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Mémoire

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Thème

Exponential stabilization of a neutrally delayed wave equation

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CONTENTS

DEDICATION

To you alone, O owner of a fragrant biography and an enlightened thought, for you are the only one who had the first credit for me to attain higher education, to the pure soul of my father.
You are the one who put me on the path of life, you are the one who made me calm, and you who took care of me until I grew up, to you, my dear mother.
To all my sisters who have been credited with removing many obstacles and difficulties from my path.
To you my dear professors, you have always helped me.
I dedicate it to all of you.

INTRODUCTION

During the last decades, the interest for systems of differential equations depending on the past history has been increasing. In fact, the introduction of the delay in the models allows a better description of the real phenomena and a more reliable prediction of their behavior. Such delay models, also called systems with memory or aftereffect, hereditary or time delay systems, are mathematically descibed by Retarded Funnctional Differential Equations (RFDEs). Their dynamics is significantly influenced by the presence of the delay terms and oscillations, instability, chaos and loss of performance as well as improved stability can occur. The reason for this more complex dynamics is that, opposite to Ordinary Differential Equations (ODEs), RFDEs are infinite dimensional dynamical systems, Krasovskii was the first to emphasize the importance of considering the state of a system defined by a functional differential equation as a function.

The scientific community is witnessing a considerable growth of interest in problems involving time delays. This is due mainly to the widespread appearances of such phenomena. Time delays are peculiar to the dependence of the rate of change on the past history of the system. This is the case, for instance, whenever there is a displacement of material or transmission of information. The class of differential equations treating delays is known as functional differential equations (FDEs). Models that necessitate the incorporation of the history of the (highest) derivative are commonly known as neutral delay differential equations (NDDEs).

NDDEs have been shown to be very useful in describing complicated phenomena in many fields including control theory, mechanical systems, chemical processes, oscillation theory, and biosciences.

It was established that differential equations are sensitive to the presence of delays. Many researchers have demonstrated that even initially stable systems may be destabilized when taking into account delays. This has forced scientists to find appropriate ways to fix this matter. We note here that delays may play a positive role in many cases. It has been well established that, in contrast to the sensitivity issue raised above, large neutral delays may stabilize systems. As a matter of fact, for better achievements, engineers have been adding neutral delays premeditatedly in the models.

The example model of neutrally retarded viscoelastic Timoshenko system

$$\begin{cases} \varphi_{tt} = (\varphi_x + \psi)_x, \\ \left[\psi_t + \int_0^t K(t-s)\psi_t(s)ds\right]' = \psi_{xx} - \int_0^t (g(t) - s)\psi_{xx}(s)ds - (\varphi_x + \psi), \end{cases}$$

for $t > 0$, $0 < x < 1$ with initial and boundary conditions

$$\begin{cases} \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0,\ t \geq 0 \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x),\ 0 < x < 1, \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x),\ 0 < x < 1, \end{cases}$$

where $\varphi_0(x), \varphi_1(x),\ \psi_0(x),$ and $\psi_1(x)$ are given initial data. Here φ is the transversal displacement of the beam from its equilibrium and ψ is the rotational displacement of the beam.

Second order NDDEs appear, in general, in the study of vibrating masses attached to an elastic bar and also (as the Euler equation) in some variational problems.

The existence and uniqueness of a mild solution. The mild solution is strong or classical when the initial data are regular. This is proved, in fact, for more general abstract problems. The operator A (generalizing the Laplacian, with domain: the set of functions v such that $v,\ v_t \in H^2(\Omega)$ are absolutely continuous with Dirichlet boundary conditions) is assumed to be an infinitesimal generator of a strongly continuous cosine family of bounded linear operators. The results there are obtained using fixed point theorems. Therefore, we shall assume that the solution (and the initial data) is regular enough to justify our computation.

We brief the reader that there are many other results but for either ordinary differential equations, or first order partial differential equations. The few treated second order problems are concerned with delays in the state function or its first derivative rather than in the second derivative (which represents here the real challenge).

It is also worth noting that several investigations appeared dealing with the oscillation phenomena for problems of exactly the same type.
We do not report here these references (although one can transform second order equations into first order systems) because of the size of the paper (and to avoid being biased).
We establish a range of values for the coefficient "p" for which solutions decay to zero exponentially in time.

Another problem of neutral delay with a viscoelastic terme is

$$\begin{cases} u_{tt}(t,x) - p\int_0^x u_{tt}(t-\tau,y)dy = u_{xx}(t,x) - \int_0^t g(t-s)u_{xx}(s,x)ds \\ in \quad (0,\infty) \times (0,1), \\ u(t,x) = 0, \quad t \in (0,\infty), \quad x = 0,1, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in (0,1), \end{cases} \tag{1}$$

where the delay in the first integral in the equation is of neutral type and the second integral is a viscoelastic term (or a memory term). The (relaxation) function g is a (nonincreasing) continuously differentiable function and p is a real number. The functions $u_0(x)$ and $u_1(x)$ are given initial data and $\tau > 0$.

Here, A strang damped wave equation with neutral delay is considered

$$\begin{cases} [u(t) - pu(t-\tau)]'' = \Delta u + \Delta u_t \quad in \quad (0,\infty) \times \Omega, \\ u(t,x) = 0, \quad t \in (\tau,\infty), \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \\ u(t,x) = \varphi(t,x), \quad t \in [-\tau,0], \quad x \in \Omega, \end{cases} \tag{2}$$

where Ω is a bounded regular domain of $\mathbb{R}^n$, $u_0(x)$, $u_1(x)$ and $\varphi(t,x)$ are given functions and p, $\tau > 0$. The primes, as well as the subscripts "t", denote time derivatives.
We will assume the compatibility condition $\varphi(0,x) = u_0(x)$ and $\varphi_t(0,x) = u_1(x),\ x \in \Omega$.

The first chapter contains an definitions and the mentary properties of L^p space, convolution, functional differential equation and lyapunov stability .

In the last chapter, we studied the stability for the damped wave equation with neutral delay.

CHAPTER 1

PRELIMINARY

1.1 Definition and elementary properties of L^p Spaces

Definition 1.1.1 *We denoted by $L^1(\Omega,\mu)$, or simply $L^1(\Omega)$ (or just L^1), the space of integrable functions from Ω into $\mathbb{R}$.*

$$\|f\|_{L^1} = \|f\|_1 = \int_\Omega |f| d\mu = \int |f|.$$

Definition 1.1.2 *we set [2]*
Let $p \in \mathbb{R}$ with $1 < p < \infty$;

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,|\, f \text{ is measurable and } |\, f \,|^p \in L^1(\Omega) \right\}$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p d\mu \right]^{1/p}.$$

Definition 1.1.3 *when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) g(x) dx,$$

is a Hilbert space.

Definition 1.1.4 *We set*

$$L^\infty(\Omega) = \left\{ f \,:\, \Omega \to \mathbb{R},\ f \text{ is measurable and there is a constant } C \text{ such that } |f(x)| \leq C \text{ a.e. on } \Omega \right\}$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{C;\ |f(x)| \leq C \text{ a.e. on } \Omega\}.$$

1-Reflexivity

Theorem 1 L^p is reflexive for any p, $2 \leq p < \infty$.

Step 1 : (Clarkson's first inequality). *Let* $1 < p < \infty$. *We claim that*

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \leq \frac{1}{2}(\|f\|_p^p + \|g\|_p^p) \quad \forall f,g \in L^p.$$

Step 2 : L^p *is* uniformly convex, and thus reflexive for $2 \leq p < \infty$. *Indeed, let* $\varepsilon > 0$ *and let* $f,g \in L^p$ *with* $\|f\|_p \leq 1$, $\|g\|_p \leq 1$, *and* $\|f-g\|_p > \varepsilon$. *We deduce that*

$$\left\|\frac{f+g}{2}\right\|_p^p < 1 - \left(\frac{\varepsilon}{2}\right)^2$$

and thus $\left\|\frac{f+g}{2}\right\|_p < 1 - \delta$ *with* $\delta = 1 - [1 - (\frac{\varepsilon}{2})^p]^{1/p} > 0$. *Therefore,* L^p *is uniformly convex and thus reflexive.*

Step 3 : L^p *is reflexive for* $1 < p \leq 2$.

2-Separability

Theorem 2 Assume that Ω is a separable measure space. Then $L^p(\Omega)$ is separable for any p, $1 \leq p < \infty$.
We shall consider only the case $\Omega = \mathbb{R}^N$, *since the general case is somewhat tricky. Note that as a consequence,* $L^p(\Omega)$ *is also separable for any measurable set* $\Omega \subset \mathbb{R}^N$. *Indeed, there is a canonical isometry from* $L^p(\Omega)$ *into* $L^p(\mathbb{R}^N)$ *(the extension by 0 outside* Ω*); therefore* $L^p(\Omega)$ *may be identified with a subspace of* $L^p(\mathbb{R}^N)$ *and hence* $L^p(\Omega)$ *is separable.*

3-Dual of L^P

Theorem 3 *(Riesz representation theorem).*
Let $\phi \in (L^1)^\star$. Then there exists a unique function $u \in L^\infty$ such that

$$\langle \phi, f \rangle = \int uf, \forall f \in L^1.$$

Moreover,

$$\|u\|_\infty = \|\phi\|_{(L^1)^\star}.$$

1.2 Some results about integration

Theorem 4 *(monotone convergence theorem, Beppo Levi).*
Let (f_n) be a sequence of functions in L^1 that satisfy
(a) $f_1 \leq f_2 \leq \cdots \leq f_n \leq f_{n+1} \leq \cdots$ *a.e. on* Ω,
(b) $\sup_n \int f_n < \infty$.

Then $f_n(x)$ converges a.e. on Ω to a finite limit, which we denote by $f(x)$; the function f belongs to L^1 and $\|f_n - f\|_1 \to 0$.

Theorem 5 *(dominated convergence theorem, Lebesgue).*
Let (f_n) be a sequence of functions in L^1 that satisfy
(a) $f_n(x) \to f(x)$ *a.e. on* Ω,
(b) there is a function $g \in L^1$ such that for all n, $|f_n(x)| \leq g(x)$ *a.e. on* Ω.

Then $f \in L^1$ and $\|f_n - f\|_1 \to 0$.

Lemma 1 (Fatou's lemma). *Let* (f_n) *be a sequence of functions in* L^1 *that satisfy*
(a) *for all* n, $f_n \geq 0$ *a.e.*
(b) $\sup_n \int f_n < \infty$.

For almost all $x \in \Omega$ *we set* $f(x) = \liminf_{n\to\infty} f_n(x) \leq +\infty$. *Then* $f \in L^1$ *and*

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

A basic example is the case in which $\Omega = \mathbb{R}^N$, M consists of the Lebesgue measurable sets, and μ is the Lebesgue measure on $\mathbb{R}^N$.
Notation. We denote by $C_c(\mathbb{R}^N)$ the space of all continuous functions on $\mathbb{R}^N)$; i.e. with compact support, i.e,

$$C_c(\mathbb{R}^N) = f \in C(\mathbb{R}^N);\ f(x) = 0 \ \ \forall x \in \mathbb{R}^N\ K, where\ K\ is\ compact.$$

Theorem 6 *(density).*
The space $C_c(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N)$; i.e.,

$$\forall f \in L^1(\mathbb{R}^N)\ \forall \varepsilon > 0\ \exists f_1 \in C_c(\mathbb{R}^N) such\ that \|f - f_1\|_1 \leq \varepsilon.$$

Let (Ω_1, M_1, μ_1) *and* (Ω_2, M_2, μ_2) *be two measure spaces that are* $\delta - finite$*. One can define in a standard way the structure of measure space* (Ω, M, μ) *on the Cartesian product* $\Omega = \Omega_1 x\, \Omega_2$.

Theorem 7 *(Tonelli).*
Let $F(x,y) : \Omega_1 \text{ x } \Omega_2 \to \mathbb{R}$ be a measurable function satisfying

$$(a) \int_{\Omega_2} |F(x,y)| d\mu_2 < \infty \ for\ a.e.\ x \in \Omega_1$$

$$and$$

$$(b) \int_{\Omega_1} d\mu_1 \int_{\Omega_2} |F(x,y)| d\mu_2 < \infty$$

.

Then $F \in L^1(\Omega_1 \text{ x } \Omega_2)$.

Theorem 8 *(Fubini).*
Assume that $F \in L^1(\Omega_1 \text{ x } \Omega_2)$. Then for a.e. $x \in \Omega_1$, $F(x,y) \in L^1_y(\Omega_2)$ and $\int_{\Omega_2} F(x,y) d\mu_2 \in L^1_x(\Omega_1)$. Similarly, for a.e. $y \in \Omega_2$, $F(x,y) \in L^1_x(\Omega_1)$ and $\int_{\Omega_1} F(x,y) d\mu_1 \in L^1_y(\Omega_2)$.
Moreover, one has

$$\int_{\Omega_1} d\mu_1 \int_{\Omega_2} F(x,y) d\mu_2 = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} F(x,y) d\mu_1 = \int\int_{\Omega_1 \text{ x } \Omega_2} F(x,y) d\mu_1 d\mu_2.$$

Theorem 9 *(Hölder's inequality).*
Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \leq p \leq \infty$. Then $fg \in L^1$ and

$$\int |fg| \leq \|f\|_p \, \|g\|_{p'}. \tag{1}$$

Proof. *The conclusion is obvious if $p = 1$ or $p = \infty$; therefore we assume that $1 \leq p \leq \infty$. We recall Young's inequality :*

$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'} \quad \forall a \geq 0, \quad \forall b \geq 0. \tag{2}$$

Inequality (2) is a straightforward consequence of the concavity of the function log on $(0,\infty)$:

$$\log\left(\frac{1}{p}a^p + \frac{1}{p'}b^{p'}\right) \geq \frac{1}{p}\log a^p + \frac{1}{p'}\log b^{p'} = \log ab.$$

we have

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{p'}|g(x)|^{p'} \ \textit{a.e. } x \in \Omega.$$

It follows that $fg \in L^1$ and

$$\int |fg| \leq \frac{1}{p}\|f\|_p^p + \frac{1}{p'}\|g\|_{p'}^{p'}. \tag{3}$$

Replacing f by $\lambda f (\lambda > 0)$ in(3), yields

$$\int |fg| \leq \frac{\lambda^{p-1}}{p}\|f\|_p^p + \frac{1}{\lambda p'}\|g\|_{p'}^{p'}. \tag{4}$$

Choosing $\lambda = \|f\|_p^{-1}\|g\|_p^{p'/p}$ (so as to minimize the right hand side in (4)). We obtain (1).[2]

1.3 Convolution

We first define the convolution product of a function $f \in L^1(\mathbb{R}^N)$ with a function $g \in L^p(\mathbb{R}^N)$.
•**Young's Inequality.** *Let $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$ with $1 \leq p \leq \infty$. Then for $x \in \mathbb{R}^N$ the function $y \to f(x-y)g(y)$ is integrable on $\mathbb{R}^N$ and we define*

$$(f \star g)(x) = \int_{\mathbb{R}^N} f(x-y)g(y)dy.$$

In addition $f \star g \in L^p(\mathbb{R}^N$ and

$$\|f \star g\| \leq \|f\|_1 \|g\|_p.$$

Proof. The conclusion is obvious when $p = \infty$. We consider two cases :
(i) $p = 1$,
(ii) $1 < p < \infty$.

Case (i) : $p = 1$. Set $F(x,y) = f(x-y)g(y)$.
For a.e. $y \in \mathbb{R}^N$ we have

$$\int_{\mathbb{R}^N} |F(x,y)|dx = |g(y)| \int_{\mathbb{R}^N} |f(x-y)|dx = |g(y)| \|f\|_1 < \infty$$

and, moreover,

$$\int_{\mathbb{R}^N} dy \int_{\mathbb{R}^N} |F(x,y)| dx = \|g\|_1 \|f\|_1 < \infty.$$

we deduce from $Theorem$ $(Tonelli)$. that $F \in L^1(\mathbb{R}^N \text{ x } \mathbb{R}^N)$. Applying $Theorem$ $(Fubini)$, we see that

$$\int_{\mathbb{R}^N} |F(x,y)| dy < \infty \; for \; a.e. \; x \in \mathbb{R}^N$$

and, moreover,

$$\int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} |F(x,y)| dy = \int_{\mathbb{R}^N} dy \int_{\mathbb{R}^N} |F(x,y)| dx = \|g\|_1 \|f\|_1.$$

This is precisely the conclusion of $(Young's\ Inequality)$. when $p = 1$.

Case (ii) : $1 < p < \infty$. By Case (i) we know that for a.e. fixed $x \in \mathbb{R}^N$ the function $y \to |f(x-y)| \, |g(y)|^p$ is integrable on $\mathbb{R}^N$, that is,

$$|f(x-y)|^{1/p} \, |g(y)| \in L^p_y(\mathbb{R}^N).$$

Since $|f(x,y)|^{1/p'} \in L^{p'}_y(\mathbb{R}^N)$, we deduce from $Theorem 6 (Holder's\ inequality)$. that

$$|f(x-y)| \, |g(y)| = |f(x-y)|^{1/p'} |f(x-y)|^{1/p} \, |g(y)| \; \in L^1_y(\mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} |f(x-y)| \, |g(y)| dy \le \|f\|_1^{1/p'} \left(\int_{\mathbb{R}^N} |f(x-y)| \, |g(y)|^p dy \right)^{1/p}.$$

that is,

$$|(f \star g)(x)|^p \le \|f\|_1^{1/p'} (|f| \star |g|^p)(x).$$

We conclude, by Case (i), that $f \star g \in L^p(\mathbb{R}^N)$ and

$$\|f \star g\|_p^p \le \|f\|_1^{p/p'} \|f\|_1 \|g\|_p^p,$$

that is,

$$\|f \star g\|_p \le \|f\|_1 \|g\|_1.$$

Lemma 1 *(Poincaré Inequality)*
For any w, continuously differentiable on $[0,1]$,

$$\int_0^1 w^2 \, dx \le 2w^2(1) + 4 \int_0^1 w_x^2 \, dx,$$
$$\int_0^1 w^2 \, dx \le 2w^2(0) + 4 \int_0^1 w_x^2 \, dx. \tag{1.1}$$

Proof of Lemma 1

$$\begin{aligned} \int_0^1 w^2 dx &= xw^2|_0^1 - 2 \int_0^1 x \, w \, w_x \, dx \; (integration \; by \; parts) \\ &= w^2(1) - 2 \int_0^1 x \, w \, w_x \, dx \\ &\le w^2(1) - \frac{1}{2} \int_0^1 w^2 \, dx + 2 \int_0^1 x^2 \, w_x^2 \, dx. \end{aligned}$$

Subtracting the second term from both sides of the inequality, we get the first inequality in (1.1) :

$$\begin{aligned} \frac{1}{2}\int_0^1 w^2\,dx &\leq w^2(1) + 2\int_0^1 x^2\,w_x^2\,dx \\ &\leq w^2(1) + 2\int_0^1 w_x^2\,dx. \end{aligned}$$

The second inequality in (1.1) is obtained in a similar fashion.

Using the Poincaré inequality along with boundary conditions, we get

$$\dot{V} = -\int_0^1 w_x^2\,dx \leq -\frac{1}{4}\int_0^1 w^2 \leq -\frac{1}{2}V \tag{1.2}$$

which, by the basic comparison principle for first order differential inequalities, implies that the energy decay rate is bounded by

$$V(t) \leq V(0)\,e^{-t/2} \tag{1.3}$$

or by

$$\|w(t)\| \leq e^{-t/4}\|w_0\|, \tag{1.4}$$

where

$$w_0(x) = w(x,0)$$

is the initial condition and $\|.\|$ denotes the $L_2 - norm$ of a function of x, namely,

$$\|w(t)\| = \left(\int_0^1 w(x,t)^2 dx\right)^{1/2}. \tag{1.5}$$

Lemma 2 *(Agmon's Inequality)*
*[5]*For a function $w \in H_1$, the following inequalities hold :

$$\begin{aligned} \max_{x\in[0,1]} |w(x,t)|^2 &\leq w(0)^2 + 2\|w(t)\|\|w_x(t)\|, \\ \max_{x\in[0,1]} |w(x,t)|^2 &\leq w(1)^2 + 2\|w(t)\|\|w_x(t)\|. \end{aligned} \tag{1.6}$$

Proof.

$$\begin{aligned} \int_0^x w\,w_x\,dx &= \int_0^x \partial_x \frac{1}{2}w^2\,dx \\ &= \frac{1}{2}w^2|_0^1 \\ &= \frac{1}{2}w(1)^2 - \frac{1}{2}w(0)^2. \end{aligned}$$

Taking the absolute value on both sides and using the triangle inequality gives

$$\frac{1}{2}|w(x)^2| \leq \int_0^x |w||w_x|dx + \frac{1}{2}w(0)^2. \tag{1.7}$$

Using the fact that an integral of a positive function is an increasing function of its upper limit, we can rewrite the last inequality as

$$|w(x)^2| \leq w(0)^2 + 2\int_0^1 |w(x)||w_x|dx. \tag{1.8}$$

The right hand side of this inequality does not depend on x, and therefore

$$\max_{x\in[0,1]} |w(x)|^2 \leq w(x)^2 + 2\int_0^1 |w(x)||w_x(x)|dx. \tag{1.9}$$

Using the Cauchy Schwarz inequality we get the first inequality of (1.9). The second inequality is obtained in a similar fashion.[5]

The simplest way is to use the following Lyapunov function :

$$V_1 = \frac{1}{2}\int_0^1 w^2\, dx + \frac{1}{2}\int_0^1 w_x^2\, dx. \tag{1.10}$$

The time derivative of (1.10) is given by

$$\begin{aligned}
\dot{V}_1 &\leq -\|w_x\|^2 - \|w_{xx}\|^2 \leq -\|w_x\|^2 \\
&\leq -\frac{1}{2}\|w_x\|^2 - \frac{1}{2}\|w_x\|^2 \\
&\leq -\frac{1}{8}\|w\|^2 - \frac{1}{2}\|w_x\|^2 \\
&\leq -\frac{1}{4}V_1.
\end{aligned}$$

Therefore,

$$\|w\|^2 + \|w_x\|^2 \leq e^{-t/2}(\|w_0\|^2 + \|w_{0,x}\|^2). \tag{1.11}$$

and using Young's and Agmon's inequalities, we get

$$\begin{aligned}
\max_{x\in[0,1]} |w(x,t)|^2 &\leq 2\|w\|\|w_x\| \\
&\leq \|w\|^2 + \|w_x\|^2 \\
&\leq e^{-t/2}(\|w_0\|^2 + \|w_{0,x}\|^2).
\end{aligned}$$

We have thus showed that

$$w(x,t) \to 0 \; as \; t \to \infty$$

for all $x \in [0,1]$.

• **The Cauchy Schwartz inequality** : Every inner product satisfies the Cauchy Schwarz inequality

$$\langle x_1, x_2\rangle \leq \|x_1\|\|x_2\|.$$

The equality sign holds if and only if x_1 and x_2 are dependent.

• **Gronwall's inequality** :

Lemma : *Let $T > 0$, $g \in L^1(0,T)$, $g \geq 0$ a.e and c_1, c_2 are positives constants. Let $\varphi \in L^1(0,T)$ $\varphi \geq 0$ a.e such that $g\varphi \in L^1(0,T)$ and*

$$\varphi(t) \leq c_1 + c_2\int_0^t g(s)\varphi(s)ds \;\; a.e \; in \; (0,T).$$

then, we have

$$\varphi(t) \leq c_1 exp\left(c_2\int_0^t g(s)ds\right) \; a.e \; in \; (0,T).$$

1.4 Functional differential equations (FDEs)

Some classes of FDES

We consider equations with an unknown scalar or vector function depending on a continuous argument t, which may be treated as time. The equations may be scalar or vector equations, and should have the same dimension as the unknown function. Unless stated otherwise, all variables under consideration are real.
As it well known, an ordinary differential equation (ODE) is an equation connecting the values of an unknown function and some of its derivatives for one and the same argument value.[3]
For example, the equation

$$F(t, x, dx/dt, d^2x/dt^2) = 0$$

may be written as $F(t, x(t), \dot{x}(t), \ddot{x}(t)) = 0$, where dots indicate derivatives:

$$\dot{x}(t) = dx(t)/dt$$

.

A functional equation FE is an equation involving an unknown function for different argument values.
The equations $x(2t) + 2x(3t) = 1, x(t) = t^2x(t+1) - [x(t-2)]^2,\ x(x(t)) = x(t) + 1$, are examples of FEs. The differences between the argument values of an unknown function and t in a FE are called argument deviations. If all argument deviations are constant (as in the second example above), then the FE is called a difference equation.

Above we gave some examples of FEs with discrete (or concentrated) argument deviations. By increasing in the equation the number of summands and simultaneously decreasing the differences between neighbouring argument values, one naturally arrives at FEs with continuous (or distributed) and mixed (both continuous and discrete) argument deviations. These are called integral and integral functional equations (in particular, integraldifference equations). However, it is meaningless to give these classes in detail, since they only serve as orientation in investigations.

Combining the notions of differential and functional equations, we obtain the notion of functional differential equation (FDE), or, equivalently, differential equation with deviating argument. Thus, this is an equation connecting the unknown function and some of its derivatives for, in general, different argument values. Here also the argument values can be discrete, continuous or mixed. Correspondingly one introduce the notions of differential difference equation (DDE), integra differential equation (IDE), etc.

1.5 Cauchy problem for FDES

The Cauchy problem (also called initial problem or basic initial problem) for a first order FDE is to find the solution of this equation subjected to a given initial function and initial value.[3]

1.6 Retarded differential difference equations

The simplest linear retarded differential difference equation has the form

$$\dot{x}(t) = Ax(t) + Bx(t-r) + f(t)$$

where A, B, and r are constants with $r > 0$, f is a given continuous function on $\mathbb{R}$, and x is a scalar.
The first question is the following : what is the initial value problem for this equation ? More specifically, what is the minimum amount of initial data that must be specified in order for this equation to define a function for $t \geq 0$? A moment of reflection indicates that a function must be specified on the entire interval $[-r, 0]$. In fact, let us prove.

1.7 Neutral differential difference equations

In this section, we introduce another class of equations depending on past as well as present values but which involve derivatives with delays as well as the function itself. Such equations historically have been referred to as *neutral differential difference equations.* The presentation will not be as detailed as the one for the retarded equations of the previous section. We concentrate only on those proofs which are significantly different from the ones for retarded equations. We also point out some of the differences between neutral equations and retarded equations.

The model nonhomogeneous equation is

$$\dot{x}(t) - C\dot{x}(t-r) = Ax(t) + Bx(t-r) + f(t)$$

where A, B, C, and r are constants with $r > 0$, $C \neq 0$ and f is a continuous function on $\mathbb{R}$. The corresponding homogeneous equation is

$$\dot{x}(t) - C\dot{x}(t-r) = Ax(t) + Bx(t-r).$$

1.8 Cauchy problem for NDEs

Hale's form of NDES

Consider NDEs in the form proposed by J. Hale

$$[x(t) - g(t, x_t)]^{\bullet} = f(t, x_t) \tag{1.12}$$

The former can be brought to the form (1.1), the latter-not. Nevertheless, many NDEs (including those important for applications), and even whole classes of them can be brought to the form (1.1). Moreover, such form of the equation should follow from the actual sense of the problem. This and the simpler formulation of the thorems are responsible for the fact that the form (1.1) has been widely used during recent years. In the theory of NDEs the form (1.1) takes about the same place as does the divergent form (or the form with divergent principal part) in the theory of partial differential equations.[3]

1.9 Lyapunov Stability and Basic Theorems

The stability of a system generally refers to its ability to return to its initial state when an external disturbance ceases. Stability is the primary condition for the normal operation of a control system.

The Lyapunov stability theorem defines the stability of a system in terms of energy, the biggest advantage of which is that the stability can be determined without the need to solve the motion equation of the system.

Types of Stability

This subsection defines various types of stability for *continuous time*.

1- Continuous Time Systems

Consider the following *continuous time system* :

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.13}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $f : \bar{\mathbb{R}}_+ \text{ x } \mathbb{R}^n \to \mathbb{R}^n$; and t is a continuous time variable.

A point $x_e \in \mathbb{R}^n$ is called an equilibrium point of system (1.13) if $f(t, x_e) = 0$ for all $t \geq 0$. The system remains at that point as long as there is no external action on it. The question is, if there is an external action, will the system remain near the equilibrium point or will it move farther and farther away?

The problem of stability at an equilibrium point is discussed below.

Shifting the origin of the system allows us to move the equilibrium point to $x_e = 0$. If there are multiple equilibrium points, the stability of each must be studied by an appropriate shift of the origin.

Various types of stability are defined below for system (1.13) at the equilibrium point, $x_e = 0$.

Definition 1.9.1

(1) *If, for any $t_0 \geq 0$ and $\varepsilon > 0$, there exists a $\delta_1 = \delta(t_0, \varepsilon) > 0$ such that*

$$\|x(t_0)\| < \delta(t_0, \varepsilon) \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0, \tag{1.14}$$

then the system is stable (in the Lyapunov sense) at the equilibrium point, $x_e = 0$.

(2) *If the system is stable at the equilibrium point $x_e = 0$ and if there exists $\delta_2 = \delta(t_0) > 0$ such that*

$$\|x(t_0)\| < \delta(t_0) \Rightarrow \lim_{t \longrightarrow \infty} x(t) = 0, \tag{1.15}$$

then the system is asymptotically stable at the equilibrium point, $x_e = 0$.

(3) *If there exist constants $\delta_3 > 0$, $\alpha > 0$, and $\beta > 0$ such that*

$$\|x(t_0)\| < \delta_3 \Rightarrow \|x(t)\| \leq \beta \|x(t_0)\| e^{-\alpha(t - t_0)}, \tag{1.16}$$

then the system is exponentially stable at the equilibrium point, $x_e = 0$.

(4) *If δ_1 in (1) (or δ_2 in (2)) can be chosen independently of t_0, then the system is uniformly stable (or uniformly asymptotically stable) at the equilibrium point, $x_e = 0$.*

(5) *If δ_2 in (2) (or δ_3 in (3)) can be an arbitrarily large, finite number, then the system is globally asymptotically stable (or globally exponentially stable) at the equilibrium point, $x_e = 0$.*

Remark 1.9.1 The requirement of uniform asymptotic stability is stronger than that of asymptotic stability. But for autonomous or periodic equations (1.2), a symptotic stability of the trivial solution implies uniform as ymptotic stability of this solution.
A similar relation holds for stability and uniform stability.

Example 1.9.1 The solution of the scalar Cauchy problem

$$\dot{x}(t) = e^{-t}x(t), \quad x(t_0) = x_0,$$

is

$$x(t) = x_0 exp(e^{-t_0} - e^{-t}).$$

Therefore,for any $x_0 \neq 0$ we find unimprovable estimate

$$|x(t)| < |x_0| exp\big(e^{-t_0}\big),\ t_0 \leq t < \infty,$$

and for any $\varepsilon > 0$ the unimprovable value of $\delta(\varepsilon, t_0)$ is $\varepsilon\ exp(-e^{-t_0})$. We find that the trivial solution is stable, but not uniformly stable because $\delta(\varepsilon, t_0) \longrightarrow 0$ as $t_0 \longrightarrow -\infty$.

Theorem 1.9.1 **(Lyapunov stability theorem for continuous time system)**
Consider system (1.13). Let $f(t, 0) = 0,\ \ \forall t$, which means that the equilibrium point of the system is $x_e = 0$.

$$\begin{array}{lll}
\bullet\ If & (1) & there\ exists\ a\ positive\ definite\ function\ V(t, x(t))\ and \\
 & (2) & \dot{V}(t, x(t)) = \dfrac{d}{dt}V(t, x(t))\ is\ negative\ semi-definite,\ then\ the\ system\ is\ stable \\
 & & at\ the\ equilibrium\ point,\ x_e = 0. \\
\bullet\ If & (1) & there\ exists\ a\ positive\ definite\ function\ V(t, x(t))\ and \\
 & (2) & \dot{V}(t, x(t)) = \dfrac{d}{dt}V(t, x(t))\ is\ negative\ definite,\ then\ the\ system\ is\ asymptotically\ stable \\
 & & at\ the\ equilibrium\ point,\ x_e = 0. \\
\bullet\ If & (1) & the\ system\ is\ asymptotically\ stable\ at\ x_e = 0\ and \\
 & (2) & V(t, x(t)) \to \infty\ as \|x\| \to \infty, \\
 & & then\ the\ system\ is\ globally\ asymptotically\ at\ the\ equilibrium\ point,\ x_e = 0.
\end{array}$$

1.10 Stability of Neutral Systems

A neutral system is a system with a delay in both the state and the derivative of the state, with the one in the derivative being called a neutral delay. That makes it more complicated than a system with a delay in only the state. Neutral delays occur not only in physical systems, but also in control systems, where they are sometimes artificially added to boost the performance.

CHAPTER 2

STABILITY FOR THE DAMPED WAVE EQUATION WITHE NEUTRAL DELAY

We consider the problem

$$\begin{cases} u_{tt} = u_{xx} - u_t - \int_0^t h(t-s)u_{tt}(s)ds, \quad x \in (0,1), \quad t > 0, \\ u(t,0) = u(t,1) = 0, \quad t \geqslant 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in (0,1) \end{cases} \tag{2.1}$$

where h is a nonnegative nonincreasing differentiable function and u_0, u_1 are two given initial data.[6]

Second order NDDEs where the neutral delay is in the second derivative appear in many applications. They arise, for instance, in the study of vibrating masses attached to an elastic bar and (as the Euler equation) in some variational problems. They arise also in the study of wave propagation in viscoelastic media

$$u_{tt} + K * u_{tt} = v^2 \nabla^2 u + \delta(t)\delta(x).$$

We can find them also as poroacoustic models in noise control (acoustic waves propagation)

$$\rho * u_{tt} = \nabla.[K * \nabla u]$$

The existence and uniqueness of a mild solution for this type of problems.

It is proved in a more general abstract problems for an operator A generalizing the Laplacian, with domain : the set of functions v such that v, $v_t \in H^2(\Omega)$ are absolutely continuous with Dirichlet boundary conditions. The operator A is assumed to be an infinitesimal generator of a strongly continuous cosine family of bounded linear operators. We shall assume that the solution (and the initial data) is regular enough to justify our computation.

2.1 Positive(negative) kernels

A- Case of a positive kernel

In this section we will treat the case when the kernel $h(t)$ is positive. We may write (2.1) as

$$u_{tt} = u_{xx} - u_t - \int_0^t h(t-s)u_{tt}(s)ds$$

we use partial integration

$$\begin{aligned} u_{tt} &= u_{xx} - u_t - \big([h(t-s)u_t(s)]_0^t + \int_0^t h'(t-s)u_t(s)ds\big) \\ &= u_{xx} - u_t - \big(h(0)u_t(0) - h(t)u_t(t) + \int_0^t h'(t-s)u_t(s)ds\big) \\ &= u_{xx} - u_t - h(0)u_t(0) + h(t)u_t(0) - \int_0^t h'(t-s)u_t(s)ds \\ &= u_{xx} - [1+h(0)]u_t - h(t)u_t(0) - \int_0^t h'(t-s)u_t(s)ds, \qquad t \geq 0. \end{aligned} \tag{2.2}$$

Theorem 2.1 *Let $u_0 \in H^1(0,1)$,$u_1 \in L^2(0,1)$, $\int_0^\infty e^{v_1 s}h(s)ds < \infty$ for some $v_1 > 0$. Then, there exist $M > 0$ and $a > 0$ such that*

$$E(t) = \frac{1}{2}(\|u_t\|^2 + \|u_x\|^2) \leq Me^{-at}, \quad t \geq 0.$$

Notice that if $\int_0^\infty e^{v_1 s}h(s)ds < \infty$ and $\lim_{t\to\infty} e^{v_1 t}h(t) < \infty$ (in fact, $\lim_{t\to\infty} e^{v_1 t}h(t) = 0$ as h is differentiable), then

$$\int_0^\infty e^{v_1 s}|h'(s)|ds = -\int_0^\infty e^{v_1 s}h'(s)ds = -e^{v_1 s}h(s)|_0^\infty + v_1\int_0^\infty e^{v_1 s}h(s)ds < +\infty$$

in case $h' \leq 0$.

Proof. We shall adopt the energy method with appropriate functionals. Let us multiply both sides of (2.2) by u_t and integrate over $(0,1)$

$$\begin{aligned} u_{tt} &= u_{xx} - u_t - \int_0^t h(t-s)u_{tt}ds \qquad x \in (0,1) \\ u_{tt}u_t &= u_t u_{xx} - u_t u_t - u_t\int_0^t h(t-s)u_{tt}(s)ds \\ \int_0^1 u_{tt}u_t dx &= \int_0^1 u_t u_{xx}dx - \int_0^1 u_t^2 dx - \int_0^1 u_t \int_0^t h(t-s)u_{tt}(s)ds\,dx. \end{aligned}$$

We use the following law

$$\begin{aligned} \int_0^1 f(x)f'(x)dx &= \frac{1}{2}\int_0^1 \frac{d}{dt}f^2 dx = \frac{1}{2}\frac{d}{dt}\int_0^1 f(x)^2 dx = \frac{1}{2}\frac{d}{dt}\|f\|_2^2 \\ \int_0^1 \frac{1}{2}\frac{d}{dt}u_t^2 dx &= \int_0^1 u_t u_{xx}dx - \int_0^1 u_t^2 dx - \int_0^1 u_t\int_0^t h(t-s)u_{tt}(s)ds\,dx \end{aligned}$$

we use partial integration of $\int_0^1 u_t u_{xx} dx$

$$\begin{aligned}
\int_0^1 \frac{1}{2}\frac{d}{dt} u_t^2 dx &= [u_t u_x]_0^1 - \int_0^1 u_x u_{tx} dx - \int_0^1 u_t^2 dx - \int_0^1 u_t \int_0^t h(t-s) u_{tt}(s) ds\, dx \\
\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 &= u_t(1)u_x(1) - u_t(0)u_x(0) - \int_0^1 u_x u_{tx} dx - \int_0^1 u_t \int_0^t h(t-s) u_{tt}(s) ds\, dx
\end{aligned}$$

we use the initial conditions

$$\begin{aligned}
u(t,1) &= 0 \qquad u(t,0) = 0 \\
u_t(1)u_x(1) &= 0 \qquad u_t(0)u_x(0) = 0
\end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 = -\int_0^1 u_x u_{tx} dx - \int_0^1 u_t^2 dx - \int_0^1 u_t \int_0^t h(t-s) u_{tt}(s) ds\, dx$$

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 u_x^2 = -\int_0^1 u_t^2 dx - \int_0^1 u_t \int_0^t h(s) u_{tt}(t-s) ds\, dx$$

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t\|_2^2 + \|u_x\|_2^2\right) = -\int_0^1 u_t^2 dx - \int_0^1 u_t \int_0^t h(s) u_{tt}(t-s) ds\, dx$$

we use integral by part

$$\int_0^t h(s) u_{tt}(t-s) ds = [h(s)u_t(t-s)] + \int_0^t h'(s) u_{tt}(t-s) ds$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|u_t\|_2^2 + \|u_x\|_2^2) &= -\int_0^1 u_t^2 dx - \int_0^1 u_t \left([h(s)u_t(t-s)]_0^t + \int_0^t h'(s) u_t(t-s) ds \right) ds \\
&= -\int_0^1 u_t^2 dx + \int_0^1 u_t h(t) u_t(0) dx - \int_0^1 h(0) u_t^2 dx - \int_0^1 u_t \int_0^t h'(s) u(t-s) ds\, dx \\
&= -\int_0^1 u_t^2 dx + h(t)\int_0^1 u_t u_t(0) dx - h(0)\int_0^1 u_t^2 dx - \int_0^1 u \int_0^t h'(s) u(t-s) ds\, dx \\
&= -[1+h(0)]\int_0^1 u_t^2 dx + h(t)\int_0^1 u_t u_t(0) dx - \int_0^1 u_t \int_0^t h'(s) u(t-s) ds\, dx
\end{aligned}$$

applining Yong's inquality

$$\begin{aligned}
h(t)\int_0^1 u_t u_t(0) dx &\leq h(t)\delta_1 \int_0^1 u_t^2 dx + \frac{h(t)}{4\delta_1}\int_0^1 u_1^2(0) dx \\
\frac{1}{2}\frac{d}{dt}(\|u_t\|^2 + \|u_x\|^2) &\leq -[1+h(0)]\|u_t\|^2 + h(t)\delta_1 \int_0^1 u_t^2 dx + \frac{h(t)}{4\delta_1}\int_0^1 u_t^2(0) dx \\
&- \int_0^1 u_t \int_0^t h'(s) u(t-s) ds\, dx \\
&\leq -[1+h(0)]\|u_t\|^2 + h(t)\delta_1\|u_t\|^2 + \frac{h(t)}{4\delta_1}\|u_t(0)\|_2^2 \\
&- \int_0^1 u_t \int_0^t h'(s) u(t-s) ds\, dx
\end{aligned}$$

we use Cauchy chwartz inequality

$$\begin{aligned}
-\int_0^1 u_t \int_0^t h'(s)u(t-s)ds\,dx &\le -\Big(\int_0^1 u_t^2 dx\Big)^{1/2}\Big(\int_0^t (h'(s)u(t-s))^2 ds\Big)^{1/2} \\
\frac{1}{2}\frac{d}{dt}(\|u_t\|^2+\|u_x\|^2) &\le -[1+h(0)]\|u_t\|_2^2 + h(t)\delta_1\|u_t\|^2 + \frac{h(t)}{4\delta_1}\|u_t(0)\|^2 \\
&- \Big(\int_0^1 u_t\,dx\Big)^{1/2}\Big(\int_0^t (h'(s)u(t-s))^2 ds\Big)^{1/2}
\end{aligned}$$

utilising Yong's inequality

$$\Big(\int_0^1 u_t dx\Big)^{1/2}\Big(\int_0^t (h'(s)u(t-s)^2 ds\Big)^{1/2} \le \delta_2\int_0^1 u_t^2 dx + \frac{1}{4\delta_2}\int_0^1 u_t \int_0^t \big(h'(s)u(t-s)\big)^2 ds\,dx$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|u_t\|^2+\|u_x\|^2) &\le -[1+h(0)]\|u_t\|_2^2 + h(t)\delta_1\|u_t\|_2^2 + \frac{h(t)}{4\delta_1}\|u_t(0)\|^2 \\
&+\delta_2\int_0^1 u_t^2 dx + \frac{1}{4\delta_2}\int_0^1 u_t\int_0^t \big(h'(s)u(t-s)\big)^2 ds \\
&\le -[1+h(0)]\|u_t\|^2 + h(t)\delta_1\|u_t\|_2^2 + \frac{h(t)}{4\delta_1}\|u_t(0)\|_2^2 + \delta_2\|u_t\|_2^2 \\
&+\frac{1}{4\delta_2}\int_0^1 u_t\int_0^t \big(h'(s)u(t-s)\big)^2\,ds\,dx \\
&\le -[1+h(0)]\|u_t\|_2^2 + h(t)\delta_1\|u_t\|_2^2 + \frac{h(t)}{4\delta_1}\|u_t(0)\|_2^2 \\
&+\delta_2\|u_t\|_2^2 + \frac{1}{4\delta_2}\big(\int_0^1 h(t-s)^{1/2}ds\big)\big(\int_0^t (h(t-s))^{1/2}u_t(s)^2 ds\,dx\big) \\
&\le [-1-h(0)+h(t)\delta_1+\delta_2]\|u_t\|^2 + \frac{h(t)}{4\delta_1}\|u_1)\|^2 \\
&+\frac{1}{4\delta_2}\big(\int_0^t (h'(s))ds\big)\int_0^1\int_0^t \big(h'(t-s)u_t(s)\big)^2\,ds\,dx. \qquad (2.3)
\end{aligned}$$

Next, we introduce

$$\phi_1(t) = \int_0^1 u_t\,u\,dx, \quad t \ge 0,$$

and

$$\psi_1(t) = e^{-\gamma t}\int_0^1\int_0^t e^{\gamma s}\tilde{H}(t-s)|u_t(s)|^2 ds\,dx, \quad t \ge 0,$$

where

$$\tilde{H}(t) = \int_t^\infty e^{\gamma s}|h'(s)|ds, \quad t \ge 0, \gamma > 0.$$

Note that $\tilde{H}(t)$ is well defined for small values of $\gamma(\gamma < v_1)$ by our assumption on $h(t)$. The derivative of $\phi_1(t)$, along solutions of (2.2), is given by

$$\begin{aligned}\phi_1'(t) &= \|u_t\|^2 + \textstyle\int_0^1 u\{u_{xx} - [1+h(0)]u_t + h(t)u_t(0) - \int_0^t h'(t-s)u_t(s)ds\}dx \\ &= \|u_t\|^2 - \|u_x\|^2 - [1+h(0)]\textstyle\int_0^1 u_t\, u\, dx + h(t)\int_0^1 u_t(0)u\, dx \\ &\quad - \textstyle\int_0^1 u \int_0^t h'(t-s)u_t(s)ds\, dx, \quad t \geq 0.\end{aligned}$$

we use Young's inequality

$$-[1+h(0)]\int_0^1 u_t\, u\, dx \leq [1+h(0)](\delta_3\|u_t\|^2 + \frac{1}{4\delta_3}\|u_x\|^2.$$

$$h(t)\int_0^1 u_t(0)\; u\; dx \leq \delta_4 h(t)\|u_x\|^2 + \frac{h(t)}{4\delta_4}\|u_1\|^2.$$

$$-\int_0^1 u\int_0^t h'(t-s)u_t(s)ds\, dx \leq \delta_5\|u_x\|^2 + \frac{1}{4\delta_5}\Big(\int_0^1 |h'(s)|ds \int_0^1\int_0^t |h'|(t-s)u_t(s)\Big)ds\, dx.$$

ϕ_1' it is estimated as follows

$$\phi_1'(t) \leq \|u_t\|^2 - \|u_x\|^2 + [1+h(0)](\delta_3\|u_t\|^2 + \frac{1}{4\delta_3}\|u_x\|^2) + \delta_4 h(t)\|u_x\|^2$$

$$+\frac{h(t)}{4\delta_4}\|u_1\|^2 + \delta_5\|u_x\|^2 + \frac{1}{4\delta_5}\Big(\int_0^t |h'(s)|ds\Big)\int_0^1\int_0^t |h'|(t-s)|u_t|^2(s)ds\, dx \tag{2.4}$$

$$\leq \Big\{1+\delta_3[1+h(0)]\Big\}\|u_t\|^2 + \big[\frac{1+h(0)}{4\delta_3} + h(t)\delta_4 + \delta_5 - 1\big]\|u_x\|^2 + \frac{h(t)}{4\delta_4}\|u_1\|^2$$

$$+\frac{1}{4\delta_5}\Big(\int_0^t |h'(s)|ds\Big)\int_0^1\int_0^t |h'|(t-s)|u_t(s)|^2 ds\, dx, \quad t \geq 0.$$

For $\psi_1(t)$, we obtain

$$\psi_1'(t) = -\gamma e^{-\gamma t}\int_0^1\int_0^t e^{\gamma s}\tilde{H}(t-s)|u_t(s)|^2 ds\, dx$$

$$+\Big[\int_0^1 e^{\gamma t}\tilde{H}(0)|u_t(t)|^2 + \int_0^1\int_0^t e^{\gamma s}\tilde{H}'(t-s)|u_t(s)|^2 ds\, dx\Big]e^{-\gamma t}. \tag{2.5}$$

$$= -\gamma\psi(t) + \tilde{H}(0)\|u_t\|^2 + e^{-\gamma t}\int_0^1\int_0^t e^{\gamma s}H'(t-s)|u_t(s)|^2 ds\, dx$$

$$\tilde{H}'(t) = -e^{\gamma t}|h'(t)|$$

$$\tilde{H}'(t-s) = -e^{\gamma(t-s)}|h'(t-s)| \; = -e^{\gamma t}e^{-\gamma s}|h'(t-s)|$$

$$\psi_1'(t) = -\gamma\psi(t) + \tilde{H}(0)\|u_t\|^2 - e^{-\gamma t}\int_0^1\int_0^t e^{\gamma s}e^{\gamma t}e^{-\gamma s}|h'(t-s)||u_t(s)|^2 ds\, dx$$

$$\psi_1'(t) = -\gamma\psi(t) + \tilde{H}(0)\|u_t\|^2 - \int_0^1\int_0^t |h'(t-s)||u_t(s)|^2 ds\, dx, \quad t \geq 0.$$

Therefore, if

$$V(t) = E(t) + \lambda\phi_1(t) + \mu\psi_1(t), \quad t \geq 0$$

for some λ, $\mu > 0$, we find, in view of (2.4)-(2.5),

$$V'(t) \leq \Big\{ -1 - h(0) + h(t)\delta_1 + \delta_2 + \lambda[1 + (1+h(0))\delta_3] + \mu\tilde{H}(0)\Big\}\|u_t\|^2 \tag{2.6}$$

$$+\lambda\left[\frac{1+h(0)}{4\delta_3}+h(t)\delta_4+\delta_5-1\right]\|u_x\|^2+\frac{h(t)}{4}(\frac{1}{\delta_1}+\frac{\lambda}{\delta_4})\|u_1\|^2$$
$$+\left[\frac{1}{4}(\frac{1}{\delta_2}+\frac{\lambda}{\delta_5})\int_0^t|h'(s)|ds-\mu\right]\int_0^1\int_0^t|h'(t-s)||u_t(s)|^2dsdx-\lambda\mu\psi(t),\quad t\geq 0.$$

We need the first two coefficients (in the right hand side of (2.6)) to be negative, and the fourth one to be nonpositive. Let us first pick $\delta_1=\delta_4=1$ and forget about $h(t)$. We need

$$\begin{cases}\delta_2+\lambda[1+(1+h(0))\delta_3]+\mu\tilde{H}(0)<1+h(0),\\ \frac{1+h(0)}{4\delta_3}+\delta_5<1,\\ \frac{1}{4}(\frac{1}{\delta_2}+\frac{\lambda}{\delta_5})\int_0^\infty|h'(s)|ds<\mu.\end{cases}\tag{2.7}$$

We choose $\delta_2=h(0)$, $\delta_3=1+h(0)$ and $\delta_5=\frac{1}{2}$. Then, we select $\mu=\frac{1}{2}(1+2\lambda h(0))$. Next, we choose λ and γ small enough so that the first relation in (2.7) is fulfilled. Finally, we pick t_* so large that $h(t)$ becomes small enough and the first two coefficients are negative. We end up with

$$V'(t)\leq -C_1E(t)-\lambda\mu\psi_1(t)+\frac{h(t)}{2}\|u_1\|^2,\ t\geq t_*\ for\ some\ C_1>0.$$

As $E(t)+\lambda\phi_1(t)$ is equivalent to $E(t)$, we may write

$$V'(t)\leq -C_2V(t)+C_3h(t),\quad t\geq t_*,\quad C_3=\frac{\|u_1\|^2}{2}.\tag{2.8}$$

An integration of this inequality (2.7) leads to

$$V(t)\leq (V(t_*)+C_3\int_{t_*}^t e^{c_2 s}h(s)ds)e^{-C_2(t-t_*)},\ t\geq t_*.$$

To be able to use our assumption, we choose C_2 so small that $C_2\leq v_1$. Observe also that for t_* large enough

$$V(t)\leq (V(t_*)+C_3\int_0^\infty e^{C_2 s}h(s)ds)e^{C_2t_*}e^{-C_2t},\ t\geq t_*$$

and a similar estimation holds on $[0,t_*]$. Thus

$$V(t)\leq C_4e^{-at},\quad a,\ C_4>0,\ t\geq 0.$$

Remark 2.1 Observe that we do not really need $e^{C_2t}h(t)$ to be summable. All we need is

$$\int_0^t e^{C_2 s}h(s)ds\leq Ce^{\alpha t}\quad with\quad C>0\quad and\quad \alpha<C_2.$$

Remark 2.2 Notice that, in the present situation, we do not need the frictional damping as $-h(0)u_t(t)$ (with $h(0)>0$) is enough.

Remark 2.3 For a similar problem to (2.2), although the situation there is a little different and the conditions on h are related to some operator (in addition to the presence of a viscoelastic damping there).

Remark 2.4 In line with Remark 2.2 above, if $h(t)$ is rather negative then we need some kind of damping (like the frictional one). We need also to impose some extra 'smallness' condition on $h(t)$. Namely, we assume that $|h(0)| < 1$.

B- Case of a negative kernel

Let us consider the problem

$$u_{tt} = u_{xx} - u_t - \int_0^t h(t-s)u_{tt}(s)ds, \quad t \geq 0, \quad 0 < x < 1. \tag{2.9}$$

with $h(t) \leq 0$. We may rewrite (2.9) as

$$u_{tt} = u_{xx} - (1 + h(0))u_t + h(t)u_t(0) - \int_0^t h'(t-s)u_t(s)ds. \tag{2.10}$$

The proof of Theorem (2.1) carries out almost verbatim

Theorem 2.2 *Let $u_0 \in H^1(0,1)$, $u_1 \in L^2(0,1)$, $e^{v_1 t}h(t) \in L^1(0,1)$ for some $v_1 > 0$ and $|h(0)| < 1$.*
Then, there exist $M > 0$ and $a > 0$ such that

$$E(t) \leq Me{-}at, \quad t \geq 0.$$

2.2 Case of non-regularity

If h is not differentiable then we cannot rewrite (2.1) in the form (2.2). However, as

$$\int_0^t h(t-s)u_{tt}(s)ds = \int_0^t h(s)u_{tt}(t-s)ds = -h(t)u_t(0) + [\int_0^t h(s)u_t(t-s)ds]'$$

we can rewrite (2.2) as

$$u_{tt} - h(t)u_t(0) + [\int_0^t h(t-s)u_t(s)ds]' = u_{xx} - u_t.$$

The idea here is to work with the 'difference operator' in the left hand side of

$$[u_t + \int_0^t h(t-s)u_t(s)ds]' = u_{xx} - u_t + h(t)u(0), \quad t \geq 0, \, 0 < x < 1. \tag{2.11}$$

Remark 2.5 Notice that, setting (2.11) as our reference equation, it is not equivalent to (2.1). In (2.11), we do not require u_t to be differentiable but rather the whole expression in square brackets. These facts justify the importance of studying this kind of equation. In the next result, however, we will assume the differentiability of the kernel on $(0, +\infty)$.

Theorem 2.3 *Let $u_0 \in H^1(0,1)$, $u_1 \in L^2(0,1)$, $h \geq 0$, $\int_0^\infty h(s)ds = \bar{h} < \frac{1}{2}$, $h'(t) \leq -\xi h(t)$, $t > 0$ for some $\xi > 0$ and $2\bar{h}\int_0^\infty e^{\theta s}h(s)ds < 1$ for some $\theta > 0$ Then, there exists A, $c > 0$ such that*

$$E(t) \leq Ae^{-ct}, \quad t \geq 0.$$

Proof. Multiplying both sides of (2.11) by the expression $u_t+\int_0^t h(t-s)u_t(s)\,ds$ and integrating over $(0,1)$, gives

$$\int_0^1 [u_t+\int_0^t h(t-s)u_t(s)ds][u_t+\int_0^t h(t-s)u_t(s)ds]dx =$$
$$\int_0^1 [u_{xx}-u_t+h(t)u_t(0)][u_t+\int_0^t h(t-s)u_t(s)ds]dx$$
$$\frac{1}{2}\frac{d}{dt}\int_0^1 [u_t+\int_0^t h(t-s)u_t(s)ds]^2dx =$$
$$\int_0^1 u_{xx}u_tdx+\int_0^1 u_{xx}\int_0^t h(t-s)u_t(s)ds\,dx-\int_0^1 u_t^2dx$$
$$-\int_0^1 u_t\int_0^t h(t-s)u_t(s)ds\,dx+h(t)\int_0^1 u_t\,u_t(0)dx+\int_0^1 h(t)u_t(0)\int_0^t h(t-s)u_t(s)ds\,dx$$

we use the integral by parts

$$\int_0^1 u_{xx}\,u_t\,dx=[u_t\,u_x]_0^1-\int_0^1 u_x\,u_{tx}\,dx$$
$$\int_0^1 u_{xx}\int_0^t h(t-s)u_t(s)ds\,dx=[\int_0^1 h(t-s)u_t(s)ds]_0^1-\int_0^1\int_0^t h(t-s)u_{tx}(s)ds\,dx$$
$$\frac{1}{2}\frac{d}{dt}\int_0^1 [u_t+\int_0^t h(t-s)u_t(s)ds]^2dx=\int_0^1 u_{xx}[u_t+\int_0^t h(t-s)u_t(s)ds]dx$$
$$-\int_0^1 u_t[u_t+\int_0^t h(t-s)u_t(s)ds]dx+h(t)\int_0^1 u_t(0)[u_t+\int_0^t h(t-s)u_t(s)ds]dx$$

we use young's inequality

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [u_t+\int_0^t h(t-s)u_t(s)ds]^2dx\le-\int_0^1 u_xu_{xt}dx-\int_0^1 u_x\int_0^t h(t-s)u_{xt}(s)dsdx-\|u_t\|^2$$
$$-\int_0^1 u_t\int_0^t h(t-s)u_t(s)ds\,dx+\frac{h(t)}{4\delta_1}\|u_1\|^2+\delta_1h(t)\|u_t+\int_0^t h(t-s)u_t(s)ds\|^2,\quad \delta_1>0,\,t\ge0. \qquad (2.11)$$

In view of the identity

$$\frac{d}{dt}\int_0^t h(t-s)u_x(s)ds=\frac{d}{dt}\int_0^t h(s)u_x(t-s)dx=h(t)u_x(0)+\int_0^t h(s)u_{xx}(t-s)ds,$$

for $t\ge0$, we may write (2.11) as

$$\frac{1}{2}\frac{d}{dt}\left\{\|u_t+\int_0^t h(t-s)u_t(s)ds\|^2+\|u_x\|^2\right\}$$
$$\le-\|u_t\|^2-\int_0^1 u_x\frac{d}{dt}\left(\int_0^t h(t-s)u_x(s)ds\right)dx+h(t)\int_0^1 u_x(0)u_xdx$$
$$-\int_0^1 u_t\int_0^t h(t-s)u_t(s)dsdx+2\delta_1h(t)\|u_t\|^2+\frac{h(t)}{4\delta_1}\|u_1\|^2$$

$$+2\delta_1 h(t)\big(\int_0^t h(s)ds\big)\int_0^t h(t-s)\|u_t(s)\|^2 ds, \quad t \geq 0.$$

In fact, we can go a little further

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\|u_t + \int_0^t h(t-s)u_t(s)ds\|^2 + \|u_x\|^2\Big\} \\
\leq &-\|u_t\|^2 - \int_0^1 u_x \frac{d}{dt}\Big(\int_0^t h(t-s)u_x(s)ds\Big)dx + \delta_2 h(t)\|u_x\|^2 + \frac{h(t)}{4\delta_2}\|u_{0x}\|^2 \\
&+\delta_3\|u_t\|^2 + \frac{1}{4\delta_3}\big(\int_0^t h(s)ds\big)\int_0^t h(t-s)\|u_t(s)\|^2 ds + \frac{h(t)}{4\delta_1}\|u_1\|^2 \\
&+2\delta_1\|u_t\|^2 + 2\delta_1 h(t)\big(\int_0^t h(s)ds\big)\int_0^t h(t-s)\|u_t(s)\|^2 ds
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\|u_t + \int_0^t h(t-s)u_t(s)ds\|^2 + \|u_x\|^2\Big\} \qquad (2.13)\\
\leq &-\int_0^1 u_x \frac{d}{dt}\Big(\int_0^t h(t-s)u_x(s)ds\Big)dx + [-1+\delta_3+2\delta_1 h(t)]\|u_t\|^2 \\
&+\big(\frac{1}{4\delta_3} + 2\delta_1 h(t)\big)\big(\int_0^t h(s)ds\big)\int_0^t h(t-s)\|u_t(s)\|^2 ds \\
&+\delta_2 h(t)\|u_x\|^2 + \frac{h(t)}{4}\Big(\frac{\|u_1\|^2}{\delta_1} + \frac{\|u_{0x}\|^2}{\delta_2}\Big), \quad t \geq 0.
\end{aligned}$$

Now, we need to take care of two terms:
the first term and the third term in the right hand side of (2.13). The third term will be treated with the help of

$$\phi_2(t) = e^{-\beta t}\int_0^1\int_0^t e^{\beta s}H(t-s)|u_t|^2(s)ds\,dx, \quad \xi > \beta > 0, t \geq 0$$

where

$$H(t) = \int_t^{+\infty} e^{\beta s}h(s)\,ds,\ t \geq 0.$$

Its derivative is equal to

$$\begin{aligned}
\phi_2'(t) &= -\beta\, e^{-\beta t}\int_0^1\int_0^t e^{\beta t}H(t-s)|u_t|^2(s)ds\,dx \\
&+e^{-\beta t}\Big[\int_0^1 e^{\beta t}H(0)|u_t|^2(s)ds + \int_0^1\int_0^t e^{\beta t}H'(t-s)|u_t|^2(s)ds\,dx\Big] \\
&H'(t) = -e^{\beta t}h(t),\ H'(t-s) = -e^{\beta(t-s)}h(t-s)
\end{aligned}$$

$$\phi_2'(t) = -\beta\phi_2(t) + H(0)\|u_t\|^2 - \int_0^t h(t-s)\|u_t(s)\|^2 ds,\ t \geq 0. \qquad (2.14)$$

Regarding the first term in the right hand side, we introduce the functional

$$\psi_2(t) = \int_0^1 u_x \int_0^t h(t-s)u_x(s)ds\, dx,\ t \geq 0.$$

When differentiating it, we get

$$\psi_2'(t) = \int_0^1 u_{xt} \int_0^t h(t-s)u_x(s)\, ds\, dx + \int_0^1 u_x \frac{d}{dt}\int_0^t h(t-s)u_x(s)\, ds\, dx,\ t \geq 0. \tag{2.15}$$

In turn, the first term in the right hand side of (2.15), is treated in the following manner

$$\int_0^1 u_{xt} \int_0^t h(t-s)u_x(s)\, ds\, dx = \frac{1}{2}\int_0^1 (h' \circ u_x)\, dx - \frac{1}{2}h(t)\|u_x\|^2$$

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 (h \circ u_x)\, dx - (\int_0^t h(s)ds)\|u_x\|^2,\ t \geq 0. \tag{2.16}$$

where

$$\int_0^1 (h \circ u_x)dx = \int_0^1 \int_0^t h(t-s)|u_x(t) - u_x(s)|^2 ds\, dx,\ t \geq 0.$$

Gathering these functionals in

$$W(t) = \varepsilon(t) + \eta\phi_2(t) + \psi_2(t),\ t \geq 0. \tag{2.17}$$

where

$$\varepsilon(t) = \frac{1}{2}\left\{\|u_t + \int_0^t h(t-s)u_t(s)ds\|^2 + (1 - \int_0^t h(s)\, ds)\|u_x\|^2 + \int_0^1 (h \circ u_x)dx\right\}$$

and taking into account (2.13)-(2.17), we infer

$$W'(t) \leq \frac{1}{2}\int_0^1 (h' \circ u_x)dx - \frac{1}{2}h(t)\|u_x\|^2 + [-1 + \delta_3 + 2\delta_1 h(t)]\|u_t\|^2 + (\int_0^t h(s)ds)$$
$$\times(\frac{1}{4\delta_3} + 2\delta_1 h(t))\int_0^t h(t-s)\|u_t(s)\|^2 ds + \delta_2 h(t)\|u_x\|^2 + \frac{h(t)}{4}(\frac{\|u_1\|^2}{\delta_1} + \frac{\|u_{0x}\|^2}{\delta_2})$$
$$-\beta\eta\phi_2(t) + \eta H(0)\|u_t\|^2 - \eta\int_0^t h(t-s)\|u_t(s)\|^2 ds,\ t \geq 0$$

or

$$W'(t) \leq -\frac{\xi}{2}\int_0^1 (h \circ u_x)dx + [-1 + \delta_3 + 2\delta_1 h(t) + \eta H(0)]\|u_t\|^2$$
$$-h(t)(\frac{1}{2} - \delta_2)\|u_x\|^2 + \frac{h(t)}{4}(\frac{\|u_1\|^2}{\delta_1} + \frac{\|u_{0x}\|^2}{\delta_2}) - \beta\eta\phi_2(t)$$
$$-\left[\eta - (\frac{1}{4\delta_3} + 2\delta_1 h(t))\int_0^t h(s)ds\right]\int_0^t h(t-s)\|u_t(s)\|^2 ds,\ t \geq 0. \tag{2.18}$$

Our objective is to get $-CW(t)$ is the right hand side of $\phi_2(t)$.
Clearly, the term

$$-h(t)(\frac{1}{2} - \delta_2)\|u_x\|^2$$

does not help fulfilling our objective as $h(t)$ goes to zero as time goes to infinity. Therefore, we need to add another functional

$$\Lambda(t) = \int_0^1 u[u_t + \int_0^t h(t-s)u_t(s)ds]dx, \; t \geq 0.$$

The role of this functional is to provide $-\|u_x\|^2$ through its derivative

$$\begin{aligned}
\Lambda'(t) &= \|u_t\|^2 + \int_0^1 u_t \int_0^t h(t-s)u_t(s)\, ds\, dx + \int_0^1 u[u_{xx} - u_t + h(t)u_t(0)]dx \\
&= \|u_t\|^2 + \int_0^1 u_t \int_0^t h(t-s)u_t(s)\, ds\, dx - \|u_x\|^2 - \int_0^1 u_t\, u\, dx + h(t)\int_0^1 u_t(0)\, u\, dx \\
&\leq \|u_t\|^2 - \|u_x\|^2 + \delta_4\|u_t\|^2 + \frac{1}{4\delta_4}(\int_0^t h(s)ds)\int_0^t h(t-s)\|u_t(s)\|^2 ds \\
&\quad + \delta_5\|u_t\|^2 + \frac{1}{4\delta_5}\|u_x\|^2 + \delta_6 h(t)\|u_x\|^2 + \frac{h(t)}{4\delta_6}\|u_1\|^2
\end{aligned} \tag{2.19}$$

That is

$$\begin{aligned}
\Lambda'(t) &\leq (1+\delta_4+\delta_5)\|u_t\|^2 + (\frac{1}{4\delta_5} + \delta_6 h(t) - 1)\|u_x\|^2 + \frac{h(t)}{4\delta_6}\|u_1\|^2 \\
&\quad + \frac{1}{4\delta_4}(\int_0^t h(s)ds)\int_0^t h(t-s)\|u_t(s)\|^2 ds, \; t \geq 0.
\end{aligned}$$

Let

$$L(t) = W(t) + \rho\Lambda(t), \; t \geq 0. \tag{2.20}$$

for some $\rho > 0$. In view of (2.18)-(2.20), we entail

$$\begin{aligned}
L'(t) &\leq -\frac{\xi}{2}\int_0^1 (h \circ u_x)dx + \Big[-1 + \delta_3 + 2\delta_1 h(t) + \eta H(0)\Big]\|u_t\|^2 \\
&\quad - h(t)(\frac{1}{2} - \delta_2)\|u_x\|^2 + \frac{h(t)}{4}(\frac{\|u_1\|^2}{\delta_1} + \frac{\|u_{0x}\|^2}{\delta_2}) - \beta\eta\phi_2(t) \\
&\quad - \Big[\eta - (\frac{1}{4\delta_3} + 2\delta_1 h(t))\int_0^t h(s)ds\Big]\int_0^t h(t-s)\|u_t(s)\|^2 ds + \rho(1+\delta_4+\delta_5)\|u_t\|^2 \\
&\quad + \rho(\frac{1}{4\delta_5} + \delta_6 h(t) - 1)\|u_x\|^2 + \frac{\rho}{4\delta_4}(\int_0^t h(s)ds)\int_0^t h(t-s)\|u_t - s)\|^2 ds
\end{aligned}$$

or

$$\begin{aligned}
L'(t) &\leq -\frac{\xi}{2}\int_0^1 (h \circ u_x)dx + \Big[-1 + \delta_3 + 2\delta_1 h(t) + \eta H(0) + \rho(1+\delta_4+\delta_5)\Big]\|u_t\|^2 \\
&\quad + \Big[-h(t)(\frac{1}{2} - \delta_2) + \rho(\frac{1}{4\delta_5} + \delta_6 h(t) - 1)\Big]\|u_x\|^2 + \frac{h(t)}{4}(\frac{\|u_1\|^2}{\delta_1} + \frac{\|u_{0x}\|^2}{\delta_2} + \frac{\rho\|u_1\|^2}{\delta_6}) \\
&\quad - \Big[\eta - \Big(\frac{1}{4\delta_3} + 2\delta_1 h(t) + \frac{\rho}{4\delta_4}\Big)\Big(\int_0^t h(s)ds\Big)\Big]\int_0^t h(t-s)\|u_t(s)\|^2 ds - \beta\eta\phi_2(t), \; t \geq 0.
\end{aligned}$$

Let us take $\delta_1 = \delta_6 = 1$, $\delta_2 = \delta_5 = \frac{1}{2}$. The second and fifth coefficients in the right hand side are negative provided that

$$\begin{cases} \delta_3 + \eta H(0) + \rho(\frac{3}{2} + \delta_4) < 1, \\ (\frac{1}{\delta_3} + \frac{\rho}{\delta_4})\bar{h} < 4\eta. \end{cases}$$

Notice that we have ignored h(t) as we can make t_* large enough.
We pick $\delta_3 = \delta_4 = \frac{\sqrt{hH(0)}}{2}$ and $\rho < (1-\bar{h})/(\bar{h} + \frac{3}{2})$. Then, these two inequalities hold for some η provided that β is small enough. We end up with

$$L'(t) \le -C_1 \left\{ \|u_t\|^2 + \|u_x\|^2 + \phi_2(t) + \int_0^1 (h \circ u_x)dx + \int_0^t h(t-s)\|u_t(s)\|^2 ds \right\} + C_2 h(t)$$

for some C_1, $C_2 > 0$ and $t \ge t_*$. It is not difficult to see that $L(t)$ is equivalent to

$$\|u_t\|^2 + \|u_x\|^2 + \phi_2(t) + \int_0^1 (h \circ u_x)dx + \int_0^t h(t-s)\|u_t(s)\|^2 ds.$$

Consequently,

$$L'(t) \le -C_3 L(t) + C_2 h(t),\ t \ge 0$$

for some $C_3 > 0$. Then, we proceed as in the proof of Theorem 2.1 to show that there exist $N, b > 0$ such that

$$L(t) \le N e^{-bt},\ t \ge 0.$$

Now we have to pass to the classical energy. To this end we notice that

$$\begin{aligned} \|u_t\| &\le \|u_t + \int_0^t h(t-s)u_t(s)ds\| + \|\int_0^t h(t-s)u_t(s)ds\| \\ &\le \sqrt{2N}e^{-bt/2} + \sqrt{\bar{h}}\left(\int_0^t h(t-s)\|u_t(s)\|^2 ds\right)^{1/2} \end{aligned}$$

or

$$\|u_t\|^2 \le 4Ne^{-bt} + 2\bar{h}\int_0^t h(t-s)\|u_t(s)\|^2 ds,\ t \ge 0$$

Therefore

$$\begin{aligned} e^{bt}\|u_t\|^2 &\le 4N + 2\bar{h}\int_0^t e^{b(t-s)}h(t-s)e^{bs}\|u_t(s)\|^2 ds \\ &\le 4N + 2\bar{h}\int_0^\infty e^{bs}h(s)ds) \sup_{0\le s\le t} e^{bs}\|u_t(s)\|^2,\ t \ge 0. \end{aligned}$$

Notice that, we may assume that $b < min\{\xi, \theta\}$ and

$$[e^{bt}h(t)]' = be^{bt}h(t) + e^{bt}h'(t) = e^{bt}[bh(t) + h'(t)] < 0$$

as

$$h'(t) \le -\xi h(t) \le -bh(t),\ t \ge 0.$$

We select b smaller that $\xi/2$ so that

$$\int_0^\infty e^{bs}h(s)ds \le h(0)\int_0^\infty e^{-(\xi-b)s}ds < \infty.$$

Passing to the sup in both sides of (2.20) (change t into s and then take the sup), we find

$$\sup_{0\le s\le t} e^{bs}\|u_t(s)\|^2 \le 4N + 2\bar{h}\left(\int_0^\infty e^{bs}h(s)ds\right)\sup_{0\le s\le t} e^{bs}\|u_t(s)\|^2,\ t\ge 0$$

By our assumptions it appears that

$$\|u_t\|^2 \le C_4 e^{-bt},\ t\ge 0$$

for some $C_4 > 0$. The proof is complete.[6]

Remark 2.6 The condition on $h\ :\ h'(t)\le -\xi h(t),\ t>0$ for some $\xi>0$, may be relaxed.

Remark 2.7 The condition $2\bar{h}\int_0^\infty e^{\theta s}h(s)ds < 1$ is satisfied if $\xi > 1$. The argument holds also under the stronger but simpler condition $h(0) < \xi/2$.

Remark 2.8 Observe that, by Gronwall inequality

$$e^{bt}\|u_t\|^2 \le 4Ne^{\int_0^t 2\bar{h}ds},\ t\ge 0$$

and, if $2\bar{h} < b < \xi$, then

$$\|u_t\|^2 \le 4Ne^{-(b-2\bar{h})t},\ t\ge 0.$$

The result stated in Theorem 2.3 looks much weaker than the one in Theorem 2.1. This is supported by the facts that the first condition on $h(t)$ in Theorem 2.1 is satisfied through our condition $h'(t)\le -\xi h(t)$. This later condition also implies the condition on $h'(t)$ in Theorem 2.1 (because $h'(t)\le 0$). However, the argument in the proof of Theorem 2.3 allows us to treat problems which cannot be transformed into the form (2.2). Namely, situations where we cannot write the derivative of $\int_0^t h(t-s)u_t(s)ds$ in terms of $\int_0^t h'(t-s)u_t(s)ds$ because of lack of regularity of the state u (not twice differentiable). The problem is worth studying also in case of non-regularity (non-differentiability) of the kernel h.

CONCLUSION

We study of the memory a wave equation with a distributed neutral delay. We prove that, despite the destructive nature of delays in general, solutions may go back to the equilibriun state in an exponential manner as time goes to infinity. Reasonable conditions on the distributed neutral delay are established. This type of problems appear in the study of wave propagation in viscoelatic media and in acoustic wave propagation. It is not well studied so far.

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