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Approximate solution of partial integro-differential equation using pseudo-spectral method

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Dedication

To my dear parents, to my brothers and sisters.

To all those who have supported

me in my university career.

To all my friends.:)

I dedicate this work.

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Introduction

In this dissertation, we present the essential aspects of spectral methods and their applications to the numerical solution of partial differential equations.

Spectral methods have been used extensively during the last decades for the numerical solution of partial differential equations (PDE)[11, 13, 8], they were initially proposed by *Bilnova* 1944, and it was first applied to PDEs by *Silberman* 1954 for metreological modeling, and was virtually abandoned in the mid-1960s, revived in 1969-1970 by *Orszag*, *Eliason*, *Machenhauer* and *Rasmussen*, it was developed for specialized applications in the 1970s. Then, in the middle of the last decadde 1977 *Gottlieb* and *Orszag* summarized the state of the art in the theory and application of spectral methods. Due to their bigger accuracy when compared to finite differences (FD) and finite elements (FE) methods, the rate of convergence of spectral approximations depends only on the smoothness of the solution, yielding the ability to achieve high precision with a small number of data.

Partial integro-differential equations are used in many problems in the applied science to model dynamical systems[18]. It also can be found in financial mathematic, biological models and industrial mathematics etc. The numerical solution is expressed as a finite expansion of some set of basis functions. When the PDE is written in terms of the coefficients of this expansion, the method is known as a Galerkin spectral method. Spectral collocation methods, also known as pseudo-spectral methods, is another subclass of spectral methods and are similar to finite differences methods due to direct use of a set of grid points, which are called "collocation points".

In the past several years, the activity on both theory and application of spectral methods have been concentrated on collocation spectral methods. One of the reasons is that collocation methods deals with nonlinear terms more easily than Galerkin methods. Spectral methods achieve a greater precision with a smaller number of points than finite difference methods.

Pseudo-spectral methods are characterized by the following very desirable properties for an analytic function u the rate of convergence of the truncated expansion u_N to u is exponential (indeed this is called spectral convergence) instead of linear of polynomial as differences method and elements methods. Even for non-smooth function this approach reveals to be protable, provided that the singularities are not too strong. Due to the fast convergence, in the most of applications a relatively coarse computational grid succes to achieve a rather good accuracy.

This work is divided into three chapter:

- **Chapter.1**: This chapter studies the foundation of spectral methods, and provides a general theory on how to apply these methods to integral equations, it also studies the orthogonality and the convergence between the approximate and exact solution according to any approved method.

- **Chapter.2**: In this chapter, Gauss-Legendre quadrature and Chebyshev cardinal functions are defined and relied upon in the application of the pseudo-spectral method to one-dimensional parabolic partial integro-differential equations. Error analysis and convergence properties are also presented in this chapter.

- **Chapter.3**: This chapter presents some numerical examples for the pseudo-spectral method based on Chebychev cardinal functions using a special algorithm.

Finaly, a breifly conclusion is given in the last section of this dissertation.

Chapter 1

Introduction to spectral methods

1.1 Foundations of spectral methods

In this chapter, we give a brief summary of the spectral methods, these methods are one of the most popular numerical methods for solving functional equations. The basic idea is to approximate any function by polynomial. So a function u will be approximated by $\tilde{u} = \sum_{n=0}^{N} c_n \Phi_n$ where the Φ_n are polynomials and called the trial functions. Depending on the choise of trial functions, one can generate various classes of numerical techniques.

Spectral methods are very powerful tools for solving many kinds of differential equations (and recently integral equations). High accuracy (often so-called "exponential convergence") and ease of applying these methods for infinite domains are two effective properties which have encouraged many authors to use them for different differential and partial differential equations.

In all the following, we will be interested in spectral methods where the Φ_n are global polynomials. Spectral methods are basically more evolved than finite difference schemes because they have long prove their ability to tackle a wide variety of problems for solving PDE. To illustrate these techniques, let given the linear Fredholm integral equation

$$\lambda x(t) - \int_D k(t,s)x(s)\,ds = y(t). \quad t \in D, \tag{1.1}$$

where D is a compact set, $k(t,s)$ is a function called kernel and $x(t)$ is unknown solution. The subject here is to apply spectral method for solving (1.1), so the principle idea is to choose a finite dimensional family of functions that is believed to contain a function $\tilde{x}(s)$ close to the true solution $x(s)$, and these lead to different types of methods. The most popular and important of these are projections methods (collocation and Galerkin methods).

The integral equation (1.1) can be written as

$$(\lambda - \mathcal{K})(x) = y,$$

where the operator $\mathcal{K}$ is assumed to be compact on a Banach space X to X. The most popular choices are $C(D)$, $L^2(D)$ or Sobolev spaces $H^r(D)$.

In practice, we choose a sequence of finite dimensional subspaces $X_n \subset X$, $n \geq 1$ with X_n having dimension $d_n = d$. Let X_n have a basis $\{\varphi_1, \ldots \varphi_d\}$. We choose a function $x_n \in X_n$ and we can writte

$$x_n(t) = \sum_{j=1}^{d} c_j \varphi_j(t), \quad t \in D. \tag{1.2}$$

Let introduce

$$\begin{aligned} r_n(t) &= \lambda x_n(t) - \int_D k(t,s) x_n(s)\, ds - y(t), \\ &= \sum_{j=1}^{d} c_j \Big\{ \lambda \varphi_j(t) - \int_D k(t,s) \varphi_j(s)\, ds \Big\} - y(t), \quad t \in D, \end{aligned} \tag{1.3}$$

this is called the residual in the approximation of the equation. The residual can be written as follows

$$r_n = (\lambda - \mathcal{K}) x_n - y,$$

the coefficients $\{c_1, \ldots, c_d\}$ are chosen by forcing $r_n(t)$ to be approximately zero in some sense.

Now, we give the most popular projection methods.

1.1.1 Collocation method

Let given a node points $t_1 \ldots t_d \in D$, and require

$$r_n(t_i) = 0, \quad i = 1, \ldots, d_n, \tag{1.4}$$

this leads to find the coefficients $\{c_1, \ldots, c_d\}$ as the solution of the linear system

$$\sum_{j=1}^{d} c_j \Big\{ \lambda \varphi_j(t_i) - \int_D k(t_i, s) \varphi_j(s)\, ds \Big\} = y(t_i), \quad i = 1, \ldots, d_n, \tag{1.5}$$

so, in this step, we ask the question if this system has a solution, and if so, whether it is unique?

Define $P_n x$ to be the element of X_n that interpolates $x \in X$ at the nodes $\{t_1, \ldots, t_d\}$ this means that

$$P_n x(t) = \sum_{j=1}^{d} c_j \varphi_j(t),$$

for which P_n is the projection operator that maps $X = C(D)$ on to X_n.

Given $x \in C(D)$. The coefficients $\{c_j\}$ can be determined by solving the linear system

$$\sum_{j=1}^{d} c_j \varphi_j(t_i) = x(t_i), \quad i = 1, \dots, d_n,$$

this linear system has a unique solution if

$$\det\left[\varphi_j(t_i)\right] \neq 0, \tag{1.6}$$

this condition also implies that the functions $\{\varphi_1, \dots, \varphi_d\}$ are an independent set over D.

Remark 1.1. *In the case of polynomial interpolation for functions of one variable, the determinant in (1.6) is called the Vandermonde determinant.*

Example 1.1. *Let given $\ell_i \in X_n$, $i, 1 \leq i \leq d_n$ a basis functions, that satisfies the interpolation conditions*

$$\ell_i(t_j) = \delta_{ij}, \quad j = 1. \dots, d_n,$$

where δ_{ij} is the Kronecker δ-function and the basis $\{\ell_1, \dots, \ell_d\}$ of X_n is called Lagrange basis functions, we have

$$P_n x(t) = \sum_{j=1}^{d} x(t_j)\ell_j(t), \quad t \in D, \tag{1.7}$$

clearly, P_n is a linear and finite rank operator from $C(D)$ to $C(D)$. In addition, we have

$$\|P_n\| = \max_{t \in D} \sum_{j=1}^{d} |\ell_j(t)|,$$

and

$$\ell_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{t - t_j}{t_i - t_j} \right),$$

the formule (1.7) is called Lagrange's form of the polynomial interpolation. We note that

$$P_n z = 0 \Longleftrightarrow z(t_i) = 0, \quad i = 1, \dots, d_n,$$

the condition (1.4) can now be written as

$$P_n r_n = 0,$$

or equivalently

$$P_n(\lambda - \mathcal{K})x_n = P_n y, \quad x_n \in X_n. \tag{1.8}$$

1.1.2 Galerkin's method

Let $X = L^2(D)$ or some other Hilbert space, and let $(.,.)$ denote the inner product for X. For this method the residual r_n satisfy the following condition

$$(r_n, \varphi_i) = 0, \quad i = 1, \ldots, d_n, \tag{1.9}$$

to find x_n, apply (1.9) to (1.3). This yields to the linear system

$$\sum_{i=1}^{d} c_j \Big\{ \lambda(\varphi_j, \varphi_i) - (\mathcal{K}\varphi_j, \varphi_i) \Big\} = (y, \varphi_i), \quad i = 1, \ldots, d_n. \tag{1.10}$$

Now, let introduce a projection operator P_n that maps X to X_n. For $x \in X$, we define $P_n x$ be the solution of the following minimization problem

$$\|x - P_n x\| = \min_{z \in X_n} \|x - z\|, \tag{1.11}$$

since X_n is finite dimensional, it can be shown that this problem has a solution, and by X_n being an inner product space, the solution can be shown to be unique.

To obtain a better understanding of P_n, we give an explicit formula for $P_n x$. Introduce a new basis $\{\psi_1, \ldots, \psi_d\}$ and using the *Gram-Schmidt process* to create an orthonormal basis from $\{\varphi_1, \ldots, \varphi_d\}$, the element ψ_i is linear combination of $\{\varphi_1, \ldots, \varphi_i\}$, and moreover

$$(\psi_i, \psi_j) = \delta_{ij}, \quad i, j = 1, \ldots, d_n,$$

and

$$P_n x = \sum_{i=1}^{d} (x, \psi_i)\psi_i. \tag{1.12}$$

The operator P_n is a linear operator, and it satisfy the following properties

$$\begin{aligned}
&(1) \quad \|x\|^2 = \|P_n x\|^2 + \|x - P_n x\|^2, \\
&(2) \quad \|P_n x\|^2 = \sum_{i=1}^{d} |(x, \psi_i)|^2, \\
&(3) \quad (P_n x, y) = (x, P_n y), \quad x, y \in X, \\
&(4) \quad ((I - P_n)x, P_n y) = 0. \quad x, y \in X.
\end{aligned}$$

$P_n x$ is called the orthogonal projection of x onto X_n. And we have

$$\|P_n\| = 1,$$

using (4), we can show that

$$\|x - z\|^2 = \|x - P_n x\|^2 + \|P_n x - z\|^2, \quad z \in X_n,$$

then $P_n x$ is the unique solution to (1.11), we have

$$P_n z = 0 \Longleftrightarrow (z, \varphi_i) = 0, \quad i = 1, \ldots, d_n,$$

equation (1.9) can be written as

$$P_n r_n = 0,$$

or equivalently,

$$P_n(\lambda - \mathcal{K})x_n = P_n y. \quad x_n \in X_n, \tag{1.13}$$

which is similar to (1.8).

1.1.3 The general case

Let X be a Banach space and let $\{X_n | n \geq 1\}$ be a sequence of finite dimensional subspaces, of dimension d_n. Let $P_n : X \to X_n$ be a bounded projection operator. This means that P_n is a bounded linear operator with

(1) $P_n x = x, \quad x \in X_n,$

(2) $P_n^2 = P_n,$

(3) $\|P_n\| = \|P_n^2\| \leq \|P_n\|^2,$

(4) $\|P_n\| \geq 1.$

Remark 1.2. *As an example of this operator P_n, we have the interpolatory projection operator of (1.7) and the orthogonal projection operator(1.12). Also, the interpolatory projection operator associated with piecewise linear interpolation on an interval $[a, b]$.*

Motivated by (1.8) and (1.13), we approximate (1.1) by the approximate problem

$$P_n(\lambda - \mathcal{K})x_n = P_n y, \quad x_n \in X_n, \tag{1.14}$$

equation(1.14) leads directly to equivalent finite linear systems such as (1.5) and (1.10). For the error analysis, however, we write (1.14) in an equivalent but more convenient form.

If x_n is a solution of (1.14), then by using $P_n x_n = x_n$, this equation can be written as

$$(\lambda - P_n\mathcal{K})x_n = P_n y, \quad x_n \in X_n, \tag{1.15}$$

if (1.15) has a solution $x_n \in X_n$, then

$$x_n = \frac{1}{\lambda}\left[P_n y + P_n \mathcal{K} x_n\right] \in X_n.$$

Thus $P_n x_n = x_n$, we have

$$(\lambda - P_n\mathcal{K})x_n = P_n(\lambda - \mathcal{K})x_n,$$

and hence (1.15) implies (1.14).

For the error analysis, we compare (1.15) with the original equation

$$(\lambda - \mathcal{K})x = y, \tag{1.16}$$

because both equations are defined on the original space X. The subject is based on the approximation of $\lambda - P_n\mathcal{K}$ by $\lambda - \mathcal{K}$, so, we can writte

$$\begin{aligned} \lambda - P_n\mathcal{K} &= (\lambda - \mathcal{K}) + (\mathcal{K} - P_n\mathcal{K}), \\ &= (\lambda - \mathcal{K})\left[I + (\lambda - \mathcal{K})^{-1}(\mathcal{K} - P_n\mathcal{K})\right]. \end{aligned} \tag{1.17}$$

Theorem 1.1. *Assume that $\mathcal{K} : X \to X$ is bounded, with X is a Banach space, and assume $\lambda - \mathcal{K} : X \to X$. Further assume*

$$\|\mathcal{K} - P_n\mathcal{K}\| \to 0, \quad as \quad n \to \infty.$$

Then for sufficiently large n, say $n \geq N$, the operator $(\lambda - P_n\mathcal{K})^{-1}$ exists as a bounded operator from X to X. Moreover, it is uniformly bounded

$$\sup_{n \geq N} \|(\lambda - P_n\mathcal{K})^{-1}\| < \infty. \tag{1.18}$$

For the solutions of (1.15) and (1.16), we have

$$x - x_n = \lambda(\lambda - P_n\mathcal{K})^{-1}(x - P_n x), \tag{1.19}$$

$$\frac{|\lambda|}{\|\lambda - P_n\mathcal{K}\|}\|x - P_n x\| \leq \|x - x_n\| \leq |\lambda| \|(\lambda - P_n\mathcal{K})^{-1}\| \|x - P_n x\|. \tag{1.20}$$

This leads to $\|x - x_n\|$ converging to zero at exactly the same speed as $\|x - P_n x\|$.

Proof 1. *(a) Let given N such that*

$$\epsilon_N = \sup_{n \geq N} \|\mathcal{K} - P_n\mathcal{K}\| < \frac{1}{\|(\lambda - \mathcal{K})^{-1}\|}.$$

Then the inverse $[I + (\lambda - \mathcal{K})^{-1}(\mathcal{K} - P_n\mathcal{K})]^{-1}$ *exists and is uniformly bounded*

$$\| \left[I + (\lambda - \mathcal{K})^{-1}(\mathcal{K} - P_n\mathcal{K})\right]^{-1} \| \leq \frac{1}{1 - \epsilon_N \|(\lambda - \mathcal{K})^{-1}\|}.$$

Using (1.17), $(\lambda - P_n\mathcal{K})^{-1}$*, exists*

$$\begin{aligned} (\lambda - P_n\mathcal{K})^{-1} &= \left[I + (\lambda - \mathcal{K})^{-1}(\mathcal{K} - P_n\mathcal{K})\right]^{-1} (\lambda - \mathcal{K})^{-1}, \\ \|(\lambda - P_n\mathcal{K})^{-1}\| &\leq \frac{\|(\lambda - \mathcal{K})^{-1}\|}{1 - \epsilon_N \|(\lambda - \mathcal{K})^{-1}\|} \equiv M. \end{aligned} \tag{1.21}$$

This shows (1.18).

(b) For the formula (1.19), multiply $(\lambda - \mathcal{K})x = y$ *by* P_n*, and then, we obtain*

$$(\lambda - P_n\mathcal{K})x = P_n y + \lambda(x - P_n x),$$

subtract $(\lambda - P_n\mathcal{K})x_n = P_n y$ *to get*

$$\begin{aligned} (\lambda - P_n\mathcal{K})(x - x_n) &= \lambda(x - P_n x), \qquad (1.22) \\ x - x_n &= \lambda(\lambda - P_n\mathcal{K})^{-1}(x - P_n x), \end{aligned}$$

which is (1.19). Taking norms and using (1.21),

$$\|x - x_n\| \leq |\lambda| M \|x - P_n x\|. \tag{1.23}$$

Thus if $P_n x \longrightarrow x$*, then* $x_n \longrightarrow x$ *as* $n \longrightarrow \infty$*.*

(c) The upper bound in (1.20) follows from (1.19). The lower bound follows by taking bounds in (1.22), to obtain

$$|\lambda| \|x - P_n x\| \leq \|\lambda - P_n\mathcal{K}\| \|x - x_n\|.$$

This is equivalent to (1.20). Note that for $n \geq N$*, we have*

$$\begin{aligned} \|\lambda - P_n\mathcal{K}\| &\leq \|\lambda - \mathcal{K}\| + \|\mathcal{K} - P_n\mathcal{K}\|, \\ &\leq \|\lambda - \mathcal{K}\| + \epsilon_N, \end{aligned}$$

then, (1.20) can be replaced by

$$\frac{|\lambda|}{\|\lambda - \mathcal{K}\| + \epsilon_N} \|x - P_n x\| \leq \|x - x_n\|.$$

Taking the last inequality and (1.23), we have

$$\frac{|\lambda|}{\|\lambda - \mathcal{K}\| + \epsilon_N}\|x - P_n x\| \leq \|x - x_n\| \leq |\lambda| M \|x - P_n x\|. \tag{1.24}$$

This means that x_n converge to x if and only if $P_n x$ converge to x. We can see that $\|x - P_n x\|$ and $\|x - x_n\|$ tend to zero with exactly the same speed.

To apply the above theorem, we need to know if $\|\mathcal{K} - P_n\mathcal{K}\| \to 0$ as $n \to \infty$. For this, we give the following two lemmas

Lemma 1.1. *Let X, Y be a Banach spaces, and let $A_n : X \to Y, n \geq 1$ be a sequence of bounded linear operators. Assume that $\{A_n x\}$ converges for all $x \in X$. Then the convergence is uniform on compact subsets of X.*

Proof 2. *The operators A_n are uniformly bounded (By the principle of uniform boundedness), so we get*

$$M = \sup_{n \geq 1} \|A_n\| < \infty.$$

Then

$$\|A_n x - A_n y\| \leq M\|x - y\|,$$

which shows that the functions A_n are also equicontinuous. Then $\{A_n\}$ is a uniformly bounded and equicontinuous family of functions on a compact set $S \subset X$, then $\{A_n x\}$ is uniformly convergent for $x \in S$.

Lemma 1.2. *Let X be a Banach space and let $\{P_n\}$ be a family of bounded projections on X with the condition*

$$P_n x \longrightarrow x, \quad as \quad n \longrightarrow \infty, \quad x \in X. \tag{1.25}$$

Let $\mathcal{K} : X \longrightarrow X$ be compact. Hence

$$\|\mathcal{K} - P_n\mathcal{K}\| \to 0, \quad as \quad n \to \infty.$$

Proof 3. *We have*

$$\|\mathcal{K} - P_n\mathcal{K}\| = \sup_{\|x\| \leq 1} \|\mathcal{K}x - P_n\mathcal{K}x\| = \sup_{z \in \mathcal{K}(U)} \|z - P_n z\|,$$

with $\mathcal{K}(U) = \{\mathcal{K}x \quad | \quad \|x\| \leq 1\}$. The set $\mathcal{K}(U)$ is compact. Therefore, by the previous Lemma and the assumption (1.25), we get

$$\sup_{z \in \mathcal{K}(U)} \|z - P_n z\| \to 0. \quad as \quad n \to \infty.$$

Then, we obtain the results.

Remark 1.3. *There are a situation where $P_n x \to x$ for most $x \in X$, but not all x. In such cases, it is necessery to show directly that $\|\mathcal{K} - P_n\mathcal{K}\| \to 0$. In such cases, of course, we see from (1.24) that $x_n \to x$ if and only if $P_n x \to x$, and thus the method is not convergent for some solutions x, for example, if we take X_n is the set of polynomials of degree $\leq n$ and $X = C[a, b]$.*

1.2 Orthogonal projection and discussion of convergence

Let $I = [x_{\min}, x_{\max}]$ be an interval and w be a positive function, one can define the scalar product of two functions f and g, with respect the measure w as follows

$$(f, g)_w = \int_I f(x)g(x)w(x)\,dx.$$

The set of orthogonal polynomials p_n composed of those polynomials, up to given degree N is a basis of $\mathbb{P}_N$.
We hope to represent any function u defined on I by its projection on the polynomials p_n. So, we define the projection of u by

$$P_N u = \sum_{n=0}^{N} c_n p_n(x),$$

where the coefficients of the projection are given by $c_n = \dfrac{(u, p_n)}{(p_n, p_n)}$. The difference between u and its projection is called the *truncation error*, then, we want that

$$\|u - P_N u\| \longrightarrow 0, \quad \text{when} \quad N \longrightarrow \infty,$$

which show that the error goes to zero when the ordre of the approximation increases. Our goal is to find an approximate solution in X_n, using a sequence of projectors $P_n : X \to X_n$ in the form

$$P_n\left(\sum_{j=0}^{\infty} c_j\varphi_j(x)\right) = \sum_{j=0}^{n} c_j\varphi_j(x) = \varphi(x). \tag{1.26}$$

By giving a function weight $w = w(x)$ sur I. The orthogonality is defined by

$$\int_I w(x)\varphi_i(x)\varphi_j(x)\,dx = \delta_{ij},$$

then, the coefficients c_j in (1.26) are given by

$$c_j = \frac{1}{\|\varphi_j\|_w^2}\left(\int_I w(x)\varphi(x)\varphi_j(x)\,dx\right),$$

with

$$\|\varphi_j\|_w = \left(\int_I w(x)\varphi_j(x)\varphi_j(x)\,dx\right)^{\frac{1}{2}}. \tag{1.27}$$

Let $X = L^2_w(I)$ and $X_n = \mathbb{P}_n$, the subspace of polynomials of degree at most n. Then, we have the following theorems

Theorem 1.2. *Let $\varphi \in L^2_w(I)$ and $n \in \mathbb{N}$. Then $P_n\varphi$ is the best approximation in the space L^2, that is*

$$\|\varphi - P_n\varphi\|_{L^2_w} = \inf_{\psi\in\mathbb{P}_n} \|\varphi - \psi\|_{L^2_w}. \tag{1.28}$$

Proof 4. *As $\psi \in \mathbb{P}_n$, there exist coefficients $c_j, 0 \le j \le n$, such that $\psi = \sum_{j=0}^{n} c_j\psi_j$. To minimize $\|\varphi-\psi\|_{L^2_w}$ is equal to minimizing $\|\varphi-\psi\|^2_{L^2_w}$, we have*

$$\begin{aligned}\frac{\partial}{\partial_{c_k}}\|\varphi-\psi\|^2_{L^2_w} &= \frac{\partial}{\partial_{c_k}}\left(\|\varphi\|^2_{L^2_w} - 2\sum_{j=0}^{n} c_j\langle\varphi,\psi_j\rangle_{L^2_w} + \sum_{j=0}^{n} c_j^2\|\psi_j\|^2_{L^2_w}\right)\\ &= -2\langle\varphi,\psi_k\rangle_{L^2_w} + 2c_k\|\psi_k\|^2_{L^2_w}, \quad 0\le k\le n,\end{aligned}$$

then, we have

$$c_k = \frac{\langle\varphi,\psi\rangle_{L^2_w}}{\|\psi_k\|^2_{L_w 2}}, \quad 0 \le j \le n,$$

which completes the proof.

Theorem 1.3. *For all $\varphi \in L^2_w(I)$*

$$\lim_{n\to\infty} \|\varphi - P_n\varphi\|_{L^2_w} = 0. \tag{1.29}$$

Proof 5. *see [9]*

1.2.1 Orthogonal polynomials

In mathematics, orthogonal polynomials sequence are a family of polynomials such that any two different polynomials are orthogonal to each other under some inner product. The most used orthogonal polynomials are Legendre and Chebyshev polynomials.

Legendre polynomials

Definition 1.1. *The Legendre polynomial $P_n(x)$ is a polynomial in x of degree n defined by the following relation of recurence*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \tag{1.30}$$

where

$$P_0(x) = 1, \quad P_1(x) = x,$$

it is easy to prove this by the generating function, the first polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned}$$

Properites of Legendre polynomials

1. The Legendre polynomials $P_n(x)$ are orthogonal in the interval $[-1, 1]$, with respect to the function weight $w(x) = 1$,

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0 & m \neq n, \\ \frac{2}{2n+1} & m = n. \end{cases}$$

2. The Legendre polynomials $P_n(x)$ satisfy the following differential equation

$$(1 - x^2)y'' - 2y' + n(n+1)y = 0, \quad n = 0, 1, 2, \ldots.$$

3. An especially compact expression for the Legendre polynomials is given by Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left((x^2 - 1)^n \right),$$

so, we have

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}^2 (x-1)^{(n-k)} (x+1)^k, \text{ where } \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{(n-k)!k!}.$$

Chebyshev polynomials of first kind

Definition 1.2. *The Chebyshev polynomials of the first kind $T_n(x)$ satisfying the following relation of recurrence*

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots,$$

where

$$T_0(x) = 1, \quad T_1(x) = x,$$

so, the first Chebyshev polynomials of the first kind are

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1. \end{aligned}$$

Properites of Chebyshev polynomials of first kind

1. The Chebyshev polynomials of the first kind can be defined as the unique polynomials satisfying

$$T_n(x) = \cos(n\theta), \quad \text{where} \quad x = \cos(\theta).$$

2. The Chebychev polynomials of the first kind $T_n(x)$ are orthogonal in the interval $[-1, 1]$ with respect to the function weight $w(x) = 1/\sqrt{1-x^2}$

$$\int_{-1}^{1} T_n(x) T_m(x) \left(1/\sqrt{1-x^2}\right) dx = \frac{\pi}{2} \delta_{n,m}.$$

3. The Chebyshev polynomials of the first kind $T_n(x)$ are the solutions of the following differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

4. For any strictly positive integer n, we have

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k},$$

where $\lfloor . \rfloor$ note the integer part.

5. $\forall x \in [-1, 1]$, we have

$$\left| T_n(x) \right| \leq 1.$$

6. The zeros in $[0, 1]$ of $T_n(x)$ must be correspond to the zeros for θ in $[0, \pi]$, of $\cos(n\theta)$, so that

$$n\theta = (k - \frac{1}{2})\pi, \quad k = 1, 2, \ldots, n,$$

hence, the zeros of $T_n(x)$ are

$$x = x_k = \cos\left(\frac{(k-\frac{1}{2})\pi}{n}\right). \quad k = 1, 2, \ldots, n.$$

Example 1.2. *For $n = 3$, the zeros are*

$$x_1 = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad x_2 = \cos\left(\frac{3\pi}{6}\right) = 0, \quad x_3 = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$

1.2.2 Convergence properties

Let us consider a C^m function φ. Upper bounds on the difference between φ and the interpolant $I_N\varphi$ can be found for various norms and choice of polynomials, for example,

- For Legendre :

$$\|I_N\varphi - \varphi\|_{L^2_w} \leq \frac{C}{N^{m-1/2}} \sum_{k=0}^{m} \|\varphi^{(k)}\|_{L^2}. \tag{1.31}$$

- For Chebyshev :

$$\begin{aligned} \|I_N\varphi - \varphi\|_{L^2_w} &\leq \frac{C}{N^m} \sum_{k=0}^{m} \|\varphi^{(k)}\|_{L^2_w}, \\ \|I_N\varphi - \varphi\|_{L^\infty} &\leq \frac{C}{N^{m-1/2}} \sum_{k=0}^{m} \|\varphi^{(k)}\|_{L^2_w}. \end{aligned} \tag{1.32}$$

Remark 1.4. *From the inequality (1.31) and (1.32), it can be seen that for sufficiently regular functions, the difference between u and its interpolant $I_N u$ goes to zero exponentialy.*

In this subsection, we give some numerical examples applying collocation and Galerkin methods.

1.3 Numerical examples

Example 1.3. *Consider the integral equation[11]*

$$\varphi(x,t) = f(x,t) + \lambda \int_0^t \tau^2 \varphi(x,\tau)\, d\tau + \lambda \int_a^b e^{x+y} \varphi(y,t)\, dy, \tag{1.33}$$

where the exact solution is given by

$$\varphi(x,t) = t^2 e^x.$$

The errors between exact and approximate solutions are summarized in Table(1.1).

Using collacation method

In Eq.(1.33), we take $a = 0$, $b = 1$, $\lambda = 1$ *and* $f(x,t) = \frac{3}{2}t^2e^x - \frac{1}{5}e^xt^5 - \frac{1}{2}t^2e^xe^2$.
Let the approximate solution has the following forme

$$S(x,t_i) = \sum_{k=1}^{N} c_k(t_i)\psi_k(x), \tag{1.34}$$

We take the basis $\psi_1(x) = 1$, $\psi_2(x) = x$, $\psi_3(x) = x^2$. *Then solving the equation when* $x = 0$, 0.5, 1, *we get*

$$\begin{aligned}
c_1(t_0) &= 0, \quad c_2(t_0) = 0, \quad c_3(t_0) = 0,\\
c_1(t_1) &= 0.0005130967957, \quad c_2(t_1) = 0.000432839493,\\
c_3(t_1) &= 0.000415594769,\\
c_1(t_2) &= 0.002104378751, \quad c_2(t_2) = 0.001731361786,\\
c_3(t_2) &= 0.001662382707.
\end{aligned}$$

So, the approximate solution for $t \in [0, 0.03]$ *is written as follows*

$$\begin{aligned}
S(x,t_0) &= 0,\\
S(x,t_1) &= 0.0005130967957 + 0.000432839493x + 0.000415594769x^2,\\
S(x,t_2) &= 0.002104378751 + 0.001731361786x + 0.001662382707x^2.
\end{aligned}$$

Using Galerkin's method

As in collocation method, we choose the same basis $\psi_1(x) = 1$, $\psi_2(x) = x$, $\psi_3(x) = x^2$, *and three points* $x = 0$, 0.5, 1, *we assume that* $t \in [0, 0.03]$. *Then, we have*

$$\begin{aligned}
c_1(t_0) &= 0, \quad c_2(t_0) = 0, \quad c_3(t_0) = 0,\\
c_1(t_1) &= 0.00051949274, \quad c_2(t_1) = 0.00042025907,\\
c_3(t_1) &= 0.00041436290,\\
c_1(t_2) &= 0.00212991168, \quad c_2(t_2) = 0.0016810402,\\
c_3(t_2) &= 0.0016574556.
\end{aligned}$$

Hence, the solution for $t \in [0, 0.03]$, *is given by*

$$\begin{aligned}
S(x,t_0) &= 0,\\
S(x,t_1) &= 0.00051949274 + 0.00042025907x + 0.00041436290x^2,\\
S(x,t_2) &= 0.002104378751 + 0.001731361786x + 0.001662382707x^2.
\end{aligned}$$

x	$\varphi(x,t)$	E^C	E^G
t=0			
0	0	0	0
0.5	0	0	0
1.00	0	0	0
$t = 0.01500000000$			
0	0.000225	0.0002880967957	0.00029449274
0.5	0.0003709622860	0.0004624529484	0.0004622507140
1.00	0.0006116134113	0.0007499176467	0.0007425012987
$t = 0.030000000$			
0	0.0009	0.001204378714	0.00122991170
0.5	0.001483849144	0.001901806122	0.001900946656
1.00	0.002446453645	0.0030516695518	0.003021953955

Table 1.1: Values of the errors E^C and E^G using collocation and Galerkin methods.

Example 1.4. *Consider the integral equation[11]*

$$\varphi(x,t) = f(x,t) + \lambda \int_0^t t\tau\varphi(x,\tau)\, d\tau + \lambda \int_a^b e^{-y}\varphi(y,t)\, dy, \tag{1.35}$$

where the exact solution is given by $\varphi(x,t) = te^{-x}$. The errors between exact and approximate solutions are summarized in Table(1.2).

Using collacation method

In Eq (1.35) we take $a = 0$, $b = 1$, $\lambda = 1$ and $f(x,t) = -0.00432t + e^{-x}t - 0.333e^{-x}t^4$. We choose three independent functions $\psi_1(x) = 1$, $\psi_2(x) = x$ and $\psi_3(x) = x^2$. Then, we take $x = 0,\ 0.5,\ 1$, we get

$$\begin{aligned}
c_1(t_0) &= 0, \quad c_2(t_0) = 0, \quad c_3(t_0) = 0,\\
c_1(t_1) &= 0.008541939634, \quad c_2(t_1) = 0.01412633660,\\
c_3(t_1) &= 0.004644538564,\\
c_1(t_2) &= 0.01708389624, \quad c_2(t_2) = 0.02825268914,\\
c_3(t_2) &= 0.009289082405.
\end{aligned}$$

So, the solution for $t \in [0, 0.03]$ takes the form

$$\begin{aligned} S(x,t_0) &= 0, \\ S(x,t_1) &= 0.008541939634 - 0.01412633660x + 0.004644538564x^2, \\ S(x,t_2) &= 0.01708389624 - 0.02825268914x + 0.009289082405x^2. \end{aligned}$$

Using Galerkin's method

We choose the same independent functions, and three points $x = 0,\ 0.5,\ 1$ and we suppose that $t \in [0, 0.03]$. Then, we have

$$\begin{aligned} c_1(t_0) &= 0, \quad c_2(t_0) = 0, \quad c_3(t_0) = 0, \\ c_1(t_1) &= 0.008459192683, \quad c_2(t_1) = -0.01395821152, \\ c_3(t_1) &= 0.00463077286, \\ c_1(t_2) &= 0.01691840427, \quad c_2(t_2) = -0.02791644232, \\ c_3(t_2) &= 0.00926155092. \end{aligned}$$

So, the solution is given as follows

$$\begin{aligned} S(x,t_0) &= 0, \\ S(x,t_1) &= 0.008459192683 - 0.01395821152x + 0.00463077286x^2, \\ S(x,t_2) &= 0.01691840427 - 0.02791644232x + 0.00926155092x^2. \end{aligned}$$

x	$\varphi(x,t)$	E^C	E^G
t=0			
0	0	0	0
0.5	0	0	0
1.00	0	0	0
$t = 0.01500000000$			
0	0.015	0.006458060366	0.006540807317
0.5	0.009097959896	0.006458053921	0.006460179758
1.00	0.005518191618	0.006458050020	0.006386437595
$t = 0.030000000$			
0	0.03	0.01291610376	0.01308159573
0.5	0.01819591979	0.01291609752	0.01292034895
1.00	0.01103638324	0.01291609374	0.01277287037

Table 1.2: Values of the errors E^C and E^G using collocation and Galerkin methods.

Chapter 2

Pseudo-spectral method based on Chebyshev cardinal functions

2.1 Definition of the problem

In this chapter, we apply the pseudo-spectral method based on Chebyshev cardinal functions to solve one-dimensional parabolic partial integro-differential equations (PIDEs)

$$u_t(x,t) + \alpha u_{xx}(x,t) = \beta \int_0^t k(x,t,s,u(x,s))\, ds + f(x,t), \quad x \in [a,b], \quad t \in [0,T], \tag{2.1}$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) &= g(x), \quad x \in [a,b], &(2.2)\\ u(0,t) &= h_0(t), \quad u(1,t) = h_1(t), \quad t \in [0,T], &(2.3) \end{aligned}$$

where α, β are constants and the functions $f(x,t)$ and $k(x,t,s,u)$ are assumed to be sufficiently smooth on $D = [a.b] \times [0.T]$ and S with $S = \{(x,t,s) : x \in [a,b], s,t \in [0,T]\}$ respectively, we suppose that equation (2.1) has a unique solution $u(x,t) \in C(D)$.

In addition, we assume that the kernel function is of diffusion type which is given by

$$k(x,t,s,u(x,s)) = k_1(x,t-s,u(x,s)),$$

and satisfies the Lipschitz condition as follows

$$|k(x,t,s,u(x,s)) - k(x,t,s,v(x,s))| \le \mathcal{A}|u(x,s) - v(x,s)|,$$

where $\mathcal{A} \ge 0$ is the Lipschitz constant.

2.2 Chebyshev cardinal functions

Given $M \in \mathbb{N}$, assume that $\mathcal{M} = \{1, 2, \cdots M+1,\}$ and $X = \{x_i : T_{M+1}(x_i) = 0, i \in \mathcal{M}\}$ where T_{M+1} is the first kind Chebyshev function of order $M+1$ on $[-1.1]$. Recall that the Chebyshev grid is obtained by

$$x_i = \cos\left(\frac{(2i-1)\pi}{2M+2}\right), \quad \forall i \in \mathcal{M}. \tag{2.4}$$

To use the Chebyshev functions of any arbitrary interval $[a, b]$, one can apply the change of variable $x = \left(\frac{2(t-a)}{b-a} - 1\right)$ to obtain the shifted Chebyshev functions,

$$T^*_{M+1}(t) = T_{M+1}\left(\frac{2(t-a)}{b-a} - 1\right), \quad t \in [a, b], \tag{2.5}$$

note that it is easy to show that the grids of shifted Chebyshev function T^*_{M+1} are equal to

$$t_i = \frac{(b-a)}{2} x_i + \frac{(b+a)}{2}. \tag{2.6}$$

A significant example of the cardinal functions for orthogonal polynomials is the Chebyshev cardinal functions. The cardinal Chebyshev functions of order $M+1$ are defind as

$$C_i(x) = \frac{T_{M+1}(x)}{T_{M+1,x}(x_i)(x - x_i)}, \quad i \in \mathcal{M}, \tag{2.7}$$

where the subscript x denotes x-differentiation. It is obvious that the functions $C_i(x)$ are polynomials of degree M which satisfy the condition

$$C_i(x_j) = \delta_{ij}. \tag{2.8}$$

For an arbitrary function $p(t)$, it can be approximated by

$$p(t) \approx \sum_{i=1}^{M+1} p(t_i) C_i(t). \tag{2.9}$$

2.3 Gauss quadrature

We want to evaluate numerically the integral $\int_a^b f(x)w(x)\,dx$ by Gauss quadrature. We have

$$\forall f \in \mathbb{P}_{2N+\delta}, \quad \int_a^b f(x)w(x)\,dx = \sum_{n=0}^{N} f(x_n)w_n,$$

where w_n are called the weights and x_n the collocation points. The exacte degree of applicability depend on the quadrature. The three usual choices are

- Gauss : $\delta = 1$.
- Gauss-Radau : $\delta = 0$ and $x_0 = x_{min}$.
- Gauss-Lobatto : $\delta = -1$, $x_0 = x_{min}$ and $x_N = x_{max}$.

Change of the interval

An integral over $[a, b]$ must be changed into an integral over $[-1, 1]$ before applying the Gaussian quadrature rule. This change of interval can be done in the following way

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{b-a}{2}\xi + \frac{b+a}{2}\right)\frac{dx}{d\xi}d\xi,$$

with $\frac{dx}{d\xi} = \frac{b-a}{2}$. Applying n points (ξ, w) of Gaussian quadrature, then, we get the following approximation

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}\xi_i + \frac{b+a}{2}\right).$$

2.3.1 Gauss-Legendre quadrature

Definition 2.1. *For integrating f over $[-1, 1]$ with Gauss-Legendre quadrature, the associated orthogonal polynomials are Legendre polynomials, denoted by $P_n x$, the i-th Gauss node, x_i is the i-th root of $P_n x$ and the weights are given by the formula*

$$w_i = \frac{-2}{(n+1)P'_n(x_i)P_{n+1}(x_i)} = \frac{2}{(1-x_i^2)P'_n(x_i)^2}. \tag{2.10}$$

Some low-order quadrature rules are given in Table (2.1).

Number of points, n	Weights	Points, x_i	Legendre polynomial
1	2	0	x
2	1.1	-$\sqrt{1/3}$,$\sqrt{1/3}$	$(3x^2-1)/2$
3	$8/9, 5/9$	$-\sqrt{3/5}, 0, \sqrt{3/5}$	$(5x^3-3x)/2$

Table 2.1: Some nodes and weights of Gauss-Legendre quadrature.

2.3.2 Gauss–Lobatto rules

It is similar to Gaussian quadrature with the following differences :

1. The integration points include the end points of the integration interval.

2. It is accurate for polynomials up to degree $2n-3$, where n is the number of integration points.

Lobatto quadrature of function $f(x)$ on the interval $[-1,1]$ is given as follows

$$\int_{-1}^{1} f(x)\,dx = \frac{2}{n(n-1)}\Big(f(1)+f(-1)\Big) + \sum_{i=2}^{n-1} w_i f(x_i) + R_n.$$

Abscissas x_i, $i = 2 : n-1$ are the roots of $P'_{n-1}(x)$, where $P_m(x)$ denotes the standard Legendre polynomial of m-th degree and the dash denotes the derivative, the weights are given by

$$w_i = \frac{2}{n(n-1)\Big(P_{n-1}(x_i)\Big)^2}, \quad x_i \neq \pm 1.$$

Some of the weights and nodes are summarized in Table (2.2).

Number of points, n	Points, x_i	Weights
3	0, ± 1	$\frac{4}{3}$, $\frac{1}{3}$
4	$\pm\sqrt{\frac{1}{5}}$, ± 1	$\frac{5}{6}$, $\frac{1}{6}$
5	0,$\pm\sqrt{\frac{3}{7}}$, $\pm$ 1	$\frac{32}{45}$, $\frac{49}{90}$, $\frac{1}{10}$

Table 2.2: Some nodes and weights of Gauss-Labatto-Legendre quadrature.

2.3.3 Error estimation

In numerical integration, we use a finite summation to approximate the value of an integral

$$I(f) = \int_a^b f(x)\,dx \cong \sum_{i=0}^{n} w_i f(x_i) + E. \tag{2.11}$$

There are two type of errors in numerical integration [15, 4]. These are truncation error and round off error. Without effective evaluation of error $|I(f) - A(f)|$, a quadrature rule is of no importance. Round off error comes from the fact that we can only compute the summation to finite precision, due to the limited accuracy of a computer's representation of floating point numbers. The round off error in general, is insignificant compared with truncation error.

Error in Gaussian quadrature

The error in the Gaussian quadrature rule[1] is

$$E_n = I(f) - G_n(f) = \frac{(b-a)^{2n+1}(n!)^4 f^{2n}(\xi_n)}{(2n+1)((2n)!)^3}, \tag{2.12}$$

where $a \leq \xi_n \leq b$ and $I(f)$ and $G_n(f)$ denotes the exact value and approximate estimation using the Gaussian quadrature formula. Obviously it is computationally expensive or even difficult to evaluate $f(x)$ at many points. Consequently it is important to consider how the error E_n depends on n. Clearly one would like to get small E_n when n becomes large. There are two firmly linked ways to describe the reliance of E_n and n. (1) the order of accuracy (2) the degree of the quadrature rule. The order of accuracy shows how fast $|E_n|$ decays to zero when n becomes large. Whereas the degree of the quadrature rule shows or which polynomials the quadrature rule G_n is exact. A quadrature formula of degree d integrates exactly to all polynomials up to degree d.

Example 2.1. *Let approximate the integral*

$$I = \int_0^1 \sin(x)\, dx,$$

by using the Trapezoidal rule, Simpson rule and Gaussian quadrature formula. Exact value (analytical solution) is calculated by simple integration rules which is 4.5970×10^{-1}.

Errors given by Trapezoidal rule, Simpson's rule and Gaussian quadrature are summarized in table (2.3).

N-values	Trapezodial rule	Simpson rule	Gaussian quadrature rule
2	9.61×10^{-3}	1.65×10^{-3}	6.42×10^{-3}
4	2.40×10^{-3}	1.01×10^{-4}	1.56×10^{-5}
6	1.06×10^{-3}	1.98×10^{-6}	5.23×10^{-10}
8	5.99×10^{-4}	6.25×10^{-7}	4.89×10^{-15}
10	3.83×10^{-4}	2.56×10^{-7}	2.22×10^{-16}
20	9.85×10^{-5}	1.60×10^{-8}	4.44×10^{-16}
50	1.53×10^{-5}	4.09×10^{-9}	1.58×10^{-11}
100	3.83×10^{-6}	2.55×10^{-11}	1.33×10^{-15}

Table 2.3: Errors given by some quadrature rules.

2.4 Pseudo-spectral method

In this section, we apply a pseudo-spectral method to solve PIDEs (2.1) based on Chebyshev cardinal functions. Let us consider the partial integro-differential equation (2.1) on the region $D = [0, 1] \times [0, 1]$. We introduce differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \alpha \frac{\partial^2}{\partial x^2},$$

and integral operator

$$\mathcal{I} = \beta \int_0^t k\left(x, t, s, u(x, s)\right)\, ds,$$

applying these operators, PIDEs (2.1) can be rewritten in the operator form

$$(\mathcal{L}+\mathcal{I})(u)=f. \tag{2.13}$$

The spectral method is based on the approximation of the solution of (2.1) as follows

$$\tilde{u}(x,t)=\sum_{i=1}^{M+1}\sum_{j=1}^{M+1}u(t_i,t_j)C_i(x)C_j(t). \tag{2.14}$$

If we define a matrix U of dimension $(M+1)\times(M+1)$ whose (i,j)-th element are $u(t_i,t_j)$, then equation (2.14) is written in the matrix form

$$\tilde{u}(x,t)=C^T(x)UC(t), \tag{2.15}$$

where the vector elements of $C(x)$ are the Chebyshev cardinal functions $\{C_i(x)\}_{i=1,\ldots,M+1}$.

Since the Chebyshev cardinal functions are polynomials, it is easy to evaluate their derivatives. Then, from (2.14), we can write

$$\tilde{u}_x(x,t)=\sum_{i=1}^{M+1}\sum_{j=1}^{M+1}u(t_i,t_j)C_{i,x}(x)C_j(t)=C_x^T(x)UC(t),$$

where $C_x(x)$ is a vector of dimension $(M+1)$ whose i-th element is $C_{i,x}(x)$, and also, we have

$$\tilde{u}_t(x,t)=\sum_{i=1}^{M+1}\sum_{j=1}^{M+1}u(t_i,t_j)C_i(x)C_{j,t}(t)=C^T(x)UC_t(t),$$

where $C_t(t)$ is a vector of dimension $(M+1)$ whose i-th element is $C_{i,t}(t)$.
Suppose that $D\in\mathbb{R}^{M+1,M+1}$ is the operational matrix of derivative whose (i,j)-th element is $D_{i,j}=C_{i,t}(t_j)$. Then, the differentiation of vectors $\{C(x)\}$ can be expressed as

$$C'(x)=DC(x), \tag{2.16}$$

the matrix D can be obtained by the following process. Let

$$C'(x)=[C_1'(x),C_2'(x),\cdots,C_{M+1}'(x)], \tag{2.17}$$

using (2.9), any function $C_j'(x)$ can be approximated as

$$C_j'(x)=\sum_{k=1}^{M+1}C_j'(x_k)C_k(x), \tag{2.18}$$

comparing (2.16) and (2.18), we obtain

$$D = \begin{pmatrix} C_1'(x_1) & \cdots & C_1'(x_{M+1}) \\ \vdots & & \vdots \\ C_{M+1}'(x_1) & \cdots & C_{M+1}'(x_{M+1}) \end{pmatrix} \tag{2.19}$$

To calculate the entires $C_j'(x_k), j, k = 1, 2, \ldots, M+1$, we have

$$\frac{T_{M+1}(x)}{x - x_j} = \alpha \times \prod_{\substack{k=1\\k\neq j}}^{M+1} (x - x_k), \tag{2.20}$$

where $\alpha = 2^{2M+1}$ is the coefficient of x^{M+1} in the shifted Chebyshev polynomial function $T_{M+1}^*(x)$. Using (2.20), we obtain

$$\frac{d}{dx}\left(\frac{T_{M+1}^*(x)}{x - x_j}\right) = \alpha \times \sum_{\substack{i=1\\i\neq j}}^{M+1} \prod_{\substack{k=1\\k\neq i,j}}^{M+1} (x - x_k) = \sum_{\substack{i=1\\i\neq j}}^{M+1} \frac{T_{M+1}^*(x)}{(x - x_j)(x - x_i)}, \tag{2.21}$$

so, we have

$$C_j'(x) = \frac{1}{T_{M+1,x}^*(x_j)} \times \frac{d}{dx}\left(\frac{T_{M+1}^*(x)}{x - x_j}\right) = \frac{1}{T_{M+1,x}^*(x_j)} \sum_{\substack{i=1\\i\neq j}}^{M+1} \frac{T_{M+1}^*(x)}{(x - x_j)(x - x_i)} = C_j(x) \sum_{\substack{i=1\\i\neq j}}^{M+1} \frac{1}{x - x_i}, \tag{2.22}$$

for $k = j$ and using (2.22), we obtain

$$C_j'(x_j) = \sum_{\substack{i=1\\i\neq j}}^{M+1} \frac{1}{x_j - x_i}, \tag{2.23}$$

for $k \neq j$ using (2.22), we have

$$C_j'(x_k) = \frac{\alpha}{T_{M+1,x}^*(x_j)} \prod_{\substack{l=1\\l\neq k,j}}^{M+1} (x_k - x_l), \tag{2.24}$$

so the coefficients of the matrix D can be found using (2.23) and (2.24).

Thus, it follows from $C_x(x) = DC(x)$ that

$$\tilde{u}_x(x,t) = C^T(x) D^T U C(t), \tag{2.25}$$

$$\tilde{u}_t(x,t) = C^T(x) U D C(t), \tag{2.26}$$

it can easily be shown that $\tilde{u}_{xx}(x,t)$ is approximated as follows

$$\tilde{u}_{xx}(x,t) = C^T(x) D^{T^2} U C(t). \tag{2.27}$$

Thus, by substituting (2.26) and (2.27) into the differential part of desired equation (2.13), we can approximate the differential operator $\mathcal{L}$ in (2.13), via

$$\mathcal{L}(u) \approx C^T(x)UDC(t) + \alpha C^T(x)D^{T^2}UC(t). \tag{2.28}$$

Then, the partial integro-differential equation (2.1) is replaced by the following approximation

$$C^T(x)UDC(t) + \alpha C^T(x)D^{T^2}UC(t) - \beta \int_0^t k\Big(x, t, s, C^T(x)UC(s)\Big)\, ds - f(x,t) = 0. \tag{2.29}$$

Let $\{t_i\}$ be the Gauss Legendre nodes with $\{i = 1, \dots, M+1\}$ on $[0,1]$. We collocate equation (2.29) by Gauss-Labatto-Legendre nodes, we get

$$C^T(x)UC(t_i) + \alpha C^T(x)D^{T^2}UC(t_i) - \beta \int_0^{t_i} k\Big(x, t_i, s, C^T(x)UC(s)\Big)\, ds - f(x,t_i) = 0. \tag{2.30}$$

For the integral

$$\int_0^t k\Big(x, t, s, C^T(x)UC(s)\Big)\, ds, \tag{2.31}$$

we use Gauss quadrature as follow

$$\int_0^{t_i} k(x, t_i, s, C^T(x)UC(s))\, ds = \frac{t_i}{2}\sum_{i=1}^{r_1} w_j k\left(x, t_i, \frac{t_i}{2}s_j + \frac{t_i}{2}, C^T(x)UC\Big(\frac{t_i}{2}s_j + \frac{t_i}{2}\Big)\right), \tag{2.32}$$

where s_j are the Gauss Legendre nodes on $[-1,1]$. From equation (2.32), we get

$$C^T(x)UDC(t_i)+\alpha C^T(x)D^{T^2}UC(t_i)-\beta\frac{t_i}{2}\sum_{j=1}^{r_1} w_j k\left(x, t_i, \frac{t_i}{2}s_j + \frac{t_i}{2}, C^T(x)UC\Big(\frac{t_i}{2}s_j + \frac{t_i}{2}\Big)\right)-f(x,t_i) = 0, \tag{2.33}$$

also, we collocate equations of inital and boundary conditions, we obtain

- $C^T(x)UC(0) - g(x) = 0,$
- $C^T(0)UC(t_i) - h_0(t_i) = 0, \quad i = 1, \dots, M$
- $C^T(1)UC(t_i) - h_1(t_i) = 0.$

Finally, we collocate equation (2.33) and boundary conditions with Gauss Legendre nodes y_l, $\{l = 1, \dots, M+1\}$, on $[0,1]$, we get

$$C^T(y_l)UC(t_i)+\alpha C^T(y_l)D^{T^2}UC(t_i)-\beta\frac{t_i}{2}\sum_{j=1}^{M+1} w_j k\left(y_l, t_i, \frac{t_i}{2}s_j + \frac{t_i}{2}, C^T(y_l)UC\Big(\frac{t_i}{2}s_j + \frac{t_i}{2}\Big)\right)-f(y_l,t_i) = 0, \tag{2.34}$$

for $l = 2 : M$, $i = 2 : M+1$,

- $C^T(y_l)UC(0) - g(y_l) = 0, \quad l = 1, \dots, M+1,$

- $C^T(0)UC(t_i) - h_0(t_i) = 0, \quad i = 2, \ldots, M+1,$
- $C^T(1)UC(t_i) - h_1(t_i) = 0, \quad i = 2, \ldots, M+1.$

Hence, we get a nonlinear system of order $(M+1)^2$, which can be solved by iterative method, such that Newton method. By solving the system and determining U, we find a numerical solution for equation (1.1) by inserting U into relation (2.15).

2.5 Error analysis and convergence of cardinal expansion

In this section, we give the convergence analysis of the proposed method.

Theorem 2.1. *Let $f : D \to \mathbb{R}$ be a sufficiently smooth function. Then, the error on Chebyshev cardinal approximation to function f can be given by*

$$\|f - \tilde{f}\| \approx O(2^{-M}). \tag{2.35}$$

Proof. From [7], we have

$$\begin{aligned} |f(x,t) - P_M(x,t)| &= \frac{\partial^{M+1}}{\partial x^{M+1}} f(\zeta,t) \frac{\prod_{i=1}^{M+1}(x-t_i)}{(M+1)!} + \frac{\partial^{M+1}}{\partial t^{M+1}} f(x,\eta) \frac{\prod_{j=1}^{M+1}(t-t_j)}{(M+1)!} \\ &- \frac{\partial^{2M+2}}{\partial x^{M+1} t^{M+1}} f(\zeta',\eta') \frac{\prod_{i=1}^{M+1}(x-t_i)\prod_{j=1}^{M+1}(t-t_j)}{(M+1)!(M+1)!}, \end{aligned}$$

where, $P_M(x)$ denotes the polynomial of degree M which interpolates the function f at the $M+1$ zeros of the first kind Chebyshev polynomials. Since the leading coefficient of the first kind Chebyshev functions is 2^M, and $|T_i(x)| \leq 1, \forall i \in \mathcal{M}$. It is possible to write

$$\begin{aligned} \|f(x,t) - P_M(x,t)\| &\leq \frac{1}{2^M(M+1)!} \left(\sup_{\zeta\in[a,b]} \left| \frac{\partial^{M+1}}{\partial x^{M+1}} f(\zeta,t) \right| + \sup_{\eta\in[0,T]} \left| \frac{\partial^{M+1}}{\partial t^r} f(x,\eta) \right| \right) \\ &+ \frac{1}{4^M((M+1)!)^2} \sup_{(\zeta',\eta')\in D} \left| \frac{\partial^{2M+2}}{\partial x^r \partial t^{M+1}} f(\zeta',\eta') \right|. \\ &\leq \frac{1}{2^M(M+1)!} 2C_{\max} + \frac{1}{4^M((M+1)!)^2} C_{\max}. \\ &= \frac{1}{2^M(M+1)!} C_{\max} \left[2 + \frac{1}{2^M(M+1)!} \right]. \end{aligned}$$

where

$$C_{\max} = \max\left\{ \sup_{\zeta\in[a,b]} \left| \frac{\partial^{M+1}}{\partial x^{M+1}} f(\zeta,t) \right|, \sup_{\eta\in[0,T]} \left| \frac{\partial^{M+1}}{\partial t^r} f(x,\eta) \right|, \sup_{(\zeta',\eta')\in D} \left| \frac{\partial^{2M+2}}{\partial x^r \partial t^{M+1}} f(\zeta',\eta') \right| \right\}.$$

2.5.1 Convergence rate in the Sobolev space

In the Sobolev space definition, one can require that the function together with its distributional derivatives be square integrable with respect to a specific weight function w. This is the most natural framework in dealing with Chebyshev methods[6]. Let (a, b) be a bounded interval of the real line, w be a weight function and let $m \geq 0$ be an integer. We define $H_w^m(a, b)$ to be the vector space of the functions $u \in L_w^2(a, b)$ such that all the distributional derivatives of u of order up to m can be represented by functions in $L_w^2(a, b)$, i.e

$$H_w^m(a, b) = \{u \in L_w^2(a, b) : for \quad 0 \leq k \leq m, \quad u^{(k)}(x) \in L_w^2(a, b)\}.$$

The Sobolev space $H_w^m(a, b)$ is endowed with the weighted inner product

$$\langle u, v\rangle_{m,w} = \sum_{k=0}^{m} \int_a^b u^{(k)}(x) v^{(k)}(x) w(x)\, dx,$$

for which $H_w^m(a, b)$ is a Hilbert space with the following associated norm

$$\|u\|_{H_w^m(a,b)} = \left(\sum_{k=0}^{m} \|u^{(k)}\|^2_{L_w^2(a,b)}\right)^{1/2}. \tag{2.36}$$

Thus, it is convenient to introduce the semi-norm

$$|u|_{H_w^{m,N}(a,b)} = \left(\sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|^2_{L_w^2(a,b)}\right)^{1/2}.$$

Note that $|u|_{H_w^{m,N}(a,b)} \leq \|u\|_{H_w^m(a,b)}$ and when $m \leq N + 1$, we have

$$|u|_{H_w^{m,N}(a,b)} = \|u^{(m)}\|^2_{L_w^2(a,b)} = |u|_{H_w^m(a,b)}.$$

Remark 2.1. *The Sobolev space $H_w^m(a, b)$ verify $\ldots H_w^{m+1}(a, b) \subset H_w^m(a, b) \subset \cdots \subset H_w^0(a, b) = L_w^2(a, b)$. In other words, $C^m([a, b]) \subset H_w^m(a, b)$ with continuous inclusion.*

Lemma 2.1. *[6] Let $u \in H_w^m(-1, 1), w(x) = 1/\sqrt{1 - x^2}$ and $I_N u = \sum_{j=1}^{N+1} u(x_j)C_j(x)$, where $C_j(x)$ and x_j for $j = 1, 2, \ldots, N + 1$ are defined in (2.7) and (2.4), respectively. Then, we have*

$$\|u - I_N u\|_{L_w^2(-1,1)} \leq \bar{C} N^{-m} |u|_{H_w^m(-1,1)}, \tag{2.37}$$

where $\bar{C}$ is a positive constant, independent of N and dependent on m.

Moreover, in the maximum norm, we have

$$\begin{aligned} \|u - I_N u\|_{L^\infty(-1,1)} &\leq \tilde{C} N^{1/2-m} |u|_{H_w^{m,N}(-1,1)}, \\ \text{where} \quad \|u\|_{L^\infty(-1,1)} &= \sup_{-1 \leq x \leq 1} |u(x)|. \end{aligned} \tag{2.38}$$

Theorem 2.2. *Suppose that* $u \in H^m_{w^*}(a,b)$, $w^*(t) = w\left(\dfrac{2}{b-a}t - \dfrac{b+a}{b-a}\right)$ *and* $I_N^* u = \sum_{j=1}^{N+1} u(t_j)\psi_j(t)$, *where* $\psi_j(t)$ *and* t_j *are respectively defined in the following equations*

$$\begin{aligned} \psi_j(t) &= C_j\left(\frac{2}{b-a}t - \frac{b+a}{b-a}\right), \\ t_j &= \frac{b-a}{2}\cos\left(\frac{(2j-1)\pi}{2(N+1)}\right) + \frac{b+a}{2}, \quad j = 1, 2, \ldots, N+1. \end{aligned}$$

The truncation error $u - I_N^* u$ *satisfies the following inequality*

$$\|u - I_N^* u\|_{L^2_{w^*}(a,b)} \leq \bar{C} N^{-m} \||u|\|_{H^{m,N}_{w^*}(a,b)},$$

where $\bar{C}$ *is a positive constant, independant of* N *and dependent on* m, *and*

$$\||u|\|_{H^{m,N}_{w^*}(a,b)} = \left(\sum_{k=\min(m,N+1)}^{m} \left(\frac{b-a}{2}\right)^{2k} \|u^{(k)}\|^2_{L^2_{w^*}(a,b)}\right)^{1/2}.$$

Moreover, in the maximum norm, we have

$$\|u - I_N^* u\|_{L^\infty(a,b)} \leq \tilde{C} N^{1/2-m} \sqrt{\frac{2}{b-a}} \||u|\|_{H^{m,N}_{w^*}(a,b)},$$

where

$$\|u\|_{L^\infty(a,b)} = \sup_{a \leq t \leq b} |u(t)|.$$

Proof 6. *We have*

$$\|u - I_N^* u\|_{L^2_{w^*}(a,b)} = \left(\int_a^b \left(u(t) - \sum_{j=1}^{N+1} u(t_j)\psi_j(t)\right)^2 w^*(t)\, dt\right)^{1/2},$$

and by change of variables $t = \dfrac{b-a}{2}x + \dfrac{b+a}{2}$ *and* $dt = \dfrac{b-a}{2}dx$, *we have*

$$\|u - I_N^* u\|_{L^2_{w^*}(a,b)} = \sqrt{\frac{b-a}{2}} \left(\int_{-1}^1 \left(u\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) - \sum_{j=1}^{N+1} u\left(\frac{b-a}{2}x_j + \frac{b+a}{2}\right) C_j(x)\right)^2 w(x) dx\right)^{1/2}, \tag{2.39}$$

by defining $u\left(\frac{b-a}{2}x+\frac{b+a}{2}\right)\equiv v(x)$, *it yields*

$$\left(\int_{-1}^{1}\left(u\left(\frac{b-a}{2}x+\frac{b+a}{2}\right)-\sum_{j=1}^{N+1}u\left(\frac{b-a}{2}x_j+\frac{b+a}{2}\right)C_j(x)\right)^2 w(x)\,dx\right)^{1/2}=\|v-I_Nv\|_{L^2_w(-1,1)}.$$

From (2.37), we get

$$\|v-I_Nv\|_{L^2_w(-1,1)}\leq \bar{C}N^{-m}|v|_{H^{m,N}_w(-1,1)},$$

where

$$|v|_{H^{m,N}_w(-1,1)}=\left(\sum_{k=\min(m,N+1)}^{m}\|v^{(k)}\|_{L^2_w(-1,1)}\right)^{1/2}. \tag{2.40}$$

Also, we have

$$\|v^{(k)}\|^2_{L^2_w(-1,1)}=\int_{-1}^{1}\left(v^{(k)}(x)\right)^2 w(x)\,dx=\int_{-1}^{1}\left(\left(\frac{b-a}{2}\right)^k u^{(k)}\left(\frac{b-a}{2}x+\frac{b+a}{2}\right)\right)^2 w(x)\,dx,$$

and by change of variable $t=\frac{b-a}{2}x+\frac{b+a}{2}$ *and* $\frac{b-a}{2}dx=dt$, *we obtain*

$$\|v^{(k)}\|^2_{L^2_w(-1,1)}=\frac{2}{b-a}\left(\frac{b-a}{2}\right)^{2k}\int_a^b\left(u^{(k)}(t)\right)^2 w^*(t)\,dt=\frac{2}{b-a}\left(\frac{b-a}{2}\right)^{2k}\|u^{(k)}\|^2_{L^2_{w^*}(a,b)}, \tag{2.41}$$

$$|v|_{H^{m,N}_w(-1,1)}=\sqrt{\frac{2}{b-a}}\left(\sum_{k=\min(m,N+1)}^{m}\left(\frac{b-a}{2}\right)^{2k}\|u^{(k)}\|_{L^2_{w^*}(a,b)}\right)^{1/2}. \tag{2.42}$$

Consequently (2.39) and (2.42) yield

$$\|u-I^*_Nu\|_{L^2_{w^*}(a,b)}\leq \bar{C}N^{-m}\left(\sum_{k=\min(m,N+1)}^{m}\left(\frac{b-a}{2}\right)^{2k}\|u^{(k)}\|_{L^2_{w^*}(a,b)}\right)^{1/2}.$$

In the maximum norm, we have

$$\|u-I^*_Nu\|_{L^\infty(a,b)}=\sup_{a\leq t\leq b}|u(t)-I^*_Nu(t)|=\sup_{a\leq t\leq b}\left|u(t)-\sum_{j=1}^{N+1}u(t_j)\psi_j(t)\right|, \tag{2.43}$$

and by change of variable $t=\frac{b-a}{2}x+\frac{b+a}{2}$, *we get*

$$\sup_{a\leq t\leq b}\left|u(t)-\sum_{j=1}^{N+1}u(t_j)\psi_j(t)\right|=\sup_{-1\leq x\leq 1}\left|u\left(\frac{b-a}{2}x+\frac{b+a}{2}\right)-\sum_{j=1}^{N+1}u\left(\frac{b-a}{2}x_j+\frac{b+a}{2}\right)C_j(x)\right|.$$

By defining $u\left(\frac{b-a}{2}x+\frac{b+a}{2}\right) \equiv v(x)$, *one has*

$$\begin{aligned}\sup_{-1\leq x\leq 1}\left|u\left(\frac{b-a}{2}x+\frac{b+a}{2}\right)-\sum_{j=1}^{N+1}u\left(\frac{b-a}{2}x_j+\frac{b+a}{2}\right)C_j(x)\right| &= \sup_{-1\leq x\leq 1}\left|v(x)-\sum_{j=1}^{N+1}v(x_j)C_j(x)\right| \\ &= \|v-I_Nv\|_{L^\infty(-1,1)},\end{aligned}$$

and by considering (2.38), we have

$$\|v-I_Nv\|_{L^\infty(-1,1)} \leq \tilde{C}N^{1/2-m}|v|_{H_w^{m,N}(-1,1)}.$$

Moreover, as shown in (2.41), one has

$$\|v^{(k)}\|^2_{L^2_w(-1,1)} = \frac{2}{b-a}\left(\frac{b-a}{2}\right)^{2k}\|u^{(k)}\|^2_{L^2_{w^*}(a,b)},$$

and then

$$\begin{aligned}|v|_{H_w^{m,N}(-1,1)} &= \left(\sum_{k=\min(m,N+1)}^{m}\|v^{(k)}\|_{L^2_w(-1,1)}\right)^{1/2} \\ &= \left(\sum_{k=\min(m,N+1)}^{m}\frac{2}{b-a}\left(\frac{b-a}{2}\right)^{2k}\|u^{(k)}\|_{L^2_{w^*}(a,b)}\right)^{1/2}. \qquad (2.44)\end{aligned}$$

Finally (2.43) and (2.44) result

$$\|u-I_N^*u\|_{L^\infty(a,b)} \leq \tilde{C}N^{1/2-m}\sqrt{\frac{2}{b-a}}|||u|||_{H_{w^*}^{m,N}(a,b)},$$

wich completes the proof.

Corollary 2.1. *Suppose that the assumptions in theorem 2.2 are satisfied. Then, we have*

$$\begin{aligned}\|u-I_N^*u\|_{L^2_{w^*}(a,b)} &\longrightarrow 0, \quad \text{as} \quad N\longrightarrow\infty \quad \text{with} \quad O(N^{-m}), \\ \|u-I_N^*u\|_{L^\infty(a,b)} &\longrightarrow 0, \quad \text{as} \quad N\longrightarrow\infty \quad \text{with} \quad O(N^{1/2-m}),\end{aligned}$$

which shows the rate of convergence of the serie expansion, I_N^*u *to* u.

Chapter 3

Numerical tests

In this section, some different partial-integro differential equations are solved by applying the present method. Accuracy, efficiency and applicability of the present method is confirmed. All problems considered have continuous solution and can be solved analytically, this allows verification and validation of the method by comparing with the results of exact solutions. Furthermore, comparison between the exact and the approximate solution is given at different choices of points and the errors are computed for infinity norm in different times.

3.1 Algorithm of the proposed method

- **Input**
  $M, n \in \mathbb{N}, \alpha, \beta, f(t,x), k(x,t,u(x,t)), g, h_0, h_1$.

  $>$ Define the basis cardinal functions C similair to relation (2.7).

  $>$ Introduce the matrix U given in the approximation by Cardinal Chebyshev.

  $>$ Compute the vector $F = f(t_i, x_j)$, $i, j = 1 : M + 1$.

  $>$ Compute Gauss Legendre nodes and weights in relation(2.10).

  $>$ Compute the integral oppeared in the equation(1.1) by Gauss quadrature.

  $>$ Extract the algebraic system(2.34) and solve it.

  $>$ Replace the acheive U into relation(2.15).

- **Output**
  The cardinal relation $u(x,t) = C(t)U^T C(x)$.

3.2 Numerical examples

Example 3.1. *Consider a linear diffusion equation [13]*

$$u_t(x,t) - u_{xx}(x,t) = f(x,t) - \int_0^t e^{x(t-s)} u(x,s)\, ds, \tag{3.1}$$

with initial and boundary conditions

$$\begin{cases} u(x,0) = 0, \quad x \in [0,1], \\ u(0,t) = \sin(t), \quad u(1,t) = 0, \quad t \in [0,1]. \end{cases}$$

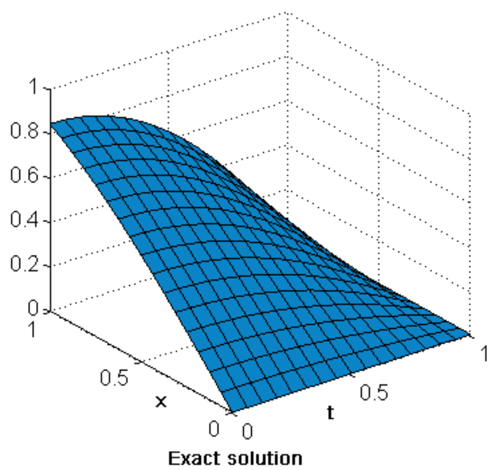
The exact solution of the problem is given by

$$u(x,t) = (1 - x^2)\sin(t),$$

and the function $f(x,t)$ is calculated using the above exact solution. The numerical results are reported in table (3.1) for different values of M.

Numerical results for the proposed Method			
t	M=3	M=4	M=5
0	1.0477×10^{-17}	1.0392×10^{-17}	1.2836×10^{-17}
0.0625	2.1378×10^{-6}	8.7322×10^{-7}	1.0930×10^{-7}
0.1875	3.0452×10^{-5}	1.0049×10^{-5}	1.4469×10^{-5}
0.3125	1.1101×10^{-4}	9.8094×10^{-5}	9.5423×10^{-5}
0.4375	2.9375×10^{-4}	3.2515×10^{-4}	3.1374×10^{-4}
0.5625	6.7471×10^{-4}	7.5365×10^{-4}	7.5125×10^{-4}
0.6875	1.4218×10^{-3}	1.4706×10^{-3}	1.4947×10^{-3}
0.8125	2.7974×10^{-3}	2.6009×10^{-2}	2.6287×10^{-3}

Table 3.1: Comparison of the maximum absolute errors at different times for Example (3.1).



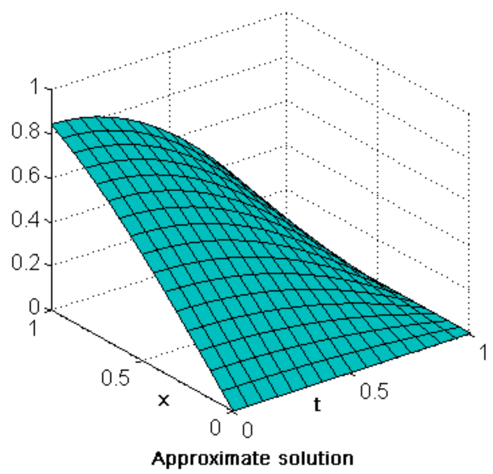


Figure 3.1: Plots of the exact and approximate solution for M=3 for Example (3.1).

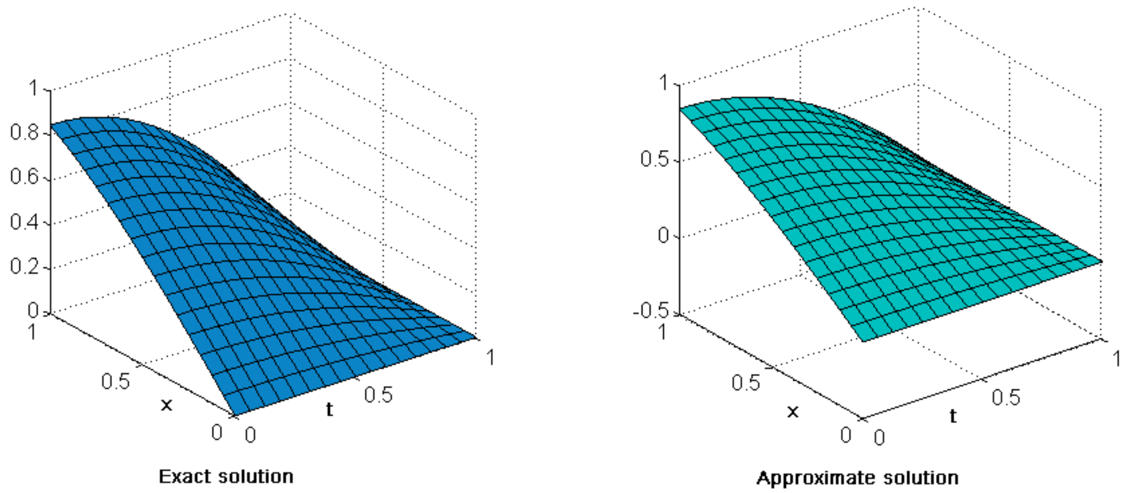


Figure 3.2: Plots of the exact and approximate solution for M=4 for Example (3.1).

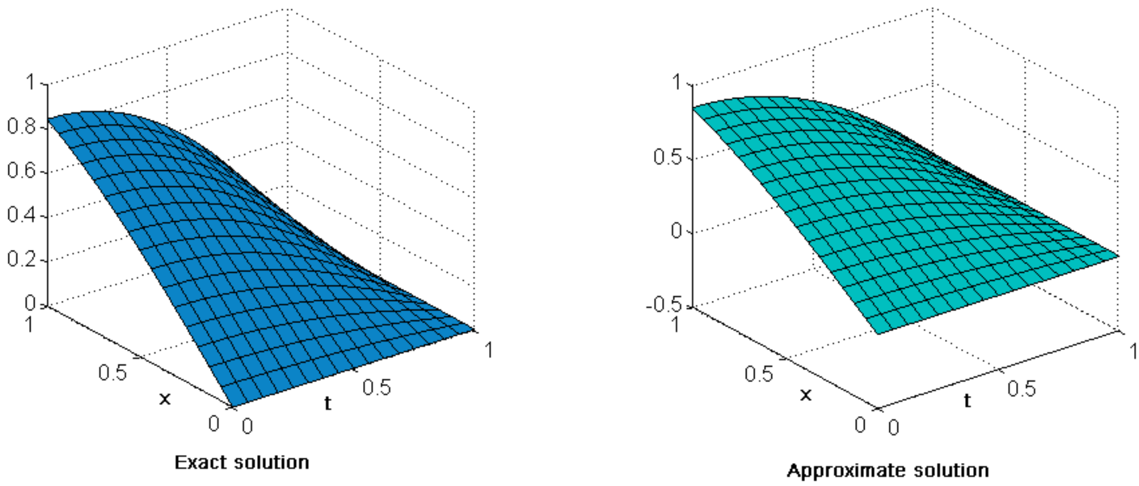


Figure 3.3: Plots of the exact and approximate solution for M=5 for Example (3.1).

Example 3.2. *To show the ability of the proposed method for solving nonlinear PIDEs, we consider the following equation*

$$u_t(x,t) + u_{xx}(x,t) = \int_0^t e^{x+t+s} u^2(x,s)\, ds + h(x,t), \tag{3.2}$$

subject to the following initial and boundary conditions

$$\begin{cases} u(x,0) = x, & x \in [0,1], \\ u(0,t) = 0, \quad u(1,t) = \cos(t), & t \in [0,1]. \end{cases}$$

The function $h(x,t)$ is chosen such that the exact solution of this problem is given by

$$u(x,t) = x\cos(t).$$

Table (3.2) illustrates the numerical results at differnt time levels.

	Numerical results for the proposed method		
t	M=3	M=4	M=5
0	2.4810×10^{-7}	9.2396×10^{-9}	9.0041×10^{-7}
0.0625	4.6881×10^{-3}	4.4045×10^{-3}	9.1125×10^{-3}
0.1875	1.2844×10^{-2}	2.2534×10^{-2}	3.2366×10^{-2}
0.3125	3.3645×10^{-2}	3.9466×10^{-2}	2.6809×10^{-2}
0.4375	5.1379×10^{-2}	4.5346×10^{-2}	1.4196×10^{-2}
0.5625	5.2479×10^{-2}	7.0263×10^{-2}	3.0276×10^{-2}
0.6875	3.3197×10^{-2}	1.6023×10^{-1}	7.2199×10^{-2}
0.8125	1.0475×10^{-1}	4.2583×10^{-1}	5.0962×10^{-1}

Table 3.2: Comparison of the maximum absolute errors at different times for Example (3.2).

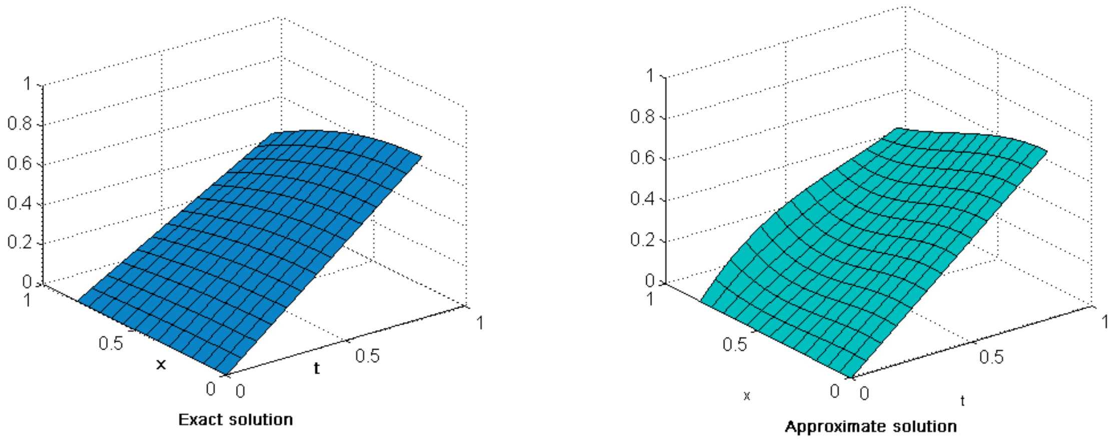


Figure 3.4: Plots of the exact and approximate solution for M=3 for Example (3.2).

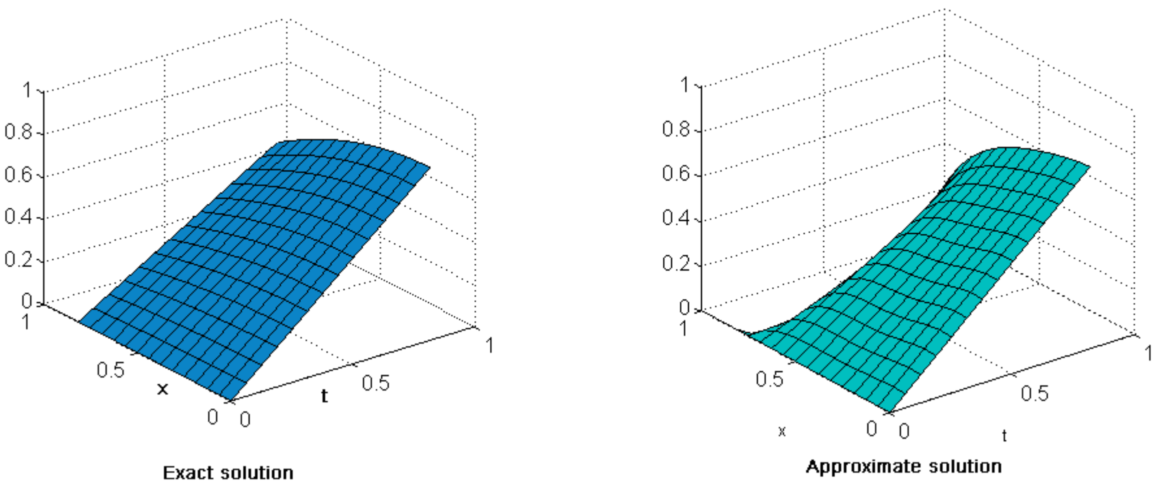


Figure 3.5: Plots of the exact and approximate solution for M=4 for Example (3.2).

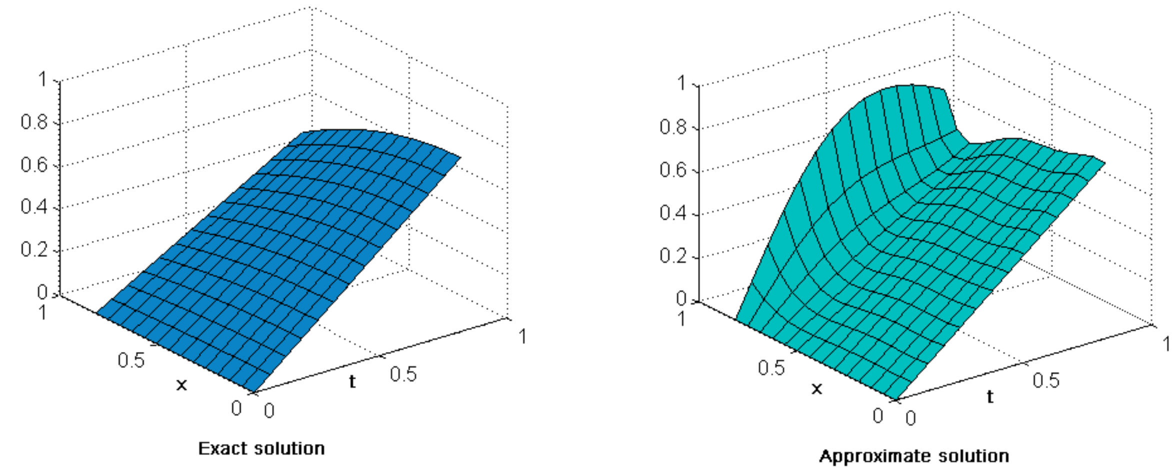


Figure 3.6: Plots of the exact and approximate solution for M=5 for Example (3.2).

Example 3.3. *Let given another linear diffusion equation[8]*

$$u_t(x,t) - u_{xx}(x,t) = h(x,t) - \int_0^t u(x,s)\,ds, \tag{3.3}$$

with the initial and boundary conditions

$$\begin{cases} u(x,0) = \dfrac{1-x^2}{2}, & x \in [0,1], \\ u(0,t) = \dfrac{\cosh(t)}{2+\sinh^2(t)}, \quad u(1,t) = 0, & t \in [0,1]. \end{cases}$$

The exact solution of this PIDE is given by

$$u(x,t) = \frac{(1-x^2)\cosh(t)}{2+\sinh^2(t)}.$$

The function $h(x,t)$ is calculated using the exact solution of this problem. Table(3.3) presentes the numerical results of the maximum absolute errors at different time levels.

	Numerical results for the proposed method		
t	M=3	M=4	M=5
0	9.5154×10^{-8}	1.6511×10^{-9}	4.3895×10^{-7}
0.0625	1.0534×10^{-5}	3.3011×10^{-6}	8.3636×10^{-7}
0.1875	7.7843×10^{-5}	1.5232×10^{-5}	9.9175×10^{-7}
0.3125	6.3607×10^{-5}	2.9635×10^{-5}	2.5959×10^{-6}
0.4375	1.9061×10^{-4}	5.1192×10^{-5}	3.4572×10^{-6}
0.5625	2.7660×10^{-4}	5.8453×10^{-5}	5.2551×10^{-6}
0.6875	1.4739×10^{-4}	1.6891×10^{-4}	7.2499×10^{-6}
0.8125	7.1222×10^{-4}	1.1200×10^{-4}	1.1422×10^{-5}

Table 3.3: Comparison of the maximum absolute errors at different times for Example (3.3).

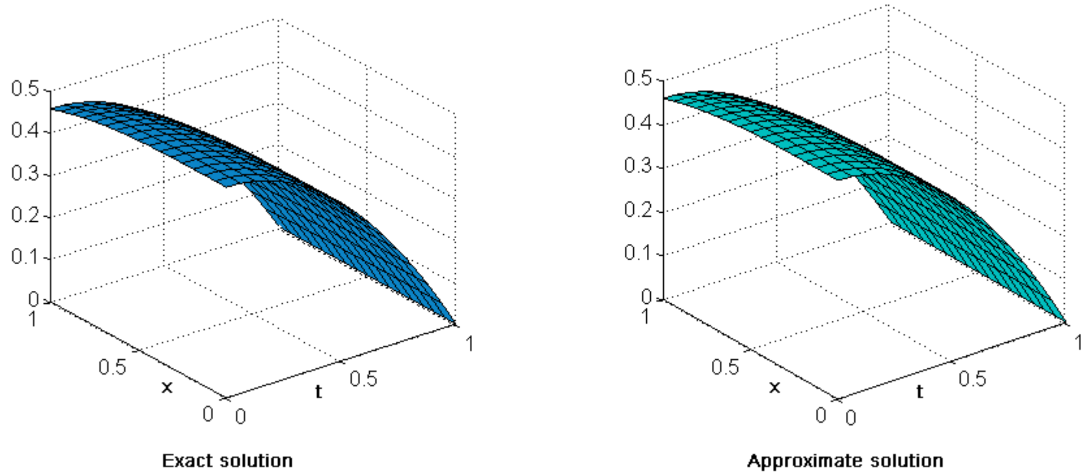


Figure 3.7: Plots of the exact and approximate solution for M=3 for Example (3.3).

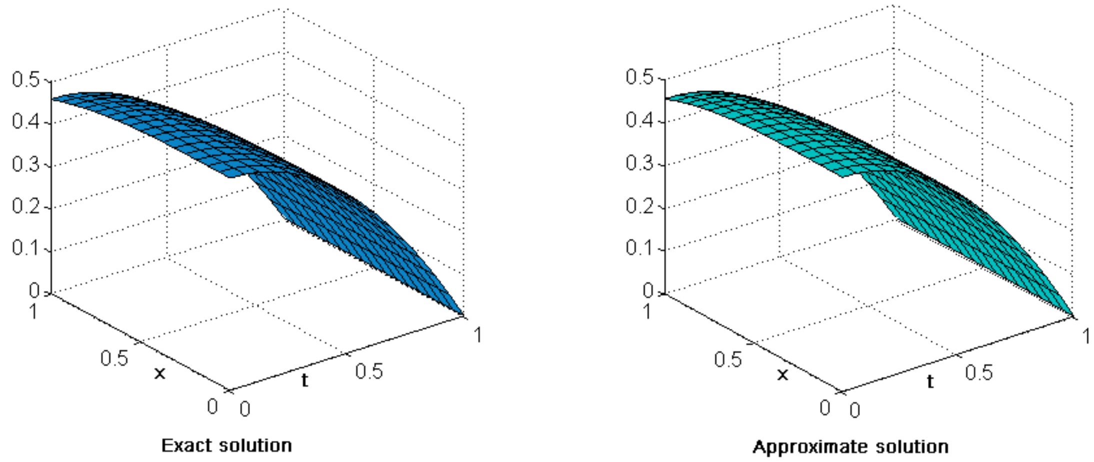


Figure 3.8: Plots of the exact and approximate solution for M=4 for Example (3.3).

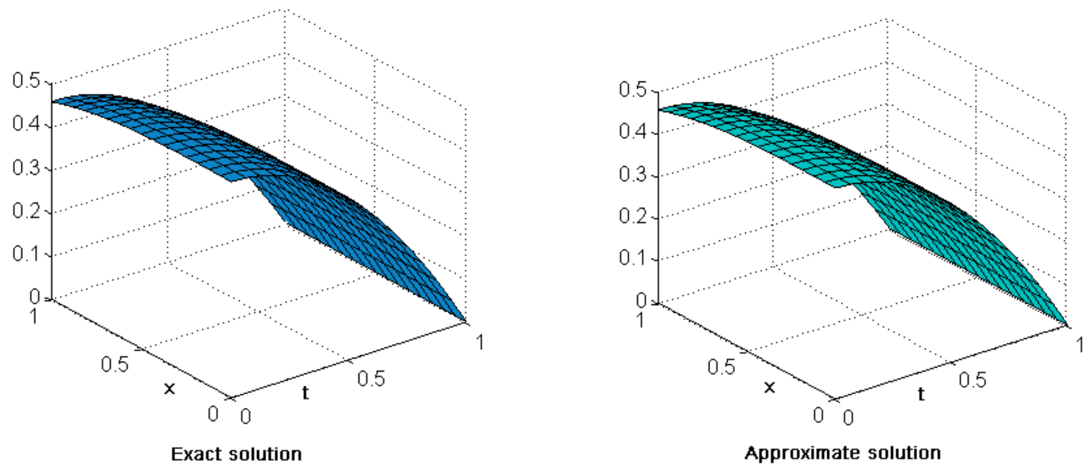


Figure 3.9: Plots of the exact and approximate solution for M=5 for Example (3.3).

Example 3.4. *In this example, we consider another nonlinear diffusion equation*

$$u_t(x,t) - u_{xx}(x,t) = f(x,t) - \int_0^t u^3(x,s)\,ds, \tag{3.4}$$

with initial and boundary conditions

$$\begin{cases} u(x,0) = 1 - x^2, & x \in [0,1], \\ u(0,t) = \cosh(t), \quad u(1,t) = 0, & t \in [0,1], \end{cases}$$

the function f is chosen such that $u(x,t) = (1-x^2)\cosh(t)$ is the exact solution of this example, numerical results at different time levels are reported in table(3.4) .

	Numerical results for the proposed method		
t	M=3	M=4	M=5
0	1.9031×10^{-7}	3.3023×10^{-9}	8.7790×10^{-7}
0.0625	5.9992×10^{-4}	5.4024×10^{-4}	5.4947×10^{-4}
0.1875	3.1701×10^{-3}	3.3606×10^{-3}	3.4829×10^{-3}
0.3125	7.0461×10^{-3}	7.2146×10^{-3}	7.2225×10^{-3}
0.4375	1.1469×10^{-2}	1.0943×10^{-2}	1.0777×10^{-2}
0.5625	1.5522×10^{-2}	1.3853×10^{-2}	1.3830×10^{-2}
0.6875	1.8129×10^{-2}	1.5843×10^{-2}	1.6391×10^{-2}
0.8125	1.8312×10^{-2}	1.7266×10^{-2}	1.7924×10^{-2}

Table 3.4: Comparison of the maximum absolute errors at different times for Example (3.4).

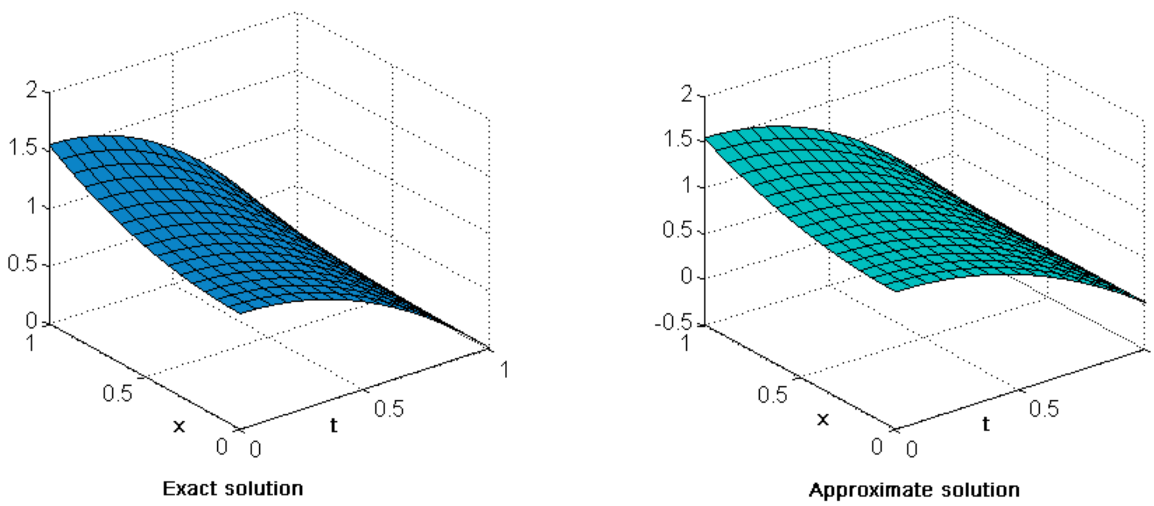


Figure 3.10: Plots of the exact and approximate solution for M=3 for Example (3.4).

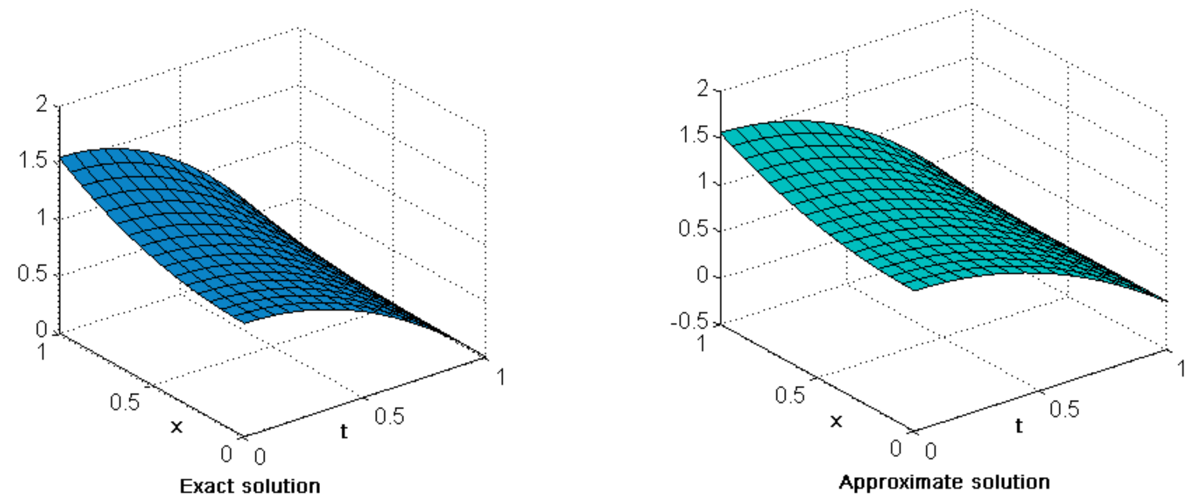


Figure 3.11: Plots of the exact and approximate solution for M=4 for Example (3.4).

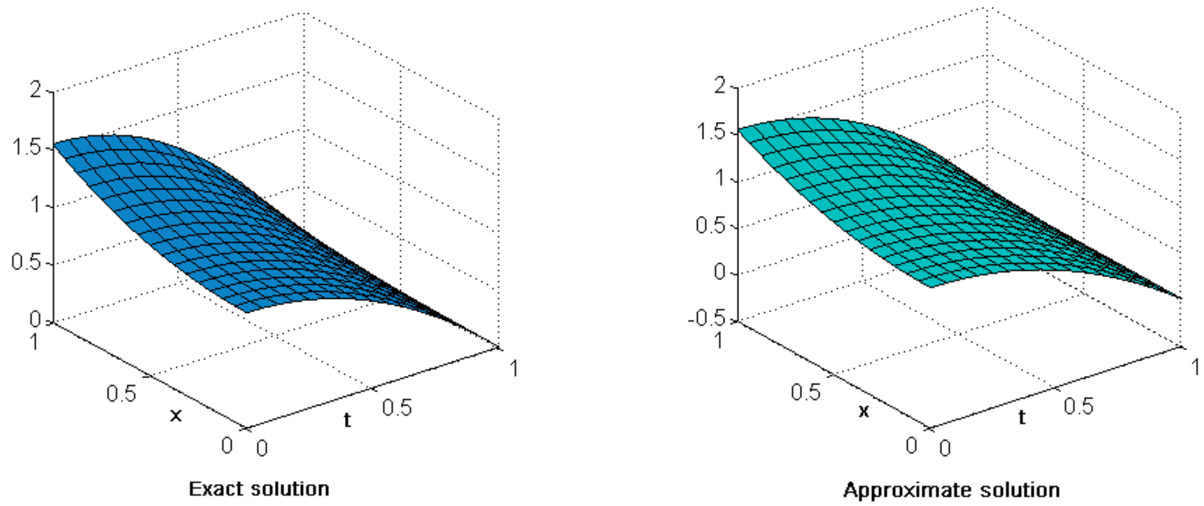


Figure 3.12: Plots of the exact and approximate solution for M=5 for Example (3.4).

Example 3.5. *Let consider the following equation[8]*

$$u_t(x,t) + u_{xx}(x,t) = h(x,t) - \int_0^t e^{s-t} u(x,s)\, ds, \tag{3.5}$$

subject to the initial and boundary conditions

$$\begin{cases} u(x,0) = x, \quad x \in [0,1], \\ u(0,t) = 0, \quad u(1,t) = e^{-t}, \quad t \in [0,1], \end{cases}$$

the function h is chosen such that $u(x,t) = xe^{-xt}$ is the exact solution of this example. The numerical tests are summarized in table (3.5).

	Numerical results for the proposed method		
t	M=3	M=4	M=5
0	2.4810×10^{-7}	9.2396×10^{-9}	9.0041×10^{-7}
0.0625	5.0469×10^{-4}	1.0564×10^{-4}	5.7548×10^{-4}
0.1875	2.1096×10^{-3}	3.9036×10^{-3}	4.1568×10^{-3}
0.3125	6.6068×10^{-3}	7.0043×10^{-3}	3.1865×10^{-3}
0.4375	9.8470×10^{-3}	7.7375×10^{-3}	1.4541×10^{-3}
0.5625	8.6462×10^{-3}	1.0788×10^{-2}	1.5115×10^{-3}
0.6875	1.4372×10^{-3}	2.6913×10^{-2}	1.6981×10^{-2}
0.8125	2.1133×10^{-2}	7.3939×10^{-2}	1.0777×10^{-1}

Table 3.5: Comparison of the maximum absolute errors at different times for Example (3.5).

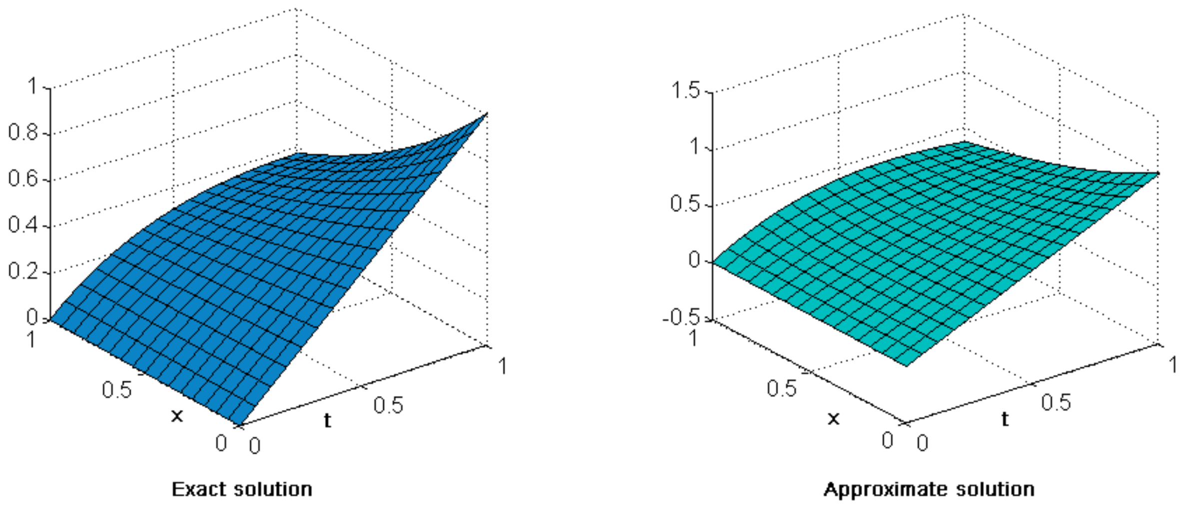


Figure 3.13: Plots of the exact and approximate solution for M=3 for Example (3.5).

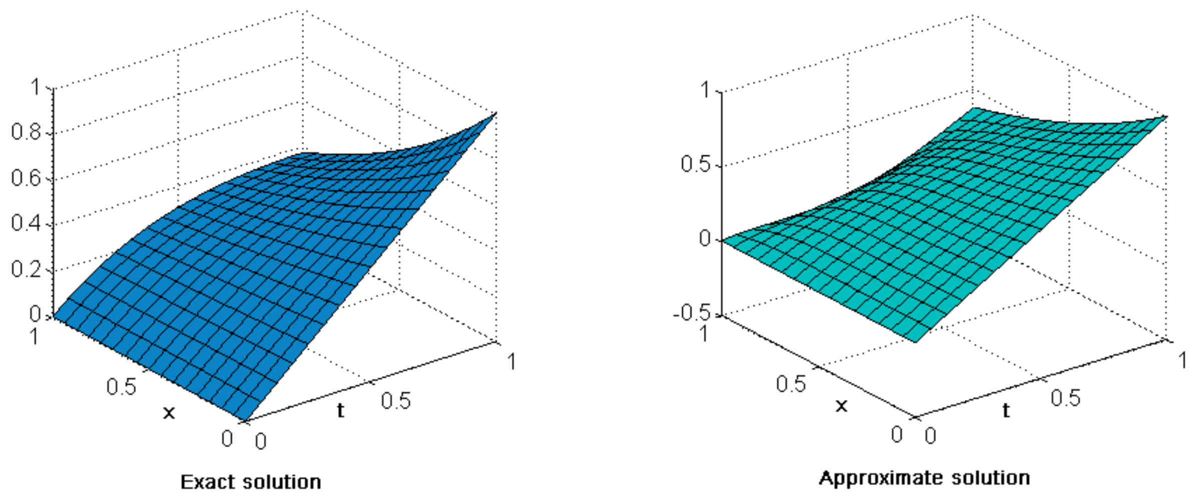


Figure 3.14: Plots of the exact and approximate solution for M=4 for Example (3.5).

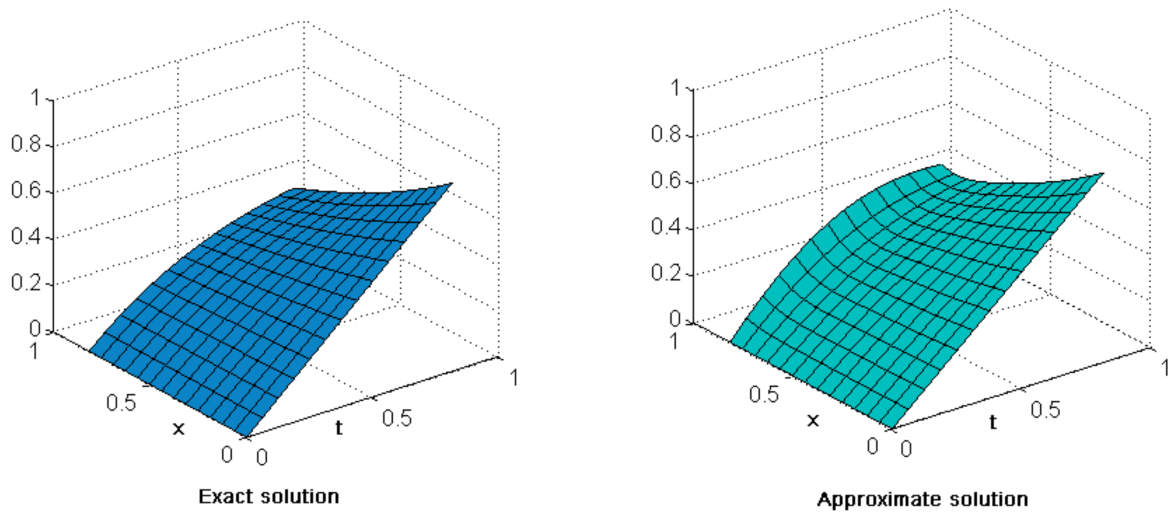


Figure 3.15: Plots of the exact and approximate solution for M=5 for Example (3.5).

Example 3.6. *As a last example of the diffusion equation[13] for the linear case we have*

$$u_t(x,t) - u_{xx}(x,t) = f(x,t) - \int_0^t \sin(x)(t-s)u(x,s)\,ds, \tag{3.6}$$

with initial and boundary conditions

$$\begin{cases} u(x,0) = 1 - x^2, & x \in [0,1], \\ u(0,t) = e^t, \quad u(1,t) = 0, & t \in [0,1], \end{cases}$$

the exact solution is given by

$$u(x,t) = (1-x^2)e^t,$$

where the function $f(x,t)$ is calculated using the above exact solution. Numerical results for this test problems are depicted in figure(3.16) and figure(3.17).

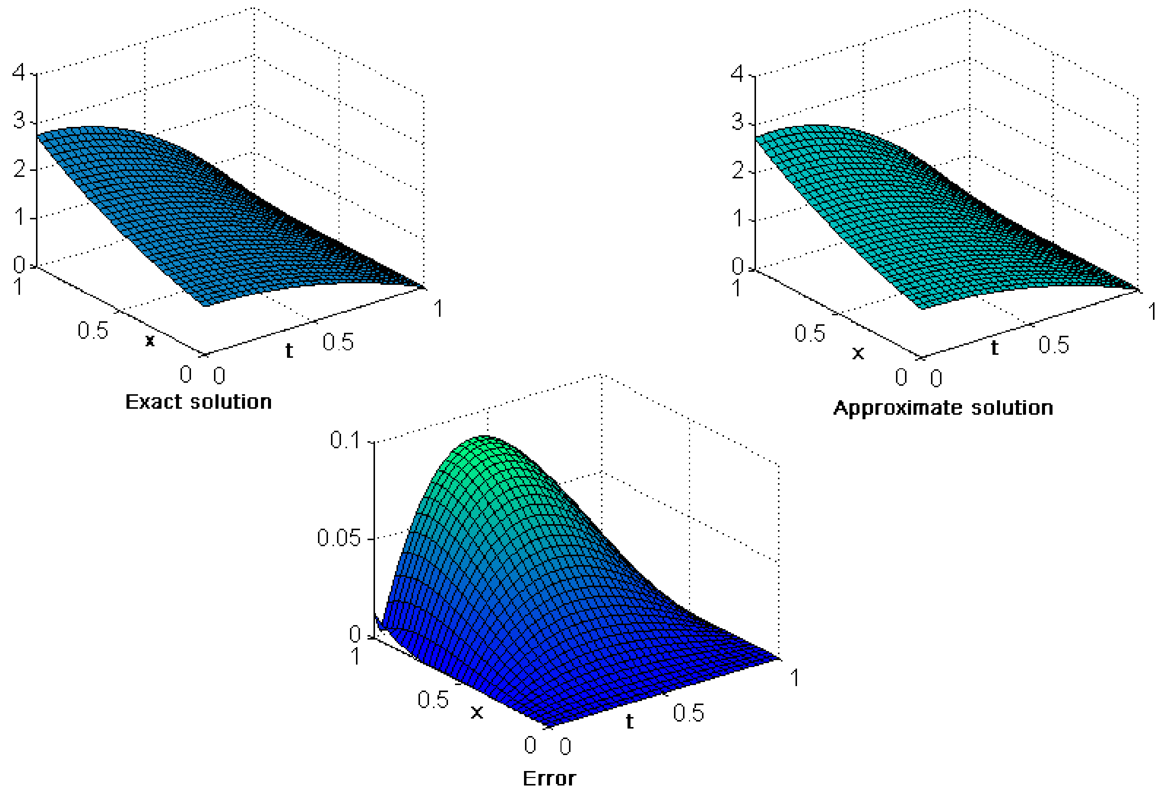


Figure 3.16: Plots of the error, exact and approximate solution for M=3 for Example (3.6).

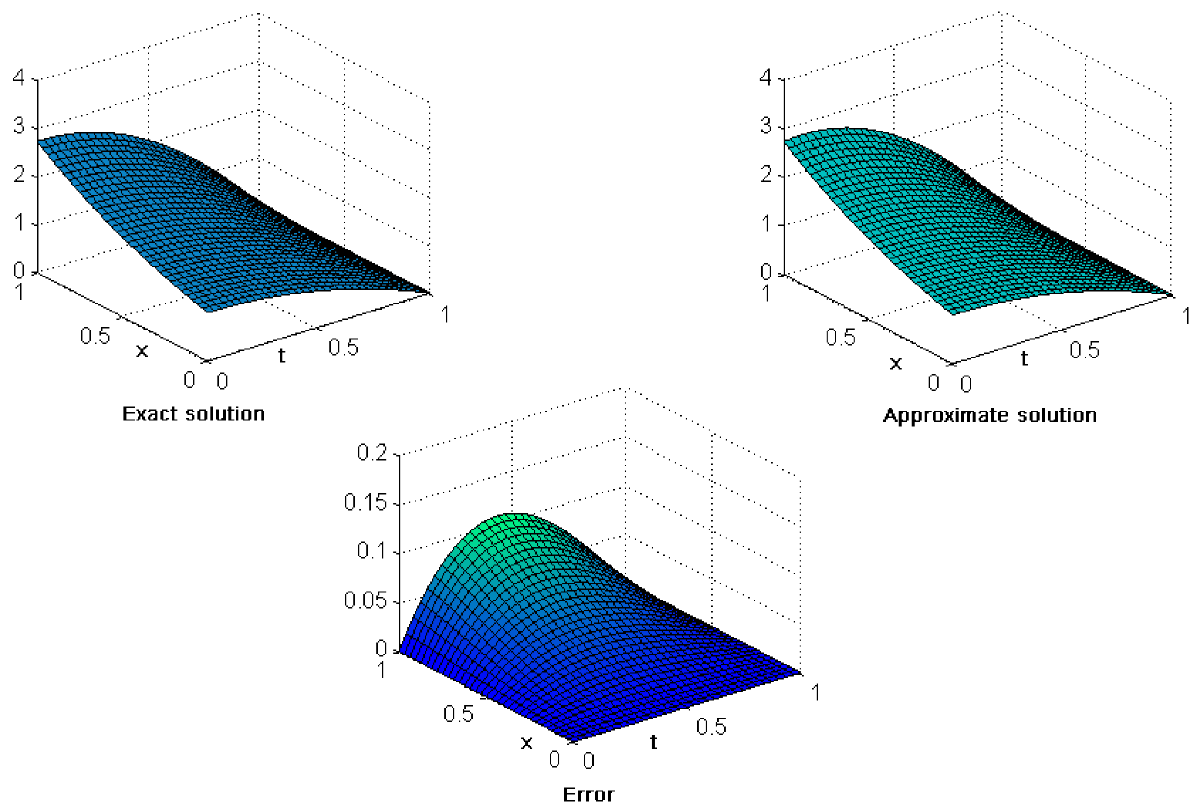


Figure 3.17: Plots of the error, exact and approximate solution for M=5 for Example (3.6).

Conclusion

This work deals with the spectral methods for the approximate solution of partial Volterra integro-differential equations on one-dimensional bounded domains. The technique is based on expanding our unknown function by cardinal Chebyshev functions and also their derivatives. Then, we use the Gauss Legendre-quadrature to approximate the integral oppeared in the PDEs which leads to the conversion of the problem to a system of linear or nonlinear algebraic equations. Finally, we demonstrate the efficiency and accuracy of the proposed method for some numerical examples. We note that the presented method is easily implementable and can be utilized efficiently to find the numerical solutions for other kinds of problems such as hyperbolic and elliptic problems.

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Abstract:

The subject of this dissertation is to apply a pseudo-spectral method based on Chebyshev cardinal functions to solve parabolic partial integro-differential equations (PIDEs). Since these equations play an essential role in mathematics, physics, and engineering. Finding an approximate solution of the equation is important. The numerical technique is based on the combination between the approximation of the solution by the Chebyshev cardinal functions and using Gauss quadrature to approximate the integral appeared in the equation. The problem is reduced to a nonlinear system of algebraic equations. The convergence analysis is investigated and some numerical examples are given to guaranted the efficiency of the proposed algorithm.

Key-Words: Spectral method, partial integro-differential equations, Chebyshev cardinal functions, Gauss quadrature, approximate solution.

Résumé :

L'objectif de ce mémoire est d'appliquer la méthode pseudo-spectrale basée sur les fonctions cardinales de Chebyshev pour résoudre les équations intégro-différentielles partielles paraboliques (EIDP). Comme ces équations jouent un rôle essentiel en mathématiques, en physique et en ingénierie, il est important de trouver une solution approximative. La technique numérique est basée sur la combinaison entre l'approximation de la solution par des fonctions cardinales de Chebyshev et l'utilisation de la quadrature de Gauss pour approximer l'intégrale qui figure dans l'équation. Le problème se réduit à un système non linéaire d'équations algébriques. L'analyse de convergence est obtenue et quelques exemples numériques sont donnés pour garantir l'efficacité de l'algorithme proposée.

Mots-Clés : Méthode spectrale, équations intégro-différentielles partielles, fonction cardinale de Chebyshev, quadrature de Gauss, solution approximative.

ملخص:

الهدف من هذه الأطروحة هو تطبيق الطريقة الطيفية الزائفة التي تعتمد على الوظائف الاساسية لتشيبيشيف لحل المعادلات التفاضلية التكاملية الجزئية المكافئة. نظرا لأن هذه المعادلات تلعب دورا أساسيا في الرياضيات والفيزياء والهندسة، فمن المهم إيجاد حل مناسب. تعتمد التقنية العددية على الجمع بين تقريب الحل بواسطة وظائف تشيبيشيف الأساسية واستخدام تربيع غوسيان لتقريب التكامل الموجود في هذه المعادلات. يتم اختزال المشكلة الى نظام غير خطي من المعادلات الجبرية. تم تحليل التقارب وبعض الأمثلة العددية تم تقديمها لضمان فعالية الخوارزمية المقترحة.

كلمات مفتاحية: طرق طيفية، المعادلات التفاضلية التكاملية الجزئية، الوظائف الأساسية لتشيبيشيف ، تربيع غوسيان، حل تقريبي.