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Mémoire

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Thème

On the number of limit cycles of a class of polynomial differential systems.

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Introduction

Differential systems have important role in the study of many problems in mechanical, sciences and technology. Direct solution of a differential system is usually difficult or impossible. However, another way out it possible. This is the qualitative study of differential systems. This study makes it possible to provide information on the behavior of the solutions of a differential system without the need to solve it explicitly, and it consists in examining the properties and the characteristics of the solutions of this systems, and to justify among these solution, the existence or non existence of an isolated closed curve form called limit cycle.

An important problem of the qualitative theory of differential systems is do determine the limit cycle of system polynomial differential system.

The limit cycles introduced by *Henri Poincaré* in 1881 (see for instance [8]) in his "Dissertation on the curves defined by a differential equations". *Poincaré* was interested in the qualitative study of the solutions of the differential systems, i.e. points equilibrium, limit cycles and their stability. This makes it possible to have an overall idea of the other orbits of studied system.
The mathematician *David Hilbert* Presented at the second international congress of mathematics (1900) (see [5]), 23 problems whose future awaits resolution through new methods that will be discovered in the centry that begins. The problem number 16 is to know the maximum number and relative position of the limit cycles of a planar polynomial differential system of degree n. We denote H_n this maximum number. *Dulac* in 1923 (see for instance [2]), offered a proof that H_n is finite. In recent years, several papers have studied the limit cycles of planar polynomial differential systems. The main reason for this study is *Hilbert* 16-th unsolved problem. Later on *Van der Pol* in 1962 (see for instance [9]), *Liénard* (see [6]) in 1928 and *Andronov* (see for instance[1]) in 1929 shown that the periodic solution of self-sustained oscillation os a circuit in a vacuum tube was a limit cycle in the sense defined by *Poincaré*.

In our memory, we will use the qualitative theory of ordinary differential equations to treat a class of polynomial planar differential systems of the form

$$\begin{cases} \dot{x} = \left(\gamma x - x\alpha\left(x^2+y^2\right)^2 - 4\gamma y\right) Q^2\left(x,y\right) - x\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)^2, \\ \dot{y} = \left(\gamma y - y\alpha((x^2+y^2)^2 + 4\gamma x\right) Q^2\left(x,y\right) - y\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)^2, \end{cases} \tag{1}$$

where $Q\left(x,y\right)$ is homogeneous polynomial of degrees 2 and γ, α, real constants. The main motivation of this dissertation is to prove that these systems are integrable. Moreover, we determine sufficient conditions for a polynomial differential system to possess at most two limit cycles, one of them algebraic and the other one non-algebraic, or two explicit algebraic limit cycles.

This work has been structured in three chapters:

The first chapter contains reminders of classical preliminary notions and tools that we have used in this work to demonstrate our results.

In the second chapter, we study the nonexistence and existence of an algebraic limit cycles for a class of septic polynomial planar differential systems of the form

$$\begin{cases} \dot{x} = w^2 \left(\gamma x - x\alpha \left(x^2 + y^2\right)^2 - 4\gamma y\right) \left(x^2 + y^2\right)^2 - x \left(\alpha \left(x^2 + y^2\right)^2 - \gamma\right)^2, \\ \dot{y} = w^2 \left(\gamma y - y\alpha((x^2 + y^2)^2 + 4\gamma x\right) \left(x^2 + y^2\right)^2 - y \left(\alpha \left(x^2 + y^2\right)^2 - \gamma\right)^2. \end{cases}$$

More precisely, we give sufficient conditions for the existence of two or one hyperbolic algebraic limit cycles.

The third chapter deals with the study the integrability and the search of limit cycles for a multi-parameter septic polynomial differential system (1) in the case when $Q\left(x, y\right) = ax^2 + by^2 + cxy.$

More precisely, we determine sufficient conditions for a polynomial differential system to polynomial differential system of degree 7, (1) exhibiting simultaneously two explicit limit cycles one algebraic and another non-algebraic.

Chapter 1

Preliminary concepts

1.1 Introduction

In this chapter, we discuss some definition of polynomial differential systems: phase portrait, equilibrium point, solution and periodic solution, limit cycle...
We will also introduce a reminder on the Poincaré theorem and others theorems.

1.2 Polynomial differential systems

Definition 1.1 *A polynomial differential system is a system of the form:*

$$\begin{cases} \dot{x} = P\left(x\left(t\right), y\left(t\right)\right), \\ \dot{y} = Q\left(x\left(t\right), y\left(t\right)\right), \end{cases} \tag{1.1}$$

where $P\left(x, y\right)$ and $Q\left(x, y\right)$ are a real polynomials in the variables x and y. The degree n of the system $\left(1.1\right)$ is the maximum of the degrees of the polynomial P and Q. As usual the dot denotes derivative with respect to the independent variable t.

Definition 1.2 *A differential system of the form*

$$\frac{dx}{dt} = f\left(t, x\right)$$

is said to be autonomous if the function f depends only on the vector variable x. Otherwise, it is not autonomous, an autonomous system is written in the form

$$\dot{x} = f\left(x\right),$$

where $\dot{x} = \frac{dx}{dt}$.

1.3 Vector field

Definition 1.3 *We call vector field a region of the plane in which exists in any point a vector* $\overrightarrow{V}(M,t)$. *Suppose that we have a* C^1 *vector field in* $\Omega \subset R^2$, *that is to say the application:*

$$M : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \overrightarrow{V}(M) = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix},$$

where F_1, F_2 *are* C^1 *in* Ω .
We consider the vector field χ *associated to the system* (1.1)

$$\frac{d\overrightarrow{M}}{dt} = \overrightarrow{V} \Leftrightarrow \begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

which means that system (1.1) *is equivalent to the vector field* $\chi(P,Q)$, *we can also write:*

$$\chi = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

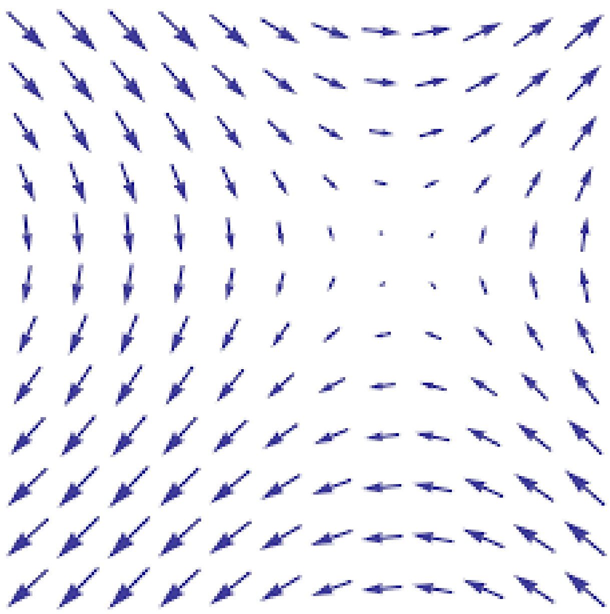

Figure 1.1: Vector field

1.4 Phase portrait

The plane $\mathbb{R}^2$ is called phase plane and the solutions of a vector field χ, represent in the phase plane of the orbits or the trajectories, the phase portrait of a vector fields χ is the set of the solutions in the phase plane.

Definition 1.4 *A phase portrait is a geometric representation of the orbits of a dynamical nonlinear system in the phase plane, at each set of initial conditions corresponds a curve or a point.*

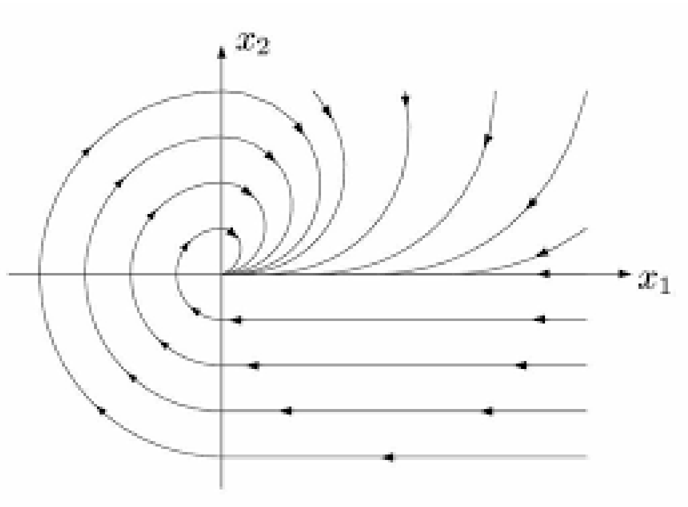

Figure 1.2: Phase portrait

1.5 Equilibrium point

The fixed points or play a vital role in the study of dynamic systems, Henri Poincaré $(1854 - 1912)$ showed that to characterize a dynamic system with multiple variables it is not necessary to calculate the detailed solutions, it is enough to know equilibriums points and their stabilities.

Definition 1.5 *Consider the system* (1.1)*, then the system* $\dot{x} = Ax$ *with*

$$A = \left(\frac{\partial f_i}{\partial x_j}(x_0)\right) = Df(x_0), 1 \leqslant i, j \leqslant n.$$

And since $f(x_0) = 0$*, is called the linearized of* (1.1) *in* x_0*.*

1.5.1 Stability of equilibrium point

Any non-linear system may have several equilibrium positions that may be stable or unstable. Let (x_0, y_0) be an equilibrium point of system (1.1). Note by $X = (P(x,y), Q(x,y))$ and $X(t) = (P(x(t), y(t)), Q(x(t), y(t))), X_0 = (P(x_0, y_0), Q(x_0, y_0))$.

Definition 1.6 *We say that:*

(x_0, y_0) *is stable if and only if*

$$\forall \epsilon > 0, \exists \eta > 0 : \| (x,y) - (x_0, y_0) \| < \eta \Rightarrow \| X(t) - X_0 \| < \epsilon, \forall t > 0.$$

(x_0, y_0) *is asymptotically stable if and only if*

$$lim_{t \to +\infty} = \| X(t) - X_0 \| = 0.$$

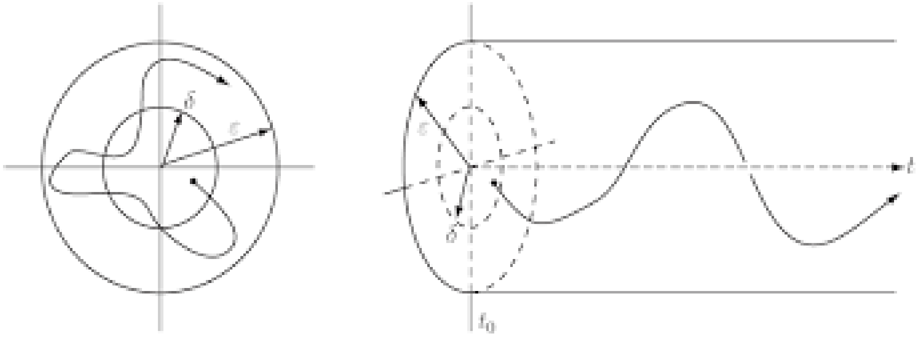

Figure 1.3: Stability of equilibrium point

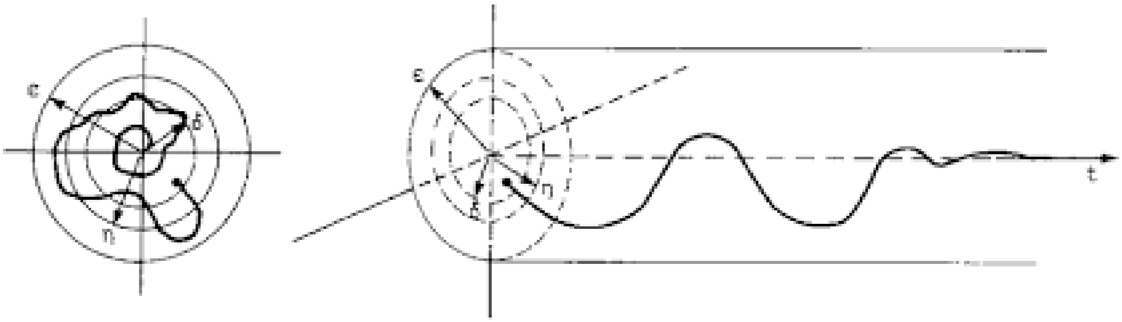

Figure 1.4: Asymptotic stability

1.6 Invariant curve

Invariant algebraic curves play an important role in the integrability of differential planar polynomial systems, and are also used in the study of the existence and non-existence of periodic solutions and consequently the existence and non-existence of limit cycle.

Definition 1.7 *Let $f \in C[x, y]$ not identically zero. The algebraic curve $f(x, y) = 0$ is an invariant algebraic curve of the polynomial system (1.1) if for some polynomial $K \in C[x, y]$ we have:*

$$\chi f = P(x, y)\frac{\partial f}{\partial x}(x, y) + Q(x, y)\frac{\partial f}{\partial y}(x, y) = K(x, y)\, f(x, y), \tag{1.2}$$

for all $(x, y) \in f$. The polynomial K is called the cofactor of the invariant algebraic curve $f = 0$. We note that since the polynomial system has degree m, any cofactor has degree at most $m - 1$.

Example 1.1 *The curve defined by equation $ay + b$ is an invariant curve for the following system*

$$\begin{cases} \dot{x} = -y(ay + b) - (x^2 + y^2 - 1), \\ \dot{y} = x(ay + b). \end{cases} \tag{1.3}$$

Let $f(x, y) = ay + b$, then

$$\begin{aligned} \dot{x}\tfrac{\partial f}{\partial x} + \dot{y}\tfrac{\partial f}{\partial y} &= -y(ay + b) - (x^2 + y^2 - 1) + a(x(ay + b)) \\ &= ax(ay + b). \end{aligned}$$

Thus, $f(x,y)=0$ is invariant curve with cofactor $K(x,y)=ax$.

Proposition 1.1 *[3]*
Suppose $f\in\mathbb{C}[x,y]$ and let $f=f_1^{n_1}...f_r^{n_r}$ be its factorization into irreducible factors over $\mathbb{C}[x,y]$. Then for a polynomial system (1.1), $f=0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i=0$ is an invariant curve for each $i=1,...,r$ with cofactor K_{f_i}. Moreover $K_f=n_1K_{f_1}+...+n1_rK_{f_r}$.

1.7 First integral

The notion of integrability for a differential system is a based on the existence of first integrals, so the question that arises: If we have a differential system, how can we know if it has a first integral?
Definition 1.8 *A function $H:f\to\mathbb{R}$ of class C^j and which is constant on each trajectory of (1.1) and not locally constant is called the first integral of the system (1.1) of class C^j on $U\in\mathbb{R}^2$.*
The equation $H(x,y)=c$ fixed for $c\in\mathbb{R}$, gives a set trajectories of the system in an implicit way.
When $j=1$, these condition are equivalent to

$$P(x,y)\frac{\partial H}{\partial x}+Q(x,y)\frac{\partial H}{\partial 0}\equiv 0$$

and H not locally constant.
The search for an explicit expression of a first integral and the determination of its functional class is called the integrability problem.

Remark 1.1 *- We say that the differential system (1.1) is integrable on an open subset Ω if it admits a first integral on Ω of $\mathbb{R}^2$.*
-It is well know that for differential systems defined on the plan $\mathbb{R}^2$ the existence of a first integral determines their phase portrait.

1.8 Solution and periodic solution

Definition 1.9 *We say that $(x(t),y(t))_{t\in\mathbb{R}}$ is a solution of system (1.1) if the vector field $X=(P,Q)$ is always tangent to the trajectory representing this solution in the phase plane, in other words*

$$\forall t\in\mathbb{R}, P(x(t),y(t))\dot{x}+Q(x(t),y(t))\dot{y}=0.$$

Definition 1.10 *Called periodic solution of system* ([1.1](#)), *all solution* $(x(t), y(t))$ *for which there exists a real* $T > 0$ *such that:*

$$\forall t \in \mathbb{R}, \quad \begin{aligned} x(t+T) &= x(t) \\ y(t+T) &= y(t) \end{aligned} .$$

The smallest number $T > 0$ *is called the period of this solution.*

1.9 Limit cycle

We have seen that the solution tend towards a singular point, another possible behavior for a trajectory is to tend towards a periodic movement in the case of a planar system, that means that the trajectories tend towards what is called a limits cycles.

Definition 1.11 *A limit cycle is an isolated closed orbit of* (1.1), *i.e., we can not find another closed orbit in its neighborhood.*

A periodic orbit Γ *is called stable if for each* $\epsilon > 0$ *there is a neighborhood* U *of* Γ *such that for all* $x \in U$ *and* $t > 0$*:*

$$d(\Phi(t,x), \Gamma) > \epsilon.$$

A periodic orbit Γ *is called unstable if it is not stable, and* Γ *is called a asymptotically stable if it is stable and if for all points* x *in some neighborhood* U *of* Γ

$$lim_{x\to\infty} d(\Phi(t,x), \Gamma) = 0.$$

Example 1.2 *The system*

$$\begin{cases} \dot{x} = -4y + x\left(1 - \frac{x^2}{4} - y^2\right), \\ \dot{y} = x + y\left(1 - \frac{x^2}{4} - y^2\right)), \end{cases}$$

has a limit cycle $\Gamma(t)$ *represent by*

$$\Gamma(t) = (2\cos(2t), \sin(2t))$$

and

$$div(P,Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2 - x^2 - 4y^2,$$

let's calculates now $\int_0^\pi div(\Gamma(t))\, dt$

$$\begin{aligned} \int_0^\pi (2\cos(2t)), \sin(2t))\, dt &= \left(2 - (2\cos(2t))^2 - 4(\sin(2t))^2\right) \\ &= \int_0^2 \pi - 2dt \\ &= -2\pi < 0. \end{aligned}$$

So the cycle $\Gamma(t) = (2\cos(2t), \sin(2t))$ *is a an stable limit cycle.*

Remark 1.2 *The limit cycle appear only in non-linear differential systems.*

1.9.1 Stability of limit cycle

Theorem 1.1 *[7]*
Consider $\Gamma(t)$ *a periodic orbit of system* (1.1) *of period* T.
$i)$ *If* $\int_0^T div\,(T(t))\,dt < 0$, *then* Γ *is a stable limit cycle.*
$ii)$ *If* $\int_0^T div\,(T(t))\,dt > 0$, *then* Γ *is a unstable limit cycle.*
And if $\int_0^T div\,(T(t))\,dt = 0$, *then* Γ *may be a stable, unstable or semi-stable limit cycle or it may belong to a continuous band of cycles.*

Definition 1.12 *We say that the limit cycle* Γ *is hyperbolic* $\int_0^T div\,(T(t))\,dt \neq 0.$

Theorem 1.2 *[4]*
Let us consider a system (1.1) *and* $\Gamma(t)$ *a periodic orbit of period* $\Gamma > 0$. *Assume that* $U : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ *is an invariant curve with* $\Gamma(t) \subseteq \{(x,y) \mid U(x,y) = 0\}$ *and let* $K(x,y)$ *be the cofactor of* $K(x,y)$ *as given in* (1.2). *We assume that* $\bigtriangledown U(p) \neq 0$ *for any* $p \in \Gamma$. *Then*

$$\int_0^T K(\Gamma(t))\,dt = \int_0^T div\,(\Gamma(t))\,dt.$$

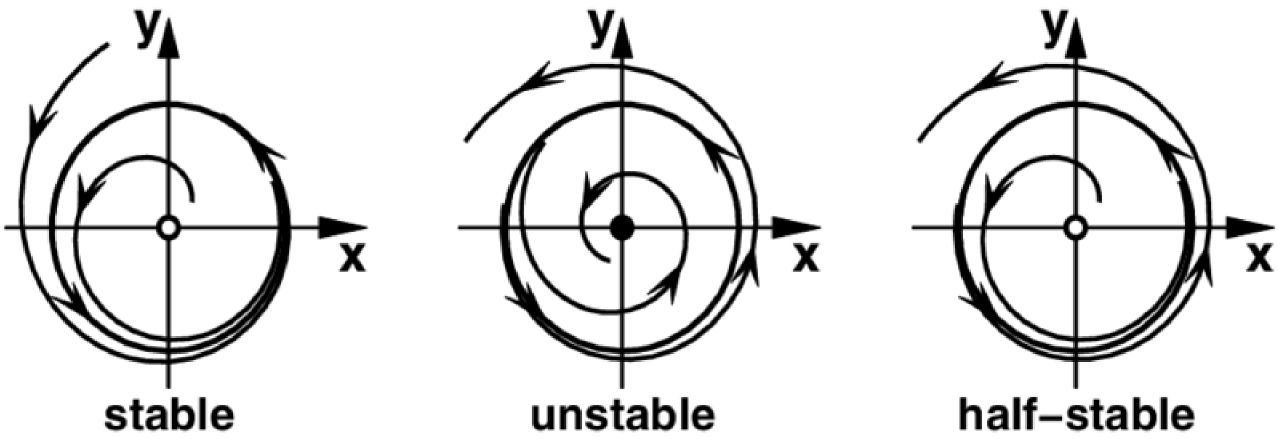

Figure 1.5: Limit cycles

1.10 The first return map

Probably the most basic tool for studying the stability of periodic orbits is the Poincaré map or first return map, defined by Henri Poincaré in 1881. The idea of poincaré map is quite simple: if Γ is a periodic orbit of system (1.1), through the point (x_0, y_0) and Σ is a hyperplane perpendicular to Γ at (x_0, y_0), then for any point $(x,y) \in \Sigma$ sufficiently near (x_0, y_0), the solution of (1.1) through (x,y) at $t = 0$, $\phi_t(x,y)$ will cross Σ again at a point $\Pi(x,y)$ near (x_0, y_0), the mapping $(x,y) \to \Pi(x,y)$ is called the Poincaré map.
The next theorem establishes the existence and continuity of the Poincaré map $\Pi(x,y)$ and of its first derivative $D\Pi(x,y)$.

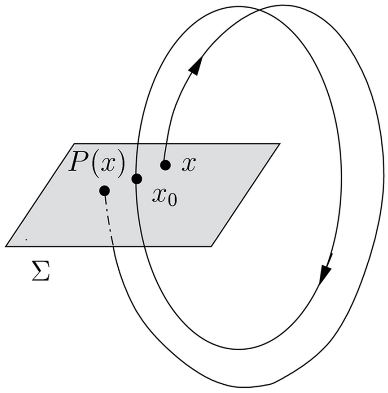

Figure 1.6: The Poincaré map

Theorem 1.3 *[7]*
Let $\boldsymbol{E}$ be open subset of $\mathbb{R}^2$ and let $(P(x,y), Q(x,y)) \in C^1(E)$. Suppose that $\phi_t(x_0, y_0)$ is a periodic solution of (1.1) of period T and that the cycle

$$\Gamma = \left\{(x,y) \in \mathbb{R}^2 \mid (x,y) = \phi_t(x_0,y_0), 0 \leq t \leq T\right\}$$

is contained in E. Let Σ be the hyperplane orthogonal to Σ at (x_0, y_0), i.e., let

$$\Sigma = \left\{(x,y) \in \mathbb{R}^2 \mid (x - x_0, y - y_0) \cdot (P(x_0,y_0), Q(x_0,y_0)) = 0\right\}.$$

Then there is a $\delta > 0$ and a unique function $\tau(x,y)$, defined and continuously differentiable for $(x,y) \in N_\delta(x_0,y_0)$, such that

$$\tau(x_0, y_0) = T,$$

and

$$\phi_{\tau(x,y)}(x,y) \in \Sigma,$$

for all $(x,y) \in N_\delta(x_0,y_0)$.

Definition 1.13 *Let Γ, Σ, δ and $\tau(x,y)$ be defined as in Theorem 1.3. Then for $(x,y) \in N_\delta(x_0,y_0) \cap \Sigma$, the function*

$$\Pi(x,y) = \phi_{\tau(x,y)}(x,y)$$

is called Poincaré map for Γ at (x_0, y_0).
The following theorem gives the formula of $\Pi'(0,0)$.

Theorem 1.4 *[7]*
Let $\gamma(t)$ be a periodic solution of (2.1) of period T. Then the derivative of the Poincaré map $\Pi(s)$ along a straight line Σ normal to
$\Gamma = \{(x,y) \in \mathbb{R}^2 \mid (x,y) = \gamma(t) - \gamma(0), 0 \leq t \leq T\}$ at $(x,y) = (0,0)$ is given by

$$\Pi'(0) = \exp \int_0^T \nabla \cdot (P(\gamma(t)), Q(\gamma(t)))\, dt.$$

Corollary 1.1 *[7]*
Under the hypotheses of **1.4**, *the periodic solution* $\gamma(t)$ *is a stable limit cycle if*

$$\int_0^T \nabla.\left(P\left(\gamma\left(t\right)\right),Q\left(\gamma\left(t\right)\right)\right)dt<0,$$

and it is an unstable limit cycle if

$$\int_0^T \nabla.\left(P\left(\gamma\left(t\right)\right),Q\left(\gamma\left(t\right)\right)\right)dt>0.$$

Chapter 2

Existence of two algebraic limit cycles

2.1 Introduction

We consider the family of the polynomial system of the from

$$\begin{cases} \dot{x} = (\gamma x - x\alpha(x^2+y^2)^2 - 4\gamma y)w^2\left(x^2+y^2\right)^2 - x(\alpha(x^2+y^2)^2-\gamma)^2, \\ \dot{y} = (\gamma y - y\alpha(x^2+y^2)^2 + 4\gamma x)w^2\left(x^2+y^2\right)^2 - y(\alpha(x^2+y^2)^2-\gamma)^2, \end{cases} \tag{2.1}$$

where γ, α and w real constant, $\gamma \neq 0$, $w \neq 0$. We prove that this system is integrable. Moreover, we determine sufficient conditions for a polynomial differential system to possess at most two algebraic limit cycle, one algebraic limit cycle,and non existence of limit cycles . Concrete examples exhibiting the applicability of our result are introduced.

2.2 Integrability

Theorem 2.1 *Consider a polynomial differential system* (2.1). *Then the following statements hold.*
$1)$ *If* $w \neq 0$, *then system* (2.1) *has the first integral*

$$F(x,y) = \left(\frac{\gamma}{\alpha\left(x^2+y^2\right)^2-\gamma} + 1 + \frac{\alpha}{w^2}\right)e^{-\arctan\frac{y}{x}}.$$

$2)$ *If* $w \equiv 0$, *then system* (2.1) *has the first integral*

$$F(x,y) = \frac{y}{x}.$$

Proof 1 $1)$ *In order to prove our results we write the polynomial differential system* (2.1) *in polar coordinates* (r,θ), *defined by* $x = r\cos\theta$ *and* $y = r\sin\theta$, *then the system* (2.1) *become*

$$\begin{cases} \dot{r} = -r\left(r^4\alpha - \gamma\right)\left(-\gamma + r^4\alpha + w^2r^4\right), \\ \dot{\theta} = 4w^2r^4\gamma, \end{cases} \tag{2.2}$$

where $\dot{\theta} = \frac{d\theta}{dt}$, $\dot{r} = \frac{dr}{dt}$.
Taking as new independent variable the coordinate θ. *The differential system* (2.2) *where* $\gamma w \neq 0$

can be written as the equivalent differential equation

$$4r^3\frac{dr}{dt} = -\frac{\left(r^4\alpha - \gamma\right)\left(-\gamma + r^4\alpha + w^2r^4\right)}{\gamma w^2}, \tag{2.3}$$

we not that the differential equation (2.3) *is a Bernoulli equation.*
Via the change of variable $\rho = r^4$, *then the equation* (2.3) *is transformed into the linear equation*

$$\frac{d\rho}{d\theta} = -\frac{\left(\rho^4\alpha - \gamma\right)\left(\left(\alpha + w^2\right)\rho - \gamma\right)}{\gamma w^2}.$$

Solving it we find the first integral.

$$F\left(\rho, \theta\right) = \left(\frac{\gamma}{\alpha\rho - \gamma} + 1 + \frac{\alpha}{w^2}\right)e^{-\theta}.$$

Then, the first integral of system (2.2) *is*

$$F\left(r, \theta\right) = \left(\frac{\gamma}{\alpha r^4 - \gamma} + 1 + \frac{\alpha}{w^2}\right)e^{-\theta}.$$

Going bake thought the changes of variables $r^4 = \left(x^2 + y^2\right)^2$ *and* $\theta = \arctan\frac{y}{x}$, *we obtain*

$$F\left(x, y\right) = \left(\frac{\gamma}{\alpha\left(x^2 + y^2\right)^2 - \gamma} + 1 + \frac{\alpha}{w^2}\right)e^{-\arctan\frac{y}{x}}.$$

Hence the statement (1) *of Theorem* 2.1 *is proved.*
2) *If* $w = 0$, *the system* (2.1) *read as*

$$\begin{cases} \dot{x} = -x\left(\alpha\left(x^2 + y^2\right)^2 - \gamma\right)^2, \\ \dot{y} = -y\left(\alpha\left(x^2 + y^2\right)^2 - \gamma\right)^2. \end{cases} \tag{2.4}$$

This system equivalent the differential equation

$$\frac{dx}{dy} = \frac{x}{y}.$$

The general solution of this equation is given by

$$y = kx.$$

Where $k \in \mathbb{R}$, *then the first integral of* (2.4) *is*

$$F\left(x, y\right) = \frac{y}{x}.$$

Hence the statement (2) *of the Theorem* 2.1 *is proved.*

Lemma 2.1 *If* $w \neq 0$ *and* $\gamma \neq 0$, *then* $(0, 0)$ *is a unique equilibrium points of system* (2.1)

Proof 2 *We have*

$$\dot{x}y - \dot{y}x = -4w^2\gamma\left(x^2 + y^2\right)^3,$$

thus, the equilibrium points of system (1) *is the origin of coordinates.*

Lemma 2.2 *The curve* $U_1(x,y) = (x^2+y^2)^2 - \frac{\gamma}{\alpha} = 0$ *is an invariant algebraic for system* (2.1) *withe cofactor*

$$K_1(x,y) = -4\alpha\left(x^2+y^2\right)^2\left(\left(w^2+\alpha\right)\left(x^2+y^2\right)^2-\gamma\right).$$

Proof 3 *The algebraic curve* $U_1(x,y) = 0$ *is an invariant algebraic curve for* (2.1) *if for some polynomial* $K_1 \in C[x,y]$, *we have*

$$\dot{x}\frac{\partial U_1(x,y)}{\partial x} + \dot{y}\frac{\partial U_1(x,y)}{\partial y} = K_1(x,y)\,U_1(x,y).$$

Immediately we have

$$\begin{aligned} K_1(x,y) &= \frac{\dot{x}\frac{\partial U_1(x,y)}{\partial x} + \dot{y}\frac{\partial U_1(x,y)}{\partial y}}{U_1(x,y)} \\ &= \frac{4w^2(\gamma-\alpha(x^2+y^2)^2)\left(x^2+y^2\right)^4 + 4\left(x^2+y^2\right)^2(\gamma-\alpha(x^2+y^2)^2)^2}{\left(x^2+y^2\right)^2-\frac{\gamma}{\alpha}} \\ &= -4\alpha\left(x^2+y^2\right)^2\left(\left(w^2+\alpha\right)\left(x^2+y^2\right)^2-\gamma\right), \end{aligned}$$

then $U_1(x,y) = 0$ *is an invariant algebraic curve the polynomial system* (2.1) *.This completes the proof of Lemma* 2.2.

Lemma 2.3 *The curve* $U_2(x,y) = (x^2+y^2)^2 - \frac{\gamma}{\alpha+w^2} = 0$ *is an invariant algebraic for system* (2.1) *withe cofactor*

$$K_2(x,y) = -4\left(w^2+\alpha\right)\left(x^2+y^2\right)^2\left(\alpha\left(x^2+y^2\right)^2-\gamma\right).$$

Proof 4 *The algebraic curve* $U_2(x,y) = 0$ *is an invariant algebraic curve for* (2.1) *if for some polynomial* $K_2 \in C[x,y]$, *we have*

$$\dot{x}\frac{\partial U_2(x,y)}{\partial x} + \dot{y}\frac{\partial U_2(x,y)}{\partial y} = K_2(x,y)\,U_2(x,y).$$

Immediately we have

$$\begin{aligned} K_2(x,y) &= \frac{\dot{x}\frac{\partial U_2(x,y)}{\partial x} + \dot{y}\frac{\partial U_2(x,y)}{\partial y}}{U_2(x,y)} \\ &= -4\left(w^2+\alpha\right)\left(x^2+y^2\right)^2\left(\alpha\left(x^2+y^2\right)^2-\gamma\right), \end{aligned}$$

then $U_2(x,y) = 0$ *is an invariant algebraic curve the polynomial system* (2.1) *.*

Lemma 2.4 *1)* $K_1(x,y) = 0$ *does not intersect the orbit* $U_1(x,y) = 0$.
2) $K_2(x,y) = 0$ *does not intersect the orbit* $U_2(x,y) = 0$.

Proof 5 *1) To show this, we prove that*

$$\begin{cases} \left(x^2+y^2\right)^2 - \frac{\gamma}{\alpha} = 0, \\ \left(w^2+\alpha\right)\left(x^2+y^2\right)^2 - \gamma = 0, \end{cases}$$

has no solutions.
Indeed, if we replace $\left(x^2+y^2\right)^2 = \frac{\gamma}{\alpha}$ *in the second equation we obtain* $\frac{w^2}{\alpha}\gamma = 0,$ *Since* $w\gamma \neq 0$ *then, the curve* $K_1(x, y) = 0$ *do not cross* $U_1(x, y) = 0.$
2) we prove that

$$\begin{cases} \left(x^2+y^2\right)^2 - \dfrac{\gamma}{\alpha+w^2} = 0, \\ \alpha\left(x^2+y^2\right)^2 - \gamma = 0, \end{cases}$$

has no solutions.
Indeed, if we replace $\left(x^2+y^2\right)^2 = \frac{\gamma}{\alpha+w^2}$ *in the second equation we obtain* $-w^2\frac{\gamma}{w^2+\alpha} = 0.$ *Since* $w\gamma \neq 0$ *then, the curve* $K_2(x, y) = 0$ *do not cross* $U_2(x, y) = 0.$

2.3 Existence of two algebraic limit cycles

The theorem below is a result of the existence of two limit cycles of system (2.1)
Theorem 2.2 *If one of the following statements hold.*
1) *If* $\alpha > 0, \gamma > 0,$
2) *If* $\alpha < 0, \gamma < 0, \alpha + w^2 < 0,$
the septic polynomial differential system (2.1) *in which* $w \neq 0$*, possesses exactly two hyperbolic algebraic limit cycles, whose expression in cartesian coordinates* (x, y) *is*

$$\left(x^2+y^2\right)^2 - \frac{\gamma}{\alpha} = 0,$$

$$\left(x^2+y^2\right)^2 - \frac{\gamma}{\alpha+w^2} = 0.$$

Proof 6 1) *If* $w \neq 0$*, we have*

$$\begin{cases} \dot{r} = -r\left(r^4\alpha - \gamma\right)\left(-\gamma + r^4\alpha + w^2 r^4\right), \\ \dot{\theta} = 4\gamma w^2 r^4. \end{cases} \tag{2.5}$$

Taking as independent variable the coordinate θ*, this differential system write*

$$4r^3\frac{dr}{d\theta} = -\frac{\left(r^4\alpha - \gamma\right)\left(-\gamma + r^4\alpha + w^2 r^4\right)}{\gamma w^2}. \tag{2.6}$$

Via the change of variables $\rho = r^4$*, this equation is transformed into the Riccati equation*

$$\frac{d\rho}{d\theta} = -\frac{\left(\rho\alpha - \gamma\right)\left(-\gamma + \rho\alpha + w^2\rho\right)}{\gamma w^2}. \tag{2.7}$$

Fortunately, this is integrable, since it possesses the particular solution $\rho\alpha - \gamma = 0$, corresponding of course to the limit cycle (Γ_1). Then the general solution of this equation is given by

$$\rho = \frac{\gamma}{\alpha} + \frac{1}{R}.$$

Indeed, substituting the solution $\rho = \left(\frac{\gamma}{\alpha} + \frac{1}{R}\right)$ into the Riccati equation, we obtain the linear equation

$$-\frac{1}{R^2}\frac{dR}{d\theta} = -\frac{\left(\left(\frac{\gamma}{\alpha} + \frac{1}{R}\right)\alpha - \gamma\right)\left(-\gamma + (\alpha + w^2)(\frac{\gamma}{\alpha} + \frac{1}{R})\right)}{\gamma w^2},$$

thus

$$\begin{aligned} \tfrac{dR}{d\theta} &= \frac{1}{w^2\gamma}(w^2\alpha + R\gamma w^2 + \alpha^2) \\ &= \frac{\alpha}{\gamma} + R + \frac{\alpha^2}{w^2\gamma}. \end{aligned} \tag{2.8}$$

The general solution of linear equation (2.8) is

$$R(\theta, k) = e^{\theta}\left(k - \frac{\alpha}{\gamma}\left(e^{-\theta} - 1\right) + \frac{\alpha^2}{\gamma}\int_0^{\theta}\frac{e^{-s}}{w^2}ds\right),$$

with $f(\theta) = \int_0^{\theta}\frac{e^{-s}}{w^2}ds$, we have

$$\begin{aligned} \textstyle\int_0^{\theta}\frac{e^{-s}}{w^2}ds &= \frac{1}{w^2}\left(-e^{-\theta} + 1\right) \\ &= -\frac{1}{w^2}\left(e^{-\theta} - 1\right). \end{aligned}$$

After we substitution the value of $\int_0^{\theta}\frac{e^{-s}}{w^2}ds$ into $R(\theta, k)$, we obtain

$$R(\theta, k) = e^{\theta}\left(k - \frac{\alpha}{\gamma}\left(e^{-\theta} - 1\right) - \frac{\alpha^2}{w^2\gamma}\left(e^{-\theta} - 1\right)\right).$$

Where $k \in \mathbb{R}$. Going back through the changes of variables, we obtain

$$\rho(\theta, k) = \frac{\gamma}{\alpha} + \gamma\frac{e^{-\theta}}{\left(\gamma k - \alpha\left(e^{-\theta} - 1\right) - \frac{\alpha^2}{w^2}\left(e^{-\theta} - 1\right)\right)},$$

if we take $h = \gamma k + 1$, the general solution of Riccati equation (2.7) is

$$\rho(\theta, h) = \frac{\gamma}{\alpha} + \gamma\frac{e^{-\theta}}{h - \alpha e^{-\theta} - \frac{\alpha^2}{w^2}\left(e^{-\theta} - 1\right)}.$$

Consequently, the general solution of (2.5) is

$$r(\theta, h) = \left(\frac{\gamma}{\alpha} + \gamma\frac{e^{-\theta}}{h - \alpha e^{-\theta} - \frac{\alpha^2}{w^2}\left(e^{-\theta} - 1\right)}\right)^{\frac{1}{4}}.$$

To go a steep further, we remark that the solution such as $r(0, r_0) = r_0 > 0$, corresponds to the value $h = \frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma}$ provided a rewriting of the general solution of (2.5)

$$r(\theta, r_0) = \left(\frac{\gamma}{\alpha} + \gamma\frac{e^{-\theta}}{\frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha e^{-\theta} - \frac{\alpha^2}{w^2}\left(e^{-\theta} - 1\right)}\right)^{\frac{1}{4}}.$$

Any periodic solution of system (2.1) *must satisfy the condition* $r(2\pi, r_0) = r_0$*, provided two distinct positive values of* $r_0 : r_1^4 = \frac{\gamma}{\alpha}$ *corresponding obviously to the algebraic limit cycle* (Γ_1)*, a well defined second value, we have*

$$r^4 = \frac{\gamma}{\alpha} + \gamma \frac{e^{-\theta}}{\frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha e^{-\theta} - \frac{\alpha^2}{w^2}\left(e^{-\theta} - 1\right)}. \tag{2.9}$$

Then the condition $r^4(2\pi, r_0) = r^4(0, r_0)$ *equivalent*

$$\frac{e^{-2\pi}}{\frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha e^{-2\pi} - \frac{\alpha^2}{w^2}\left(e^{-2\pi} - 1\right)} = \frac{1}{\frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha},$$

this imply that

$$e^{-2\pi}\left(\frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha\right) = \frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha e^{-2\pi} - \frac{\alpha^2}{w^2}\left(e^{-2\pi} - 1\right),$$

then

$$\frac{r_0^4}{\alpha r_0^4 - \gamma} = \frac{\frac{-1}{w^2}\left(e^{-2\pi} - 1\right)}{\left(e^2\pi - 1\right)},$$

so

$$r_0^4\left(1 + \frac{\alpha}{w^2}\right) = \frac{\gamma}{w^2},$$

then

$$r_0^4 = \frac{\gamma}{\alpha + w^2}.$$

So we substitution the value of r_0^4 *into* $\frac{\alpha^2 r_0^4}{\alpha r_0^2 - \gamma}$*, we obtain*

$$\frac{\alpha^2 r_0^4}{\alpha r_0^2 - \gamma} = \frac{-\alpha^2}{w^2}.$$

So we substitution the value of $\frac{\alpha^2 r_*^4}{\alpha r_*^2 - \gamma}$ *into* (2.9)*, we obtain*

$$\begin{aligned} r^4(\theta, r_*) &= \tfrac{\gamma}{\alpha} + \gamma \frac{e^{-\theta}}{\frac{\alpha^2}{w^2} - \alpha e^{-\theta} - \frac{\alpha^2}{w^2}\left(e^{-\theta} - 1\right)} \\ &= \tfrac{\gamma}{\alpha} + \gamma \frac{e^{-\theta}}{-\alpha e^{-\theta} - \frac{\alpha^2}{w^2} e^{-\theta}} \\ &= \frac{\gamma}{w^2 + \alpha}. \end{aligned}$$

To see that $\Gamma_1 : \left(x^2 + y^2\right)^2 - \frac{\gamma}{\alpha} = 0$ *and* $\Gamma_2 : \left(x^2 + y^2\right)^2 - \frac{\gamma}{\alpha + w^2} = 0$ *are in fact a hyperbolic limit cycles, we use a classic result characterizing limit cycles among other periodic orbits see for instance theorem1.1, which means that* Γ_i *is a hyperbolic limit cycle when*

$$\int_0^T Div\left(\Gamma_i(t)\right) dt \neq 0,$$

where T_i *be the period of the periodic solution* (Γ_i). *We use also a practical result of J. Giné and all see theorem 1.2 (chapter I), which asserts that*

$$\int_0^{T_i} Div\left(\Gamma_i(t)\right) dt = \int_0^{T_i} K_i(x, y)\, dt$$

Note that if a periodic curve (Γ_i) *is invariant for a differential system with a cofactor* $K_i(x,y)$ *of constant sign for* $(x,y) \in Int(\Gamma_i)$ *where* $Int(\Gamma_i)$ *denotes the interior of* (Γ_i)*, then* $\int_0^T K_i\,(x,y)\,dt$*, is automatically different from zero.*
By proposition 2.2 we have $K_1\,(x,y) = -4\alpha\,(x^2+y^2)^2\,(w^2+\alpha)\,(x^2+y^2)^2 - \gamma$ *if we take* $P_1(x,y) = (w^2+\alpha)\,(x^2+y^2)^2 - \gamma$ *we have* $P_1(0,0) = -\gamma < 0$*, hence* $P_1(x,y) < 0$ *inside* (Γ_1) *and* $K_1(x,y) = -4\alpha\,(x^2+y^2)^2\,(w^2+\alpha)\,(x^2+y^2)^2 - \gamma > 0$ *inside* $Int(\Gamma_1)\diagdown\{(0,0)\}$*, so* $\int_0^{T_1} K_1\,(x,y)\,dt > 0$*; where* T_1 *be the period of the periodic solution* (Γ_1)*. Consequently* (Γ_1) *defines a hyperbolic algebraic limit cycles for system (2.1).*
By proposition 2.3 we have $K_2\,(x,y) = -4\,(w^2+\alpha)\,(x^2+y^2)^2\left(\alpha\,(x^2+y^2)^2 - \gamma\right)$ *if we take* $P_2(x,y) = \alpha\,(x^2+y^2)^2 - \gamma$ *we have* $P_2(0,0) = -\gamma < 0$*,hence* $P_2(x,y) < 0$ *inside* (Γ_2) *and* $K_2(x,y) = -4\,(w^2+\alpha)\,(x^2+y^2)^2\left(\alpha\,(x^2+y^2)^2 - \gamma\right) > 0$ *inside* $Int(\Gamma_2)\diagdown\{(0,0)\}$*, so* $\int_0^{T_2} K_2\,(x,y)\,dt > 0$*; where* T_2 *be the period of the periodic solution* (Γ_2)*. Consequently* (Γ_2) *defines a hyperbolic algebraic limit cycles for system (2.1). This complete the proof of Theorem 2.2.*

Example 2.1 *If we take* $w = 2$*,* $\gamma = 1$*, and* $\alpha = 1$ *then system (2.1) reads*

$$\begin{cases} \dot{x} = 4(x - x(x^2+y^2)^2 - 4\gamma y)\,(x^2+y^2)^2 - x((x^2+y^2)^2-1)^2, \\ \dot{y} = 4(y - y(x^2+y^2)^2 + 4\gamma x)\,(x^2+y^2)^2 - y((x^2+y^2)^2-1)^2, \end{cases} \tag{2.10}$$

we have $w = 2$*,* $\frac{\gamma}{\alpha} = 1 > 0$ *and* $\frac{\gamma}{\alpha+w^2} = \frac{1}{5} > 0$*. So the Theorem 2.2 is satisfied and hence the system (2.10) has two hyperbolic algebraic limit cycles whose expression in cartesian coordinates* (x,y) *is*

$$\left(x^2+y^2\right)^2 = 1,$$
$$\left(x^2+y^2\right)^2 = \frac{1}{5}.$$

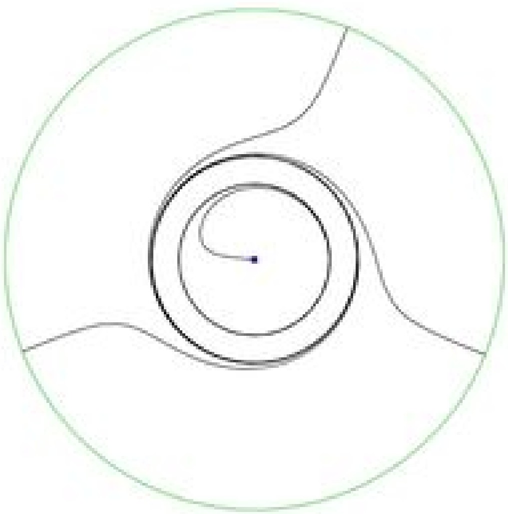

Figure 2.1: Two algebraic limit cycles for system (2.10)

2.4 Existence of one algebraic limit cycle

Theorem 2.3 *$i)$ If $w^2 \neq 0, \gamma < 0, \alpha < 0$ and $\alpha + w^2 > 0$, then the septic polynomial differential system (2.1) possesses exactly one hyperbolic algebraic limit cycle whose expression in cartesian coordinates (x, y)*

$$\left(x^2+y^2\right)^2 - \frac{\gamma}{\alpha} = 0.$$

$ii)$ If $w^2 \neq 0, \gamma > 0$, $\alpha < 0$ and $\alpha + w^2 > 0$, then the septic polynomial differential system (2.1) possesses exactly one hyperbolic algebraic limit cycle whose expression in cartesian coordinates (x, y)

$$\left(x^2+y^2\right)^2 - \frac{\gamma}{\alpha + w^2} = 0.$$

Proof 7 *$i)$ If $\alpha < 0$, $\gamma < 0$, $\alpha + w^2 > 0$, then $\frac{\gamma}{\alpha+w^2} < 0$, and $(x^2+y^2)^2 - \frac{\gamma}{\alpha+w^2} \neq 0, \forall\,(x, y) \in \mathbb{R}^2$. And if $\frac{\gamma}{\alpha} > 0$, then through the Theorem 2.2, $(x^2+y^2)^2 = \frac{\gamma}{\alpha}$ is hyperbolic algebraic limit cycle because it is polynomial.*
So the septic polynomial differential system (2.1) possesses exactly one hyperbolic algebraic limit cycle whose expression in cartesian coordinates (x, y) ;

$$\left(x^2+y^2\right)^2 = \frac{\gamma}{\alpha},$$

$ii)$ If $\alpha < 0, \gamma > 0$, $\alpha + w^2 > 0$ then $\frac{\gamma}{\alpha+w^2} > 0$, and, $(x^2+y^2)^2 - \frac{\gamma}{\alpha+w^2} = 0, \forall\,(x, y) \in \mathbb{R}^2$, is hyperbolic algebraic limit cycle because it is polynomial. And if $\frac{\gamma}{\alpha} < 0$, then $(x^2+y^2)^2 - \frac{\gamma}{\alpha} \neq 0, \forall\,(x, y) \in \mathbb{R}^2$.
So the septic polynomial differential system (2.1) possesses exactly one hyperbolic algebraic limit cycle whose expression in cartesian coordinates (x, y)

$$\left(x^2+y^2\right)^2 = \frac{\gamma}{\alpha + w^2}.$$

Hence the Theorem 2.3 is proved.

Example 2.2 *If we take $w = 3$, $\gamma = -1$ and $\alpha = -1$, then system (2.1) reads*

$$\begin{cases} \dot{x} = 9(-x + x(x^2+y^2)^2 + \gamma y)\left(x^2+y^2\right)^2 - x(-(x^2+y^2)^2+1)^2, \\ \dot{y} = 9(-y + y(x^2+y^2)^2 - 4x)\left(x^2+y^2\right)^2 - y(-(x^2+y^2)^2+1)^2, \end{cases} \tag{2.11}$$

we have $w = 3$, $\frac{\gamma}{\alpha+a^2} = -\frac{1}{8} < 0$ and $\frac{\gamma}{\alpha} = 1 > 0$. So the first condition of the Theorem 2.3 is satisfied and hence the system (2.11) has one hyperbolic algebraic limit cycles whose expression in cartesian coordinates (x, y) is

$$\left(x^2+y^2\right)^2 = 1.$$

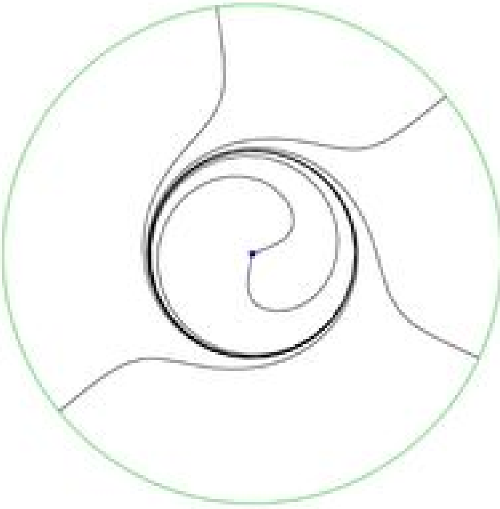

Figure 2.2: One algebraic limit cycle for system $(\mathbf{2.11})$

Example 2.3 *If we take $w = 2$, $\gamma = 1$ and $\alpha = -1$, then system $(\mathbf{2.1})$ reads*

$$\begin{cases} \dot{x} = 4(x + x(x^2+y^2)^2 - 4y)\left(x^2+y^2\right)^2 - x(-(x^2+y^2)^2-1)^2, \\ \dot{y} = 4(y + y(x^2+y^2)^2 + 4x)\left(x^2+y^2\right)^2 - y(-(x^2+y^2)^2-1)^2, \end{cases} \tag{2.12}$$

we have $w = 2$, $\frac{\gamma}{\alpha+w^2} = \frac{1}{3} > 0$ and $\frac{\gamma}{\alpha} = -1 < 0$. So the second condition of the Theorem $\mathbf{2.3}$ is satisfied and hence the system $(\mathbf{2.12})$ has one hyperbolic algebraic limit cycles whose expression in cartesian coordinates (x, y) is

$$\left(x^2+y^2\right)^2 = \frac{1}{3}.$$

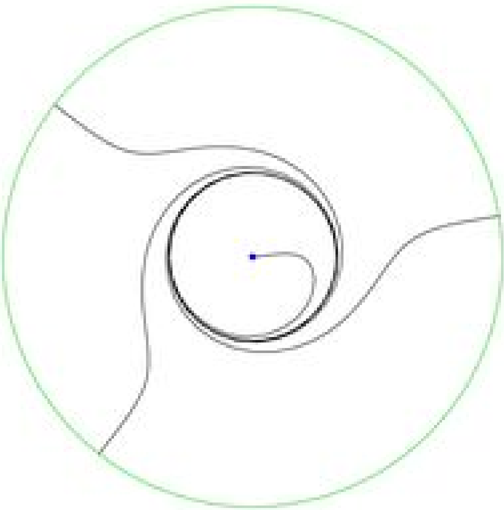

Figure 2.3: Two algebraic limit cycle for system $(\mathbf{2.12})$

2.5 Non existence of limit cycles

Theorem 2.4 *If one of the following conditions is assumed:*
$i)$ $\gamma < 0, \alpha > 0$.
$ii)$ $\gamma > 0, \alpha < 0, \alpha + w^2 < 0,$
the septic polynomial differential system (2.1) *has no limit cycles.*

Proof 8 $i)$ *if* $\gamma < 0, \alpha > 0$, *we have*

$$\left(x^2 + y^2\right)^2 - \frac{\gamma}{\alpha + w^2} \neq 0,$$

for all $(x, y) \in \mathbb{R}^2$,

$$\left(x^2 + y^2\right)^2 - \frac{\gamma}{\alpha} \neq 0,$$

for all $(x, y) \in \mathbb{R}^2$.
Or
$ii)$ *if* $\gamma > 0, \alpha < 0$, *and* $\alpha + w^2 < 0$, *we have*

$$\left(x^2 + y^2\right)^2 - \frac{\gamma}{\alpha + w^2} \neq 0,$$

for all $(x, y) \in \mathbb{R}^2$,

$$\left(x^2 + y^2\right)^2 - \frac{\gamma}{\alpha} \neq 0,$$

for all $(x, y) \in \mathbb{R}^2$.
Then, system (2.1) *has no limit cycle. Hence the Theorem 2.4 is proved.*

Example 2.4 *If we take* $w = 2$, $\gamma = -2$ *and* $\alpha = 1$, *then system* (2.1) *reads*

$$\begin{cases} \dot{x} = 4(-2x - x(x^2 + y^2)^2 + 8y)\left(x^2 + y^2\right)^2 - x((x^2 + y^2)^2 + 2)^2, \\ \dot{y} = 4(-2y - y(x^2 + y^2)^2 - 8x)\left(x^2 + y^2\right)^2 - y((x^2 + y^2)^2 + 2)^2, \end{cases} \tag{2.13}$$

we have $w = 2$, $\frac{\gamma}{\alpha + w^2} = \frac{-2}{5} < 0$ *and* $\frac{\gamma}{\alpha} = -2 < 0$. *So the first condition of Theorem 2.4 is satisfied and hence the system* (2.13) *has no limit cycle.*

Example 2.5 *If we take* $w = 1$, $\gamma = 1$ *and* $\alpha = -2$, *then system* (2.1) *reads*

$$\begin{cases} \dot{x} = (x + 2x(x^2 + y^2)^2 - 4y)\left(x^2 + y^2\right)^2 - x(-2(x^2 + y^2)^2 - 1)^2, \\ \dot{y} = (y + 2y(x^2 + y^2)^2 + 4x)\left(x^2 + y^2\right)^2 - y(-2(x^2 + y^2)^2 - 1)^2, \end{cases} \tag{2.14}$$

we have $w = 1$, $\frac{\gamma}{\alpha + w^2} = -1 < 0$ *and* $\frac{\gamma}{\alpha} = \frac{-1}{2} < 0$. *So the second condition of Theorem 2.4 is satisfied and hence the system* (2.14) *has no limit cycle.*

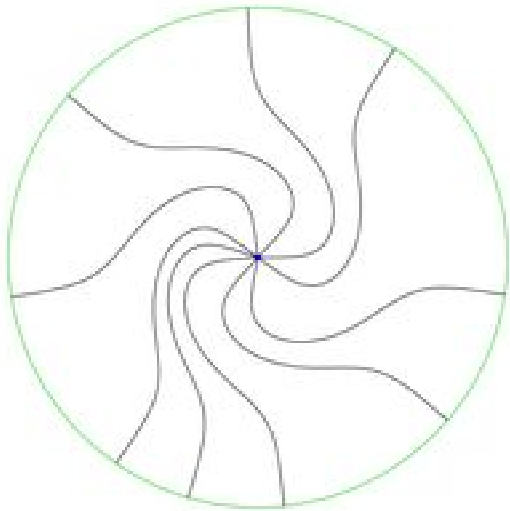

Figure 2.4: Phase portrait of system (2.13)

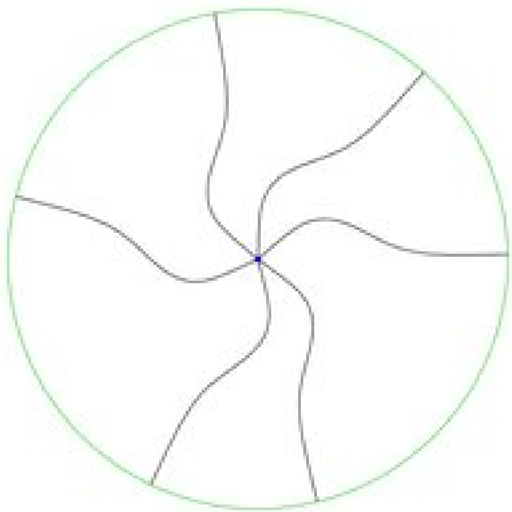

Figure 2.5: Phase portrait of system (2.14)

Chapter 3

Coexistence of algebraic and non-algebraic limit cycles

3.1 Introduction

We consider the family of the polynomial differential system of the form

$$\begin{cases} \dot{x} = \left(\gamma x - x\alpha \left(x^2+y^2\right)^2 - 4\gamma y\right) Q^2(x,y) - x\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)^2, \\ \dot{y} = \left(\gamma y - y\alpha((x^2+y^2)^2 + 4\gamma x\right) Q^2(x,y) - y\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)^2, \end{cases} \tag{3.1}$$

where $Q(x,y) = ax^2 + by^2 + cxy$ is homogeneous polynomial of degrees 2 and γ, α, a, b and c real constants $\gamma \neq 0$ and $\alpha \neq 0$. We prove that this system is integrable. Moreover, we determine sufficient conditions for a polynomial differential system to possess at most two explicit limit cycles, one of them algebraic and the other one non-algebraic. Concrete examples exhibiting the applicability of our result are introduced.

3.2 Integrability

Theorem 3.1 *Consider a polynomial differential system* (3.1) *for* $Q(\theta) \neq 0$ *for all* $\theta \in [0, 2\pi]$, *system* (3.1) *has the first integral*

$$F(x,y) = \frac{\gamma e^{-\arctan\frac{y}{x}}}{\alpha\left(x^2+y^2\right)^2-\gamma} + e^{-\arctan\frac{y}{x}} - \alpha\int_0^{\arctan\frac{y}{x}} \frac{e^{-s}}{Q^2(s)}ds.$$

Proof 9 *In polar coordinates system* (3.1) *reads as*

$$\begin{cases} \dot{r} = -r\left(r^4\alpha-\gamma\right)\left(-\gamma+r^4\alpha+Q^2r^4\right). \\ \dot{\theta} = 4Q^2r^4\gamma. \end{cases} \tag{3.2}$$

Let

$$F(x,y) = \frac{\gamma e^{-\arctan\frac{y}{x}}}{\alpha(x^2+y^2)^2-\gamma} + e^{-\arctan\frac{y}{x}} - \alpha\int_0^{\arctan\frac{y}{x}} \frac{e^{-s}}{Q^2(s)}ds. \tag{3.3}$$

In polar coordinates system (3.3) *reads as*

$$F(r,\theta)=\left(\frac{\gamma}{\left(\alpha\left(r^2\right)^2-\gamma\right)}+1\right)e^{-\theta}-\alpha\int_0^\theta\frac{e^{-s}}{Q^2(s)}ds,$$

then the derivative of F with respect to r is

$$\frac{\partial F}{dr}=-4r^3\alpha\gamma\frac{e^{-\theta}}{\left(\gamma-r^4\alpha\right)^2},$$

and the derivative of F with respect to θ is

$$\frac{\partial F}{d\theta}=\alpha\frac{e^{-\theta}}{Q^2\gamma-Q^2r^4\alpha}\left(Q^2r^4+\alpha r^4-\gamma\right).$$

By replacing the expressions of derivatives of F with respect to θ and r in

$$\frac{\partial F}{dt}=\dot{r}\frac{\partial F}{\partial r}+\dot{\theta}\frac{\partial F}{\partial\theta},$$

it follows that

$$\begin{aligned}\tfrac{dF}{dt} &= -4r^3\alpha\gamma\left(-r\left(r^4\alpha-\gamma\right)\left(-\gamma+r^4\alpha+Q^2r^4\right)\right)\frac{e^{-\theta}}{\left(\gamma-r^4\alpha\right)^2}\\ &\quad+4Q^2r^4\gamma\alpha\frac{e^{-\theta}}{Q^2\gamma-Q^2r^4\alpha}\left(Q^2r^4+\alpha r^4-\gamma\right)\\ &\equiv 0.\end{aligned}$$

So $F(r,\theta)$ is a first integral of (3.2) *, then $F(x,y)$ is a first integral of system* (3.1). *Hence the Theorem* 3.1 *is proved.*

Theorem 3.2 *The curve $U(x,y)=4Q^2\gamma\left(x^2+y^2\right)=0$ is an invariant algebraic for system* (3.1) *with cofactor*

$$\begin{aligned}K(x,y) &= -6\left(\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)\right)\left(\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)+Q^2\right)\\ &\quad-8\gamma Q\left(y\frac{\partial Q}{\partial x}-x\frac{\partial Q}{\partial y}\right).\end{aligned}$$

Proof 10 *We have*

$$\begin{aligned}\dot{x}\tfrac{\partial U}{\partial x}+\dot{y}\tfrac{\partial U}{\partial y} &= \left(\left(\gamma x-\alpha x\left(x^2+y^2\right)^2-4\gamma y\right)Q^2-x\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)^2\right)\\ &\quad\times\left(8Q^2x\gamma+8Q\frac{\partial Q}{\partial x}\gamma\left(x^2+y^2\right)\right)\\ &\quad+\left(\left(\gamma y-\alpha y\left(x^2+y^2\right)^2-4\gamma x\right)Q^2-y\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)^2\right)\\ &\quad\times\left(8Q^2y\gamma+8Q\frac{\partial Q}{\partial y}\gamma\left(x^2+y^2\right)\right)\\ &= -8Q^2\gamma\left(x^2+y^2\right)\left(-\gamma+x^4\alpha+y^4\alpha+2x^2y^2\alpha\right)\\ &\quad\times\left(-\gamma+x^4\alpha+y^4\alpha+Q^2+2x^2y^2\alpha\right)\\ &\quad+8Q^3\left(x^2+y^2\right)\left(\left(\gamma-\alpha\left(x^2+y^2\right)^2\right)\left(x\frac{\partial Q}{\partial x}+y\frac{\partial Q}{\partial y}\right)-4\gamma y\frac{\partial Q}{\partial x}\right)\\ &\quad+8Q\left(x^2+y^2\right)\left(4\gamma xQ^2\frac{\partial Q}{\partial y}-\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)^2\left(y\frac{\partial Q}{\partial y}+x\frac{\partial Q}{\partial x}\right)\right),\end{aligned}$$

due to the Euler's theorem for homogeneous function $\dot{x}\frac{\partial U}{\partial y}+\dot{y}\frac{\partial U}{\partial x}=2Q$ *, we obtain*

$$\begin{aligned}\dot{x}\frac{\partial U}{\partial x}+\dot{y}\frac{\partial U}{\partial y} \;=\; & -8Q^2\gamma\left(x^2+y^2\right)\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)\left(\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)+Q^2\right)\\ & +16Q^2\gamma\left(x^2+y^2\right)\left(\left(\gamma-\alpha\left(x^2+y^2\right)^2\right)Q^2-\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)^2\right)\\ & -8Q^2\gamma\left(x^2+y^2\right)\left(+4\gamma y\frac{\partial U}{\partial x}Q-4\gamma xQ\frac{\partial U}{\partial y}\right)\\ & -24Q^2\gamma\left(x^2+y^2\right)\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)\left(\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)+Q^2\right)\\ & +4Q^2\gamma\left(x^2+y^2\right)\left(-8\gamma Q\left(y\frac{\partial Q}{\partial x}-x\frac{\partial Q}{\partial y}\right)\right).\end{aligned}$$

Therefore, $U=0$ *is an invariant algebraic curve of the polynomial differential system* (3.1) *with cofactor*

$$\begin{aligned}K\left(x,y\right) \;=\; & -6\left(\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)\right)\left(\left(\alpha\left(x^2+y^2\right)^2-\gamma\right)+Q^2\right)\\ & -8\gamma Q\left(y\frac{\partial Q}{\partial x}-x\frac{\partial Q}{\partial y}\right).\end{aligned}$$

Hence the Theorem 3.2 is proved.

3.3 Periodic solution

Theorem 3.3 *If* $Q^2\left(\theta\right)$ *vanishes for some* $\theta\in[0,2\pi]$, *then system* (3.1) *has no periodic solutions surrounding the origin.*

Proof 11 *If* $\theta^*\in[0,2\pi]$ *is a zero of* $Q^2\left(\theta\right)=0$, *then* $(\sin\theta^*x-\cos\theta^*y)$ *is a factor of* $Q^2\left(x,y\right)$, *and consequently the straight line*

$$\sin\theta^*x-\cos\theta^*y=0$$

is invariant. It is well known that if $Q\left(\theta^*\right)=0$ *is an invariant algebraic curve, then any factor of* Q^2 *is also an invariant algebraic curve. So the straight line* $\sin\theta^*x-\cos\theta^*y=0$ *through the origin of coordinates is invariant, i.e., formed by solutions of system* (3.1). *Therefore, it can not be periodic solutions surrounding the origin. This completes the proof of Theorem 3.3.*

3.4 Coexistence of algebraic and non-algebraic limit cycles

Theorem 3.4 *The septic polynomial differential system* (3.1) *in which* $Q\left(\theta\right)\neq 0$, *for all* $\theta\in\mathbb{R}^*$ *possesses exactly two hyperbolic limit cycles, one algebraic whose expression in cartesian coordinates* (x,y) *is*

$$\left(x^2+y^2\right)^2-\frac{\gamma}{\alpha}=0,$$

and the other one non-algebraic whose expression in polar coordinates (r, θ) *is*

$$r(\theta, r_*) = \left(\frac{\gamma}{\alpha} + \gamma \frac{e^{-\theta}}{\frac{\alpha^2 r_*^2}{\alpha r_*^2 - \gamma} - \alpha e^{-\theta} + \alpha^2 f(\theta)}\right)^{\frac{1}{4}},$$

with $f(\theta) = \int_0^\theta \frac{e^{-\theta}}{Q^2(s)} ds$ *and* $r_* = \left(\gamma \frac{f(2\pi)}{\alpha f(2\pi) - e^{-2\pi} + 1}\right)^{\frac{1}{4}}$, *when the following condition is assumed:*

$$\alpha > 0, \gamma > 0.$$

Proof 12 *Firstly, we have*

$$y\dot{x} - x\dot{y} = \left(x^2 + y^2\right) Q^2(x, y),$$

thus, the equilibrium points of system (3.1) *are present in the curve*

$$U(x, y) = \left(x^2 + y^2\right) Q^2(x, y) = 0. \tag{3.4}$$

In polar coordinates (r, θ) *defined by* $x = r\cos\theta$ *and* $y = r\sin\theta$, (3.1) *reads as*

$$U(r\cos\theta, r\sin\theta) = r^2 Q^2(\theta).$$

Since $\alpha > 0$ *and* $Q(\theta) \neq 0$, *for all* $\theta \in \mathbb{R}^*$, *thus the origin is the unique critical point at finite distance.*

We prove that $(\Gamma_1) : \alpha(x^2 + y^2)^2 - \gamma = 0$ *is an invariant algebraic curve of the differential system* (3.1). *Indeed, if we put*

$$\begin{aligned} f(x, y) &= \left(\gamma x - x\alpha(x^2 + y^2)^2 - 4\gamma y\right) Q^2(x, y) - x\left(\alpha(x^2 + y^2)^2 - \gamma\right)^2, \\ g(x, y) &= \left(\gamma y - y\alpha(x^2 + y^2)^2 + 4\gamma x\right) Q^2(x, y) - y\left(\alpha(x^2 + y^2)^2 - \gamma\right)^2, \\ H(x, y) &= \alpha(x^2 + y^2)^2 - \gamma. \end{aligned}$$

Immediately, we have

$$f(x, y)\frac{\partial H}{\partial x} + g(x, y)\frac{\partial H}{\partial y} = K(x, y) H(x, y),$$

Where

$$K(x, y) = -4\alpha\left(x^2 + y^2\right)^2 \left(Q^2(x, y) + \left(\alpha\left(x^2 + y^2\right)^2 - \gamma\right)\right).$$

To see that

$$\begin{cases} -4\alpha(x^2 + y^2)^2\left(Q^2 + \alpha(x^2 + y^2)^2 - \gamma\right) = 0, \\ \alpha(x^2 + y^2)^2 - \gamma = 0 \end{cases} \tag{3.5}$$

has no solutions. Indeed in polar coordinates (r, θ), *system* (3.5) *reads as*

$$\begin{cases} -4\alpha r^4(Q^2(\theta) + \alpha r^4 - \gamma) = 0, \\ \alpha r^4 - \gamma = 0, \end{cases}$$

this system can be written as

$$-4\alpha r^4 Q^2(\theta) = 0.$$

Since $\alpha > 0$. *Then the curve* $K(x,y) = 0$ *do not cross* (Γ_1). *But* $P(0,0) = -\gamma$ *where* $P(x,y) = \left(Q^2 + \alpha\left(x^2+y^2\right)^2 - \gamma\right)$.
If $\gamma > 0$, *we have* $P(x,y) < 0$ *inside* (Γ_1), *then*

$$K(x,y) = -4\alpha(x^2+y^2)^2\left(Q^2 + \left(\alpha(x^2+y^2)^2 - \gamma\right)\right) > 0$$

inside $(\Gamma_1) \setminus \{(0,0)\}$ *because* $\alpha > 0$.
Since $K(x,y) = -4\alpha\left(x^2+y^2\right)^2\left(Q^2 + \left(\alpha\left(x^2+y^2\right)^2 - \gamma\right)\right) > 0$ *inside* $(\Gamma_1) \setminus \{(0,0)\}$ *so* $\int_0^T K(x,y)\,dt \neq 0$, *where* T *be the period of the periodic solution* (Γ_1).
Consequently (Γ_1) *defines a stable algebraic cycle for system* (3.1).
The search for the non-algebraic limit cycle, requires the integration of our system. Taking into account (3.4), *then in polar coordinates* (r,θ), *defined by* $x = r\cos\theta$ *and* $y = r\sin\theta$, *the system* (3.1) *can be written as the system.*

$$\begin{cases} \dot{r} = -r\left(r^4\alpha - \gamma\right)\left(-\gamma + r^4\alpha + Q^2r^4\right). \\ \dot{\theta} = 4Q^2r^4\gamma. \end{cases} \tag{3.6}$$

This differential system where $4Q^2r^4\gamma \neq 0$ *can be written as the equivalent differential equation*

$$4r^3\frac{dr}{d\theta} = -\frac{\left(r^4\alpha - \gamma\right)\left(-\gamma + r^4\alpha + Q^2(\theta)\,r^4\right)}{\gamma Q^2(\theta)}. \tag{3.7}$$

Via the change of variables $\rho = r^4$, *this equation is transformed into the Riccati equation*

$$\frac{d\rho}{d\theta} = -\frac{(\rho\alpha - \gamma)\left(\left(\alpha + Q^2(\theta)\right)\rho - \gamma\right)}{\gamma Q^2(\theta)}. \tag{3.8}$$

Fortunately, this is integrable, since it possesses the particular solution $\rho\alpha - \gamma = 0$, *corresponding of course to the limit cycle* (Γ_1). *Then the general solution of this equation is given by*

$$\rho = \frac{\gamma}{\alpha} + \frac{1}{R}.$$

Indeed, substituting the solution $\rho = \left(\frac{\gamma}{\alpha} + \frac{1}{R}\right)$ *into the Riccati equation, we obtain the linear equation*

$$-\frac{1}{R^2}\frac{dR}{d\theta} = -\frac{\left(\left(\frac{\gamma}{\alpha} + \frac{1}{R}\right)\alpha - \gamma\right)\left(-\gamma + (\alpha + Q^2(\theta))(\frac{\gamma}{\alpha} + \frac{1}{R})\right)}{\gamma Q^2(\theta)},$$

thus

$$\begin{aligned} \frac{dR}{d\theta} &= \frac{1}{Q^2(\theta)\,\gamma}\left(Q^2(\theta)\,\alpha + R\gamma Q^2(\theta) + \alpha^2\right) \\ &= \frac{\alpha}{\gamma} + R + \frac{\alpha^2}{Q^2(\theta)\,\gamma}. \end{aligned} \tag{3.9}$$

The general solution of linear equation (3.9) *is*

$$R(\theta,k) = e^{\theta}\left(k - \frac{\alpha}{\gamma}\left(e^{-\theta} - 1\right) + \frac{\alpha^2}{\gamma}\int_0^{\theta}\frac{e^{-\theta}}{Q^2(s)}ds\right),$$

Where $k \in \mathbb{R}$. *Going back through the changes of variables, we obtain*

$$\rho(\theta,k) = \frac{\gamma}{\alpha} + \gamma\frac{e^{-\theta}}{\left(k\gamma - \alpha\left(e^{-\theta} - 1\right) + \alpha^2\int_0^{\theta}\frac{e^{-\theta}}{Q^2(s)}ds\right)},$$

if we take $h = \gamma k + 1$, the general solution of Riccatti equation (3.8) is

$$\rho(\theta, h) = \frac{\gamma}{\alpha} + \gamma \frac{e^{-\theta}}{h - \alpha e^{-\theta} + \alpha^2 \int_0^\theta \frac{e^{-\theta}}{Q^2(s)} ds}.$$

Consequently, the general solution of (3.6) is

$$r(\theta, h) = \left(\frac{\gamma}{\alpha} + \gamma \frac{e^{-\theta}}{h - \alpha e^{-\theta} + \alpha^2 \int_0^\theta \frac{e^{-\theta}}{Q^2(s)} ds} \right)^{\frac{1}{4}}.$$

To go a steep further, we remark that the solution such as $r(0, r_0) = r_0 > 0$, corresponds to the value $h = \frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma}$ provided a rewriting of the general solution of (3.6)

$$r(\theta, r_0) = \left(\frac{\gamma}{\alpha} + \gamma \frac{e^{-\theta}}{\frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha e^{-\theta} + \alpha^2 \int_0^\theta \frac{e^{-\theta}}{Q^2(s)} ds} \right)^{\frac{1}{4}}.$$

Any periodic solution of system (3.1) must satisfy the condition $r(2\pi, r_0) = r_0$, provided two distinct positive values of $r_0 : r_1^4 = \frac{\gamma}{\alpha}$ corresponding obviously to the algebraic limit cycle (Γ_1), a well defined second value, we have

$$r^4 = \frac{\gamma}{\alpha} + \gamma \frac{e^{-\theta}}{\frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha e^{-\theta} + \alpha^2 f(\theta)}.$$

Then the condition $r^4(2\pi, r_0) = r^4(0, r_0)$ equivalent

$$\frac{e^{-}2\pi}{\frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha e^{-2\pi} + \alpha^2 f(2\pi)} = \frac{1}{\frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha},$$

this imply that

$$e^{-2\pi} \left(\frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha \right) = \frac{\alpha^2 r_0^4}{\alpha r_0^4 - \gamma} - \alpha e^{-2\pi} + \alpha^2 f(2\pi),$$

then

$$\frac{r_0^4}{\alpha r_0^4 - \gamma} = \frac{f(2\pi)}{(e^2\pi - 1)},$$

so

$$r_0^4 \left(\alpha f(2\pi) - e^{-2\pi} + 1 \right) = \gamma f(2\pi),$$

then

$$r_0^4 = \gamma \frac{f(2\pi)}{\alpha f(2\pi) - e^{-2\pi} + 1}.$$

So there are two difference values with the equation $r_0^4 = \gamma \frac{f(2\pi)}{\alpha f(2\pi) - e^{-2\pi} + 1}$, one of them is equal to

$$r_0 = - \left(\frac{\gamma f(2\pi)}{\alpha f(2\pi) - e^{-2\pi} + 1} \right)^{\frac{1}{4}},$$

and we don't consider this case because $r_0 < 0$*, we only take into consideration the following value* $r_0 = r_*$ *which satisfies* $r(2\pi, r_*) = r_* > 0$

$$r_* = \left(\frac{\gamma f(2\pi)}{\alpha f(2\pi) - e^{-2\pi} + 1}\right)^{\frac{1}{4}}.$$

Where $f(2\pi) = \int_0^2 \pi \frac{e^{-s}}{Q^2(s)} ds.$
Since $\alpha > 0$*, and* $\frac{\gamma}{\alpha} > 0$ *then* $\gamma Q^2(\theta) > 0$ *and* $\gamma \frac{f(2\pi)}{\alpha f(2\pi) - e^{-2\pi}+1} > 0$*. So it follows that*

$$r_* = \left(\gamma \frac{f(2\pi)}{\alpha f(2\pi) - e^{-2\pi} + 1}\right)^{\frac{1}{4}} > 0,$$

then

$$r(\theta, r_*) = \left(\frac{\gamma}{\alpha} + \gamma \frac{e^{-\theta}}{\frac{\alpha^2 r_*^4}{\alpha r_*^4 - \gamma} - \alpha e^{-\theta} + \alpha^2 \int_0^\theta \frac{e^{-\theta}}{Q^2(s)} ds}\right)^{\frac{1}{4}}. \tag{3.10}$$

To show that it is a periodic solution, we have to show that:
$i)$ *the function* $x \to g(\theta)$*, where in this case*

$$g(\theta) = \frac{\gamma}{\alpha} + \gamma \frac{e^{-\theta}}{\frac{\alpha^2}{e^{-2\pi}-1} f(2\pi) - \alpha e^{-\theta} + \alpha^2 f(\theta)}$$

is 2π *periodic.*
$ii)$ $g(\theta) > 0$ *for all* $\theta \in [0, 2\pi)$ *. The last condition ensures that* $r(\theta, r_*)$ *is well defined for all* $\theta \in [0, 2\pi)$ *and the periodic solution do not pass through the unique equilibrium point* $(0, 0)$ *of system* (3.1).

Periodicity. Let $\theta \in [0, 2\pi)$, then

$$g(\theta + 2\pi) = \frac{\gamma}{\alpha} + \gamma \frac{e^{-\theta - 2\pi}}{\frac{\alpha^2}{e^{-2\pi}-1} f(2\pi) - \alpha e^{-\theta-2\pi} + \alpha^2 f(\theta + 2\pi)} \tag{3.11}$$

but

$$\begin{aligned} f(\theta + 2\pi) &= \int_0^{\theta+2\pi} \frac{e^{-s}}{Q^2(s)} ds \\ &= \int_0^{2\pi} \frac{e^{-s}}{Q^2(s)} ds + \int_{2\pi}^{\theta+2\pi} \frac{e^{-s}}{Q^2(s)} ds \\ &= f(2\pi) + \int_{2\pi}^{\theta+2\pi} \frac{e^{-s}}{Q^2(s)} ds \end{aligned}$$

we make the change of variable $u = s - 2\pi$ in the integral $\int_{2\pi}^{\theta} \frac{e^{-s}}{Q^2(s)} ds$, we get

$$\begin{aligned} f(\theta + 2\pi) &= f(2\pi) + \int_0^\theta \frac{e^{-(u+2\pi)}}{Q^2(u + 2\pi)} du \\ &= f(2\pi) + e^{-2\pi} f(\theta), \end{aligned}$$

we replace $f(\theta+2\pi)$ by $f(2\pi)+e^{-2\pi}f(\theta)$ in (3.11), we obtain

$$\begin{aligned}
g(\theta+2\pi) &= \frac{\gamma}{\alpha}+\gamma\frac{e^{-\theta-2\pi}}{\frac{\alpha^2}{e^{-2\pi}-1}f(2\pi)-\alpha e^{-\theta-2\pi}+\alpha^2\left(f(2\pi)+e^{-2\pi}f(\theta)\right)}\\
&= \frac{\gamma}{\alpha}+\gamma\frac{e^{-\theta-2\pi}}{\frac{\alpha^2}{e^{-2\pi}-1}f(2\pi)-\alpha e^{-\theta-2\pi}+\alpha^2 f(2\pi)+\alpha^2 e^{-2\pi}f(\theta)}\\
&= \frac{\gamma}{\alpha}+\frac{\gamma e^{-2\pi}e^{-\theta}}{\alpha^2 f(2\pi)\left(\frac{1}{e^{-2\pi}-1}+1\right)-\alpha e^{-\theta}e^{-2\pi}+\alpha^2 e^{-2\pi}f(\theta)}\\
&= \frac{\gamma}{\alpha}+\frac{\gamma e^{-2\pi}e^{-\theta}}{e^{-2\pi}\left(\alpha^2 f(2\pi)\left(\frac{1}{e^{-2\pi}-1}\right)-\alpha e^{-\theta}+\alpha^2 f(\theta)\right)}\\
&= \frac{\gamma}{\alpha}+\gamma\frac{e^{-\theta}}{\frac{\alpha^2}{e^{-2\pi}-1}f(2\pi)-\alpha e^{-\theta}+\alpha^2 f(\theta)}\\
&= g(\theta),
\end{aligned}$$

hence g is 2π-periodic.

Strict positivity of $g(\theta)$ for all $\theta\in[0.2\pi)$, we have

$$\begin{aligned}
g(\theta) &= \frac{\gamma}{\alpha}+\frac{\gamma}{\alpha}\frac{e^{-\theta}}{\frac{\alpha}{e^{-2\pi}-1}f(2\pi)-e^{-\theta}+\alpha f(\theta)}\\
&= \frac{\gamma}{\alpha}\left(1+\frac{e^{-\theta}}{\frac{\alpha}{e^{-2\pi}-1}f(2\pi)-e^{-\theta}+\alpha f(\theta)}\right)
\end{aligned}$$

and

$$\begin{aligned}
f(2\pi) &= \int_0^{2\pi}\frac{e^{-s}}{Q^2(s)}ds\\
&= f(\theta)+\int_0^{2\pi}\frac{e^{-s}}{Q^2(s)}ds.
\end{aligned}$$

If $\alpha>0$, for all $\theta\in[0,2\pi)$, then $\alpha f(\theta)>0$, for all $\theta\in[0.2\pi)$, so

$$\alpha f(2\pi)\geq\alpha f(\theta)>0$$

for all $\theta\in[0,2\pi)$ and

$$\begin{aligned}
g(\theta) &= \frac{\gamma}{\alpha}+\frac{\gamma}{\alpha}\frac{e^{-\theta}}{\frac{\alpha}{e^{-2\pi}-1}f(2\pi)-e^{-\theta}+\alpha f(\theta)}\\
&\geq \frac{\gamma}{\alpha}+\frac{\gamma}{\alpha}\frac{e^{-\theta}}{\frac{\alpha}{e^{-2\pi}-1}f(2\pi)-e^{-\theta}+\alpha f(2\pi)}\\
&= \gamma\frac{f(2\pi)}{\alpha f(2\pi)+e^{-\theta}\left(e^{2\pi}-1\right)}>0,
\end{aligned}$$

hence $g(\theta)$ for all $\theta\in[0,2\pi)$ **.**

Finally $r(\theta,r_*)$ defines through (3.10) a periodic solution. To show that it is a limit cycle, we consider (3.10), and introduce the poincaré return map $\lambda\to\Pi(2\pi,\lambda)=r(2\pi,\lambda)$, to prove that the periodic solution is an isolated periodic orbit (see [9], it is sufficient for the function

of poincaré first return), we compute $\frac{d\Pi}{d\lambda}(2\pi,\lambda)\mid_{\lambda=r*}$.
We have

$$\begin{aligned}\frac{d\Pi}{d\lambda}(2\pi,\lambda) &= \frac{d}{d\lambda}\left(\frac{\lambda^4\gamma+\lambda^4\alpha\gamma f(2\pi)-\gamma^2 f(2\pi)}{\alpha\lambda^4-\alpha\lambda^4 e^{-2\pi}+\gamma e^{-2\pi}+(\alpha^2\lambda^4-\alpha\gamma)f(2\pi)}\right)^{\frac{1}{4}}\\ &= \frac{\lambda^3\gamma^2 e^{-2\pi}}{\left(\gamma e^{-2\pi}+\alpha\lambda^4-\alpha\lambda^4 e^{-2\pi}+f(2\pi)(\alpha^2\lambda^4-\alpha\gamma)\right)^2}\\ &\times\left(\frac{\gamma e^{-2\pi}+\alpha\lambda^4-\alpha\lambda^4 e^{-2\pi}+f(2\pi)(\alpha^2\lambda^4-\alpha\gamma)}{\lambda^4\gamma-f(2\pi)\gamma^2+f(2\pi)\alpha\lambda^4\gamma}\right)^{\frac{3}{4}}.\end{aligned}$$

By replacing λ by its value given by $r_*=\left(\gamma\frac{f(2\pi)}{\alpha f(2\pi)-e^{-2\pi}+1}\right)^{\frac{1}{4}}$, and after some calculation, we get

$$\frac{d\Pi}{d\lambda}(2\pi,\lambda)\mid_{\lambda=r*}=e^{4\pi}>1.$$

Consequently the limit cycle of the differential equation (3.7) is unstable and hyperbolic.
On the other hand, we have $\dot{\theta}=-2\gamma Q^2(\theta)r^2$, since $\gamma>0$, for all $\theta\in[0,2\pi)$, then $\dot{\theta}<0$, and the orbit $r(\theta)$ of the equation (3.7) have the opposite orientation with respect to those $(x(t),y(t))$ of system (3.1).
Consequently the limit cycle of system (3.1) is a stable hyperbolic limit cycle.

Now we prove that this limit cycle is not algebraic more precisely, in Cartesian coordinates $r^2=(\theta,r_*)=x^2+y^2$ and $\theta=\arctan\frac{y}{x}$ we have the first integral

$$F(x,y)=\frac{\gamma e^{-\arctan\frac{y}{x}}}{\alpha(x^2+y^2)^2-\gamma}+e^{-\arctan\frac{y}{x}}-\alpha\int_0^{\arctan\frac{y}{x}}\frac{e^{-s}}{Q^2(s)}ds.$$

To see that we must prove for instance that there is non value of the integer **n** for which $\frac{\partial F^n}{\partial y^n}=0$, for this purpose let us compute $\frac{\partial F}{\partial y}(x,y)$. We find that they can be put on the form :

$$\frac{\partial F}{\partial y}=\left(-\frac{(x^2+y^2)(\alpha x^5+2\alpha x^3y^2+\alpha xy^4-\gamma x+4\gamma y)}{(\alpha x^4+2\alpha x^2y^2+\alpha y^4-\gamma)^2}+\frac{1}{Q^2(x,y)}\right)\alpha e^{-\arctan\frac{y}{x}}.$$

Since $\frac{1}{Q^2(x,y)}e^{-\arctan\frac{y}{x}}$ appears again, it will remains in any order of derivation therefore the curve $F(x,y)=0$ is non-algebraic and the limit cycle will also be non algebraic.
This completes the proof of Theorem 3.4.

Example 3.1 *If we take* $a=3,b=2,c=1,\gamma=1$; *and* $\alpha=1$, *then system* (3.1) *reads*

$$\begin{cases}\dot{x}=\left(x-x(x^2+y^2)^2-4y\right)(3x^2+2y^2+xy)^2-x\left((x^2+y^2)^2-1\right)^2,\\ \dot{y}=\left(y-y((x^2+y^2)^2+4x\right)(3x^2+2y^2+xy)^2-y\left((x^2+y^2)^2-1\right)^2,\end{cases}\tag{3.12}$$

we have $\frac{\gamma}{\alpha}=1>0$.
So the condition of Theorem 3.4 *is satisfied and hence the system* (3.12) *has two hyperbolic limit*

cycles, one algebraic whose expression in polar coordinates (r, θ) *is* $r^4(\theta, r_*) = 1$, *and the other is non-algebraic limit cycle whose expression in polar coordinates* (r, θ), *is*

$$r(\theta, r_*) = \left(1 + \frac{e^{-\theta}}{-0.142\,59 - e^{-\theta} + f(\theta)}\right)^{\frac{1}{4}},$$

where $\theta \in \mathbb{R}$, *and*

$$f(2\pi) = \int_0^{2\pi} \frac{e^{-s}}{\left(3\cos^2 s + 2\sin^2 s + \sin s \cos s\right)^2} ds = 0.142\,32,$$

$$r_* = \left(\frac{f(2\pi)}{1 + f(2\pi) - e^{-2\pi}}\right)^{\frac{1}{4}} = 0.594\,36.$$

Figure 3.1: Coexistence of algebraic and non algebraic limit cycles for system (3.12)

3.5 Existence of non-algebraic limit cycle

Theorem 3.5 *If* $\gamma > 0, \alpha < 0$, *and* $\alpha f(2\pi) > e^{-2\pi} - 1$. *The septic polynomial differential system* (3.1) *in which* $\gamma \neq 0$ *possesses exactly one non-algebraic hyperbolic limit cycles, whose expression in polar coordinates* (r, θ) *is*

$$r(\theta, r_*) = \left(\frac{\gamma}{\alpha} + \gamma \frac{e^{-\theta}}{\frac{\alpha^2 r_*^4}{\alpha r_*^4 - \gamma} - \alpha e^{-\theta} + \alpha^2 f(\theta)}\right)^{\frac{1}{4}},$$

with $f(\theta) = \int_0^{\theta} \frac{e^{-\theta}}{Q^2(s)} ds$ *and* $r_* = \left(\gamma \frac{f(2\pi)}{\alpha f(2\pi) - e^{-2\pi} + 1}\right)^{\frac{1}{4}}$.

Proof 13 *For the proof of Theorem 3.5 we shall use the notation and the expressions of the proof of Theorem 3.4. The system (3.1) has two periodic solutions given by*

$$\left(x^2+y^2\right)^2=\frac{\gamma}{\alpha}$$

and we don't consider this case (because $\frac{\gamma}{\alpha}<0$*). We only take into consideration the following solution*

$$r\left(\theta,r_*\right)=\left(\frac{\gamma}{\alpha}+\gamma\frac{e^{-\theta}}{\frac{\alpha^2r_*^4}{\alpha r_*^4-\gamma}-\alpha e^{-\theta}+\alpha^2f\left(\theta\right)}\right)^{\frac{1}{4}},$$

with $f\left(\theta\right)=\int_0^\theta\frac{e^{-\theta}}{Q^2(s)}ds$ *and* $r_*=\left(\gamma\frac{f(2\pi)}{\alpha f(2\pi)-e^{-2\pi}+1}\right)^{\frac{1}{4}}$.
If $\gamma>0,\alpha<0$ *then* $\alpha f\left(2\pi\right)<0$ *since* $\alpha f\left(2\pi\right)>e^{-2\pi}-1$*, then*

$$r_*^4=\gamma\frac{f\left(2\pi\right)}{1+\alpha f\left(2\pi\right)-e^{-2\pi}}>0.$$

Moreover we have $\frac{\alpha^2r_*^4}{\alpha r_*^4-\gamma}=\frac{\alpha^2}{e^{-2\pi}-1}f\left(2\pi\right)$ *and*

$$\begin{aligned}\frac{1}{\alpha}+\frac{e^{-\theta}}{\frac{\alpha^2}{e^{-2\pi}-1}f\left(2\pi\right)-\alpha e^{-\theta}+\alpha^2f\left(\theta\right)} &> \frac{1}{\alpha}+\frac{e^{-\theta}}{\frac{\alpha^2}{e^{-2\pi}-1}f\left(2\pi\right)-\alpha e^{-\theta}+\alpha^2f\left(\theta\right)}\\ &> \frac{1}{\alpha}+\frac{\left(e^{-2\pi}-1\right)e^{-2\pi}}{f\left(2\pi\right)\alpha^2\left(e^{-2\pi}-1\right)e^{-2\pi}-\alpha e^{-\theta}}\\ &> \frac{1}{\alpha}+\frac{\left(e^{-2\pi}-1\right)e^{-2\pi}}{f\left(2\pi\right)\alpha^2\left(e^{-2\pi}-1\right)e^{-2\pi}-\alpha e^{-2\pi}}\\ &= \frac{f\left(2\pi\right)}{\alpha f\left(2\pi\right)-e^{-2\pi}+1}.\end{aligned}$$

Since $f\left(2\pi\right)>0$ *and* $\alpha f\left(2\pi\right)>e^{2\pi}-1$*, then*

$$\frac{f\left(2\pi\right)}{\alpha f\left(2\pi\right)-e^{-2\pi}+1}>0.$$

Consequently

$$r^4\left(\theta,r_*\right)=\gamma\left(\frac{1}{\alpha}+\frac{e^{-\theta}}{\frac{\alpha^2}{e^{-2\pi}-1}f\left(2\pi\right)-\alpha e^{-\theta}+\alpha^2f\left(\theta\right)}\right)>0,$$

because $\gamma>0$*. Then the curve* $r\left(\theta,r_*\right)$ *is limit cycle of system (3.1). Finally it clearly to this limit cycle is not algebraic. This completes the proof of Theorem 3.5.*

Example 3.2 *If we take* $a=-4,b=-3,c=2,\gamma=1$ *and* $\alpha=-1$*, then system (3.1) reads*

$$\begin{cases}\dot{x}=\left(x+x\left(x^2+y^2\right)^2-4y\right)\left(-4x^2-3y^2+2xy\right)^2-x\left(-\left(x^2+y^2\right)^2-1\right)^2,\\ \dot{y}=\left(y+y((x^2+y^2)^2+4x\right)\left(-4x^2-3y^2+2xy\right)^2-y\left(-\left(x^2+y^2\right)^2-1\right)^2.\end{cases} \tag{3.13}$$

So the condition of Theorem 3.5 is satisfied and hence system (3.13) has one non algebraic limit cycle whose expression in polar coordinates (r,θ) *is*

$$r\left(\theta,r_*\right)=\left(-1+\frac{e^{-\theta}}{-0.109\,1+e^{-\theta}+f\left(\theta\right)}\right)^{\frac{1}{4}}$$

where $\boldsymbol{\theta} \in \mathbb{R}$, *and*

$$f(2\pi) = \int_0^{2\pi} \frac{e^{-s}}{\left(-4\cos^2 s - 3\sin^2 s + 2\sin s\cos s\right)^2} ds = 0.108\,90$$

$$r_* = \left(\frac{f(2\pi)}{1 - f(2\pi) - e^{-2\pi}}\right)^{\frac{1}{4}} = 0.591\,57$$

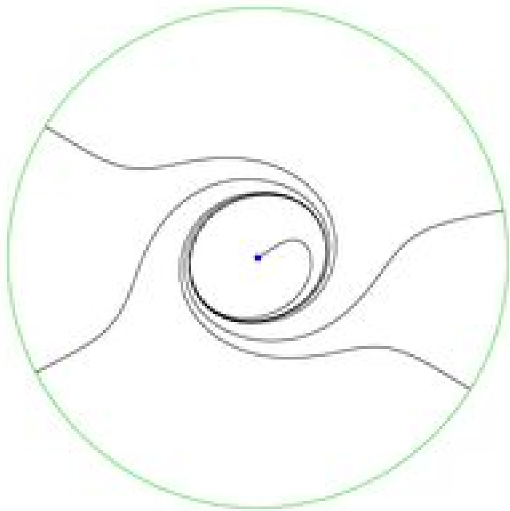

Figure 3.2: One non algebraic limit cycle for system (3.13)

3.6 Non existence of limit cycles

Theorem 3.6 *The septic polynomial differential system* (3.1) *has no limit cycle when the one of the following conditions is assumed:*
i) If $\gamma < 0, \alpha > 0, Q(\theta) \neq 0$.
ii) If $\gamma > 0, \alpha < 0, Q(\theta) \neq 0$, *and* $\alpha f(2\pi) < e^{-2\pi} - 1$.

Proof 14 $i)$ *If* $\gamma < 0, \alpha > 0, Q(\theta) \neq 0$, *for all* $\theta \in [0, 2\pi)$, *then* $\gamma f(2\pi) \leq 0$ *and* $\alpha f(2\pi) \geq 0$, *we have* $\alpha f(2\pi) \geq e^{-}2\pi - 1$, *then* $\alpha f(2\pi) - e^{-2\pi} + 1 \geq 0$ *and*

$$r_* = \left(\gamma \frac{f(2\pi)}{\alpha f(2\pi) - e^{-2\pi} + 1}\right)^{\frac{1}{4}} < 0.$$

Or
$ii)$ *If* $\gamma > 0, \alpha < 0, Q(\theta) \neq 0$, *for all* $\theta \in [0, 2\pi)$, *then* $\gamma f(2\pi) > 0$ *and* $\alpha f(2\pi) < 0$. *Since* $\alpha f(2\pi) < e^{-2\pi} - 1$, *then* $\alpha f(2\pi) - e^{-2\pi} + 1 \leq 0$ *and*

$$r_* = \left(\gamma \frac{f(2\pi)}{\alpha f(2\pi) - e^{-2\pi} + 1}\right)^{\frac{1}{4}} < 0$$

Then, system (3.1) *has no limit cycle. Hence the Theorem* 3.6 *is proved.*

Example 3.3 *If we take* $a = b = c = 1, \gamma = -1$ *and* $\alpha = 2$*, then system* (3.1) *reads*

$$\begin{cases} \dot{x} = \left(-x - 2x\left(x^2+y^2\right)^2 + 4y\right)\left(x^2+y^2+xy\right)^2 - x\left(2\left(x^2+y^2\right)^2+1\right)^2, \\ \dot{y} = \left(-y - 2y((x^2+y^2)^2 - 4x\right)\left(x^2+y^2+xy\right)^2 - y\left(2\left(x^2+y^2\right)^2+1\right)^2, \end{cases} \tag{3.14}$$

we have $\frac{\gamma}{\alpha} = -\frac{1}{2} < 0$ *and* $\alpha f(2\pi) \simeq 0.96 > e^{-2\pi} - 1 \simeq -0.99$*. So the first condition of the Theorem* 3.6 *is satisfied and hence the system* (3.14) *has no limit cycle.*

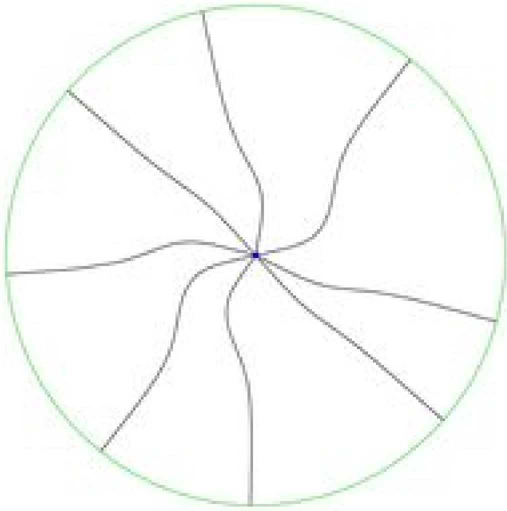

Figure 3.3: Phase portrait of system (3.14)

Conclusion

In conclusion, this work is about studying the existence and non existence of limit cycles and its number if their exists, so we have studied one of the main problems in the qualitative theory of planar differential systems.

We find the expressions of limit cycles by using the polar coordinates. In the second chapter, we analysis the existence of two limit cycle for a septic polynomial differential system (2.1). Moreover, the expression of a limit cycle is contained in a algebraic curve of the plane, the we say that it is algebraic.

As the third chapter, we study the following problems for a septic polynomial differential system (3.1):

- Coexistence of algebraic and non algebraic limit cycles.
- Existence of non algebraic limit cycle.
- Non existence of limit cycles.

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Dedications

To my darling little princess: AMIRA INSAF.

To my dear parents: Mohamed Tayeb and Zohra.

To my dear Sister: Hadjira and her children Islem and Meriem.

To my dear brother: Boubeker

To my dear friends: Ibtissam, Amani, Rayan, Ibtissam, Ikram, Amel.

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I dedicate my master's dissertation with gratitude and appreciation.

CHAFIKA

الملخص:

الهدف من هذه المذكرة هو الدراسة النوعية لصنف من الانظمة التفاضلية كثيرات الحدود المستوية. النتائج التي تحصلنا عليها في هذه الدراسة تهتم بوجود و عدد الحلول الدورية المعزولة بالتالي دورات الحد. و قد اثبتنا انها تحتوي على دورتين حديتين على الاكثر. بالإضافة الى ذلك تمكننا من تحديد التغييرات الصريحة لهذه الدورات الحدية (جبرية و غير جبرية) بالنسبة لهذه الفئة المدروسة اخيرا. قدمنا بعض الامثلة لتوضيح النتائج التي تم الحصول عليها.

كلمات مفتاحية: نظام تفاضلي، حل دوري، المنحنى الثابت، دورة حد جبرية، دورة حد غير جبرية.

Abstract :

The objective of this memory is the qualitative study of a class of polynomial planar differential systems. The results obtained in this study concern the existence and the number of periodic solutions isolated by consequent limit cycles. And we have shown we have shown that it has at most two limit cycles. Moreover, we were able to determine the explicit expressions of the limit cycles (algebraic and non-algebraic) found for the studied class.

Finally, some examples were presented to illustrate the results obtained.

Keywords: Differential system, periodic solution, invariant curve, algebraic limit cycle, non-algebraic limit cycle.

Résumé :

L'objectif de cette mémoire est l'étude qualitative d'une classe de systèmes différentiels planaires polynômiaux. Les résultats obtenus dans cette étude concernent l'existence et le nombre des solutions périodiques isolées par conséquence les cycles limites. Et nous avons montré nous avons montré qu'il a au plus deux cycles limites. De plus, nous avons pu déterminer les expressions explicites des des cycles limites (algébriques et non algébriques) trouvés pour la classe étudiée.

Enfin, quelques exemples ont été présentés pour illustrer les résultats obtenus.

Mots clés : Système différentiel, solution périodique, courbe invariante, cycle limite algébrique, cycle limite non-algébrique.