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Mémoire

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Pour l'obtention du diplôme de

Master

Filière : Mathématiques

Spécialité : Mathématiques appliqués

Thème

ON THE FUZZY FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND: THEORY AND APPLICATION

Soutenu publiquement le ... juillet 2021 devant le jury composé de

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Promotion 2020/2021

ACKNOWLEDGMENTS

We cannot begin and finish our work without thanking the greatest and the most powerful"ALLAH" for blessing us to complete this memory.

We sincerely wish to thank Mr.BOUREMEL HASSANE for the continuous support and providing all the necessary resources and facilities without which it would not have been possible to complete this project.

All the gratitude to the president of the jury and the examiner for devoting their time and their effort to read and examine our work.

DEDICATION

I dedicate this humble work,

To my parents for their prayers, encouragements and generosities which followed me the whole time.

To my brothers and sisters for their supports and passions.

to all the family Djenane.

To all my friends, and my colleague.

My sincere thanks.

Djenane Soumia

DEDICATION

I dedicate this humble work,

To the people who taught me how patience is a path to success, to those whose satisfaction is my goal.... my parents.

To my brothers and sisters for their supports and passions.

To my aunt for her support and encouragement.

To my grandfather "Taher" and my grandmother "Zineb" my god have mercy on them.

To everyone who gave me help in my life.

to all the family Maadadi.

To all my friends, and my colleague.

My sincere thanks.

Maadadi lamis

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INTRODUCTION

The first publications in fuzzy set theory by Zadeh $[1965]$ and Goguen $[1967, 1969]$ show the intention of the authors to generalize the classical notion of a set and a proposition [statement] to accommodate fuzziness.

Zadeh suggested a modified set theoretical approach in which an individual may have a degree of membership value which is ranged over a continuum grade of values ranging between 0 and 1, rather than exactly 0 or 1.

Applications of the theory of fuzzy sets can be found, for example, in artificial intelligence, computer science,medicine, control engineering, decision theory, expert systems, logic, management science, operations research, pattern recognition, and robotics.

The topics of fuzzy integral equations (FIE)which growing interest for some time, in particular in relation to fuzzy control, have been rapidely developed in recent years. Its importance appears in studying and solving a large proportion of problems, in particular in relation to biologie, physics, medecine and geography.

The concept of integration of fuzzy functions was first introduced by Dubois and Prade [6] and investigated by Goetschel and Voxman [10], Kaleva [13], Matloka [15] and others. Congxin and Ming [5] presented the first applications of fuzzy integration. They investigated the fuzzy Fredholm integral equation of the second kind (FFIE-2). One of the first applications of fuzzy integration was given by Wu and Ma [17] who investigated the fuzzy Fredholm integral equation of the second kind (FFIE-2).

This memory is organized in three chapters as follows:
In the first chapter, we give some basic notions and generalities about the fuzzy sets, their characteristic notions and $alpha$-level sets, also we define fuzzy numbers with its operations and some examples.
In the second chapter, we give definitions of kinds of fuzzy function and their integral and properties, then we define the fuzzy linear system with example.

In the last chapter, we use Adomian decomposition method to solve the fuzzy Fredholm integral equation of the second kind. Using the parametric form of fuzzy number we convert a linear fuzzy Fredholm integral equation to linear systems of integral equations of the second kind and we present a numerical examples.

CHAPTER

ONE

GENERALITIES ON FUZZY SETS

The purpose of this chapter is to provide the concept of fuzzy set, then characteristic notions, and operations on fuzzy sets, and paying particular attention to the basic properties of α-cuts.

1.1 Crisp and fuzzy sets

In this section, we present the concepts of fuzzy sets and crisp sets with examples.

Definition 1.1 ***(Crisp set)*** *[4] By a crisp set, or a classical set, or simply set we mean a collection of distinct well-defined objects. These objects are said to be elements or members of the set. We usually denote the sets by capital letters A, B, C, etc., and the members by a, b, c, etc. To denote a is an element of A we write $a \in A$. The negation of $a \in A$ is written $a \notin A$ and means that a does not belong to A. A set with no elements is called an empty set and will be denoted by $\emptyset$.*

Definition 1.2 ***(Crisp Logic)*** *The traditional approach (crisp logic) of knowledge representation does not provide an appropriate way to interpret the imprecise and non-categorical data. As its functions are based on the first order logic and classical probability theory. In another way, it can not deal with the representation of human intelligence.*

Example 1.1 *Now, let's understand the crisp logic by an example. We are supposed to find the answer to the question, Does she have a pen? The answer of the above-given question is definite Yes or No, depending on the situation. If yes is assigned a value 1 and No is assigned a 0, the outcome of the statement could have a* 0 *or* 1. *So, a logic which demands a binary* $(0/1)$ *type of handling is known as Crisp logic in the field of fuzzy set theory.*

Definition 1.3 ***(Fuzzy set)*** *[11] Fuzzy set A of a universe X is characterized by a function*

$$\mu_A : x \to [0,1],$$

called membership function.

If $\mu_A : x \to \{0,1\}$, then the set A is said to be crisp. In the nonfuzzy case, μ_A is called the characteristic function (or indicator function) and it is often denoted by χ_A. If $\chi_A(x) = 0$, then x does not belong to A, whereas if $\chi_A(x) = 1$, then x belongs to A.

Example 1.2 *We consider three fuzzy sets that represent the concepts of a young, middle-aged, and old person. A reasonable expression of these concepts by trapezoidal membership functions μ_{A1}, μ_{A2}, μ_{A3} is shown in Fig (1.1). These functions are defined on the interval $[0,80]$ as follows:*

$$\mu_{A1}(x) = \begin{cases} 1 & when \quad x \leqslant 20 \\ \dfrac{35-x}{15} & when \quad 20 < x < 35 \\ 0 & when \quad x \geqslant 35 \end{cases}, \quad \mu_{A2}(x) = \begin{cases} 0 & when\ either \quad x \leqslant 20 \ or\ x \geqslant 60 \\ \dfrac{x-20}{15} & when \quad 20 < x < 35 \\ \dfrac{60-x}{15} & when \quad 45 < x < 60 \\ 1 & when \quad 35 \leqslant x \leqslant 45 \end{cases},$$

$$\mu_{A3}(x) = \begin{cases} 0 & when \quad x \leqslant 45 \\ \dfrac{x-45}{15} & when \quad 45 < x < 60 \\ 1 & when \quad x \geqslant 60 \end{cases}.$$

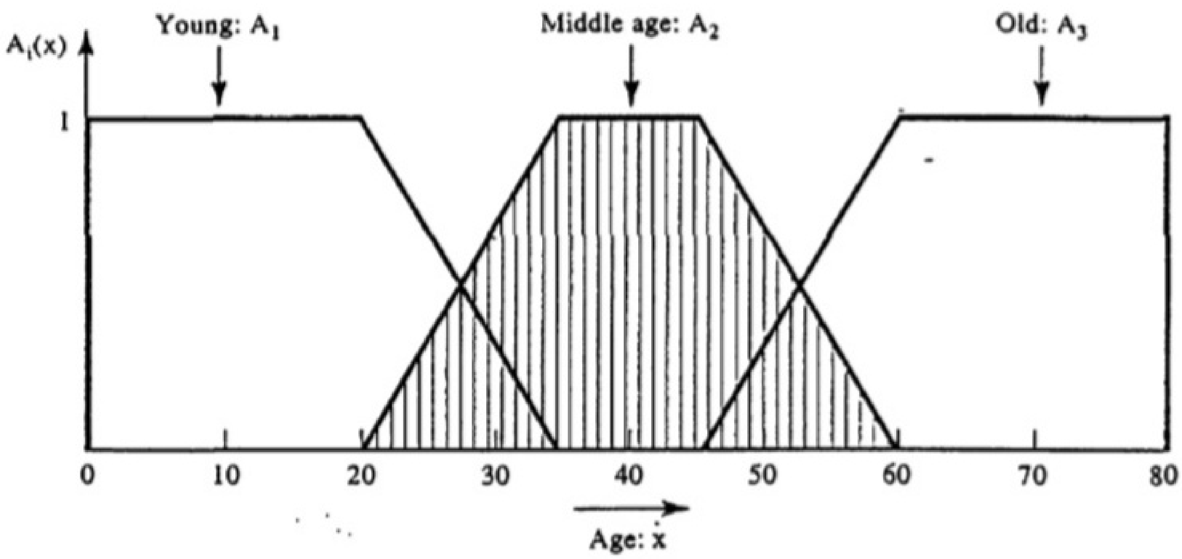


Figure 1.1: Membership function representing the concepts of a young, middle-aged, and old person

Definition 1.4 ***(Fuzzy logic)*** *Unlike crisp logic, in fuzzy logic, approximate human reasoning* *capabilities are added in order to apply it to the knowledge-based systems. But, what was the need to develop such a theory? The fuzzy logic theory provides a mathematical method to apprehend the uncertainties related to the human cognitive process, for example, thinking and reasoning and it can also handle the issue of uncertainty and lexical imprecision.*

Example 1.3 *Let's take an example to understand fuzzy logic. Suppose we need to find whether the colour of the object is blue or not. But the object can have any of the shade of blue depending on the intensity of the primary colour. So, the answer would vary accordingly, such as royal blue, navy blue, sky blue, turquoise blue, azure blue, and so on. We are assigning the darkest shade of blue a value 1 and 0 to the white colour at the lowest end of the spectrum of values. Then the other shades will range in 0 to 1 according to intensities. Therefore, this kind of situation where any of the values can be accepted in a range of 0 to 1 is termed as fuzzy.*

Key differences between fuzzy set and crisp set

1) A fuzzy set is determined by its indeterminate boundaries, there exists an uncertainty about the set boundaries. On the other hand, a crisp set is defined by crisp boundaries, and contain the precise location of the set boundaries.

2) Fuzzy set elements are permitted to be partly accommodated by the set (exhibiting gradual membership degrees). Conversely, crisp set elements can have a total membership or non-membership.

3) There are several applications of the crisp and fuzzy set theory, but both are driven towards the development of the efficient expert systems.

4) The fuzzy set follows the infinite-valued logic whereas a crisp set is based on bi-valued logic.

Notation 1.1 *The fuzzy set theory is intended to introduce the imprecision and vagueness in order to attempt to model the human brain in artificial intelligence and significance of such theory is increasing day by day in the field of expert systems. However, the crisp set theory was very effective as the initial concept to model the digital and expert systems working on binary logic.*

Characteristic notions

The characteristics of a fuzzy set A of nonempty X which describe it, are the ones that show how much it makes different than a classic set.

Definition 1.5 *[2] Let A be a fuzzy set on X.*

1) *The **support** of A denoted by $supp(A)$ is the subset whose elements are included at least a little in A, and we write:*

$$supp(A) = \{x \in X : \mu_A(x) > 0\}.$$

2) *The **core** of A denoted by $core(A)$ is the subset whose elements are included totally in A, and we write :*

$$core(A) = \{x : \mu_A(x) = 1\}.$$

3) *The **height** denoted by $H(A)$ correspond to the upper bound of the domain of its membership function, and we write:*

$$H(A) = sup\{\mu_A(x) \mid x \in X\}.$$

Example 1.4 *Let $X = \{a,b,c,d,e,f\}$. Consider the fuzzy set A on X defined by:*

$$A = \{(a,0.6),(b,1),(c,0.1),(e,0.8),(f,0.5)\}.$$

$supp(A) = \{a,b,c,e,f\}$, $core(A) = \{b\}$, $H(A) = \{b\}$.

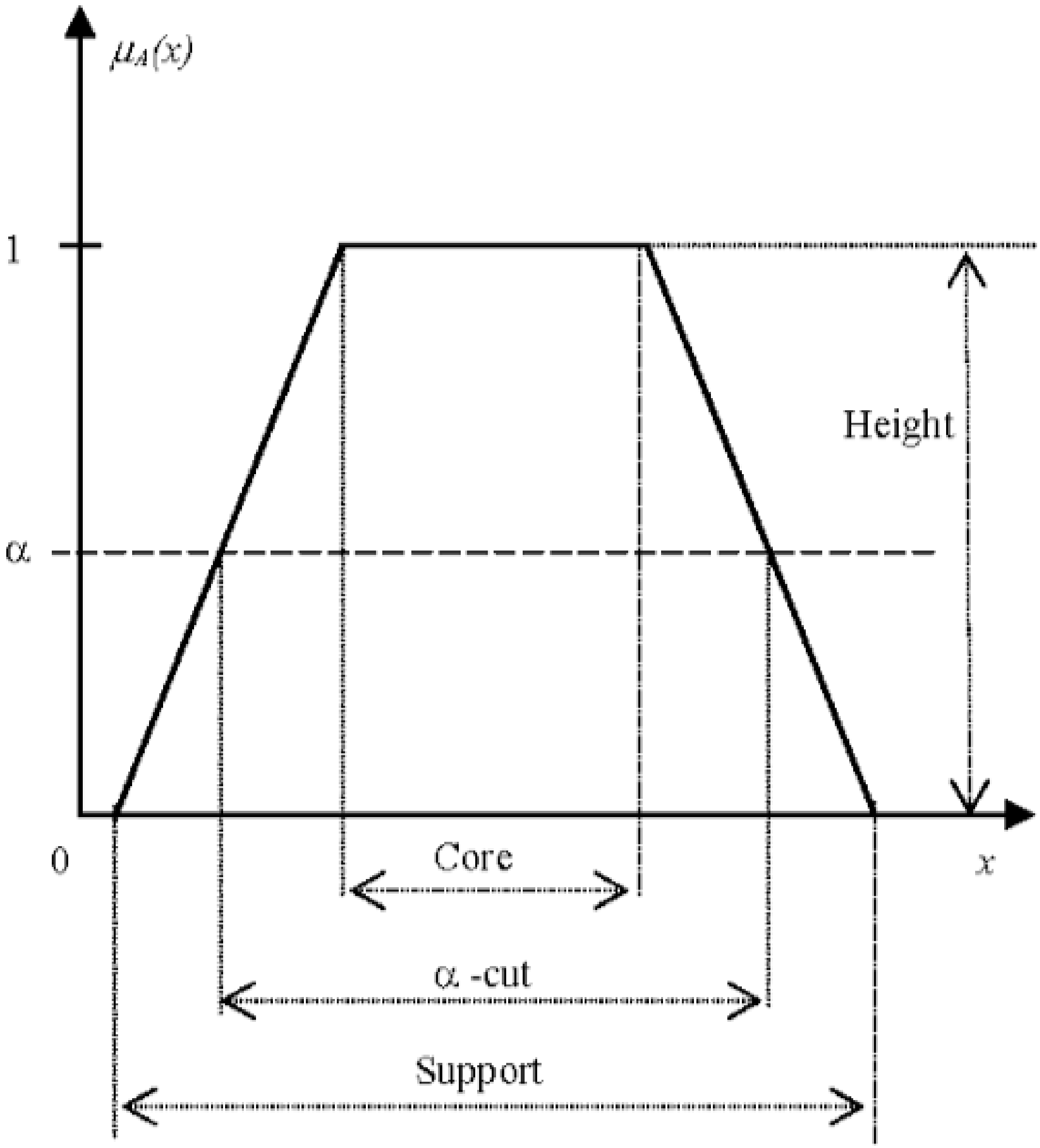


Figure 1.2: Support, core and height of fuzzy set

1.1.1 Operations on fuzzy sets

We define on fuzzy sets the same operations of the classic sets which are for each two fuzzy subsets A and B of X given by the following rules.

Definition 1.6 *[19]*

i) *A fuzzy set A is empty, we note $A = \emptyset$ if and only if*

$$\forall\, x \in X : \mu_A(x) = 0.$$

ii) *Two fuzzy sets A and B are equal, we note $A = B$ if and only if*

$$\forall\, x \in X : \mu_A(x) = \mu_B(x).$$

iii) *A fuzzy set A is contained in a fuzzy set B, we note $A \subseteq B$ if and only if*

$$\forall\, x \in X : \mu_A(x) \leq \mu_B(x).$$

Let $A, B \subset \mathscr{P}(X)$ two subset of X. As we know, there are the familiar operations of union, intersection, and complement. These are given by the rules

$$A \cup B = \{x \mid x \in A \; or \; x \in B\},$$

$$A \cap B = \{x \mid x \in A \; and \; x \in B\},$$

$$A^c = \{x \mid x \notin A\}.$$

As we have noted that a classic set A of X can be represented by a function $\chi_A : X \to \{0,1\}$ writing these rules in terms of indicator functions, we get:

$$\chi_{A\cup B}(x) = \max\{\chi_A(x), \chi_B(x)\},$$

$$\chi_{A\cap B}(x) = \min\{\chi_A(x), \chi_B(x)\},$$

$$\chi_{A^c}(x) = 1 - \chi_A(x).$$

A natural way to extend these operations to the fuzzy subsets of X is by the membership functions. Let A, B be two fuzzy subset of X.

- **Union**: $A \cup B$ is defined by the membership function

$$\mu_{A\cup B}(x) \;=\; \max\{\mu_A(x), \mu_B(x)\}.$$

- **Intersection**: $A \cap B$ is defined by the membership function

$$\mu_{A\cap B}(x) \;=\; \min\{\mu_A(x), \mu_B(x)\}.$$

- **Complement** of the fuzzy subset A, is noted by A^c and is defined by the membership function

$$\mu_{A^c}(x) \;=\; 1-\mu_A(x).$$

Remark 1.1 *For any collection $\{A_i \mid i \in I\}$ of fuzzy subsets of X, where I is a non-empty index set and μ_{A_i} its membership functions, the union and intersection of A_i are defined by the following membership functions:*

$$\mu_{\underset{i\in I}{\cup} Ai}(x) = \underset{i\in I}{sup}\, \{\mu_{A_i}(x)\} = \underset{i\in I}{\vee} A_i(x),$$

$$\mu_{\underset{i\in I}{\cap} Ai}(x) = \underset{i\in I}{inf}\, \{\mu_{A_i}(x)\} = \underset{i\in I}{\wedge} A_i(x).$$

Example 1.5 ***(Finite case):***

Let $X = \{a,b,c,d,e,r,s,t\}$ the set which represent a menu of restaurant, the patron want to classify it according to two description, tasty and cheap. Let A and B two fuzzy subset of X, such that A represent "tasty" and B "cheap". We get

$$A = \{(a,0.6),(b,1),(c,0.1),(e,0.4),(r,0.8),(s,0.5)\};$$

$$B = \{(b,0.3),(c,0.6),(d,0.5),(e,0.9),(s,1),(t,0.7)\}.$$

Which we give
$A\cup B = \{(a,0.6),(b,1),(c,0.6),(d,0.5),(e,0.9),(r,0.8),(s,1)\}$*; a fuzzy subset represent the description "tasty or cheap".*
$A\cap B = \{(b,0.3),(c,0.1),(e,0.4),(s,0.5)\}$*; a fuzzy subset represent the description "tasty and cheap".*
$A^c = \{(a,0.4),(c,0.9),(d,1),(e,0.6),(r,0.2),(s,0.5),(t,1)\}$*; a fuzzy subset represent the description "not tasty".*
$B^c = \{(a,1),(b,0.7),(c,0.4),(d,0.5),(e,0.1),(r,1),(t,0.3)\}$*; a fuzzy subset represent the description "not cheap".*

Example 1.6 ***(Infinite case):***

The set X be the positive real numbers representing the possible ages of people, the function μ_A define the fuzzy subset "young" and μ_B the fuzzy subset "have thirty old", such that:

$$\mu_A(x) = \begin{cases} 1 & if \quad x \leq 25 \\ \frac{40-x}{15} & if \quad 25 < x < 40 \\ 0 & if \quad 40 \leq x \end{cases}, \ \mu_B(x) = \begin{cases} 0 & if \quad x \leq 25 \\ \frac{x-25}{3} & if \quad 25 < x < 28 \\ 1 & if \quad 28 < x < 32 \\ \frac{35-x}{3} & if \quad 32 < x < 35 \\ 0 & if \quad 40 \leq x \end{cases}.$$

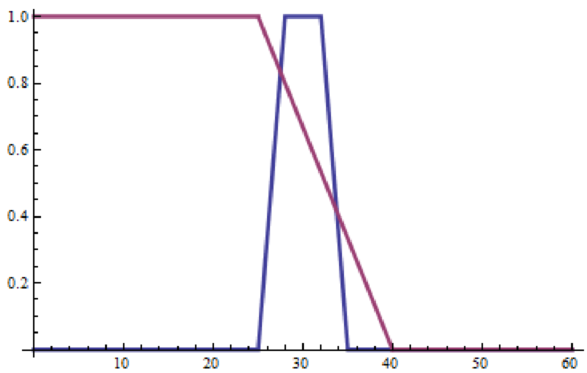
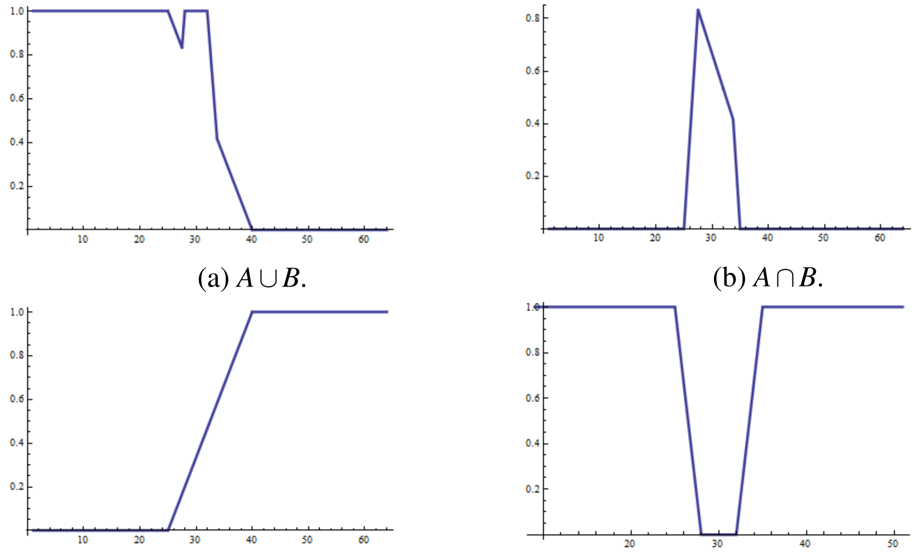


Figure 1.3: Membership function of A and B.

The following plots are the plots of union, intersection and the complement of the fuzzy subset A and B.



(a) $A \cup B$. (b) $A \cap B$.

(c) A^c. (d) B^c.

Figure 1.4: Membership functions.

1.1.2 $\alpha - cuts$

One of the characteristics of a fuzzy subset A of X is the alpha-cuts or also known as the level set. In this subsection and after given the definition of the alpha-cut, we will investigate its basic properties.

Definition 1.7 *[11] Given a fuzzy subset A of a topological space X, its $\alpha - cut$ (or $\alpha - levels$) are the subsets*

$$[A]_\alpha = \begin{cases} \{x \in X \, \mu_A(x) \geqslant \alpha\}, & if\ \alpha \in [0,1] \\ cl\{x \in X : \mu_A(x) > 0\}, & if\ \alpha = 0 \end{cases},$$

where $cl\,Z$ denotes the closure of the classical subset Z (see figure 1.2).

Theorem 1.1 *Let $A \in \mathscr{F}\mathscr{P}(X)$, μ_A its membership function and $\alpha \in [0,1]$. Then for all $x \in X$ it holds that*

$$\mu_A(x) = \sup_{\alpha \in [0,1]} (\alpha \cdot \chi_{A_\alpha}(x)).$$

Proof 1.1 *Let $x \in X$, suppose that $\mu_A(x) = \beta$ ($\beta \in [0,1]$).*

$$\begin{aligned} \mu_A(x) = \beta &\Rightarrow \mu_A(x) \geq \beta \\ &\Rightarrow x \in A_\beta \\ &\Rightarrow \chi_{A_\beta}(x) = 1. \end{aligned}$$

On the one hand, as $\mu_A(x) = \beta.1 = \beta\chi_{A_\beta}(x)$, it follows that $\mu_A(x) \leq \sup_{\alpha \in [0,1]} (\alpha.\chi_{A_\alpha}(x))$. On the other hand, we have

$$\chi_{A_\beta}(x) = \begin{cases} 1 & if \quad \mu_A(x) \geq \beta \\ 0 & if \quad \mu_A(x) \leq \beta \end{cases},$$

it follows that,

$$\beta.\chi_{A_\beta}(x) = \begin{cases} \beta & if \quad \mu_A(x) \geq \beta \\ 0 & if \quad \mu_A(x) \leq \beta \end{cases}.$$

This implies that, $\beta.\chi_{A_\beta}(x) \leq \beta$ and $\beta = \mu_A(x)$, thus, $\beta.\chi_{A_\beta}(x) \leq \mu_A(x)$. Then

$$\sup_{\alpha \in [0,1]} (\alpha.\chi_{A_\alpha}(x)) \leq \mu_A(x).$$

Therefore, it holds that $\forall x \in X,\ \mu_A(x) = \sup_{\alpha \in [0,1]} (\alpha.\chi_{A_\alpha}(x))$.

Proposition 1.1 *Let A, B be two fuzzy subset of X and $\alpha, \beta \in [0,1]$. The α-cuts satisfy the following statements:*

i) $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$,

ii) $A \subset B \Rightarrow A_\alpha \subset B_\alpha$,

iii) $\alpha < \beta \Rightarrow A_\beta \subset A_\alpha$.

Proof.

i) Let $x \in (A \cup B)_\alpha$. We have, $(A \cup B)_\alpha = \{x \in X \mid \mu_{(A \cup B)}(x) \geq \alpha\}$.

$$\begin{aligned} x \in (A \cup B)_\alpha &\Leftrightarrow \mu_{(A \cup B)}(x) \geq \alpha \\ &\Leftrightarrow max\{\mu_A(x), \mu_B(x)\} \geq \alpha \\ &\Leftrightarrow \mu_A(x) \geq \alpha \text{ } or \text{ } \mu_B(x) \geq \alpha \\ &\Leftrightarrow x \in A_\alpha \text{ } or \text{ } x \in B_\alpha \\ &\Leftrightarrow x \in (A_\alpha \cup B_\alpha). \end{aligned}$$

Then, $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.

ii) Let $x \in A_\alpha$.

$$\begin{aligned} x \in A_\alpha &\Rightarrow \mu_A(x) \geq \alpha \\ &\Rightarrow \mu_B(x) \geq \alpha \\ &\Rightarrow x \in B_\alpha. \end{aligned}$$

Then $A_\alpha \subset B_\alpha$.

iii) Let $x \in A_\beta$,

$$\begin{aligned} x \in A_\beta &\Rightarrow \mu_A(x) \geq \beta \\ &\Rightarrow \mu_A(x) \geq \alpha \\ &\Rightarrow x \in A_\alpha. \end{aligned}$$

Then $A_\beta \subset A_\alpha$.

■

The following two theorems state some basic properties of the $\alpha-$ cuts of a given fuzzy set.

Theorem 1.2 *Suppose that $\{A_i \mid i \in I\}$ is a collection of fuzzy subsets of X. Then for any $\alpha \in [0,1]$ it holds that*

i) $\underset{i \in I}{\cup}(A_i)_\alpha \subseteq (\underset{i \in I}{\cup} A_i)_\alpha$,

ii) $\underset{i \in I}{\cap}(A_i)_\alpha = (\underset{i \in I}{\cap} A_i)_\alpha$.

Moreover, if I is finite, then we have equality in (i).

Proof.

i) Let $x \in \bigcup\limits_{i\in I}(A_i)_\alpha$,

$$\begin{aligned} x \in \bigcup_{i\in I}(A_i)_\alpha &\Rightarrow \exists\, i \in I,\, x \in (A_i)_\alpha \\ &\Rightarrow \exists\, i \in I,\, \mu_{A_i}(x) \geq \alpha \\ &\Rightarrow \sup_{i\in I}\{\mu_{A_i}(x)\} \geq \alpha \\ &\Rightarrow \mu_{\bigcup\limits_{i\in I} Ai}(x) \geq \alpha \\ &\Rightarrow x \in (\bigcup_{i\in I} A_i)_\alpha. \end{aligned}$$

Then $\bigcup\limits_{i\in I}(A_i)_\alpha \subseteq (\bigcup\limits_{i\in I} A_i)_\alpha$.

ii) Let $x \in \bigcap\limits_{i\in I}(A_i)_\alpha$,

$$\begin{aligned} x \in \bigcap_{i\in I}(A_i)_\alpha &\Leftrightarrow \forall\, i \in I,\, x \in (A_i)_\alpha \\ &\Leftrightarrow \forall\, i \in I,\, \mu_{A_i}(x) \geq \alpha \\ &\Leftrightarrow \inf_{i\in I}\{\mu_{A_i}(x)\} \geq \alpha \\ &\Leftrightarrow \mu_{\bigcap\limits_{i\in I} Ai}(x) \geq \alpha \\ &\Leftrightarrow x \in (\bigcap_{i\in I} A_i)_\alpha. \end{aligned}$$

Then $\bigcap\limits_{i\in I}(A_i)_\alpha = (\bigcap\limits_{i\in I} A_i)_\alpha$.

■

1.2 Fuzzy numbers

This section describes the fundamental concept of fuzzy number, then operations on fuzzy numbers, also we introduce special kinds of fuzzy numbers such as triangular fuzzy number and trapezoidal fuzzy number.

1.2.1 Definition and examples

Definition 1.8 *[1] A fuzzy set is a fuzzy set* $u : \mathbb{R} \longrightarrow [0,1]$ *which satisfies:*

1) u is upper semicontinuous.

2) $u(x) = 0$ *outside some interval* $[c,d]$.

3) There are real numbers $a, b, c : c \leqslant a \leqslant b \leqslant d$ *for which*

i) $u(x)$ *is monotonic increasing on* $[c, a]$,

ii) $u(x)$ *is monotonic decreasing on* $[b, d]$,

iii) $u(x) = 1, \quad a \leqslant x \leqslant b$.

The set of all fuzzy numbers is denoted by E^1.

Remark 1.2 *Every real number r is a fuzzy number whose membership function is the characteristic function:*

$$\chi_r(x) = \begin{cases} 1 & \text{if} \quad x = r \\ 0 & \text{if} \quad x \neq r \end{cases} .$$

Example 1.7 (Triangular fuzzy number). *The triangular fuzzy number is represented with three points as follows*

$$A = (a_1, a_2, a_3);$$

this representation is interpreted as membership function and holds the following conditions:

i) from a_1 *to* a_2 *is increasing function.*

ii) from a_2 *to* a_3 *is decreasing function.*

iii) $a_1 \leqslant a_2 \leqslant a_3$.

$$\mu_A(x) = \begin{cases} 0 & \text{for} \quad x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & \text{if} \quad a_1 \leqslant x \leqslant a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & \text{if} \quad a_2 \leqslant x \leqslant a_3 \\ 0 & \text{for} \quad x > a_3 \end{cases} .$$

Example 1.8 (Trapezoidal fuzzy number). *We can define trapezoidal fuzzy number A as*

$$A = (a_1, a_2, a_3, a_4).$$

The membership of this fuzzy number will be interpreted as follows:

$$\mu_A(x) = \begin{cases} 0 & \text{for} \quad x < a_1 \\ \dfrac{x - a_2}{a_2 - a_1} & \text{if} \quad a_1 \leqslant x \leqslant a_2 \\ 1 & \text{if} \quad a_2 \leqslant x \leqslant a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & \text{if} \quad a_3 \leqslant x \leqslant a_4 \\ 0 & \text{for} \quad x > a_4 \end{cases} .$$

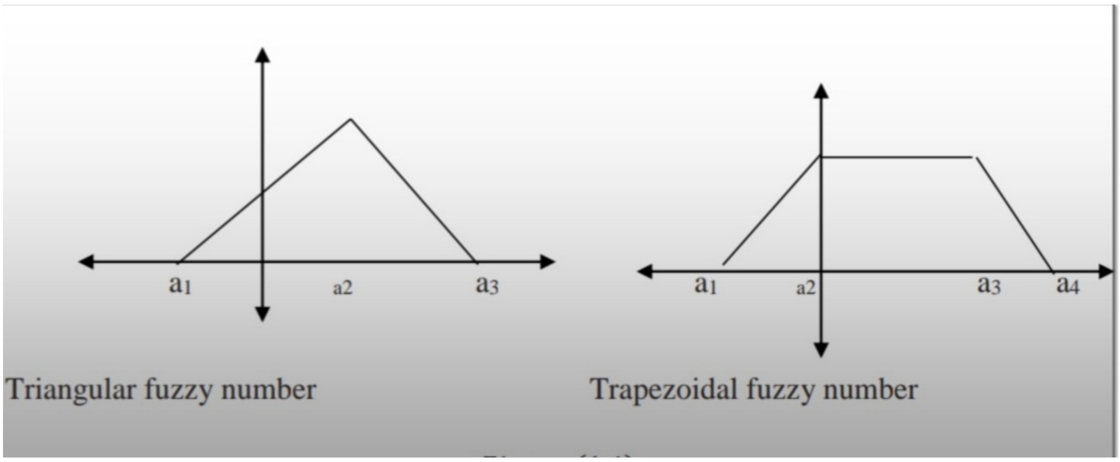


Figure 1.5: Triangular and trapezoidal fuzzy numbers

Theorem 1.3 *[11] A fuzzy number A satisfies the following conditions:*

a) *its* $\alpha-cuts$ *are non-empty closed intervals, for all* $\alpha \in [0,1]$*;*

b) *if* $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$*, then* $[A]_{\alpha 1} \subseteq [A]_{\alpha 2}$*;*

c) *for any non-decreasing sequence* (α_n) *in [0,1] converging to* $\alpha \in [0,1]$ *we have*

$$\cap_{n=1}^{\infty}[A]_{\alpha n} = [A]_{\alpha};$$

d) *for any non-increasing sequence* (α_n) *in [0,1] converging to zero we have*

$$cl(\cup_{n=1}^{\infty}[A]_{\alpha n}) = [A]_0.$$

1.2.2 Operations on fuzzy numbers

In this section we give definitions of the algebraic operations between fuzzy numbers with illustrative examples.

Definition 1.9 *[3] Let A and B be two fuzzy numbers and* λ *a real number.*

(a) *The sum of the fuzzy numbers A and B is the fuzzy number* $A+B$*, whose membership function is*

$$\varphi_{(A+B)}(Z) = \sup_{\phi(Z)} \min[\varphi_A(x), \varphi_B(y)],$$

where $\phi(Z) = \{(x,y) : x+y = Z\}$.

(b) *The multiplication of A by a scalar* λ *is the fuzzy number* λA*, whose membership function is*

$$\varphi_{\lambda A}(Z) = \begin{cases} \sup_{\{x:\lambda x = Z\}}[\varphi_A(x)], & if \lambda \neq 0 \\ \chi_{\{0\}}, & if \lambda = 0 \end{cases} = \begin{cases} \varphi_A(\lambda^{-1}Z), & if \lambda \neq 0 \\ \chi_{\{0\}}(Z), & if \lambda = 0 \end{cases},$$

where $\chi_{\{0\}}$ *is the characteristic function of* $\{0\}$.

(c) *The difference* $A - B$ *is the characteristic function of* $\{0\}$,

$$\varphi_{(A-B)}(Z) = \sup_{\phi(Z)} \min[\varphi_A(x), \varphi_B(y)],$$

where $\phi(Z) = \{(x,y) : x - y = Z\}$.

(d) *The multiplication of A by B is the fuzzy number A.B, whose membership function is given by:*

$$\varphi(A.B)(Z) = \sup_{\phi(Z)} \min[\varphi_A(x), \varphi_B(y)],$$

where $\phi(Z) = \{(x,y) : xy = Z\}$.

(e) *The quotient is the fuzzy number* A/B *whose membership function is*

$$\varphi_{(A/B)}(Z) = \sup_{\phi(Z)} \min[\varphi_A(x), \varphi_B(y)],$$

where $\phi(Z) = \{(x,y) : x/y = Z\}$.

Theorem 1.4 *[3] The* α*-levels of the fuzzy set* $A \otimes B$ *are given by:*

$$[A \otimes B]^\alpha = [A]^\alpha \otimes [B]^\alpha,$$

for all $\alpha \in [0,1]$*, where* $\otimes$ *is any arithmetic operation* $\{+,-,\times,\div\}$.
We observe again that the α*-levels of a fuzzy number is always a closed interval of* $\mathbb{R}$ *given by:*

$$[A]^\alpha = [a_1^\alpha, a_2^\alpha],$$

with $a_1^\alpha = \min\{\varphi_A^- 1(\alpha)\}$ *and* $a_2^\alpha = \max\{\varphi_A^- 1(\alpha)\}$,
where $\varphi_A^- 1(\alpha) = \{x \in \mathbb{R} : \varphi_A(x) = \alpha\}$ *is the pre-image of* α.

Proposition 1.2 *[3] Let A and B be fuzzy numbers with* α*-levels respectively given by* $[A]^\alpha = [a_1^\alpha, a_2^\alpha]$ *and* $[B]^\alpha = [b_1^\alpha, b_2^\alpha]$*. Then the following properties hold:*

(a) *The sum of A and B is the fuzzy number* $A + B$ *whose* α*-levels are:*

$$[A+B]^\alpha = [A]^\alpha + [B]^\alpha = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha].$$

(b) *The difference of A and B is the fuzzy number $A-B$ whose α-levels are:*

$$[A-B]^{\alpha} = [A]^{\alpha} - [B]^{\alpha} = [a_1^{\alpha} - b_2^{\alpha}, a_2^{\alpha} - b_1^{\alpha}].$$

(c) *The multiplication of A by a scalar λ is the fuzzy number λA whose α-levels are:*

$$[\lambda A]^{\alpha} = \lambda [A]^{\alpha} = \begin{cases} [\lambda a_1^{\alpha}, \lambda a_2^{\alpha}], & if \lambda \geqslant 0 \\ [\lambda a_2^{\alpha}, \lambda a_1^{\alpha}], & if \lambda < 0 \end{cases}$$

(d) *The multiplication of A by B is the fuzzy number A.B whose α-levels are:*

$$[A.B]^{\alpha} = [A]^{\alpha}[B]^{\alpha} = [\min P^{\alpha}, \max P^{\alpha}],$$

where $P^{\alpha} = \{a_1^{\alpha} b_1^{\alpha},\ a_2^{\alpha} b_2^{\alpha},\ a_2^{\alpha} b_1^{\alpha},\ a_2^{\alpha} b_2^{\alpha}\}$.

(e) *The division of A by B, if $0 \notin suppB$, is the fuzzy number whose α-levels are:*

$$\left[\frac{A}{B}\right]^{\alpha} = \frac{[A]^{\alpha}}{[B]^{\alpha}} = [a_1^{\alpha}, a_2^{\alpha}]\left[\frac{1}{b_2^{\alpha}}, \frac{1}{b_1^{\alpha}}\right].$$

Example 1.9 *Consider the expressions nearly 2 and nearly 4 and let A and B be the triangular fuzzy numbers that indicate these expressions. Thus, we define $A = (1;2;3)$ and $B = (3;4;5)$. The results of $A \otimes B$ for each of the arithmetic operations between fuzzy numbers are shown next. First, let us notice that:*

$$[A]^{\alpha} = [1+\alpha, 3-\alpha] \text{ and } [B]^{\alpha} = [3+\alpha, 5-\alpha].$$

Then by Proposition (1.2) we get

(a) $[A+B]^{\alpha} = [A]^{\alpha} + [B]^{\alpha} = [4+2\alpha, 8-2\alpha]$. *Thus,* $A+B = (4;6;8)$*;*

(b) $[A-B]^{\alpha} = [A]^{\alpha} - [B]^{\alpha} = [-4+2\alpha, -2\alpha]$. *Thus,* $A-B = (-4;-2;0)$*;*

(c) $[4.A]^{\alpha} = 4[A]^{\alpha} = [4+4\alpha, 12-4\alpha]$. *Thus,* $4A = (4;8;12)$*;*

(d) $[A.B]^{\alpha} = [A]^{\alpha}[B]^{\alpha} = [(1+\alpha)(3+\alpha), (3-\alpha)(5-\alpha)]$*;*

(e) $\left[\frac{A}{B}\right]^{\alpha} = \frac{[A]^{\alpha}}{[B]^{\alpha}} = [(1+\alpha)/(5-\alpha), (3-\alpha)/(3+\alpha)]$.

Notice that the fuzzy numbers obtained in (d) and (e) are not triangular. However, it is easy to verify that with triangular fuzzy numbers, the sum, the difference and the multiplication by *a scalar results in a triangular fuzzy number. To see this, it suffices to consider the numbers $A = (a_1; u; a_2)$ and $B = (b_1; v; b_3)$. Then, we have:*

$$[A]^{\alpha} = [(u - a_1)\alpha + a_1, (u - a_2)\alpha + a_2],$$

$$[B]^{\alpha} = [(v - b - 1)\alpha + b_1, (v - b_2)\alpha + b_2].$$

Thus

$$[A+B]^{\alpha} = [A]^{\alpha} + [B]^{\alpha},$$

and then

$[A+B]^{\alpha} + [\{(u+v) - (a_1+b_1)\}\alpha + (a_1+b_1), \{(u+v) - (a_2+b_2)\}\alpha + (a_2+b_2)]$.
We see that these intervals are the α-levels of the following triangular fuzzy number:

$$((a_1+b_1); (u+v); (a_2+b_2)).$$

Finally, it is possible to conclude that $(A-B)+B \neq A$ so that it follows that $A - A \neq 0$. That is, the space of fuzzy numbers is not a vector space since there are no additive (nor multiplication) inverses.

CHAPTER

TWO

FUZZY INTEGRAL AND ITS PROPERTIES

In this chapter we focus on the parametric form of fuzzy number, the fuzzy integral and their properties in order to use them in the main results in the last chapter, also we define the fuzzy linear system with example.

2.1 Parametric form of a fuzzy number

In the following, we give another definition equivalent to fuzzy number called parametric form.

2.1.1 Definition and Examples

Definition 2.1 *[1] A fuzzy number u is a pair* $(\underline{u},\overline{u})$ *of function* $\underline{u}(\alpha),\overline{u}(\alpha);0\leqslant\alpha\leqslant 1$ *which satisfying the following requirements:*

1) $\underline{u}(\alpha)$ *is a bounded left-continuous non-decreasing function over* $[0,1]$.

2) $\overline{u}(\alpha)$ *is a bounded left-continuous non-increasing function over* $[0,1]$.

3) $\underline{u}(\alpha)\leqslant\overline{u}(\alpha)$, $\quad 0\leqslant\alpha\leqslant 1$.

Remark 2.1 *If* $(\underline{u},\overline{u})$ *is the parametric form of u then*

$$\mu(x)=\sup\{\alpha\mid\underline{u}(\alpha)\leqslant x\leqslant\overline{u}(\alpha)\}.$$

Example 2.1 *The parametric form of triangular fuzzy number defined in Example (1.7) is*

$$(\underline{u}(\alpha),\overline{u}(\alpha)) = (a_1+\alpha(a_2-a_1), a_3-\alpha(a_3-a_2)).$$

Example 2.2 *The parametric form of trapezoidal fuzzy number defined in Example (1.8) is*

$$(\underline{u}(\alpha),\overline{u}(\alpha)) = (a_1+\alpha(a_2-a_1), a_4-\alpha(a_4-a_3)).$$

Properties 2.1 *[1] For arbitrary $u=(\underline{u},\overline{u})$, $v=(\underline{v},\overline{v})$ and $K>0$ we define addition $(u+v)$ and multiplication by k as*

$$(\underline{u+v})(\alpha) = \underline{u}(\alpha)+\underline{v}(\alpha), \quad (\overline{u+v})(\alpha) = \overline{u}(\alpha)+\overline{v}(\alpha), \tag{2.1}$$

$$ku(\alpha) = \begin{cases} k\underline{u}(\alpha), k\overline{u}(\alpha), & k \geqslant 0 \\ k\overline{u}(\alpha), k\underline{u}(\alpha), & k<0 \end{cases}. \tag{2.2}$$

The collection of all fuzzy numbers with addition and multiplication as defined by Eq. (2.1) and (2.2) is denoted by E^1.

2.1.2 Distance between two fuzzy numbers

In this section we will try to find an application $d: E^1 \to \mathbb{R}^+$ that should satisfy the properties of distance.

Proposition 2.1 *Let*

$$\begin{array}{rcl} d: E^1\times E^1 & \to & \mathbb{R}^+ \\ (u,v) & \mapsto & d(u,v) = \max\{\sup\limits_{0\leqslant\alpha\leqslant1} |\underline{u}(\alpha)-\underline{v}(\alpha)|, \sup\limits_{0\leqslant\alpha\leqslant1} |\overline{u}(\alpha)-\overline{v}(\alpha)|\}. \end{array}$$

Then d is distance on E^1.

Proof 2.1 *Clearly that $d(u,u)=0$ and $d(u,v)=d(v,u)$. It reminds to show that $d(u,w)\leqslant d(u,v)+d(v,w)$.*

we have

$$\begin{aligned}
d(u,w) &= \max\{\sup_{0\leqslant\alpha\leqslant 1}|\underline{u}-\underline{w}|, \sup_{0\leqslant\alpha\leqslant 1}|\overline{u}-\overline{w}|\} \\
&= \max\{\sup_{0\leqslant\alpha\leqslant 1}|(\underline{u}-\underline{v})+(\underline{v}-\underline{w})|, \sup_{0\leqslant\alpha\leqslant 1}|(\overline{u}-\overline{v})+(\overline{v}-\overline{w})|\} \\
&\leqslant \max\{\sup_{0\leqslant\alpha\leqslant 1}(|\underline{u}-\underline{v}|+|\underline{v}-\underline{w}|), \sup_{0\leqslant\alpha\leqslant 1}(|\overline{u}-\overline{v}|+|\overline{v}-\overline{w}|)\} \\
&\leqslant \max\{(\sup_{0\leqslant\alpha\leqslant 1}|\underline{u}-\underline{v}|,|\overline{u}-\overline{v}|)\}+\max\{(\sup_{0\leqslant\alpha\leqslant 1}|\underline{v}-\underline{w}|,|\overline{v}-\overline{w}|)\} \\
&= d(u,v)+d(v,w).
\end{aligned}$$

Definition 2.2 *[1] For arbitrary Fuzzy numbers $u=(\underline{u},\overline{u})$ and $v(\underline{v},\overline{v})$ the quantity*

$$D(u,v) = \max\{\sup_{0\leqslant r\leqslant 1}|\underline{u}(r)-\underline{v}(r)|, \sup_{0\leqslant r\leqslant 1}|\overline{u}(r)-\overline{v}(r)|\}, \tag{2.3}$$

is the distance between u and v.

2.2 Fuzzy function

Fuzzy function can be classified into following three groups according to which aspect of the crisp function the fuzzy concept was applied.

2.2.1 Kinds of fuzzy function

1) Crisp function with fuzzy constraint.

2) Crisp function which propagates the fuzziness of independent variable to dependent variable.

3) Function that is itself fuzzy. This fuzzifying function blurs the image of a crisp independent variable.

1) Function with fuzzy constraint

Definition 2.3 *[14] Let X and Y be crisp sets, and f be a crisp function. A and B are fuzzy sets defined on universal sets X and Y respectively. Then the function satisfying the condition $\mu_A(x)\leqslant\mu_B(f(x))$ is called a function with constraints on fuzzy domain A and fuzzy range B.*

Example 2.3 *There is a function $f : X \to Y$ assume that the function f has fuzzy constraint like this,*
"if x is a member of A, Then y is a member in B".
"The membership degree $\mu_A(x)$ of x for A is less than that $\mu_B(y)$ of y for B"
or "$\mu_A(x) \leqslant \mu_B(y)$".
The previous fuzzy constraints denote the sufficient fuzzy condition for y to be a member of B.
"If membership degree of x for A is α, then that of y for B would be no less than α".

Example 2.4 *Consider two fuzzy sets:*
$A = \{(1,0.5),(2,0.8)\},\ B = \{(2,0.7),(4,0.9)\},$
and a function
$y = f(x) = 2x\ ,\ \ for\ x \in A\ ,\ y \in B.$
We see the function f satisfies the condition, $\mu_A(x) \leqslant \mu_B(y)$.

2) Propagation of fuzziness by crisp function

Definition 2.4 ***(Fuzzy extension function)*** *[14] Let X and Y be two universes and $f : X \to Y$ a classical function. For each $A \in \mathscr{F}\mathscr{P}(X)$ we define the extension of f as $\hat{f}(A) \in \mathscr{F}\mathscr{P}(Y)$ such that*

$$\mu_{\hat{f}(A)}(y) = \begin{cases} \sup_{s \in f^{-1}(y)} \mu_A(s), & if\, f^{-1}(y) \neq \emptyset \\ 0, & if\, f^{-1}(y) = \emptyset \end{cases},$$

for all $y \in Y$, where $f^{-1}(y) = \{x \in X : f(x) = y\}$.

Example 2.5 *Let $f(x) = ax + b$ with $a, b \in \mathbb{R}, a \neq 0$.*
Since

$$f^{-1}(y) = \frac{(y-b)}{a},$$

the extension of f is the fuzzy function $\hat{f}$ such that, given $X \in \mathscr{F}\mathscr{P}(\mathbb{R})$,

$$\mu_{\hat{f}(x)}(y) = \sup_{x = \frac{(y-b)}{a}} \mu_X(x) = \mu_X(\frac{(y-b)}{a}),$$

for all $y \in \mathbb{R}$. Or

$$\mu_{\hat{f}(x)}(ax+b) = \mu_X(x),$$

that is,

$$\hat{f}(X) = aX + b.$$

3) Fuzzifying function of crisp variable

Fuzzifying function of crisp variable is a function which produces image of crisp domain in a fuzzy set.

Definition 2.5 ***(Single fuzzifying function)*** *[14] Fuzzifying function from X to Y is the mapping of X in fuzzy power set $\mathscr{F}\mathscr{P}(Y)$,*

$$f : X \to \mathscr{F}\mathscr{P}(Y).$$

That is to say, the fuzzifying function is a mapping from domain to fuzzy set of range. Fuzzifying function and the fuzzy relation coincides with each other in the mathematical manner. So to speak, fuzzifying function can be interpreted as fuzzy relation R defined as following:

$$\forall (x,y) \in X \times Y, \quad \mu_{f(x)}(y) = \mu_R(x,y).$$

Example 2.6 *Consider two crisp sets $A = \{2,3,4\}$ and $B = \{2,3,4,6,8,9,12\}$*
A fuzzifying function f maps the elements in A to power set $\mathscr{F}\mathscr{P}(B)$ in the following manner:

$$f(2) = B_1, \quad f(3) = B_2, \quad f(4) = B_3,$$

where

$$\mathscr{F}\mathscr{P}(B) = \{B_1, B_2, B_3\},$$

$B_1 = \{(2,0.5),(4,1),(6,0.5)\}$, $B_2 = \{(3,0.5),(6,1),(9,0.5)\}$, $B_3 = \{(4,0.5),(8,1),(12,0.5)\}$.
The function f maps element $2 \in A$ to element $2 \in B_1$ with degree 0.5, to element $4 \in B_1$ with $0,1$, and to element $6 \in B_1$ with 0.5. Now we apply $\alpha-$cut operation to the fuzzifying function.

$$f : 2 \to \{2,4,6\} \quad for\, \alpha = 0.5,$$

$$f : 2 \to \{4\} \quad for\, \alpha = 1.$$

In the same manner

$$f : 3 \to \{3,6,9\} \quad for\, \alpha = 0.5,$$

$$f : 3 \to \{6\} \quad for\, \alpha = 1.$$

Again

$$f : 4 \rightarrow \{4, 8, 12\} \quad for\, \alpha = 0.5,$$

$$f : 4 \rightarrow \{8\} \quad for\, \alpha = 1.$$

Definition 2.6 *A function $f : \mathbb{R} \longrightarrow E^1$ is called a fuzzy function. If for arbitrary fixed $t_0 \in \mathbb{R}$, and $\varepsilon > 0$, a $\delta > 0$ such that*

$$| t - t_0 | < \delta \Rightarrow D[f(t), f(t_0)] < \varepsilon,$$

exist, f is said to be continuous.

2.3 Integral of fuzzy function

In this section, we present a definition of fuzzy integral using Riemann integral concept.

Goetschel and Voxman approach

Definition 2.7 *[1] Goetschel and Voxman defined the integral of fuzzy function using the Riemann integral concept.*
Let $f : [a, b] \longrightarrow E^1$. For each partition $p = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ with $h = max|t_1 \longrightarrow t_1 - 1|$ and for arbitrary points $\xi_i : t_{i-1} \leqslant \xi_i \leqslant t_i, 1 \leqslant i \leqslant n$ let

$$R_n = \sum_{i=0}^{n} f(\xi_i)(t_i - t_{i-1}). \tag{2.4}$$

The definite integral of $f(t)$ over $[a, b]$ is

$$\int_a^b f(t)\, dt = lim R_n,\, h \longrightarrow 0, \tag{2.5}$$

provided that this limit exists in the metric D (and is independent of the partition and the selected points ξ_i).
If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists.

Theorem 2.1 *If the fuzzy function $f : [a, b] \longrightarrow E^1$ is continuous (with respect to the metric D) and if for each $t \in [a, b], f(t)$ has the parametric form*

$$(\underline{f}(\alpha,t), \overline{f}(\alpha,t)),$$

then $\int_a^b f(t)dt$ *exists, belongs to* E^1*, and is parametrized by*

$$\left(\int_a^b \underline{f}(\alpha,t)\,dt, \int_a^b \overline{f}(\alpha,t)\,dt\right). \tag{2.6}$$

Proof 2.2 *That* $\int_a^b f(t)\,dt$ *exists and is parametrized by Eq.(2.6) is an emmediate consequence of equicontinuity of the families of functions under the metric D. To see that* $\int_a^b f(t)\,dt$ *is a fuzzy number note that for each* t, $a \leqslant t \leqslant b$*, and for* $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$*, we have*

$$\underline{f}(\alpha_1,t) \leqslant \underline{f}(\alpha_2,t) \leqslant \overline{f}(\alpha_2,t) \leqslant \overline{f}(\alpha_1,t).$$

Therefore,

$$\int_a^b \underline{f}(\alpha_1,t)\,dt \leqslant \int_a^b \underline{f}(\alpha_2,t)\,dt \leqslant \int_a^b \overline{f}(\alpha_2,t)\,dt \leqslant \int_a^b \overline{f}(\alpha_1,t)\,dt, \tag{2.7}$$

and it follows from Eq.(2.7) and earlier remarks that $\int_a^b f(t)\,dt$ *satisfies conditions* $(1),(2)$*, and* (3) *of definition (2.1).*
Observe also that for $a \leqslant t \leqslant b$ *and* $0 < k \leqslant 1$*,*

$$\lim_{\alpha \to k^{-1}} \underline{f}(\alpha,t) = \underline{f}(k,t), \quad \lim_{\alpha \to k^{-1}} \overline{f}(\alpha,t) = \overline{f}(k,t),$$

and

$$\lim_{\alpha \to 0+} \underline{f}(\alpha,t) = \underline{f}(0,t), \quad \lim_{\alpha \to 0+} \overline{f}(\alpha,t) = \overline{f}(0,t).$$

It now follows from the monotone convergence theorem for integrable functions that $\int_a^b f(t)\,dt$ *also satisfies conditions of defintion (2.1). Consequently,* $\int_a^b f(t)\,dt \in E^1$*, as was to be shown.*

Remark 2.2 *if* $f(t)$ *is continuous, Lebesgue approach [13] yield the same value. Moreover, the representation of the fuzzy integral using Eqs. (2.4) and (2.5) is more convenient for numerical calculations.*

2.3.1 Properties of integral of fuzzy function

Next we provide some properties of integral of fuzzy function.

Properties 2.2 *([7],[9]) Let* $f,g : [a,b] \to E^1$ *be integrable and* $\lambda \in \mathbb{R}$*, then the following are satisfied:*

1) $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx.$

2) If $f:[a,b]\to E^1$ *is a continuous fuzzy function, then for each triple of real numbers* a,b,c*,*

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

3) If $f:[c,d]\to E^1$ *and* $g:[c,d]\to E^1$ *are integrable fuzzy functions, and if* α *and* β *are real numbers, then*

$$\int_c^d (\alpha f(x)+\beta g(x))\,dx = \alpha\int_c^d f(x)\,dx + \beta\int_c^d g(x)\,dx.$$

2.4 Fuzzy linear systems

In this section, we will define the fuzzy linear system and give its solution with example.

Definition 2.8 *[8] The* $n\times n$ *linear system*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = y_2 \\ \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = y_n \end{cases}$$

where the coefficient matrix $A=(a_{ij})$, $1\le i,j\le n$ *is a crisp* $n\times n$ *matrix and* $x_i, y_i \in E^1$, $1\le i\le n$ *is called a fuzzy linear system(FLS).*

Definition 2.9 ***(Solution of fuzzy linear system)****:[8] a fuzzy number vector* $(x_1,x_2,\ldots,x_n)^T$ *given by* $x_i = (\underline{x}_i(r),\overline{x}_i(r))$, $1\le i\le n$, $0\le r\le 1$*, is called a solution of the fuzzy system if*

$$\underline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \underline{a_{ij}x_j} = \underline{y}_i, \quad \overline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \overline{a_{ij}x_j} = \overline{y}_i. \tag{2.8}$$

If, for a particular i*,* $a_{ij}>0$*,* $1\le j\le n$*, we simply get*

$$\sum_{j=1}^n a_{ij}\underline{x}_j = \underline{y}_i, \quad \sum_{j=1}^n a_{ij}\overline{x}_j = \overline{y}_i. \tag{2.9}$$

Let us now rearrange the linear system of Eq. (2.8) so that the unknowns are $\underline{x}_i, (-\overline{x}_i), 1\le i\le n$*, and the right-hand side column is* $Y=(\underline{y}_1,\underline{y}_2,\ldots,\underline{y}_n,-\overline{y}_1,-\overline{y}_2,\ldots,-\overline{y}_n)^T$*. We get the*

$(2n) \times (2n)$ *linear system*

$$\begin{cases} s_{11}\underline{x}_1 + s_{12}\underline{x}_2 + \dots + s_{1n}\underline{x}_n + s_{1,n+1}(-\overline{x}_1) + s_{1,n+2}(-\overline{x}_2) + \dots + s_{1,2n}(-\overline{x}_n) = \underline{y}_1 \\ \vdots \\ s_{n1}\underline{x}_1 + s_{n2}\underline{x}_2 + \dots + s_{nn}\underline{x}_n + s_{n,n+1}(-\overline{x}_1) + s_{n,n+2}(-\overline{x}_2) + \dots + s_{n,2n}(-\overline{x}_n) = \underline{y}_n \\ s_{n+1,1}\underline{x}_1 + s_{n+1,2}\underline{x}_2 + \dots + s_{n+1,n}\underline{x}_n + s_{n+1,n+1}(-\overline{x}_1) + s_{n+1,n+2}(-\overline{x}_2) + \dots + s_{n+1,2n}(-\overline{x}_n) = -\overline{y}_1 \\ \vdots \\ s_{2n,1}\underline{x}_1 + s_{2n,2}\underline{x}_2 + \dots + s_{2n,n}\underline{x}_n + s_{2n,n+1}(-\overline{x}_1) + s_{2n,n+2}(-\overline{x}_2) + \dots + s_{2n,2n}(-\overline{x}_n) = -\overline{y}_n \end{cases}$$

where s_{ij} *are determined as follows:*

$$\begin{aligned} a_{ij} &\geq 0 \ \Rightarrow s_{ij} = a_{ij},\ s_{i+n,j} = a_{ij}, \\ a_{ij} &< 0 \ \Rightarrow s_{i,j+n} = -a_{ij},\ s_{i+n,j} = -a_{ij}, \end{aligned} \tag{2.10}$$

and any s_{ij} *which is not determined by Eq. (2.10) is zero. Using matrix notation we get*

$$SX = Y, \tag{2.11}$$

where $S = (S_{ij})$, $1 \leq i, j \leq 2n$ *and*

$$X = \begin{pmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{pmatrix} \quad Y = \begin{pmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{pmatrix}.$$

The following theorem guarantees the existence of a fuzzy solution for general case.

Consider the dual fuzzy linear system, and transform its $n \times n$ coefficient matrix A and B into $(2n) \times (2n)$ matrices as:

$$
\begin{cases}
s_{11}\underline{x}_1+...+s_{1n}\underline{x}_n+s_{1,n+1}(-\overline{x}_1)+...+s_{1,2n}(-\overline{x}_n)=\underline{y}_1\\
+t_{11}\underline{x}_1+...+t_{1n}\underline{x}_n+t_{1,n+1}(-\overline{x}_1)+...+t_{1,2n}(-\overline{x}_n),\\
\vdots\\
s_{n1}\underline{x}_1+...+s_{nn}\underline{x}_n+s_{n,n+1}(-\overline{x}_1)+...+s_{n,2n}(-\overline{x}_n)=\underline{y}_n\\
+t_{n1}\underline{x}_1+...+t_{nn}\underline{x}_n+t_{n,n+1}(-\overline{x}_1)+...+t_{n,2n}(-\overline{x}_n),\\
s_{n+1,1}\underline{x}_1+...+s_{n+1,n}\underline{x}_n+s_{n+1,n+1}(-\overline{x}_1)+...+s_{n+1,2n}(-\overline{x}_n)=-\overline{y}_1\\
+t_{n+1,1}\underline{x}_1+...+t_{n+1,n}\underline{x}_n+t_{n+1,n+1}(-\overline{x}_1)+...+t_{n+1,2n}(-\overline{x}_n),\\
\vdots\\
s_{2n,1}\underline{x}_1+...+s_{2n,n}\underline{x}_n+s_{2n,n+1}(-\overline{x}_1)+...+s_{2n,2n}(-\overline{x}_n)=-\overline{y}_n\\
+t_{2n,1}\underline{x}_1+...+t_{2n,n}\underline{x}_n+t_{2n,n+1}(-\overline{x}_1)+...+t_{2n,2n}(-\overline{x}_n),
\end{cases}
$$

where s_{ij} and t_{ij} are determined as follows:

$$
\begin{aligned}
a_{ij} &\geqslant 0 \quad \Rightarrow s_{ij}=a_{ij}, \quad s_{i+n,j+n}=a_{ij},\\
a_{ij} &< 0 \quad \Rightarrow s_{i,j+n}=-a_{ij}, \quad s_{i+n,j}=-a_{ij},\\
b_{ij} &\geqslant 0 \quad \Rightarrow t_{ij}=b_{ij}, \quad t_{i+n,j+n}=b_{ij},\\
b_{ij} &< 0 \quad \Rightarrow t_{i,j+n}=-b_{ij}, \quad t_{i+n,j}=-b_{ij},
\end{aligned}
\tag{2.12}
$$

and any s_{ij} and t_{ij} which is not determined by Eq (2.12) is zero. Using matrix notation we get

$$SX=Y+TX, \tag{2.13}$$

therefore, we have:

$$(S-T)X=Y, \tag{2.14}$$

where $S=(s_{ij})\geqslant 0$ and $T=(t_{ij})\geqslant 0$, $1\leqslant i,j\leqslant 2n$, and

$$
X=\begin{bmatrix}\underline{x}_1\\ \vdots\\ \underline{x}_n\\ -\overline{x}_1\\ \vdots\\ -\overline{x}_n\end{bmatrix}, \quad Y=\begin{bmatrix}\underline{y}_1\\ \vdots\\ \underline{y}_n\\ -\overline{y}_1\\ \vdots\\ -\overline{y}_n\end{bmatrix}.
$$

Example 2.7 *Consider the dual fuzzy linear system:*

$$\begin{cases} x_1 - x_2 = y_1 + 2x_1 + x_2, \\ x_1 + 2x_2 = y_2 + x_1 - 2x_2. \end{cases} \tag{2.15}$$

Let $\underline{y}_1 = \alpha$, $\bar{y}_1 = 2 - \alpha$ *and* $\underline{y}_2 = 4 + \alpha$, $\bar{y}_2 = 7 - 2\alpha$, *the extended* 4×4 *matrices are*

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad T = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix},$$

and

$$Y = \begin{bmatrix} \alpha \\ 4+\alpha \\ \alpha - 2 \\ 2\alpha - 7 \end{bmatrix}, \quad X = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{bmatrix}.$$

We obtain that the system Eq.(2.15) is equivalent to the function equation system

$$SX = Y + TY,$$

Consequently

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ 4+\alpha \\ \alpha - 2 \\ 2\alpha - 7 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{bmatrix},$$

Also

$$(S - T)X = Y.$$

The structure of S *and* T *implies that:*

$$S = \begin{pmatrix} C & D \\ D & C \end{pmatrix}, T = \begin{pmatrix} E & F \\ F & E \end{pmatrix},$$

where C *and* E *contains the positive entries of* A *and* B *respectively, and* D *and* F *the absolute* *values of the negative entries of A and B,i.e.* $A = C - D$ *and* $B = E - F$. *Therefore*

$$S - T = \begin{pmatrix} C-E & D-F \\ D-F & C-E \end{pmatrix}.$$

Theorem 2.2 *1) [12] The matrix* $S-T$ *is nonsingular if and if the matrix* $(C+D)-(E+F)$ *and* $(C+F)-(E+D)$ *are both nonsingular.*

Proof 2.3 *Assuming that* $S-T$ *is nonsingular we obtain of the Eq.(2.14)*

$$X = (S-T)^{-1}Y. \tag{2.16}$$

2) If $(S-T)^{-1}$ *exists it must have the same structure as S,i.e.*

$$(S-T)^{-1} = \begin{pmatrix} G & H \\ H & G \end{pmatrix},$$

and

$$G = \frac{1}{2}[((C+D)-(E+F))^{-1} + ((C+F)-(E+D))^{-1}],$$
$$H = \frac{1}{2}[((C+D)-(E+F))^{-1} - ((C+F)-(E+D))^{-1}].$$

3) The unique solution X of Eq.(2.16) is a fuzzy vector for arbitrary Y if and only if $(S-T)^{-1}$ *is nonnegative, i.e.*

$$((S-T)^{-1})_{ij} \geqslant 0, \quad 1 \leqslant i \leqslant 2n, \quad 1 \leqslant j \leqslant 2n.$$

Definition 2.10 *[12] Let* $X = \{(\underline{x}_i(\alpha), \overline{x}_i(\alpha)),\ 1 \leqslant i \leqslant n\}$ *denotes the unique solution of Eq.(2.13), if* $\underline{y}_i(\alpha), \overline{y}_i(\alpha)$ *are linear functions of* α, *then the fuzzy number vector*

$$U = (\underline{u}_i(\alpha), \overline{u}_i(\alpha),\ 1 \leqslant i \leqslant n),$$

defined by

$$\underline{u}_i(\alpha) = \min\{\underline{x}_i(\alpha), \overline{x}_i(\alpha), \underline{x}_i(1)\}, \quad \overline{u}_i(\alpha) = \max\{\underline{x}_i(\alpha), \overline{x}_i(\alpha), \underline{x}_i(1)\}.$$

Is called the fuzzy solution of Eq. (2.13). If $(\underline{x}_i(\alpha), \overline{x}_i(\alpha)), 1 \leqslant i \leqslant n,$ *are all fuzzy numbers then* $\underline{u}_i(\alpha) = \underline{x}_i(\alpha), \quad \overline{u}_i(\alpha) = \overline{x}_i(\alpha)$, *and then U is called a strong fuzzy solution. Otherwise, U is a weak fuzzy solution.*

CHAPTER

THREE

THE DECOMPOSITION METHOD APPLIED TO FUZZY FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

A fuzzy integral equation is the equation in which the unknown fuzzy function appears under an integral sign. One of the simplest integral equations is the fuzzy Fredholm integral equation of the first type:

$$U(t) = \int_a^b K(s,t)U(s)ds.$$

If the unknown fuzzy function occurs both inside and outside the integral, then it is the fuzzy Fredholm integral equation of the second type (FFIE-2 for short).

Therefore, in this chapter we will use the parametric form of fuzzy number to convert a linear FFIE-2 to a linear system of integral equation of the second kind in crisp case.

3.1 Fuzzy integral equation

The fuzzy integral equation which are discussed in this section are the fuzzy Fredholm equations of the second kind. The Fredholm integral equation of the second kind is [6]

$$U(t) = f(t) + \beta \int_a^b k(s,t)U(s)\,ds, \tag{3.1}$$

where $\beta > 0$, $k(s,t)$ is an arbitrary kernel function over the square $a \leqslant s,t \leqslant b$ and $f(t)$ is a function of $t : a \leqslant t \leqslant b$. If $f(t)$ is a crisp function then the solutions of Eq.(3.1) are crisp as well. However, if $f(t)$ is a fuzzy function these equations may only possess fuzzy solutions. Sufficient conditions for the existence of a unique solution to the fuzzy Fredholm integral equation of the second kind where $f(t)$ is a fuzzy function, are given in [17].

3.1.1 Transforme FFIE-2 to a system of Fredholm integral equation in crisp case

Let $(\underline{f}(t,\alpha),\overline{f}(t,\alpha))$ and $(\underline{u}(t,\alpha),\overline{u}(t,\alpha))$, $0 \leqslant \alpha \leqslant 1$ and $t \in [a,b]$ are parametric form of $f(t)$ and $u(t)$, respectively then, parametric form of FFIE-2 (3.1), is as follows:

$$\begin{cases} \underline{u}(t,\alpha) = \underline{f}(t,\alpha) + \beta \int_a^b v_1(s,t,\underline{u}(s,\alpha),\overline{u}(s,\alpha))\,ds, \\ \overline{u}(t,\alpha) = \overline{f}(t,\alpha) + \beta \int_a^b v_2(s,t,\underline{u}(s,\alpha),\overline{u}(s,\alpha))\,ds, \end{cases} \tag{3.2}$$

where

$$v_1(s,t,\underline{u}(s,\alpha),\overline{u}(s,\alpha)) = \begin{cases} k(s,t)\underline{u}(s,\alpha), & k(s,t) \geqslant 0 \\ k(s,t)\overline{u}(s,\alpha), & k(s,t) < 0 \end{cases}, \tag{3.3}$$

and

$$v_2(s,t,\underline{u}(s,t),\overline{u}(s,t)) = \begin{cases} k(s,t)\overline{u}(s,\alpha), & k(s,t) \geqslant 0 \\ k(s,t)\underline{u}(s,\alpha), & k(s,t) < 0 \end{cases}, \tag{3.4}$$

for each $0 \leqslant \alpha \leqslant 1$ and $a \leqslant t \leqslant b$.

We can see that (3.2) is a system of linear Fredholm integral equation in crisp case for each $0 \leqslant \alpha \leqslant 1$ and $a \leqslant t \leqslant b$.

In the following section, we use Adomian method to solve system of linear Fredholm integral equation in crisp case then, we find approximating solution for $\underline{u}(t,\alpha)$ and $\overline{u}(t,\alpha)$ for each $0 \leqslant \alpha \leqslant 1$ and $a \leqslant t \leqslant b$.

3.2 Adomian decompostion method

The Adomian decomposition method, decomposes each solution as an infinite sum of components, where these components are determined recurrently. We explain the main algorithm of Adomian decomposition method that applied to a system of linear Fredholm integral equation of the form

$$U(t) = F(t) + \int_a^b k(s,t)U(s)ds, \tag{3.5}$$

$$U(t) = (u_1(t),u_2(t))^T,$$

$$F(t) = (f_1(t),f_2(t))^T,$$

$$K(s,t) = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

Where the unknown functions $u_1(t)$ and $u_2(t)$ appear only under the integral sign, a and b are constants. However, for systems of Fredholm integral equations of the second kind, the unknown functions $u_1(t)$ and $u_2(t)$ appear inside and outside the integral sign. The second kind is represented by the form

$$u_1(t) = f_1(t) + \int_a^b \sum_{j=1}^{2} K_{1j}(s,t)u_j\,ds, \tag{3.6}$$

$$u_1(t) = f_1(t) + \int_a^b K_{11}(s,t)u_1 + K_{12}(s,t)u_2 ds,$$

$$u_2(t) = f_2(t) + \int_a^b \sum_{j=1}^{2} K_{2j}(s,t)u_j\,ds, \tag{3.7}$$

$$u_2(t) = f_2(t) + \int_a^b K_{21}(s,t)u_1 + K_{22}(s,t)u_2\,ds,$$

Eq.(3.6),(3.7) can be written as:

$$u_1(t) = f_1(t) + N_1(u_1,u_2), \tag{3.8}$$

$$u_2(t) = f_2(t) + N_2(u_1,u_2), \tag{3.9}$$

where

$$N_i(u_1,u_2)(t) = \int_a^b \sum_{j=1}^{2} K_{ij}u_j\,ds, \quad i = 1,2 \tag{3.10}$$

$$N_1(u_1,u_2) = \int_a^b K_{11}(s,t)u_1 + K_{12}(s,t)u_2\,ds,$$

$$N_2(u_1,u_2) = \int_a^b K_{21}(s,t)u_1 K_{22}(s,t)u_2\,ds,$$

then new form of Eq.(3.5) is:

$$u_1 = f_1 + N_1(u_1,u_2), \tag{3.11}$$

$$u_2 = f_2 + N_2(u_1,u_2). \tag{3.12}$$

To use Adomian decomposition method, let

$$u_1 = \sum_{m=0}^{\infty} u_{1m}, \tag{3.13}$$

$$\sum_{m=0}^{\infty} u_{1m} = u_{10} + u_{11} + ... + ...$$

$$u_2 = \sum_{m=0}^{\infty} u_{2m}, \tag{3.14}$$

$$\sum_{m=0}^{\infty} u_{2m} = u_{20} + u_{21} + ... + ...$$

and

$$N_1(u_1,u_2) = \int_a^b \sum_{j=1}^{2} K_{1j}(s,t)(\sum_{m=0}^{\infty} u_{1m})\,ds, \tag{3.15}$$

$$N_1(u_1,u_2) = \int_a^b K_{11}(s,t)(u_{10}+u_{11}+..+..)+K_{12}(s,t)(u_{20}+u_{21}+..+..)\,ds.$$

$$N_2(u_1,u_2) = \int_a^b \sum_{j=1}^{2} K_{2j}(s,t)(\sum_{m=0}^{\infty} u_{2m})\,ds, \tag{3.16}$$

$$N_2(u_1,u_2) = \int_a^b K_{21}(u_{10}+u_{11}+..+..)+K_{22}(u_{20}+u_{21}+..+..)\,ds.$$

Substituting Eq.(3.13),(3.14) in (3.8),(3.9) we get:

$$\sum_{m=0}^{\infty} u_{1m} = f_1 + \int_a^b \sum_{j=1}^{2} K_{1j}(s,t)(\sum_{m=0}^{\infty} u_{jm})\,ds, \tag{3.17}$$

$$u_{10}+u_{11}+..+.. = f_1 + \int_a^b K_{11}(u_{10}+u_{11}+..+..)+K_{12}(u_{20}+u_{21}+..+..)\,ds.$$

$$\sum_{m=0}^{\infty} u_{2m} = f_2 + \int_a^b \sum_{j=1}^{2} k_{2j}(s,t)(\sum_{m=0}^{\infty} u_{jm})\,ds, \tag{3.18}$$

$$u_{20}+u_{21}+..+.. = f_2 + \int_a^b K_{21}(u_{10}+u_{11}+..+..)+K_{22}(u_{20}+u_{21}+..+..)\,ds.$$

Equating powers of Eq.(3.17),(3.18) gives:

$u_{10} = f_1,$
$u_{11} = \int_a^b K_{11}(s,t)u_{10} + K_{12}(s,t)u_{20},$
$u_{12} = \int_a^b K_{11}(s,t)u_{11} + K_{12}(s,t)u_{21},$
$\vdots$
$u_{1k} = \int_a^b K_{11}(s,t)u_{1k-1} + K_{12}(s,t)u_{2k-1}.$
$u_{20} = f_2,$
$u_{21} = \int_a^b K_{21}(s,t)u_{10} + K_{22}(s,t)u_{20},$
$u_{22} = \int_a^b K_{21}(s,t)u_{11} + K_{22}(s,t)u_{21},$
$\vdots$

$u_{2k} = \int_a^b K_{21}(s,t)u_{1k-1} + K_{22}(s,t)u_{2k-1}$.
We approximate u_1 by:
$\varphi_{1K} = \sum_{m=0}^{k-1} u_{1m}$ where $\lim_{k\to\infty} \varphi_{1K} = u_1$,
and we approximate u_2 by:
$\varphi_{2k} = \sum_{m=0}^{K-1} u_{2m}$ where $\lim_{k\to\infty} \varphi_{2K} = u_2$.

Theorem 3.1 *[16] Let $K(s,t)$ be continuous for $a \leqslant s,\ t \leqslant b$ and $f(t)$ a fuzzy continuous function of t, $a \leqslant t \leqslant b$. If $1 < \frac{1}{M(b-a)}$, where $M = \max_{a\leqslant s,t\leqslant b} |K(s,t)|$, then the iterative procedure*
$u_{10} = f_1$,
$u_{1k} = \int_a^b K_{11}(s,t)u_{1k-1} + K_{12}(s,t)u_{2k-1}$,
and
$u_{20} = f_2$,
$u_{2k} = \int_a^b K_{21}(s,t)u_{1k-1} + K_{22}(s,t)u_{2k-1}$,
converges to the unique solution.

In the following we discuss the general case.
General case: The general form of Fredholm integral equation of the second kind is:

$$U(t) = (u_1(t), ..., u_n(t))^T,$$

$$F(t) = (f_1(t), ..., f_n(t))^T,$$

$$k(s,t) = [k_{ij}(s,t)],\ i = 1,...,n,\ j = 1,...,n.$$

The second kind of (3.5) is represented by the form:

$$u_i(t) = f_i(t) + \int_a^b \sum_{j=1}^{n} k_{ij}(s,t)u_j(s)\,ds, \tag{3.19}$$

Eq.(3.19) can be written as:

$$u_i = f_i + N_i(u_1, u_2, ..., u_n)(t), \tag{3.20}$$

where

$$N_i(u_1, u_2, ..., u_n)(t) = \int_a^b \sum_{j=1}^{n} k_{ij}(s,t)u_j(s)ds, \tag{3.21}$$

then new form of Eq. (3.5) is:

$$u_i = f_i + N_i(u_1, u_2, ..., u_n). \tag{3.22}$$

To use Adomian decomposition method, let

$$u_i = \sum_{m=0}^{\infty} u_{im}, \tag{3.23}$$

and

$$N_i(u_1,...,u_n) = \int_a^b \sum_{j=1}^{n} k_{ij}(s,t)(\sum_{m=0}^{\infty} u_{jm})ds. \tag{3.24}$$

Substituting (3.23) and (3.24) in (3.20) we have:

$$\sum_{m=0}^{\infty} u_{im} = f_i + \int_a^b \sum_{j=1}^{n} k_{ij}(s,t)(\sum_{m=0}^{\infty} u_{jm})ds. \tag{3.25}$$

Equating powers on both sides of Eq.(3.25) gives:

$$u_{i0} = f_i,$$

$$u_{i,k+1} = \int_a^b \sum_{j=1}^{n} k_{ij}(s,t)u_{jk}ds, \quad i = 1,...,n, \quad k = 0,...$$

We usually approximate u_i by $\varphi_{ik} = \sum_{m=0}^{k-1} u_{im}$, where $\lim_{k\to\infty} \varphi_{ik} = u_i$.

Example 3.1 *Consider the following system of linear Fredholm integral equations with the exact solutions $f_1(t) = t+1$ and $f_2(t) = t^2+1$*

$$\begin{cases} f_1(t) = \dfrac{t}{18} + \dfrac{17}{36} + \displaystyle\int_0^1 \dfrac{s+t}{3}(f_1(s)+f_2(s))\,ds, \\ f_2(t) = t^2 - \dfrac{19}{12}t + 1 + \displaystyle\int_0^1 st(f_1(s)+f_2(s))\,ds. \end{cases}$$

To derive the solutions by using the decomposition method, we can use the following Adomian scheme:

$$\begin{cases} f_{10}(t) = \dfrac{t}{18} + \dfrac{17}{36} \simeq 0.0556t + 0.4722, \\ f_{20}(t) = t^2 - \dfrac{19}{12}t + 1 \simeq t^2 - 1.5833t + 1, \end{cases}$$

and

$$\begin{cases} f_{1,m+1}(t) = \displaystyle\int_0^1 \dfrac{(s+t)}{3}(f_{1m}(s)+f_{2m}(s))\,ds, \\ f_{2,m+1}(t) = \displaystyle\int_0^1 st(f_{1m}(s)+f_{2m}(s))\,ds, \quad m = 0,1,2,... \end{cases}$$

For the first iteration, we have:

$$\begin{cases} f_{11}(t) = \int_0^1 \frac{(s+t)}{3}(f_{10}(s)+f_{20}(s))\,ds = \frac{25}{72}t + \frac{103}{648} \simeq 0.3472t + 0.1590t, \\ f_{21}(t) = \int_0^1 st(f_{10}(s)+f_{20}(s))\,ds = \int_0^1 st(f_{1m}(s)+f_{2m}(s))\,ds = \frac{103}{216}t \simeq 04769t. \end{cases}$$

In practice, all terms of the series $f_i(t) = \sum_{m=0}^{\infty} f_{im}(t)$ cannot be determined and so we use an approximation of the solution by the following truncated series:

$$\varphi_{ik}(t) = \sum_{m=0}^{k-1} f_{im}(t), \textit{with} \lim_{k\to\infty} \varphi_{ik}(t) = f_i(t).$$

The approximated solutions with two terms are:

$$\begin{cases} \varphi_{12}(t) = f_{10}(t) + f_{11}(t) \simeq 0.4028t + 0.6312, \\ \varphi_{22}(t) = f_{20}(t) + f_{21}(t) \simeq t^2 - 1.1065t + 1. \end{cases}$$

Next terms are:

$$\begin{cases} f_{12}(t) = \int_0^1 \frac{(s+t)}{3}(f_{11}(s)+f_{21}(s))\,ds = \frac{185}{972}t + \frac{17}{144} \simeq 0.1903t + 0.1181, \\ f_{22}(t) = \int_0^1 st(f_{11}(s)+f_{21}(s))\,ds = \frac{17}{48}t \simeq 0.3542t. \end{cases}$$

Solutions with three terms are:

$$\begin{cases} \varphi_{13}(t) = f_{10}(t) + f_{11}(t) + f_{12}(t) \simeq 0.5931t + 0.7492, \\ \varphi_{23}(t) = f_{20}(t) + f_{21}(t) + f_{22}(t) \simeq t^2 - 0.7523t + 1. \end{cases}$$

In the same way, the components $\varphi_{1k}(t)$ and $\varphi_{2k}(t)$ can be calculated for $K = 3,4,\ldots$ The solutions with eleven terms are given as:

$$\begin{cases} \varphi_{1,11}(t) = f_{10}(t) + f_{11}(t) + \ldots + f_{1,10}(t) \simeq 0.9813t + 0.9885, \\ \varphi_{2,11}(t) = f_{20}(t) + f_{21}(t) + \ldots + f_{2,10}(t) \simeq t^2 - 0.0345t + 1. \end{cases}$$

3.3 Numerical results

In this section, we apply the previous algorithm to two examples. We compare results with exact solutions using the metric of Definition (2.2) (see Tables 1 and 2). The approximate solutions and exact solutions are compared in Figs.(3.1), (3.2) and (3.3) for a fixed t.

Example 3.2 *Consider the fuzzy Fredholm integral equation with*

$$\underline{f}(t,\alpha) = \sin(t/2)(13/15(\alpha^2+\alpha) + 2/15(4-\alpha^3-\alpha)),$$

$$\overline{f}(t,\alpha) = \sin(t/2)(2\backslash 15(\alpha^2+\alpha)+13/15(4-\alpha^3-\alpha)),$$

and kernel

$$K(s,t) = 0.1\sin(s)\sin(t/2), \quad 0 \leqslant s,t \leqslant 2\pi,$$

and $a=0,\ b=2\pi$. *The exact solution in this case is given by:*

$$\underline{u}(t,\alpha) = (\alpha^2+\alpha)\sin(t/2),$$

$$\overline{u}(t,\alpha) = (4-\alpha^3-\alpha)\sin(t/2).$$

Some first terms of Adomian decomposition series are:

$$\underline{u}_0(t,\alpha) = \underline{f}(t,\alpha) = \frac{1}{15}\sin(t/2)(13\alpha^2+11\alpha+8-2\alpha^3),$$

since $\begin{cases} K(s,t) = 0.1sin(s)sin(t/2) \geqslant 0, & if\, 0 \leqslant s \leqslant \pi, \\ K(s,t) = 0.1sin(s)sin(t/2) < 0, & if\, \pi \leqslant s \leqslant 2\pi, \end{cases}$

then from (3.3), it follows that:

$$\begin{aligned}
\underline{u}_1(t,\alpha) &= \int_0^{\pi} k(s,t)\underline{u}_0(s,\alpha)\,ds + \int_{\pi}^{2\pi} K(s,t)\overline{u}_0(s,\alpha)\,ds \\
&= \frac{\sin(t/2)}{10}\left[\int_0^{\pi} 1/15\sin(\frac{s}{2})\sin(s)(13\alpha^2+11\alpha+8-2\alpha^3)\,ds\right. \\
&+ \left.\int_{\pi}^{2\pi} 1/15\sin(s)\sin(\frac{s}{2})(2\alpha^2-11\alpha+52-13\alpha^3\,ds\right] \\
&= \frac{\sin(t/2)}{150}\left[(13\alpha^2+11\alpha+8-2\alpha^3)\int_0^{\pi}\sin(s)\sin(\frac{s}{2})\,ds\right. \\
&- \left.(2\alpha^2-11\alpha+52-13\alpha^3)\int_0^{\pi}\sin(s)\sin(\frac{s}{2})\,ds\right] \\
&= \frac{\sin(t/2)}{150}(11\alpha^2+22\alpha-44+11\alpha^3)\int_0^{\pi}\sin(s)\sin(\frac{s}{2})\,ds,
\end{aligned}$$

we have

$$I = \int_0^{\pi}\sin(s)\sin(\frac{s}{2})\,ds = -\cos(s)\sin(\frac{s}{2})\Big]_0^{\pi} + \frac{1}{2}\int_0^{\pi}\cos(s)\cos(\frac{s}{2})\,ds,$$

and

$$\int_0^{\pi}\cos(s)\cos(\frac{s}{2})\,ds = \sin(s)\cos(\frac{s}{2})\Big]_0^{\pi} + \frac{1}{2}\int_0^{\pi}\sin(s)\cos(\frac{s}{2})\,ds,$$

then

$$I = -\cos(s)\sin(\frac{s}{2})\Big]_0^{\pi} + \frac{1}{2}\sin(s)\cos(\frac{s}{2})\Big]_0^{\pi} + \frac{1}{4}I,$$

this implies that $\frac{3}{4}I = 1$. *Hence,* $I = \frac{4}{3}$. *We conclude that*

$$\begin{aligned}\underline{u}_1(t,\alpha) &= \frac{2}{225}\sin(t/2)(11\alpha^2+22\alpha-44+11\alpha^3)\\ &= \frac{22}{225}\sin(t/2)(\alpha^2+2\alpha-4+\alpha^3).\end{aligned}$$

By the same method we find:

$$\underline{u}_2(t,\alpha) = \frac{88}{3375}\sin(t/2)(\alpha^2+2\alpha-4+\alpha^3),$$

$$\underline{u}_3(t,\alpha) = \frac{352}{50625}\sin(t/2)(\alpha^2+2\alpha-4+\alpha^3).$$

And

$$\overline{u}_0(t,\alpha) = \overline{f}(t,\alpha) = 1/15\sin(t/2)(2\alpha^2-11\alpha+52-13\alpha^3),$$

since $\begin{cases} K(s,t) = 0.1\sin(s)\sin(t/2) \geqslant 0, & if\, 0 \leqslant s \leqslant \pi,\\ K(s,t) = 0.1\sin(s)\sin(t/2) < 0, & if\, \pi \leqslant s \leqslant 2\pi,\end{cases}$

then from (3.4), it follows that:

$$\begin{aligned}\overline{u}_1(t,\alpha) &= \int_0^{\Pi} K(s,t)\overline{u}_0(s,\alpha)\,ds + \int_{\pi}^{2\pi} K(s,t)\underline{u}_0(s,\alpha)\,ds\\ &= \frac{\sin(t/2)}{10}\Big[\int_0^{\pi} 1/15\sin(s)\sin(\frac{s}{2})(2\alpha^2-11\alpha+52-13\alpha^3)\,ds\\ &+ \int_{\pi}^{2\pi} 1/15\sin(s)\sin(\frac{s}{2})(13\alpha^2+11\alpha+8-2\alpha^3)\,ds\Big]\\ &= \frac{\sin(t/2)}{150}\Big[(2\alpha^2-11\alpha+52-13\alpha^3)\int_0^{\pi}\sin(s)\sin(\frac{s}{2})\,ds\\ &- (13\alpha^2+11\alpha+8-2\alpha^3)\int_{\pi}^{2\pi}\sin(s)\sin(\frac{s}{2})\,ds\Big]\\ &= \frac{\sin(t/2)}{150}(-11\alpha^2-22\alpha+44-11\alpha^3)\int_0^{\pi}\sin(s)\sin(\frac{s}{2})\,ds,\end{aligned}$$

we have

$$I = \int_0^{\pi}\sin(s)\sin(\frac{s}{2})\,ds = -\cos(s)\sin(\frac{s}{2})\Big]_0^{\pi} + \frac{1}{2}\int_0^{\pi}\cos(s)\cos(\frac{s}{2})\,ds,$$

and

$$\int_0^{\pi}\cos(s)\cos(\frac{s}{2})\,ds = \sin(s)\cos(\frac{s}{2})\Big]_0^{\pi} + \frac{1}{2}\int_0^{\pi}\sin(s)\cos(\frac{s}{2})\,ds,$$

then

$$I = -\cos(s)\sin(\frac{s}{2})\Big]_0^{\pi} + \frac{1}{2}\sin(s)\cos(\frac{s}{2})\Big]_0^{\pi} + \frac{1}{4}I,$$

this implies that $\frac{3}{4}I = 1$. *Hence* $I = \frac{4}{3}$. *We conclude that*

$$\begin{aligned}\overline{u}_1(t,\alpha) &= \frac{2}{225}\sin(t/2)(-11\alpha - 22\alpha + 44 - 11\alpha^3)\\ &= -\frac{22}{225}\sin(t/2)(\alpha^2 + 2\alpha - 4 + \alpha^3).\end{aligned}$$

By the same method we find:

$$\overline{u}_2(t,\alpha) = -\frac{88}{3375}\sin(t/2)(\alpha^2 + 2\alpha - 4 + \alpha^3),$$

$$\overline{u}_3(t,\alpha) = -\frac{352}{50625}\sin(t/2)(\alpha^2 + 2\alpha - 4 + \alpha^3).$$

We approximate

$\underline{u}(t,\alpha)$ *with* $\underline{\varphi}_4(t,\alpha) = -\frac{1}{50625}(128\alpha^3 - 50497\alpha^2 - 50369\alpha - 512) \times \sin(t/2).$

and

$\overline{u}(t,\alpha)$ *with* $\overline{\varphi}_4(t,\alpha) = -\frac{1}{50625}(50497\alpha^3 - 128\alpha^2 + 50369\alpha - 201988) \times \sin(t/2).$

Table 1: The absolute error between the exact and approximate solution.for Example (3.2)

t	0	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000	1.0000
Error	0	0.0005	0.0010	0.0015	0.0020	0.0025	0.0030	0.0035	0.0039	0.0044	0.0048

1.1000	1.2000	1.3000	1.4000	1.5000
0.0053	0.0057	0.0061	0.0065	0.0069

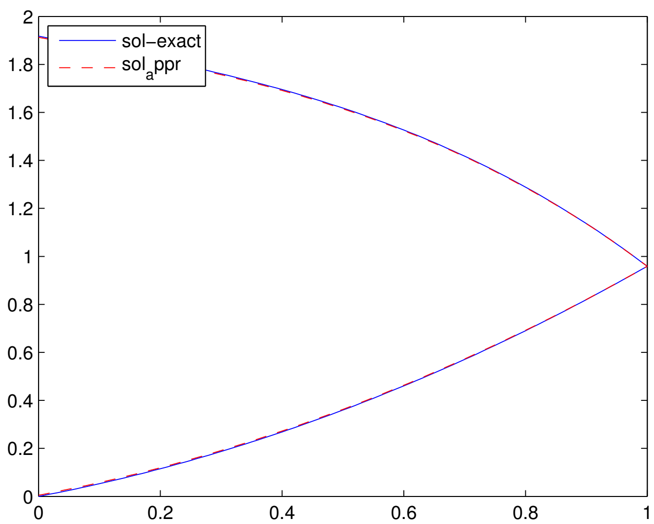


Figure 3.1: Exact solution and approximate solution for Example 3.2 (t=1)

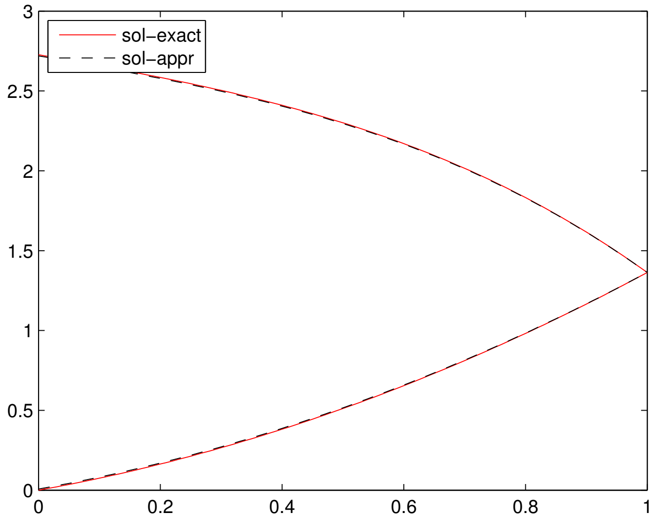


Figure 3.2: Exact solution and approximate solution for Example 3.2 (t=1.5)

Example 3.3 *Consider the following fuzzy Fredholm integral equation with*

$$\underline{f}(t,\alpha) = \alpha t + 3/26 - 3/26\alpha - 1/13t^2 - 1/13t^2\alpha,$$

$$\overline{f}(t,\alpha) = 2t - \alpha t + 3/26\alpha + 1/13t^2\alpha - 3/26 - 3/13t^2,$$

and kernel

$$K(s,t) = (s^2 + t^2 - 2)/13, \quad 0 \leqslant s,t \leqslant 2,$$

and $a = 0, b = 2$. The exact solution in this case is given by

$$\underline{u}(t,\alpha) = \alpha t$$

$$\overline{u}(t,\alpha) = (2-\alpha)t.$$

We can see that, some first terms of Adomian decomposition series are as follows:

$$\underline{u}_0(t,\alpha) = \underline{f}(t,\alpha) = \alpha t + 3/26 - 3/26\alpha - 1/13t^2 - 1/13t^2\alpha,$$
$$\underline{u}_1(t,\alpha) = 29/338\alpha - 499/5070 + 11/169t^2\alpha + 29/507t^2,$$
$$\underline{u}_2(t,\alpha) = \frac{-235}{19773} + \frac{51}{2197}\alpha + \frac{1298}{98865}t^2 + \frac{22}{2197}t^2\alpha,$$
$$\underline{u}_3(t,\alpha) = \frac{146}{28561}\alpha - \frac{28114}{6426225} + \frac{44}{28561}t^2\alpha + \frac{1468}{296595}t^2,$$
$$\underline{u}_4(t,\alpha) = \frac{-425948}{751868325} + \frac{380}{371293}\alpha + \frac{908008}{751868325}t^2 + \frac{88}{371293}t^2\alpha,$$
$$\underline{u}_5(t,\alpha) = \frac{72}{371293}\alpha - \frac{17922712}{146614323375} + \frac{176}{4826809}t^2\alpha + \frac{744752}{2255604975}t^2,$$

and

$$\begin{aligned}
\overline{u}_0(t,\alpha) &= \overline{f}(t,\alpha) = 2t - \alpha t + 3/26\alpha + 1/13t^2\alpha - 3/26 - 3/13t^2, \\
\overline{u}_1(t,\alpha) &= 371/5070 - 29/338\alpha + 95/507t^2 - 11/169t^2\alpha, \\
\overline{u}_2(t,\alpha) &= \frac{-51}{2197}\alpha + \frac{683}{19773} - \frac{22}{2197}t^2\alpha + \frac{3278}{98865}t^2, \\
\overline{u}_3(t,\alpha) &= \frac{37586}{6426225} - \frac{146}{28561}\alpha + \frac{30964}{3855735}t^2 - \frac{44}{28561}t^2\alpha, \\
\overline{u}_4(t,\alpha) &= \frac{-380}{371293}\alpha + \frac{1113052}{7518687325} - \frac{88}{371293}t^2\alpha + \frac{1264408}{751868325}t^2, \\
\overline{u}_5(t,\alpha) &= \frac{38939288}{146614323375} - \frac{72}{371293}\alpha + \frac{11820176}{293228646675}t^2 - \frac{176}{4826809}t^2\alpha.
\end{aligned}$$

Then

$$\underline{\varphi}_6(t,\alpha) = \frac{32}{4826809}t^2\alpha - \frac{3540832}{29322864675}t^2 - t\alpha + 2t + \frac{16}{371293}\alpha - \frac{10586032}{146614323375}$$

and

$$\overline{\varphi}_6(t,\alpha) = -\frac{32}{4826809}t^2\alpha - \frac{242464}{2255604975}t^2 + t\alpha - \frac{16}{371293}\alpha + \frac{2049968}{146614323375}$$

are aproximation for $\underline{u}(t,\alpha)$ *and* $\overline{u}(t,\alpha)$*, respectively.*

Table 2: The absolute error between the exact and approximate solution for Example (3.3)

t	0	0.1000	0.2000	0.3000	0.4000	0.5000
$Error$	$7.22\times10^{-}5$	$7.34\times10^{-}5$	$7.70\times10^{-}5$	$8.31\times10^{-}5$	$9.15\times10^{-}5$	$1.024\times10^{-}4$

0.6000	0.7000	0.8000	0.9000	1.0000
$1.157\times10^{-}4$	$1.314\times10^{-}4$	$1.495\times10^{-}4$	$1.700\times10^{-}4$	$1.930\times10^{-}4$

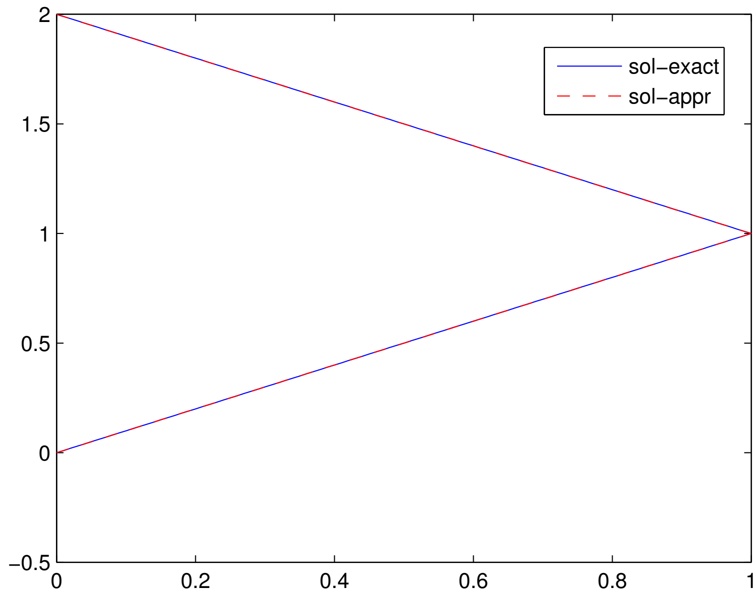


Figure 3.3: Exact solution and approximate solution for Example 3.3 (t=1)

CONCLUSION

In this work, we have introduced the concepts of fuzzy sets, fuzzy numbers with examples. We have solved the linear fuzzy Fredholm integral equation of the second kind. Using the Adomian method, it is possible to find the exact solution or the approximate solution of the problem in the form of a series, the proposed method is illustrated by solving some examples. The solution of the fuzzy Fredholm integral equation can be found directly from the crisp solution without going through the complexity of fuzziness.

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Abstract

After introducing basic concepts in fuzzy mathematics, we focused our attention to solve the linear fuzzy Fredholm integral equation of the second kind by Adomian method. Using parametric form of fuzzy numbers a linear FFIE-2 is converted to a linear system of integral equations of the second kind in crisp case. We illustrated by a numerical algorithm applied to some examples.

Résumé

Dans ce travail, on a utilisé la forme paramétrique des nombres flous pour convertir une équation intégrale floue de Fredholm linéaire du deuxième type en un système linéaire d'équations intégrales du deuxième type.
On a utilisé la méthode Adomian pour trouver la solution approximative de ce système et donc obtenir une approximation pour la solution floue de l'équation intégrale linéaire floue de Fredholm du deuxième type.
Enfin, on a illustré cette méthode en l'appliquant à quelque exemples.

ملخص

في هذا العمل، استخدمنا الشكل الوسيطي للأعداد الضبابية لتحويل المعادلات التكاملية الخطية الضبابية لفريدهولم من النوع الثاني الى جملة خطية من المعادلات التكاملية الخطية الكلاسيكية من النوع الثاني. لقد استخدمنا طريقة ادوميان لإيجاد الحل التقريبي لهذه الجملة الخطية ومن ثم حصلنا على تقدير تقريبي للحل الضبابي للمعادلة التكاملية الخطية الضبابية لفريدهولم من النوع الثاني.

في الاخير لتوضيح هذه الطريقة قمنا بتطبيقها على بعض الامثلة.