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DEDICATION

I dedicate this humble work,

To my parents for their prayers, encouragements and generosities which followed me the whole time.

To my brothers and sisters for their supports and passions.

To all my friends, my partner of work and colleagues.

Mys sincere thanks.

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Saadallah Sara

Dédicace

Je dèdie ce travail,

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Bessai Nesrine

TABLE DES MATIÈRES

INTRODUCTION

Most of problems encountered can be modelled mathematically, but these models require assumptions that are sometimes too restrictive, making application to the real world difficult. Real-world problems must take into account imprecise, uncertain information. The concept of fuzzy set was introduced in 1965 by A. Zadeh [3], many authors were interested by this concept [1, 4, 13]. The main problem in fuzzy mathematics is how to carry out the ordinary concepts to the fuzzy case.

The partially ordered algebraic systems play an important role in algebra. Some important concepts in partially ordered systems are ordered groups and lattice ordered groups. These concepts play a major role in many branches of Algebra.

In 1971, A. Rosenfeld applied the notion of fuzzy set theory on group theory in his book [2], he introduced the concept of fuzzy subgroup and show that many theorem can be extended to develop the fuzzy group theory. Next, many authors worked on fuzzy theory and introduced the concept of fuzzy orders, fuzzy cosets and fuzzy lattice [7, 9, 15].

Convexity play an important role in the study of compatible orders, ordered groups and especially in lattice-ordered groups. Our main aim in this work is to investigate some properties and characterizations theorems of the fuzzy convex subgroup (resp. fuzzy convex lattice-ordered subgroup) of an ordered group (resp. lattice-ordered group). Some more results related to this topic are also derived.

This memory is organized in three chapter as follows :
In the first chapter, we recall some definitions and well-known about the ordered sets, coset, groups, and ordered groups. This chapter also focuses on lattice, lattice-ordered group and some related concept which we will need in the sequel.
In the second chapter, we give some basic notions and generalities about the fuzzy sets, their characteristic notion and level sets. Also, we define fuzzy subgroup and give some properties.

In the last chapter, we specified our searches about convexity in fuzzy case more precisely fuzzy convex subgroups and fuzzy convex lattice-ordered subgroups.

CHAPITRE

ONE

GENERALITIES ON ORDERED GROUPS AND LATTICE-ORDERED SUBGROUP

The notion of a group play a fundamental role in mathematics,it is one of the main algebraic structures. So in this chapter we will recall the basic notions of the ordered sets, lattices, subgroups and normal subgroups. Next, we investigate some basic properties of ordered groups and lattice ordered groups.

1.1 Ordered sets, Lattices

The purpose of this section is to provide a basic introduction to the theory of ordered structures, and we mention the concept of ordered groups, lattices and lattice-ordered groups.
For more details on Ordered sets and Lattices, we refer to [5, 6, 7, 12].

Definition 1.1 *[7]Let X be non-empty set, an* ***order*** *(or a* ***partial order****) is a binary relation $\leqslant$ on X which is :*

- ***i) Reflexive****, i.e., for all $x \in X$, $x \leqslant x$.*
- ***ii) Antisymmetric****, i.e., for all $x, y \in X$, if $x \leqslant y$ and $y \leqslant x$ imply $x = y$.*
- ***iii) Transitive****, i.e., for all $x, y, z \in X$, if $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$.*

Definition 1.2 *[7] A set X equipped with an order relation $\leqslant$ is called an* ***ordered set*** *(poset), denoted $(X, \leqslant)$.*

We note that if $(X, \leqslant))$ is a poset and $A \subset X$, then A^u and A^l denote the set of all upper bounds and the set of all lower bounds of A, respectively.

$$A^u = \{u \in X \mid x \leqslant u,\ \forall x \in A\}, \quad A^l = \{l \in X \mid l \leqslant x,\ \forall x \in A\}.$$

Let A be a subset of a poset $(X, \leqslant)$. An element $u_0 \in A^u$ is the **least upper bound** of A, denoted by $supA$ or $\vee A$, if $u_0 \leqslant u$, $\forall u \in A^u$.
An element $l_0 \in A^l$ is the **greatest lower bound** of A, denoted by $infA$ or $\wedge A$, if $l \leqslant l_0, \forall l \in A^l$.

Example 1.1 ***i)*** *On every set the relation of equality is an order.*

ii) *On the set $\mathbb{N}$ of natural numbers the relation $|$ of divisibility, defined by $m|n$ if and only if there exist $k \in \mathbb{N}$ such that $n = km$ is an order. Then $(\mathbb{N}, |)$ is a poset.*

iii) *The power set of a given set X is ordered by inclusion. Then $(\mathbb{P}(X), \subseteq)$ is a poset.*

iv) *Let E and F be two ordered sets. Then the set $Map(E,F)$ of all mappings $f : E \to F$ can be ordered by defining $f \leqslant g \Leftrightarrow (\forall x \in E) f(x) \leqslant g(x)$.*

Example 1.2 *Consider X is the additive set $\mathbb{R} \times \mathbb{R}$, we defined the relation $\leqslant$ by*

$$(x,y) \leqslant (z,w) \Leftrightarrow x = z,\ y = w \text{ or } x \leq z,\ y < w$$

It is easy to check that " $\leqslant$" is reflexive, transitive and antisymmetric, then $\mathbb{R} \times \mathbb{R}$ is aposet

Definition 1.3 *We say that two elements x, y of an ordered set $(X, \leqslant)$ are **comparable** and we write $x \parallel y$ if either $x \leqslant y$ or $y \leqslant x$. If all pairs of elements of X are comparable then we say that X forms a **chain**, and that $\leqslant$ is a total order.*

Example 1.3 *The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ of natural numbers, integers, rationales, and real numbers form chains under their usual orders $\leq$.*

Example 1.4 *Let $X = \{1,2,3,4,6,12\}$ be the set of positive divisors of 12. If we order X in the usual way, we obtain a chain. If we order X by divisibility, we obtain the Hasse diagram of Figure1.1.*

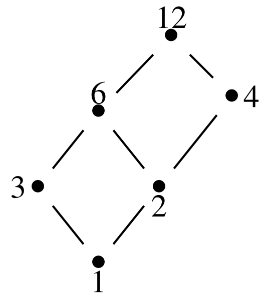


FIGURE 1.1 – Hasse diagram of a poset $(D_{12}, |)$.

Definition 1.4 *[8] Let* $(X, \leqslant)$ *be an ordered set.*

- *For all* $x, y \in X$ *with* $x \leqslant y$*, an* ***interval*** *in* X *is the subset denoted by,*

$$[x,y] = \{z \in X \mid x \leqslant z \leqslant y\}.$$

- *A non-empty subset* A *of* X *is* ***convex*** *if for all* $x, y \in A$ *with* $x \leqslant y$*,*

$$[x,y] \subseteq A.$$

We can also define a convex subset A *by,*

$$\forall x, y \in A,\ if\ x \leqslant z \leqslant y \Rightarrow z \in A.$$

Definition 1.5 *Let* X *be a non-empty ordered set* $(X, \leqslant)$ *is called a* ***lattice*** *if for all* $x, y \in X$*,* $x \vee y$ *and* $x \wedge y$ *exist.*

Example 1.5 *Let us consider Examples 1.1. Then*

- $(\mathbb{N}^*, |)$ *is a lattice such that for all* $x, y \in \mathbb{N}^*$*,* $x \vee y = lcm(x,y)$ *and* $x \wedge y = gcd(x,y)$*.*
- $(\mathbb{P}(X), \subseteq)$ *is a lattice such that for all* $A, B \in \mathbb{P}(X)$*,* $A \vee B = A \cup B$ *and* $A \wedge B = A \cap B$*.*

1.2 Ordered groups, Lattice-ordered groups

1.2.1 Some reminders of the classical groups

In this subsection we recall some basic definitions and properties of the classical groups.

Definition 1.6 *Let* G *be a non-empty set, A pair* $(G, \cdot)$ *associated with the inner operation " $\cdot$ " is called a* ***group*** *if is verifying this three proprieties :*

i) Associativity *:*

$$\forall\, a, b \in G, a \cdot (b \cdot c) = (a \cdot b) \cdot c,$$

ii) *Have an* ***identity element*** *:*

$$\exists\, e \in G, \forall\, a \in G, a \cdot e = e \cdot a = a,$$

iii) *All element of G have an* ***inverse*** *:*

$$\forall\, a \in G, \exists\, b \in G, a \cdot b = b \cdot a = e.$$

Example 1.6 *One of the most commons groups is* $(\mathbb{Z}, +)$ *which is constituted from the integer set and the inner operation "addition".*

The following example is useful in the next of our work.

Example 1.7 *The Klein* 4*-group* $G = \{e, a, b, c\}$ *(the Klein four-group is a group with four elements, in which each element is self-inverse (composing it with itself produces the identity) and in which composing any two of the three non-identity elements produces the third one). The Klein* [1] 4*-group G is defined by the table of Figure 1.2.*

.	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

FIGURE 1.2 – The Klein 4-group.

Notation 1.1 ***(Vocabulary note) :***

- *If the group is noted* ***additively*** $(\mathbb{R}, +)$, *(which we note* $a + b$ *for* $a \cdot b$*) :*
  - *The identity element is zero noted by 0.*
  - *The inverse element of a (called also the opposite) is -a.*
- *If the group is noted* ***multiplically*** $(\mathbb{R}, \times)$, *(which we note ab for* $a \cdot b$*) :*
  - *The identity element is 1.*
  - *The inverse element of a is* a^{-1}.

Definition 1.7 *Let* $(G, \cdot)$ *be a group. We call a* ***commutative group****, or* ***abelian*** *group, all group G in which the operation "* $\cdot$*" satisfies also the the condition of commutativity :*

$$\forall a, b \in G,\ a \cdot b = b \cdot a$$

1. Felix Klein. 1849 – 1925 was a German mathematician and mathematics educator, known for his work with group theory, complex analysis, non-Euclidean geometry, and on the associations between geometry and group theory

Definition 1.8 *Let $(G,\cdot)$ be a group, and $H \subset G$ non-empty subset. H is called a* ***subgroup*** *of G if*

i) *for all $a,b \in H \Rightarrow a \cdot b \in H$,*

ii) *$a \in H \Rightarrow a^{-1} \in H$.*

Remark 1.1 *The two conditions (i) and (ii) can be combined into one equivalent condition is*

$$\forall a,b \in H \Rightarrow a \cdot b^{-1} \in H.$$

Example 1.8
- *Let $(G,\cdot)$ be a group of identity element e, then G and $\{e\}$ are subgroups of G.*
- *$(\mathbb{Z},+)$ is a subgroup of $(\mathbb{Q},+)$ which is a subgroup of $(\mathbb{R},+)$ which is a subgroup of $(\mathbb{C},+)$.*
- *$(\mathbb{Q}^{*},\times)$ is a subgroup of $(\mathbb{R}^{*},\times)$ which is a subgroup of $(\mathbb{C}^{*},\times)$.*
- *$(\mathbb{N},+)$ is not a subgroup of any group because the opposite of an element in $\mathbb{N}$ is not in $\mathbb{N}$*

1.2.2 Cosets and normal subgroups

The notion of a normal subgroup is one of the central concepts of classical group theory. It plays an important role in the study of the general structure of groups. Just as a normal subgroup plays an important role in the classical group theory, a normal fuzzy subgroup plays a similar role in the theory of fuzzy subgroups.
Let $(G,\cdot)$ be group with identity element e, and H a subgroup of G.

Definition 1.9 *[10] Let A and B be subsets of a group G.* ***The product*** *AB of the sets A and B is defined by*

$$AB = \{xy \mid x \in A \text{ and } y \in B\}.$$

For all elements x of G, we denote the product $\{x\}A$ and $A\{x\}$ by xA and Ax, respectively.

Definition 1.10 *[10] For all element x of G and subgroup H, we note the* ***left coset*** *of H in G by the set xH defined by*

$$xH = \{xh \mid h \in H\},$$

Similarly, we note the ***right coset*** *of H in G by the set Hx defined by*

$$Hx = \{hx \mid h \in H\}.$$

Example 1.9 *Consider $H = 3\mathbb{Z}$ the subgroup of the group $(\mathbb{Z},+)$. Then the right cosets of H are the only three sets $3\mathbb{Z}, 3\mathbb{Z}+1, 3\mathbb{Z}+2$, where for all $a \in \{0,1,2\}$,*

$$3\mathbb{Z}+a = \{\cdots,-6+a,-3+a,0,3+a,6+a,\cdots\}.$$

Due the commutativity of addition, it holds that every left coset of $(3\mathbb{Z},+)$ is also a right coset.

Definition 1.11 *Let $(G,\cdot)$ be a group, and H subgroup of G. H is called a* ***normal subgroup*** *of G if*

$$\forall h \in H, \forall x \in G \mid xhx^{-1} \in H.$$

Note that every subgroup H of an Abelian group G is a normal subgroup. Indeed, for all $x \in G$ and $y \in H$, $xyx^{-1} = (xy)x^{-1} = x^{-1}(xy) = (x^{-1}x)y = ey = y \in H$

Proposition 1.1 *H is a normal subgroup of G if and only if $\forall x \in G$,*

$$xHx^{-1} = H.$$

Proof.

$\Rightarrow$) Suppose that H is a normal subgroup of G. Let $x \in G$.

$$\begin{aligned} x \in G \quad &\Rightarrow \quad xhx^{-1} \in H,\ \forall\ h \in H \\ &\Rightarrow \quad xHx^{-1} \subset H \end{aligned}$$

We see also that $x^{-1}Hx \subset H$ (by replacing x with x^{-1}), thus

$$H = x(x^{-1}Hx)x^{-1} \subset xHx^{-1}.$$

$$\Rightarrow H \subset xHx^{-1}$$

and therefore, each of the sets N and xHx^{-1} is contained in the other. Then, $xHx^{-1} = H$.

$\Leftarrow$) Conversely, it's clear that a subgroup H of G which satisfies the property $xHx^{-1} = H$ for all x from G is a normal subgroup of G.

■

Corollary 1.1 *A subgroup H of a group G is a normal subgroup of G if and only if $xH = Hx$ for all $x \in G$.*

Proof.

Let $x \in G$, if $H = xHx^{-1}$ we have

$$H = xHx^{-1} \;\;\Rightarrow\;\; Hx = (xHx^{-1})x = xH$$

conversely, if $xH = Hx$

$$xH = Hx \;\;\Rightarrow\;\; H = x^{-1}Hx$$

Thus,

$$H = x^{-1}Hx \Leftrightarrow xH = Hx$$

And therefore, from Proposition 1.1 the subgroup H of G is normal if and only if $xH = Hx$ for all $x \in G$. ■

Lemma 1.1 *Let H be a normal subgroup of G, and let x and y be elements of G. Then*

$$(xH)(yH) = (xy)H.$$

Proof.

Let G be a group, we know that if H is a normal subgroup of G then $Hy = yH$.

$$\begin{aligned}
(xH)(yH) &= x(Hy)H \\
&= x(yH)H \\
&= (xy)HH \\
&= (xy)H \;(Since\ H\ is\ a\ subgroup\ of\ G\ (HH = H))
\end{aligned}$$

Then $(xH)(yH) = (xy)H$. ■

Proposition 1.2 *Let H be a normal subgroup of G. Then the set of all cosets of H in G is a group under the operation of multiplication.*

Remark 1.2 *The identity element of the set of all cosets of H in G is H itself, and the inverse of a coset xH is the coset $x^{-1}H$ for all element x of G.*

Proof.

Let x and y be an elements of G. According to Lemma 1.1, we have $(xH)(yH) = (xy)H$, and The subgroup H is itself a coset of H in G, since $H = eH$. So we will prove that The set of all cosets of H in G is a group. Moreover

1) Let x $\in G$,

$$\begin{aligned} (xH)H &= (xH)(eH) = (xe)H = xH \\ H(xH) &= (He)(xH) = (ex)H = xH \end{aligned}$$

2) Let x $\in G$ and x^{-1} the inverse of x in G,

$$\begin{aligned} (xH)(x^{-1}H) &= (xx^{-1})H = H \\ (x^{-1}H)(xH) &= (x^{-1}x)H = H \end{aligned}$$

3) Let $x, y, z \in G$,

$$(xH\,yH)\,zH = (xy)H\,zH = (xy)\,zH = x\,(yz)H = xH\,(yz)H = xH\,(yH\,zH).$$

Then, for all elements x of G, the group axioms are satisfied. ■

Definition 1.12 *Let H be a normal subgroup of a group G. The group of cosets of H in G under the operation of multiplication is called* ***the quotient group****, denoted by* G/H.

1.2.3 Ordered groups

Definition 1.13 *[8] Let* $(G,\cdot)$ *be a group. We say that a partial order* $\leqslant$ *on G is* ***compatible*** *if,*

$$\forall x, y\; z, z' \in G,\; x \leqslant y \;\Rightarrow\; z\cdot x\cdot z' \leqslant z\cdot y\cdot z'$$

We say also that the operation " $\cdot$" is compatible with the order " $\leqslant$".

Definition 1.14 *[9] An* ***ordered group*** *is the triple* $(G,\cdot,\leqslant)$ *which is verifying the following three axioms :*

i) $(G,\cdot)$*is a group,*

ii) *"* $\leqslant$*" is an order,*

iii) *"* $\cdot$*" is compatible with the order "* $\leqslant$*".*

Example 1.10 $(\mathbb{Z},+,\leq)$, $(\mathbb{Q},+,\leq)$, $(\mathbb{R},+,\leq)$ *are all ordered groups.*

Remark 1.3 *Every group can be made into an ordered group only by placing the trivial ordering on the set.*

Proposition 1.3 *Let $(G,\cdot,\leqslant)$ be an ordered group, then,*

$$x \leqslant y \Leftrightarrow x^{-1} \geqslant y^{-1}.$$

Proof.

Let $x,y \in G$ such that$x \leqslant y$. Then, we have

$$\begin{aligned} x \leqslant y &\Leftrightarrow x^{-1}\cdot x \leqslant x^{-1}\cdot y \\ &\Leftrightarrow e \leqslant x^{-1}\cdot y \\ &\Leftrightarrow y^{-1}\cdot e \leqslant x^{-1}\cdot yy^{-1} \\ &\Leftrightarrow y^{-1} \leqslant x^{-1}\cdot e \\ &\Leftrightarrow x^{-1} \geqslant y^{-1}. \end{aligned}$$

■

Definition 1.15 *[12] Let $(E,\leqslant)$ be an ordered group.*

- *D is called a **down-set** if $x \in D$ and $y \leqslant x$ imply $y \in D$.*

*A down-set D is called a **principal down-set**, denoted by $x^{\downarrow}$, if $\exists\, x \in D$ such that,*

$$x^{\downarrow} = \{y \in E \mid y \leqslant x\}.$$

- *U is called an **upper-set** if $x \in U$ and $x \leqslant y$ imply $y \in U$.*

*An upper-set U is called a **principal upper-set**, denoted by $x^{\uparrow}$, if $\exists\, x \in U$ such that,*

$$x^{\uparrow} = \{y \in E \mid x \leqslant y\}.$$

Definition 1.16 *[12] Let $(G,\leqslant)$ and $(H,\leqslant)$ two ordered sets. We say that a mapping $A : G \to H$ is **isotone** (or **order-preserving**) if $\forall\, x,y \in G,\ x \leqslant y \Rightarrow A(x) \leqslant A(y)$,*

*And is **antitone** (or **order-reversing**) if $\forall\, x,y \in G,\ x \leqslant y \Rightarrow A(x) \geqslant A(y)$.*

Definition 1.17 *[8] An element x of a group G is called **positive element** if $e \leqslant x$,then the set $P = \{x \in G \mid x \geqslant e\}$ is called the **positive cone** of G.*
*x is called **negative element** if $e \geqslant x$, then the set $N = \{x \in G \mid e \geqslant x\}$ is called the **negative cone** of G.*
We often denoted with G^{+} and G^{-} for P and N, respectively.

If A is a subset of an ordered group then we will use the notation
$A^{-1} = \{x^{-1} \in A \mid x \in A\}$ and $A^2 = \{xy \mid x,y \in A\}$.

The following theorem provides the necessary and the sufficient conditions for a subset P of a group G to be a positive cone relative to some compatible order on G.

Theorem 1.1 *[12] A subset P of a group G is the positive cone relative to some compatible order on G if and only if*

(1) $P \cap P^{-1} = \{e\}$;

(2) $P^2 = P$;

(3) $\forall x \in G, \quad xPx^{-1} = P$.

Moreover, this order is a total order if, in addition, $P \cup P^{-1} = G$.

1.2.4 Lattice-ordered groups

Definition 1.18 *[8] A **lattice-ordered group** written ℓ-group is an ordered group $(G, \cdot, \leqslant)$ such that $(G, \leqslant)$ is a lattice.*

Example 1.11 *Let $(\mathbb{C}, +)$ be the additive group of complex numbers. We define on $(\mathbb{C}, +)$ the partial order : $x + iy \leq u + iv$ if $x \leq u$ and $y \leq v$. Then, $(\mathbb{C}, +, \leq)$ is an ℓ-group.*
Note that $(x+iy) \vee (u+iv) = (x \vee u) + i(y \vee v)$ and $(x+iy) \wedge (u+iv) = (x \wedge u) + i(y \wedge v)$.

The following three propositions will be helpful in the proofs of the next results especially in the last chapter .

Proposition 1.4 *Let $(G, \cdot, \leqslant)$ be an ordered group.It holds that*

i) *If $(G, \leqslant)$ is a $\vee$-semilattice. Then for all $x, y, z \in G$, it holds that*

$$x(y \vee z) = xy \vee xz \text{ and } (y \vee z)x = yx \vee zx.$$

ii) *If $(G, \leqslant)$ is a $\wedge$-semilattice. Then for all $x, y, z \in G$, it holds that*

$$x(y \wedge z) = xy \wedge xz \text{ and } (y \wedge z)x = yx \wedge zx.$$

Proof.

We will prove that $x(y \vee z) = xy \vee xz$ (the other cases where $(y \vee z)x = yx \vee zx$, $x(y \wedge z) = xy \wedge xz$ and $(y \wedge z)x = yx \wedge zx$. are similarly proved).
As $y \leqslant y \vee z$, $z \leqslant y \vee z$ and $\leqslant$ is a compatible order on G, it follows that $ay \leqslant a(y \vee z)$ and $az \leqslant a(y \vee z)$. This implies that $a(y \vee z) \in \{ay, az\}^u$, i.e., $a(y \vee z)$ is an upper bound of ay and az. Let us know consider $c \in \{ay, az\}^u$, it follows that $ay \leqslant c$ and $az \leqslant c$. This implies that

$y \leqslant a^{-1}c$ and $z \leqslant a^{-1}c$, it follows that $y \vee z \leqslant a^{-1}c$ implies $a(y \vee z) \leqslant c$. Hence $a(y \vee z)$ is the least upper bound of ay and az. We conclude that $x(y \vee z) = xy \vee xz$ ∎

Proposition 1.5 *Let $(G, \cdot, \leqslant)$ be an ordered group. if $(G, \leqslant)$ is a semilattice, then $(G, \leqslant)$ is a lattice.*

Proof.

We suppose that $(G, \leqslant)$ is a $\vee$-semilattice (the proof is similar if $(G, \leqslant)$ is a $\wedge$-semilattice) We will prove that G is a lattice in which for all $a, b \in G$,

$$a \wedge b = a(a \vee b)^{-1}b = b(a \vee b)^{-1})a.$$

We have $a \leqslant a \vee b$ gives $(a \vee b)^{-1} \leqslant a^{-1}$ and therefore $a(a \vee b)^{-1}b \leqslant b$, and similarly $b \leqslant a \vee b$ gives $a(a \vee b)^{-1}b \leqslant a$. Thus $a(a \vee b)^{-1}b \in \{a, b\}^l$. Suppose now that $c \in G$ is any lower bound of $\{a, b\}$ ($c \in \{a, b\}^l$). Then $c \leqslant a$ and $c \leqslant b$ give $a^{-1} \vee b^{-1} \leqslant c^{-1}$, and therefore by Proposition 1.4, $b^{-1}(a \vee b)a^{-1} = a^{-1} \vee b^{-1} \leqslant c^{-1}$ hence, $c \leqslant [b^{-1}(a \vee b)a^{-1}]^{-1} = a(a \vee b)^{-1}b$. Hence, $a \wedge b$ exists and is $a \wedge b = a(a \vee b)^{-1}b$. $a \wedge b = b \wedge a$, it follows that $a \wedge b = b(a \vee b)^{-1}a$.
By the same way if $(G, \leqslant)$ is a $\wedge$-semilattice, we proof that G is a lattice in which for all $a, b \in G$,

$$a \vee b = a(a \wedge b)^{-1}b = b(a \wedge b)^{-1}a.$$

∎

Proposition 1.6 *Let G be a ℓ-group, and $a, b, x \in G$, then,*

i) *$(a \vee b)x = ax \vee bx$, $x(a \vee b) = xa \vee xb$, $(a \wedge b)x = ax \wedge bx$, $x(a \wedge b)$ and $x(a \wedge b) = xa \wedge xb$.*

ii) *$(a \vee b)^{-1} = a^{-1} \wedge b^{-1}$ and $(a \wedge b)^{-1} = a^{-1} \vee b^{-1}$,*

iii) *$a \wedge b = b(a \vee b)^{-1}a$ and $a \vee b = b(a \wedge b)^{-1}a$*

Proof.

i) It follows directly from Proposition1.4.

ii) From (i) and the proof of Proposition 1.5, it follows that, $(a \vee b)^{-1} = [a(a \wedge b)^{-1}b]^{-1} = b^{-1}(a \wedge b)a^{-1} = a^{-1} \wedge b^{-1}$. The other part $(a \wedge b)^{-1} = a^{-1} \vee b^{-1}$ is proved similarly.

iii) It follows directly from Proposition 1.5.

∎

Definition 1.19 *[8] Let G be a group, for every $x \in G$, the positive part of x and dually, the negative part of x is defined by $x_+ = x \vee e \in G^+$ and $x_- = x \wedge e \in G^-$, respectively.*

Theorem 1.2 *Let $(G, \cdot, \leqslant)$ be an ℓ-group, then for all $x, y \in G$, the following hold :*

i) $(x_+)^{-1} = (x^{-1})_-$ *and* $(x_-)^{-1} = (x^{-1})_+$,

ii) $x \vee y = (yx^{-1})_+ x$ *and* $x \wedge y = x(x^{-1}y)_-$.

Proof.

i) Let $x \in G$, then we have $x_+ = x \vee e$ and $x_- = x \wedge e$.

$$\begin{aligned} (x_+)^{-1} &= (x \vee e)^{-1} \\ &= x^{-1} \wedge e \\ &= (x^{-1})_-. \end{aligned}$$

Similarly $(x_-)^{-1} = (x^{-1})_+$.

ii) Let $x, y \in G$, then we have

$$\begin{aligned} (yx^{-1})_+ x &= (yx^{-1} \vee e)x \\ &= (yx^{-1}x \vee x) \\ &= y \vee x. \end{aligned}$$

Then $x \vee y = (yx^{-1})_+ x$.
Similarly $x \wedge y = x(x^{-1}y)_-$.

■

Lemma 1.2 *Let $(G, \cdot, \leqslant)$ be a ℓ-group, then for any positive integer n and $z \in G$,*

i) $(z \wedge e)^n = z^n \wedge z^{n-1} \wedge \cdots \wedge z \wedge e$,

ii) $(z \vee e)^n = z^n \vee z^{n-1} \vee \cdots \vee z \vee e$.

Proof.

Let G be a group. We see that the statements are obviously true if $n = 1$, we assume that they are true for $n - 1$, and we will prove that are true for n.

Then, for $n-1$ we get, $(z\wedge e)^{n-1} = z^{n-1}\wedge z^{n-2}\wedge\cdots\wedge z\wedge e$.

$$\begin{aligned}
(z\wedge e)^n &= (z\wedge e)^{n-1}(z\wedge e). \\
&= (z\wedge e)^{n-1}z\wedge(z\wedge e)^{n-1}e \ \text{ (see Proposition 1.6)} \\
&= (z^{n-1}\wedge z^{n-2}\wedge\cdots\wedge z\wedge e)z\wedge(z^{n-1}\wedge z^{n-2}\wedge\cdots\wedge z\wedge e) \\
&= (z^n\wedge z^{n-1}\wedge\cdots\wedge z)\wedge(z^{n-1}\wedge z^{n-2}\wedge\cdots\wedge z\wedge e) \\
&= z^n\wedge z^{n-1}\wedge\cdots\wedge z\wedge z^{n-1}\wedge z^{n-2}\wedge\cdots\wedge z\wedge e \\
&= z^n\wedge z^{n-1}\wedge z^{n-2}\wedge\cdots\wedge z\wedge e.
\end{aligned}$$

The statement is true for n.

Then $(z\wedge e)^n = z^n\wedge z^{n-1}\wedge\cdots\wedge z\wedge e$.

Similarly $(z\vee e)^n = z^n\vee z^{n-1}\vee\cdots\vee z\vee e$. ■

CHAPITRE

TWO

PROPERTIES OF FUZZY SUBSETS AND FUZZY SUBGROUPS

The purpose of this second chapter is to provide a basic introduction to the fuzzy set , its characteristic notions, some operations on fuzzy sets and the basic properties of $\alpha-$cuts of a fuzzy set paying particular attention to characterize a fuzzy set by means of its $\alpha-$cuts. Next, we investigate the notion of fuzzy subgroups and normal fuzzy subgroups.

2.1 Fuzzy Subsets

2.1.1 Definition and Examples

In this subsection, we present the concepts of fuzzy set theory with illustrative examples.

Definition 2.1 *[1] Let X be a reference set. A* ***fuzzy subset*** *A is a function defined from X to the interval* $[0,1]$.

• It is customary in the fuzzy literature to have two notations for fuzzy sets, the letter A and the notation μ_A.

• We can describe a fuzzy subset A by the pair $\{(x,\mu_A(x)) \mid x\in X\}$.

• The function $\mu_A : X \to [0,1]$ is called the membership function, and the value $\mu_A(x)$ is the degree of membership of x to the fuzzy set A.

Remark 2.1 *Those function whose images are contained in the set* $\{0,1\}$ *correspond to the classic set for which* μ_A *is the indicator function* χ_A. *So the classic subsets are special cases of fuzzy subsets.*

Example 2.1 ***(Finite case) :***
Let $X = \{a,b,c,d,e\}$ and $A = \{(a,0.6),(b,0.9),(c,0.2),(d,0),(e,0.5)\}$. A is a fuzzy subset of X.

Example 2.2 ***(Infinite case) :***
Suppose that we want to model the fuzzy concept "young", let the set X be the positive real numbers representing the possible ages of people. People with age less than 25 take the degree 1, between 25 and 40 their degree is included in the interval $]0,1[$, and people with age more than 40 take the 0. We can represent this subset with the following membership function :

$$\mu_A(x) = \begin{cases} 1 & if \quad x \leq 25, \\ \frac{40-x}{15} & if \quad 25 < x < 40, \\ 0 & if \quad 40 \leq x. \end{cases}$$

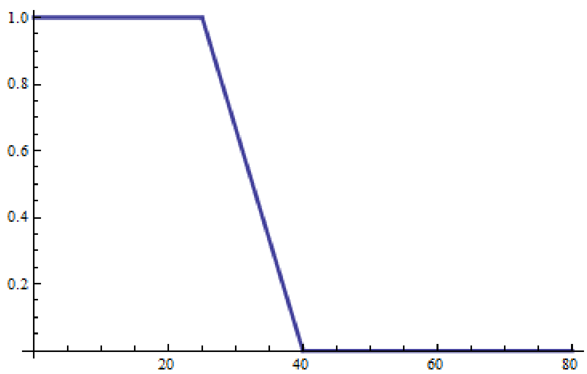


FIGURE 2.1 – The fuzzy subset "young".

Notation 2.1 *The set of all fuzzy subsets of X is called the fuzzy power set of X and is denoted by $\mathscr{F}\mathscr{P}(X)$.*

2.1.2 Characteristic notions

The characteristics of a fuzzy subset A of non empty X which describe it, are the ones that show how much it makes it different than a classic subset.

Definition 2.2 *[4] Let $A \in \mathscr{F}\mathscr{P}(X)$, and μ_A its membership function.*

i) *The **height** of A, denoted by $H(A)$ correspond to the upper bound of the codomain of its membership function, and we write :*

$$H(A) = sup\{\mu_A(x) \mid x \in X\}.$$

ii) *The* ***support*** *of A, denoted by* $supp(A)$*(it is also denoted by* A^* *in some references) is the subset whose elements are included at least a little in A, and we write :*

$$supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

iii) *The* ***core*** *of A, denoted by* $core(A)$ *is the subset whose elements are included totally in A, and we write :*

$$core(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

Remark 2.2 *Let A a fuzzy subset of X.*

i) *A is called* ***normalized*** *if and only if* $H(A) = 1$*, in practice it is extremely rare to work on non normalized fuzzy subset.*

ii) *A is called a finite fuzzy set if* A^* *is a finite subset, and an infinite fuzzy subset otherwise.*

From the above remark, it is clear that if X is finite, then every fuzzy subset A of X is finite.

Example 2.3 *Let us consider the fuzzy subset A of Example 2.1. It obviously holds that :*

$$H(A) = 0.9,\ \ supp(A) = \{a,b,c,e\},\ \ core(A) = \emptyset.$$

Example 2.4 *Consider the fuzzy subset "young" of Example 2.2. It obviously holds that :*

$$H(A) = 1,\ \ supp(A) = [0,40[,\ \ core(A) = [0,25].$$

2.1.3 Operations on fuzzy sets

We define on fuzzy sets the same operations of the classic sets which are for each two fuzzy subsets A and B of X given by the following rules.

Definition 2.3 *[3]*

i) *A fuzzy set A is empty, we note* $A = \emptyset$ *if and only if*

$$\forall\, x \in X : \mu_A(x) = 0.$$

ii) *Two fuzzy sets A and B are equal, we note* $A = B$ *if and only if*

$$\forall\, x \in X : \mu_A(x) = \mu_B(x)$$

iii) *A fuzzy set A is contained in a fuzzy set B, we note $A \subseteq B$ if and only if*

$$\forall\, x \in X : \mu_A(x) \leq \mu_B(x).$$

Let $A, B \subset \mathscr{P}(X)$ two subset of X. As we know, there are the familiar operations of union, intersection, and complement. These are given by the rules

$$A \cup B = \{x \mid x \in A \ or \ x \in B\},$$

$$A \cap B = \{x \mid x \in A \ and \ x \in B\},$$

$$A^c = \{x \mid x \notin A\}.$$

As we have noted that a classic set A of X can be represented by a function $\chi_A : X \to \{0,1\}$ writing these rules in terms of indicator functions, we get :

$$\chi_{A\cup B}(x) = \max\{\chi_A(x), \chi_B(x)\},$$

$$\chi_{A\cap B}(x) = \min\{\chi_A(x), \chi_B(x)\},$$

$$\chi_{A^c}(x) = 1 - \chi_A(x).$$

A natural way to extend these operations to the fuzzy subsets of X is by the membership functions. Let A, B be two fuzzy subset of X.

i) **Union** : $A \cup B$ is defined by the membership function

$$\mu_{A\cup B}(x) \;=\; \max\{\mu_A(x), \mu_B(x)\}.$$

ii) **Intersection** : $A \cap B$ is defined by by the membership function

$$\mu_{A\cap B}(x) \;=\; \min\{\mu_A(x), \mu_B(x)\}.$$

iii) **Complement** of the fuzzy subset A, is noted by A^c and is defined by the membership function

$$\mu_{A^c}(x) \;=\; 1 - \mu_A(x).$$

Remark 2.3 *For any collection $\{A_i \mid i \in I\}$ of fuzzy subsets of X, where I is a non-empty index set and μ_{A_i} its membership functions, the union and intersection of A_i are defined by the* *following membership functions :*

$$\mu_{\underset{i\in I}{\cup}Ai}(x) = \underset{i\in I}{sup}\,\{\mu_{A_i}(x)\} = \underset{i\in I}{\vee}A_i(x),$$

$$\mu_{\underset{i\in I}{\cap}Ai}(x) = \underset{i\in I}{inf}\,\{\mu_{A_i}(x)\} = \underset{i\in I}{\wedge}A_i(x).$$

Example 2.5 ***(Finite case) :***
Let $X = \{a,b,c,d,e,r,s,t\}$ the set which represent a menu of restaurant, the patron want to classify it according to two description, tasty and cheap. Let A and B two fuzzy subset of X, such that A represent "tasty" and B "cheap". We get

$$A = \{(a,0.6),(b,1),(c,0.1),(e,0.4),(r,0.8),(s,0.5)\};$$

$$B = \{(b,0.3),(c,0.6),(d,0.5),(e,0.9),(s,1),(t,0.7)\}.$$

Which we give
$A\cup B = \{(a,0.6),(b,1),(c,0.6),(d,0.5),(e,0.9),(r,0.8),(s,1)\}$*; a fuzzy subset represent the description "tasty or cheap".*
$A\cap B = \{(b,0.3),(c,0.1),(e,0.4),(s,0.5)\}$*; a fuzzy subset represent the description "tasty and cheap".*
$A^c = \{(a,0.4),(c,0.9),(d,1),(e,0.6),(r,0.2),(s,0.5),(t,1)\}$*; a fuzzy subset represent the description "not tasty".*
$B^c = \{(a,1),(b,0.7),(c,0.4),(d,0.5),(e,0.1),(r,1),(t,0.3)\}$*; a fuzzy subset represent the description "not cheap".*

Example 2.6 ***(Infinite case) :***
We consider the example (2.2). the set X be the positive real numbers representing the possible ages of people, the function μ_A define the fuzzy subset "young" and μ_B the fuzzy subset "have thirty old", such that :

$$\mu_A(x) = \begin{cases} 1 & if \quad x \leq 25 \\ \frac{40-x}{15} & if \quad 25 < x < 40 \\ 0 & if \quad 40 \leq x \end{cases},\ \mu_B(x) = \begin{cases} 0 & if \quad x \leq 25 \\ \frac{x-25}{3} & if \quad 25 < x < 28 \\ 1 & if \quad 28 < x < 32 \\ \frac{35-x}{3} & if \quad 32 < x < 35 \\ 0 & if \quad 40 \leq x \end{cases}.$$

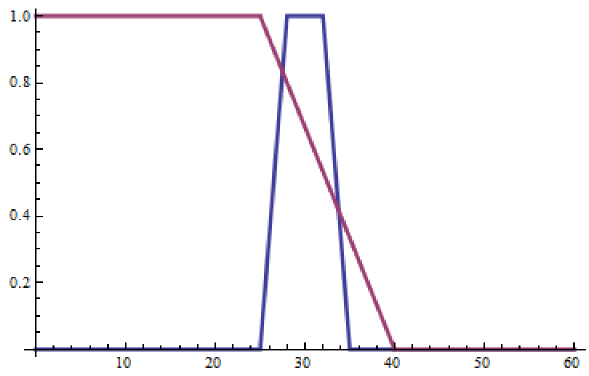
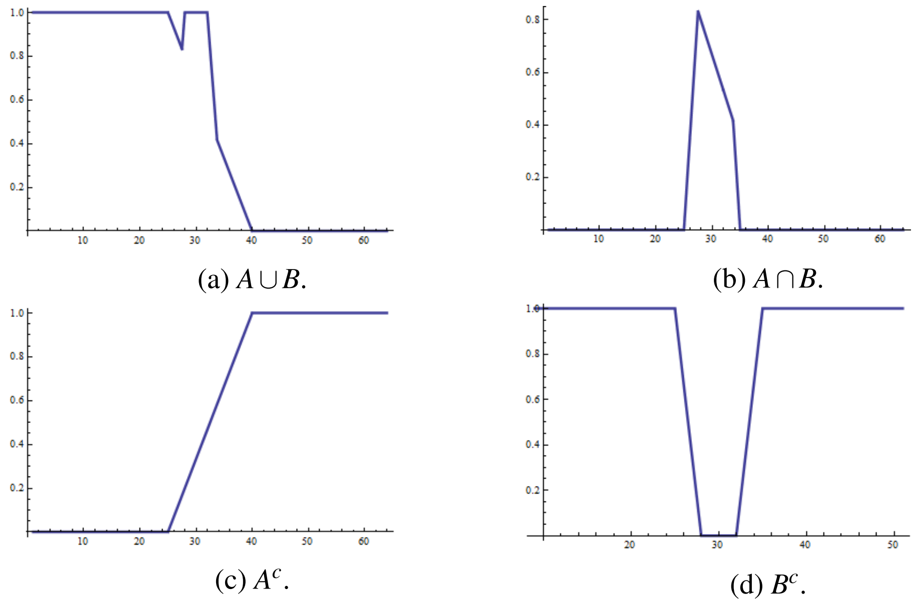


FIGURE 2.2 – Membership function of A and B.

The following plots are the plots of union, intersection and the complement of the fuzzy subset A and B.



(a) $A \cup B$.

(b) $A \cap B$.

(c) A^c.

(d) B^c.

FIGURE 2.3 – Membership functions.

2.1.4 Alpha-cuts

One of the characteristics of a fuzzy subset A of X is the alpha-cuts or also known as the level set. In this subsection and after given the definition of the alpha-cut, we will investigate its basic properties, paying particular attention to characterize a fuzzy set by means of its alpha-cuts.

Definition 2.4 *[4] For all $\alpha \in [0,1]$, we construct the ordinary subset A_α of X associates to $A \in \mathscr{F}\mathscr{P}(X)$, by selecting all element of X belonging to A with a degree at least equal to α, we write :*

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

The characteristic function of A_α is χ_{A_α} such that $\chi_{A_\alpha}(x) = 1$ if and only if $\mu_A(x) \geq \alpha$.

Example 2.7 ***(Finite case) :****Let $X = \{a,b,c,d,e\}$ and A a fuzzy subset of X such that $A = \{(a,0.6),(b,0.1),(c,0.8),(d,0.4),(e,0.3)\}$. We have :*
$A_{0.2} = X,\ \ A_{0.4} = \{a,c,d\},\ \ A_{0.7} = \{c\}$ *and* $A_{0.9} = \emptyset$.

Example 2.8 *Consider the fuzzy subset A "young" of X of Example 2.2. It is easy to verify that :* $A_{0.4} = [0,34]$, *and* $A_{0.7} = [0, \frac{295}{10}]$.

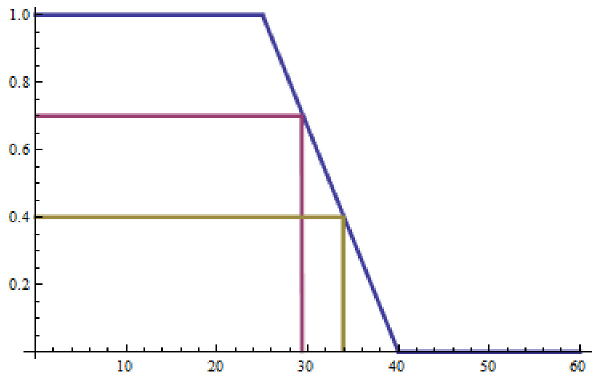


FIGURE 2.4 – The α-cuts of A

Proposition 2.1 *[2] Let A,B be two fuzzy subset of X and $\alpha,\beta \in [0,1]$. The α-cuts satisfy the following statements :*

- ***i)*** $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$,
- ***ii)*** $A \subset B \Rightarrow A_\alpha \subset B_\alpha$,
- ***iii)*** $\alpha < \beta \Rightarrow A_\beta \subset A_\alpha$.

Proof.

i) Let $x \in (A \cup B)_\alpha$. We have, $(A \cup B)_\alpha = \{x \in X \mid \mu_{(A \cup B)}(x) \geq \alpha\}$.

$$\begin{aligned} x \in (A \cup B)_\alpha &\Leftrightarrow \mu_{(A \cup B)}(x) \geq \alpha \\ &\Leftrightarrow max\{\mu_A(x), \mu_B(x)\} \geq \alpha \\ &\Leftrightarrow \mu_A(x) \geq \alpha \text{ or } \mu_B(x) \geq \alpha \\ &\Leftrightarrow x \in A_\alpha \text{ or } x \in B_\alpha \\ &\Leftrightarrow x \in (A_\alpha \cup B_\alpha). \end{aligned}$$

Then, $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.

ii) Let $x \in A_\alpha$.

$$\begin{aligned} x \in A_\alpha &\Rightarrow \mu_A(x) \geq \alpha \\ &\Rightarrow \mu_B(x) \geq \alpha \\ &\Rightarrow x \in B_\alpha. \end{aligned}$$

Then $A_\alpha \subset B_\alpha$.

iii) Let $x \in A_\beta$,

$$\begin{aligned} x \in A_\beta &\Rightarrow \mu_A(x) \geq \beta \\ &\Rightarrow \mu_A(x) \geq \alpha \\ &\Rightarrow x \in A_\alpha. \end{aligned}$$

Then $A_\beta \subset A_\alpha$.

■

The following two theorems state some basic properties of the $\alpha-$ cuts of a given fuzzy set.

Theorem 2.1 *[2] Suppose that $\{A_i \mid i \in I\}$ is a collection of fuzzy subsets of X. Then for any $\alpha \in [0,1]$ it holds that*

i) $\bigcup\limits_{i \in I} (A_i)_\alpha \subseteq (\bigcup\limits_{i \in I} A_i)_\alpha$,

ii) $\bigcap\limits_{i \in I} (A_i)_\alpha = (\bigcap\limits_{i \in I} A_i)_\alpha$.

Moreover, if I is finite, then we have equality in (i).

Proof.

i) Let $x \in \underset{i\in I}{\cup}(A_i)_\alpha$,

$$\begin{aligned} x \in \underset{i\in I}{\cup}(A_i)_\alpha &\Rightarrow \exists\, i \in I,\ x \in (A_i)_\alpha \\ &\Rightarrow \exists\, i \in I,\ \mu_{A_i}(x) \geq \alpha \\ &\Rightarrow \underset{i\in I}{sup}\ \{\mu_{A_i}(x)\} \geq \alpha \\ &\Rightarrow \mu_{\underset{i\in I}{\cup}Ai}(x) \geq \alpha \\ &\Rightarrow x \in (\underset{i\in I}{\cup}A_i)_\alpha. \end{aligned}$$

Then $\underset{i\in I}{\cup}(A_i)_\alpha \subseteq (\underset{i\in I}{\cup}A_i)_\alpha$.

ii) Let $x \in \underset{i\in I}{\cap}(A_i)_\alpha$,

$$\begin{aligned} x \in \underset{i\in I}{\cap}(A_i)_\alpha &\Leftrightarrow \forall\, i \in I,\ x \in (A_i)_\alpha \\ &\Leftrightarrow \forall\, i \in I,\ \mu_{A_i}(x) \geq \alpha \\ &\Leftrightarrow \underset{i\in I}{inf}\ \{\mu_{A_i}(x)\} \geq \alpha \\ &\Leftrightarrow \mu_{\underset{i\in I}{\cap}Ai}(x) \geq \alpha \\ &\Leftrightarrow x \in (\underset{i\in I}{\cap}A_i)_\alpha. \end{aligned}$$

Then $\underset{i\in I}{\cap}(A_i)_\alpha = (\underset{i\in I}{\cap}A_i)_\alpha$.

■

Theorem 2.2 *[2] Let $A \in \mathscr{F}\mathscr{P}(X)$, and $\alpha_i \in [0,1]$ ($i \in I$). Then it holds that*

i) $\underset{i\in I}{\cup}A_{\alpha_i} \subseteq A_{\underset{i\in I}{\wedge}\alpha_i}$,

ii) $\underset{i\in I}{\cap}A_{\alpha_i} = A_{\underset{i\in I}{\vee}\alpha_i}$.

Proof.

i) Let $x \in \underset{i\in I}{\cup}A_{\alpha_i}$,

$$\begin{aligned} x \in \underset{i\in I}{\cup}A_{\alpha_i} &\Rightarrow \exists\, i \in I,\ x \in A_{\alpha_i} \\ &\Rightarrow \exists\, i \in I,\ \mu_A(x) \geq \alpha_i \\ &\Rightarrow \mu_A(x) \geq \underset{i\in I}{inf}\ \{\alpha_i\} \\ &\Rightarrow \mu_A(x) \geq \underset{i\in I}{\wedge}\alpha_i \\ &\Rightarrow x \in A_{\underset{i\in I}{\wedge}\alpha_i}. \end{aligned}$$

Then $\underset{i\in I}{\cup} A_{\alpha_i} \subseteq A_{\underset{i\in I}{\wedge}\alpha_i}$.

ii) Let $x \in \underset{i\in I}{\cap} A_{\alpha_i}$,

$$\begin{aligned} x \in \underset{i\in I}{\cap} A_{\alpha_i} &\Leftrightarrow \forall\, i\in I,\ x\in A_{\alpha_i} \\ &\Leftrightarrow \forall\, i\in I,\ \mu_A(x) \geq \alpha_i \\ &\Leftrightarrow \mu_A(x) \geq \underset{i\in I}{sup}\ \{\alpha_i\} \\ &\Leftrightarrow \mu_A(x) \geq \underset{i\in I}{\vee} \alpha_i \\ &\Leftrightarrow x \in A_{\underset{i\in I}{\vee}\alpha_i}. \end{aligned}$$

Then $\underset{i\in I}{\cap} A_{\alpha_i} = A_{\underset{i\in I}{\vee}\alpha_i}$.

■

The following theorem characterize any fuzzy subset A of X by means of their α-cuts.

Theorem 2.3 *Let $A \in \mathscr{F}\mathscr{P}(X)$, μ_A its membership function and $\alpha \in [0,1]$. Then for all $x \in X$ it holds that*

$$\mu_A(x) = \underset{\alpha\in[0,1]}{sup} \ (\alpha \cdot \chi_{A_\alpha}(x)).$$

Proof.

Let $x \in X$, suppose that $\mu_A(x) = \beta$ ($\beta \in [0,1]$).

$$\begin{aligned} \mu_A(x) = \beta &\Rightarrow \mu_A(x) \geq \beta \\ &\Rightarrow x \in A_\beta \\ &\Rightarrow \chi_{A_\beta}(x) = 1. \end{aligned}$$

On the one hand, as $\mu_A(x) = \beta.1 = \beta\chi_{A_\beta}(x)$, it follows that $\mu_A(x) \leq \underset{\alpha\in[0,1]}{sup} \ (\alpha.\chi_{A_\alpha}(x))$.

On the other hand, we have $\chi_{A_\beta}(x) = \begin{cases} 1 & if \quad \mu_A(x) \geq \beta \\ 0 & if \quad \mu_A(x) \leq \beta \end{cases}$, it follows that, $\beta.\chi_{A_\beta}(x) = \begin{cases} \beta & if \quad \mu_A(x) \geq \beta \\ 0 & if \quad \mu_A(x) \leq \beta \end{cases}$. This implies that, $\beta.\chi_{A_\beta}(x) \leq \beta$ and $\beta = \mu_A(x)$, thus, $\beta.\chi_{A_\beta}(x) \leq \mu_A(x)$. Then, $\underset{\alpha\in[0,1]}{sup} \ (\alpha.\chi_{A_\alpha}(x)) \leq \mu_A(x)$.

Therefore, it holds that $\forall x \in X,\ \mu_A(x) = \underset{\alpha\in[0,1]}{sup} \ (\alpha.\chi_{A_\alpha}(x))$. ■

Example 2.9 *Let $X = \{a,b,c,d,e\}$ be a reference set, we will describe the fuzzy subset A of X from the following α-cuts,*

$A_{0.1} = \{a,b,c,d,e\}, A_{0.3} = \{b,c,d,e\}, A_{0.8} = \{c,d\}, A_1 = \{c\}$.

We find, $\mu_A(a) = sup\ \{0.1\chi_{A_{0.1}}(a), ..., 1\chi_{A_1}(a)\}$,
$\mu_A(a) = sup\ \{0.1 \times 1, 0.3 \times 0, 0.8 \times 0, 1 \times 0\} = 0.1$,
$\mu_A(b) = sup\ \{0.1 \times 1, 0.3 \times 1, 0.8 \times 0, 1 \times 0\} = 0.3$,
$\mu_A(c) = sup\ \{0.1 \times 1, 0.3 \times 1, 0.8 \times 1, 1 \times 1\} = 1$,
$\mu_A(d) = sup\ \{0.1 \times 1, 0.3 \times 1, 0.8 \times 1, 1 \times 0\} = 0.8$,
$\mu_A(e) = sup\ \{0.1 \times 1, 0.3 \times 1, 0.8 \times 0, 1 \times 0\} = 0.3$.
Then, we get $A = \{(a, 0.1), (b, 0.3), (c, 1), (d, 0.8), (e, 0.3)\}$.

2.2 Fuzzy subgroups, Normal Fuzzy subgroups

In this section, we will present the definition of fuzzy subgroup, fuzzy normal subgroup and investigate their basic properties which we need in the next chapter.
In what follows, we will note the product of two element of $(G, \cdot)$ by xy, instead of $x \cdot y$.

2.2.1 Fuzzy subgroups

Definition 2.5 *[2] Let* $H \in \mathscr{F}\mathscr{P}(G)$. *H is called a* ***fuzzy subgroup*** *of G if*

i) $\mu_H(xy) \geq \min\{\mu_H(x), \mu_H(y)\}$,

ii) $\mu_H(x^{-1}) \geq \mu_H(x)$.

Remark 2.4 *It is easy to show that, the conditions (i) and (ii) of Definition 2.5 are equivalents to the unique following condition :*

$$\mu_H(xy^{-1}) \geq \min\{\mu_H(x), \mu_H(y)\}, \quad \forall x, y \in G.$$

Notation 2.2 *We denote by* $\mathscr{F}(G)$ *the set of all fuzzy subgroups of G.*

In what follows, G denotes a group and e its identity element.

Proposition 2.2 *Let H be fuzzy subgroup of G. Then for all* $x, y \in G$, *it hold that*

i) $\mu_H(x) \leq \mu_H(e)$,

ii) $\mu_H(xy^{-1}) = \mu_H(e) \Rightarrow \mu_H(x) = \mu_H(y)$,

iii) $\mu_H(x) \leq \mu_H(x^n)$,

iv) $\mu_H(x) = \mu_H(x^{-1})$.

Proof.

i) Let $x \in$ G. We have $\mu_H(e) = \mu_H(xx^{-1}) \geq \min(\mu_H(x)\ ,\ \mu_H(x^{-1})) = \mu_H(x)$. Thus, $\mu_H(x) \leq \mu_H(e)$, for all $x \in G$.

ii) Let $x, y \in G$ and suppose that $\mu_H(xy^{-1}) = \mu_H(e)$. We have

$$\begin{aligned} \mu_H(x) = \mu_H(xy^{-1}y) &\geq \min(\mu_H(xy^{-1})\ ,\ \mu_H(y)) \\ &\geq \min(\mu_H(e)\ ,\ \mu_H(y)) \\ &\geq \mu_H(y). \end{aligned}$$

In similar way, we show that $\mu_H(y) \geq \mu_H(x)$. We conclude that $\mu_H(x) = \mu_H(y)$.

iii) By recurrence. For $n = 1$, $P(1) : \mu_H(x) \leq \mu_H(x)$ is true.
For $n > 1$, we suppose that $\mu_H(x) \leq \mu_H(x^n)$ and prove that $\mu_H(x) \leq \mu_H(x^{n+1})$.
For all $x \in G$, $\mu_H(x^{n+1}) = \mu_H(x^n.x) \geq \min(\mu_H(x^n), \mu_H(x)) \geq \mu_H(x)$.
Hence, $\mu_H(x) \leq \mu_H(x^n)$, for all $n \in \mathbb{N}$.

iv) By definition, we have $\mu_H(x) \leq \mu_H(x^{-1})$. It remains to show that $\mu_H(x) \geq \mu_H(x^{-1})$. We put $y = x^{-1}$, it follows by definition that $\mu_H(y^{-1}) \geq \mu_H(y)$, this implies that $\mu_H(x) \geq \mu_H(x^{-1})$. Hence, $\mu_H(x) = \mu_H(x^{-1})$.

■

Proposition 2.3 *The intersection of two fuzzy subgroups of a group G is a fuzzy subgroup of G.*

Proof.

Let $H, K \in \mathscr{F}(G)$ and $x, y \in G$. We have, $\mu_{H\cap K}(xy) = \min(\mu_H(xy), \mu_K(xy))$, it follows that

$$\begin{aligned} \mu_{H\cap K}(xy) &\geq \min(\min(\mu_H(x), \mu_H(y)), \min(\mu_K(x), \mu_K(y))) \\ &\geq \min(\min(\mu_H(x), \mu_K(x)), \min(\mu_H(y), \mu_K(y))) \\ &\geq \min(\mu_{H\cap K}(x), \mu_{H\cap K}(y)). \end{aligned}$$

And we have, $\mu_{H\cap K}(x^{-1}) = \min(\mu_H(x^{-1}), \mu_K(x^{-1}))$. Since H, K both are fuzzy subgroup, it follows that $\mu_{H\cap K}(x^{-1}) \geq \min(\mu_H(x), \mu_K(x)) = \mu_{H\cap K}(x)$. Consequently, $H \cap K$ is a fuzzy subgroup of G. ■

Note that the union of two fuzzy subgroups of a group G does not necessarily need to be a fuzzy subgroup of G as can be seen in the following example.

Let us consider the group of integers $\mathbb{Z}$ under addition. We define the fuzzy subgroups A

and B by,

$$\mu_A(x) = \begin{cases} 1 & if\ x \in 2\mathbb{Z}, \\ \frac{1}{4} & otherwise. \end{cases} \quad and\ \mu_B(x) = \begin{cases} 1 & if\ x \in 3\mathbb{Z}, \\ \frac{1}{5} & otherwise. \end{cases}$$

We can easily find the subgroup $A \cup B$ defined by,

$$\mu_{A\cup B}(x) = \begin{cases} 1 & if\ x \in 2\mathbb{Z} \cup 3\mathbb{Z}, \\ \frac{1}{4} & otherwise. \end{cases}$$

Let $x = 2$ and $y = 3$, then $\mu_{A\cup B}(2) = \mu_{A\cup B}(3) = 1$, and
$\mu_{A\cup B}(x+y) = \mu_{A\cup B}(5) = \frac{1}{4}$,
this implies that, $\mu_{A\cup B}(2+3) = \frac{1}{4} < \min\{\mu_{A\cup B}(2), \mu_{A\cup B}(3)\} = 1$ which shows that $A \cup B$ is not a subgroup of $\mathbb{Z}$.

We can extend the above proposition as following

Proposition 2.4 *The intersection of family of subgroups of a group G is a fuzzy subgroup of G.*

Proof.

Let $(A_i)_{i\in I}$ be a family of subgroups of G and $x, y \in G$. We have, $\mu_{\underset{i\in I}{\cap} A_i}(xy) = \underset{i\in I}{inf} A_i(xy)$. Since A_i is a subgroup for all $i \in I$, it follows that

$$\begin{aligned} \mu_{\underset{i\in I}{\cap} A_i}(xy) &\geq \underset{i\in I}{inf} \min(\mu_{A_i}(x), \mu_{A_i}(y)) \\ &\geq \min(\underset{i\in I}{inf} \mu\mu_{A_i}(x)\ ,\ \underset{i\in I}{inf} \mu_{A_i}(y)) \\ &= \min(\underset{i\in I}{\cap} A_i(x)\ ,\ \underset{i\in I}{\cap} A_i(y)). \end{aligned}$$

And we have, $\mu_{\underset{i\in I}{\cap} A_i}(x^{-1}) = \underset{i\in I}{inf} A_i(x^{-1})$. Since for all $i \in I$, A_i is a fuzzy subgroup, it follows that $\mu_{\underset{i\in I}{\cap} A_i}(x^{-1}) \geq \underset{i\in I}{inf} A_i(x^{-1}) = \mu_{\underset{i\in I}{\cap} A_i}(x)$. Consequently, $\underset{i\in I}{\cap} A_i$ is a fuzzy subgroup of G. ■

2.2.2 Normal fuzzy subgroup

Definition 2.6 *[2] Let $N \in \mathscr{F}(G)$. N is called a* ***normal fuzzy subgroup*** *of G if $\forall x, y \in G$,*

$$\mu_N(xy) = \mu_N(yx).$$

The following proposition identifies the basic properties of a fuzzy subgroup, which will be helpful in the rest of our work .

Proposition 2.5 *Let N be a fuzzy subgroup of the group G. Then the following statements are equivalent for all $x, y \in G$,* *(1)* $\mu_N(xy) = \mu_N(yx)$. *(N is called an Abelian fuzzy subset of G).*

(2) $\mu_N(xyx^{-1}) = \mu_N(y)$,

(3) $\mu_N(xy) \geq \mu_N(yx)$,

(4) $\mu_N(xy) \leq \mu_N(yx)$.

Proof.

Let $x, y \in G$.
(1) $\Rightarrow$ **(2)** We have $\mu_N(xyx^{-1}) = \mu_N((xy)x^{-1}) = \mu_N(x^{-1}(xy)) = \mu_N((x^{-1}x)y) = \mu_N(y)$.
(2) $\Rightarrow$ **(3)** We have $\mu_N(xy) = \mu_N(x(yx)x^{-1}) = \mu_N(xy)$. Hence, $\mu_N(xy) \geq \mu_N(yx)$.
(3) $\Rightarrow$ **(4)** According to (3), $\mu_N(xy) \geq \mu_N(yx) \geq \mu_N(xy)$. This implies that $\mu_N(xy) \leq \mu_N(yx)$.
(4) $\Rightarrow$ **(1)** Similar to the previous implication. ■

The following corollary is a direct result of the previous proposition.

Corollary 2.1 *Let G be a group. A fuzzy subgroup N of G is a fuzzy normal subgroup if it satisfies the following equivalent conditions.*

(1) $\forall x, y \in G, \mu_N(xy) = \mu_N(yx)$.

(2) $\forall x, y \in G, \mu_N(xyx^{-1}) = \mu_N(y)$,

(3) $\forall x, y \in G, \mu_N(xy) \geq \mu_N(yx)$,

(4) $\forall x, y \in G, \mu_N(xy) \leq \mu_N(yx)$.

CHAPITRE

THREE

FUZZY LATTICE-ORDERED SUBGROUPS

After we introduced the general notions of fuzzy sets of reference set X, we will put a condition on X to be once a lattice, an ordered subgroup or an ℓ-subgroup in order to be able to apply the notion of convexity in fuzzy case.

In this section we will note the degree of membership of an element x to the fuzzy set A by $A(x)$ instead of $\mu_A(x)$.

3.1 Fuzzy Cosets and Fuzzy Normal Subgroups

Definition 3.1 *[15] Let $H \in \mathscr{F}(G)$, for any $x \in G$ we define a map $H_x : G \to [0,1]$ by,*

$$H_x(y) = H(yx^{-1}), \ \ \forall y \in G.$$

H_x is called the fuzzy coset of G determined by x and H.

Denote the set of all fuzzy cosets of H by G/H (i.e., $G/H = \{H_x \mid x \in G\}$.)

From the above definition it follows that $H_e = H$. Indeed, for all $y \in G$, we have $H_e(y) = H(ye^{-1}) = H(y)$. Hence, $H_e = H$.

Example 3.1 *Consider the Klein 4-group [1] $G = \{e,a,b,c\}$ of Example 1.7 (see table of Figure 1.2.*

Let us consider the fuzzy set H of G defined by $H(e) = H(c) = t_0$ and $H(a) = H(b) = t_1$, where $t_0 > t_1$. It is easy to see that H is a fuzzy group of G.

For instance if we take $x = a$, then H_a is defined as following :

1. The Klein four-group is a group with four elements, in which each element is self-inverse and in which composing any two of the three non-identity elements produces the third one.

$H_a(e) = H(ea^{-1}) = H(a^{-1}) = H(a) = t_1$, $H_a(a) = H(aa^{-1}) = H(e) = t_0$,
$H_a(b) = H(ba^{-1}) = H(ba) = H(c) = t_0$, $H_a(c) = H(ca^{-1}) = H(ca) = H(b) = t_1$. *Hence,*

$$H_a = \{(e,t_1),(a,t_0),(b,t_0),(c,t_1)\}.$$

By the same way we find, $H_b = \{(e,t_1),(a,t_0),(b,t_0),(c,t_1)\}, H_c = \{(e,t_0),(a,t_1),(b,t_1),(c,t_0)\}$ *and* $H_e = H = \{(e,t_0),(a,t_1),(b,t_1),(c,t_0)\}$.
Finally, we conclude that $G/H = \{H_a, H\}$.

Properties 3.1 *If H is a fuzzy normal subgroup of a group G, then the set of all fuzzy cosets of H form a group under the inner operation $\circ$ defined, for all $H_x, H_y \in G/H$, $H_x \circ H_y = H_{xy}$.*

Proof.

i) The identity element is H itself. Indeed, for all $x \in G$,
$H_x \circ H = H_x \circ H_e = H_{xe} = H_x$ and $H \circ H_x = H_e \circ H_x = H_{ex} = H_x$.

ii) "$\circ$" is associative. Indeed $\forall x,y,z \in G$ we have,
$H_x \circ (H_y \circ H_z) = H_x \circ H_{yz} = H_{x(yz)} = H_{(xy)z} = H_{xy} \circ H_z = (H_x \circ H_y) \circ Hz$.

ii) For all $x \in G$, the inverse of the coset H_x is $H_{x^{-1}}$. Indeed, we have
$H_x \circ H_{x^{-1}} = H_{xx^{-1}} = H_e = H$.

We, conclude that $(G/H, \circ)$ the set of all fuzzy cosets of H is a group. ∎

Theorem 3.1 *Let G be a group and N a fuzzy normal subgroup of G. Then the two following statements hold*

1) *$N_{N(e)}$ is a normal subgroup of G. (for illustrate we can put $N(e) = \alpha$, we get $N_{N(e)} = N_\alpha$ it is a classic subset of G).*

2) *$N_x = N_y$ if and only if $N(x) = N(y)$.*

Proof.

1) Let $x \in G$ and $y \in N_{N(e)}$. We need to show that $xyx^{-1} \in N_{N(e)}$.
As N is a fuzzy normal subgroup and $y \in N_{N(e)}$, it follows that $N(xyx^{-1}) = N(y) \geq N(e)$. Hence, $xyx^{-1} \in N_{N(e)}$. Therefore, $N_{N(e)}$ is a normal subgroup

2) Let $x,y \in G$. We have

$$\begin{aligned} Nx = Ny &\Leftrightarrow \forall z \in G, \quad Nx(z) = Ny(z) \\ &\Leftrightarrow \forall z \in G, \quad N(x^{-1}z) = N(y^{-1}z) \\ &\Rightarrow N(x^{-1}e) = N(y^{-1}e) (\text{ we take } z = e) \\ &\Rightarrow N(x^{-1}) = N(y^{-1}) \\ &\Rightarrow N(x) = N(y). \end{aligned}$$

■

Proposition 3.1 *Let H be a fuzzy normal subgroup of a group G. Then it holds that*

$$H_x(xy) = H_x(yx) = H(y), \forall x,y \in G.$$

Proof.

Let $x,y \in G$. Since H is a fuzzy normal subgroup, it follows that $H(xyx^{-1}) = H(x)$, this implies that $H_x(xy) = H((xy)x^{-1}) = H(xyx^{-1}) = H(y) = H((yx)x^{-1}) = H_x(yx)$. ■

The following theorem extends Theorem1.1 to the fuzzy case.

Theorem 3.2 *[8] Let N be a fuzzy normal subgroup of G. A subset P of the quotient group G/N is the positive cone of a compatible order on G/N if and only if :*

(1) $P \cap P^{-1} = \{N\}$;

(2) $P^2 = P$;

(3) $\forall x \in G, \quad N_x \circ P \circ N_{x^{-1}} = P.$

Moreover, $P \cup P^{-1} = G/N$ if and only if this order is total.

Proof.

We must indicate that in the quotient groups the associated operation is " $\circ$", and its identity element is N. Then, its positive cone is determined by $P(G/N) = \{N_x \geq N \mid N_x \in G/N\}$, and $P^2(G/N) = \{N_x \circ N_y \mid N_x, N_y \in P(G/N)\}$.

⇒) Suppose that " $\leq$" is a compatible order on G/N, and $P(G/N)$ is the associated positive cone.

(1)

$$\begin{aligned} N_x \in P \cap P^{-1} &\Rightarrow N_x \in P \text{ and } N_x \in P^{-1} \\ &\Rightarrow N_x \geq N \text{ and } N \geq N_x \\ &\Rightarrow N_x = N \\ &\Rightarrow P \cap P^{-1} = \{N\}. \end{aligned}$$

(2) • Let $N_x \circ N_y \in P^2$ such that $N_x, N_y \in P$. Recall that $N_x \circ N_y = N_{xy}$.

$$\begin{aligned} N_x, N_y \in P(G/N) &\Rightarrow N_x \geq N \text{ and } N_y \geq N \\ &\Rightarrow N_x \circ N_y \geq N \circ N_y \text{ and } N \circ N_y \geq N \circ N = N \\ &\Rightarrow N_{xy} \geqslant N_y \geqslant N \\ &\Rightarrow N_{xy} \in P. \end{aligned}$$

Then $P^2(G/N) \subseteq P(G/N)$.

• Conversely, let $N_x \in P(G/N)$. We know that,

$$N_x = N_{xe} = N_x \circ N_e \in P^2(G/N).$$

Then, $P(G/N) \subseteq P^2(G/N)$.

Thus, $P^2 = P$.

(3) • Let for all $x \in G$, $N_x \in G/N$ and $N_y \in P(G/N)$, then

$$\begin{aligned} N_y \geqslant N &\Rightarrow N_x \circ N_y \circ N_{x^{-1}} \geqslant N_x \circ N \circ N_{x^{-1}} \\ &\Rightarrow N_x \circ N_y \circ N_{x^{-1}} \geqslant N \\ &\Rightarrow N_x \circ N_y \circ N_{x^{-1}} \in P(G/N), \quad \forall N_y \in P \\ &\Rightarrow N_x \circ P \circ N_{x^{-1}} \subseteq P(G/N). \end{aligned}$$

• Conversely, let $N_y \in P(G/N)$. We know that $N_y = N_e \circ N_y \circ N_{e^{-1}} \in N_x \circ P \circ N_{x^{-1}}$. Then $P(G/N) \subseteq N_x \circ P \circ N_{x^{-1}}$.

Thus, $N_x \circ P \circ N_{x^{-1}} = P(G/N)$.

$\Leftarrow$) Now, suppose that P is a subset of G/N that satisfies the conditions $(1), (2)$ *and* (3). We define a relation " $\leqslant$" on G/N by

$$N_x \leqslant N_y \;\Leftrightarrow\; N_{yx^{-1}} \in P.$$

i) First, we prove that " $\leqslant$" is an order.

- It is clear that " $\leqslant$" is reflexive cause, $\forall\, N_x \in G/N,\ N_x \leqslant N_x$ since $N_{xx^{-1}} = N \in P$ (From **(1)**, then $N \in P$ *and* $N \in P^{-1}$).
- Let $N_x, N_y \in P/N$ such that, $N_x \leqslant N_y$ *and* $N_y \leqslant N_x \;\Rightarrow\; N_{yx^{-1}} \in P$ *and* $N_{xy^{-1}} \in P$. We have, $(N_{yx^{-1}})^{-1} = N_{xy^{-1}} \in P$. Hence, $N_{yx^{-1}} \in P \cap P^{-1} = \{N\}$ then, $N_{yx^{-1}} = N \Rightarrow N_y = N_x$ which shows that " $\leqslant$" is antisymmetric.
- Let $N_x, N_y \in P/N$ such that, $N_x \leqslant N_y$ *and* $N_y \leqslant N_z \;\Rightarrow\; N_{yx^{-1}} \in P$ *and* $N_{zy^{-1}} \in P$.

$$\begin{aligned} N_{zx^{-1}} &= N_{zy^{-1}yx^{-1}} \\ &= N_{zy^{-1}} \circ N_{yx^{-1}} \in P^2(G/N) \end{aligned}$$

From **(2)**, then $N_{zx^{-1}} \in P \;\Rightarrow\; N_x \leqslant N_z$. Thus, " $\leqslant$" is transitive.

ii) Now for compatibility, let $N_x \leqslant N_y \;\Rightarrow\; N_{yx^{-1}} \in P$. From the condition **(3)**, we find $\forall a, b \in G$ and $N_{yx^{-1}} \in P$

$$\begin{aligned} N_{yx^{-1}} &= N_{ab} \circ N_{yx^{-1}} \circ N_{(ab)^{-1}} \in P \\ &\Rightarrow N_{abyx^{-1}a^{-1}b^{-1}} \in P \\ &\Rightarrow N_{(ayb)(a^{-1}x^{-1}b^{-1})} \in P \\ &\Rightarrow N_{(ayb)(axb)^{-1}} \in P \\ &\Rightarrow N_{(axb)} \leqslant N_{ayb} \end{aligned}$$

Then , " $\leqslant$" is a compatible order on G/N.

iii) For all $N_x \in P$, to be P the positive cone of G/N it must be $N \leqslant N_x \;\Rightarrow\; N_x \in P$ which is verified.

Assume that $P \cup P^{-1} = G/N$, let $N_x, N_y \in G/N$, then $N_{xy^{-1}} \in G/N$ this implies that $N_{xy^{-1}} \in P$ or $N_{xy^{-1}} \in P^{-1}$

$$\begin{aligned} N_{xy^{-1}} \in G/N &\Rightarrow N_{xy^{-1}} \in P \text{ or } N_{xy^{-1}} \in P^{-1}, \\ &\Rightarrow N_{xy^{-1}} \geqslant N \text{ or } N_{xy^{-1}} \leqslant N, \\ &\Rightarrow N_{xy^{-1}} \circ N_y \geqslant N \circ N_y \text{ or } N_{xy^{-1}} \circ N_y \leqslant N \circ N_y, \\ &\Rightarrow N_x \geqslant N_y \text{ or } N_x \leqslant N_y. \end{aligned}$$

Then " $\leqslant$" is total.

Conversely, suppose that " $\leqslant$" is total. Let $N_x \in G/N$, then $N_x \leqslant N$ or $N_x \geqslant N$, this implies that $N_x \in P$ or $N_x \in P^{-1}$. Hence, $G/N = P \cup P^{-1}$. ■

3.2 Fuzzy Convex Subgroups

We can define on fuzzy group theory the concept of convexity that we will give it in what follows with some of its properties. First, we will give in the following subsection some reminders of convex subgroup and basic properties which we will need in the sequel.

3.2.1 Convex subgroups

Definition 3.2 *A **sublattice** E of a lattice L is a subset that is both a $\wedge$-subsemilattice and a $\vee$-subsemilattice, i.e., if $x,y \in E$ then $x \wedge y \in E$ and $x \vee y \in E$.*

Example 3.2 *For every ordered set E the lattice $\mathscr{O}(E)$ of down-sets of E is a sublattice of the lattice $\mathbb{P}(E)$.*

Definition 3.3 *[12] Let G be an ordered group. A **convex subgroup** of G is a subgroup which, under the order of G, is a convex subset.*

Example 3.3 *Let us consider the ordered set $(\mathbb{R}^2, \leqslant)$ defined in Example (1.2), and $A = \{(x,0) \mid x \in \mathbb{R}\}$. It is clear that A is a subgroup of the group $(\mathbb{R}^2, +)$ (recall that in $\mathbb{R}^2$, $(x,y)+(x',y') = (x+x', y+y')$. It reminds to show that A is convex.*
Let $(x,0)$, $(z,0) \in A$ such that $(x,0) \leqslant (y,0)$, and $(\alpha,\beta) \in \mathbb{R}^2$ with, $(x,0) \leqslant (\alpha,\beta) \leqslant (y,0)$.

- *If $\alpha = y$, $\beta = 0$, then $(\alpha, 0) \in A$.*
- *If $\alpha \leq y$, $\beta < 0$ then,*
  - *Either $x = \alpha$ and $\beta = 0$, then $(\alpha,\beta) \in A$.*
  - *Or, $x \leq \alpha$ and $0 < \beta$ (contradiction).*

As $(\alpha,\beta) \in A$, then $[(x,0),(z,0)] \subseteq A$ implies that A is convex subgroup.

Example 3.4 *In the additive group $\mathbb{Z}$, $3\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ but it is not convex.*

Proposition 3.2 *[12] Let G be an ordered group. The intersection of any family of convex subgroups of G is a convex subgroup of G.*

Proof.

Let $\{A_i,\ i \in I\}$ be a family of convex subgroup of G, we know that $\bigcap\limits_{i \in I} A_i$ is subgroup, it reminds to prove that $\bigcap\limits_{i \in I} A_i$ is convex.
Let $a,b \in \bigcap\limits_{i \in I} A_i$, such that $a \leq b$, as A_i are convex, it follows that $[a,b] \subseteq Ai,\ \ \forall i \in I$.
This implies that $[a,b] \subseteq \bigcap\limits_{i \in I} A_i$.
Hence $\bigcap\limits_{i \in I} A_i$ is a convex subgroup of G. ■

Theorem 3.3 *[12] If H is a subgroup of an ordered group G then $P_H = H \cap P_G$. Moreover, the following two statements are equivalent :*

(1) H is convex ;

(2) P_H is a down-set of P_G.

Proof.

We know that $e_G = e_H$, let $y \in P_H$,

$$\begin{aligned} y \in P_H &\Rightarrow y \in \{x \in H \mid x \geqslant e_H\}, \\ &\Rightarrow y \geqslant e_H,\ y \in H \\ &\Rightarrow y \geqslant e_G,\ y \in H \\ &\Rightarrow y \in P_G \end{aligned}$$

It is clear that $y \in H \cap P_G$, and so $P_H = H \cap P_G$.

(1) $\Rightarrow$**(2)** Suppose that H is convex.
Let $x, y \in P_G$ with $e_H \leqslant y \leqslant x$ such that $e_H, x \in P_H$, then $e_H, x \in H$ and since H is convex then $y \in H$ and $y \in P_G$ this implies that $y \in H \cap P_G = P_H$. And so P_H is a down set of P_G.

(2) $\Rightarrow$ **(1)** Now we suppose that P_H is a down set of P_G.
Let $x \leqslant y \leqslant z$ such that $x, z \in H$, then $e_H \leqslant x^{-1}y \leqslant x^{-1}z$, we easily remark that $x^{-1}z \in P_H$ and as it is down set then $x^{-1}y \in P_H \subset H$.
Whence $y \in xH = H$,then H is convex.

■

Theorem 3.4 *[12] Let G be an ordered group and H be a normal subgroup of G. Then $Q = \{pH/p \in P_G\}$ is the positive cone of a compatible order on the quotient group G/H if and only if H is convex.*

3.2.2 Fuzzy Convex Subgroups

Definition 3.4 *[8] A fuzzy subset A of a lattice $(L, \leqslant)$ is said to be a* ***fuzzy sublattice*** *if, $\forall x, y$ in L,*

i) $A(x \vee y) \geq A(x) \wedge A(y)$,

ii) $A(x \wedge y) \geq A(x) \wedge A(y)$.

Example 3.5 *Consider the Hass digram of Figure 3.1 of the poset* $L = \{1,2,3,6,12\}$ *under division.*
It is easy to see that L is a lattice. Let us consider the fuzzy set A on L given by

$$A(x) = \begin{cases} \frac{1}{3} & si\ x \neq 12, \\ \frac{1}{2} & si\ x = 12. \end{cases}$$

If we take $x = 1$ *and* $y = 2$*, we find,*
$x \vee y = lcm(1,2) = 2 \Rightarrow A(2) = \frac{1}{3}$, *and* $A(1) = A(2) = \frac{1}{3} \Rightarrow A(1) \wedge A(2) = \frac{1}{3}$
Then $A(1 \vee 2) \geq A(1) \wedge A(2)$
$x \wedge y = lcd(1,2) = 1 \Rightarrow A(1) = \frac{1}{3}$, *and* $A(1) \wedge A(2) = \frac{1}{3}$, *then* $A(1 \wedge 2) \geq A(1) \wedge A(2)$.
The same for all element of L.

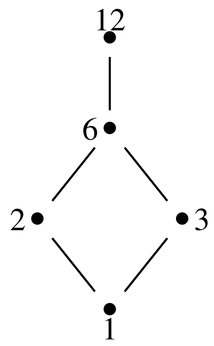


FIGURE 3.1 – Hasse diagram of a poset $(L, |)$.

The following proposition characterize any fuzzy sublattice of a lattice L by means of the crisp α-cut sublattices.

Proposition 3.3 *Let A be a fuzzy subset of a lattice* $(L, \leqslant)$*. Then, A is a fuzzy sublattice in L if and only if every nonempty* α*-cut* A_α *with* $\alpha \in [0,1]$ *of A is a sublattice of L.*

Proof.

$\Rightarrow$) Suppose that A is a fuzzy sublattice of L. Let A_α be a nonempty α-cut of A, and let x,y $\in A_\alpha$.

$$\begin{aligned} x,y \in A_\alpha &\Rightarrow A(x) \geq \alpha \text{ and } A(y) \geq \alpha \\ &\Rightarrow \min\{A(x),A(y)\} \geq \alpha \end{aligned}$$

Since A is a fuzzy sublattice, it follows that

$A(x \vee y) \geq A(x) \wedge A(y) \geq \alpha$ and $A(x \wedge y) \geq A(x) \wedge A(y) \geq \alpha$. This implies that

$$x \vee y \in A_\alpha \text{ and } x \wedge y \in A_\alpha.$$

Hence, A_α is a sublattice of L.

$\Leftarrow$) Suppose that each non-empty A_α, is a sublattice of L. In order to show that A is a fuzzy sublattice of L let us consider $x, y \in L$. We put $A(x) = \alpha_1$, and $A(y) = \alpha_2$, and assume that $\alpha_2 \leq \alpha_1$, it follows from Proposition 2.1 that $A_{\alpha_1} \subset A_{\alpha_2}$. As $x \in A_{\alpha_1}$ and $y \in A_{\alpha_2}$, it follows that $x, y \in A_{\alpha_2}$. Since A_{α_2} is a sublattice of L, it obviously holds that $x \vee y \in A_{\alpha_2}$ and $x \wedge y \in A_{\alpha_2}$. This implies that

$$A(x \vee y) \geq \alpha_2 \text{ and } A(x \wedge y) \geq \alpha_2.$$

On the other hand as $\alpha_2 = \min\{A(x), A(y)\}$, it follows that

$$A(x \vee y) \geq \min\{A(x), A(y)\} \;\text{ and }\; A(x \wedge y) \geq \min\{A(x), A(y)\}.$$

Therefore, A is a fuzzy sublattice of L.

■

Example 3.6 *By considering the fuzzy set A of L given in Example 3.5. We can prove that A is a fuzzy sublattice according to their α-cuts.*

- *If $\alpha \in [0, \frac{1}{3}[$, $A_\alpha = \{1, 2, 3, 6, 12\} = L$ is a sublattice.*
- *If $\alpha \in]\frac{1}{3}, \frac{1}{2}]$, $A_\alpha = \{12\}$ is a sublattice.*
- *If $\alpha \in]\frac{1}{2}, 1]$, $A_\alpha = \emptyset$.*

by Proposition 3.3, then A is a fuzzy sublattice.

Definition 3.5 *[8]Let G be an ordered group and A is a fuzzy subgroup of G. A is called a* ***fuzzy convex subgroup*** *if for all x, y, z from G, with $x \leq z \leq y$*

$$A(z) \geq A(x) \wedge A(y).$$

The following proposition characterize the fuzzy convex subgroup of a given group by means of its α-cut.

Proposition 3.4 *[8] Let A be a fuzzy subset of G. A is a fuzzy convex subgroup if and only if every nonempty α-cut A_α with $\alpha \in [0, 1]$ is a convex subgroup of G.*

Proof.

$\Rightarrow$) Suppose that $A \in \mathscr{F}(G)$ such that is a fuzzy convex subgroup. Let $x, y \in A_\alpha$ such that $x \leq y$.

$$\begin{aligned} x, y \in A_\alpha &\Rightarrow A(x) \geq \alpha,\ A(y) \geq \alpha \\ &\Rightarrow A(x) \wedge A(y) \geq \alpha \end{aligned}$$

From A is fuzzy convex subgroup then, $\forall x, y, z \in A$,

$$\begin{aligned} x \leq z \leq y &\Rightarrow A(z) \geq A(x) \wedge A(y) \\ &\Rightarrow A(z) \geq \alpha \\ &\Rightarrow z \in A_\alpha \\ &\Rightarrow [x, y] \subseteq A_\alpha. \end{aligned}$$

Then A_α is convex.

$\Leftarrow$) Conversely, we suppose that A_α is a convex subgroup. Let $a, b, c \in G$, such that $a \leq c \leq b$. Since

$$A(a) \geq A(a) \wedge A(b) \ and \ A(b) \geq A(a) \wedge A(b)$$

It follows that $a, b \in A_{A(a)\wedge A(b)}$. (with $\alpha = A(a) \wedge A(b)$)
As A_α is convex, $a, b \in A_\alpha$ and $a \leq c \leq b$, it follows that $c \in A_\alpha$.
This implies that

$$A(c) \geq \alpha$$

$$A(c) \geq A(a) \wedge A(b)$$

Then, A is fuzzy convex subgroup of G.

■

Example 3.7 *Consider the additive abelian group $\mathbb{R} \times \mathbb{R}$ equipped with the relation defined in Example 1.2. We define the fuzzy subgroup H of $\mathbb{R}^2$ by,* $H(x,y) = \begin{cases} b & if \ \ y = 0, \\ 0 & otherwise. \end{cases}$ *such that* $b \in [0,1]$. *Let $\alpha \in [0,1]$.*

- *If $\alpha = 0$, then we find $H_\alpha = \mathbb{R} \times \mathbb{R}$.*
- *If $\alpha \in]0, b]$, then we find $H_\alpha = \{(x,0) \mid x \in \mathbb{R}\}$.*
- *If $\alpha \in]b, 1]$, then we find $H_\alpha = \emptyset$.*

We find that the non-empty α-cuts of H are $\mathbb{R} \times \mathbb{R}$ (which is convex) and the subset A defined in Example 3.3 which we have already proved that is convex subgroups of $\mathbb{R} \times \mathbb{R}$. Then from Proposition 3.4, H is a fuzzy convex subgroup of $\mathbb{R} \times \mathbb{R}$.

Proposition 3.5 *Let G be an ordered group, any fuzzy convex subgroup satisfies : $\forall x, y \in G$,*

- ***i)*** *if $e \leq x \leq y$ then $A(x) \geq A(y)$,*
- ***ii)*** *if $y \leq x \leq e$ then $A(x) \geq A(y)$.*

Proof.

Let A be a fuzzy convex subgroup of G. From Proposition 2.2, it follows that $A(y) \leq A(e)$, for all $y \in G$. This implies that, $A(e) \wedge A(y) = A(y)$

- **i)** Suppose that $e \leq x \leq y$. From Definition 3.5, it holds that $A(x) \geq A(e) \wedge A(y \geq A(y)$.
- **ii)** Suppose that $y \leq x \leq e$. Again from Definition 3.5, it holds s that $A(x) \geq A(y) \wedge A(e) \geq A(y)$.

■

Definition 3.6 *[8] Let A be a fuzzy subset of G. A is called a **fuzzy down-set** if every nonempty α-cuts A_α is down-set of G.*

Example 3.8 *By considering the poset given in Example 3.5.*
We define the fuzzy set A by $A = \{(1, \frac{1}{3}), (2, \frac{1}{3}), (3, \frac{1}{3}), (6, \frac{1}{3}), (12, \frac{1}{2})\}$ then,

- *$\forall \alpha \in [0, \frac{1}{3}[$, $A_\alpha = \{1, 2, 3, 6, 12\} = L$ is a down set.*
- *$\forall \alpha \in]\frac{1}{3}, \frac{1}{2}]$, $A_\alpha = \{12\}$ is a down set.*
- *$\forall \alpha \in]\frac{1}{2}, 1]$, $A_\alpha = \emptyset$ is a down set.*

Then A is a fuzzy down set.

Properties 3.2 *Let $A \in \mathscr{F}(G)$. A is a fuzzy down-set of G if and only if it is order-reversing.*

Proof.

$\Rightarrow$) Suppose that A is fuzzy down-set of G, then $\forall \alpha \in [0, 1]$, A_α is a down-set of G.
Let $x, y \in G$, such that $x \lessdot y$. We have $y \in A_{A(y)}$ (Since $A(y) \in [0, 1]$ we put $\alpha = A_y$).
As $A_{A(y)}$ is a down-set, then

$$\begin{aligned} x \leq y \text{ and } y \in A_{A(y)} &\Rightarrow x \in A_{A(y)} \\ &\Rightarrow A(x) \geq A(y) \\ &\Rightarrow A \text{ is an order} - \text{reversing} \end{aligned}$$

$\Leftarrow$) Let A is a fuzzy set of G, such that A is an order-reversing.

We have, $\forall \alpha \in [0,1]$ $y \in A_\alpha \Rightarrow A(y) \geq \alpha$.

$$\begin{aligned} x \leq y \text{ and } y \in A_\alpha &\Rightarrow A(x) \geq A(y) \\ &\Rightarrow A(x) \geq \alpha \\ &\Rightarrow x \in A_\alpha \\ &\Rightarrow A_\alpha \text{ is a down} - set. \end{aligned}$$

Then A is a fuzzy down-set of G.

■

3.3 Fuzzy convex ℓ-subgroups

In this section, we describe some properties of a fuzzy lattice-ordered subgroup by extending the ones of the previous subsection. In this subsection G will denote Lattice-ordered group.

Definition 3.7 *[12] A **lattice-ordered subgroup** (ℓ-subgroup, for short) of a lattice-ordered group G is a subgroup H of G that is also a sublattice of G.*

Not that in general, a subgroup of a lattice-ordered group need not be a sublattice. For example, in the lattice-ordered additive abelian group $G = \mathbb{R} \times \mathbb{R}$ the subset $H = \{(n,-n), n \in \mathbb{Z}\}$ is a subgroup but is not a sublattice since $(0,0) \vee (1,-1) = (1,0) \notin H$.

Theorem 3.5 *[12] A subgroup H of a lattice-ordered group G is an ℓ-subgroup of G if and only if $x \vee e \in H$ for every $x \in H$.*

Proof.

The condition is clearly necessary. Conversely, if it holds then for all $x, y \in H$ we have $x \vee y = (xy^{-1} \vee e)y \in H$. We have also from Proposition1.6 that $x \wedge y = x(x \vee y)^{-1}y \in H$ and so H is a sublattice of G. ■

Definition 3.8 *An ℓ-subgroup of G which is convex is said to be a **convex ℓ-subgroup**.*

Definition 3.9 *[8] A fuzzy subset A of the lattice-ordered group G is said to be a **fuzzy lattice-ordered subgroup** (fuzzy ℓ-subgroup, for short) if, for all x, y in G,*

i) $A(xy^{-1}) \geq A(x) \wedge A(y)$,

ii) $A(x \vee y) \wedge A(x \wedge y) \geq A(x) \wedge A(y.)$

The flowing proposition characterize the fuzzy ℓ-subgroup.

Properties 3.3 *[8] A fuzzy subgroup A of G is a fuzzy ℓ-subgroup if and only if,*

$$A(x \vee e) \geq A(x), \forall x \in G.$$

Proof.

$\Rightarrow$) Suppose that $A(x \vee y) \wedge A(x \wedge y) \geq \min(A(x), A(y))$, then we have,

$$A(x \vee y) \geq \min(A(x), A(y)) \text{ and } A(x \wedge y) \geq \min(A(x), A(y))$$

We put $y = e$, then $A(x \vee e) \geq \min(A(x), A(e))$. Since $A(e) \geq A(x))$, it follows that

$$A(x \vee e) \geq A(x).$$

$\Leftarrow$) Now let us consider that $A(x \vee e) \geq A(x)$. From Proposition 1.6, it holds that $A(x \vee y) = A((xy^{-1} \vee e)y)$. This implies that

$$\begin{aligned} A(x \vee y) &\geq \min(A(xy^{-1} \vee e), A(y)) \\ &\geq \min(A(xy^{-1}), A(y)) \\ &\geq \min(\min(A(x), A(y)), A(y)) \\ &\geq \min(A(x), A(y)). \end{aligned}$$

By the same way we find $A(x \wedge y) \geq \min(A(x), A(y))$, then A is a fuzzy sublattice of G. Hence, A is a fuzzy ℓ-subgroup.

■

Example 3.9 *Let us consider the fuzzy subgroup H given in Example 3.7. Let $(x,y) \in \mathbb{R} \times \mathbb{R}$, and $(0,0)$ the identity element of $\mathbb{R} \times \mathbb{R}$.*

- *If $y = 0$, $H((x,0) \vee (0,0)) = b \geq H(x,0) = b$,*
- *If $y \neq 0$,*
  - *Either, $H((x,y) \vee (0,0)) = H(x,y) \geq H((x,y)$,*
  - *Or, $H((x,y) \vee (0,0)) = H(0,0) = b \geq H((x,y) = 0$.*

Then $\forall X, e \in \mathbb{R} \times \mathbb{R}$, $H(X \vee e) \geq H(X)$. Hence, by Proposition 3.3 H is fuzzy ℓ-subgroup.

Definition 3.10 *Let A be a fuzzy ℓ-subgroup of G which is convex is said to be a* ***fuzzy convex ℓ-subgroup****, i.e., a fuzzy subgroup A of G is said to be a fuzzy convex ℓ-subgroup if,*

i) *For all $x \in G$, $A(x \vee e) \leq A(x)$,*

ii) *For all $a,b,c \in G$ with $a \leq c \leq b$ we have*

$$A(c) \geq A(a) \wedge A(b).$$

• The set of all fuzzy convex ℓ-subgroups of G is noted by $\mathscr{FCL}(G)$.

Properties 3.4 *Let $\{A_i : i \in I\}$ be a family of fuzzy convex ℓ-subgroups of G and $A = \bigcap\limits_{i\in I} A_i$. Then, A is a fuzzy convex ℓ-subgroup of G.*

Proof.

From Proposition 2.4, $\bigcap\limits_{i\in I} A_i$ is a fuzzy subgroup. It reminds to prove that $\bigcap\limits_{i\in I} A_i$ is a fuzzy ℓ-subgroup and is a fuzzy convex.
Since $\forall i \in I, A_i$ is an ℓ-subgroup, it follows that $A_i(x \vee e) \geq A_i(x), \forall x \in G, \forall i \in I$. This implies that $\inf\limits_{i\in I} A_i(x \vee e) \geq \inf\limits_{i\in I} A_i(x), \forall x \in G$, it follows from Proposition 3.3 that ,$\bigcap\limits_{i\in I} A_i$ is a fuzzy ℓ-subgroup of G.
Now suppose that $a,b,c \in G$ with $a \leq c \leq b$. We need to show that $\bigcap\limits_{i\in I} A_i(c) \geq \bigcap\limits_{i\in I} A_i(a) \wedge \bigcap\limits_{i\in I} A_i(b))$. As for all $i \in I$, A_i is a fuzzy convex, it follows that

$$\begin{aligned}
a \leq c \leq b \quad &\Rightarrow \quad A_i(c) \geq A_i(a) \wedge A_i(b), \forall i \in I, \\
&\Rightarrow \quad \inf_{i\in I} A_i(c) \geq \inf_{i\in I}(A_i(a) \wedge A_i(b)), \\
&\Rightarrow \quad \inf_{i\in I} A_i(c) \geq (\inf_{i\in I} A_i(a) \wedge \inf_{i\in I} A_i(b)), \\
&\Rightarrow \quad \bigcap_{i\in I} A_i(c) \geq \bigcap_{i\in I} A_i(a) \wedge \bigcap_{i\in I} A_i(b).
\end{aligned}$$

Then $A = \bigcap\limits_{i\in I} A_i$ is fuzzy convex.
We conclude that the intersection of a family of fuzzy convex ℓ-subgroups is fuzzy convex ℓ-subgroup. ■

CONCLUSION

In this work, we have introduced the concept of fuzzy sets, fuzzy subgroups and normal subgroup of a given reference set, and we used them to introduce the notion of fuzzy lattices and fuzzy lattice-ordered subgroups and we have discussed various related properties and theorems. We have also investigated some properties and characterizations theorems of the fuzzy convex subgroup (resp. fuzzy convex lattice-ordered subgroup) of an ordered group (resp. lattice-ordered group).

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