q(s)} e^{-as} ds \end{array} \right) \\ &= e^{a\theta} \left(\frac{e^{a2\pi}}{1-e^{a2\pi}} f(2\pi) + \int_0^{\theta} \frac{1}{\gamma Q(s)} e^{-as} ds \right) \\ &= g(\theta). \end{aligned}$$

Hence g is 2π-periodic.

Finally, it remains to show that $r(\theta, r_*)$ is hyperbolic limit cycle. For that we consider $r(\theta, r_*)$, and introduce the Poincaré return map $\lambda \longrightarrow \Pi(2\pi, \lambda) = r(2\pi, \lambda)$.
We compute $\dfrac{d\Pi}{d\lambda}(2\pi, \lambda) \mid_{\lambda=r_*}$. So

$$\frac{d\Pi}{d\lambda}(2\pi, \lambda) \mid_{\lambda=r_*} = r_* e^{a2\pi} \frac{1}{\sqrt{e^{a2\pi} \left(\lambda^2 + \displaystyle\int_0^{2\pi} \frac{e^{-as}}{\gamma Q(s)} ds \right)}},$$

by replacing r_* by its value given by $r_* = \sqrt{\dfrac{e^{2a\pi}}{1-e^{2a\pi}} f(2\pi)}$, and after some calculation, we get

$$\frac{d\Pi}{d\lambda}(2\pi, \lambda) \mid_{\lambda=r_*} = e^{a2\pi}. \tag{2.9}$$

1) If $a > 0$, then $e^{a2\pi} > 1$. Therefore, $r(\theta, r_*)$ is an unstable and hyperbolic limit cycle of the differential equation (2.3).
2) If $a < 0$, then $e^{a2\pi} < 1$. Therefore, $r(\theta, r_*)$ is an stable and hyperbolic limit cycle of the differential equation (2.3) .

Proof of statement (2): $r_0 > 0$ we only take into consideration the following value $r_0 = r_*$ which satisfies $r(2\pi, r_*) = r_*$

$$r_* = \sqrt{\frac{e^{a2\pi}}{1 - e^{a2\pi}} f(2\pi)},$$

where $f(2\pi) = \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds$. After the substitution of this value r^* into $r(\theta, r_0)$ we obtain

$$r(\theta, r_*) = \sqrt{e^{a\theta} \left(\frac{e^{a2\pi}}{1 - e^{a2\pi}} \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds + \int_0^{\theta} \frac{1}{\gamma Q(s)} e^{-as} ds \right)}.$$

Since $\gamma Q(\theta) < 0$, then $f(\theta) = \int_0^{\theta} \frac{1}{\gamma Q(s)} e^{-as} ds < 0$ and $a > 0$ then $1 - e^{2a\pi} < 0$, so it follows that

$$r_* = \sqrt{\frac{e^{a2\pi}}{1 - e^{a2\pi}} f(2\pi)} > 0,$$

then system (2.1) can have one limit cycle.

Proof of statement (3): If $\gamma Q(\theta) > 0$ for all $\theta \in \mathbb{R}$ then

$$f(\theta) = \int_0^{\theta} \frac{1}{\gamma Q(s)} e^{-as} ds > 0,$$

for all $\theta \in [0, 2\pi]$, in particular

$$f(2\pi) = \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds > 0.$$

If $a > 0$, so it follows that $1 - e^{a2\pi} < 0$

$$r_*^2 = \frac{e^{a2\pi}}{1 - e^{a2\pi}} f(2\pi) < 0,$$

then r_*does not exist, and the system (2.1) has no limit cycle.

Proof of statement(4): If $\gamma Q(\theta) < 0$ for all $\theta \in \mathbb{R}$, then

$$f(\theta) = \int_0^{\theta} \frac{1}{\gamma Q(s)} e^{-as} ds < 0,$$

for all $\theta \in [0, 2\pi]$, in particular

$$f(2\pi) = \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds < 0.$$

If $a < 0$. So it follows that $1 - e^{2a\pi} > 0$, thus

$$r_*^2 = \frac{e^{a2\pi}}{1 - e^{a2\pi}} f(2\pi) < 0,$$

then r_* does not exist, and the system (2.1) has no limit cycle. Hence the Theorem 2.4 is proved. ❐

2.5 Algebraic and non-algebraic limit cycle

In this section, we give the conditions of existence of an algebraic limit cycle or non algebraic and their exact expression for the polynomial differential system (2.1).

Theorem 2.4 *Consider a polynomial differential system* (2.1) *with* $\gamma Q(\theta) > 0$ *for all* $\theta \in [0, 2\pi]$ *. Then the following statements hold.*
i) If $Q(\theta) \not\equiv \omega$ *, where* $\omega \in \mathbb{R}$ *, then the limit cycle of system* (2.1) *is non algebraic.*
ii) If $Q(\theta) \equiv \omega$*, where* $\omega \in \mathbb{R}$ *, then the limit cycle of system* (2.1) *is algebraic.*

Proof**.** **Proof of statement (i)**: By Theorem 2.3 the curve $r(\theta)$ where $Q(\theta) \neq constant$ defined by the limit cycle of system (2.1) is

$$r^2(\theta, r_*) = e^{a\theta} \left(\frac{e^{2a\pi}}{1 - e^{a2\pi}} \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds + \int_0^{\theta} \frac{1}{\gamma Q(s)} e^{-as} ds \right),$$

is not algebraic, due the expression

$$e^{a\theta} \left(\frac{e^{a2\pi}}{1 - e^{a2\pi}} \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds \right).$$

More precisely in cartesian coordinates $r^2(\theta, r_*) = x^2 + y^2$ and $\theta = \arctan(\frac{y}{x})$ the curve defined by this limit cycle is

$$f(x, y) = x^2 + y^2 - e^{\arctan \frac{y}{x}} \left(\begin{array}{c} \frac{e^{a2\pi}}{1 - e^{a2\pi}} \int_0^{2\pi} \frac{1}{\gamma Q(s)} e^{-as} ds \\ + \int_0^{\arctan \frac{y}{x}} \frac{1}{\gamma Q(s)} e^{-as} ds \end{array} \right) = 0.$$

But there is no integer n for which both $\dfrac{\partial^n f}{\partial x^n}$ and $\dfrac{\partial^n f}{\partial y^n}$ vanish identically to be convinced by this fact one has to compute for example $\dfrac{\partial f}{\partial x}$, that is

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x + \frac{ye^{a\arctan\frac{y}{x}}}{x^2+y^2}\left(\begin{array}{c}\dfrac{e^{a2\pi}}{1-e^{a2\pi}}\displaystyle\int_0^{2\pi}\frac{1}{\gamma Q(s)}e^{-as}ds \\ +\displaystyle\int_0^{\arctan\frac{y}{x}}\frac{1}{\gamma Q(s)}e^{-as}ds\end{array}\right) \\ &\quad -\frac{y}{\gamma Q(\arctan\frac{y}{x})(x^2+y^2)}.\end{aligned}$$

Since $f(x,y)$ appears again, it will remains in any order of derivation therefore the curve $f(x,y)=0$ is non algebraic and the limit cycle will also be non algebraic.

Proof of statement(ii): If $Q(\theta)\equiv\omega$, we have

$$\begin{aligned} r^2(\theta,r_*) &= e^{a\theta}\left(\frac{e^{a2\pi}}{1-e^{a2\pi}}\int_0^{2\pi}\frac{1}{\gamma Q(s)}e^{-as}ds+\int_0^{\theta}\frac{1}{\gamma Q(s)}e^{-as}ds\right) \\ &= e^{a\theta}\left(\frac{e^{a2\pi}}{1-e^{a2\pi}}\int_0^{2\pi}\frac{1}{\gamma\omega}e^{-as}ds+\int_0^{\theta}\frac{1}{\gamma\omega}e^{-as}ds\right).\end{aligned}$$

We calculate $f(2\pi)$ and $f(\theta)$

$$f(2\pi)=\int_0^{2\pi}\frac{1}{\gamma\omega}e^{-as}ds=\frac{1}{\gamma\omega}\left(-\frac{1}{a}e^{-a2\pi}+\frac{1}{a}\right)$$

and

$$f(\theta)=\int_0^{\theta}\frac{1}{\gamma\omega}e^{-as}ds=\frac{1}{\gamma\omega}\left(-\frac{1}{a}e^{-a\theta}+\frac{1}{a}\right)$$

then,

$$\begin{aligned} r^2(\theta,r_*) &= \frac{e^{a\theta}}{\gamma\omega}\left(\frac{e^{a2\pi}}{1-e^{a2\pi}}\left(\frac{-1}{a}e^{-a2\pi}+\frac{1}{a}\right)+\left(\frac{-1}{a}e^{-a\theta}+\frac{1}{a}\right)\right) \\ &= \frac{e^{a\theta}}{\gamma\omega}\left(\frac{-1}{a(1-e^{a2\pi})}+\frac{e^{a2\pi}}{a(1-e^{a2\pi})}-\frac{e^{-a\theta}}{a}+\frac{1}{a}\right) \\ &= \frac{e^{a\theta}}{\gamma\omega}\left(\frac{-(1-e^{a2\pi})}{a(1-e^{a2\pi})}-\frac{e^{-a\theta}}{a}+\frac{1}{a}\right) \\ &= \frac{e^{a\theta}}{\gamma\omega}\left(\frac{-1}{a}-\frac{e^{-a\theta}}{a}+\frac{1}{a}\right)=\frac{-1}{a\gamma\omega},\end{aligned}$$

we precisely in cartesian coordinates

$$x^2+y^2=-\frac{1}{a\gamma\omega},$$

in the latter this curve is a polynomial and thus is algebraic limit cycle of system (2.1). This completes the proof of Theorem 2.4. ❒

2.6 Special Cases

In this section we are interested in studying the limit cycles of system (2.1) when $Q(x, y) = \beta x^2 + cy^2 + bxy$ where β, b and c are real constants. Moreover, we determine sufficient conditions for a polynomial differential system (2.1) to possess at most one explicit limit cycle.

Lemma 2.2 *Consider a polynomial differential system* (2.1)

$$\begin{cases} \dot{x} = x - \gamma(2y - ax)(\beta x^2 + bxy + cy^2), \\ \dot{y} = y - \gamma(-2x - ay)(\beta x^2 + bxy + cy^2), \end{cases} \tag{2.10}$$

where β, b, c are constants in $\mathbb{R}$. Then the following statements hold.
1) In each one of the followings cases

I) $\mid \beta + c \mid > \mid \beta - c \mid + \mid b \mid, \mid \beta + c > 0, \gamma > 0$ *and* $a < 0.$

II) $\mid \beta + c \mid > \mid \beta - c \mid + \mid b \mid, \mid \beta + c < 0, \gamma < 0$ *and* $a > 0.$

System (2.1) *has exactly one limit cycle.*

Moreover if $\beta \neq c$, and $b \neq 0$, this limit cycle is non algebraic and if $\beta = c$ and $b = 0$ the limit cycle is algebraic whose expression in cartesian coordinates is

$$x^2 + y^2 = -\frac{1}{a\gamma\beta}.$$

2) In each one of the followings cases

III) $\mid \beta + c \mid > \mid \beta - c \mid + \mid b \mid, \mid \beta + c > 0, \gamma < 0$ *and* $a < 0.$

IV) $\mid \beta + c \mid > \mid \beta - c \mid + \mid b \mid, \mid \beta + c < 0, \gamma > 0$ *and* $a > 0.$

System (2.1) *has no limit cycle.*

***Proof*.** **Proof of statement (1)** : In polar coordinate (r, θ); $Q(x, y)$ reads as

$$\begin{aligned} Q(\theta) &= \beta\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta \\ &= \frac{1}{2}\left((\beta + c) + (\beta - c)\cos 2\theta + b\sin 2\theta\right). \end{aligned}$$

If
$\mid \beta + c \mid > \mid \beta - c \mid + \mid b \mid, \beta + c > 0, \gamma > 0$ and $a < 0.$
Or

$\mid \beta + c \mid > \mid \beta - c \mid + \mid b \mid$, $\beta + c < 0, \gamma < 0$ and $a > 0.$
We have

$$\gamma Q(\theta) > 0,$$

through the Theorem 2.3, system (2.1) has one limit cycle.
If $\beta = c$ and $b = 0$ then,

$$Q(\theta) = \beta$$

for all $\theta \in \mathbb{R}$ is constant, through the Theorem 2.4, the limit cycle is algebraic.
And because $\omega = \beta$ then whose expression is

$$x^2 + y^2 = -\frac{1}{a\gamma\beta}.$$

And if the opposite $\beta \neq c$ and $b \neq 0$, then

$$Q(\theta) = \frac{1}{2}\left((\beta + c) + (\beta - c)\cos 2\theta + b \sin 2\theta\right).$$

is a homogeneous trigonometric polynomial in the variables $\cos\theta$ and $\sin\theta$ and is non identically a non-zero real constants in $\mathbb{R}$, then through the Theorem 2.4, the limit cycle is non algebraic.
Proof of statement (2): If
$\mid \beta + c \mid > \mid \beta - c \mid + \mid b \mid, \mid \beta + c > 0, \gamma < 0$ and $a < 0.$
Or
$\mid \beta + c \mid > \mid \beta - c \mid + \mid b \mid, \mid \beta + c < 0, \gamma > 0$ and $a > 0.$
We have

$$\gamma Q(\theta) < 0,$$

through the Theorem 2.3 ,system (2.1) has no limit cycle. Hence the Lemma 2.2 is proved.
❒

2.7 Examples

Example 2.1 *If we take $\beta = c = 1, b = 0$ and $\gamma = 1, a = 2$, then system (2.1) reads*

$$\begin{cases} \dot{x} = x - (2y + x)(x^2 + y^2), \\ \dot{y} = y - (-2x + y)(x^2 + y^2). \end{cases} \tag{2.11}$$

So the first hypothesis of Lemma 2.2 is satisfied and hence the system (2.11) has exactly one algebraic limit cycle whose expression in cartesian coordinates (x, y) is

$$x^2 + y^2 = \frac{1}{2}.$$

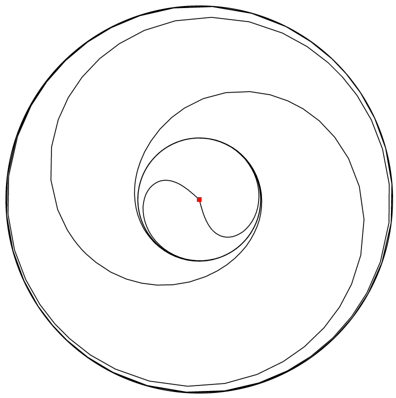

Figure 2.1: Algebraic limit cycle of the differential system (2.11)

Example 2.2 *If we take* $\beta = 4, b = c = 2$ *and* $\gamma = 1, a = -2$*, then system* (2.1) *reads*

$$\begin{cases} \dot{x} = x - (2y - 2x)(4x^2 + 2xy + 2y^2), \\ \dot{y} = y - (-2x - 2y)(4x^2 + 2xy + 2y^2), \end{cases} \tag{2.12}$$

it is easy to verify that the first conditions of Lemma 2.2 *are satisfied and hence the system* (2.12) *has exactly one non algebraic limit cycle whose expression in polar coordinates* (r, θ) *is*

$$r(\theta, r_*) = \sqrt{e^{2\theta}\left(\begin{array}{c} \frac{e^{4\pi}}{1 - e^{4\pi}} \int_0^{2\pi} \frac{2}{5 + 3\cos 2\theta + \sin 2\theta} e^{-2s} ds \\ + \int_0^{\theta} \frac{2}{5 + 3\cos 2\theta + \sin 2\theta} e^{-2s} ds \end{array}\right)}.$$

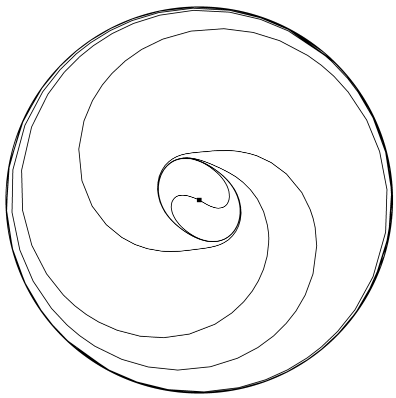

Figure 2.2: Non algebraic limit cycle of the differential system (2.12)

Example 2.3 *If we take* $\beta = -6, b = 1, c = -8$ *and* $\gamma = 1, a = -2$*, then system*

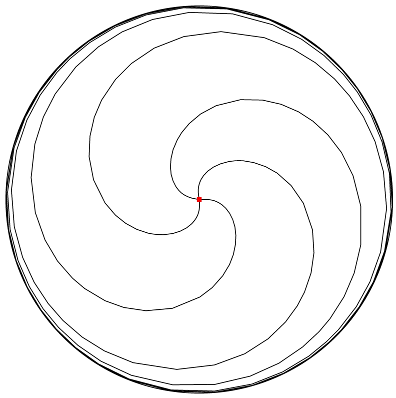

Figure 2.3: Phase portrait of the differential system (2.13)

(2.1) *reads*

$$\begin{cases} \dot{x} = x - (2y + 2x)(-6x^2 + xy - 8y^2), \\ \dot{y} = y - (-2x + 2y)(-6x^2 + xy - 8y^2), \end{cases} \tag{2.13}$$

it is easy to verify that the second condition of Lemma 2.2 is satisfied and hence the system (2.13) has no limit cycle.

Chapter 3

A Quintic polynomial differential systems with two explicit limit cycles

3.1 Introduction.

We consider the family of the polynomial differential system of the form

$$\begin{cases} \dot{x} = x - \left(\gamma\left(2y - ax\right) + \alpha\left(x^2+y^2\right)\left(ax - 4y\right)\right) Q\left(x,y\right), \\ \dot{y} = y - \left(-\gamma\left(2x + ay\right) + \alpha\left(x^2+y^2\right)\left(4x + ay\right)\right) Q\left(x,y\right), \end{cases} \tag{3.1}$$

where $Q(x, y)$ is homogeneous polynomial of degrees 2 and γ, α, a real constants. We prove that these systems are integrable. Moreover, we determine sufficient conditions for a polynomial differential system to possess at most two limit cycles. Concrete examples exhibiting the applicability of our result are introduced.

3.2 Equilibrium points

Lemma 3.1 *If $Q(x, y) \neq 0$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$, then the equilibrium points of system (3.1) are present on the curve*

$$\left(x^2+y^2\right)\left(2\alpha x^2 + 2\alpha y^2 - \gamma\right) = 0.$$

Proof. We have

$$\dot{x}y - \dot{y}x = 2\left(x^2+y^2\right)\left(2\alpha x^2 + 2\alpha y^2 - \gamma\right) Q\left(x,y\right) = 0,$$

thus, the equilibrium points of system (3.1) if exists, must present in the curve

$$2\left(x^2+y^2\right)\left(2\alpha x^2 + 2\alpha y^2 - \gamma\right) Q\left(x,y\right) = 0.$$

From the condition $Q(x,y) \neq 0$ for all $(x,y) \in \mathbb{R} \times \mathbb{R}$ then, the equilibrium points of system (3.1) are present on the curve

$$\left(x^2+y^2\right)\left(2\alpha x^2+2\alpha y^2-\gamma\right)=0,$$

so the equilibrium points of system (3.1) are present on the curve

$$\left(2\alpha x^2+2\alpha y^2-\gamma\right)=0,$$

and the origin of coordinates.
Since the Jacobian matrix of the vector field defined in (3.1) at $(0,0)$ is given by:

$$J(0,0)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The origin is an unstable node because its eigenvalues are $\lambda_1=\lambda_2=1>0.$ ❐

3.3 integrability

Theorem 3.1 *Consider a polynomial differential system* (3.1)*. Then the following statement hold.*
If $\alpha a \gamma \neq 0$ *and* $Q(\theta) \neq 0$ *for all* $\theta \in [0, 2\pi)$*, then system* (3.1) *has the first integral*

$$H(x,y)=\left(x^2+y^2\right)\left(\alpha\left(x^2+y^2\right)-\gamma\right)e^{-a\arctan\frac{y}{x}}+\int_0^{\arctan\frac{y}{x}}\frac{e^{-as}}{Q(s)}ds.$$

Proof. In polar coordinates system (3.1) reads as

$$\begin{cases} \dot{r}=r+Q(\theta)ar^3\gamma-Q(\theta)ar^5\alpha, \\ \dot{\theta}=-2Q(\theta)r^2\left(2r^2\alpha-\gamma\right). \end{cases} \tag{3.2}$$

Let

$$H(x,y)=\left(x^2+y^2\right)\left(\alpha\left(x^2+y^2\right)-\gamma\right)e^{-a\arctan\frac{y}{x}}+\int_0^{\arctan\frac{y}{x}}\frac{e^{-as}}{Q(s)}ds.. \tag{3.3}$$

In polar coordinates, (3.3) reads as

$$H(x,y)=\left(r^2\right)\left(\alpha\left(r^2\right)-\gamma\right)e^{-a\theta}+\int_0^{\theta}\frac{e^{-as}}{Q(s)}ds.$$

Then, the derivative of H with respect to r is

$$\frac{dH(r,\theta)}{dr}=2e^{-a\theta}r\left(-\gamma+2r^2\alpha\right).$$

And the derivative of H with respect to θ is

$$\frac{dH(r,\theta)}{d\theta} = \left(\frac{1}{Q(\theta)}e^{-a\theta}\left(-Q(\theta)a\alpha r^4 + Q(\theta)a\gamma r^2 + 1\right)\right).$$

By replacing the expressions of derivatives of H with respect to and r in

$$\frac{dH}{dt} = \dot{r}\frac{\partial H(r,\theta)}{\partial r} + \dot{\theta}\frac{\partial H(r,\theta)}{\partial \theta},$$

it follows that :

$$\begin{aligned}
\frac{dH}{dt} &= \left(r + Q(\theta)ar^3\gamma - Q(\theta)ar^5\alpha\right)\left(2e^{a\theta}r\left(-\gamma + 2r^2\alpha\right)\right) \\
&+ \left(-2Qr^2\left(2r^2\alpha - \gamma\right)\right)\left(\frac{1}{Q(\theta)}e^{a\theta}\left(-Q(\theta)a\alpha r^4 + Q(\theta)a\gamma r^2 + 1\right)\right) \\
&\equiv 0.
\end{aligned}$$

So $H(r,\theta)$ is a first integral of system (3.2). Consequently $H(x,y)$ is a first integral of system (3.1). ❒

3.4 Non existence of limit cycles

Theorem 3.2 *The quintic polynomial differential system* (3.1) *has no limit cycle when the one of the following conditions is assumed:*

1) $\gamma^2 + \dfrac{4\alpha e^{2\pi a}}{e^{2\pi a} - 1}g(2\pi) < 0.$

2) $\alpha\gamma < 0$ *and* $-\gamma^2 < \dfrac{4\alpha e^{2\pi a}}{e^{2\pi a} - 1}g(2\pi) < 0.$

Proof. The differential system (3.2) where $-2Q(\theta)r^2(2r^2\alpha - \gamma) \neq 0$ can be written as the equivalent differential equation

$$\frac{dr}{d\theta} = -\frac{r + Q(\theta)ar^3\gamma - Q(\theta)ar^5\alpha}{2Q(\theta)r^2(2r^2\alpha - \gamma)}. \tag{3.4}$$

Note that since $\dot{\theta}(t)$ is positive for all t, the orbit $r(\theta)$ of the differential equation (3.4) has preserved their orientation with respect to the orbits $(r(t), \theta(t))$ or $(x(t), y(t))$ the differential systems (3.1) or (3.2).
Via the change of variables $(r^2(\alpha r^2 - \gamma)) = \rho$, we have

$$\frac{d\left(r^2\left(\alpha r^2 - \gamma\right)\right)}{dr} = -2r\left(\gamma - 2r^2\alpha\right).$$

Thus

$$-2r\left(\gamma - 2r^2\alpha\right)\frac{dr}{d\theta} = -\frac{r + Q(\theta)ar^3\gamma - Q(\theta)ar^5\alpha}{2Q(\theta)r^2(2r^2\alpha - \gamma)}\left(-2r\left(\gamma - 2r^2\alpha\right)\right),$$

then and the differential equation (3.4) becomes the linear differential equation

$$\frac{d\rho}{d\theta} = a\rho(\theta) - \frac{1}{Q(\theta)}. \tag{3.5}$$

The general solution of linear equation (3.5) is

$$\rho(\theta) = e^{a\theta}\left(h - \int_0^\theta \frac{e^{-as}}{Q(s)} ds\right).$$

Where $h \in \mathbb{R}$

Consequently, the implicit form of the solution of the differential equation (3.4) is

$$F(r,\theta) = \left(r^2\left(\alpha r^2 - \gamma\right)\right) - e^{a\theta}\left(h - \int_0^\theta \frac{e^{-as}}{Q(s)} ds\right) = 0.$$

Notice the system (3.1) has a periodic orbit if and only is equation (3.4) has a strictly positive $2\pi-$ periodic solution $r(\theta)$. This, moreover, this is equivalent to the existence of a solution of (3.4) that satisfies $r(0, r_*) = r(2\pi, r_*)$ and $r(\theta, r_*) > 0$ for any in $[0, 2\pi]$. we remark that the solution $r(\theta, r_0)$ of the differential equation (3.4) such that $r(\theta, r_0) = r_0 > 0$, we have since $(r^2(\alpha r^2 - \gamma)) = \rho$, then there are two different values with the property $r(2\pi, r_0) = r_0$, so one of them is equal to

$$\begin{cases} r_1^2(\theta) = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right), \\ r_2^2(\theta) = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right). \end{cases} \tag{3.6}$$

Since the change of variable is $(r^2(\alpha r^2 - \gamma)) = \rho$, it is clear that $r_i(2\pi, r_0) = r_i(0, r_0)$ if and only if $\rho(2\pi, r_0) = \rho(0, r_0) = \rho_0$. To go a steep further, we remark that the solution such as $r(0, r_0) = r_0 > 0$, corresponds to the value

$$h = \rho_0 = \rho(0, r_0) = \left(r_0^2\left(\alpha r_0^2 - \gamma\right)\right),$$

we have

$$\rho(2\pi, r_0) = e^{a2\pi}\left(\left(r_0^2\left(\alpha r_0^2 - \gamma\right)\right) - \int_0^{2\pi} \frac{e^{-as}}{Q(s)} ds\right).$$

Then, the condition $\rho(2\pi, r_0) = \rho(0, r_0)$ implies that

$$\left(r_0^2\left(\alpha r_0^2 - \gamma\right)\right) = e^{a2\pi}\left(\left(r_0^2\left(\alpha r_0^2 - \gamma\right)\right) - \int_0^{2\pi} \frac{e^{-as}}{Q(s)} ds\right),$$

the solution of this last equation are

$$r_{*i}^2 = \frac{1}{2\alpha}\left(\gamma \pm \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a} - 1} g(2\pi)}\right), \tag{3.7}$$

where $g(2\pi) = \displaystyle\int_0^{2\pi} \frac{e^{-as}}{Q(s)} ds.$

1) If $\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) < 0$, then r_{*i}^2 are not exists. Then, system (3.1) has no limit cycle.

2) If $\alpha > 0, \gamma < 0$ and $-\gamma^2 < \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) < 0$ then

$$\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) > 0,$$

and

$$\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)} < \sqrt{\gamma^2},$$

we have $\frac{1}{2\alpha} > 0$ and

$$\frac{1}{2\alpha}\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)} < \frac{1}{2\alpha}\sqrt{\gamma^2} = \frac{-1}{2\alpha}\gamma,$$

then

$$\begin{aligned} r_{1*}^2 &= \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)}\right) \\ &< \frac{1}{2\alpha}(\gamma - \gamma) = 0. \end{aligned}$$

On the other hand we have
If $\alpha > 0, \gamma < 0$ and $-\gamma^2 < \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) < 0$

$$\frac{1}{2\alpha}\gamma < 0 < \frac{1}{2\alpha}\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)},$$

thus

$$r_{2*}^2 = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)}\right) < 0.$$

-If $\alpha < 0, \gamma > 0$ and $-\gamma^2 < \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) < 0$, we have $\frac{1}{2\alpha} < 0$ and

$$\frac{1}{2\alpha}\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)} > \frac{1}{2\alpha}\sqrt{\gamma^2} = \frac{1}{2\alpha}\gamma,$$

then

$$r_{1*}^2 = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)}\right) < \frac{\gamma}{\alpha} < 0.$$

On the other hand we have

$$\gamma = \sqrt{\gamma^2} > \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g\left(2\pi\right)},$$

thus

$$\frac{1}{2\alpha}\gamma < \frac{1}{2\alpha}\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{e^{2\pi a}-1} g\left(2\pi\right)},$$

so

$$r_{2*}^2 = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{\left(e^{2\pi a}-1\right)} g\left(2\pi\right)}\right) < 0.$$

Then, system (3.1) has no limit cycle. Hence the Theorem 3.2 is proved. ❒

3.5 Existence of two limit cycles

Theorem 3.3 *Consider polynomial differential system* (3.1). *Then the following statements hold*

a) If $\alpha < 0, \gamma < 0$ *and one of the following conditions are holds.*

1) $a < 0, Q\left(\theta\right) < 0$ *for all* θ *and* $\frac{e^{2\pi a}}{e^{2\pi a}-1} g\left(2\pi\right) < \left(-\frac{\gamma^2}{4\alpha}\right)$.

2) $a > 0, Q\left(\theta\right) > 0$ *for all* $\theta, e^{a2\pi}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g\left(2\pi\right)\right) < \left(-\frac{\gamma^2}{4\alpha}\right)$.

b) If $\alpha > 0, \gamma > 0$ *and and one of the following conditions are holds.*

1) $a < 0, Q\left(\theta\right) > 0$ *for all* θ *and* $\frac{e^{2\pi a}}{e^{2\pi a}-1} g\left(2\pi\right) > \frac{-\gamma^2}{4\alpha}$.

2) $a > 0, Q\left(\theta\right) < 0$ *for all* θ *and* $\frac{e^{2\pi a}}{e^{2\pi a}-1} g\left(2\pi\right) > \frac{-\gamma^2}{4\alpha}$.

Then, system (3.1) *has two explicit limit cycle, given in polar coordinates* (r, θ) *by*

$$\begin{cases} r_1^2\left(\theta\right) = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho\left(\theta\right)}\right), \\ r_2^2\left(\theta\right) = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho\left(\theta\right)}\right), \end{cases}$$

where $g\left(\theta\right) = \int_0^\theta \frac{e^{-as}}{Q\left(s\right)} ds$ *and*

$$\rho\left(\theta\right) = e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g\left(2\pi\right) - g\left(\theta\right)\right),$$

and

$$\begin{cases} r_{1*}^2 = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{\left(e^{2\pi a}-1\right)} g\left(2\pi\right)}\right), \\ r_{2*}^2 = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{\left(e^{2\pi a}-1\right)} g\left(2\pi\right)}\right). \end{cases}$$

Next Lemma collects some results which we need to show the statements of theorem 3.3.

Lemma 3.2 *If one of then the following statements hold*

1) If $\alpha < 0, \gamma < 0$ *and* $a < 0, Q(\theta) < 0$ *for all* θ *and* $\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) < \left(-\frac{\gamma^2}{4\alpha}\right)$.

2) If $\alpha < 0, \gamma < 0, a > 0, Q(\theta) > 0$ *for all* $\theta, e^{a2\pi}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)\right) < \left(-\frac{\gamma^2}{4\alpha}\right)$.

then

$$-\gamma^2 < 4\alpha\rho(\theta) < 0$$

for all θ*. Where* $\rho(\theta) = e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(\theta)\right)$ *and* $g(\theta) = \int_0^\theta \frac{e^{-as}}{Q(s)} ds$.

Proof**.** **Proof of statement (1) of Lemma 3.2 :** Since $\alpha < 0, \gamma < 0$ and $a < 0, Q(\theta) < 0$ for all θ, then $\frac{e^{2\pi a}}{e^{2\pi a}-1} < 0$ and $g(2\pi) < g(\theta) < 0$, it follows that

$$\begin{aligned} \rho(\theta) &= e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(\theta)\right) \\ &> e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)\right) > 0. \end{aligned}$$

We prove that

$$\rho(\theta) = e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(\theta)\right) < \frac{-\gamma^2}{4\alpha},$$

we have

$$\begin{aligned} 0 &< e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(\theta)\right) \\ &< e^{a2\pi}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(2\pi)\right) \\ &= \frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi). \end{aligned}$$

Since $\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) < -\frac{\gamma^2}{4\alpha}$, then

$$0 < \rho(\theta) < \left(-\frac{\gamma^2}{4\alpha}\right).$$

Proof of statement (2) of Lemma 3.2 : Since $\alpha < 0, \gamma < 0, a > 0$ and $Q(\theta) > 0$

for all θ, then $g(2\pi) > g(\theta) > 0$ and $\frac{g}{e^{2\pi a}-1} > 0$, it follows that

$$\begin{aligned}
\rho(\theta) &= e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g(\theta)\right) \\
&> e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g(0)\right) \\
&= \frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) > 0,
\end{aligned}$$

we prove that

$$\rho(\theta) = e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g(\theta)\right) < \frac{-\gamma^2}{4\alpha},$$

we have $a > 0$ and $\rho(\theta) > 0$, thus $\dfrac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g(\theta) > 0$ so

$$\begin{aligned}
\rho(\theta) &= e^{a\theta}\left(g(2\pi)\frac{e^{2\pi a}}{e^{2\pi a}-1} - g(\theta)\right) \\
&< e^{a2\pi}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g(\theta)\right) \\
&< e^{a2\pi}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi)\right).
\end{aligned}$$

Since $e^{a2\pi}\left(\dfrac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi)\right) < -\dfrac{\gamma^2}{4\alpha}$ then

$$0 < \rho(\theta) < \left(-\frac{\gamma^2}{4\alpha}\right).$$

This complete the proof of lemma 3.2. ❐

Lemma 3.3 *If one of then the following statements hold*
1) $\alpha > 0, \gamma > 0$ *and* $a < 0, Q(\theta) > 0$ *for all* θ *and* $\dfrac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) > \dfrac{-\gamma^2}{4\alpha}$.
2) $\alpha > 0, \gamma > 0$ *and* $a > 0, Q(\theta) < 0$ *for all* θ *and* $\dfrac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) > \dfrac{-\gamma^2}{4\alpha}$.
Then for all θ *we have*

$$-\gamma^2 < 4\alpha\rho(\theta) < 0$$

Where $\rho(\theta) = e^{-a\theta}\left(\dfrac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) + g(\theta)\right)$ *and* $g(\theta) = \displaystyle\int_0^\theta \frac{e^{-as}}{Q(s)}ds$

Proof**. Proof of statement (1) of Lemma 3.3 :** If $\alpha > 0, \gamma > 0$, $a < 0, Q(\theta) > 0$ for all θ and $\dfrac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) > \dfrac{-\gamma^2}{4\alpha}$ we prove that

$$0 > \rho(\theta) > \frac{-\gamma^2}{4\alpha}.$$

Since $Q(\theta) > 0$ for all θ then $g(0) = 0 < g(\theta) < g(2\pi)$ and

$$\begin{aligned} \rho(\theta) &= e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g(\theta)\right) \\ &< e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi)\right) \\ &= e^{a\theta}\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi), \end{aligned}$$

since $a < 0,$ then $e^{2\pi a} - 1 < 0$ so

$$\rho(\theta) < e^{a\theta}\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) < 0.$$

On the other hand we have $0 > \rho(\theta)$ for all $\theta,$ which implies that $\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g(\theta)\right) < 0,$ so

$$\begin{aligned} \rho(\theta) &= e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g(\theta)\right) \\ &> e^{a2\pi}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g(\theta)\right), \end{aligned}$$

since $Q(\theta) > 0$ for all $\theta,$ thus $-g(\theta) > -g(2\pi)$ and

$$\rho(\theta) > e^{a2\pi}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g(2\pi)\right) = \frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi),$$

by hypotheses $\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) > \frac{-\gamma^2}{4\alpha}$, it follows that

$$\rho(\theta) > \frac{-\gamma^2}{4\alpha},$$

the statement (1) holds.

Proof of statement (2) of Lemma 3.3 : If $\alpha > 0, \gamma > 0$ and $a > 0, Q(\theta) < 0$ for all θ and $\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) > \frac{-\gamma^2}{4\alpha}$, we have

$$g(0) = 0 > g(\theta) > g(2\pi),$$

thus

$$\begin{aligned} \rho(\theta) &= e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g(\theta)\right) \\ &< e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) - g\right) \\ &= \frac{e^{a\theta}}{e^{2\pi a}-1}g(2\pi), \end{aligned}$$

if $a > 0$ we have $(e^{2\pi a} - 1) > 0,$ then

$$\rho(\theta) < 0$$

for all $\theta.$
On the other hand we have $a > 0, 0 > g(\theta) > g(2\pi)$ and $\left(\frac{1}{e^{2\pi a}-1} g(2\pi) + g(\theta)\right) < 0,$ then

$$\begin{aligned} \rho(\theta) &= e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(\theta)\right) \\ &> \left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(\theta)\right) \\ &> \frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi), \end{aligned}$$

by hypotheses we have $\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) > \frac{-\gamma^2}{4\alpha}$ so

$$\rho(\theta) > \frac{-\gamma^2}{4\alpha}$$

for all $\theta,$ the statement (2) holds. This completes the proof of lemma 3.3. ❐

Lemma 3.4 *The curve*

$$\left(r^2\left(\alpha r^2 - \gamma\right)\right) - e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - \int_0^\theta \frac{e^{-as}}{Q(s)} ds\right) = 0 \qquad (3.8)$$

does not intersect the orbit

$$\left(2\alpha x^2 + 2\alpha y^2 - \gamma\right) = 0. \qquad (3.9)$$

Proof. The curve (3.9) in polar coordinates becomes $(2\alpha r^2 - \gamma) = 0$. All this is equivalent to show that the system

$$\begin{aligned} \left(r^2\left(\alpha r^2 - \gamma\right)\right) - e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - \int_0^\theta \frac{e^{-as}}{Q(s)} ds\right) &= 0, \\ \left(2\alpha r^2 - \gamma\right) &= 0. \end{aligned} \qquad (3.10)$$

From the second equation of this system, we get that $r^2 = \left(\frac{\gamma}{2\alpha}\right)$; we replace this value in the first equation we obtain

$$-\frac{1}{4\alpha}\gamma^2 = e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - \int_0^\theta \frac{e^{-as}}{Q(s)} ds\right),$$

which is a contradiction because $e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g\left(2\pi\right)+\int_0^\theta \frac{e^{-as}}{Q\left(s\right)}ds\right)=\rho\left(\theta\right)$ and from lemma 3.2 and 3.3 we have

$$\frac{-\gamma^2}{4\alpha}\neq\rho\left(\theta\right),$$

for all θ. So the system (3.10) has no solutions, hence there does not exist any singular point of (3.8). This completes the proof of lemma 3.4. ❒

Lemma 3.5 *The functions $\theta\longrightarrow\rho\left(\theta\right)$ is 2π-periodic,*

Proof. We have

$$\rho\left(\theta+2\pi\right)=e^{a(\theta+2\pi)}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g\left(2\pi\right)-g\left(\theta+2\pi\right)\right).$$

However,

$$\begin{aligned} g\left(\theta+2\pi\right) &= \int_0^{\theta+2\pi}\frac{e^{-as}}{Q\left(s\right)}ds \\ &= g\left(2\pi\right)+\int_{2\pi}^{\theta+2\pi}\frac{e^{-as}}{Q\left(s\right)}ds. \end{aligned}$$

In the integral $\int_{2\pi}^{\theta+2\pi}\frac{e^{-as}}{Q\left(s\right)}ds$, we use the change of variable $w=s-2\pi$, we obtain

$$\begin{aligned} g\left(\theta+2\pi\right) &= g\left(2\pi\right)+\int_0^{\theta}\frac{e^{-a(w+2\pi)}}{Q\left(w+2\pi\right)}dw \\ &= g\left(2\pi\right)+e^{-2\pi a}g\left(\theta\right). \end{aligned}$$

In (3.8) we replace $g\left(\theta+2\pi\right)$ by $g\left(2\pi\right)+e^{-2\pi a}g\left(\theta\right)$, and after some calculations we obtain $\rho\left(\theta+2\pi\right)=\rho\left(\theta\right)$ hence $\rho\left(\theta\right)$ is 2π-periodic. This completes the proof of lemma 3.5. ❒

***Proof*. of Theorem 3.3** . To show that $r(\theta,r_*)$ is a periodic solution, we have to show that:

i) the function $\theta\rightarrow r_i(\theta,r_*)$ is 2π-periodic.

ii) $r_i(\theta,r_*)>0$. for all $\theta\in[0,2\pi[$.

This last condition ensures that $r(\theta,r_*)$ is well defined for all $\theta\in[0,2\pi[$ and the periodic solution do not pass through the equilibrium point $(0,0)$ of system (3.1).

Periodicity. We say that by lemma 3.5 we have $\rho\left(\theta\right)$ is 2π-periodic, then $r_i\left(\theta\right),i=1,2$ are 2π-periodic.

Strict positivity of $r_i(\theta,r_*)$:

Proof of statement 1: If $\alpha < 0, \gamma < 0$ and $a < 0, Q(\theta) < 0$ then $\dfrac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a}-1)} < 0$ and since $\dfrac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) < \left(-\dfrac{\gamma^2}{4\alpha}\right)$ from the statement (1) of Lemma 3.2 we have $-\gamma^2 < 4\alpha\rho(\theta) < 0,$ then the two solutions

$$\begin{cases} r_1^2(\theta) = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right), \\ r_2^2(\theta) = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right), \end{cases}$$

are strictly positive

$$\begin{cases} r_{1*}^2 = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right), \\ r_{2*}^2 = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right). \end{cases}$$

On the other hand, we have $\dfrac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi) < 0$ and $\dfrac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) < \left(-\dfrac{\gamma^2}{4\alpha}\right)$ for all $\theta \in \mathbb{R},$ which implies that

$$\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi) > 0,$$

and

$$\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)} < \sqrt{\gamma^2} = -\gamma,$$

thus

$$\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2a\pi}}{(e^{2\pi a}-1)} g(2\pi)}\right) < 0$$

and

$$\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2a\pi}}{(e^{2\pi a}-1)} g(2\pi)}\right) < 2\gamma < 0,$$

since $\alpha < 0,$ then

$$\begin{cases} r_{1*}^2 = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right) > 0, \\ r_{2*}^2 = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right) > 0. \end{cases}$$

Proof of statement 2: If $\alpha < 0, \gamma < 0, a > 0, Q(\theta) > 0$ for all θ and $\dfrac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) < \left(-\dfrac{\gamma^2}{4\alpha}\right)$ from the statement (1) of Lemma 3.2 we have $-\gamma^2 < 4\alpha\rho < 0,$ then

$$\sqrt{\gamma^2 + 4\alpha\rho(\theta)} < \sqrt{\gamma^2} = -\gamma,$$

thus

$$\gamma < -\sqrt{\gamma^2 + 4\alpha\rho(\theta)}$$

$$\left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) < -2\sqrt{\gamma^2 + 4\alpha\rho(\theta)} < 0,$$

and

$$\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) < 0,$$

since $\alpha < 0,$ thus the two solutions

$$\begin{cases} r_1^2(\theta) = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right), \\ r_2^2(\theta) = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right), \end{cases}$$

are strictly positive.

On the other hand, we have $e^{a2\pi}\left(\dfrac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi)\right) < \left(-\dfrac{\gamma^2}{4\alpha}\right)$ and since $a > 0$ and $\left(\dfrac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi)\right) > 0$, then

$$g\frac{e^{2\pi a}}{e^{2\pi a}-1} < e^{a2\pi}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi)\right)$$

thus

$$\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi) < \left(-\frac{\gamma^2}{4\alpha}\right)$$

and $a > 0$, then

$$\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)}g(2\pi) > 0$$

and $(e^{2\pi a} - 1) > 0,$ thus $\frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)}g(2\pi) < 0$ for all $\theta \in \mathbb{R},$ which implies that

$$\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}g}{(e^{2\pi a}-1)}} < \sqrt{\gamma^2} = -\gamma,$$

thus

$$\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)}g(2\pi)}\right) < 0$$

and

$$\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)}g(2\pi)}\right) < -2\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)}g(2\pi)} < 0,$$

since $\alpha < 0,$ then

$$\begin{cases} r_{1*}^2 = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)}g(2\pi)}\right) > 0, \\ r_{2*}^2 = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}g}{(e^{2\pi a}-1)}g(2\pi)}\right) > 0. \end{cases}$$

Proof of statement 3: $\alpha > 0, \gamma > 0$ and $a < 0, Q(\theta) > 0$ for all θ and $\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) > \frac{-\gamma^2}{4\alpha}$ from the statement (1) of Lemma 3.3 we have $-\gamma^2 < 4\alpha\rho(\theta) < 0$,then

$$\sqrt{\gamma^2 + 4\alpha\rho(\theta)} < \sqrt{\gamma^2} = \gamma.$$

Since $\alpha < 0$, then

$$\left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) > 0,$$

and

$$\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) > 2\sqrt{\gamma^2 + 4\alpha\rho(\theta)} > 0,$$

the two solutions

$$\begin{cases} r_1^2(\theta) = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right), \\ r_2^2(\theta) = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right), \end{cases}$$

are strictly positive.
On the other hand, we have

$$\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) > \frac{-\gamma^2}{4\alpha},$$

for all $\theta \in \mathbb{R}$, then,

$$\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi) > 0,$$

since $a < 0, Q(\theta) > 0$ for all θ then $\frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi) < 0$ and

$$\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)} < \sqrt{\gamma^2} = \gamma.$$

Thus

$$\begin{aligned} r_{1*}^2 &= \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 - \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right) \\ &> \frac{1}{2\alpha}\left(2\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a}-1)}}\right) > 0, \\ r_{2*}^2 &= \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 - \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right) > 0. \end{aligned}$$

Proof of statement 4: $\alpha > 0, \gamma > 0$ and $a > 0, Q < 0$ for all θ and $\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) > \frac{-\gamma^2}{4\alpha}$, from the statement (2) of Lemma 3.3 we have $-\gamma^2 < 4\alpha\rho(\theta) < 0$ for all θ, then the two solutions

$$\begin{cases} r_1^2(\theta) = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right), \\ r_2^2(\theta) = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right), \end{cases}$$

are strictly positive.
On the other hand, because $a > 0$ and $Q(\theta) < 0$ for all θ we have $(e^{2\pi a} - 1) > 0$ and $\frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi) < 0$, since $\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) > \frac{-\gamma^2}{4\alpha}$, thus

$$-\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi) > 0,$$

so

$$\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)} < \sqrt{\gamma^2} = \gamma,$$

then

$$\begin{aligned} r_{1*}^2 &= \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right) \\ &> \frac{1}{2\alpha}\left(2\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a}-1)}}\right) > 0, \\ r_{2*}^2 &= \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a}-1)}}\right) > 0. \end{aligned}$$

Finally $r(\theta, r_*)$ defines through (3.6) a periodic solution. To show that it is a limit cycle, we consider (3.6), and introduce the Poincaré return map $\Pi_i : (2\pi, \lambda) \to r_i(2\pi, \lambda)$, to prove that the periodic solution is an isolated periodic orbit, (see [5], it is sufficient for the function of Poincaré first return), we compute

$$\left.\frac{d\Pi_i(2\pi, \lambda)}{d\lambda}\right|_{\lambda = r_{*i}}$$

by replacing r_{*i} by its value given by

$$\begin{cases} r_{1*}^2 = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 - \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right) \\ r_{2*}^2 = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 - \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right) > 0. \end{cases}$$

and after some calculation, we get

$$\left.\frac{dr_1(2\pi;\lambda)}{d\lambda}\right|_{\lambda = r_1^*} = \left.\frac{dr_2(2\pi;\lambda)}{d\lambda}\right|_{\lambda = r_2^*} = e^{2\pi a}.$$

If $a < 0$ then $e^{2\pi a} < 1$

Consequently the limit cycles of thedifferential equation (3.4) are hyperbolic and stable.

If $a < 0$ then $e^{2\pi a} > 1$

Consequently the limit cycles of the differential equation (3.4) are hyperbolic and unstable. ❒

3.5.1 Tow non algebraic limit cycles

Theorem 3.4 *Assume that one of the conditions (a) and (b) of theorem 3.3 is hold and* $Q(\theta) \neq const$. *Then the orbits* $r_1^2(\theta)$ *and* $r_2^2(\theta)$ *are two non-algebraic hyperbolic limit cycles for system (2).*

Proof. More precisely, in Cartesian coordinates, $r^2 = x^2 + y^2, \theta = \arctan \frac{y}{x}$, the implicit solution of equation (6) can be written as

$$\begin{gathered} G(x,y) = \left(\left(x^2+y^2\right)\left(\alpha\left(x^2+y^2\right)-\gamma\right)\right) \\ -e^{a \arctan \frac{y}{x}} \left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - \int_0^{\arctan \frac{y}{x}} \frac{e^{-as}}{Q(s)} ds\right) = 0. \end{gathered} \tag{3.11}$$

We remark that the curve $G(x,y) = 0$ does not a polynomial because there is no integer n for which both $\dfrac{\partial^n G}{\partial x^n}$ and $\dfrac{\partial^n G}{\partial y^n}$ vanish identically, for example when calculating $\dfrac{\partial G}{\partial x}$ note that the expression $e^{a \arctan \frac{y}{x}} \left(\dfrac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - \displaystyle\int_0^{\arctan \frac{y}{x}} \dfrac{e^{-as}}{Q(s)} ds\right)$ appear again, So for any order of derivation this expression will appear. Therefore the curve $G(x,y) = 0$ is non-algebraic and the limit cycles of the system (3.1) will also be non-algebraic. ❐

3.5.2 Tow algebraic limit cycles

Theorem 3.5 *Assume that one of the conditions (a) and (b) of theorem 3.3 is holds and* $Q(\theta) = w = const$ *for all* θ, *then the orbits* $r_1^2(\theta)$ *and* $r_2^2(\theta)$ *are two algebraic hyperbolic limit cycles for system* (3.1) *whose expression in cartesian coordinates* (x,y) *are*

$$\alpha x^4 + \alpha y^4 + 2\alpha x^2 y^2 - \gamma x^2 - \gamma y^2 - \frac{1}{aw} = 0.$$

Proof. if $Q(\theta) = w$ for all θ we have

$$g(\theta) = \int_0^\theta \frac{e^{-as}}{w} ds = \frac{-1}{aw}\left(e^{-a\theta} - 1\right),$$

and

$$g(2\pi) = \int_0^{2\pi} \frac{e^{-as}}{w} ds = \frac{-1}{aw}\left(e^{-a2\pi} - 1\right).$$

Then

$$\begin{aligned} \rho(\theta) &= e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}\left(\frac{-1}{aw}\left(e^{-a2\pi}-1\right)\right) - \left(\frac{-1}{aw}\left(e^{-a\theta}-1\right)\right)\right) \\ &= \frac{1}{aw}, \end{aligned}$$

through the previous change of variable $\left(r^2\left(\alpha r^2-\gamma\right)\right)=\rho$ and by passing to Cartesian coordinates (x, y), we obtain that the system (3.1) has two algebraic limit cycles, these two algebraic limit cycles given by

$$\left(\left(x^2+y^2\right)\left(\alpha\left(x^2+y^2\right)-\gamma\right)\right)-\frac{1}{aw}=0.$$

This complete the proof of Theorem 3.3. ❐

3.5.3 Examples

Example 3.1 *If we take*

$$Q\left(x, y\right)=\left(cx^2+cy^2-bxy\right),$$

and $\alpha=a=1, b=2, \gamma=3, c=-2$ *then system* (3.1) *reads*

$$\begin{cases}\dot{x}=x+\left(-6y-x^3+4x^2y+4y^3+3x-xy^2\right)\left(-2x^2-2y^2-2xy\right), \\ \dot{y}=y-\left(4x^3+x^2y+4xy^2-6x+y^3-3y\right)\left(-2x^2-2xy-2y^2\right),\end{cases} \tag{3.12}$$

this system has two non algebraic limit cycles whose expressions in polar coordinates (r, θ) *is*

$$\begin{cases}r_1^2\left(\theta\right)=\frac{1}{2}\left(3-\sqrt{9+4\rho\left(\theta\right)}\right), \\ r_2^2\left(\theta\right)=\frac{1}{2}\left(3+\sqrt{9+4\rho\left(\theta\right)}\right),\end{cases}$$

where $g\left(\theta\right)=\int_0^\theta \frac{e^{-s}}{Q\left(s\right)}ds$ *and*

$$\begin{cases}\rho\left(\theta\right)=e^\theta\left(g\frac{e^{2\pi}}{e^{2\pi}-1}-g\left(\theta\right)\right), \\ Q\left(\theta\right)=-2-\sin 2\theta,\end{cases}$$

and

$$\begin{cases}r_{1*}^2=\frac{1}{2}\left(3-\sqrt{9+\frac{4e^{2\pi}}{\left(e^{2\pi}-1\right)}\int_0^{2\pi}\frac{e^{-s}}{\left(-2-\sin 2s\right)}ds}\right)=0.159\,20, \\ r_{2*}^2=\frac{1}{2}\left(3+\sqrt{9+\frac{4e^{2\pi}}{\left(e^{2\pi}-1\right)}\int_0^{2\pi}\frac{e^{-s}}{\left(-2-\sin 2s\right)}ds}\right)=2.\,840\,8.\end{cases}$$

Example 3.2 *Let*

$$Q\left(x, y\right)=wx^2+wy^2,$$

and $\alpha=a=1, \gamma=3, w=-2,$ *then the system* (3.1) *becomes*

$$\begin{cases}\dot{x}=x+\left(-2x^2-2y^2-2xy\right)\left(-6y-x^3+4x^2y+4y^3+3x-xy^2\right), \\ \dot{y}=y-\left(-2x^2-2xy-2y^2\right)\left(4x^3+x^2y+4xy^2-6x+y^3-3y\right),\end{cases} \tag{3.13}$$

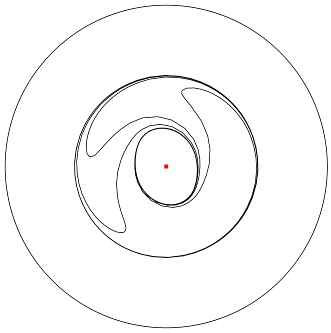

Figure 3.1: Two non algebriac limit cycles of differential system (3.12)

this system possess two algebraic limit cycles, these two limit cycles given in Cartesian coordinates by the expression

$$\left(x^2+y^2\right)\left(\left(x^2+y^2\right)-3\right)+\frac{1}{2}=0.$$

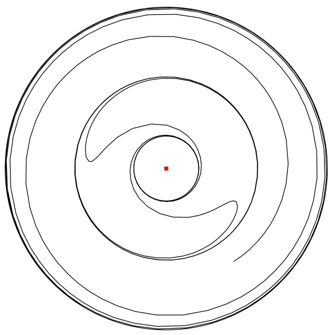

Figure 3.2: Two algebriac limit cycles of differential system (3.13)

3.6 Existence of one limit cycle

Theorem 3.6 *1) If* $\alpha<0,\gamma<0$ *and* $aQ(\theta)<0,$ *then system* (3.1) *has two explicit limit cycles, given in polar coordinates* (r,θ) *by*

$$r_1^2(\theta)=\frac{1}{2\alpha}\left(\gamma-\sqrt{\gamma^2+4\alpha\rho(\theta)}\right),$$

where $g(\theta)=\int_0^\theta\frac{e^{-as}}{Q(s)}ds$ *and*

$$\rho(\theta)=e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1}g(2\pi)-g(\theta)\right),$$

and

$$r_{1*}^2 = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right).$$

Moreover, if $Q(\theta) = w = constant$, *this limit cycle is algebraic and if* $Q(\theta) \neq const$, *the limit cycle is non algebraic.*
2)If $\alpha > 0, \gamma > 0$ *and* $aQ(\theta) > 0,$ *for all* θ *then, system* (3.1) *has two explicit limit cycle, given in polar coordinates* (r, θ) *by*

$$r_2^2(\theta) = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right),$$

where $g(\theta) = \int_0^\theta \frac{e^{-as}}{Q(s)} ds$ *and*

$$\rho(\theta) = e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(\theta)\right),$$

and

$$r_{2*}^2 = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right).$$

Moreover, if $Q(\theta) = w = constant$, *this limit cycle is algebraic and if* $Q(\theta) \neq constant$, *the limit cycle is non algebraic.*

Next Lemma collects some results which we need to show the statements of Theorem 3.6.

Lemma 3.6 *Consider polynomial differential system* (3.1). *Then the following three statements hold*
1) If $\alpha < 0, \gamma < 0$ *and* $aQ(\theta) < 0$ *for all* θ

$$\rho(\theta) < 0.$$

2) If $\alpha > 0, \gamma > 0$ *and* $aQ(\theta) > 0$ *for all* θ

$$\rho(\theta) > 0.$$

Proof. 1) If $\alpha < 0, \gamma < 0$ and $a < 0, Q(\theta) > 0$, for all θ then

$$\begin{aligned} \rho(\theta) &= e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(\theta)\right) \\ &< e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)\right), \end{aligned}$$

by the hypotheses we have $a < 0, Q(\theta) > 0$ for all θ thus $\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)\right) < 0$ so

$$\rho(\theta) < 0.$$

-If $\alpha < 0, \gamma < 0$ and $a > 0, Q(\theta) < 0,$ for all θ then $g(2\pi) < g(\theta) < 0$ and

$$\begin{aligned} \rho(\theta) &= e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(\theta)\right) \\ &< e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(2\pi)\right) \\ &= \frac{g(2\pi)}{e^{2\pi a}-1}, \end{aligned}$$

by the hypotheses we have $a < 0, Q(\theta) > 0$ for all θ thus $\frac{g(2\pi)}{e^{2\pi a}-1} < 0$ so

$$\rho(\theta) < 0.$$

2) -If $\alpha > 0, \gamma > 0$ and $a > 0, Q(\theta) > 0$, for all θ then $g(2\pi) > g(\theta)$ and

$$\begin{aligned} \rho(\theta) &= e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(\theta)\right) \\ &> e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(2\pi)\right) \\ &= \frac{e^{a\theta}}{e^{2\pi a}-1} g(2\pi). \end{aligned}$$

By the hypotheses we have $a < 0, Q(\theta) > 0$ for all θ thus $\frac{e^{a\theta}}{e^{2\pi a}-1} g(2\pi) > 0$ so

$$\rho(\theta) > 0.$$

-If $\alpha > 0, \gamma > 0$ and $a < 0, Q(\theta) < 0$, for all θ then, then $g(2\pi) < g(\theta) < 0$ and

$$\begin{aligned} \rho(\theta) &= e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) - g(\theta)\right) \\ &> e^{a\theta}\left(\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi)\right). \end{aligned}$$

By the hypotheses we have $a < 0, Q(\theta) > 0$ for all θ thus $\frac{e^{2\pi a}}{e^{2\pi a}-1} g(2\pi) > 0$ so

$$\rho(\theta) > 0.$$

❐

Proof.

of Theorem 3.6: If $\alpha < 0, \gamma < 0$ and $a < 0, Q(\theta) > 0$ for all θ then $\frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi) > 0$, it follows that

$$\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)} > \sqrt{\gamma^2} = -\gamma,$$

so

$$-\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)} < \gamma,$$

and

$$\begin{aligned} \left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right) &< 2\gamma < 0, \\ \left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right) &> -\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a}-1)}} + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a}-1)}} = 0. \end{aligned}$$

Since $\alpha < 0$, then

$$r_{2*}^2 = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right) < 0,$$

and we do not consider this case. We only take into consideration the following value r_* which satisfies $r(2\pi, r_*) = r_* > 0$

$$r_{1*}^2 = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a}}{(e^{2\pi a}-1)} g(2\pi)}\right) > 0.$$

On other hand by Lemma 3.6 we have

$$\rho(\theta) < 0.$$

Consequently

$$4\alpha\rho(\theta) > 0,$$

this imply that

$$\sqrt{\gamma^2 + 4\alpha\rho(\theta)} > \sqrt{\gamma^2} = -\gamma,$$

thus

$$-\sqrt{\gamma^2 + 4\alpha\rho(\theta)} < \gamma,$$

and

$$\begin{aligned} \left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) &< 2\gamma < 0, \\ \left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) &> -\sqrt{\gamma^2 + 4\alpha\rho(\theta)} + \sqrt{\gamma^2 + 4\alpha\rho(\theta)} = 0. \end{aligned}$$

Since $\alpha < 0,$ it follows that

$$r_1^2(\theta) = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) > 0,$$

and we do not consider this case. We only take into consideration the following value $r(\theta)$

$$r_2^2(\theta) = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) < 0.$$

2) If $\alpha > 0, \gamma > 0$ and $aQ(\theta) > 0$ for all θ then $\dfrac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a} - 1)} > 0,$ thus

$$\sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a} - 1)}} > \sqrt{\gamma^2} = \gamma,$$

and

$$\begin{aligned}
\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a} - 1)}}\right) &< \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a} - 1)}} - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a} - 1)}} < 0,\\
\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a} - 1)}}\right) &> 2\gamma = 0.
\end{aligned}$$

Since $\alpha > 0,$ then

$$r_{1*}^2 = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a} - 1)}}\right) < 0,$$

and we do not consider this case. We only take into consideration the following value r_* which satisfies $r(2\pi, r_*) = r_* > 0$

$$r_{2*}^2 = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + \frac{4\alpha e^{2\pi a} g(2\pi)}{(e^{2\pi a} - 1)}}\right) > 0.$$

On other hand by Lemma 3.6 we have

$$\rho(\theta) > 0.$$

Consequently

$$4\alpha\rho(\theta) > 0,$$

this imply that

$$\sqrt{\gamma^2 + 4\alpha\rho(\theta)} > \sqrt{\gamma^2} = \gamma,$$

and

$$\begin{aligned}
\left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) &< \sqrt{\gamma^2 + 4\alpha\rho(\theta)} - \sqrt{\gamma^2 + 4\alpha\rho(\theta)} = 0,\\
\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) &> 2\gamma > 0.
\end{aligned}$$

Since $\alpha > 0$, it follows that

$$r_1^2(\theta) = \frac{1}{2\alpha}\left(\gamma - \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) < 0,$$

for all θ and we do not consider this case. We only take into consideration the following value $r(\theta)$

$$r_2^2(\theta) = \frac{1}{2\alpha}\left(\gamma + \sqrt{\gamma^2 + 4\alpha\rho(\theta)}\right) > 0.$$

❐

3.6.1 Examples

Let

$$Q(x,y) = \left(cx^2 + cy^2 - bxy\right),$$

Example 3.3 *If we take* $\alpha = -1, a = 1, b = 2, \gamma = -3, c = -2$ *then system* (3.1) *reads*

$$\begin{cases} \dot{x} = x - \left(-2\left(x^2+y^2\right) - 2xy\right)\left(-3\left(2y-x\right) - \left(x^2+y^2\right)\left(x-4y\right)\right), \\ \dot{y} = y - \left(-2\left(x^2+y^2\right) - 2xy\right)\left(3\left(2x+y\right) - \left(x^2+y^2\right)\left(4x+y\right)\right), \end{cases} \tag{3.14}$$

this system has one non algebraic limit cycle whose expressions in polar coordinates (r,θ) *is*

$$r_1^2(\theta) = \frac{-1}{2}\left(-3 - \sqrt{9 - 4\rho(\theta)}\right),$$

where $g(\theta) = \displaystyle\int_0^\theta \frac{e^{-s}}{-2-\sin 2s}ds$ *and*

$$\rho(\theta) = e^\theta\left(\frac{e^{2\pi}}{e^{2\pi}-1}g(2\pi) - g(\theta)\right),$$

and

$$r_{1*}^2 = \frac{-1}{2}\left(-3 - \sqrt{9 + \frac{-4e^{2\pi}}{\left(e^{2\pi}-1\right)}\int_0^{2\pi}\frac{e^{-s}}{-2-\sin 2s}ds}\right) = 3.1439.$$

.

Example 3.4 *If we take* $\alpha = 1, a = -1, b = 2, \gamma = 3, c = -2$ *then system* (3.1) *reads*

$$\begin{cases} \dot{x} = x - \left(3\left(2y+x\right) + \left(x^2+y^2\right)\left(-x-4y\right)\right)\left(-2\left(x^2+y^2\right) - 2xy\right), \\ \dot{y} = y - \left(-3\left(2x-y\right) + \left(x^2+y^2\right)\left(4x-y\right)\right)\left(-2\left(x^2+y^2\right) - 2xy\right), \end{cases} \tag{3.15}$$

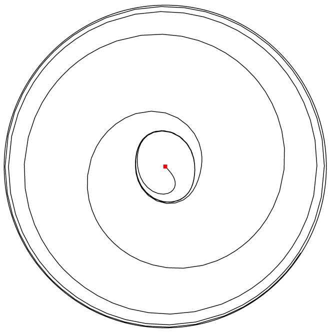

Figure 3.3: One non algebriac limit cycle of differential system (3.14)

this system has one non algebraic limit cycle whose expressions in polar coordinates (r, θ) *is*

$$r_2^2(\theta) = \frac{1}{2}\left(3 + \sqrt{9 + 4\rho(\theta)}\right),$$

where $g(\theta) = \int_0^\theta \frac{e^s}{-2 - \sin 2s} ds$ *and*

$$\rho(\theta) = e^{-\theta}\left(\frac{e^{-2\pi}}{e^{-2\pi} - 1} g(2\pi) - g(\theta)\right),$$

and

$$r_{2*}^2 = \frac{1}{2}\left(3 + \sqrt{9 + \frac{4e^{-2\pi}}{(e^{2\pi} - 1)} \int_0^{2\pi} \frac{e^s}{-2 - \sin 2s} ds}\right) = 2.999\,6.$$

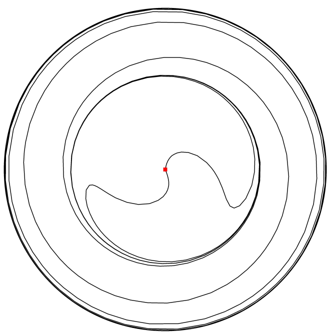

Figure 3.4: One non algebriac limit cycle of differential system (3.15)

Example 3.5 *If we take* $\alpha = -1, a = 1, b = 0, \gamma = -3, c = -2$ *then system* (3.1) *reads*

$$\begin{cases} \dot{x} = x - \left(-2\left(x^2 + y^2\right)\right)\left(-3\left(2y - x\right) - \left(x^2 + y^2\right)\left(x - 4y\right)\right), \\ \dot{y} = y - \left(-2\left(x^2 + y^2\right)\right)\left(3\left(2x + y\right) - \left(x^2 + y^2\right)\left(4x + y\right)\right), \end{cases} \tag{3.16}$$

since

$$\rho(\theta) = e^{\theta}\left(\frac{e^{2\pi}}{e^{2\pi}-1}\int_0^{2\pi}\frac{e^{-s}}{-2}ds - \int_0^{\theta}\frac{e^{-s}}{-2}ds\right) = -\frac{1}{2}$$

this system possess one algebraic limit cycle, this limit cycle given in Cartesian coordinates by the expression

$$\left(x^2+y^2\right)\left(-\left(x^2+y^2\right)+3\right)+\frac{1}{2}=0$$

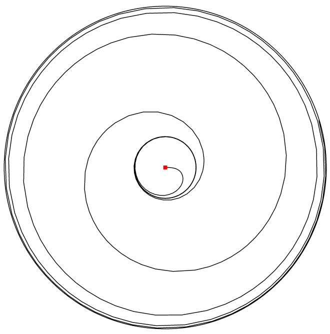

Figure 3.5: One algebriac limit cycle of differential system (3.16)

Example 3.6 *If we take* $\alpha = 1, a = -1, b = 0, \gamma = 3, c = -2$ *then system* (3.1) *reads*

$$\begin{cases} \dot{x} = x - \left(3\left(2y+x\right)+\left(x^2+y^2\right)\left(-x-4y\right)\right)\left(-2\left(x^2+y^2\right)\right), \\ \dot{y} = y - \left(-3\left(2x-y\right)+\left(x^2+y^2\right)\left(4x-y\right)\right)\left(-2\left(x^2+y^2\right)\right), \end{cases} \quad (3.17)$$

since

$$\rho(\theta) = e^{-\theta}\left(\frac{e^{-2\pi}}{e^{-2\pi}-1}\int_0^{2\pi}\frac{e^{s}}{-2}ds - \int_0^{\theta}\frac{e^{s}}{-2}ds\right) = \frac{1}{2}$$

this system possess one algebraic limit cycle, this limit cycle given in Cartesian coordinates by the expression

$$\left(x^2+y^2\right)\left(\left(x^2+y^2\right)-3\right)-\frac{1}{2}=0$$

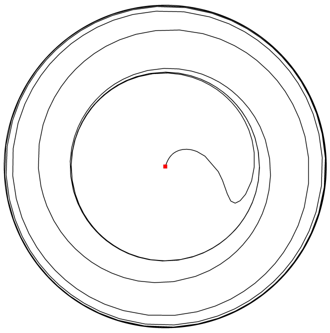

Figure 3.6: One algebriac limit cycle of differential system (3.17)

CONCLUSION

In this dissertation, we focused on studying the limit cycles and its number of quintic differential systems. Moreover, if there exists we distinguish when it is algebriac or not. In the first chapter, we presented some basic results of the necessary qualitative theory of differential systems. Whereas in the second one we analysis the existence and non existence of one limit cycle for a cubic polynomial differential system(2.1). Moreover, if the expression of a limit cycle is contained in algebraic curve of the plan, then we say that it is algebriac otherwise it is called non algebriac.

As for the third chapter, we study the following problems for a quintic polynomial differential system(3.1):

- Non existence of limit cycles
- Existence of two limit cycles
- Two non algebriac limit cycles
- Tow algebriac limit cycles
- Existense of one limit cycle

The question asked if we replaced the homogeneous polynomial $Q(x, y)$ of degree $\mathbf{2}$ of the system (2.1) and (3.1) with an homogeneous polynomial of degree more than $\mathbf{2}$, will we able to find expressions of limit cycles by using the polar coordinates or not? and if we can be, what is her number?.

Bibliography

[1] Andronov. A. A. Les cycles limites de Poincaré et la théorie des oxillations autoentretenues, C. R, 89. 559-561(1929).

[2] Dulac. H. Sur les cycles limites, Bull. Soc. Math. France, 51. 45-188 (1923).

[3] Hilbert. D. Mathematische Probleme, Lecture, in, Second Interant. Congr. Math, Paris (1990) Nachr. Ges. Wiss. Goett. Math-phus. KI,253-297 (1900) English transl, Bull. Amer. Math. Soc, 8. 437-479 (1902)

[4] Liénard. A. Etude des oscillation entrenues, Reve Génerale de l'Electricité, 23. 946-954 (1928).

[5] Perko, L. – Differential equations and dynamical systems, Third edition. Texts in Applied Mathematics, 7. Springer Verlag, New York, 2001.

[6] Poncaré. H. Mémoire sur les courbes définies par une équation différentielle, J. Math. Pures. Appl, (3)7. 375-422(1881).

[7] Van der Pol. B. On relaxtion-oscillation, Philos. Mag, 2. 978-992 (1926).