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Memory

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Theme

ON THE SOLUTION OF THE VAN DER POL EQUATION

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Benterki rbiha	Président
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Dedicate

I dedicate the letters of my memo to :

My support in my father's life " Abd El Ouahab ".

My mother's beloved soul "Saâdia".

Companions in my path, my brothers "Âlaa", and "Iyad".

My happiness in life my sisters "Afrah", "Ikram", and "Aya"

My cousins "Salim, Abd El Djalil, Saif Eddine , Zakariya, Anes", and "Rouge..."

To the whole family "CHERIET" et " BELMILOUD ".

My best friends "Islem, Achraf, Rafik, Lakhdar, Badi, Amine, Bachir ", and "Djalil..."

my friends "BOUKHARI Nor El Imane, Hadjer, Chahira, Silya, Zahra, Mouna,... "

CHERIET Selmane

Dedicate

I dedicate the letters of my memo to :

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Contents

Introduction

Ordinary differential equations play a major and prominent role in all fields of science due to their wide of applications in physics, engineering, and biology as an approximate model or as a result of physical laws for a phenomenon. The nonlinear behavior of the real world makes the nonlinear ordinary differential equations the key that determines the relationship between all the variables that describe that phenomenon. Closed-form solution of nonlinear ordinary differential problems are rarely obtainable or impossible, in this case, mathematician deals with qualitative methods to show stability, periodicity and different properties of solutions without solving these nonlinear ordinary differential equations, another different point of view is to deal with numerical techniques to get an approximate solution, and finally searching for an analytical approximate solution is another way to deal with the subject, which is the main goal of this thesis taking into account the classical well known Van der Pol second order non–linear ordinary differential equation (VDPDE) which appears in the study of nonlinear damping.

The Van der Pol oscillator was first introduced by the Dutch engineer and physicist Balthasar van der Pol. It has been used in the analysis of a vacuum-tube circuit among other practical problems in engineering as a basic model for oscillatory processes in physics, electronics, biology, neurology, sociology, and economics. In this work, we will study the behavior of the Van der Pol equation mathematically, and see some analytical approximate methods to solve the Van der Pol equation. Van der Pol oscillator is a non-conservative oscillator with nonlinear damping governed by the following second-order ordinary differential equation

$$\frac{d^2x}{dt^2} - \varepsilon(1-x^2)\frac{dx}{dt} + x = 0,$$

this equation is equivalent to the first order nonlinear system

$$\dot{x} = y, \quad \dot{y} = \varepsilon(1-x^2)y - x.$$

The first chapter is devoted to basic concepts, definitions and some prerequisites in the subject of ordinary differential equations and their solutions, and we give also some necessary tools which are needed for the next chapters of our thesis to solve the Van der Pol equation.

In the second chapter,contains three sections the first one we apply the G'/G expansion method to determine some general solutions to the Van der Pol equation $x'' + \varepsilon\,(x^2 - \lambda)\,x' + \alpha\,x = 0$ where α, λ and ε are real parameters. Using the change of variable $x = \sqrt{w}$, we convert the Van der Pol equation into a second order nonlinear differential equation with respect to w. Finally, we apply the G'/G method to the new equation to find two families of solutions.

In the second section we use the first order approximation perturbation method to find the approximate solution by substituting $x(t) = x_0 + \alpha x_1$ in the van der Pol equation $x'' - \alpha(1 - x^2)x' + x = 0$, then we collocate the terms with same power of α and we equating the terms which multiplying by higher power α to zero we get two coupled second order differential equations with respect to $x_0(t)$ and $x_1(t)$ to get the solution of the van der Pol equation on the form $x(t) = x_0 + \alpha x_1$.

The averaging method is used in the last section to prove that the van der Pol equation has a period solution and studying its stability, furthermore the periodic orbits in this case is an isolated one which means that the Van der Pol oscillator has a *limit cycle*.

In the final chapter the homotopy perturbation method is considered and used to give an approximate analytical solution for the Van der Pol differential equation with different boundary conditions.

Chapter 1

Basic concepts and prerequisites

1.1 Differential Equations

Definition 1.1 *A differential equation is an equation that contains one or more functions with its derivatives.*

Order of Differential Equation

The order of the differential equation is the order of the highest order derivative present in the equation, as an example $\frac{dy}{dx} = 5x + 4$ is a first order differential equation, while the differential equation $\frac{d^2y}{d^2x} - 3\frac{dy}{dx} + y = 0$ is a second order one.

Linear Differential Equations

Definition 1.2 *A linear differential equation is any differential equation that can be written in the following form.*

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + ...a_1(t)y' + a_0(t)y(t) = g(t).$$

The coefficients $a_n, a_{n-1}...a_0$ and $g(t)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function $y(t)$, and its derivatives are used in determining if a differential equation is linear.

1.2 Differential system

Definition 1.3 *A differential system is a set of coupled differential equations, that can not be solved separately , these are usually ordinary differential equations:*

$$\begin{cases} X' = P(x,y), \\ Y' = Q(x,y). \end{cases}$$

where P and Q polynomials in x and y with real coefficients of degree d, the dependent variable x and y, the independent variable t.

Linear and non linear Differential system

A linear Differential system consists of linear Differential Equations. And a non linear differential system consists of non linear differential equations.

1.3 Solution of Differential system

Definition 1.4 *A solution of differential system, are the applications;*

$\phi_i : I \to \Omega$ where Ω is an open set of IR^2 such that:

$$\begin{cases} \dfrac{d\phi_1(t)}{dt} = P(\phi_1(t), \phi_2(t)), \\ \dfrac{d\phi_2(t)}{dt} = Q(\phi_1(t), \phi_2(t)). \end{cases} \tag{1.1}$$

1.3.1 Periodic solutions

Definition 1.5 *A periodic solution of the system* (1.1) *is a solution such that: for ol $t > 0$*

$$(\phi_1(T+t), \phi_2(T+t)) = (\phi_1(t), \phi_2(t)).$$

For $T > 0$, to any periodic solution corresponds a closed orbit in the space of phases.

1.3.2 Limit cycle

Definition 1.6 *A limite cycle is an isolated closed orbite of* (1.1)*, and we can not find another closed orbit in it's neighborhood.*

Stable and unstable limit cycle

A periodic orbit Γ is called stable if for each $\varepsilon > 0$ there is a neighborhood U of Γ such that for all $x \in U$ and $t > 0$ we have

$$d(\phi(t,x),\Gamma) < \varepsilon.$$

A periodic orbit Γ is called unstable if it is not stable.

1.4 Homotopy perturbation method

A homotopy between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0,1] \to Y$ from the product of the space X with the unit interval $[0,1]$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

1.4.1 Homotopy perturbation method

Consider the following non-linear differential equation [3, 6].

$$M(y) - q(x) = 0, \quad x \in \varphi. \tag{1.2}$$

With boundary conditions,

$$N\left(y, \frac{dy}{dx}\right) = 0, \quad x \in \Phi. \tag{1.3}$$

Where M is a general differential operator, N is a boundary operator, $q(x)$ is known analytic function the operator M has two parts (linear and non linear), where Ln is linear and Nln the non linear part. Therefor, can rewritten as follows

$$Ln(y) + NLn(y) - q(x) = 0. \tag{1.4}$$

by the homotopy technique, we introduced a homotopy $g(r,t) : \Phi \times [0,1] \to \mathbb{R}$
which satisfy

$$H(g,t) = (1-t)[Ln(g) - Ln(y_0)] + t[M(y) - q(x)] = 0, \; t = \in [0,1], \; x \in \phi. \tag{1.5}$$

Which is equivalent to:

$$H(g,t) = Ln(g) - Ln(y_0) + tLn(y_0) + t[NLn(g)] = 0. \tag{1.6}$$

where $t \in [0,1]$ is an embedding parameter, y_0 is an initial approximation of (1.2), which satisfies the boundary condition,
then from equation (1.5) we have:

$$\begin{aligned} H(g,0) &= Ln(g) - Ln(y_0) = 0, &(1.7)\\ H(g,1) &= R(g) - q(x) = 0. &(1.8)\end{aligned}$$

the changing process of t from zero to unity is just that of $g(x,t)$ from $y_0(x)$ to $g(x)$. In topology, this is called deformation, and $Ln(g) - Ln(y_0)$,$R(g) - q(x)$ are called homotopic here, t is very small and assume that the solution of Eqequation (1.5) can be written as a power series in p

$$g = g_0 + tg_1 + t^2g^2 + ... \tag{1.9}$$

Setting $t = 1$, approximate solution of Equation (1.2) can be obtained as,

$$g = g_0 + g_1 + g^2 + ... \tag{1.10}$$

Example 1 (see [3]) *We consider equation as follows*

$$(x + \varepsilon y)\frac{dy}{dx} + y = 0, \qquad y(1) = 1. \tag{1.11}$$

We can readily construct a homotopy which satisfies

$$(1-p)\left[\varepsilon Y\frac{dY}{dx} - \varepsilon y_0\frac{dy_0}{dx}\right] + t\left[(x+\varepsilon Y)\frac{dY}{dx} + Y\right] = 0, \qquad p \in [0,1]. \tag{1.12}$$

One may now try to obtain a solution of (1.2) *in the forme:*

$$Y(x) = Y_0(x) + tY_1(x) + t^2Y_2(x) + ... \tag{1.13}$$

where the $Y_i(x)$ are functions not yet to be determined,
by substitution of (1.13) *into* (1.12)

$$\varepsilon Y_0\frac{dY_0}{dx} - \varepsilon y_0\frac{dy_0}{dx} = 0, \tag{1.14}$$

$$\varepsilon Y_1 \frac{dY_1}{dx} + \left[(x + \varepsilon Y_0) \frac{dY_0}{dx} + Y_0 \right] = 0. \tag{1.15}$$

The initial approximation $Y_0(x)$ or $y_0(x)$ can be chosen , here we set

$$Y_0(x) = y_0(x) = \frac{-x}{\varepsilon}, \qquad Y_0(1) = \frac{-1}{\varepsilon}. \tag{1.16}$$

The substitution of Eq (1.16) *into* (1.15)*) gives;*

$$\varepsilon Y_1 \frac{dY_1}{dx} - \frac{x}{\varepsilon} = 0, \qquad Y_1(1) = 1 + \frac{1}{\varepsilon}. \tag{1.17}$$

the solution of (1.17) *written as follows:*

$$Y_1(x) = \frac{1}{\varepsilon} \sqrt{x^2 + 2\varepsilon + \varepsilon^2}.$$

If the first approximation is sufficient , then we obtain:

$$y_1(x) = Y_0(x) + Y_1(x) = \frac{-x}{\varepsilon} + \frac{1}{\varepsilon} \sqrt{x^2 + 2\varepsilon + \varepsilon^2}.$$

Wiche is the solution.

1.5 Asymptotic Expansion

To determine the asymptotic behaviour of the singular root by the expansion method [2] we simply pose a formal power series expansion for the solution $x(\varepsilon)$;

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + ... \tag{1.18}$$

Where the coefficient, x_1, x_2, x_3,... are a-priori unknown. We then substitute this expansion into the quadratic equation and formally equate powers of ε.

Example 2 *Consider the following quadratic equation for x which involves the small parameter ε:*

$$x^2 + \varepsilon x - 1 = 0. \tag{1.19}$$

Where $0 < \varepsilon < 1$ in this simple case we can solve the equation exactly so the solution exacte is:

$$x(t) = \begin{cases} \frac{-1}{2}\varepsilon + \sqrt{1 + \frac{1}{4}\varepsilon^2}, \\ \frac{-1}{2}\varepsilon - \sqrt{1 + \frac{1}{4}\varepsilon^2}. \end{cases}$$

We substitute this expansion (1.18) *into the quadratic equation* (1.19) *and formally equate powers of* ε*, we get:*

$$(1+\varepsilon x_1+\varepsilon^2 x_2^2+\varepsilon^3 x_3^3+...)^2+\varepsilon(1+\varepsilon x_1+\varepsilon^2 x_2^2+\varepsilon^3 x_3^3+...)-1=0,$$

then for

$$(1+\varepsilon(2x_1)+\varepsilon^2(2x_2+x_1^2)+\varepsilon^3(2x_3 2x_1x_2)+...)+(\varepsilon+\varepsilon^2 x_1+\varepsilon^3 x_2+...)-1=0.$$

Now equating the powers of ε *on both sides of the equation:*

$$1-1=0,$$

$$2x_1+1=0\Rightarrow x_1=-\frac{1}{2},$$

$$2x_2+x_1^2+x_1=0\Rightarrow x_2=\frac{1}{2},$$

$$2x_3+2x_1x_2+x_2=0\Rightarrow x_3=0.$$

Note that the first equation is trivial since we actually expanded about the $\varepsilon=0$ *solution, namely* $x_0=1$ *so:*

$$x(\varepsilon)=1-\frac{1}{2}\varepsilon+\frac{1}{8}\varepsilon^2+\varphi(\varepsilon^4).$$

For ε *small, this expansion truncated after the third term is a good approximation to the actual positive root of* (1.19)*, and We say it is an order 4 approximation , the error is a term* $\varphi(\varepsilon^4)$.

1.6 First order averaging method for periodic orbits

The method of averaging is a classical tool allowing us to study the dynamics of the nonlinear differential systems under periodic forcing, the method of averaging has a long history starting with the classical works of Lagrange and Laplace, who provided an intuitive justification of the method. The first formalization of this theory was done in 1928 by Fatou [4].

In general, to obtain analytically periodic solutions of a differential system is a very difficult problem, many times a problem impossible to solve. As we shall see when we can apply the averaging theory, this difficult problem for differential systems

$$x'=\varepsilon F(t,x)+\varepsilon^2 R(t,x,\varepsilon). \tag{1.20}$$

Is reduced to finding the zeros of a non-linear function of dimension at most n, i.e., now the problem has the same difficulty as the problem of finding the singular or equilibrium points of a differential system. An important problem for studying periodic solutions of differential systems of the form

$$x' = F(t,x) \qquad or \qquad x' = F(x) \tag{1.21}$$

using averaging theory is to transform them into systems written in the normal form of the averaging theory, i.e., as a system (1.20). Note that systems (1.21), in general, are not periodic in the independent variable t and do not have any small parameter ε. So, we must find changes of variables which allow us to write the differential systems (1.21) into the form (1.20).
We consider the differential equation:

$$x' = \varepsilon F(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{1.22}$$

with $x \in D \subset R^n$, D a bounded domain, and $t \geq 0$. Moreover we assume that $F(t,x)$ and $R(t,x,\varepsilon)$ are T-periodic in t.
The averaged system associated to the system(1.22) is defined by:

$$Y' = \varepsilon f^0(y), \tag{1.23}$$

where

$$f^0(y) = \frac{1}{T}\int_0^T F(s,y)ds. \tag{1.24}$$

The next theorem says under what conditions the singular points of the averaged system (1.23) provide T-periodic orbits for the system (1.22).

Theorem 1.1 *We consider system* (1.22) *and assume that the vector functions* F, $R, D_xF, D_x^2\, F$ *and* D_XR *are continuous and bounded by a constant* M *(independent of* ε *in* $[0,\infty) \times D$*, with* $-\varepsilon_O < \varepsilon < \varepsilon_0$*. Moreover, we suppose that* F *and* R *are* T*-periodic in* t*, with* T *independent of* ε*.*

(i) *If* $p \in D$ *is a singular point of the averaged system* (1.23) *such that:*

$$det(D_x f^0(p)) \neq 0, \tag{1.25}$$

then, for $|\varepsilon| > 0$ *sufficiently small, there exists a* T*-periodic solution* $x(t,\varepsilon)$ *of system* (1.22) *such that* $x(0,\varepsilon) \to p$ *as* $\varepsilon \to 0.$

(ii) *If the singular point* $y = p$ *of the averaged system* (1.23) *has all its eigenvalues with negative real part then, for* $|\varepsilon| > 0$ *sufficiently small, the corresponding periodic solution* $x(t,\varepsilon)$ *of system* (1.22) *is asymptotically stable and, if one of the eigenvalues has positive real part* $x(t,\varepsilon)$*, it is unstable.*

Chapter 2

The Van der Pol differential equation using perturbation method

2.1 G'/G Expansion method

In this section we apply the G'/G expansion method to determine the solution of the van der pol equation [1]. We consider the well known Van der Pol's differential equation:

$$x'' + \varepsilon\,(x^2 - \lambda)\,x' + \alpha\,x = 0, \tag{2.1}$$

where x is the unknown function and t is the independent variable which represents time. The derivative is with respect to t. We use the following change of variable,

$$x = \sqrt{w}, \qquad w \equiv w(t). \tag{2.2}$$

Since we have:

$$x' = \frac{w'}{2\sqrt{w}},$$

and

$$x'' = \frac{2w''\sqrt{w} - (w')^2}{4w\sqrt{w}} = \frac{1}{2\sqrt{w}}\left(w'' - \frac{1}{2}\frac{(w')^2}{w}\right).$$

Equation (2.1) becomes:

$$\frac{1}{2\sqrt{w}}\left(w'' - \frac{1}{2}\frac{(w')^2}{w}\right) + \varepsilon(w-\lambda)\frac{w'}{2\sqrt{w}} + \alpha\sqrt{w} = 0. \tag{2.3}$$

We multiply this last equation by $2\,w\,\sqrt{w}$, we get:

$$w\,w'' - \frac{1}{2}\left(w'\right)^2 + \varepsilon w^2 w' - \lambda\varepsilon w w' + 2\alpha w^2 = 0. \tag{2.4}$$

Now we solve (2.4) by using the G'/G expansion method. We consider the two terms of the power series with respect to G'/G, i.e.;

$$w = a_0 + a_1\left(\frac{G'}{G}\right), \tag{2.5}$$

where G is a function and a_0, a_1 are constant coefficients.

The function G is forced to satisfy the second order linear differential equation:

$$G'' + m\,G' + n\,G = 0, \tag{2.6}$$

where m and n are constant coefficients . Differentiation of (2.5) with respect to t gives

$$\begin{aligned} w' &= a_1\left(\frac{G''G - (G')^2}{G^2}\right), \\ &= a_1\left(\frac{G''}{G} - \left(\frac{G'}{G}\right)^2\right). \end{aligned} \tag{2.7}$$

dividing (2.6) by G we get:

$$\frac{G''}{G} + m\frac{G'}{G} + n = 0,$$

which means that:

$$\frac{G''}{G} = -m\left(\frac{G'}{G}\right) - n,$$

and then we substitute $\dfrac{G''}{G}$ by $-m\left(\dfrac{G'}{G}\right) - n$, which gives:

$$\begin{aligned} w' &= a_1\left(-m\left(\frac{G'}{G}\right) - n - \left(\frac{G'}{G}\right)^2\right), \\ &= -a_1 n - a_1 m\left(\frac{G'}{G}\right) - a_1\left(\frac{G'}{G}\right)^2. \end{aligned} \tag{2.8}$$

Differentiation of the above equation with respect to time and making the substitution $\frac{G''}{G} \rightarrow$

$-m(\frac{G'}{G})-n$, we obtain:

$$\begin{aligned} w'' &= -a_1 m\left(\frac{G''}{G}-\left(\frac{G'}{G}\right)^2\right)-2a_1\left(\frac{G''}{G}-\left(\frac{G'}{G}\right)^2\right)\left(\frac{G'}{G}\right) \\ &= a_1 n(m+2)+a_1 m^2(\frac{G'}{G})+3a_1 m(\frac{G'}{G})^2+2a_1\left(\frac{G'}{G}\right)^3. \end{aligned} \quad (2.9)$$

Substitution of (2.5), (2.8) and (2.9) into (2.4) rearranging, we obtain the equation:

$$\begin{aligned} & a_1^2\left(\frac{3}{2}-\varepsilon a_1\right)\left(\frac{G'}{G}\right)^4+a_1[a_1(\lambda\varepsilon+2m-2\varepsilon a_0)-\varepsilon m a_1^2+2a_0]\left(\frac{G'}{G}\right)^3 \\ & +[-\varepsilon n a_1^2+a_1(2\alpha+m\lambda\varepsilon-n+\frac{m^2}{2}-2\varepsilon m a_0)+a_0(\lambda\varepsilon-\varepsilon a_0+3m)]\left(\frac{G'}{G}\right)^2 \\ & +a_1[a_1(n\lambda\varepsilon-2\varepsilon n a_0+2n)+a_0(4\alpha+\lambda\varepsilon m+m^2-\varepsilon m a_0)]\left(\frac{G'}{G}\right) \\ & +a_1[n(m+2)a_0-\varepsilon n a_0^2-n^2\frac{a_1}{2}+n\lambda\varepsilon a_0]+2\alpha a_0^2=0. \end{aligned} \quad (2.10)$$

Equation the coefficients of the different power of $(\frac{G'}{G})^k, k=1,2,3,4$ to zero we obtain the following system of algebraic equation $(a_1\neq 0)$:

$$\frac{3}{2}-\varepsilon a_1=0$$

$$a_1(\lambda\varepsilon+2m-2\varepsilon a_0)-\varepsilon m a_1^2+2a_0)=0$$

$$-\varepsilon n a_1^2+a_1(2\alpha+m\lambda\varepsilon-n+\frac{m^2}{2}-2\varepsilon m a_0)+a_0(\lambda\varepsilon-\varepsilon a_0+3m)=0$$

$$[a_1(n\lambda\varepsilon-2\varepsilon n a_0+2n)+a_0(4\alpha+\lambda\varepsilon m+m^2-\varepsilon m a_0)=0$$

$$a_1[n(m+2)a_0-\varepsilon n a_0^2-n^2\frac{a_1}{2}+n\lambda\varepsilon a_0]+2\alpha a_0^2)=0$$

solving the above system of algebraic equations,we obtain the following two families of solutions:

Solution 2.1

$$\begin{aligned} a_0 &= -\frac{3}{2\varepsilon}, \\ a_1 &= \frac{3}{2\varepsilon}, \\ m &= -2(\lambda\varepsilon+1), \\ n &= 2\lambda\varepsilon+1, \\ \alpha &= 2\lambda\varepsilon+1. \end{aligned} \quad (2.11)$$

Solution 2.2

$$
\begin{aligned}
a_0 &= -\frac{3\lambda}{\alpha-1},\\
a1 &= \frac{3\lambda}{\alpha-1},\\
m &= -2(\alpha+1),\\
n &= \alpha,\\
\varepsilon &= \frac{\alpha-1}{2\lambda}.
\end{aligned}
\tag{2.12}
$$

Now we use the polynomial $r^2+mr+n=0$ to find the general solution of $G''+mG'+nG=0$. Because the discriminant is given by

$$\Delta = m^2-4n \Rightarrow \sqrt{\Delta} = \sqrt{m^2-4n}.$$

Then we have two real roots

$$\begin{cases} G_1 = \dfrac{-m+\sqrt{m^2-4\varepsilon}}{2},\\ G_2 = \dfrac{-m-\sqrt{m^2-4\varepsilon}}{2}, \end{cases}$$

and the general solution is given by

$$G(t) = k_1\exp(G_1t) + k_2\exp(G_2t).$$

where k_1, k_2 are constants. Hence

$$G(t) = k_1\exp\left(\frac{-m}{2}+\frac{\sqrt{\Delta}}{2}\right)t + k_2\exp\left(\frac{-m}{2}-\frac{\sqrt{\Delta}}{2}\right)t).$$

We differentiate G with respect to t to get

$$
\begin{aligned}
G'(t) &= k_1\left(\frac{-m}{2}+\frac{\sqrt{\Delta}}{2}\right)\exp\left(\frac{-m}{2}+\frac{\sqrt{\Delta}}{2}\right)t\\
&+ k_2\left(\frac{-m}{2}-\frac{\sqrt{\Delta}}{2}\right)\exp\left(\frac{-m}{2}-\frac{\sqrt{\Delta}}{2}\right)t.
\end{aligned}
$$

So

$$\begin{aligned}
\frac{G'}{G} &= \frac{\exp(\frac{-m}{2}t)\left[k_1\left(\frac{-m}{2}+\frac{\sqrt{\Delta}}{2}\right)\exp(\frac{\sqrt{\Delta}}{2}t)+k_2\left(\frac{-m}{2}-\frac{\sqrt{\Delta}}{2}\right)\exp(\frac{-\sqrt{\Delta}}{2}t)\right]}{\exp(\frac{-m}{2}t)\left[k_1\exp(\frac{\sqrt{\Delta}}{2}t)+k_2\exp(\frac{-\sqrt{\Delta}}{2}t)\right]},\\
&= \frac{\frac{k_1}{2}(\sqrt{\Delta}-m)\exp(\frac{\sqrt{\Delta}}{2}t)-\frac{k_2}{2}(\sqrt{\Delta}+m)\exp(-\frac{\sqrt{\Delta}}{2}t)}{k_1\exp(\frac{\sqrt{\Delta}}{2}t)+k_2\exp(-\frac{\sqrt{\Delta}}{2}t)},\\
&= \frac{-m}{2}+\frac{\sqrt{\Delta}}{2}\left(\frac{k_1\exp(\frac{\sqrt{\Delta}}{2}t)-k_2\exp(\frac{-\sqrt{\Delta}}{2}t)}{k_1\exp(\frac{\sqrt{\Delta}}{2}t)+k_2\exp(\frac{-\sqrt{\Delta}}{2}t)}\right),
\end{aligned}$$

Since we have

$\exp(\frac{\sqrt{\Delta}}{2}t)=\cosh\left(\frac{\sqrt{\Delta}}{2}t\right)+\sinh\left(\frac{\sqrt{\Delta}}{2}t\right)$ and

$\exp(-\frac{\sqrt{\Delta}}{2}t)=\cosh\left(\frac{\sqrt{\Delta}}{2}t\right)-\sinh\left(\frac{\sqrt{\Delta}}{2}t\right).$

We conclude that

$$\begin{aligned}
\frac{G'}{G} &= \frac{\sqrt{\Delta}}{2}\left(\frac{k_1\cosh(\frac{\sqrt{\Delta}}{2}t)+k_1\sinh(\frac{\sqrt{\Delta}}{2}t)-k_2\cosh(\frac{\sqrt{\Delta}}{2}t)+k_2\sinh(\frac{\sqrt{\Delta}}{2}t)}{k_1\cosh(\frac{\sqrt{\Delta}}{2}t)+k_1\sinh(\frac{\sqrt{\Delta}}{2}t)+k_2\cosh(\frac{\sqrt{\Delta}}{2}t)-k_2\sinh(\frac{\sqrt{\Delta}}{2}t)}\right)\\
&- \frac{m}{2},\\
&= \frac{-m}{2}+\frac{\sqrt{\Delta}}{2}\left(\frac{(k_1-k_2)\cosh(\frac{\sqrt{\Delta}}{2}t)+(k_1+k_2)\sinh(\frac{\sqrt{\Delta}}{2}t)}{(k_1+k_2)\cosh(\frac{\sqrt{\Delta}}{2}t)+(k_1-k_2)\sinh(\frac{\sqrt{\Delta}}{2}t)}\right),
\end{aligned}$$

Assuming that $k_1+k_2=c_1, \quad k_1-k_2=c_2$

$$\frac{G'}{G}=\frac{-m}{2}+\frac{\sqrt{\Delta}}{2}\left(\frac{c_1\sinh(\frac{\sqrt{\Delta}}{2}t)+c_2\cosh(\frac{\sqrt{\Delta}}{2}t)}{c_1\cosh(\frac{\sqrt{\Delta}}{2}t)+c_2\sinh(\frac{\sqrt{\Delta}}{2}t)}\right),$$

and from solution 1 we have

$$m=-2(\lambda\varepsilon+1),$$

and

$$n=2\lambda\varepsilon+1.$$

Replacing m and n in $\sqrt{\Delta}$ gives

$$\begin{aligned}
\sqrt{\Delta} &= \sqrt{m^2 - 4n}, \\
&= \sqrt{(-2(\lambda\varepsilon + 1)) - 4(2\lambda\varepsilon + 1)}, \\
&= 2\sqrt{\lambda^2\varepsilon^2 + 1 + 2\lambda\varepsilon - 2\lambda\varepsilon - 1}, \\
&= 2\lambda\varepsilon.
\end{aligned}$$

substitute $\sqrt{\Delta}$ by $2\lambda\varepsilon$ in $\frac{G'}{G}$ we get:

$$\frac{G'}{G} = \lambda\varepsilon + 1 + \lambda\varepsilon\left(\frac{c_1 \sinh(\lambda\varepsilon t) + c_2 \cosh(\lambda\varepsilon t)}{c_1 \cosh(\lambda\varepsilon t) + c_2 \sinh(\lambda\varepsilon t)}\right).$$

since we have determined the coefficients m and n ,we can evaluate the function G from (2.6) becomes;

$$G'' - 2(\lambda\varepsilon + 1)G' + (2\lambda\varepsilon + 1)G = 0, \tag{2.13}$$

equation (2.13) admets the general solution:

$$G(t) = [A \cosh(\lambda\varepsilon t) + B \sinh(\lambda\varepsilon t)]\, e^{(1+\lambda\varepsilon)t}, \tag{2.14}$$

we thus obtain:

$$\frac{G'}{G} = (1 + \lambda\varepsilon) + \frac{H'}{H}, \tag{2.15}$$

where $H = H(t)$is given by:

$$H = A \cosh(\lambda\varepsilon t) + B \sinh(\lambda\varepsilon t), \tag{2.16}$$

on the other hand, equation (2.5)gives:

$$W = -\frac{3}{2\varepsilon} + \frac{3}{2\varepsilon}\left(\frac{G'}{G}\right), \tag{2.17}$$

therefor, using (2.15) and (2.17),we have:

$$W = \frac{3}{2\varepsilon}\left(\lambda\varepsilon + \frac{H'}{H}\right), \tag{2.18}$$

we thus find that the first family of solutions of the van der pol equation is given by:

$$x(t) = \sqrt{\frac{3}{2\varepsilon}\left(\lambda\varepsilon + \frac{H'}{H}\right)}, \tag{2.19}$$

in this cace we shold take into sccount the relation

$$\alpha = -1 + 2\lambda\varepsilon$$

for the solution 2, equation becomes;

$$G'' - (\alpha + 1)G' + \alpha G = 0, \tag{2.20}$$

which admits general solution:

$$G = C_1 \exp(\alpha t) + C_2 \exp(t), \tag{2.21}$$

and thus:

$$\frac{G'}{G} = \frac{\alpha \exp(\alpha t) + C \exp(t)}{\exp(\alpha t) + C \exp(t)}, \qquad C = \frac{C_1}{C_2}, \tag{2.22}$$

on the other hand,equation (2.5) gives:

$$W = -\frac{3\lambda}{\alpha - 1} + \frac{3\lambda}{\alpha - 1\alpha}\left(\frac{G'}{G}\right), \tag{2.23}$$

therefore using (2.22) and (2.23), we have:

$$W = 3\lambda \frac{\exp(\alpha t)}{\exp(\alpha t) + C \exp(t)}, \tag{2.24}$$

we thus find that the second family of solutions of the vander pol equation is given by:

$$x(t) = \sqrt{3\lambda \frac{e^{\alpha t}}{e^{\alpha t} + Ce^{t}}}. \tag{2.25}$$

In this case we should also take into account the relation;

$$\varepsilon = \frac{(\alpha - 1)}{2\lambda}, \qquad \alpha = 1 + 2\lambda\varepsilon, \tag{2.26}$$

we thus arrive at the following theorem:

Theorem 2.1 *The van der pol equation :*

$$x'' + \varepsilon(x^2 - \lambda)x' + \alpha x = 0. \tag{2.27}$$

Under the substitution $x = \sqrt{w}$ *transforms into the equation:*

$$ww'' - \frac{1}{2}(w')^2 + \varepsilon w^2 w' - \lambda\varepsilon w w' + 2\alpha w^2 = 0. \tag{2.28}$$

The above equation is solved using the $\left(\frac{G'}{G}\right)$ *- method, considering the expansion* $w = a_0 + a_1\left(\frac{G'}{G}\right)$*, where* $G \equiv G(t)$ *is a function satisfying the linear second order differential equation* $G'' + mG' + nG = 0$ *and* a_0, a_1*,* m*,* n *are constant coeffcients. There are two families of solutions the first family is given by*

$$x(t) = \sqrt{\frac{3}{2\varepsilon}\left(\lambda\varepsilon + \frac{H'}{H}\right)}, \quad H = A\cosh(\lambda\varepsilon t) + B\sinh(\lambda\varepsilon t), \quad \alpha = 1 + 2\lambda\varepsilon. \tag{2.29}$$

the second family is given by

$$x(t) = \sqrt{3\lambda\frac{e(\alpha t)}{e(\alpha t) + C\exp(\alpha t)}}, \qquad \alpha = 1 + \lambda\varepsilon. \tag{2.30}$$

2.2 The Van der Pol non linear differential equation using a first order approximation perturbation method

In this section we use the first order approximation perturbation method to find the approximate solution [8].We consider the The Van der Pol equation :

$$x'' - \varepsilon(1 - x^2)x' + x = 0. \tag{2.31}$$

And assuming initial conditions are $x(0) = \varphi$ and $x'(0) = \xi$. With a restriction on initial condition, $x(0)^2 + x'^2(0) = 4$, which means, $\xi^2 + \varphi^2 = 4$,with $\varepsilon \ll 1$.

We assume that the solution $x(t)$ of the Van der Pol equation is given as the following power series

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots$$

Now we neglect the higher powers of ε because they're too small, we get

$$x(t) = x_0(t) + \varepsilon x_1(t). \tag{2.32}$$

To determine $x_0(t)$ and $x_1(t)$, we substituting (2.32) into (2.31) which gives

$$\begin{aligned} &x''_0 + \varepsilon x''_1 - \varepsilon x'_0 + \varepsilon x_0^2 x'_0 + \varepsilon^3 x_1^2 x'_0 + 2\varepsilon^2 x_1 x_0 x'_0 - \varepsilon^2 x'_1 \\ &+ \varepsilon^2 x_0^2 x'_1 + \varepsilon^4 x_1^2 x'_1 + 2\varepsilon^3 x_1 x_0 x'_1 + x_0 + \varepsilon x_1 = 0. \end{aligned}$$

Collecting terms with same power of ε gives

$$\varepsilon^0(x''_0 + x_0) + \varepsilon(x''_1 - x'_0 + x_0^2 x'_0 + x_1) + \varepsilon^2(x_0^2 x'_1 + 2x_1 x_0 x'_0 - x'_1)$$
$$+\varepsilon^3(x_1^2 x'_0 + 2x_1 x_0 x'_1) + \varepsilon^4 x_1^2 x'_1 = 0.$$

Setting terms which multiply by higher power ε to zero and since it is assumed that ε is very small ($\varepsilon^n \simeq 0, n > 2$) therefore:

$$(x''_0 + x_0) + \varepsilon(x''_1 - x'_0 + x_0^2 x'_0 + x_1) = 0.$$

For the left hand side to be zero implies that

$$x''_0 + x_0 = 0, \tag{2.33}$$

and

$$(x''_1 - x'_0 + x_0^2 x'_0 + x_1) = 0, \tag{2.34}$$

the solution of equation (2.33) for x_0 is :

$$x_0(t) = A_0 \cos t + B_0 \sin t.$$

Assuming that the initial conditions for $x_1(t)$ at $t = 0$ are zero , then initial conditions for $x_0(t)$ can be taken to be those given for $x(t)$:

$$\begin{cases} x_0(0) = \varphi, \ x'_0(0) = \xi, \\ x_1(0) = 0, \ x'_1(0) = 0. \end{cases}$$

Which gives

$$\begin{cases} x(0) = x_0(0) + \varepsilon x_1(0) = \varphi, \\ x'(0) = x'_0(0) + \varepsilon x'_1(0) = \xi. \end{cases}$$

Solving for A_0 , B_0 gives:

$$x_0(t) = \varphi \cos t + \xi \sin t.$$

Since we determined $x_0(t)$ we can solve (2.34),

$$x''_1 + x_1 = x'_0 - x_0^2 x'_0.$$

The substituting of the solution $x_0(t)$ and its derivative into (2.34) gives:

$$\begin{aligned} x''_1 + x_1 &= -\varphi\cos t + \xi\sin t - \xi^3\cos t\sin^2 t - 2\xi^2\varphi\cos^2 t\sin t + \xi^2\varphi\sin^3 t \\ &- \xi\varphi^2\cos^3 t + 2\xi\varphi^2\cos t\sin^2 t + \varphi^3\cos^2 t\sin t. \end{aligned}$$

Using the fact that :

$$\sin t\cos^2 t = \frac{1}{4}(\sin t + \sin 3t),$$

and

$$\cos t\sin^2 t = \frac{1}{4}(\cos t - \cos 3t),$$

the above equation can be simplified to:

$$\begin{aligned} x''_1 + x_1 &= \left(-\varphi - \frac{\xi^2\varphi}{2} + \frac{\varphi^3}{4}\right)\sin t + \left(\xi - \frac{\xi^3}{4} + \frac{\xi\varphi^2}{2}\right)\cos t \\ &+ \left(\frac{\xi^3}{4} - \frac{\xi\varphi^2}{2}\right)\cos 3t + \left(-\frac{\xi^2\varphi}{2} + \frac{\varphi^3}{4}\right)\sin 3t + \xi\varphi\left(\xi\sin^3 t - \varphi\cos^3 t\right). \end{aligned}$$

Using the following equality:

$$\xi\sin^3 t - \varphi\cos^3 t = \frac{1}{4}(-3\varphi\cos t - \varphi\cos 3t + 3\xi\sin t - \xi\sin 3t),$$

the above can be simplified further to

$$\begin{aligned} x''_1 + x_1 &= (-\varphi - \frac{\xi^2\varphi}{2} + \frac{\varphi^3}{4} + \frac{3\xi^2\varphi}{4})\sin t + (\xi - \frac{\xi^3}{4} + \frac{\xi\varphi^2}{2} - \frac{3\xi\varphi^2}{4})\cos t \\ &+ (\frac{\xi^3}{4} - \frac{\xi\varphi^2}{2} - \frac{\xi\varphi^2}{4})\cos 3t + (-\frac{\xi^2\varphi}{2} + \frac{\varphi^3}{4} - \frac{\xi^2\varphi}{4})\sin 3t, \end{aligned} \tag{2.35}$$

using the restriction regarding the initial condition, $\xi^2 + \varphi^2 = 4$, we obtain

$$(-\varphi - \frac{\xi^2\varphi}{2} + \frac{\varphi^3}{4} + \frac{3\xi^2\varphi}{4}) = 0,$$

and

$$(\xi - \frac{\xi^3}{4} + \frac{\xi\varphi^2}{2} - \frac{3\xi\varphi^2}{4}) = 0.$$

Which means that the equation (2.35) becomes:

$$x''_1 + x_1 = (\frac{\xi^3}{4} - \frac{\xi\varphi^2}{2} - \frac{\xi\varphi^2}{4})\cos 3t + (-\frac{\xi^2\varphi}{2} + \frac{\varphi^3}{4} - \frac{\xi^2\varphi}{4})sin3t.$$

Then, for

$$x''_1 + x_1 = \xi(\frac{\xi^2}{4} - \frac{3\varphi^2}{4})cos3t + \varphi(\frac{\varphi^2}{4} - \frac{3\xi^2}{4})sin3t,$$

with $\varphi^2 = 4 - \xi^2$ and $\xi^2 = 4 - \varphi^2$, the previous equation becomes;

$$x''_1 + x_1 = \xi(\xi^2 - 3)cos3t + \varphi(\varphi^2 - 3)sin3t. \tag{2.36}$$

The homogeneous solution of equation (2.36) is:

$$x_{1;h} = c_1 cost + c_2 sint,$$

and the two particular solutions $x_{1,p1}(t), x_{1,p2}(t)$ are given by

$$x_{1;p_1} = a_1 cos3t + a_2 sin3t,$$

and

$$x_{1;p_2} = d_1 cos3t + d_2 sin3t.$$

By substituting each of these particular solutions into (2.36) one by one, we can determine a_1 , a_2, d_1, and d_2. We obtain the following result:

$$x_{1;p_1} = \frac{\xi(3 - \xi^2)}{8} cos3t,$$

similarly,

$$x_{1;p_2} = \frac{\varphi(3 - \varphi^2)}{8} sin3t.$$

the solution of (2.36) becomes;

$$\begin{aligned} x_1(t) &= x_{1,h} + x_{1,p_1} + x_{1,p_2}, \\ &= (c_1 cos(t) + c_2 sin(t)) + \frac{\xi(3 - \xi^2)}{8} cos(3t) + \frac{\varphi(3 - \varphi^2)}{8} sin(3t). \end{aligned} \tag{2.37}$$

By using the initial conditions $x_1(0) = 0$ and $x_1'(0) = 0$, to determine the coefficients c_1 and c_2 in equation (2.37), and we have:

$$x_1(t) = \frac{\xi(\xi^2 - 3)}{8} cost + \frac{3}{8}\varphi(\varphi^2 - 3)sint + \frac{\xi(3 - \xi^2)}{8} cos3t + \frac{\varphi(3 - \varphi^2)}{8} sin3t.$$

Therefore we now have the final perturbation solution wiche is:

$$\begin{aligned}
x(t) &= x_0(t) + \varepsilon x_1(t), \\
&= \varphi cost + \xi sint + \varepsilon \left(\frac{\xi(\xi^2 - 3)}{8} cost + \frac{3}{8}\varphi(\varphi^2 - 3) sint \right) \\
&+ \left(\frac{\xi(3 - \xi^2)}{8} cos3t + \frac{\varphi(3 - \varphi^2)}{8} sin3t \right).
\end{aligned}$$

Where $x(0) = \varphi$ and $x'(0) = \xi$ and the above solution is valid for ε small under the restriction $x^2(0) + x'^2(0) = 4$.

2.3 Periodic solution of the Van der Pol equation using the first order averaging method

In this last section we use the averaging method to prove that the Van der Pol equation has a periodic solution and studying its stability [4].

The Van der Pol differential equation

$$x'' + x = \varepsilon\,(1 - x^2)\,x',$$

can be written in the following equivalent differential system form

$$\begin{aligned} x' &= y, \\ y' &= -x + \varepsilon\,(1 - x^2)\,y. \end{aligned} \tag{2.38}$$

Using the polar coordinates (r, θ) we have

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta).$$

Differentiate both r and θ with respect to t, it follows that

$$\begin{aligned} r' &= (xx' + yy')/r, \\ &= (xy - xy + \varepsilon(1 - x^2)y^2)/r, \\ &= (r^2\sin(\theta)^2(1 - r^2\cos(\theta)^2)\varepsilon)/r, \\ &= r\varepsilon(1 - r^2\cos(\theta)^2)\sin(\theta)^2. \end{aligned} \tag{2.39}$$

And

$$\begin{aligned} \theta' &= (x\,y' - x'\,y)/r^2, \\ &= (-(x^2 + y^2) + x\,y\,\varepsilon(1 - x^2))/r^2, \\ &= [-r^2 + r^2\varepsilon(1 - r^2\cos(\theta)^2)\cos(\theta)\sin(\theta)]/r^2, \\ &= -1 + \varepsilon(1 - r^2\cos(\theta)^2)\cos(\theta)\sin(\theta). \end{aligned}$$

Since

$$\cos(\theta)\sin(\theta) = \sin(2\theta)/2,$$

after simplification, we get

$$\theta' = -1 + \frac{\varepsilon}{2}(1 - r^2\cos(\theta)^2)\sin(2\theta), \tag{2.40}$$

dividing (2.39) on (2.40) we find:

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{r\varepsilon(1-r^2\cos(\theta)^2)\sin(\theta)^2}{-1+\frac{\varepsilon}{2}(1-r^2\cos(\theta)^2)\sin(2\theta)},\\ &= \frac{-r\varepsilon(1-r^2\cos(\theta)^2)\sin(\theta)^2}{1-\frac{\varepsilon}{2}(1-r^2\cos(\theta)^2)\sin(2\theta)}.\end{aligned}$$

Now, we use the Taylor expansion of second order

$$\begin{aligned}\frac{dr}{d\theta} &= -r\varepsilon(1-r^2\cos(\theta)^2)\sin(\theta)^2(1+\varepsilon\frac{1}{2}(1-r^2\cos(\theta)^2)\sin(2\theta)),\\ &= \varepsilon[-r(1-r^2\cos(\theta)^2)\sin(\theta)^2]+\varepsilon^2[-\frac{1}{2}(1-r^2\cos(\theta)^2)\sin(2\theta))\sin(\theta)^2\\ &- r(1-r^2\cos(\theta)^2)].\end{aligned}$$

This differential system is in the normal form $x' = \varepsilon F(t,x)+\varepsilon^2 R(t,x,\varepsilon)$ for applying the averaging theory, we take $x=r$, $t=\theta$, $T=2\pi$.
$F(r,\theta) = -r(1-r^2\cos(\theta)^2)\sin(\theta)^2$, and
$R(r,\theta) = -\frac{1}{2}(1-r^2\cos(\theta)^2)\sin(2\theta))\sin(\theta)^2 - r(1-r^2\cos(\theta)^2)$,

$F(t,x)$ is a periodic function with period 2π, continuous and bounded. From (1.24) we get

$$\begin{aligned}f^0(r) &= \frac{1}{T}\int_0^T -r(1-r^2\cos(\theta)^2)\sin(\theta)^2 d\theta,\\ &= \frac{1}{2\pi}\int_0^{2\pi} -r(1-r^2\cos(\theta)^2)\sin(\theta)^2 d\theta,\\ &= \frac{-r}{2\pi}\int_0^{2\pi}\sin(\theta)^2 d\theta+\frac{r^3}{2\pi}\int_0^{2\pi}(\cos(\theta)\sin(\theta))^2 d\theta,\\ &= \frac{-r}{2\pi}\int_0^{2\pi}\frac{1-\cos(2\theta)}{2}d\theta+\frac{r^3}{8\pi}\int_0^{2\pi}\sin(2\theta)^2 d\theta,\\ &= \frac{-r}{2\pi}\int_0^{2\pi}\frac{1-\cos(2\theta)}{2}d\theta+\frac{r^3}{8\pi}\int_0^{2\pi}\frac{1-\cos(4\theta)}{2}d\theta,\\ &= \frac{-r}{2\pi}\int_0^{2\pi}\frac{1}{2}d\theta+\frac{r}{2\pi}\int_0^{2\pi}\frac{\cos(2\theta)}{2}d\theta+\frac{r^3}{8\pi}\int_0^{2\pi}\frac{1}{2}d\theta-\frac{r^3}{8\pi}\int_0^{2\pi}\frac{\cos(4\theta)}{2}d\theta,\\ &= \frac{-r}{2}+\frac{r^3}{8},\\ &= \frac{r}{8}(r^2-4).\end{aligned}$$

Then for $y'=0$, we get:

$$f^0(r)=0 \Rightarrow \frac{r}{8}(r^2-4)=0 \Rightarrow r=0 \text{ or } r=2.$$

And taking the derivative of f^0.

$$df^0(r) = \frac{r^2}{4} - \frac{1}{2} \Rightarrow df^0(2) = \frac{1}{2} > 0.$$

The unique positive root of $f^0(r)$ is $r = 2$. Since $df^0(2) = \dfrac{1}{2}$, according to statement **(i)** of the theorem (1.1), we know that the system has a 2π periodic solution, for $\mid \varepsilon \mid = 0$ sufficiently small, a limit cycle bifurcating from the periodic orbit of radius 2 of the unperturbed system (2.38) with $\varepsilon = 0$. Moreover, since $df^0 2 = \dfrac{1}{2} > 0$. By using statement **(ii)**, of theorem (1.1) we conclude that the limit cycle bifurcating from system is unstable [**?**, **?**], Figure 2.1 shows limit cycles of the Van der Pol equation for two different values of ε.

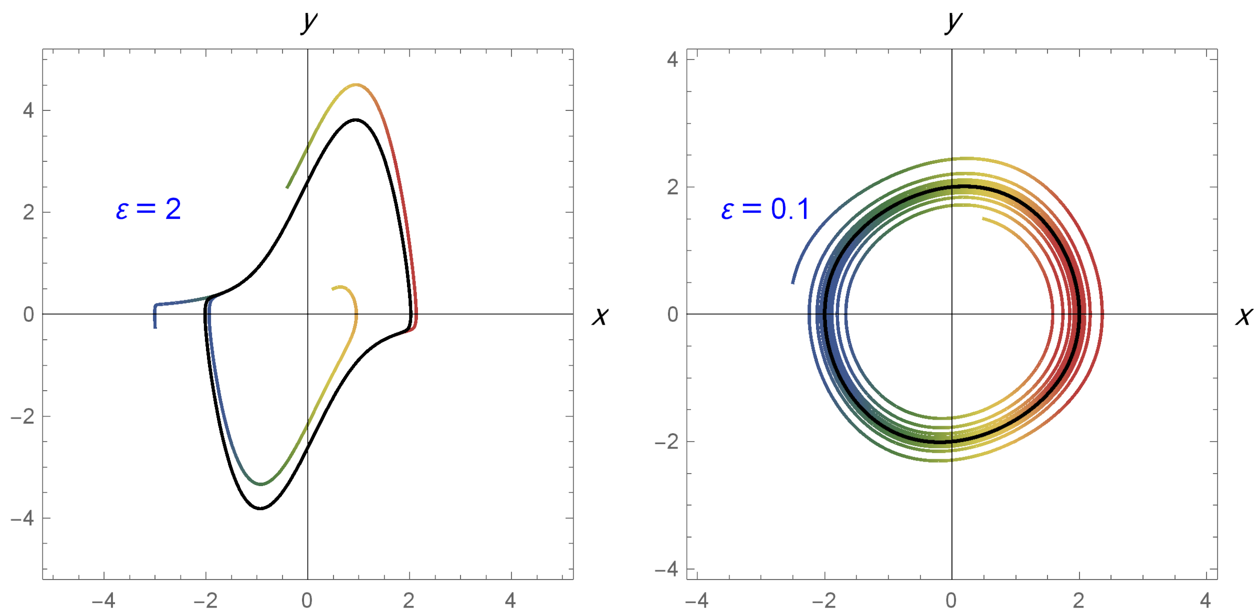

Figure 2.1: The Van der Pol oscillator with limit cycles included for different values of ε.

Chapter 3

Analytical solution of Van der Pol's differential equation using homotoy perturbation method

In this chapter we use the homotopy perturbation method to find the analytical solution of the Van der Pol equation withe a different boundary conditions [6].

3.1 The Van der Pol differential equation with Dirichlet boundary conditions

We consider the boundary value problem for the Van der Pol equation as follows

$$y'' + \varepsilon(y^2 - 1)y' + y = 0, \;\; y(0) = 0, \;\; y(20) = 1,$$

where $y'' + y$ is the linear part and $\varepsilon(y^2 - 1)y'$ are the non linear part, we get the following Homotopy

$$Y'' + Y - (y_0'' + y_0) + t(y_0'' + y_0) + t[\varepsilon(Y^2 - 1)Y'] = 0. \tag{3.1}$$

Suppose that

$$Y = y_0 + ty_1 + t^2 y_2 \tag{3.2}$$

we assume that the solution of (3.2) can be written as a power series in t, substituting (3.2) into (3.2) and equating the coefficients of t from both sides, we get:

$$y_0'' + y_0 = 0, \qquad y_0(0) = 0, \qquad y_0(20) = 1, \tag{3.3}$$

$$y_1'' + y_1 + \varepsilon y_0^2 y_0' - \varepsilon y_0' = 0, \qquad y_1(0) = 0, \qquad y_1(20) = 0, \tag{3.4}$$

and

$$y_2'' + y_2 + 2\varepsilon y_0 y_1 y_0' - \varepsilon y_1' + \varepsilon y_0^2 y_1' = 0, \qquad y_2(0) = 0, \qquad y_2(20) = 0. \tag{3.5}$$

The first boundary value problem (3.3) has the following a solution

$$y_0(x) = A\cos(x) + B\sin(x).$$

From the boundary conditions we obtain $A = 0$ and $B = \dfrac{1}{\sin(20)}$. Replacing by these two parameters, then $y_0(x)$ becomes

$$y_0 = csc(20)\sin(x), \tag{3.6}$$

right now we substituting the solution (3.6) into (3.4) to obtain

$$y_1(x) = \frac{1}{16}\varepsilon\csc^3(20)\sin(x)(\sin(2x) - 2(x-20)(2\cos(40) - 1) - \sin(40)),$$

since we have y_0 and y_1 we can find y_2 from (3.5):

$$\begin{aligned} y_2 \; = \; & \frac{1}{1536}\epsilon^2\csc^5(20)[9\cos(3x)(2(x-20)(2\cos(40)-1)+\sin(40)) \\ & -3\cos(x)(4(x+30)+4(x-60)\cos(40)+8x\cos(80)+3\sin(40)) \\ & +\sin(x)((19-12\cos(40))\cos(2x)-5\cos(4x)+12x(x+2(x-40) \\ & \cos(80)-40+\sin(40)+\sin(80))+14-9621\cos(40)+240\cot(20) \\ & +\csc(20)(4801\sin(100)+360\cos(100)))]. \end{aligned}$$

Thus the two term solution by HPM is $Y = y_0 + y_1$ and the three terms solution by HPM is

$$Y = y_0 + y_1 + y_2$$

Table 3.1 shows that there is a negligible difference between the results of two and three terms HPM solutions comparing with numerical result given by **NDSolve** which is a function of **MATHEMATICA** language that gives numerical solutions for differential equations.

Figures 3.1 and 3.2 show an insignificant difference between the solution given by **ND-Solve[]** function of **MATHEMATICA** and the solutions given by the homotopy method with

x	HPM 2 terms	HPM 3 terms	NDSolve[]	HPM 2 Error	HPM 3 Error
0	0	0	0	0	0
1	0.92171	0.860729	0.860855	0.000329678	0.000126869
2	0.996004	0.932566	0.932669	0.000539139	0.000102668
3	0.154577	0.145325	0.145341	0.0000808258	0.0000155596
4	-0.82897	-0.7828	-0.78287	0.000175989	0.0000699884
5	-1.05036	-0.99452	-0.994591	0.000390629	0.0000661627
6	-0.30606	-0.29088	-0.290896	0.000128774	0.0000185554
7	0.719634	0.687061	0.687094	0.0000729565	0.0000325515
8	1.083699	1.037556	1.03759	0.000252919	0.000038365
9	0.451416	0.433663	0.433679	0.000149599	0.0000157133
10	-0.5959	-0.57511	-0.575121	7.46084E-06	0.0000104587
11	-1.09535	-1.06034	-1.06036	0.000140108	0.0000193591
12	-0.58774	-0.57069	-0.570702	0.000149015	0.0000117983
13	0.460232	0.448929	0.448928	0.0000338741	9.37587E-7
14	1.085068	1.061946	1.06195	0.000060492	8.08511E-6
15	0.712297	0.699029	0.699038	0.000133416	9.19814E-6
16	-0.31536	-0.31084	-0.310833	0.0000621931	5.94807E-6
17	-1.05307	-1.04186	-1.04186	0.0000154493	2.45705E-6
18	-0.8226	-0.81587	-0.815879	0.00011073	8.50523E-6
19	0.164169	0.163462	0.163454	0.0000854309	8.28424E-6
20	1.	1.	1.	1.18091E-07	1.18091E-7

Table 3.1: Relative errors for example 1 with $\varepsilon = \mathbf{0.01}$.

different values of ε.

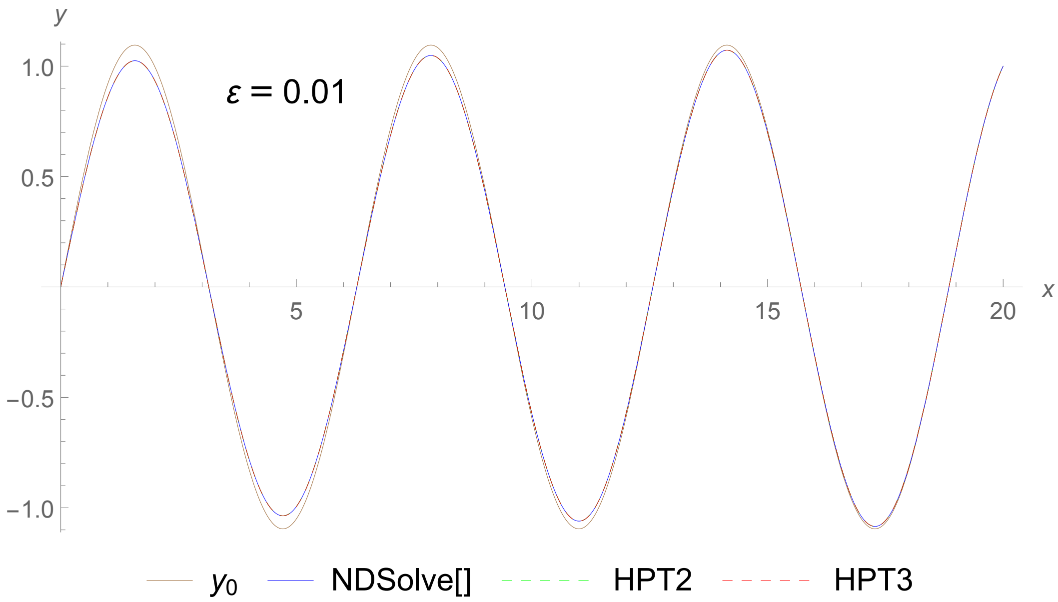

Figure 3.1: The graph of Van der Pol equation showing the difference between numerical and analytical solution with $\boldsymbol{\varepsilon = 0.01}$.

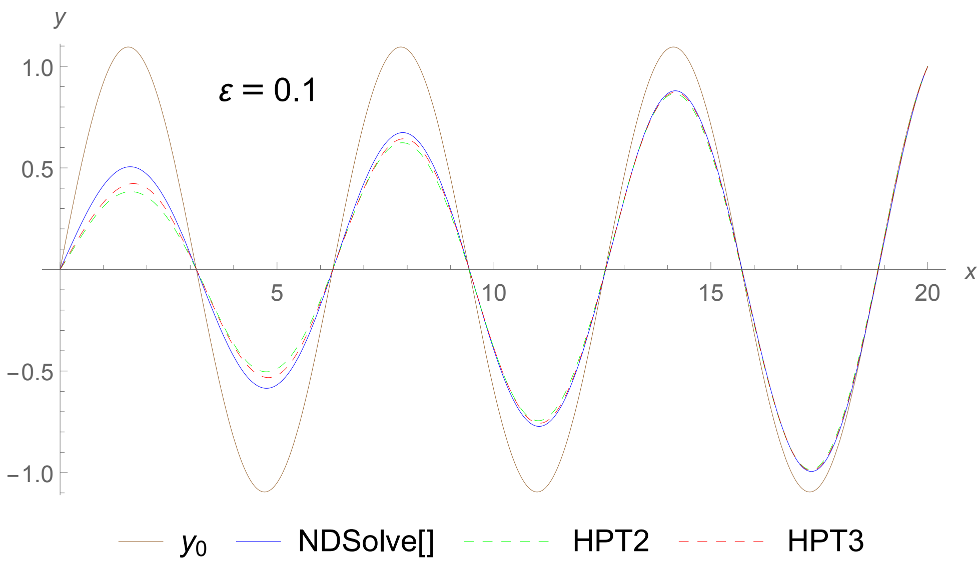

Figure 3.2: The graph of Van der Pol equation showing the difference between numerical and analytical solution with $\boldsymbol{\varepsilon = 0.1}$.

3.2 The Van der Pol equation with first Robin type boundary condition

Consider the VDP equation with first Robin type boundary condition

$$y'' + \varepsilon(y2 - 1)y' + y = 0, \qquad y(0) = 0, \qquad y'(20) = 1.$$

Using the new variable Y from (3.2), replacing it into and equating the coefficients of t from both sides, we get

$$y_0'' + y_0 = 0, \qquad y_0(0) = 0, \qquad y_0'(20) = 1,$$

$$y_1'' + y_1 + u y_0^2 y_0' - u y_0' = 0, \qquad y_1(0) = 0, \qquad y_1'(20) = 0,$$

and

$$y_2'' + y_2 + 2u y_0 y_1 y_0' - u y_1' + u y_0^2 y_0' = 0, \qquad y_2(0) = 0, \qquad y_2'(20) = 0.$$

Solving the first boundary value problem we get $y_0 = \sec(20)\sin(x)$. Then, it follows from the second boundary value problem that y_1 be in the following form

$$\begin{aligned} y_1 \;=\;& \frac{1}{32}\varepsilon \sec^3(20)\sin(x)[x(4+8\cos(40)) + 4\sin(x)\cos(x) \\ & -160 + \tan(20) - \sec(20)(7\sin(60) + 80\cos(60))], \end{aligned}$$

and finally, we determine y_2 by solving the third boundary value problem, therefore

$$\begin{aligned} y_2(x) \;=\;& -\frac{1}{6144}\varepsilon^2 \sec^5(20)[\sin(x)[-\sec^2(20)(24x^2(1+2\cos(40)+2\cos(80)) \\ & +\cos(120)) - 12x(80+\sin(40) - \sin(80) + 7\sin(120) + 160\cos(40) \\ & +160\cos(80) + 80\cos(120)) + 9930 - 480\sin(40) - 720\sin(80) \\ & +1200\sin(120) + 18891\cos(40) + 19495\cos(80) + 9382\cos(120)) \\ & -81\sin^2(2x) + 9720\sin(2x)] - 20\sin^7(x) - 100\sin(x)\cos^6(x) \\ & -2\sin(x)\cos^4(x)(25\cos(2x) - 118 + 72\cos(40)) + 6(3+8\cos(40))\sin^5(x) \\ & -12(4\cos(80)-3)\sin^3(x) + 36(x-20)\sin^3(2x)\csc(x) + 12\sin(x)\cos^2(x) \\ & (15\sin^4(x) - 4\cos(60)\sec(20)\sin^2(x) - 17 + 8\cos(40) - 4\cos(80)) \\ & +3\sec(20)\cos^3(x)[-16x(-5\cos(20) - 2\cos(60) + \cos(100)) + 12\sin^2(x) \\ & (4x\cos(60) + \sin(20) - 7\sin(60) - 80\cos(60)) + 48\sin(20) - 83\sin(60) \\ & +28\sin(100) - 1200\cos(20) - 640\cos(60) + 320\cos(100)] - 18\cos^5(x) \\ & (x(4+8\cos(40)) - 160 + \tan(20) - 80\cos(60)\sec(20) - 7\sin(60)\sec(20)) \\ & +3\cos(x)[-\sec(20)\sin^2(x)(16x(19\cos(20) + 10\cos(60) + \cos(100)) \\ & -253\sin(60) - 28\sin(100) - 3200\cos(60) - 320\cos(100)) + \sec(20) \\ & (8x(3\cos(20) + 3\cos(60) + 4\cos(100)) - 42\sin(20) + 41\sin(60) \\ & -28\sin(100) + 240\cos(20) + 160\cos(60) - 320\cos(100)) + 18\sin^4(x) \\ & (x(4+8\cos(40)) - 160 + \tan(20) - 80\cos(60)\sec(20) - 7\sin(60)\sec(20))]]. \end{aligned}$$

The comparison between numerical and analytical solutions is displayed in Figure 3.3.

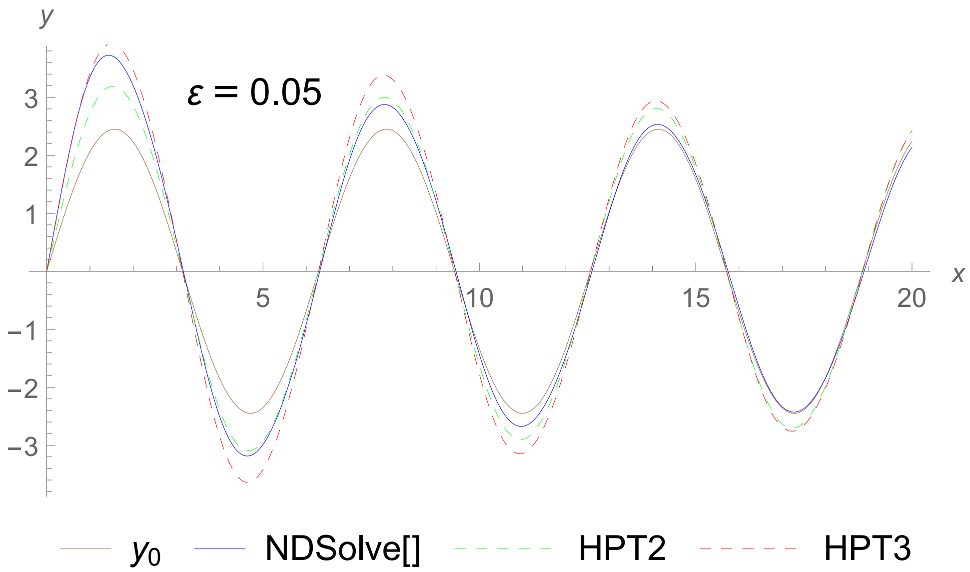

Figure 3.3: The graph of Van der Pol equation showing the difference between numerical and analytical solution with $\varepsilon = \mathbf{0.05}$.

3.3 The Van der Pol equation with second Robin type boundary condition

Consider the VDP equation with second Robin type boundary condition,

$$y'' + u(y^2 - 1)y' + y = 0,\ \ y'(0) = 0,\ \ y(20) = 1$$

The Homotopy is,

$$Y'' + Y - (y_0'' + y_0) + t(y_0'' - y_0) + t[u(Y^2 - 1)Y'] = 0$$

From (3.2) we get

$$Y'' + Y - (y_0'' + y_0) + t(y_0'' - y_0) + t[u(Y^2 - 1)Y'] = 0$$

and equating the coefficients of t from both sides, we get

$$y''_0 + y_0 = 0, y_0'(0) = 0,\ \ y_0(20) = 1,$$

$$y''_1 + y_1 + uy_0^2 y_0' - uy' = 0,\ \ y_1'(0) = 0,\ \ y_1(20) = 0,$$

and

$$y''_2 + y_2 + 2uy_0y_1y_0' - uy_1' + uy_0'^2 y_1' = 0,\ \ y_2'(0) = 0,\ \ y_2(20) = 0$$

Solving the first boundary value problem, we get $y_0 = \sec(20)\cos(x)$, and

$$\begin{aligned} y_1(x) &= \frac{1}{64}\varepsilon\sec^3(20)[-7\sin(x)\cos^2(x)+\sin(x)[\sin^2(x)-1-16\cos(40)]+2\cos(x) \\ &\quad (x(4+8\cos(40))-160-3\tan(20)+5\sec(20)(\sin(60)-16\cos(60)))]. \end{aligned}$$

And finally

$$\begin{aligned} y_2(x) &= \frac{1}{6144}\varepsilon^2\sec^5(20)[\cos(x)[48x^2(1+2\cos(80))+100\sin^6(x)-6(31+48\cos(40)) \\ &\quad \sin^4(x)+12(9+56\cos(40)-4\cos(80))\sin^2(x)-24x\sec(20)(-11\sin(20) \\ &\quad +8\sin(60)-5\sin(100)+80\cos(20)+80\cos(60)+80\cos(100))+19455 \\ &\quad +477\tan^2(20)+6720\tan(20)+19404\cos(60)\sec(20)+19214\cos(100)\sec(20) \\ &\quad -7\cos(140)\sec(20)+3840\sin(60)\sec(20)-2400\sin(100)\sec(20) \\ &\quad +5760\cos(60)\tan(20)\sec(20)-960\cos(100)\tan(20)\sec(20)-576\sin(60) \\ &\quad \tan(20)\sec(20)+92\sin(100)\tan(20)\sec(20)-7\sin(140)\tan(20)\sec(20)] \\ &\quad +20\cos^7(x)+6(15\cos(2x)-18+16\cos(40))\cos^5(x)-4\cos^3(x)[25\sin^4(x) \\ &\quad +(48\cos(40)-69)\sin^2(x)+33+96\cos(40)+12\cos(80)]+12\cos^2(x) \\ &\quad [3\sec(20)\sin^3(x)(4x\cos(60)-3\sin(20)+5\sin(60)-160\cos(20) \\ &\quad -80\cos(60))-2(4+\cos(40))\sin(x)(x(4+8\cos(40))-160-3\tan(20) \\ &\quad -80\cos(60)\sec(20)+5\sin(60)\sec(20))]+54\sin(x)\cos^4(x)(x(4+8\cos(40)) \\ &\quad -160-3\tan(20)-80\cos(60)\sec(20)+5\sin(60)\sec(20))-6\sin(x) \\ &\quad [36x-12x\sin^2(2x)+20x\cos(60)\sec(20)+3\sin^4(x)(x(4+8\cos(40))-160 \\ &\quad -3\tan(20)-80\cos(60)\sec(20)+5\sin(60)\sec(20))+4(\cos(40)-2)\sin^2(x) \\ &\quad (x(4+8\cos(40))-160-3\tan(20)-80\cos(60)\sec(20)+5\sin(60)\sec(20)) \\ &\quad +1280+49\tan(20)+560\cos(60)\sec(20)-160\cos(100)\sec(20)-61\sin(60) \\ &\quad \sec(20)+10\sin(100)\sec(20)]]. \end{aligned}$$

Figure 3.4 shows an insignificant difference between the solution given by **NDSolve[]** function of **MATHEMATICA** and the solutions given by the homotopy method for $\varepsilon = 0.06$.

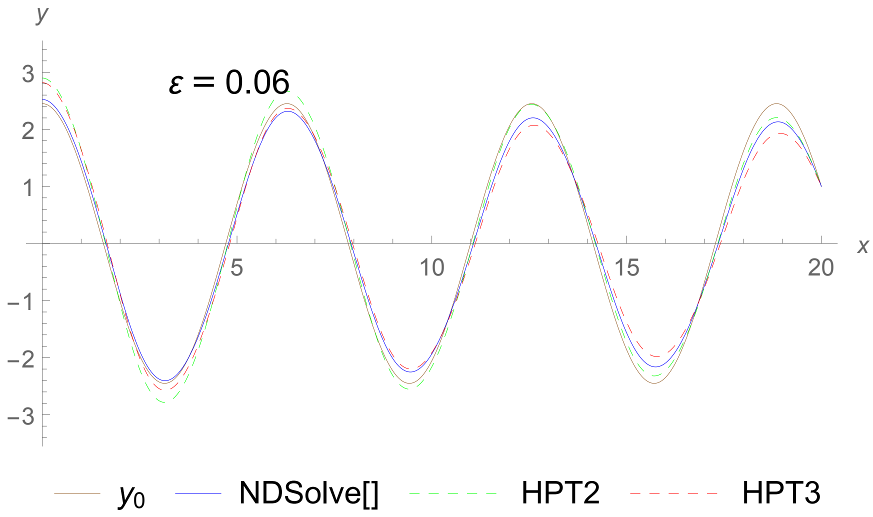

Figure 3.4: The graph of Van der Pol equation showing the difference between numerical and analytical solution with $\varepsilon = 0.06$.

3.4 The Van der Pol equation with Neumann boundary condition

Consider the VDP equation with Neumann type boundary condition

$$y'' + u(y^2 - 1)y' + y = 0, \;\; y'(0) = 0, \;\; y'(20) = 1$$

The Homotopy is,

$$Y'' + Y - (y_0'' + y_0) + t(y_0'' - y_0) + t[u(Y^2 - 1)Y'] = 0.$$

Working as in previous subsection and equating the coefficients of t from both sides, we get

$$y''_0 + y_0 = 0, \;\; y'_0(0) = 0, \;\; y'_0(20) = 1$$

$$y''_1 + y_1 + \varepsilon y_0^2 y_0' - \varepsilon y' = 0, \;\; y'_1(0) = 0, \;\; y'_1(20) = 0$$

$$y''_2 + y_2 + 2\varepsilon y_0 y_1 y_0' - \varepsilon y_1' + \varepsilon y'2_0 y'_1 = 0, \;\; y'_2(0) = 0, \;\; y'_2(20) = 0$$

Solving the first equation, we get $y_0 = \csc(20)(-\cos(x))$.

Hence the second equation gives

$$\begin{aligned} y_1(x) &= \frac{1}{32}\varepsilon \csc^3(20)[\sin(x) + \sin(3x) + 4(x-20)(2\cos(40) - 1)\cos(x) \\ &\quad -6\sin(40)\cos(x) - 8\cos(40)\sin(x)]. \end{aligned}$$

From the third equation we can easily obtain

$$\begin{aligned}
y_2(x) &= \frac{1}{3072}\varepsilon^2\csc^5(20)[\cos(x)[-\csc(20)[24x^2(\sin(20)-\sin(60)+\sin(100))\\
&+12x(-80(\sin(20)-\sin(60)+\sin(100))-3\cos(20)+3\cos(100))\\
&+9540\sin(20)-9399\sin(60)+9430\sin(100)+1440\cos(20)+3840\cos(60)\\
&-240\cos(100)]+25\sin^4(x)+36(7\cos(40)-2)\sin^2(x)]+30\cos^5(x)\\
&-\cos^3(x)(25\sin^2(x)+1+84\cos(40))-63\sin(x)\cos^2(x)(x(4\cos(40)-2)\\
&+40-3\sin(40)-80\cos(40))+3\sin(x)[2x(9-14\cos(40)+8\cos(80))\\
&+3\sin^2(x)(x(4\cos(40)-2)+40-3\sin(40)-80\cos(40))+520\\
&-87\sin(40)-24\sin(80)-1680\cos(40)-640\cos(80)]].
\end{aligned}$$

Figure 3.5 gives a comparison between numerical solution given by **NDSolve[]** function of **MATHEMATICA** and analytical solutions given by homotopy perturbation method with $\varepsilon = 0.07$.

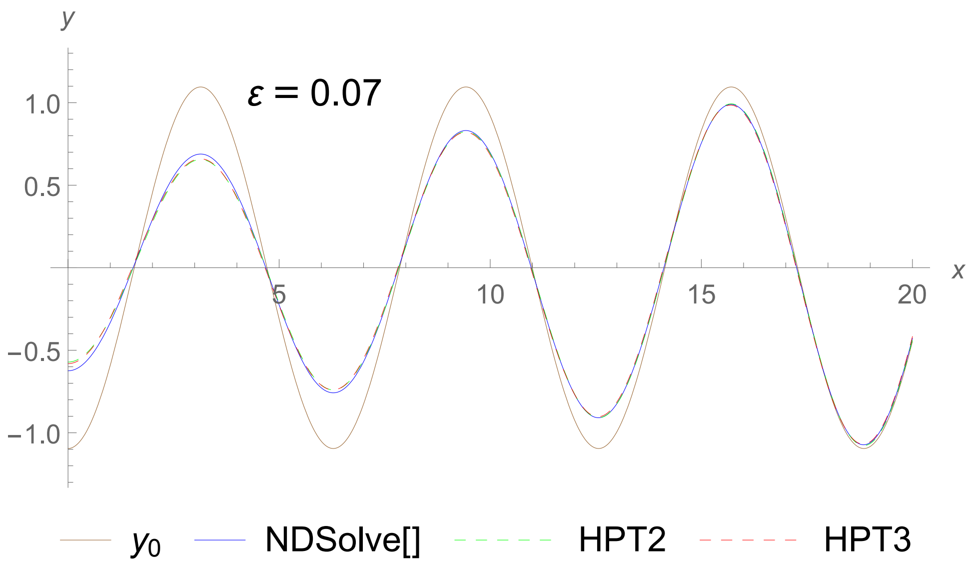

Figure 3.5: The graph of Van der Pol equation showing the difference between numerical and analytical solution with $\varepsilon = 0.07$.

Conclusion

In this work, we have studied the Van der Pol differential equation with different techniques. First, the G'/G expansion method is used to find a general approximate analytical solution, and secondly using the averaging method we have proved that the Van der Pol oscillator has an isolated periodic orbit. Finally, the Van der Pol equation is solved by the perturbation and the homotopy perturbation technique to find approximate analytical solutions.

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