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Integrability and piecewise planar differential systems

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Dedication

I dedicate this dissertation work to my family and my friends

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Dedication

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Introduction

In general, a dynamical system is everything changes over time, in mathematics is the formalization of the general scientific concept of a deterministic process. The future and past states of many physical, chemical, biological, ecological, economical, and even social systems can be predicted to a certain extent by knowing their present state and the laws governing their evolution. On the condition that these laws do not change in time, the behavior of such a system could be considered as completely defined by its initial condition. Thus, the notion of a dynamical system includes a set of its possible situations (state space) and a law of the evolution of the state in time.

Modelling dynamical systems by ordinary differential equations has a long history, see for instance [1]. Over the last hundred years, many techniques have been developed for the solution of ordinary differential equations but most of the non linear differential equations cannot be solved by the calculus methods we know at present. The qualitative theory of differential equations is being used to examine differential equations whose explicit solutions cannot be determined.These tools are originated by Henri Poincaré in his work on differential equations at the end of the nineteenth century [30, 31].

In this thesis, we use the qualitative theory of differential equations to study a differential system in two-dimensional and we treat the most important solution of differential equations which is the limit cycle introduced by H. Poincaré [31] and reported by his work the most sought solutions in the modeling of physical systems in the plane.

Our first aim in this thesis is to study the integrability of ordinary differential equations or simply differential systems in two real variables

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

where P and Q are polynomials of degree two. The second aim is to determine the number of limit cycles of the piecewise differential system of the form

$$\begin{cases} \dot{x} = f_1(x, y), \\ \dot{y} = f_2(x, y), \end{cases} \quad H(x, y) > 0, \text{ and } \begin{cases} \dot{x} = g_1(x, y), \\ \dot{y} = g_2(x, y), \end{cases} \quad H(x, y) < 0,$$

separated by $\Sigma = \{(x, y) \in \mathbb{R}^2 / H(x, y) = 0\}$.

Now, we describe the structure of this thesis which is divided into three chapters, in the first one we present the necessary definitions, lemmas and theorems used in our study as fixed points and their nature, Hartman-Grobman theorem, Poincaré map, piecewise differential system, invariant, first integral..., (see [22, 23]).

In chapter 2, we start to present our work by classifying quadratic differential systems having a special invariant of the form $ax^2 + bxy + cy^2 + dx + ey + c_1 t$, we prove that there are 21 different families of quadratic systems having invariants of this form. As far as we know this is the first time that quadratic differential systems having an invariant different from a Darboux invariant have been classified. In the second part of this chapter, we study the limit cycles of piecewise planar differential systems formed by quadratic systems that have the first integral of the form $ax^2 + bxy + cy^2$ and linear center. We prove that piecewise systems have a continuum of periodic orbits, so no limit cycles.

In chapter 3, we tackle the number of limit cycles of the piecewise planar differential system formed by the quadratic or cubic systems with uniform isochronous center and linear center separated the straight-line $x = 0$ by treating the two cases continuous and discontinuous. We prove that piecewise systems have at most one limit cycle for discontinuous piecewise systems, we give an example for the quadratic case and we show that no limit cycles for continuous piecewise systems.

Chapter 1

Preliminaries

In this chapter, we give some notations and definitions of the geometric theory of integrability of planar differential system, and lemmas that are used along of the thesis.
By definition a planar differential system is:

$$\begin{cases} \dot{x} = P(x(t), y(t)), \\ \dot{y} = Q(x(t), y(t)), \end{cases} \tag{1.1}$$

be a two dimensional (real) differential systems where the dependent variables x and y are real, P and Q are $\mathbb{C}^r$ functions from an open subset **U** of $\mathbb{R}^2$ in $\mathbb{R}$. As usual we denote the vector field associated to differential system as:

$$\chi = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

If **P** and **Q** are polynomials then system (1) is called polynomial differential system, we denote by $m = max(deg(P), deg(Q))$ the degree of the polynomial system, and we always assume that the polynomials **P** and **Q** are relatively prime in the ring of complex polynomials in the variables **x** and **y**.

1.1 Vector field

***Definition* 1.1.** *We call vector field a region of the plane in which exists in any point a vector* $\vec{V}(M, t)$. *Suppose that we have a* C^1 *vector field in* $\Omega \subset \mathbb{R}^2$, *that is to say the application:*

$$M: \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \vec{V}(M) = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix},$$

where F_1, F_2 *are* C^1 *in* Ω.
We consider the vector field χ *associated to the system* (1.1)

$$\frac{d\vec{M}}{dt} = \vec{V} \Leftrightarrow \begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

which means that system (1.1) *is equivalent to the vector field* $\chi(P,Q)$*, we can also write:*

$$\chi = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

1.2 Phase portrait

Definition 1.2. *A phase portrait is a geometric representation of the trajectories of a dynamic system in phase space; to each set of initial conditions corresponds a curve or a point.*

1.2.1 Fixed point

Definition 1.3. *Let* $\dot{x} = f(x)$ *be a differential system, with f of class* $\mathbb{C}^1$ *of* $E = \mathbb{R}^n$ *in it self,* x_0 *is a fixed point (also called equilibrium point, stationary point, critical point), if and only if* $f(x_0) = 0$*. A regular point* x_1 *if it is not a fixed point i.e.,* $f(x_1) \neq 0$.

Definition 1.4. *We say that the point* $(x^*; y^*)$ *is a fixed point of the system (1.1), if it is a solution of the following algebraic system*

$$\begin{cases} P(x,y) = 0, \\ Q(x,y) = 0. \end{cases}$$

1.2.2 Stability of fixed point

Definition 1.5. *We say that* (x^*, y^*) *is stable if and only if*

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \| \ (x,y) - (x^*, y^*) \ \| < \delta \Rightarrow \forall t > 0,\ \|x(t) - x^*\| < \varepsilon.$$

The fixed point (x^*, y^*) *is asymptotically stable if and only if is stable and* $\lim_{t \longrightarrow \infty} \|x(t) - x^*\| = 0$.

1.2.3 Classification of fixed point

The flow of (1.1) in the neighbourhood of a fixed point (x^*, y^*) is classified according to the eigenvalues of the matrix $A_\chi((x^*, y^*))$ its determinant, as well as its trace. The eigenvalues of A_χ are solutions of the characteristic equation

$$\lambda^2 - tr(A_\chi)\lambda + det(A_\chi) = 0,$$

with

$$tr(A_\chi) = \lambda_1 + \lambda_2 \text{ and } det(A_\chi) = \lambda_1\lambda_2.$$

The nature of eigenvalues depends or the sign of the discriminant

$$\Delta = (tr(A_\chi))^2 - 4det(A_\chi).$$

Three cases arise.

cas 1: $\Delta = 0$. We then have $\lambda_1 = \lambda_2 = \lambda$, that is to say

$$det(A_\chi) = \lambda^2 > 0 \text{ and } tr(A_\chi) = 2\lambda.$$

Therefore, if the trace is positive $\lambda > 0$, we have an unstable star node, if the trace is negative $\lambda < 0$, we have a stable star.

cas 2: $\Delta > 0$. We then have two distinct real eigenvalues, so:

- $det(A_\chi) < 0$, λ_1 and λ_2 are of opposite sign, the origin is a saddle.
- $det(A_\chi) > 0$ and $tr(A_\chi) > 0$, $\lambda_1, \lambda_2 > 0$, the origin is an unstable node.
- $det(A_\chi) > 0$ and $tr(A_\chi) < 0$, $\lambda_1, \lambda_2 < 0$, the originis an stable node.

cas 3: $\Delta < 0$. We then have two conjugate complex eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$, so we get $det(A_\chi) = \alpha^2 + \beta^2 > 0$ and $tr(A_\chi) = 2\alpha$.

- $tr(A_\chi) < 0$ the origin is a stable spiral.
- $tr(A_\chi) > 0$ the origin is a unstable spiral.
- $tr(A_\chi) = 0$ the origin is a centre.

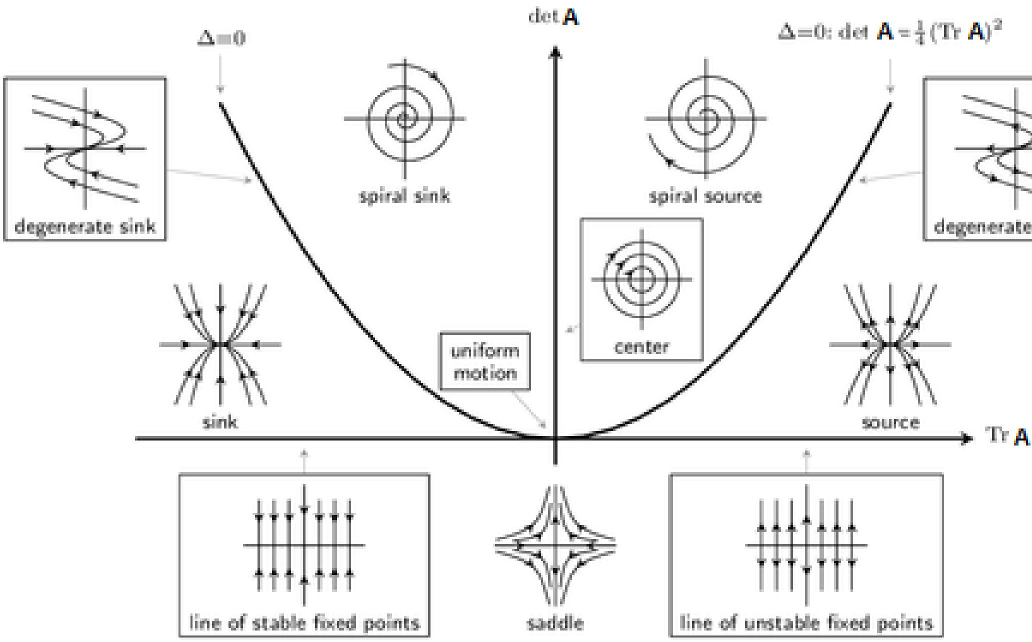

Figure 1.1: Classification of phase portraits in the (detA, trA).

1.3 Periodic orbits

***Definition* 1.6.** *We call periodic orbit of (1.1), any solution $(x(t), y(t))$ for which there exists a real $T > 0$ such that:*

$$\forall t \in \mathbb{R}, x(t+T) = x(t) \text{ and } y(t+T) = y(t).$$

- *The samallest number $T > 0$ which is suitable, is then called period of this solution.*
- *To any periodic solution corresponds a closed orbit in phase space.*

1.4 Hartman-Grobman theorem

The Hartman-Grobman Theorem shows that near a hyperbolic fixed point x_0, the non linear system

$$\dot{x} = f(x) \tag{1.2}$$

has the same qualitative structure as the linear system

$$\dot{x} = A(x) \tag{1.3}$$

with $A = Df(x_0)$: Throughout this section we shall assume that the fixed point x_0 has been translated to the origin.

Definition 1.7. *(Topologically equivalent) Two autonomous systems of differential equations are said to be topologically equivalent in a neighbourhood $N_\delta(0)$ or have the same qualitative structure near the origin if there is a homeomorphism H mapping an open U containing the origin onto an open set V containing the origin which map trajectories of the first system in U to the second one in V and preserves their orientation by time, for more details see [29].*

Theorem 1.1. *Let E be an open sub set of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and ϕ_t be the flow of the non linear system (1.2). Suppose that the origin is an equilibrium point of (1.2) which mean $f(0) = 0$ and that the matrix $Df(0)$ has no eigenvalue with zero real part. Then there exists $H : U \longrightarrow V$ Homomorphism such that for all $x_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x_0 \in U$ and $t \in I_0$*

$$H \circ \phi(x_0) = e^{At} H(x_0);$$

i.e., H maps trajectories of (1.2) near the origin onto trajectories of (1.3) near the origin and preserves the parametrization by time.

1.5 Limit cycle

Definition 1.8. *A limit cycle is an isolated closed trajectory ("isolated" means that neighbouring trajectories are not closed).*

Definition 1.9. *We call limit cycle ω -limit a periodic orbit γ which is the limit set of at least one point not belonging in γ, one calls cycle α - limit of the mirror system $\dot{x} = f(x)$.*

Example 1.1. *A simple limit cycle consider*

$$\begin{cases} \dot{r} = r(1 - r^2) & , r \geqslant 0 \\ \dot{\theta} = 1, \end{cases}$$

$r = 0$ is an unstable fixed point and $r = 1$ is a periodic orbit, hence all trajectories in the phase plane (except $r = 0$) approach to the unit circle $r = 1$ monotonically.

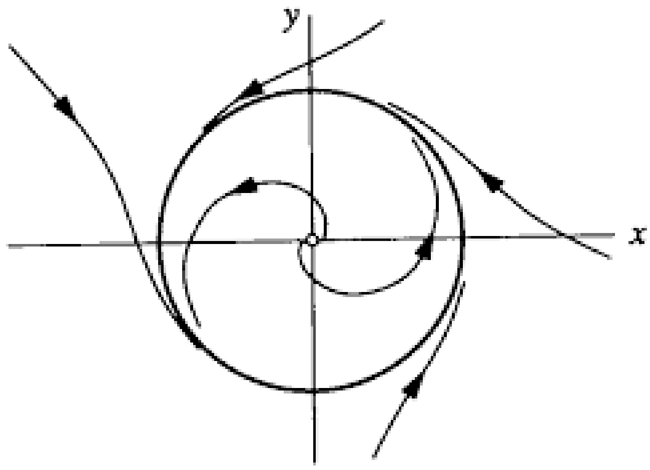

Figure 1.2: Limit cycle.

1.5.1 Classification of limit cycles

- **Stable limit cycles** i.e., all trajectories in the vicinity of the limit cycle converging to it as $t \longrightarrow \infty$.
- **Unstable limit cycles** i.e., all trajectories in the vicinity of the limit cycle diverging from it as $t \longrightarrow \infty$.
- **Semi-stable limit cycles** i.e., some of the trajectories in the vicinity converging to the limit cycle while others diverging from it as $t \longrightarrow \infty$.

The most important kind of limit cycle is the stable limit cycle, where nearby curves spiral towards Γ on both sides.

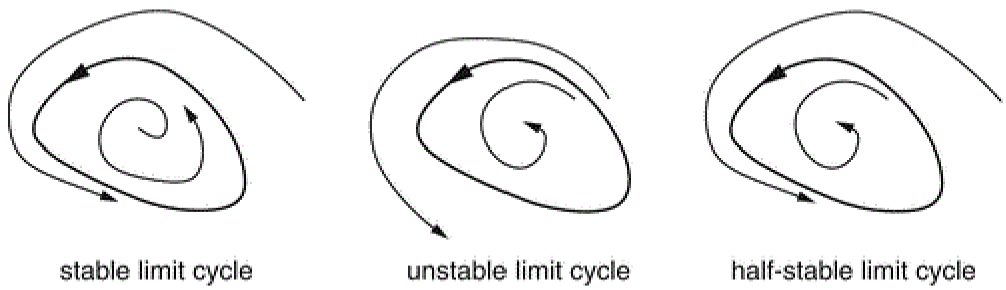

Figure 1.3: Classification of limit cycles

1.6 The Poincaré map

Probably the most basic tool for studying the stability and bifurcation of the periodic orbits is the Poincaré map. The idea of the Poincaré map when $\boldsymbol{\Gamma}$ is a periodic orbit of the system (1.2) through $\boldsymbol{x_0}$, with $\boldsymbol{\Sigma}$ is a hyperplane perpendicular to $\boldsymbol{\Gamma}$ at $\boldsymbol{x_0}$, then for any point $\boldsymbol{x} \in \boldsymbol{\Sigma}$ sufficiently near $\boldsymbol{x_0}$ the solution of (1.2) through $\boldsymbol{x}$ at $\boldsymbol{t} = \boldsymbol{0}$, $\boldsymbol{\Phi_t(x)}$, will cross $\boldsymbol{\Sigma}$ again at $\boldsymbol{P(x)}$ near $\boldsymbol{x_0}$; cf.Figure (1.2), the mapping $\boldsymbol{x \to P(x)}$ is called the Poincaré map. The Poincaré map can also be defined when $\boldsymbol{\Sigma}$ is a smooth surface.

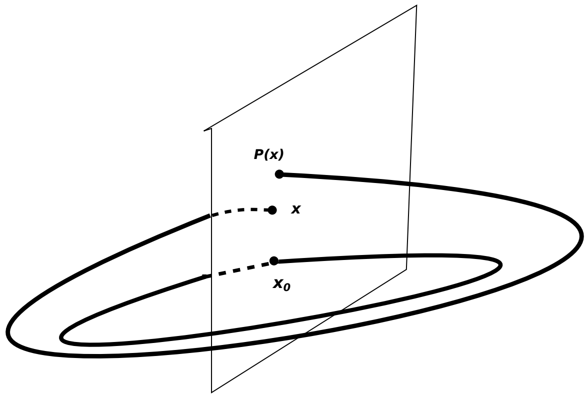

Figure 1.4: The Poincaré map.

Theorem 1.2 (The existence and continuity of the Poincaré map and its first derivative). *Let $\boldsymbol{E}$ be an open subset of $\mathbb{R}^n$ and let $\boldsymbol{f} \in \boldsymbol{C^1(E)}$. Suppose that $\boldsymbol{\Phi_t(x_0)}$ is a periodic solution of (1.2) of period $\boldsymbol{T}$ and that the cycle*

$$\boldsymbol{\Gamma = \{x \in \mathbb{R}^n \mid x = \Phi_t(x_0),\ \ 0 \leq t \leq T\}},$$

is contained in $\boldsymbol{E}$. Let $\sum$ be the hyperplane orthogonal to $\boldsymbol{\Gamma}$ at $\boldsymbol{x_0}$; i.e., let

$$\boldsymbol{\Sigma = \{x \in \mathbb{R}^n \mid (x - x_0) \cdot f(x_0) = 0\}},$$

then $\exists \boldsymbol{\delta > 0}$ and $\exists!$ function $\boldsymbol{\tau(x)}$ defined and continuously differentiable for $\boldsymbol{x \in N_\delta(x_0)}$ such that

$$\begin{cases} \boldsymbol{\tau(x_0) = T} \\ \boldsymbol{\Phi_{\tau(x)}(x) \in \Sigma} \end{cases} \quad \textit{for all } \boldsymbol{x \in N_\delta(x_0)}.$$

Proof. *The proof of this theorem is an immediate application of the implicit function theorem, by the supposition of*

$$\boldsymbol{F(t,x) = (\Phi_t(x) - x_0) \cdot f(x_0)}, \quad \textit{for a given } \boldsymbol{x_0 \in \Gamma \subset E}.$$

for more details see [29].

***Definition* 1.10** (The Poincaré map). *Let* Γ, Σ, δ, *and* $\tau(x)$ *be defined as in theorem* (1.1). *Then, for* $x \in N_\delta(x_0) \cap \Sigma$, *the function*

$$P(x) = \Phi_{\tau(x)},$$

is called the Poincaré map for Γ *at* x_0.

***Remark* 1.1.** *It follows from theorem* (1.1) *that* $P \in C^1(U)$ *where* $U = N_{\delta(x_0)} \cap \Sigma$.

1. *If* f *analytic in* $E \Rightarrow P$ *analytic in* U ,
2. *Fixed points of the Poincaré map, i.e.,(points* $x \in \Sigma : P(x) = x$*) are periodic orbits of* (1.2),
3. *By considering the system* (1.2) *with* $t \to -t$, *we can show that the Poincaré map* P *has a* C^1*-inverse,* $P^{-1}(x) = \Phi_{-\tau(x)}(x)$. *Thus,* P *is a diffeomorphism; i.e., a smooth function with a smooth inverse.*

1.6.1 The Poincaré map of planar systems

Now, we are going to cite some specific results for the Poincaré map for planar systems. For planar systems, if we translate the origin to the point $x_0 \in \Sigma \cap \Gamma$. The point $0 \in \Gamma \cap \Sigma$ divide Σ on two open segments $\Sigma^+ \wedge \Sigma^-$; cf. Figure (1.1) below. Let s be the signed distance along Σ with $s > 0$ for points in Σ^+ .

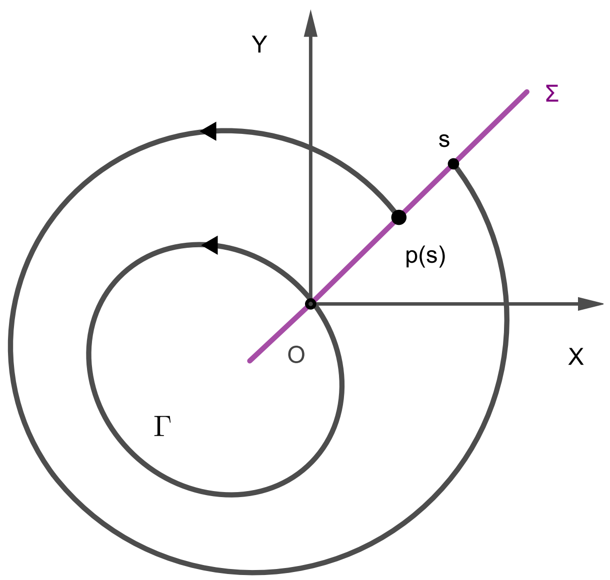

Figure 1.5: The straight line normal Σ to Γ at 0

By theorem (1.3), the Poincaré map $P(s)$ defined for $|s| < \delta$ and we have $P(0)$. In order to see how the stability of the cycle Γ is determined by $P'(0)$, let us introduce the displacement function, which defined for all $|s| < \delta$ by

$$d(s) = P(s) - s. \tag{1.4}$$

with $P(0) = 0$ and $d'(s) = P'(s) - 1$. From the mean value theorem, for $|s| < \delta$, $\exists\, \Sigma \in [0, s]$ such that $: d(s) = d'(\Sigma)s$. Since $d'(s)$ is continuous, the sign of $d'(s)$ will be the same as $d'(0)$ for $|s|$ sufficiently small as long as $d'(0) \neq 0$. Thus, if $d'(0) < 0$ implie that $d(s) < 0$ for $s > 0$ and $d(s) > 0$ for $s < 0$ and that $s < 0\ in\ \Sigma^-$; i.e., the cycle Γ is a stable limit cycle Cf. Figure (1.1). Similarly, if $d'(0) > 0$ then Γ is an unstable limit cycle. So, we have the corresponding results that if $P(0) = 0$ and $P'(0) < 1$, then Γ is stable limit cycle and if $P(0) = 0$ and $P'(0) > 1$, then Γ is an unstable limit cycle.

1.7 Integrability

There is no general approach to obtain an explicit form for the solution of a system, furthermore, for most system it can be shown that such forms do not exist, therefore, we lower our expectations and attempt to identify global invariants.

1.7.1 Global invariant

Definition 1.11. *let Ω be an open and dense subset of $\mathbb{R}^2$, an invariant of system* **(1.1)** *in Ω is a non-contant $\mathbb{C}^1$ function I in the variable x, y and t such that $I(x(t), y(t), t)$ is constant on all solution curves $(x(t), y(t))$ of system (1) contained in Ω, i,e:*

$$P\frac{\partial I}{\partial x} + Q\frac{\partial I}{\partial y} + \frac{\partial I}{\partial t} = 0.$$

1.7.2 First integral

Definition 1.12. *When I is independent of t we say I is first integral i,e $I(x, y) = C$ on solution:*

$$P\frac{\partial I}{\partial x} + Q\frac{\partial I}{\partial y} = 0$$

In the case of a two -dimensional system, having one first integral is enough to obtain a global picture of the solution in the (x, y) phase space. Moreover, if I is time-independent, then the solution curves lie on the level set $I(x, y) = c$. Furthermore I completely characterizes the phase portrait.

1.7.3 Invariant algbraic curve

Definition 1.13. *For $f \in \mathbb{C}[x, y]$, the curve $f(x, y) = 0$ is an invariant algebraic curve of system (1) if there exists $K \in \mathbb{C}[x, y]$, such that:*

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the cofactor of the real invariant algebraic curve $f = 0$.

1.8 Piecewise linear differential system

Definition 1.14 (Piecewise linear system). *A differential system defined on an open region $S \subseteq \mathbb{R}^2$ is said to be a piecewise linear differential system (PWLS) on S if there exists a set of 3-tuplesm (A_i, b_i, S_i) such that A_i is a 2×2 real matrix, $b_i \in \mathbb{R}^n$, $S_i \subseteq S$ is an open set in $\mathbb{R}^n$ satisfying that $S_i \cap S_j = \phi$ if $i \neq j$ and $\cup_{i \in \mathbb{I}} Cl(S_i) = S$ and $A_i x + b_i$ is the vector field defined by the system when $x \in S_i$. As usual $Cl(S_i)$ denotes the closure of S_i.*

Example 1.2.

$$\dot{x} = |x| = \begin{cases} x, & x > 0, \\ -x, & x < 0, \end{cases}$$

is (PWLS): $\{(A_i, b_i, s_i)\} = \{(1, 0, s_1), (1, 0, s_2)\}$ *such that:*

$$S_1 = \{x \in \mathbb{R}/x > 0\},$$
$$S_2 = \{x \in \mathbb{R}/x < 0\},$$
$$\Sigma = \{x \in \mathbb{R}/x = 0\},$$

$s_1 \bigcap s_2 = \phi$, $\bar{s_1} \cup \bar{s_2} = \{x \in \mathbb{R}/x \geq 0\} \cup \{x \in \mathbb{R}/x \leqslant 0\} = \mathbb{R}$.

Thus the vector field defined by a PWLS is a linear map on each of the disjoint regions S_i, but is not globally linear on the whole S.

1.9 Linearisation method

From a given planar differential system $\dot{x}= f(x)$ with a differentiable vector field f can construct a set of different PWLS. For instance, let us suppose that p_1 and p_2 are two zeros of f, and let k be a vector in $\mathbb{R}^2$ such that: $k^T p_1 < 0$, and $k^T p_2 > 0$. The straight line $\Sigma = \{x \in \mathbb{R}^2 : k^T x = 0\}$

divides $\mathbb{R}^2$ into the two open regions: $S_1 = \{x \in \mathbb{R}^2 : k^T x < 0\}$ and $S_2 = \{x \in \mathbb{R}^2 : k^T x > 0\}$.

Denoting by $Df(p_i)$ the Jacobian matrix of the vector field f at the point p_i, it follows that $\{((Df(p_i), -Df(p_i)p_i, S_i))\}_{i=1,2}$ is a piecewise differential system on the whole $\mathbb{R}^2$.

1.10 Continuous and discontinuous piecewise linear differential system

***Definition* 1.15.** *Let $\Sigma_{ij} = \partial \bar{s}_i \cap \partial \bar{s}_j$ be the common boundary of the regions $\bar{s}_i$ and $\bar{s}_j$. If $A_i p + b_i = A_j p + b_j$ for every $p \in \Sigma_{ij}$, is said to be continuous, otherwise the piecewise linear differential system is said to be discontinuous. In discontinuous piecewise linear differential system, two different vector $\dot{x}$, namely $f_i(x)$ and $f_j(x)$, can be associated to a point $x \in \Sigma_{ij}$. If the transversal components of $f_i(x)$ and $f_j(x)$ have the same sign, the orbit crosses the boundary and has, at that point, a discontinuity in its tangent vector. On the contrary, if the transversal components of $f_i(x)$ and $f_j(x)$ are of opposite sign, i.e., if the two vector fields are "pushing" in opposite directions, the state of the system is forced to remain on the boundary and slide on it. Although, in principle, motions on the boundary could be defined in different ways, the most natural one is Filippov convex method.*

1.11 Solution of continuous piecewise linear differential system

Since the piecewise linear differential system is formed by linear differential systems in each region $\bar{s}_i$, then the solution of linear differential system $\dot{x}_i = A_i x + b_i$ starting at p_0 is given by

$$X(s, p_0) = e^{A_i s} p_0 \int_0^s e^{A_i(s-r)} b_i dr.$$

Since, for a continuous piecewise linear differential system we have $A_i p + b_i = A_j p + b_j$ at any point of the boundary Σ_{ij} separating two adjacent regions S_i and S_j, then for these systems the vector $A_i p + b_i$ is uniquely defined at any point of the state space and the orbits in region S_i approaching transersely the boundary Σ_{ij}, cross it and enter into the adjacent region S_j. In particular if the vector field

$$\dot{X} = \begin{cases} f_1(X) = A_1 X + b_1, \; if H(x) > 0, \\ f_2(X) = A_2 X + b_2, \; if H(x) < 0, \end{cases} \tag{1.5}$$

with the boundary

$$\Sigma = \{x \in \mathbb{R}^2 : H(x) = 0\},$$

and two regions

$$S_1 = \{x \in \mathbb{R}^2 : H(x) > 0\},\ S_2 = \{x \in \mathbb{R}^2 : H(x) < 0\},$$

is continuous Let $(x_1(t); y_1(t))$ *and* $(x_2(t); y_2(t))$ are solutions of system (1.5) on S_1 and S_2 respectively. Then, the trajectory corresponding to the initial condition $X_0 = (x_{01}, y_{01})$ of the system (1.5) on S_1 is crossed the curve $H(x) = 0$, at the instance t^* in this case the initial condition of the second system (on S_2) is $(x_{02}, y_{02}) = (x_2(t^*), y_2(t^*))$. Furthermore, for continuous piecewise linear differential system (1.5), we have if

$$X(s, p_0) = e^{A_1 s} p_0 \int_0^s e^{A_1(s-r)} b_1 dr,$$

is a solution of linear differential system piecewise linear differential system starting at p_0 in S_1; then there exist a point $q = (x_1; y_1) \in \Sigma$ and the finite time t^* such that the orbit of linear differential system in S_1 starting at the point p is crossed the curve $H(x) = 0$, at the instance t^* at the point

$$q_0 = (x_1, y_1) = e^{A_1 t^*} p_0 \int_0^{t^*} e^{A_1(t^*-r)} b_1 dr,$$

by the continuity of piecewise linear differential system, the solution of this system in S_2 is

$$X(s, q_0) = e^{A_2 s} p_0 \int_0^s e^{A_2(s-r)} b_2 dr.$$

1.12 Solution of discontinuous piecewise linear differential system

We consider planar Filippov systems and assume, for simplicity, that there are only two regions S_i,

$$\dot{x} = \begin{cases} f_1(x), x \in S_1, \\ f_2(x), x \in S_2. \end{cases} \tag{1.6}$$

Moreover, the discontinuity boundary separating these two regions is described as

$$\Sigma = \{x \in \mathbb{R}^2 : H(x) = 0\},$$

where H is a smooth scalar function with non vanishing gradient $\nabla H(x) = (\frac{\partial H}{\partial X_i})^T$ on Σ , and

$$S_1 = \{x \in \mathbb{R}^2 : H(x) > 0\},$$

$$S_2 = \{x \in \mathbb{R}^2 : H(x) < 0\}.$$

The boundary is either closed or goes to infinity in both directions and $f_1 \neq f_2$ on Σ.

1.12.1 sliding solutions

The sliding solutions on Σ obtained with the well-known Filippov convex method.
Let

$$\delta(x) = \langle \nabla H(x), F_1(x) \rangle \langle \nabla H(x), F_2(x) \rangle,$$

where $\langle .. \rangle$ denotes the standard scalar product.

***Definition* 1.16.** *We define the crossing set Σ_c as*

$$\Sigma_c = \{x \in \Sigma : \delta(x) > 0\} \subset \Sigma.$$

It is the set of all points $x \in \Sigma$, where the two vectors $f_i(x)$ have non trivial normal components of the same sign. By definition, at these points the orbit of (1.6) crosses Σ.
We define the sliding set Σ_s as the complement to Σ_c in Σ,

$$\Sigma_s = \{x \in \Sigma : \delta(x) \leq 0\} \subset \Sigma.$$

***Remark* 1.2.** *In general we define Escaping region (unstable sliding):*

$$\Sigma_{es} = \{x \in \Sigma : \langle \nabla H(x), f_1(x) \rangle > 0 \; and \; \langle \nabla H(x), f_2(x) \rangle < 0\}.$$

Attractive sliding region (stable sliding):

$$\Sigma_{as} = \{x \in \Sigma : \langle \nabla H(x), f_1(x) \rangle < 0 \; and \; \langle \nabla H(x), f_2(x) \rangle > 0\}.$$

Then, the sliding set

$$\begin{aligned} &\Sigma_s = \{x \in \Sigma : \delta(x) \leqslant 0\} \\ &= \{x \in \Sigma : \langle \nabla H(x), f_1(x) \rangle > 0 \; and \; \langle \nabla H(x), f_2(x) \rangle < 0\} \\ &\cup \{x \in \Sigma : \langle \nabla H(x), f_1(x) \rangle < 0 \; and \; \langle \nabla H(x), f_2(x) \rangle > 0\} \\ &\cup \{x \in \Sigma : \langle \nabla H(x), f_1(x) \rangle \langle \nabla H(x), f_2(x) \rangle = 0\}. \end{aligned}$$

1.13 Filippov method

Within the sliding set, the Filippov method can be used to construct solutions, to be considered as extensions for solutions of (1.6). Such a method consists in defining a new vector field computed from an adequate convex combination $g(x)$ of the two original vector fields $f_i(x)$ to each non singular sliding point $x \in \Sigma_s$, namely

$$g(x) = \lambda f_1(x) + (1 - \lambda) f_2(x),$$

where for each $x \in \Sigma$ the value of λ is selected such that $\langle \nabla H(x), f_2(x) - f_1(x) \rangle \neq 0$ A simple computation shows that

$$\lambda = \lambda(x) = \frac{\langle \nabla H(x), f_2(x) \rangle}{\langle \nabla H(x), f_2(x) - f_1(x) \rangle},$$

provided the above denominator does not vanish and then, by using the definition of Σ_s, one concludes that

$$0 \leqslant \lambda(x) \leqslant 1.$$

Therefore, we have a explicit definition for the sliding vector field, namely

$$g(x) = \frac{\langle \nabla H(x), f_1(x) \rangle f_2(x) - \langle \nabla H(x), f_2(x) \rangle f_1(x)}{\langle \nabla H(x), f_2(x) - f_1(x) \rangle} \,. \tag{1.7}$$

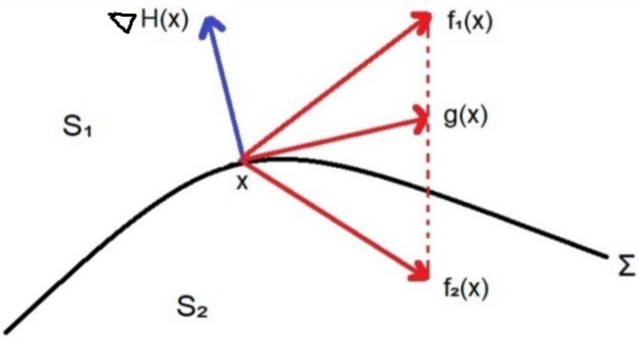

Figure 1.6: Filippov method

Chapter 2

Integrability of quadratic differential systems $\mathbb{R}^2$

In this chapter we shall work with polynomial differential systems

$$\dot{x} = P(x, y), \;\; \dot{y} = Q(x, y), \tag{2.1}$$

where P and Q are polynomials of degree 2, called simply *quadratic systems*.

Many natural phenomena in various branches of the sciences are modelized using quadratic systems. We can find in the literature more than one thousand published papers studying the quadratic systems.

Let U be an open and dense subset of $\mathbb{R}^2$, an *invariant* of system (1) in U is a non-constant C^1 function $I : U \times \mathbb{R} \to \mathbb{R}$ depending on t such that $I(x(t), y(t), t)$ is constant on all the solution curves $(x(t), y(t))$ of system (1.6.1) contained in U, i.e.

$$\frac{dI}{dt} = \frac{\partial I}{\partial x}P + \frac{\partial I}{\partial x}Q + \frac{\partial I}{\partial t} = 0, \tag{2.2}$$

for all $(x, y) \in U$.

The objective of this work is to classify all quadratic systems

$$\begin{aligned} \dot{x} &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2, \\ \dot{y} &= b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2, \end{aligned} \tag{2.3}$$

having invariants of the form

$$I(x, y, t) = ax^2 + bxy + cy^2 + dx + ey + c_1 t, \tag{2.4}$$

with $c_1 \neq 0$.

We note that many different classes of quadratic systems have been classified as the structurally stable, their centers, their isochronous centers, their Hopf bifurcations, their Lotka-Volterra, their Bernoulli, their Abel, their quadratic-linear, their with a unique finite singularity, their having a polynomial first integral, their having a Hamiltonian first integral, their homogeneous, ..., see [2]. But there are few works on the quadratic systems having invariants, see [5, 23, 24, 26, 27] and all thiese invariants are Darboux invariants, i.e. invariants of the form $f(x, y)e^{st}$ with $s \in \mathbb{R} \setminus \{0\}$. As far as we know this is the first time that quadratic systems having an invariant different from a Darboux invariant are studied.

If the function $I(x, y, t) = dx + ey + ax^2 + bxy + cy^2 + c_1 t$ is an invariant of system (2.3), we must verify that the equation (2.2). Thus the following polynomial must be the zero polynomial

$$
\begin{aligned}
&a_0 d + b_0 e + c_1 + (a_1 d + 2aa_0 + b_1 e + bb_0)x + (a_0 b + a_2 d + 2b_0 c + b_2 e)y+\\
&(a_3 d + 2aa_1 + b_3 e + bb_1)x^2 + (a_1 b + a_4 d + 2aa_2 + 2b_1 c + b_4 e + bb_2)xy+\\
&+(a_2 b + a_5 d + 2b_2 c + b_5 e)y^2 + (2aa_3 + bb_3)x^3+\\
&(a_3 b + 2aa_4 + 2b_3 c + bb_4)x^2 y + (a_4 b + 2aa_5 + 2b_4 c + bb_5)xy^2 + (a_5 b + 2b_5 c)y^3.
\end{aligned}
\tag{2.5}
$$

Therefore we must solve the following algebraic system

$$
\begin{aligned}
&a_0 d + b_0 e + c_1 = 0,\\
&a_1 d + 2aa_0 + b_1 e + bb_0 = 0,\\
&a_0 b + a_2 d + 2b_0 c + b_2 e = 0,\\
&a_3 d + 2aa_1 + b_3 e + bb_1 = 0,\\
&a_1 b + a_4 d + 2aa_2 + 2b_1 c + b_4 e + bb_2 = 0,\\
&a_2 b + a_5 d + 2b_2 c + b_5 e = 0,\\
&2aa_3 + bb_3 = 0,\\
&a_3 b + 2aa_4 + 2b_3 c + bb_4 = 0,\\
&a_4 b + 2aa_5 + 2b_4 c + bb_5 = 0,\\
&a_5 b + 2b_5 c = 0.
\end{aligned}
\tag{2.6}
$$

With the help of the Mathematica software we solved this algebraic system.

Then, the quadratic systems (2.3) admitting an invariant of the form (2.4) are one of the following **23** families of quadratic systems (set of independent solutions with $c_1 \neq 0$):

$$
\begin{aligned}
&\dot{x} = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,\\
&\dot{y} = b_0 - \frac{d}{e}(a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2),\\
&I = dx + ey - (da_0 + eb_0)t,
\end{aligned}
\tag{2.7}
$$

where $e \neq 0$. Sets of solutions yield this system are

$$\begin{aligned}
s_1 = \;& \left\{ b_1 = -\frac{d}{e}a_1,\ b_2 = -\frac{d}{e}a_2,\ b_3 = -\frac{d}{e}a_3,\ b_4 = -\frac{d}{e}a_4,\ b_5 = -\frac{d}{e}a_5,\ a = 0,\right.\\
& \left. b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0)\right\},\\
s_2 = \;& \left\{ a_2 = 0,\ b_1 = -\frac{d}{e}a_1,\ b_2 = 0,\ b_3 = -\frac{d}{e}a_3,\ b_4 = -\frac{d}{e}a_4,\ b_5 = -\frac{d}{e}a_5,\right.\\
& \left. a = 0,\ b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0)\right\},\\
s_3 = \;& \{b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c = 0,\ d = 0,\\
& c_1 = -eb_0\},\\
s_4 = \;& \left\{ a_2 = 0,\ a_4 = 0,\ b_1 = -\frac{d}{e}a_1,\ b_2 = 0,\ b_3 = -\frac{d}{e}a_3,\ b_4 = 0,\ b_5 = -\frac{d}{e}a_5,\right.\\
& \left. a = 0,\ b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0)\right\},\\
s_5 = \;& \left\{ a_3 = 0,\ b_1 = -\frac{d}{e}a_1,\ b_2 = -\frac{d}{e}a_2,\ b_3 = 0,\ b_4 = -\frac{d}{e}a_4,\ b_5 = -\frac{d}{e}a_5,\right.\\
& \left. a = 0,\ b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0)\right\},\\
s_6 = \;& \left\{ a_2 = 0,\ a_3 = 0,\ b_1 = -\frac{d}{e}a_1,\ b_2 = 0,\ b_3 = 0,\ b_4 = -\frac{d}{e}a_4,\ b_5 = -\frac{d}{e}a_5,\right.\\
& \left. a = 0,\ b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0),\right\}\\
s_7 = \;& \{a_3 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c = 0,\\
& d = 0,\ c_1 = -eb_0\},\\
s_8 = \;& \left\{ a_1 = 0,\ a_3 = 0,\ a_4 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = -\frac{d}{e}a_5,\right.\\
& \left. a = 0,\ b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0)\right\},\\
s_9 = \;& \{b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c = 0,\ d = 0,\\
& c_1 = -eb_0\},\\
s_{10} = \;& \{\ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a = 0,\ b = 0,\ c = 0,\ e = 0,\\
& c_1 = -da_0\},\\
s_{14} = \;& \{a_3 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\\
& c = 0,\ d = 0,\ c_1 = -eb_0\},\\
s_{17} = \;& \left\{ a_2 = -\frac{e}{d}b_2,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_1 = -\frac{d}{e}a_1,\ b_3 = 0,\ b_4 = 0,\right.\\
& \left. b_5 = 0,\ a = 0,\ b = 0,\ c = 0,\ c_1 = -(da_0 + eb_0)\right\},\\
s_{18} = \;& \{a_3 = 0,\ a_4 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,
\end{aligned}$$

$$\begin{aligned}
& c = 0,\ d = 0,\ c_1 = -eb_0\},\\
s_{19} = \;& \{b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c = 0,\ d = 0,\\
& c_1 = -eb_0\}.
\end{aligned}$$

$$
\begin{aligned}
\dot{x} = &\ a_0 + a_1 x + \frac{1}{b^2}(2bca_1 + bea_4 - 4cea_3)y + a_3 x^2 + a_4 xy + \frac{2}{b^2}c(ba_4 - 2ca_3)y^2,\\
\dot{y} = &\ \frac{1}{2b^2c}((ba_1 - ea_3)(be - 2cd) - b^3 a_0) - \frac{1}{2bc}(b^2 a_1 - bea_3 + 2cda_3)x +\\
&\ \frac{1}{b^2}(bea_3 + 2cda_3 - b^2 a_1 - bda_4)y - \frac{2}{b}aa_3 x^2 + \frac{1}{b^2}((4ac - b^2)a_3 - 2aba_4)xy +\\
&\ \frac{1}{b}(2ca_3 - ba_4)y^2,\\
I = &\ dx + ey + \frac{b^2}{4c}x^2 + bxy + cy^2 - \frac{1}{b^3}(bd - 2ae)(b^2 a_0 + e(ea_3 - ba_1))t,
\end{aligned}
\tag{2.8}
$$

with $cb \neq 0$. This system is given by the set of solutions

$$
\begin{aligned}
s_{11} = &\ \Big\{ a_2 = \frac{1}{b^2}(2bca_1 + bea_4 - 4cea_3),\ a_5 = \frac{2}{b^2}c(ba_4 - 2ca_3),\ b_3 = -\frac{b}{2c}a_3,\\
&\ b_0 = \frac{1}{2b^2c}((ba_1 - ea_3)(be - 2cd) - b^3 a_0),\ b_5 = \frac{1}{b}(2ca_3 - ba_4),\\
&\ b_1 = -\frac{1}{2bc}(b^2 a_1 - bea_3 + 2cda_3),\ b_4 = -\frac{b}{2c}a_4,\\
&\ a = \frac{b^2}{4c},\ b_2 = \frac{1}{b^2}(bea_3 + 2cda_3 - b^2 a_1 - bda_4),\\
&\ c_1 = \frac{1}{2b^2c}(be - 2cd)(b^2 a_0 + e(ea_3 - ba_1)) \Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} = &\ a_0 + \frac{1}{b(bd - 2ae)}(db(2aa_2 + bb_2) + (ea_3 - bb_2)(bd - 2ae))x + a_2 y + a_3 x^2 +\\
&\ \frac{b}{2a(2ae - bd)}(2a(2aa_2 + bb_2) + (2ae - bd)a_3)xy - \frac{2aa_2 + bb_2}{2a(bd - 2ae)}b^2 y^2,\\
\dot{y} = &\ -\frac{2}{b^2}a(ba_0 + a_2 d + eb_2) + \frac{1}{b(bd - 2ae)}(4a^2(da_2 + eb_2) + bd^2 a_3 - 2adea_3)x +\\
&\ b_2 y - \frac{2}{b}aa_3 x^2 + \frac{1}{bd - 2ae}(2a(2aa_2 + bb_2) - (bd - 2ae)a_3)xy + \frac{2aa_2 + bb_2}{bd - 2ae}by^2,\\
I = &\ dx + ey + ax^2 + bxy + \frac{b^2}{4a}y^2 - \frac{1}{b^2}((bd - 2ae)ba_0 - 2ae(da_2 + eb_2))t,
\end{aligned}
\tag{2.9}
$$

where $ab(bd - 2ae) \neq 0$. This system is given by the sets

$$
\begin{aligned}
s_{12} = \ & \Big\{ a_1 = \frac{1}{b^2 d - 2abe}(bdea_3 + 2abda_2 + 2abeb_2 - 2ae^2 a_3),\ b_3 = -\frac{2}{b} a a_3, \\
& a_4 = \frac{b}{2a(2ae - bd)}(4a^2 a_2 - bda_3 + 2a(ea_3 + bb_2)), \\
& b_0 = -\frac{2}{b^2}(ba_0 + da_2 + eb_2)a,\ b_4 = \frac{1}{2ae - bd}(bda_3 - 4a^2 a_2 - 2a(ea_3 + bb_2)), \\
& b_1 = \frac{1}{b(bd - 2ae)}(-4a^2(da_2 + eb_2) + 2adea_3 - bd^2 a_3), \\
& a_5 = \frac{b^2}{2a(2ae - bd)}(2aa_2 + bb_2),\ c = \frac{b^2}{4a},\ b_5 = \frac{b}{bd - 2ae}(2aa_2 + bb_2), \\
& c_1 = \frac{1}{b^2}((2ae - bd)ba_0 + 2(da_2 + eb_2)ae) \Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} &= a_0 + a_1 x + a_2 y + \frac{2}{e} c a_1 xy - \frac{2}{d} c b_2 y^2, \\
\dot{y} &= -\frac{1}{2c}(a_2 d + b_2 e) - \frac{d}{e} a_1 x + b_2 y, \\
I &= dx + ey + cy^2 + \frac{1}{2c}(e(da_2 + eb_2) - 2cda_0)t,
\end{aligned}
\tag{2.10}
$$

with $ced \neq 0$. Sets of solutions provide this system are

$$
\begin{aligned}
s_{13} = \ & \Big\{ a_3 = 0,\ a_4 = \frac{2}{e} c a_1,\ a_5 = -\frac{2}{d} c b_2,\ b_1 = -\frac{d}{e} a_1,\ b_0 = -\frac{1}{2c}(da_2 + eb_2), \\
& b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c_1 = \frac{e}{2c}(da_2 + eb_2) - da_0 \Big\}, \\
s_{15} = \ & \Big\{ a_1 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = -\frac{2}{d} b_2 c,\ b_0 = -\frac{1}{2c}(da_2 + eb_2), \\
& b_1 = -\frac{d}{e} a_1,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c_1 = -(da_0 + eb_0) \Big\}, \\
s_{16} = \ & \Big\{ a_3 = 0,\ a_4 = \frac{2}{e} c a_1,\ a_5 = 0,\ b_0 = -\frac{d}{2c} a_2,\ b_2 = 0,\ b_1 = -\frac{d}{e} a_1,\ b_3 = 0, \\
& b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c_1 = \frac{d}{2c} e a_2 - da_0 \Big\}, \\
s_{32} = \ & \Big\{ a_1 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_0 = -\frac{d}{2c} a_2,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0, \\
& b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ c_1 = da_0 - \frac{d}{2c} e a_2, \Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} = \ & a_0 - \frac{e}{2a} b_3 x + a_2 y, \\
\dot{y} = \ & b_0 - \frac{1}{2aea_2}(4(aa_0)^2 + e^2 b_2 b_3)x + b_2 y - \frac{2}{e} a a_2 xy + b_3 x^2, \\
I = \ & -\frac{e}{a_2} b_2 x + ey + ax^2 + (\frac{e}{a_2} a_0 b_2 - eb_0)t,
\end{aligned}
\tag{2.11}
$$

where $a_2ae \neq 0$. This system is given by the set

$$s_{20} = \left\{a_1 = -\frac{e}{2a}b_3,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_1 = -\frac{1}{2aea_2}(e^2b_3b_2 + 4a^2a_2a_0),\right.$$
$$\left. b_4 = -\frac{2}{e}aa_2,\ b_5 = 0,\ b = 0,\ c = 0,\ d = -\frac{e}{a_2}b_2,\ c_1 = \frac{e}{a_2}a_0b_2 - eb_0\right\}.$$

$$\begin{aligned} \dot{x} &= a_0 - \frac{e}{2a}b_3x, \\ \dot{y} &= b_0 + \frac{1}{2ae}(deb_3 - 4a^2a_0)x + b_3x^2, \\ I &= dx + ey + ax^2 - (da_0 + eb_0)t, \end{aligned} \tag{2.12}$$

with $ae \neq 0$. The set give this system is

$$s_{21} = \left\{a_1 = -\frac{e}{2a}b_3,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_1 = \frac{1}{2ae}(db_3 - 4a^2a_0),\right.$$
$$\left. b_2 = 0,\ b_4 = 0,\ b_5 = 0,\ c_1 = -(da_0 + eb_0)\right\}.$$

$$\begin{aligned} \dot{x} =&\ a_0 + \frac{1}{bd - 2ae}(2a(da_2 + eb_2)x + (bd - 2ae)a_2y - b(2aa_2 + bb_2)xy- \\ &\frac{b^2}{2a}(2aa_2 + bb_2)y^2), \\ \dot{y} =&\ -\frac{2}{b^2}(ba_0 + da_2 + eb_2)a - \frac{1}{bd - 2ae}(\frac{4}{b}a^2(da_2 + eb_2)x + (bd - 2ae)b_2y+ \\ &2a(2aa_2 + bb_2)xy + b(2aa_2 + bb_2)y^2), \\ I =&\ dx + ey + ax^2 + bxy + \frac{b^2}{4a}y^2 + \frac{1}{b^2}(b(2ae - bd)a_0 + 2ae(da_2 + eb_2))t, \end{aligned} \tag{2.13}$$

where $ab(bd - 2ae) \neq 0$. This system is given by the sets

$$s_{22} = \left\{ a_1 = \frac{2a}{bd - 2ae}(da_2 + eb_2),\ a_3 = 0,\ a_4 = -\frac{b}{bd - 2ae}(2aa_2 + bb_2),\right.$$
$$a_5 = -\frac{b^2}{2a(bd - 2ae)}(2aa_2 + bb_2),\ b_0 = -\frac{2}{b^2}(ba_0 + da_2 + eb_2)a,\ b_3 = 0,$$
$$b_1 = -\frac{4}{b(bd - 2ae)}a^2(da_2 + eb_2),\ b_4 = \frac{2}{bd - 2ae}a(2aa_2 + bb_2),\ c = \frac{b^2}{4a},$$
$$\left. b_5 = \frac{b}{bd - 2ae}(2aa_2 + bb_2),\ c_1 = \frac{1}{b^2}(b(2ae - bd)a_0 + 2ae(da_2 + eb_2))\right\},$$
$$s_{23} = \left\{a_1 = \frac{2}{b}aa_2,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_0 = \frac{1}{b^3}(4a^2ea_2 - 2ab(ba_0 + da_2)),\right.$$
$$b_1 = -\frac{4}{b^2}a^2a_2,\ b_2 = -\frac{2}{b}aa_2,\ b_3 = 0,\ b_4 = 0,\ c = \frac{b^2}{4a},\ b_5 = 0,$$
$$\left. c_1 = \frac{1}{b^3}(2ae - bd)(b^2a_0 - 2aea_2)\right\}.$$

$$
\begin{aligned}
\dot{x} = \ & a_0 + \frac{2}{b}aa_2x + a_2y + a_3x^2 + \frac{1}{2ad}(bda_3 - 4a^2a_2 - 2bab_2)xy - \\
& \frac{b}{2ad}(2aa_2 + bb_2)y^2, \\
\dot{y} = \ & -\frac{2}{b^2}(ba_0 + da_2)a - \frac{1}{b^2}(4a^2a_2 + bda_3)x + b_2y - \frac{2}{b}aa_3x^2 + \\
& \frac{1}{bd}(4a^2a_2 - bda_3 + 2abb_2)xy + \frac{1}{d}(2aa_2 + bb_2)y^2, \\
I = \ & dx + ax^2 + bxy + \frac{b^2}{4a}y^2 - da_0t,
\end{aligned}
\tag{2.14}
$$

with $bad \neq 0$. The set provide this system is

$$
\begin{aligned}
s_{24} = \ & \Big\{a_1 = \frac{2}{b}aa_2,\ a_4 = \frac{1}{2ad}(ba_3 - 2aa_2 - bb_2),\ a_5 = -\frac{b}{2ad}(2aa_2 + bb_2), \\
& b_0 = -\frac{2}{b^2}(ba_0 + da_2)a,\ b_1 = -\frac{1}{b^2}(4a^2a_2 + bda_3),\ b_3 = -\frac{2}{b}aa_3, \\
& b_4 = \frac{1}{bd}(4a^2a_2 + 2abb_2 - bda_3),\ b_5 = \frac{1}{d}(2aa_2 + bb_2),\ c = \frac{b^2}{4a},\ e = 0, \\
& c_1 = -da_0\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} = \ & a_0 + \frac{2}{b}aa_2x + a_2y + a_3x^2 + \frac{b}{2a}a_3xy, \\
\dot{y} = \ & -\frac{2}{b^2}(ba_0 + da_2)a - \frac{1}{b^2}(4a^2a_2 + bda_3)x - \frac{2}{b}aa_2y - \frac{2}{b}aa_3x^2 - a_3xy, \\
I = \ & dx + ax^2 + bxy + \frac{b^2}{4a}y^2 - da_0t,
\end{aligned}
\tag{2.15}
$$

where $ba \neq 0$. This system is given by the sets

$$
\begin{aligned}
s_{25} = \ & \Big\{a_1 = \frac{2}{b}aa_2,\ a_4 = \frac{b}{2a}a_3,\ a_5 = 0,\ b_0 = -\frac{2}{b^2}(ba_0 + da_2)a, \\
& b_1 = -\frac{1}{b^2}(4a^2a_2 + bda_3),\ b_2 = -\frac{2}{b}aa_2,\ b_3 = -\frac{2}{b}aa_3,\ b_4 = -a_3,\ b_5 = 0, \\
& c = \frac{b^2}{4a},\ e = 0,\ c_1 = -da_0,\Big\}, \\
s_{26} = \ & \Big\{a_1 = \frac{2}{b}aa_2,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_0 = -\frac{2}{b^2}(ba_0 + da_2)a, \\
& b_1 = -\frac{4}{b^2}a^2a_2,\ b_2 = -\frac{2}{b}aa_2,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ c = \frac{b^2}{4a},\ e = 0, \\
& c_1 = -da_0\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} = {} & a_0 + \frac{b}{2c}a_2x + a_2y + a_3x^2 + a_4xy + \frac{2}{b^2}(ba_4 - 2ca_3)cy^2, \\
\dot{y} = {} & -\frac{1}{2c}((ba_0 + da_2) + \frac{1}{2bc}(b^3a_2 + 4c^2da_3)x + \frac{1}{b^2}(2(ba_4 - 2ca_3)dc + b^3a_2)y + \\
& ba_3x^2 + ba_4xy) + \frac{1}{b}(2ca_3 - ba_4)y^2, \\
I = {} & dx + \frac{b^2}{4c}x^2 + bxy + cy^2 - da_0t,
\end{aligned}
\tag{2.16}
$$

where $cb \neq 0$. This system is given by the sets

$$
\begin{aligned}
s_{27} = {} & \Big\{a_1 = \frac{b}{2c}a_2,\ a_5 = \frac{2}{b^2}(ba_4 - 2ca_3)c,\ b_0 = -\frac{1}{2c}(ba_0 + da_2), \\
& b_1 = -\frac{1}{4c^2}(b^3a_2 + 4c^2da_3),\ b_3 = -\frac{b}{2c}a_3,\ b_4 = -\frac{b}{2c}a_4,\ b_5 = \frac{2}{b}ca_3 - a_4, \\
& b_2 = -\frac{d}{b^2}(ba_4 - 2ca_3) - \frac{b}{2c}a_2,\ a = \frac{b^2}{4c},\ e = 0,\ c_1 = -da_0\Big\}, \\
s_{28} = {} & \Big\{a_1 = 0,\ a_2 = 0,\ a_5 = \frac{2}{b^2}(ba_4 - 2ca_3)c,\ b_0 = -\frac{b}{2c}a_0,\ b_1 = -\frac{d}{b}a_3, \\
& b_2 = -\frac{d}{b^2}(ba_4 - 2ca_3),\ b_3 = -\frac{b}{2c}a_3,\ b_4 = -\frac{b}{2c}a_4,\ b_5 = \frac{2}{b}ca_3 - a_4, \\
& a = \frac{b^2}{4c},\ e = 0,\ c_1 = -da_0\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} &= a_0 + a_2y + a_4xy + a_5y^2, \\
\dot{y} &= -\frac{d}{2c}(a_2 + a_4x + a_5y), \\
I &= dx + cy^2 - da_0t,
\end{aligned}
\tag{2.17}
$$

where $c \neq 0$. This system is given by the sets

$$
\begin{aligned}
s_{29} = {} & \Big\{a_1 = 0,\ a_3 = 0,\ b_0 = -\frac{d}{2c}a_2,\ b_1 = -\frac{d}{2c}a_4,\ b_2 = -\frac{d}{2c}a_5,\ b_3 = 0, \\
& b_4 = 0,\ b_5 = 0,\ a = 0,\ b = 0,\ e = 0,\ c_1 = -da_0\}, \\
s_{30} = {} & \{a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0, \\
& a = 0,\ b = 0,\ c = 0,\ e = 0,\ c_1 = -da_0\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} = {} & a_0 + a_2y + \frac{b}{e}a_2xy + \frac{b^2}{2e}a_2y^2, \\
\dot{y} = {} & -\frac{2}{b}aa_0 - \frac{a_2}{e}(dy + 2axy + by^2), \\
I = {} & dx + ey + ax^2 + bxy + \frac{b^2}{4a}y^2 - \frac{a_0}{b}(db - 2ae)t,
\end{aligned}
\tag{2.18}
$$

where $eab \neq 0$. The sets of solutions provide this system are

$$
\begin{aligned}
s_{33} = \ & \Big\{ a_1 = 0,\ a_3 = 0,\ a_4 = \frac{b}{e}a_2,\ a_5 = \frac{b^2}{2ae}a_2,\ b_0 = -\frac{2}{b}aa_0,\ b_1 = 0,\ b_3 = 0, \\
& b_2 = -\frac{d}{e}a_2,\ b_4 = -\frac{2}{e}aa_2,\ b_5 = -\frac{b}{e}a_2,\ c = \frac{b^2}{4a},\ c_1 = a_0(\frac{2}{b}ae - d) \Big\}, \\
s_{34} = \ & \Big\{ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_0 = -\frac{2}{b}aa_0,\ b_1 = 0,\ b_2 = 0, \\
& b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ c = \frac{b^2}{4a},\ c_1 = a_0(\frac{2}{b}ae - d) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
s_{31} = \ & \Big\{ a_1 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = -\frac{2}{d}cb_2,\ b_1 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0, \\
& a = 0,\ b = 0,\ e = -\frac{1}{b_2}(da_2 + 2cb_0),\ c_1 = \frac{b_0}{b_2}(da_2 + 2cb_0) - da_0 \Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} = \ & a_0 - \frac{b}{2ad}b_2(2axy + by^2), \\
\dot{y} = \ & -\frac{2}{b}aa_0 + \frac{b_2}{d}(dy + 2axy + by^2), \\
I = \ & dx + ax^2 + bxy + \frac{b^2}{4a}y^2 - da_0t,
\end{aligned}
\tag{2.19}
$$

where $dab \neq 0$. The set of solutions provide this system are

$$
\begin{aligned}
s_{35} = \ & \Big\{ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = -\frac{b}{d}b_2,\ a_5 = -\frac{b^2}{2ad}b_2,\ b_0 = -\frac{2}{b}aa_0, \\
& b_1 = 0,\ b_3 = 0,\ b_4 = \frac{2}{d}ab_2,\ b_5 = \frac{b}{d}b_2,\ c = \frac{b^2}{4a},\ e = 0,\ c_1 = -da_0 \Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} = \ & a_0 + \frac{2}{b}aa_2x + a_2y - \frac{1}{d}(2aa_2 + bb_2)xy - \frac{b}{2ad}(2aa_2 + bb_2)y^2, \\
\dot{y} = \ & -\frac{2}{b^2}(ba_0 + da_2)a - \frac{4}{b^2}a^2a_2x + \frac{2}{bd}(2aa_2 + bb_2)axy + \frac{1}{d}(2aa_2 + bb_2)y^2 + \\
& b_2y, \\
I = \ & dx + ax^2 + bxy + \frac{b^2}{4a}y^2 - da_0t,
\end{aligned}
\tag{2.20}
$$

where $bda \neq 0$. The sets of solutions provide this system are

$$
\begin{aligned}
s_{36} = \ & \Big\{ a_1 = \frac{2}{b}aa_2,\ a_3 = 0,\ a_4 = -\frac{1}{d}(2aa_2 + bb_2),\ a_5 = -\frac{b}{2ad}(2aa_2 + bb_2), \\
& b_0 = -\frac{2}{b^2}a(ba_0 + da_2),\ b_1 = -\frac{4}{b^2}a^2a_2,\ b_3 = 0,\ b_4 = \frac{2}{bd}a(2aa_2 + bb_2), \\
& b_5 = \frac{1}{d}(2aa_2 + bb_2),\ e = 0,\ c = \frac{b^2}{4a},\ c_1 = -da_0 \Big\},
\end{aligned}
$$

$$
\begin{aligned}
s_{37} = \ & \Big\{ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = -\frac{b}{d} b_2,\ a_5 = -\frac{b^2}{2ad} b_2,\ b_0 = -\frac{2}{b} a a_0, \\
& b_1 = 0,\ b_3 = 0,\ b_4 = \frac{2}{d} a b_2,\ b_5 = \frac{b}{d} b_2,\ c = \frac{b^2}{4a},\ e = 0,\ c_1 = -d a_0 \Big\}, \\
s_{38} = \ & \Big\{ a_1 = \frac{2}{b} a a_2,\ a_4 = 0,\ a_3 = 0,\ a_5 = 0,\ b_0 = -\frac{2}{b^2} a (b a_0 + d a_2), \\
& b_1 = -\frac{4}{b^2} a^2 a_2,\ b_2 = -\frac{2}{b} a a_2,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ c = \frac{b^2}{4a}, e = 0, \\
& c_1 = -d a_0 \Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} = \ & a_0 + a_3 x^2 + \frac{b}{2ad} (d a_3 - 2 a b_2) x y - \frac{b^2}{2ad} b_2 y^2, \\
\dot{y} = \ & -\frac{2}{b} a a_0 - \frac{d}{b} a_3 x + b_2 y - \frac{2}{b} a a_3 x^2 - \frac{1}{d} (d a_3 - 2 a b_2) x y + \frac{b}{d} b_2 y^2, \\
I = \ & d x + a x^2 + b x y + \frac{b^2}{4a} y^2 - d a_0 t,
\end{aligned}
\tag{2.21}
$$

where $adb \neq 0$. The sets of solutions provide this system are

$$
\begin{aligned}
s_{39} = \ & \Big\{ a_1 = 0,\ a_2 = 0,\ a_4 = \frac{b}{2ad} (d a_3 - 2 a b_2),\ a_5 = -\frac{b^2}{2ad} b_2,\ b_0 = -\frac{2}{b} a a_0, \\
& b_1 = -\frac{d}{b} a_3,\ b_3 = -\frac{2}{b} a a_3,\ b_4 = \frac{2}{d} a b_2 - a_3,\ b_5 = \frac{b}{d} b_2,\ c = \frac{b^2}{4a}, e = 0, \\
& c_1 = -d a_0 \Big\}, \\
s_{40} = \ & \Big\{ a_1 = 0,\ a_2 = 0,\ a_4 = \frac{b}{2a} a_3,\ a_5 = 0,\ b_0 = -\frac{2}{b} a a_0,\ b_2 = 0, \\
& b_1 = -\frac{d}{b} a_3,\ b_3 = -\frac{2}{b} a a_3,\ b_4 = -a_3,\ b_5 = 0,\ c = \frac{b^2}{4a},\ e = 0, c_1 = -d a_0 \Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} = \ & a_0 + a_1 x + \frac{2}{b} c a_1 y + a_4 x y + \frac{2}{b} c a_4 y^2, \\
\dot{y} = \ & -\frac{1}{2cb} (b^2 a_0 + 2 c d a_1) - \frac{b}{2c} a_1 x - \frac{1}{b} (b a_1 + d a_4) y - \frac{b}{2c} a_4 x y - a_4 y^2, \\
I = \ & d x + \frac{b^2}{4c} x^2 + b x y + c y^2 - d a_0 t,
\end{aligned}
\tag{2.22}
$$

where $bc \neq 0$. This system is given by the set

$$
\begin{aligned}
s_{41} = \ & \Big\{ a_2 = \frac{2}{b} c a_1,\ a_3 = 0,\ a_5 = \frac{2}{b} c a_4,\ b_0 = -\frac{1}{2cb} (b^2 a_0 + 2 c d a_1),\ b_1 = -\frac{b}{2c} a_1, \\
& b_2 = -\frac{1}{b} (b a_1 + d a_4),\ b_3 = 0,\ b_4 = -\frac{b}{2c} a_4,\ b_5 = -a_4,\ \ a = \frac{b^2}{4c},\ e = 0, \\
& c_1 = -d a_0 \Big\}.
\end{aligned}
$$

$$\begin{aligned}
\dot{x} = &\; a_0 + a_3x^2 + a_4xy + \frac{2}{b^2}(ba_4 - 2ca_3)y^2, \\
\dot{y} = &\; -\frac{b}{2c}a_0 - \frac{d}{b}a_3x - \frac{d}{b^2}(ba_4 - 2ca_3)y - \frac{b}{2c}a_3x^2 - \frac{b}{2c}a_4xy + \frac{1}{b}(2ca_3 - ba_4)y^2, \\
I = &\; dx + \frac{b^2}{4c}x^2 + bxy + cy^2 - da_0t,
\end{aligned} \tag{2.23}$$

where $bc \neq 0$. Set of solutions yield this system are

$$\begin{aligned}
s_{42} = &\; \Big\{a_1 = 0,\ a_2 = 0,\ a_5 = \frac{2}{b^2}(ba_4 - 2ca_3)c,\ b_0 = -\frac{b}{2c}a_0,\ b_1 = -\frac{d}{b}a_3, \\
&\; b_2 = -\frac{d}{b^2}(ba_4 - 2ca_3),\ b_3 = -\frac{b}{2c}a_3,\ b_4 = -\frac{b}{2c}a_4,\ b_5 = \frac{2}{b}ca_3 - a_4, \\
&\; a = \frac{b^2}{4c},\ e = 0,\ c_1 = -da_0\Big\}.
\end{aligned}$$

$$\begin{aligned}
\dot{x} = &\; \frac{bda_2^2}{b^2a_2 + 2cda_4 + 2bcb_2} - \frac{b}{2c}a_2x + a_2y + \frac{b}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2))x^2 + \\
&\; a_4xy - \frac{b}{d}(ba_2 + 2cb_2)y^2, \\
\dot{y} = &\; -\frac{d(b^2a_2 + c(da_4 + bb_2))}{c(b^2a_2 + 2c(da_4 + bb_2))}a_2 - \frac{1}{2c^2}(b^2a_2 + cda_4 + bcb_2)x + \\
&\; b_2y - \frac{b^2}{8c^3d}(a_2b^2 + 2c(da_4 + bb_2))x^2 - \frac{b}{2c}a_4xy + \\
&\; \frac{1}{2cd}(b^2a_2 + 2c(da_4 + bb_2) - 2cda_4)y^2, \\
I = &\; dx + \frac{b^2}{4c}x^2 + bxy + cy^2 - \frac{bd^2a_2^2}{b^2a_2 + 2cda_4 + 2bcb_2}t,
\end{aligned} \tag{2.24}$$

where $cd(b^2a_2 + 2cda_4 + 2bcb_2) \neq 0$. Set of solutions yield this system are

$$\begin{aligned}
s_{43} = &\; \Big\{a_0 = \frac{bda_2^2}{b^2a_2 + 2c(da_4 + bb_2)},\ a_3 = \frac{b}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2)), \\
&\; a_1 = -\frac{b}{2c}a_2,\ \ a_5 = -\frac{1}{d}(ba_2 + 2cb_2),\ b_0 = d\Big(\frac{a_2}{2c} - \frac{2aa_2^2}{b^2a_2 + 2c(da_4 + bb_2)}\Big), \\
&\; b_1 = \frac{1}{2c}\Big(2aa_2 - \frac{1}{2c}(b^2a_2 + 2c(da_4 + bb_2))\Big),\ a = \frac{b^2}{4c},\ e = 0, \\
&\; b_4 = \frac{b}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2))\Big(\frac{4}{b^2}ac - 1\Big) - \frac{2}{b}aa_4,\ c_1 = -da_0, \\
&\; b_3 = -\frac{2}{4c^2d}(b^2a_2 + 2c(da_4 + bb_2))a,\ b_5 = \frac{1}{2cd}(b^2a_2 + 2c(da_4 + bb_2)) - a_4\Big\}.
\end{aligned}$$

$$
\begin{aligned}
\dot{x} = &\ a_0 + a_4 xy + \frac{2}{b} c a_4 y^2,\\
\dot{y} = &\ -\frac{b}{2c} a_0 - a_4(\frac{d}{b} y + \frac{b}{2c} xy + y^2),\\
I = &\ dx + \frac{b^2}{4c} x^2 + bxy + cy^2 - da_0 t,
\end{aligned} \tag{2.25}
$$

where $bc \neq 0$. This system is given by the sets

$$
\begin{aligned}
s_{44} = &\ \Big\{ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_5 = \frac{2}{b} c a_4,\ b_0 = -\frac{b}{2c} a_0,\ b_1 = 0,\ b_2 = -\frac{d}{b} a_4,\\
&\ b_3 = 0,\ b_4 = -\frac{b}{2c} a_4,\ b_5 = -a_4,\ e = 0,\ a = \frac{b^2}{4c},\ c_1 = -da_0 \Big\},\\
s_{49} = &\ \Big\{ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_5 = \frac{2}{b} c a_4,\ b_0 = -\frac{b}{2c} a_0,\ b_1 = 0,\ b_2 = -\frac{d}{b} a_4,\\
&\ b_3 = 0,\ b_4 = -\frac{b}{2c} a_4,\ b_5 = -a_4,\ a = \frac{b^2}{4c},\ e = 0,\ c_1 = -da_0 \Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} = &\ \frac{bda_2^2}{b^2 a_2 + 2cda_4 + 2bcb_2} - \frac{b}{2c} a_2 x + a_2 y + \frac{b}{4c^2 d}(b^2 a_2 + 2c(da_4 + bb_2))x^2 +\\
&\ a_4 xy - \frac{1}{d}(ba_2 + 2cb_2) y^2,\\
\dot{y} = &\ \frac{da_2(da_4 + bb_2)}{b^2 a_2 + 2c(da_4 + bb_2)} - \frac{1}{2c}(da_4 + bb_2)x - \frac{b^2}{8c^3 d}(b^2 a_2 + 2c(da_4 + bb_2))x^2 +\\
&\ b_2 y - \frac{b}{2c} a_4 xy + \frac{1}{2cd}(b^2 a_2 + 2c(da_4 + bb_2) - 2cda_4) y^2,\\
I = &\ dx + \frac{b^2}{4c} x^2 + bxy + cy^2 - \frac{bd^2 a_2^2}{b^2 a_2 + 2cda_4 + 2bcb_2} t,
\end{aligned} \tag{2.26}
$$

where $cd(b^2 a_2 + 2cda_4 + 2bcb_2) \neq 0$. The sets of solutions provide this system are

$$
\begin{aligned}
s_{45} = &\ \Big\{ a_0 = \frac{bda_2^2}{b^2 a_2 + 2dca_4 + 2bcb_2},\ a_1 = \sqrt{a_0}\sqrt{a_3},\ b_4 = (\frac{4}{b^2} ac - 1) a_3 - \frac{2}{b} a a_4,\\
&\ a_3 = \frac{b}{4c^2 d}(b^2 a_2 + 2c(da_4 + bb_2)),\ a_5 = \frac{2}{b^2} c(ba_4 - 2ca_3),\ b_3 = -\frac{2}{b} a a_3,\\
&\ b_0 = -\frac{1}{b}(d\sqrt{a_0}\sqrt{a_3} + 2aa_0),\ b_1 = -\frac{1}{b}(da_3 + 2a\sqrt{a_0}\sqrt{a_3}),\\
&\ b_5 = \frac{2}{b} c a_3 - a_4,\ a = \frac{b^2}{4c},\ e = 0,\ c_1 = -da_0 \Big\},\\
s_{46} = &\ \Big\{ a_1 = \sqrt{a_0}\sqrt{a_3},\ a_2 = 0,\ a_3 = 0,\ a_5 = \frac{2}{b^2}(ba_4 - 2ca_3)c,\\
&\ b_0 = -\frac{1}{b}(d\sqrt{a_0}\sqrt{a_3} + 2aa_0),\ b_1 = -\frac{1}{b}(da_3 + 2a\sqrt{a_0}\sqrt{a_3}),\ b_2 = -\frac{d}{b} a_4,\\
&\ b_3 = -\frac{2}{b} a a_3,\ b_4 = \frac{1}{b^2}(a_3(4ac - b^2) - 2aba_4),\ b_5 = \frac{2}{b} c a_3 - a_4,\\
&\ a = \frac{b^2}{4c},\ e = 0,\ c_1 = -da_0 \Big\}.
\end{aligned}
$$

$$\begin{aligned} \dot{x} &= a_0 - \frac{b}{d}b_2xy - \frac{b^2}{2ad}b_2y^2, \\ \dot{y} &= -\frac{2}{b}aa_0 + b_2y + \frac{2}{d}ab_2xy + \frac{b}{d}b_2y^2, \\ I &= dx + ax^2 + bxy + \frac{b^2}{4a}y^2 - a_0dt, \end{aligned} \tag{2.27}$$

where $dab \neq 0$. The sets of solutions provide this system are

$$\begin{aligned} s_{47} = &\Big\{a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = -\frac{b}{d}b_2,\ a_5 = -\frac{b^2}{2ad}b_2,\ b_0 = -\frac{2}{b}aa_0, \\ &b_1 = 0,\ b_3 = 0,\ b_4 = \frac{2}{d}ab_2,\ b_5 = \frac{b}{d}b_2,\ c = \frac{b^2}{4a},\ e = 0,\ c_1 = -da_0\Big\}, \\ s_{48} = &\Big\{a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ b_0 = -\frac{2}{b}aa_0,\ b_1 = 0, \\ &b_2 = 0,\ b_3 = 0,\ b_4 = 0,\ b_5 = 0,\ c = \frac{b^2}{4a},\ e = 0,\ c_1 = -da_0\Big\}. \end{aligned}$$

2.1 Linear centers

It is well known that the linear differential centers are isochronous and that the general expression of such centers is as follows proved by J. Llibre and M.A. Teixeira work [28].

***Lemma* 2.1.** *A linear differential system having a center can be in the following form written below*

$$\dot{x} = -\beta x - \frac{4\beta^2 + \omega^2}{4\alpha}y + \delta, \qquad \dot{y} = \alpha x + \beta y + \gamma, \tag{2.28}$$

with $\alpha > 0$ and $\omega > 0$.

The linear differential system (2.28) has the first integral

$$H_L(x, y) = 4(\alpha x + \beta y)^2 + 8\alpha(\gamma x - \delta y) + \omega^2 y^2.$$

2.1.1 Quadratic system with first integral $ax^2 + bxy + cy^2$

The aim is to study the periodic orbits of continuous and discontinuous piecewise differential systems formed by the following quadratic system and the linear isochronous center (2.28).

$$\begin{cases} \dot{x} = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2, \\ \dot{y} = b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2, \end{cases} \tag{2.29}$$

we want to integrate this system by the following quadratic function

$$H = ax^2 + bxy + cy^2,$$

so we must verify the algebraic equation $H_x\dot{x} + H_y\dot{y} = 0$, which gives the following sets of solutions

$$\begin{aligned}
&s_{q1} = \left\{a_0 = -\tfrac{bb_0}{2a},\ a_1 = -\tfrac{bb_1}{2a},\ a_3 = -\tfrac{bb_3}{2a},\ a_2 = -\tfrac{bb_2}{2a},\ a_4 = -\tfrac{bb_4}{2a},\ a_5 = -\tfrac{bb_5}{2a},\ c = \tfrac{b^2}{4a}\right\},\\
&s_{q2} = \Big\{\ a_0 = 0,\ b_0 = 0, a_1 = -b_2,\ b_1 = \tfrac{2ab_2}{b},\ a_3 = \tfrac{2ab_5 - bb_4}{b},\ b_3 = \tfrac{2(abb_4 - 2a^2 b_5}{b^2},\\
&a_2 = -\tfrac{2b_2 c}{b},\ a_4 = \tfrac{4ab_5 c - b^2 b_5 - 2bb_4 c}{b^2},\ a_5 = -\tfrac{2b_5 c}{b}\Big\},\\
&s_{q3} = \Big\{\ a_0 = 0,\ b_0 = 0,\ a_1 = 0,\ b_1 = -\tfrac{aa_2}{c},\ a_3 = 0,\ b_3 = -\tfrac{aa_4}{c},\ b_2 = 0,\ b_4 = -\tfrac{aa_5}{c},\\
&b_5 = 0,\ b = 0\},
\end{aligned}$$

where $abc \neq 0$. Therefore, we have three cases.

Case 1: The set of solutions s_{q1} gives the following system

$$\begin{cases} \dot{x_1} = -\frac{bb_0}{2a} - \frac{bb_1}{2a}x - \frac{bb_3}{2a}x^2 - \frac{bb_2}{2a}y - \frac{bb_4}{2a}xy - \frac{bb_5}{2a}y^2, \\ \dot{y_1} = b_0 + b_1 x + b_3 x^2 + b_2 y + b_4 xy + b_5 y^2, \end{cases} \tag{2.30}$$

the first integral of this system is

$$H_1 = \frac{b^2}{4a}y^2 + ax^2 + bxy,$$

and the first integral of linear system in this case becomes. To obtain a continuous piecewise differential system formed by this quadratic system and linear center (2.28) we must verify the following algebraic system

$$\dot{x_1} - \dot{x_l} = 0 \text{ and } \dot{y_1} - \dot{y_l} = 0,$$

the solutions of this algebraic system are

$$\begin{aligned}
&s_1 = \{b_5 = 0,\ \gamma = b_0, b_2 = \beta,\ b = 0,\ a = 0\},\\
&s_2 = \left\{b_5 = 0,\ \gamma = b_0,\ \delta = -\tfrac{bb_0}{2a},\ \alpha = \tfrac{a\left(4\beta^2 + \omega^2\right)}{2b\beta},\ b_2 = \beta\right\},\\
&s_3 = \{b_5 = 0,\ \gamma = 0,\ b_0 = 0,\ b_2 = 0,\ \beta = 0,\ a = 0\},\\
&s_4 = \{b_5 = 0,\ \gamma = 0,\ b_0 = 0,\ \alpha = 0,\ b_2 = \beta,\ a = 0\},\\
&s_5 = \{b_5 = 0,\ \gamma = b_0,\ b_2 = 0,\ b = 0,\ \beta = 0,\ a = 0\},\\
&s_6 = \{b_5 = 0,\ \gamma = b_0,\ \delta = 0,\ b_2 = \beta,\ \omega = -2i\beta,\ b = 0\},\\
&s_7 = \{b_5 = 0,\ \gamma = b_0,\ \delta = 0,\ b_2 = \beta,\ \omega = 2i\beta,\ b = 0\},\\
&s_8 = \left\{b_5 = 0,\ \gamma = b_0,\ \delta = -\tfrac{bb_0}{2a},\ b_2 = 0,\ \omega = 0,\ \beta = 0\right\},
\end{aligned}$$

The only solution verify the all conditions ($abc \neq 0$ and $\omega > 0$, $\alpha > 0$) is s_2.

The system concerning the solution s_2 is

$$\begin{cases} \dot{x}_l = -\frac{bb_0}{2a} - \frac{bb_1}{2a}x - \frac{bb_3}{2a}x^2 - \frac{bb_4}{2a}xy, \\ \dot{y}_l = b_0 + b_1x + b_3x^2 + b_4xy, \end{cases} \tag{2.31}$$

and its first integral is

$$H_1 = \frac{b^2}{4a}y^2 + ax^2 + bxy.$$

The first integral of linear system becomes

$$H_l = \frac{4a\left(4\beta^2+\omega^2\right)\left(\frac{bb_0}{2a}y + b_0x\right)}{b\beta} + 4\left(\frac{ax\left(4\beta^2+\omega^2\right)}{2b\beta} + \beta y\right)^2 + y^2\omega^2.$$

To have periodic orbits we must verify the following algebraic system

$$H_1(0,y) - H_1(0,Y) = 0 \text{ and } H_l(0,y) - H_l(0,Y) = 0,$$

then we get

$$\frac{b^2(y-Y)(y+Y)}{4a} = 0 \text{ and } \frac{\left(4\beta^2+\omega^2\right)(y-Y)(2b_0+\beta y+\beta Y)}{\beta} = 0.$$

We have three solution $y = Y, y = -Y$ or $Y = -\frac{2b_0+\beta y}{\beta}$, this piecewise system has a continuum of periodic orbits. So, no limit cycle.

To obtain a periodic orbit discontinuous piecewise differential system formed by this quadratic system and linear center (2.28) we must verify the following algebraic system

$$H_1(0,y) - H_1(0,Y) = 0 \text{ and } H_l(0,y) - H_l(0,Y) = 0,$$

then we get

$$\frac{b^2(y-Y)(y+Y)}{4a} = 0 \text{ and } (y-Y)\left(-8\alpha\delta + 4\beta^2y + y\omega^2 + 4\beta^2Y + \omega^2Y\right) = 0.$$

We have three solutions $y = Y, y = -Y$ or $Y = -\frac{-8\alpha\delta+4\beta^2y+y\omega^2}{\omega^2+4\beta^2}$, then this piecewise system has a continuum of periodic orbits. So, no limit cycle.

Case 2: The set of solutions s_{q2} gives the following system

$$\begin{cases} \dot{x}_l = \frac{\left(4ab_5c - b^2b_5 - 2bb_4c\right)}{b^2}xy + \frac{(2ab_5-bb_4)}{b}x^2 - \frac{2b_2c}{b}y - \frac{2b_5c}{b}y^2 - b_2x, \\ \dot{y}_l = \frac{2\left(abb_4-2a^2b_5\right)}{b^2}x^2 + \frac{2ab_2}{b}x + b_2y + b_4xy + b_5y^2, \end{cases} \tag{2.32}$$

the first integral of this system is

$$H_1 = ax^2 + bxy + cy^2,$$

To get a continuous piecewise differential system formed by this quadratic system and linear center (2.28) we should satisfy the following algebraic system

$$\dot{x_1} - \dot{x}_l = 0 \text{ and } \dot{y_1} - \dot{y}_l = 0,$$

which gives the following solutions

$$\begin{aligned}
s_1 &= \left\{b_5 = 0,\ \gamma = 0,\ \delta = 0,\ c = \tfrac{b\left(4\beta^2+\omega^2\right)}{8\alpha\beta},\ b_2 = \beta\right\}, \\
s_2 &= \{b_5 = 0,\ \gamma = 0,\ \delta = 0,\ b_2 = 0,\ b = 0,\ \beta = 0\}, \\
s_3 &= \{b_5 = 0,\ \gamma = 0,\ \delta = 0, b_2 = 0,\ \omega = 0,\ \beta = 0\}, \\
s_4 &= \{b_5 = 0,\ \gamma = 0,\ \delta = 0,\ c = 0,\ b_2 = \beta,\ b = 0\}, \\
s_5 &= \{b_5 = 0,\ \gamma = 0,\ \delta = 0,\ c = 0,\ b_2 = \beta,\ \omega = -2i\beta\}, \\
s_6 &= \{b_5 = 0,\ \gamma = 0,\ \delta)0,\ c = 0,\ b_2 = \beta,\ \omega = 2i\beta\}, \\
s_7 &= \{b_5 = 0,\ \gamma = 0,\ \delta = 0,\ \alpha = 0,\ b_2 = \beta,\ b = 0\}, \\
s_8 &= \{b_5 = 0,\ \gamma = 0,\ \delta = 0,\ \alpha = 0,\ b_2 = \beta,\ \omega = -2i\beta\}, \\
s_9 &= \{b_5 = 0,\ \gamma = 0,\ \delta = 0,\ \alpha = 0,\ b_2 = \beta,\ \omega = 2i\beta\},
\end{aligned}$$

the unique solution satisfy the all conditions ($abc \neq 0$ and $\omega > 0$, $\alpha > 0$) is s_2. Then the system provided by this solution is

$$\begin{cases} \dot{x_1} = -b_4x^2 - \beta x - \frac{((4\beta^2+\omega^2))}{(4\alpha)}y - \frac{(b_4(4\beta^2+\omega^2))}{(4\alpha\beta)}xy, \\ \dot{y_1} = \frac{(2ab_4)}{bx^2+b_4}xy + \frac{(2a\beta)}{b}x + \beta y, \end{cases} \tag{2.33}$$

thus its first integral is

$$H_1 = ax^2 + bxy + \frac{b\left(4\beta^2 + \omega^2\right)}{8\alpha\beta}y^2,$$

the first integral of linear system is

$$H_l = 4(\alpha x + \beta y)^2 + y^2\omega^2$$

To have periodic orbits we must verify the following algebraic system

$$H_1(0, y) - H_1(0, Y) = 0 \text{ and } H_l(0, y) - H_l(0, Y) = 0,$$

which yield

$$\frac{b\left(4\beta^2+\omega^2\right)(y-Y)(y+Y)}{8\alpha\beta}=0 \text{ and } \left(4\beta^2+\omega^2\right)(y-Y)(y+Y)=0.$$

We have two solutions $y=Y, y=-Y$, then this piecewise system has a continuum of periodic orbits. So, no limit cycle.

To preform a periodic orbit of discontinuous piecewise differential system formed by this quadratic system and linear center (2.28) the following algebraic system must be verified

$$H_1(0,y)-H_1(0,Y)=0 \text{ and } H_l(0,y)-H_l(0,Y)=0,$$

thus,

$$c(y-Y)(y+Y) \text{ and } (y-Y)\left(-8\alpha\delta+4\beta^2 y+y\omega^2+4\beta^2 Y+\omega^2 Y\right).$$

which gives three solution $y=Y, y=-Y$ or $Y=-\frac{-8\alpha\delta+4\beta^2 y+y\omega^2}{\omega^2+4\beta^2}$, then this piecewise system has a continuum of periodic orbits. So, no limit cycle.

Case 3: The set of solutions s_{q3} gives the following system

$$\begin{cases} \dot{x_1}=a_2 y+a_4 xy+a_5 y^2, \\ \dot{y_1}=-\frac{aa_2}{c}x-\frac{aa_4}{c}x^2-\frac{aa_5}{c}xy, \end{cases} \tag{2.34}$$

where the first integral of this system is

$$H_1=ax^2+cy^2,$$

To obtain a continuous piecewise differential system formed by this quadratic system and linear center (2.28) we must verify the following algebraic system

$$\dot{x_1}-\dot{x_l}=0 \text{ and } \dot{y_1}-\dot{y_l}=0,$$

the solutions of this algebraic system are

$$\begin{aligned} s_1&=\left\{a5=0,\ \gamma=0,\ \delta=0,\ a2=-\tfrac{\omega^2}{4\alpha},\ \beta=0\right\}, \\ s_2&=\{a5=0,\ \gamma=0,\ \delta=0,\ \alpha=0,\ \omega=0,\ \beta=0\}, \end{aligned}$$

The only solution verify the all conditions ($abc\neq 0$ and $\omega>0$, $\alpha>0$) is s_1.

This solution gives the following system

$$\begin{cases} \dot{x}_l = a4xy - \frac{\omega^2}{4\alpha}y, \\ \dot{y}_l = \frac{a\omega^2}{4\alpha c}x - \frac{aa_4}{c}x^2, \end{cases} \tag{2.35}$$

and its first integral is

$$H_1 = ax^2 + cy^2.$$

the first integral of system linear is

$$H_l = 4\alpha^2x^2 + y^2\omega^2.$$

To have periodic orbits we must verify the following algebraic system

$$H_1(0,y) - H_1(0,Y) = 0 \text{ and } H_l(0,y) - H_l(0,Y) = 0,$$

then we get

$$c(y-Y)(y+Y) = 0 \text{ and } \omega^2(y-Y)(y+Y) = 0,$$

we have two solution $y = Y, y = -Y$, then this piecewise system has a continuum of periodic orbits. So, no limit cycle.

To obtain a periodic orbit discontinuous piecewise differential system formed by this quadratic system and linear center (2.28) we must verify the following algebraic system

$$H_1(0,y) - H_1(0,Y) = 0 \text{ and } H_l(0,y) - H_l(0,Y) = 0,$$

then we get

$$c(y-Y)(y+Y) = 0 \text{ and } (y-Y)\left(-8\alpha\delta + 4\beta^2y + y\omega^2 + 4\beta^2Y + \omega^2Y\right)$$

We have three solution $y = Y, y = -Y$ or $Y = -\frac{-8\alpha\delta+4\beta^2y+y\omega^2}{\omega^2+4\beta^2}$, then this piecewise system has a continuum of periodic orbits. So, no limit cycle.

Chapter 3

Continuous and discontinuous piecewise isochronous centers

In the theory of limit cycles in planar differential systems, most of the first examples were usually related to practical problems with mechanical and electronic systems, then appeared the periodic behavior in all fields of science. So, proving the existence or non-existence of this problem became one of the most difficult goal in the qualitative theory of planar differential equations.

Piecewise differential systems are provided as one of the most remarkable non-smooth dynamical systems and widely applied in various scientific domains of studies such as engineering, electronics, and physics [7, ?, 14, 20]. Since the 1930s, many books discussed the extension study of limit cycles [3, 4, 33] due to the applications such as mechanics and electrical circuits. This notion became important in the continuous and discontinuous piecewise differential systems separated by a straight-line.

We could say that the singular point $p \in \mathbb{R}^2$ is a center of a planar differential system if there is only a neighborhood U of p where all the orbits of $U \setminus \{p\}$ are periodic. When all the periodic orbits surrounding a center have the same period, this center is called isochronous. p is called a uniform isochronous center or rigid center, only when the angular velocity is constant. The centers had been firstly studied by Poincaré [30] and Dulac [11], and the notion of isochronocity was reported by Huygens [16] in 1673.

In our present work, we focused on continuous and discontinuous piecewise differential systems formed by the linear isochronous center and the quadratic or cubic uniform isochronous center separated by the straight-line $x = 0$ to study the non-existence and the existence of crossing periodic orbits and crossing limit cycles defining the maximum number of crossing limit cycles for these systems.

It is well-known that a crossing periodic orbit or a crossing limit cycle is defined as a periodic orbit or a limit cycle that intersects the discontinuity line $x = 0$ in two different points.

The meaning of the continuity of a piecewise differential system separated by the straight-line $x = 0$ formed by two centers is that the vector fields defined by these two centers (linear, quadratic or cubic) coincide on the discontinuity line $x = 0$. We can conclude that a continuous piecewise differential system is both continuous in $\mathbb{R}^2$ and analytic in $\mathbb{R}^2/\{x = 0\}$.

3.1 Quadratic and cubic uniform isochronous centers

The objective to accomplish is to study the periodic orbits of continuous and discontinuous piecewise differential systems formed by the following quadratic or cubic uniform isochronous centers (3.1), (3.2), and the linear isochronous center (2.28).

$$\dot{x} = -y + x^2, \quad \dot{y} = x + xy, \tag{3.1}$$

$$\dot{x} = -y + x^2 y, \quad \dot{y} = x + xy^2. \tag{3.2}$$

System (3.1) is the unique quadratic uniform isochronous center [21, 10] and system (3.2) is the easiest cubic uniform isochronous center [8, 18].

For getting the general expressions of the quadratic and cubic uniform isochronous centers, we transform the normal forms (3.1) and (3.2) by the following general affine change of variables for a continuous piecewise differential system

$$(x, y) \to (a_1 x + b_1 y + c_1, a_2 x + b_2 y + c_2), \tag{3.3}$$

considering

$$a_1 b_2 - a_2 b_1 \neq 0. \tag{3.4}$$

Generalized uniform isochronous system (3.1). Doing the change of variables (3.3) the quadratic system (3.1) becomes

$$\begin{aligned}
\dot{x} =& \frac{-b_2 c_1^2 + b_1 c_1 + b_1 c_2 c_1 + b_2 c_2}{a_2 b_1 - a_1 b_2} + \frac{\left(b_1^2 c_2 - b_2 b_1 c_1 + b_1^2 + b_2^2\right)}{a_2 b_1 - a_1 b_2} y \\
&+ \frac{a_2 b_1 c_1 + a_1 b_1 c_2 - 2 a_1 b_2 c_1 + a_1 b_1 + a_2 b_2}{a_2 b_1 - a_1 b_2} x + b_1 y x + a_1 x^2, \\
\dot{y} =& \frac{a_2 c_1^2 - a_1 c_1 - a_1 c_2 c_1 - a_2 c_2}{a_2 b_1 - a_1 b_2} - \frac{a_1^2 c_2 - a_2 a_1 c_1 + a_1^2 + a_2^2}{a_2 b_1 - a_1 b_2} x + \\
& \frac{2 a_2 b_1 c_1 - a_1 b_1 c_2 - a_1 b_2 c_1 - a_1 b_1 - a_2 b_2}{a_2 b_1 - a_1 b_2} y + a_1 x y + b_1 y^2.
\end{aligned} \tag{3.5}$$

After changing the variables (3.3) and knowing that $-(1+y)/(x^2+y^2)^{1/2}$ is the first integral of the system (3.1), we were able to obtain the following first integral of the generalized isochronous quadratic system (3.5).

$$H_{GQ}(x,y) = \frac{-a_2x - b_2y - c_2 - 1}{\sqrt{(a_1x+b_1y+c_1)^2 + (a_2x+b_2y+c_2)^2}}.$$

Generalized uniform isochronous system (3.2).

After the linear change of variables (3.3) is done, system (3.2) becomes equivalent to the following generalized isochronous system.

$$\begin{aligned}
\dot{x} =& \frac{b_2c_2c_1^2 - b_1c_2^2c_1 - b_1c_1 - b_2c_2}{a_1b_2 - a_2b_1} + \frac{-b_1^2c_2^2 + b_2^2c_1^2 - b_1^2 - b_2^2}{a_1b_2 - a_2b_1}y + \\
& \frac{a_2b_2c_1^2 - 2a_2b_1c_2c_1 + 2a_1b_2c_2c_1 - a_1b_1c_2^2 - a_1b_1 - a_2b_2}{a_1b_2 - a_2b_1}x + \\
& \frac{a_1^2b_2c_2 + 2a_2a_1b_2c_1 - 2a_2a_1b_1c_2 - a_2^2b_1c_1}{a_1b_2 - a_2b_1}x^2 + \frac{2a_1b_2^2c_1 - 2a_2b_1^2c_2}{a_1b_2 - a_2b_1}xy + \\
& \frac{b_1b_2^2c_1 - b_1^2b_2c_2}{a_1b_2 - a_2b_1}y^2 + \frac{a_1^2a_2b_2 - a_1a_2^2b_1}{a_1b_2 - a_2b_1}x^3 + \frac{a_1^2b_2^2 - a_2^2b_1^2}{a_1b_2 - a_2b_1}x^2y + \frac{a_1b_1b_2^2 - a_2b_1^2b_2}{a_1b_2 - a_2b_1}xy^2, \\
\dot{y} =& \frac{a_2c_2c_1^2 - a_1c_2^2c_1 - a_1c_1 - a_2c_2}{a_2b_1 - a_1b_2} + \frac{-a_1^2c_2^2 + a_2^2c_1^2 - a_1^2 - a_2^2}{a_2b_1 - a_1b_2}x + \\
& \frac{a_2b_2c_1^2 + 2a_2b_1c_2c_1 - 2a_1b_2c_2c_1 - a_1b_1c_2^2 - a_1b_1 - a_2b_2}{a_2b_1 - a_1b_2}y + \frac{a_1a_2\left(a_2c_1 - a_1c_2\right)}{a_2b_1 - a_1b_2}x^2 \\
& + \frac{2\left(a_2^2b_1c_1 - a_1^2b_2c_2\right)}{a_2b_1 - a_1b_2}xy + \frac{a_2b_1^2c_2 + 2a_2b_2b_1c_1 - 2a_1b_2b_1c_2 - a_1b_2^2c_1}{a_2b_1 - a_1b_2}y^2 + \\
& + a_1a_2x^2y + \left(a_2b_1 + a_1b_2\right)y^2x + b_1b_2y^3.
\end{aligned} \tag{3.6}$$

The cubic system (3.2) has the first integral $(x^2-1)/2\,(x^2+y^2)$. Thus, the first integral of system (3.6) will be

$$H_{GC}(x,y) = \frac{(a_1x+b_1y+c_1)^2 - 1}{2\,(a_1x+b_1y+c_1)^2 + 2\,(a_2x+b_2y+c_2)^2}.$$

3.2 Statement of the main results

In a study of J. Itikawa and his collaborators [18] for the bifurcation of limit cycles from the periodic orbits of the uniform isochronous center of the differential systems (3.1) and (3.2), the authors applied the averaging method of the first order for discontinuous differential systems, when they were perturbed inside the class of all discontinuous quadratic and cubic polynomials differential systems with four zones separated by the axes of coordinates. In our work, we studied two cases at the same time: the continuous and the discontinuous of the same system but without perturbation and separated by the line $x = 0$.

Know, we expose our first main results characterizing the existence and non-existence of crossing periodic orbits and crossing limit cycles for continuous and discontinuous piecewise differential systems formed by the linear isochronous center and the uniform isochronous quadratic or cubic center.

***Theorem* 3.1.** *The continuous piecewise differential systems formed by the linear differential center (which is isochronous) and the uniform isochronous quadratic and cubic center separated by the straight-line $x = 0$ have no crossing limit cycles.*

3.3 Proof of Theorem 3.1

Our first objective is to study limit cycles of continuous piecewise differential systems formed by the linear center (2.28) and the generalized quadratic system (3.5) or the generalized cubic system (3.6). The study of an existing crossing periodic orbits of such piecewise differential systems it need the following algebraic system must be satisfied.

$$H(x_1, y_1) - H(x_2, y_2) = 0, \qquad H_L(x_1, y_1) - H_L(x_2, y_2) = 0, \tag{3.7}$$

It should be noted that $H(x, y)$ is the first integral of the uniform isochronous quadratic or cubic system and $H_L(x, y)$ is the first integral of the linear center. The two intersection points (x_1, y_1) and (x_2, y_2) with $y_1 \neq y_2$ are the crossing periodic orbits with the straight-line $x = 0$.

In order that the piecewise differential system formed by systems (2.28) and (3.5) to be continuous, we had to impose that

- both systems coincide on $x = 0$,
- both systems must verify the following algebraic system

$$\dot{x}_{GQ} - \dot{x}_L|_{x=0} = 0, \qquad \dot{y}_{GQ} - \dot{y}_L|_{x=0} = 0, \tag{3.8}$$

where $\dot{x}_L$, $\dot{y}_L$, $\dot{x}_{GQ}$ and $\dot{y}_{GQ}$ are the derivatives respecting time t of x and y for linear system and quadratic system, respectively. Thus, the algebraic system will be as follows

$$
\begin{aligned}
&-a_2b_1\delta + a_1b_2\delta - b_2c_1^2 + b_1c_1 + b_1c_2c_1 + b_2c_2 = 0,\\
&4a_2b_1\beta^2 - 4a_1b_2\beta^2 + a_2b_1\omega^2 - a_1b_2\omega^2 + 4\alpha b_1^2 + 4\alpha b_2^2 - 4\alpha b_1b_2c_1 + 4\alpha b_1^2c_2 = 0,\\
&-a_2b_1\gamma + a_1b_2\gamma + a_2c_1^2 - a_1c_1 - a_1c_2c_1 - a_2c_2 = 0,\\
&-a_2\beta b_1 + a_1\beta b_2 + 2a_2b_1c_1 - a_1b_1c_2 - a_1b_2c_1 - a_1b_1 - a_2b_2 = 0,\\
&b_1 = 0.
\end{aligned}
$$

All real solutions for this algebraic system are

$$
\begin{aligned}
s_1 &= \{b_1 = 0,\ b_2 = 0,\ c_2 = \frac{a_2c_1^2 - a_1c_1}{a_1c_1 + a_2}\},\\
s_2 &= \{a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = 0\},\\
s_3 &= \{a_2 = 0,\ b_1 = 0,\ b_2 = 0,\ c_1 = 0\},\\
s_4 &= \{a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ c_2 = c_1^2,\ \alpha = 0\},\\
s_5 &= \{b_1 = 0,\ \alpha = \frac{4a_1^2c_1^2 + 8a_2a_1c_1 + a_1^2\omega^2 + 4a_2^2}{4a_1b_2},\ \beta = \frac{a_1c_1 + a_2}{a_1},\\
&\qquad \gamma = \frac{-a_2c_1^2 + a_1c_1 + a_1c_2c_1 + a_2c_2}{a_1b_2},\ \delta = \frac{c_1^2 - c_2}{a_1}\}.
\end{aligned}
$$

s_5 is the unique set of solutions that verify the conditions $a_2b_1 - a_1b_2 \neq 0$, $\alpha > 0$ and $\omega > 0$, therefore, in this case, we have a continuous piecewise differential system.

In order to obtain crossing periodic orbits, the algebraic system (3.7) must be solved and becomes

$$
\begin{aligned}
&\frac{b_2\left(y_2\sqrt{(b_2y_1+c_2)^2+c_1^2} - y_1\sqrt{(b_2y_2+c_2)^2+c_1^2}\right) + (c_2+1)\left(\sqrt{(b_2y_1+c_2)^2+c_1^2} - \sqrt{(b_2y_2+c_2)^2+c_1^2}\right)}{\sqrt{(b_2y_1+c_2)^2+c_1^2}\sqrt{(b_2y_2+c_2)^2+c_1^2}} = 0,\\
&\frac{(y_1-y_2)\left(4(a_1c_1+a_2)^2+a_1^2\omega^2\right)\left(b_2(y_1+y_2)-2c_1^2+2c_2\right)}{a_1^2b_2} = 0,
\end{aligned}
$$

by solving this algebraic system, we get

$$
y_1 = y_2 = \frac{{c_1}^2 - c_2}{b_2}.
$$

Consequently, this piecewise differential system has no periodic orbits and then no limit cycles.

To obtain a continuous piecewise differential system formed by systems (2.28) and (3.6), both systems must coincide on $x = 0$, which induces the verification of the following algebraic system

$$
\dot{x}_{GC} - \dot{x}_L|_{x=0} = 0, \qquad \dot{y}_{GC} - \dot{y}_L|_{x=0} = 0. \tag{3.9}
$$

Once done, we get the following algebraic system

$$
\begin{aligned}
&-a_2b_1\delta+a_1b_2\delta-b_2c_2c_1^2+b_1c_2^2c_1+b_1c_1+b_2c_2=0,\\
&4a_2b_1\beta^2-4a_1b_2\beta^2+a_2b_1\omega^2-a_1b_2\omega^2+4\alpha b_1^2+4\alpha b_2^2-4\alpha b_2^2c_1^2+4\alpha b_1^2c_2^2=0,\\
&b_1b_2\left(b_1c_2-b_2c_1\right)=0,\\
&-a_2b_1\gamma+a_1b_2\gamma+a_2c_2c_1^2-a_1c_2^2c_1-a_1c_1-a_2c_2=0,\\
&-a_2\beta b_1+a_1\beta b_2+a_2b_2c_1^2+2a_2b_1c_2c_1-2a_1b_2c_2c_1-a_1b_1c_2^2-a_1b_1-a_2b_2=0,\\
&a_2b_1^2c_2+2a_2b_2b_1c_1-2a_1b_2b_1c_2-a_1b_2^2c_1=0,\\
&b_1b_2=0,
\end{aligned}
$$

and by solving it, we get one of the following sets of real solutions

$s_1=\{a_2=\frac{a_1c_1\left(c_2^2+1\right)}{\left(c_1^2-1\right)c_2},\ b_1=0,\ b_2=0\},$

$s_2=\{a_1=0,\ b_1=0,\ c_1=-1\},$

$s_3=\{a_1=0,\ b_1=0,\ c_1=1\},$

$s_4=\{a_1=0,\ b_1=0,\ b_2=0,\ c_1=-1\},$

$s_5=\{b_1=0,\ b_2=0,\ c_1=0,\ c_2=0\},$

$s_6=\{a_1=0,\ b_1=0,\ b_2=0,\ c_1=1\},$

$s_7=\{a_1=0,\ b_1=0,\ b_2=0,\ c_2=0\},$

$s_8=\{a_1=0,\ a_2=0,\ b_1=0,\ c_2=0,\ \alpha=0\},$

$s_9=\{a_1=0,\ a_2=0,\ b_2=0,\ c_1=0,\ \alpha=0\},$

$s_{10}=\{a_1=0,\ a_2=0,\ b_1=0,\ c_1=0,\ c_2=0,\ \alpha=0\},$

$s_{11}=\{b_1=0,\ c_1=0,\ \beta=\frac{a_2}{a_1},\ \gamma=\frac{a_2c_2}{a_1b_2},\ \delta=-\frac{c_2}{a_1},\ \omega=-\frac{2\sqrt{\alpha a_1b_2-a_2^2}}{a_1}\},$

$s_{12}=\{b_1=0,\ c_1=0,\ \beta=\frac{a_2}{a_1},\ \gamma=\frac{a_2c_2}{a_1b_2},\ \delta=-\frac{c_2}{a_1},\ \omega=\frac{2\sqrt{\alpha a_1b_2-a_2^2}}{a_1}\},$

$s_{13}=\{a_1=0,\ a_2=0,\ b_2=0,\ c_1=0,\ c_2=0,\ \alpha=0\},$

$s_{14}=\{b_2=0,\ c_2=0,\ \beta=-\frac{a_1}{a_2},\ \gamma=-\frac{a_1c_1}{a_2b_1},\ \delta=\frac{c_1}{a_2},\ \omega=-\frac{2\sqrt{-\alpha a_2b_1-a_1^2}}{a_2}\},$

$s_{15}=\{b_2=0,\ c_2=0,\ \beta=-\frac{a_1}{a_2},\ \gamma=-\frac{a_1c_1}{a_2b_1},\ \delta=\frac{c_1}{a_2},\ \omega=\frac{2\sqrt{-\alpha a_2b_1-a_1^2}}{a_2}\},$

$s_{16}=\{b_2=0,\ c_1=0,\ c_2=0,\ \beta=-\frac{a_1}{a_2},\ \gamma=0,\ \delta=0,\ \omega=-\frac{2\sqrt{-\alpha a_2b_1-a_1^2}}{a_2}\},$

$s_{17}=\{b_2=0,\ c_1=0,\ c_2=0,\ \beta=-\frac{a_1}{a_2},\ \gamma=0,\ \delta=0,\ \omega=\frac{2\sqrt{-\alpha a_2b_1-a_1^2}}{a_2}\},$

$s_{18}=\{a_1=0,\ b_2=0,\ c_2=0,\ \beta=0,\ \gamma=0,\ \delta=\frac{c_1}{a_2},\ \omega=-2\sqrt{\alpha}\sqrt{-\frac{b_1}{a_2}}\},$

$s_{19}=\{a_1=0,\ b_2=0,\ c_2=0,\ \beta=0,\ \gamma=0,\ \delta=\frac{c_1}{a_2},\ \omega=2\sqrt{\alpha}\sqrt{-\frac{b_1}{a_2}}\},$

$s_{20}=\{a_1=0,\ b_2=0,\ c_1=0,\ c_2=0,\ \beta=0,\ \gamma=0,\ \delta=0,\ \omega=-2\sqrt{\alpha}\sqrt{-\frac{b_1}{a_2}}\},$

$s_{21}=\{a_1=0,\ b_2=0,\ c_1=0,\ c_2=0,\ \beta=0,\ \gamma=0,\ \delta=0,\ \omega=2\sqrt{\alpha}\sqrt{-\frac{b_1}{a_2}}\}.$

The sets of solutions verifying the conditions $a_2b_1 - a_1b_2 \neq 0$, $\alpha > 0$ and $\omega > 0$ are s_{11}, s_{12}, s_{14}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, and s_{21}. It should be pointed out that we have a continuous piecewise differential system in these cases where we must study a crossing periodic orbit in all these ones.

A crossing periodic orbit is obtained by solving the algebraic system (3.7). For the cases s_{11} and s_{12}, the algebraic system becomes

$$\frac{b_2(y_1 - y_2)\left(b_2y_1 + b_2y_2 + 2c_2\right)}{2\left(b_2y_1 + c_2\right)^2\left(b_2y_2 + c_2\right)^2} = 0, \quad \frac{4\alpha(y_1 - y_2)\left(b_2y_1 + b_2y_2 + 2c_2\right)}{a_1} = 0,$$

by solving this system, we get the trivial solution $y_1 = y_2$ and $y_1 = \frac{-2c_2 - b_2y_2}{b_2}$, $y_2 \in \mathbb{R}$. Furthermore, for the cases s_{14}, s_{15}, s_{18}, s_{19}, s_{20}, and s_{21}, system (3.7) becomes

$$\frac{b_1(y_1 - y_2)\left(b_1y_1 + b_1y_2 + 2c_1\right)}{2\left(b_1y_1 + c_1\right)^2\left(b_1y_2 + c_1\right)^2} = 0, \quad -\frac{4\alpha(y_1 - y_2)\left(b_1y_1 + b_1y_2 + 2c_1\right)}{a_2} = 0,$$

and also by solving this algebraic system, we obtain the trivial solution $y_1 = y_2$ and $y_1 = \frac{-b1y_2 - 2c_1}{b_1}$, $y_2 \in \mathbb{R}$. Finally for the cases s_{16} and s_{17} the algebraic system (3.7) will become

$$\frac{(y_1 - y_2)(y_1 + y_2)}{2b_1^2y_1^2y_2^2} = 0, \quad -\frac{4\alpha b_1(y_1 - y_2)(y_1 + y_2)}{a_2} = 0,$$

indeed, we get the solutions $y_1 = y_2$ and $y_1 = -y_2$.

Concluding that in all these cases, the continuous piecewise differential systems formed by a linear center (2.28) and a generalized cubic system (3.6) has a continuum of periodic orbits, so, no limit cycles.

***Theorem* 3.2.** *The discontinuous piecewise differential systems formed by the linear differential isochronous center and the uniform isochronous quadratic and cubic center separated by the straight-line $x = 0$ have at most one crossing limit cycles.*

3.4 Proof of Theorem 3.2

In this part, we study the crossing limit cycles for discontinuous piecewise differential systems formed by the isochronous linear center (2.1) and the uniform isochronous quadratic system (3.1) or cubic system (3.2). To simplify the calculations, we used another way a little different from that used in the proof of theorem 3.1, but the idea remains the same.

We start first by the quadratic system, by replacing in the first integral $H(x,y) = (-y - 1)/(\sqrt{x^2 + y^2}) = k_1$ of system (3.1) x by 0 and we solved the result equation for y, getting

$$y_Q = \frac{1 \pm \sqrt{k_1^2}}{k_1^2 - 1},$$

and doing the same computation for the first integral of system (2.28), we get

$$y_L = \frac{8\alpha\delta \pm \sqrt{64\alpha^2\delta^2 + 4\mathrm{k}2(4\beta^2+\omega^2)}}{2(4\beta^2+\omega^2)},$$

To obtain a crossing periodic orbit, y_L of linear center must coincide with y_Q of quadratic system, by solving the equation $y_L - y_Q = 0$ we get

$$\begin{aligned} k_1 &= -\frac{\sqrt{4\alpha\delta+4\beta^2+\omega^2}}{2\sqrt{\alpha}\sqrt{\delta}}, \\ k_2 &= 4\alpha\delta. \end{aligned}$$

Then, the discontinuous piecewise differential systems formed by the isochronous linear center (2.1) and the uniform isochronous quadratic system (3.1) have one limit cycle. A example is shown in figure 3.1.

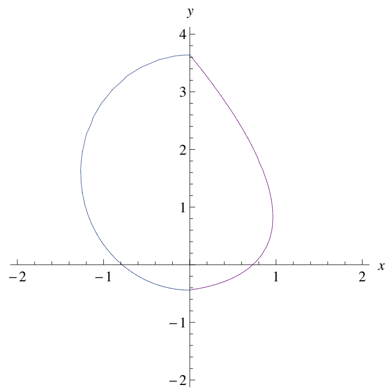

Figure 3.1: Limit cycle for $a = \delta = 2,\ b = \alpha = \beta = \gamma = \omega = 1.$

Conclusion

In this thesis, we have first given essential information as definitions, lemmas, and theorems used in our study. Second, we have classified quadratic differential systems having a special invariant of the form $ax^2 + bxy + cy^2 + dx + ey + c_1 t$ and we proved that there are 21 different families of quadratic systems having invariants of this form.

Next, we have studied the limit cycles of planar piecewise differential systems formed by quadratic systems that have the first integral of the form $ax^2 + bxy + cy^2$ and linear center. We proved that these piecewise systems have a continuum of periodic orbits and no limit cycles.

Finally, we have tackled the number of limit cycles of the piecewise planar differential system formed by the quadratic or cubic systems with uniform isochronous center and linear center separated the straight-line $x = 0$ by treating the two cases continuous and discontinuous.

We proved that piecewise systems have at most one limit cycle for discontinuous piecewise systems and we give an example for the quadratic case and no limit cycles for continuous piecewise systems.

Bibliography

[1] H.Arduin, *Modélisation mathématique des interaction entre pathogénes chez l'hote humain*, 15 mars 2018.

[2] J.C. Artés, J. Llibre, D. Schlomiuk and N. Vulpe, *Geometric configurations of singularities of planar polynomial differential systems. A global classification on the quadratic case*, to appear in Birkhäuser, 2021.

[3] A. Andronov, A. Vitt and S. Khaikin, *Theory of Oscillations*, Pergamon Press, Oxford, 1966.

[4] M. di Bernardo, C. J. Budd, A. R. Champneys and P. Kowalczyk, *Piecewise-Smooth Dynamical Systems: Theory and Applications*, Appl. Math. Sci. Series 163, Springer-Verlag, London, 2008.

[5] Y. Bolaños, J. Llibre and C. Valls, *Phase portraits of quadratic Lotka–Volterra systems with a Darboux invariant in the Poincaré disc*, Comm. in Contemporary Math. **16** (2014), 1350041, 23 pp.

[6] D. C. Braga and L. F. Mello, Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane, Non linear Dynam. 73 (2013), 1283-1288.

[7] S. Banerjee, G. Verghese, *Nonlinear Phenomena in Power Electronics: Attractors, Bifurcations, Chaos, and Nonlinear*, Control, Wiley-IEEE Press, New York, 2001.

[8] C.B. Collins, *Conditions for a center in a simple class of cubic systems*, Differential and Integral Equations, **10** (1997), 333–356.

[9] L. Cairo and J. Llibre, Darbouxian first integrals and invariants for real quadratic systems having an invariant conic, J. Phys. A, Math. Gen. 35 (2002) 589–608

[10] J. Chavarriga and M. Sabatini, *A survey of isochronous centers*, Qualitative Theory of Dynamical Systems, **1** (1999), 1–70.

[11] H. Dulac, *Détermination et integration d'une certaine classe d'équations différentielle ayant par point singulier un centre*, Bull. Sci. Math. Sér. (2) **32** (1908), 230–252.

[12] F. Dumortier, J. Llibre and J.C. Artés,, *Qualitative Theory of Planar Differential Systems*, UniversiText, Springer–Verlag, New York, 2006.

[13] R .D. Euzéebio and J. Llibre, On the number of limit cycles in discontinuous piecewise linear differential systems with two zones separated by a straight line, J. Math. Anal. Appl. 424(2015), 475-486.

[14] A.F. Filippov, *Differential Equation with Discontinuous Right-Hand Sides*, Kluwer Academic, Amsterdam, 1988.

[15] D. Hilbert, *Mathematische Probleme*, Lecture, Second Internat. Congr. Math. (Paris, 1900), *Nachr. Ges. Wiss. G"ottingen Math. Phys. KL.* (1900), 253–297; English transl., *Bull. Amer. Math. Soc.* **8** (1902), 437–479; *Bull. (New Series) Amer. Math. Soc.* **37** (2000), 407–436.

[16] C. Huygens, *Horologium Oscillatorium*, translated by Ian Bruce (July 2007), (PDF). 17thcentury-maths.com. Retrieved 2009-03-01.

[17] Yu. Ilyashenko, *Centennial history of Hilbert's* $\mathbf{16}$*th problem*, *Bull. (New Series) Amer. Math. Soc.* **39** (2002), 301–354.

[18] J. Itikawa, J. Llibre, A.C. Mereu and R. Oliveira, *Limit cycles in uniform isochronous centers of discontinuous differential systems with four zones*, Discrete and continuous dynamical systems serie B.

[19] M. Kukucka, Non-Smooth Dynamical Systems, Springer-Verlag, Berlin, Heidelberg, 2000.

[20] Y.A. Kuznetzov, *Elements of applied bifurcation theory*, second editions, Appl. Math. Sci. **112**, Springer, New York, 1998.

[21] W.S. Loud, *Behavior of the periodic of solutions of certain plane autonomous systems near centers*, Contributions to Differential Equations **3** (1964), 21–36.

[22] J. Li, *Hilbert's* $\mathbf{16}$*th problem and bifurcations of planar polynomial vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), 47–106.

[23] J. Llibre, M. Messias and A.C. Reinol, *Darboux invariants for planar polynomial differential systems having an invariant conic*, Zeitschrift fuer Angewandte Mathematik und Physik, **65** (2014), 1127–1136.

[24] J. Llibre, M. Messias and A.C. Reinol, *Normal forms for polynomial differential systems in* $\mathbb{R}^3$ *having an invariant quadric and a Darboux invariant*, Int. J. Bifurcation and Chaos **25** (2015), 1550015, 16 pp.

[25] J. Llibre, D.D. Novaes and M.A. Teixeira, *Maximum number of limit cycles for certain piecewise linear dynamical systems*, Nonlinear Dyn. **82** (2015), 1159–1175.

[26] J. Llibre and R.D.S. Oliveira, *Quadratic systems with invariant straight lines of total multiplicity two having Darboux invariants*, Comm. in Contemporary Math. **17** (2015), 1450018, 17 pp.

[27] J. Llibre and R. Oliveira, *Quadratic systems with an invariant conic having Darboux invariants*, Commun. Contemp. Math. Vol. 20, No. 4 (2018) 1750033 (15 pages).

[28] J. Llibre and M.A. Teixeira, *Piecewise linear differential systems with only centers can create limit cycles*?, Nonlinear Dynamics **91** (2018), 249–255.

[29] L. Perko, *Differential equations and dynamical systems.* Third Edition. Texts in Applied Mathematics, 7. Springer-Verlag, New York, Berlin Heidelberg, 2001.

[30] H. Poincaré, *Mémoire sur les courbes définies par les équations différentielles*, Journal de Mathématiques **37** (1881), 375–422; Oeuvres de Henri Poincaré, vol. I, Gauthier-Villars, Paris, 1951, pp 3–84.

[31] H. Poincaré, *Sur les courbes définies par une équation différentielle*, Oevres complètes, Vol. 1, 1928.

[32] J. Reyn, *Phase portraits of planar quadratic systems*, Mathematics and Its Applications **583**, Springer, New York, 2007.

[33] D.J.W. Simpson, *Bifurcations in piecewise–smooth continuous systems*, World Scientific Series on Nonlinear Science A, vol. **69**, 2010.

[34] Y. Yanqian et al., *Theory of Limit Cycles, Translations of Mathematics Monographs* , Vol. 66 (American Mathematics Society, Providence, RI, 1986).