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Problem C-F For Nonlinear Dynamical Systems

Averaging Theory

Destined to Master 2 Dynamical Systems

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Chapter 1

Basic Material and Asymptotics

The information that will be used in the theory that will be created in the next chapters is gathered in this chapter. The existence and uniqueness theorem for contraction-based starting value problems, along with results from subsequent iterations and growth estimates, make up this background information.

The equations that we will investigate have the following general form:

$$\dot{x} = f(x, t, \varepsilon),$$

where x and $f(x, t, \varepsilon)$ are vectors, elements of $\mathbb{R}^n$. All quantities used will be real except if explicitly stated otherwise. Often we shall assume $x \in D \subset \mathbb{R}^n$ with D an open, bounded set. The variable $t \in \mathbb{R}$ is usually identified with time; We assume $t \geq 0$ or $t \geq t_0$ with t_0 a constant. The parameter ε plays the part of a small parameter which characterizes the magnitude of certain perturbations. We usually take ε to satisfy either $0 \leq \varepsilon \leq \varepsilon_0$ or $|\varepsilon| \leq \varepsilon_0$, but even when $\varepsilon = 0$ is not in the domain, we may want to consider limits as $\varepsilon \downarrow 0$. We shall use $D_x f(x, t, \varepsilon)$ to indicate the derivative with respect to the spatial variable x; so $D_x f(x, t, \varepsilon)$ is the matrix with components $\partial f_i / \partial x_j (x, t, \varepsilon)$. For a vector $u \in \mathbb{R}^n$ with components u_i, $i = 1, \ldots, n$, we use the norm

$$||u|| = \sum_{i=1}^{n} |u_i|. \tag{1.1}$$

For the $n \times n-$matrix A, with elements a_{ij} we have

$$||A|| = \sum_{i,j=1}^{n} |a_{i,j}|.$$

Any pair of vector and matrix norms satisfying $||Ax|| \leq ||A||\ ||x||$ may be used instead, such as the Euclidean norm for vectors and its associated operator norm for matrices, $||A|| = Sup\{||Ax|| : ||x|| = 1\}$.

In the study of differential equations most vectors depend on variables. To estimate vector functions we shall nearly always use the sup norm. For instance for the vector functions arising in the differential equation formulated above we put

$$||f||_{sup} = \sup_{x \in D, 0 \leq t \leq T, 0 \leq \varepsilon \leq \varepsilon_0} ||f(x,t,\varepsilon)||.$$

A system of differential equations on $\mathbb{R}^{2n}$ is called a Hamiltonian system with n degrees of freedom if it has the form

$$\begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} \dfrac{\partial H}{\partial p_i} \\ -\dfrac{\partial H}{\partial q_i} \end{pmatrix},$$

where $(q_1, ..., q_n, p_1, ..., p_n)$ are the coordinates on $\mathbb{R}^{2n}$ and $H : \mathbb{R}^{2n} \to \mathbb{R}$ is a function called the Hamiltonian for the system 1. In particular, when dealing with Hamiltonian systems we often use special coordinate changes $(q,p) \Leftrightarrow (Q,P)$ that preserve the property of being Hamiltonian, and transform a system with Hamiltonian $H(q,p)$ into one with Hamiltonian $K(Q,P) = H(q(Q,P), p(Q,P))$. Such coordinate changes are associated with symplectic mappings but were known traditionally as canonical transformations.

1.1 The initial value problem: Existence, Uniqueness and Continuation

The vector functions $f(x,t,\varepsilon)$ arising in our study of differential equations will have certain properties with respect to the variables x and t and the parameter ε. With respect to the 'spatial variable' x, f will always satisfy a Lipschitz condition:

Notation 1 *Let* $G = D \times [t_0, t_0 + T] \times (0, \varepsilon_0]$.

Definition 1 *The vector function* $f : G \to \mathbb{R}^n$ *satisfies a Lipschitz condition in* x *with Lipschitz constant* λ_f *if we have*

$$||f(x_1, t, \varepsilon) - f(x_2, t, \varepsilon)|| \leq \lambda_f ||x_1 - x_2||,$$

where $\leq \lambda_f$ *is a constant. If* f *is periodic with period* T*, the Lipschitz condition will hold for all time.*

It is well known that if f is of class C^1 on an open set U in $\mathbb{R}^n$, and D is a subset of U with compact and convex closure D, f will satisfy a Lipschitz condition on D with $\leq \lambda_f = max\{Df(x) : x \in D\}$. (The proof uses the mean value theorem for the scalar functions $g_i(s) = f_i(x_1+s(x_2-x_1), t, \varepsilon)$ for $0 \leq s \leq 1$.) The following lemma shows that convexity is not necessary. (This is a rather technical issue and the reader can skip the proof of this lemma on first reading.)

Lemma 1 *Suppose that* f *is* C^1 *on* U*, as above, and* $\overline{D}$ *is compact (but not necessarily convex). Then* f *is still Lipschitz on* $\overline{D}$*.*

Proof 1 *For convenience we suppress the dependence on* t *and* ε*. Since* $\overline{D}$ *is compact, there exists* $M > 0$ *such that* $f(x_1) - f(x_2) \leq M$ *for* $x_1, x_2 \in \overline{D}$*. Again by compactness, construct a finite set of open balls Bi with centers* p_i *and radii* r_i *(in the norm* $||\ \ ||$*), such that each* B_i *is contained in* U *and such that the smaller balls* B_i' *with centers* p_i *and radii* $r_i/3$ *cover* $\overline{D}$*. Let* λ_f^i *be a Lipschitz constant for* f *in* B_i*, let* $\lambda_f^0 = max_i \lambda_f^i$*, and let* $\delta = min_i r_i/3$*. Observe that if* $x_1, x_2 \in \overline{D}$ *and* $||x_1 - x_2|| \leq \delta$*, then* x_1 *and* x_2 *belong to the same ball* B_i *(in fact* x_1 *belongs to some* B_i' *and then* $x_2 \in B_i$ *), and therefore* $||f(x_1) - f(x_2)|| < \lambda_f^0 ||x_1 - x_2||$*. Now let* $\lambda_f = max\{\lambda_f^0, M/\delta\}$*. We claim that* $||f(x_1) - f(x_2)|| < \lambda_f^0 ||x_1 - x_2||$ *for all* $x_1, x_2 \in \overline{D}$*.*

If $||x_1 - x_2|| \leq d$*, this has already been proved (since* $\lambda_f^0 \leq \lambda_f$*). If* $x_1 - x_2 > \delta$*, then*

$$||f(x_1) - f(x_2)|| \leq M = \frac{M\delta}{\delta} \leq \lambda_f \delta < \lambda_f ||x_1 - x_2||$$

This completes the proof of the lemma.

We are now able to formulate a well-known existence and uniqueness theorem for initial value problems.

Theorem 1 (Existence and uniqueness) *Consider the differential equation*

$$\dot{x} = f(x, t, \varepsilon),$$

We are interested in solutions x of this equation with initial value $x(t_0) = a$. Let $D = \{x \in \mathbb{R}^n | \; ||x - a|| < d\}$, inducing G by Notation 1, and $f : G \to \mathbb{R}^n$. We assume that

1 f is continuous on G,

2 f satisfies a Lipschitz condition as in Definition 1.

Then the initial value problem has a unique solution x which exists for $t_0 \leq t \leq t_0 + inf(T, d/M)$ where $M = sup_G||f|| = ||f||_{sup}$

Note that the theorem guarantees the existence of a solution on an interval of time which depends explicitly on the norm of f. Additional assumptions enable us to prove continuation theorems, that is, with these assumptions one can obtain existence for larger intervals or even for all time. In the sequel we shall often meet equations in the so called standard form

$$\dot{x} = \varepsilon g^1(x, t),$$

where the superscript reflects the $\varepsilon-$degree. (We often use integer superscripts in place of subscripts to avoid confusion with components of vectors. These superscripts are not to be taken as exponents.) Here, if the conditions of the existence and uniqueness theorem have been satisfied, we find that the solution exists for $t_0 \leq t \leq t_0 + inf(T, d/M)$ with

$$M = sup_{x \in D} sup_{t[t_0, t_0+T)} ||g^1||.$$

This means that the size of the interval of existence of the solution is of the order C/ε with C a constant. This conclusion, in which ε is a small parameter, involves an asymptotic estimate of the size of an interval.

1.2 The Gronwall Lemma

Closely related to contraction is the idea behind an inequality derived by Gronwall.

Lemma 2 (General Gronwall Lemma) *Suppose that for $t_0 \leq t \leq t_0 + T$ we have*

$$\varphi(t) \leq \alpha \int_{t_0}^{t} \beta(s)\varphi(s)ds,$$

where φ and β are continuous and $\beta(t) > 0$. Then

$$\varphi(t) \leq \alpha exp \int_{t_0}^{t} \beta(s)ds,$$

for $t_0 \leq t \leq t_0 + T$.

Proof 2

$$\Phi(t) = \alpha + \int_{t_0}^{t} \beta(s)\phi(s)ds.$$

Then $\phi(t) \leq \Phi(t)$ and $\dot{\Phi}(t) = \beta(t)\phi(t)$, so (since $\beta(t) > 0$) we have $\dot{\Phi}(t) - \beta(t)\Phi(t) \leq 0$. This differential inequality may be handled exactly as one would solve the corresponding differential equation (with $\leq$ replaced by $=$). That is, it may be rewritten as

$$\frac{d}{dt}\left(\Phi(t)e^{\int_{t_0}^{t}\beta(s)ds}\right) \leq 0,$$

and then integrated from t_0 to t, using $\Phi(t_0) = \alpha$, to obtain

$$\Phi(t)e^{\int_{t_0}^{t}\beta(s)ds} - \alpha \leq 0,$$

which may be rearranged into the desired result.

Remark 1 *The lemma may be generalized further to allow α to depend on t, provided we assume α is differentiable and $\alpha(t) \geq 0$, $\dot{\alpha}(t) > 0$.*

Lemma 3 ***(Specific Gronwall lemma).***
Suppose that for $t_0 \leq t \leq t_0 + T$

$$\phi(t) \leq \delta_2(t - t_0) + \delta_1 \int_{t_0}^{t} \phi(s)ds + \delta_3,$$

with $\phi(t)$ continuous for $t_0 \leq t \leq t_0 + T$ and constants $\delta_1 > 0$, $\delta_2 \geq 0$, $\delta_3 \geq 0$ then

$$\phi(t) \leq (\delta_2/\delta_1 + \delta_3)e^{\delta_1(t-t_0)} - \delta_2/\delta_1$$

for $t_0 \leq t \leq t_0 + T$.

Proof 3 *This has the form of Lemma 2 with $\alpha = \delta_1/\delta_2 + \delta_3$ and $\beta(t) = \delta_1$ for all t, and the result follows at once (changing back to $\phi(t)$.)*

1.3 Concepts of Asymptotic Approximation

In the following sections we shall discuss those concepts and elementary methods in asymptotics which are necessary prerequisites for the study of slow-time processes in nonlinear oscillations. In considering a function defined by an integral or defined as the solution of a differential equation with boundary or initial conditions, approximation techniques can be useful. In the applied mathematics literature no single theory dominates but many techniques can be found based on a great variety of concepts leading in general to different results. We mention here the methods of numerical analysis, approximation by orthonormal function series in a Hilbert space, approximation by convergent series and the theory of asymptotic approximations. Each of these methods can be suitable to understand an explicitly given problem. In this book we consider problems where the the- ory of asymptotic approximations is useful and we introduce the necessary concepts in detail. One of the first examples of an **asymptotic approximation** was discussed by Euler [6], or [[7], pp. 585-617], who studied the series

$$\sum_{n=0}^{\infty}(-1)^n n! x^n$$

with $x \in \mathscr{R}$. This series clearly diverges for all $x = 0$. We shall see in a moment why Euler would want to study such a series in the first place, but first we remark that if $x > 0$ is small, the individual terms decrease in absolute value rapidly as long as $nx < 1$. Euler used the truncated series to approximate the function given by the integral

$$\int_0^{\infty} \frac{e^{-s}}{1+sx} ds$$

Poincarée ([[21], Chapter 8]) and Stieltjes gave the mathematical foundation of using a divergent series in approximating a function. The theory of asymptotic approximations has expanded enormously ever since, but curiously enough only few authors concerned themselves with the foundations of the methods. Both the foundations and the applications of asymptotic analysis have been treated by Eckhaus [**?**]. We are interested in perturbation problems of the following kind: consider the differential equation

$$\dot{x} = f(t, x, \varepsilon) \tag{1.2}$$

As usual, let $x, \alpha \in \mathscr{R}^n$, $t \in [t_0, \infty)$ and $\varepsilon \in (0, \varepsilon_0]$ with ε_0 a small positive parameter. If the vector field f is sufficiently smooth in a neighborhood of $(\alpha, t_0) \in \mathscr{R}^n \times \mathscr{R}$, the initial value problem has a unique solution $x_\varepsilon(t)$ for small values of ε on some interval $[t_0, \tilde{t})$;

Some of the problems arising in this approximation process can be illustrated by the following examples. Consider the first-order equation with initial value

$$\dot{x} = x + \varepsilon, \;\; x_\varepsilon(0) = 1.$$

The solution is $x_\varepsilon(t) = (1+\varepsilon)e^t - \varepsilon$. We can rearrange this expression with respect to ε:

$$x_\varepsilon(t) = e^t + \varepsilon(e^t - 1).$$

This result suggests that the function e^t is an approximation in some sense for $x_\varepsilon(t)$ if t is not too large. In defining the concept of approximation one certainly needs a consideration of the domain of validity. A second simple example also shows that the solution does not always depend on the parameter ε in a smooth way:

$$\dot{x} = -\frac{\varepsilon x}{\varepsilon + t}, \;\; x_\varepsilon(0) = 1.$$

The solution reads

$$x_\varepsilon = \left(\frac{\varepsilon}{\varepsilon + t}\right)^{\varepsilon},$$

To characterize the behavior of the solution with ε for $t \geq 0$ one has to divide $\mathscr{R}^+$ into different domains. For instance, it is sometimes possible to write

$$x_\varepsilon(t) = 1 + \varepsilon \log \varepsilon + \varepsilon \log t + O(\varepsilon/t).$$

where $O(\varepsilon/t)$ is small compared to the other terms. (O will be defined more carefully below.) This expansion is possible when t is confined to an ε-dependent interval I_ε such that ε/t is small. (For instance, if $I_\varepsilon = (\sqrt{\varepsilon}, \infty)$ then $t \in I_\varepsilon$ implies $\varepsilon/t < \sqrt{\varepsilon}$.) Of course, this expansion does not satisfy the initial condition. Such problems about the domain of validity and the form of the expansions arise in classical mechanics. To discuss these problems one has to introduce several concepts.

Definition 2 *A function $\delta(\varepsilon)$ will be called an* ***order function*** *if $\delta(\varepsilon)$ is continuous and positive in $(0,\varepsilon_0)$ and if $lim_{\downarrow 0}\delta(\varepsilon)$ exists. Sometimes we use subscripts such as i in $\delta_i(\varepsilon)$, $i = 1,2,\ldots$ In many applications we shall use the set of order functions $\{\varepsilon^n\}_{n=1}^{\infty}$; however also order functions such as ε^q, $q \in \mathcal{Q}$ will play a part. To compare order functions we use Landau's symbols:*

Definition 3 *Let $\phi(t,\varepsilon)$ be a real or vector valued function defined for $\varepsilon > 0$ (or $\varepsilon \geq 0$) and for $t \in I_\varepsilon$. The expression for $\varepsilon \downarrow 0$ means that there exists an $\varepsilon_0 > 0$ such that the relevant statement holds for all $\varepsilon \in (0,\varepsilon_0])$. We define the symbols $O(.)$ and $o(.)$ as follows.*

1. *We say that $\phi(t,\varepsilon) = O(\delta(\varepsilon))$ for $\varepsilon \downarrow 0$ if there exist constants $\varepsilon_0 > 0$ and $k > 0$ such that $||\phi(t,\varepsilon)|| \leq k|\delta(\varepsilon)|$ for all $t \in I_\varepsilon$, for $0 < \varepsilon < \varepsilon_0$.*

2. *We say that $\phi(t,\varepsilon) = o(\delta(\varepsilon))$ for $\varepsilon \downarrow 0$ if*

$$lim_{\varepsilon\downarrow 0} = \frac{||\phi(t,\varepsilon)||}{\delta(\varepsilon)} = 0,$$

uniformly for $t \in I_\varepsilon$. (That is, for every $\alpha > 0$ there exists $\beta > 0$ such that $||\phi(t,\varepsilon)||/\delta(\varepsilon) < \alpha$ if $t \in I_\varepsilon$ and $0 < \varepsilon < \beta$).)

3. *We say that $\delta_1(\varepsilon) = o(\delta_2(\varepsilon))$ for $\varepsilon \downarrow 0$ if $lim_{\varepsilon\downarrow 0}\delta_1(\varepsilon)/\delta_2(\varepsilon) = 0$.*

In all problems we shall consider ordering in a neighborhood of $\varepsilon = 0$ so in estimates we shall often omit 'for $\varepsilon \downarrow 0$'.

Example 1 *The following show the usage of the symbols $O(\cdot)$ and $o(\cdot)$.*

1. *$\varepsilon^n = o(\varepsilon^m)$ for $\varepsilon \downarrow 0$ if $n > m$;*
2. *$\varepsilon\, sin(1/\varepsilon) = O(\varepsilon)$ for $\varepsilon \downarrow 0$;*
3. *$\varepsilon^2 \log\varepsilon = o(\varepsilon^2 \log^2\varepsilon)$ for $\varepsilon \downarrow 0$;*
4. *$e^{-1/\varepsilon} = o(\varepsilon^n)$ for $\varepsilon \downarrow 0$ and all $n \in \mathcal{N}$.*

Now $\delta_1(\varepsilon) = o(\delta_2(\varepsilon))$ implies $\delta_1(\varepsilon) = O(\delta_2(\varepsilon))$; for instance $\varepsilon^2 = o(\varepsilon)$ and $\varepsilon^2 = O(\varepsilon)$ as $\varepsilon \downarrow 0$. It is useful to introduce the notion of a sharp estimate of order functions:

Definition 4 ***(Eckhaus [5]).*** *We say that $\delta_1(\varepsilon) = O_{\#}(\delta_2(\varepsilon))$ for $\varepsilon \downarrow 0$ if $\delta_1(\varepsilon) = O(\delta_2(\varepsilon))$ and $\delta_1(\varepsilon) \neq o(\delta_2(\varepsilon))$ for $\varepsilon \downarrow 0$.*

Example 2 *One has $\varepsilon\ sin(1/\varepsilon) = O_{\#}(\varepsilon)$, $\varepsilon \log \varepsilon = O_{\#}(2\varepsilon \log \varepsilon + \varepsilon^3)$.*

The real variable t used in the initial value problem (2) will be called time. Extensive use shall also be made of **time-like variables** of the form $\tau = \delta(\varepsilon) t$ with $\delta(\varepsilon) = O(1)$. We are now able to estimate the order of magnitude of functions $\phi(t, \varepsilon)$, also written $\phi_\varepsilon(t)$, defined in an interval I_ε, $\varepsilon \in (0, \varepsilon_0]$.

Definition 5 *Suppose that $\phi(\varepsilon) : I_\varepsilon \to \mathscr{R}^n$ for $0 < \varepsilon \leq \varepsilon_0$. Let $||.||$ be the Euclidean metric on $\mathscr{R}^n$ and let $|.|$ be defined by $|\phi_\varepsilon| = sup\{||\phi_\varepsilon(t)|| : t \in I_\varepsilon\}$. (Notice that this norm depends on ε and could be written more precisely as $|.|_\varepsilon$.) Let δ be an order function. Then:*

1. *$\varphi_\varepsilon = O(\delta(\varepsilon))$ in I_ε if $|\varphi_\varepsilon| = O(\delta(\varepsilon))$ for $\varepsilon \downarrow 0$;*
2. *$\varphi_\varepsilon = o(\delta(\varepsilon))$ in I_ε if $lim_{\varepsilon \downarrow 0} |\phi_\varepsilon| / \delta(\varepsilon)) = 0$;*
3. *$\varphi_\varepsilon = O_\sharp(\delta(\varepsilon))$ in I_ε if $\varphi_\varepsilon = O_\sharp(\delta(\varepsilon))$ and $\varphi_\varepsilon = o(\delta(\varepsilon))$.*

It is customary to say that the estimates defined in this way are uniform or uniformly valid on I_ε, because of the use of $|.|$, which makes the estimates independent of t.

Of course, one can give the same definitions for spatial variables.

Example 3 *We wish to estimate the order of magnitude of the error we make in approximating $sin(t + \varepsilon\ t)$ by $sin(t)$ on the interval I_ε. If I_ε is $[0, 2\Pi]$ we have for the difference of the two functions $sup_{t \in [0, 2\Pi]} |sin(t + \varepsilon t) - sin(t)| = O(\varepsilon)$.*

Remark 2 *An additional complication is that in many problems the boundaries of the interval I_ε depend on ε in such a way that the interval becomes unbounded as ε tends to 0. For instance in the example above we might wish to compare $sin(t + \varepsilon t)$ with $sin(t)$ on the interval $I_\varepsilon = [0, 2\pi/\varepsilon]$. We obtain in the sup norm*

$$sin(t + \varepsilon t) - sin(t) = \mathcal{O}_{\#}(1)$$

(with $\mathcal{O}_{\#}$ as defined in Definition 4).
Suppose $\delta(\varepsilon) = o(1)$ and we wish to estimate φ_ε on $I_\sigma = [0, L/\delta(\varepsilon)]$ with L a constant independent of ε. Such an estimate will be stated as $\varphi_\varepsilon = \mathcal{O}(\delta_0(\varepsilon))$ as $\varepsilon \downarrow 0$ on I_ε, or else as $\varphi_\varepsilon(t) = \mathcal{O}(\delta_0(\varepsilon))$ as $\varepsilon \downarrow 0$ on I_ε. The first form, without the t, is preferable, but is difficult to use in an example such as

$$sin(t+\varepsilon t) - sin(t) = \mathcal{O}(1)$$

as $\varepsilon \downarrow 0$ on I_ε. We express such estimates often as follows:

Definition 6 *We say that $\varphi_\varepsilon(t) = \mathcal{O}(\delta_0(\varepsilon))$ as $\varepsilon \downarrow 0$ on the* ***time scale*** *$\delta(\varepsilon)^{-1}$ if the estimate holds for $0 \le \delta(\varepsilon) t \le L$ with L a constant independent of ε.*

An analogous definition can be given for $o(\delta_0(\varepsilon))$-estimates. Once we are able to estimate functions in terms of order functions we are able to define asymptotic approximations.

Definition 7 *We define* ***asymptotic approximations*** *as follows.*

1/ . $\psi_\varepsilon(t)$ is an asymptotic approximation of $\varphi_\varepsilon(t)$ on the interval I_ε if

$$\varphi_\varepsilon(t) - \psi_\varepsilon(t) = o(1)$$

as $\varepsilon \downarrow 0$, uniformly for $t \in I_\varepsilon$. Or rephrased for time scales:

2/ $\psi_\varepsilon(t)$ is an asymptotic approximation of $\varphi_\varepsilon(t)$ on the time scale $\delta(\varepsilon)^{-1}$ if

$$\varphi_\varepsilon - \psi_\varepsilon = o(1)$$

as $\varepsilon \downarrow 0$ on the time scale $\delta(\varepsilon)^{-1}$.

In general one obtains as approximations asymptotic series (or expansions) on some interval I_ε. An **asymptotic series** is an expression of the form

$$\varphi_\varepsilon(t,\varepsilon) \sim \sum_{j=1}^{\infty} \delta_j(\varepsilon)\varphi^j(t,\varepsilon) \tag{1.3}$$

in which $\delta_j(\varepsilon)$ are order functions with $\delta_{j+1} = o(\delta_j)$. Such a series is not expected to converge, but instead one has

$$\varphi_\varepsilon(t,\varepsilon) = \sum_{j=1}^{m} \delta_j(\varepsilon)\varphi^j(t,\varepsilon) + o(\delta_m(\varepsilon)) \quad on \quad I_\varepsilon$$

for each m in N, or, more commonly, the stronger condition

$$\varphi_\varepsilon(t,\varepsilon)=\sum_{j=1}^{m}\delta_j(\varepsilon)\varphi^j(t,\varepsilon)+\mathcal{O}(\delta_{m+1}(\varepsilon)) \quad on \quad I_\varepsilon$$

often stated as "the error is of the order of the first omitted term."

Example 4 *Consider, on $I=[0,2\pi]$,*

$$\varphi_\varepsilon(t)=sin(t+\varepsilon),$$

$$\tilde{\varphi}_\varepsilon(t)=sin(t)+\varepsilon t cos(t)-\frac{1}{2}\varepsilon^2t^2sin(t).$$

The order functions are $\delta_n(\varepsilon)=\varepsilon^{n-1}$, $n=1,2,3,\dots$ and clearly

$$\varphi_\varepsilon(t)-\tilde{\varphi}_\varepsilon(t)=o(\varepsilon^2) \quad on \quad I,$$

so that $\tilde{\varphi}_\varepsilon(t)$ is a third-order asymptotic approximation of $\varphi_\varepsilon(t)$ on I. Asymptotic approximations are not unique.

Another third-order asymptotic approximation of $\varphi_\varepsilon(t)$ on I is

$$\psi_\varepsilon(t)=sin(t)+\varepsilon\varphi_{2\varepsilon}(t)-\frac{1}{2}\varepsilon^2t^2sin(t),$$

with $\varphi_{2\varepsilon}(t)=sin(\varepsilon)cos(t)/\varepsilon$. The functions $\varphi_{n\varepsilon}=(t)$ are not determined uniquely as is immediately clear from the definition.

More serious is that for a given function different asymptotic approximations may be constructed with different sets of order functions. Consider an example given by Eckhaus ([**?**], Chapter 1]):

$$\varphi_\varepsilon(t)=(1-\frac{\varepsilon}{1+\varepsilon}t)^{-1}, \quad I=[0,1].$$

One easily shows that the following expansions are asymptotic approximations of φ_ε on I:

$$\psi_{1\varepsilon}(t)=\sum_{n=0}^{m}\left(\frac{\varepsilon}{1+\varepsilon}\right)^n t^n,$$

$$\psi_{2\varepsilon}(t)=1+\sum_{n=1}^{m}\varepsilon^n t(t-1)^{n-1}.$$

Although asymptotic series in general are not unique, special forms of asymptotic series can be unique. A series of the form (1.3) in which each φ_n is independent of ε is called a **Poincaré asymptotic series.**

Theorem 2 *If $\varphi(t,\varepsilon)$ has a Poincaré asymptotic series with order functions $\delta_1, \delta_2, \ldots$ then this series is unique.*

Proof 4 *First, $\varphi(t,\varepsilon) = \delta_1(\varepsilon)\varphi^1(t) + o(\delta_1(\varepsilon))$. Dividing by δ_1 we have $\varphi/\delta_1 = \varphi_1 + o(1)$, and letting $\varepsilon \to 0$ gives*

$$\varphi^1(t) = \lim_{\varepsilon\to 0} \frac{\varphi(t,\varepsilon)}{\delta_1(\varepsilon)}$$

which determines $\varphi_1(t)$ uniquely. Next, dividing $\varphi = \delta_1\varphi_1 + \delta_2\varphi_2 + o(\delta_2)$ by δ_2 and letting $\varepsilon \to 0$ give

$$\varphi^2(t) = \lim_{\varepsilon\to 0} \frac{\varphi(t,\varepsilon) - \delta_1(\varepsilon)\varphi_1(t)}{\delta_2(\varepsilon)}$$

*which fixes φ^2. It is clear how to continue. Because of these formulas, Poincaré asymptotic series are often called **limit process expansions***

Another special type of asymptotic series is one in which the φ_j depend on ε only through a second time variable $\tau = \varepsilon t$. The next theorem, due to Perko [20], shows that certain series of this type are unique.

Theorem 3 ***(Perko[20])**. Suppose that the function $\varphi(t,\varepsilon)$ has an asymptotic expansion of the form*

$$\varphi(t,\varepsilon) \sim \varphi_0(\tau,t) + \varepsilon\varphi 1(\tau,t) + \varepsilon^2\varphi 2(\tau,t) + \ldots, \qquad (1.4)$$

valid on an interval $0 \le t \le L/\varepsilon$ for some $L > 0$. Suppose also that each $\varphi_j(\tau,t)$ is defined for $0 \le \tau \le L$ and $t \ge 0$, and is periodic in t with some period T (for all fixed τ). Then there is only one such expansion.

Proof 5 *By considering the difference of two such expansions, it is enough to prove that if*

$$0 \sim \varphi^0(\tau,t) + \varepsilon\varphi^1(\tau,t) + \varepsilon^2\varphi^2(\tau,t) + \ldots$$

then each $\varphi_j = 0$. This asymptotic series implies that $\varphi^0(\tau,t) = o(1)$. We claim that $\varphi^0(\tau,t) = 0$ for any $t \ge 0$ and any τ with $0 \le \tau \le L$. Let $t_j = t + jT$ and $\varepsilon_j = \tau/t_j$, and note that $\varepsilon_j \to 0$ as $j \to \infty$ and that $0 \le t_j \le L/\varepsilon_j$. Now

$\|\varphi^0(\tau,t)\| = \|\varphi^0(\varepsilon_j t_j, t_j)\| \to 0$ as $j \to \infty$ (in view of the definition of $|.|$, so $\varphi^0(\tau,t) = 0$. We see that φ^0 drops out of the series,

For the sake of completeness we return to the example discussed by Euler which was mentioned at the beginning of this section. Instead of x we use the variable $\varepsilon \in (0, \varepsilon_0]$. Basic calculus can be used to show that we may define the function φ_ε by

$$\varphi_\varepsilon = \int_0^\infty \frac{e^{-s}}{1+\varepsilon s} ds, \quad \varepsilon \in (0, \varepsilon_0].$$

Transform $\varepsilon s = \tau$ to obtain

$$\varphi_\varepsilon = \frac{1}{\varepsilon} \int_0^\infty \frac{e^{-\tau/\varepsilon}}{1+\tau} d\tau,$$

and by partial integration

$$\varphi_\varepsilon = \frac{1}{\varepsilon}\Big[-\varepsilon \frac{e^{-\tau/\varepsilon}}{1+\tau} |_0^\infty - \varepsilon \int_0^\infty \frac{e^{-\tau/\varepsilon}}{(1+\tau)^2} d\tau \Big],$$

and after repeated partial integration

$$\varphi_\varepsilon = 1 - \varepsilon + 2\varepsilon \int_0^\infty \frac{e^{-\tau/\varepsilon}}{(1+\tau)^3} d\tau.$$

We may continue the process and define 1 Basic Material and Asymptotics

$$\tilde{\varphi}_\varepsilon = \sum_{n=0}^{m} (-1)^n n! \varepsilon n.$$

It is easy to see that

$$\varphi_\varepsilon = \tilde{\varphi}_\varepsilon + R_{m\varepsilon},$$

$$R_{m\varepsilon} = (-1)^{m+1}(m+1)! \varepsilon^m \int_0^\infty e^{-\tau/\varepsilon}(1+\tau)^{-(m+2)} d\tau.$$

Transforming back to t we can show that

$$R_{m\varepsilon} = \mathcal{O}(\varepsilon^{m+1}).$$

Therefore $\tilde{\varphi}_\varepsilon$ is an asymptotic approximation of $\varphi(\varepsilon)$. The expansion is in the set of order functions $\{\varepsilon^n\}_{n=1}^\infty$ and the series is divergent. A final remark concerns the case for which one is able to prove that an asymptotic series converges. This does not imply that the series converges to the function to be studied: consider the simple example

$$\varphi_\varepsilon = sin(\varepsilon) + e^{-1/\varepsilon}.$$

Taylor expansion of $sin(\varepsilon)$ produces the series

$$\tilde{\varphi}_1 = \sum_{n=0}^{m} \frac{(-1)^n \varepsilon^{2n+1}}{(2n+1)!}$$

which is convergent for $m \to \infty$; $\tilde{\varphi}_\varepsilon$ is an asymptotic approximation of φ_ε as

$$\varphi_\varepsilon - \tilde{\varphi}_\varepsilon = \mathcal{O}(\varepsilon^{2m+3}), \forall m \in \mathbb{N}.$$

However, the series does not converge to φ_ε, but instead to $sin(\varepsilon)$. The term $e^{-1/\varepsilon}$ is called **flat** or **transcendentally** small.

In the theory of nonlinear differential equations, this matter of convergence is of some practical interest. Usually, the calculation of one or a few more terms in the asymptotic expansion is all that one can do within a reasonable amount of (computer) time. But there are examples in bifurcation theory that show this flat behavior.

Chapter 2

The Averaging Theory for Computing Periodic Orbits

2.1 Preface

A classic tool for studying the dynamics of non-linear differential systems under periodic forcing is the method of averaging. A long history of the average technique may be traced back to the classical writings of Lagrange and Laplace, who gave the average method an intuitive basis. The first formalization of this theory was done in 1928 by Fatou [18]. A differential system's orbits are all homeomorphic to either a point, a circle, or a straight line. First, it is referred to as a unique point or an equilibrium point, and a periodic orbit is a term used in the second instance. There is no name for the third instance. The periodic orbits of a specific differential system are being studied analytically in the following notes.

We look at differential systems with the form

$$\dot{x} = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{2.1}$$

with x in some open subset D of $\mathbb{R}^n$, $F_i : \mathbb{R} \times D \to \mathbb{R}^n$ of class C^2 for $i = 1,2$, $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ of class C^2 with $\varepsilon_0 > 0$ small, and with the functions F_i and $\mathbb{R}$ being $T-$periodic in the variable t. Here, the dot denotes the derivative with respect to the time t.

In general, it is very difficult and usually impossible to discover analytically periodic solutions to differential systems. As we'll see when we can use the averaging theory, this challenging problem for differential systems (2.1) is reduced to finding the zeros of a nonlinear function with at most n dimensions, which means that the problem now has a similar level of difficulty to that of locating the singular or equilibrium points of a differential system. A significant challenge for researching periodic solutions to differential systems of the form

$$\dot{x} = F(t, x), \quad or \quad \dot{x} = F(x), \tag{2.2}$$

using averaging theory is to convert them into systems represented as a system eqrefk1, which is the standard form of the averaging theory, i.e., as a system (2.1). Note that systems (2.2), in general, are not periodic in the independent variable t and do not have any small ε. In order to write the differential systems of form (2.2) into (2.1), where $F0$ may eventually equal zero, we must find changes in variables.

2.2 Introduction: the conventional theory

2.2.1 A first-order averaging method for periodic orbits

We consider the differential system

$$\dot{x} = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{2.3}$$

with $x \in D \subset \mathbb{R}$, D a bounded domain, and $t \geqslant 0$. Moreover, we assume that $F(t, x)$ and $R(t, x, \varepsilon)$ are $T-$periodic in t. The averaged system associated to the system (2.3) is defined by

$$\dot{y} = \varepsilon f_(y), \tag{2.4}$$

where

$$f^0(y) = \frac{1}{T}\int_0^T F(s, y)ds. \tag{2.5}$$

The next theorem says under what conditions the singular points of the averaged system (2.4) provide $T-$periodic orbits for the system (2.3).

Theorem 4 *We consider system* (2.3) *and assume that the vector functions* F, R, D_xF, $D_x^2\,F$ *and* D_xR *are continuous and bounded by a constant* M *(independent of* ε*) in* $[0,\infty)\times D$*, with* $-\varepsilon_0<\varepsilon<\varepsilon_0$*. Moreover, we suppose that* F *and* R *are* $T-$*periodic in* t*, with* T *independent of* ε*.*

(i) *If* $p\in D$ *is a singular point of the averaged system* (2.4) *such that*

$$\det(D_x f^0(p))\neq 0 \tag{2.6}$$

then, for $\mid\varepsilon\mid>0$ *sufficiently small, there exists a* $T-$*periodic solution* $x(t,\varepsilon)$ *of system* (2.3) *such that* $x(0,\varepsilon)\to p$ *as* $\varepsilon\to 0$*.*

(ii) *If the singular point* $y=p$ *of the averaged system* (2.4) *has all its eigenvalues with negative real part then, for* $\mid\varepsilon\mid>0$ *sufficiently small, the corresponding periodic solution* $x(t,\varepsilon)$ *of system* (2.3) *is asymptotically stable and, if one of the eigenvalues has positive real part* $x(t,\varepsilon)$*, it is unstable.*

For each $z\in D$ we denote by $x(\cdot,z,\varepsilon)$ the solution of (2.3) with initial condition $x(0,z,\varepsilon)=z$. We consider also the function $\zeta : D\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^n$ defined by

$$\zeta(z,\varepsilon)=\int_0^T[\varepsilon F(t,x(t,z,\varepsilon))+\varepsilon^2R(t,x(t,z,\varepsilon),\varepsilon)]\,dt \tag{2.7}$$

From (2.3) it follows that, for every $z\in D$,

$$\zeta(z,\varepsilon)=x(T,\,z,\,\varepsilon)-x(0,\,z,\,\varepsilon). \tag{2.8}$$

The function ζ can be written in the form

$$\zeta(z,\varepsilon)=\varepsilon f_0(z)+O(\varepsilon^2), \tag{2.9}$$

where f_0 is given by (1.5). Moreover, under the assumptions of Theorem 4 the solution $x(t,\varepsilon)$, for $\mid\varepsilon\mid$ sufficiently small, satisfies that $z_\varepsilon=x(0,\varepsilon)$ tends to be an isolated zero of $\zeta(\cdot,\varepsilon)$ when $\varepsilon\to 0$. Of course, due to (2.8) the function ζ is a displacement function for system (2.3), and its fixed points are initial conditions for the $T-$periodic solutions of system (2.1).

2.2.2 Other first order averaging methods for periodic orbits

We consider the problem of bifurcation of $T-$periodic solutions from the differential system

$$\dot{x} = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{2.10}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here, the functions F_0, $F_1 : \mathbb{R} \times D \to \mathbb{R}^n$ and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are C^2 functions $T-$periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\dot{x} = F_0(t,x) \tag{2.11}$$

has a submanifold of periodic solutions.

Let $x(t,z)$ be the solution of the unperturbed system (2.11) satisfying that $x(0,z) = z$. We write the linearization of the unperturbed system along the periodic solution $x(t,z)$ as

$$\dot{y} = D_x F_0(t, x(t,z))\, y. \tag{2.12}$$

In what follows we denote by $M_z(t)$ some fundamental matrix of the linear differential system (2.12), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates, $i.e.$, $\xi(x_1, \cdots, x_n) = (x_1, \cdots, x_n)$.

Theorem 5 *Let $V \subset \mathbb{R}^k$ be open and bounded, and let $\beta_0 : Cl(V) \to \mathbb{R}^{n-k}$ be a C^2 function. We assume that*

(i) $\mathcal{Z} = \{z_\alpha = (\alpha, \beta_0(\alpha)) : \alpha \in Cl(V)\} \subset \Omega$ and that for each $z_\alpha \in \mathcal{Z}$ the solution $x(t, z\alpha)$ of (2.11) is $T-$periodic;

(ii) for each $z_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{z_\alpha}(t)$ of (2.12) such that the matrix $M_{z_\alpha}^{-1}(0) - M_{z_\alpha}^{-1}(T)$ has in the right up corner the $k \times (n-k)$ zero matrix, and in the right lower corner a $(n-k) \times (n-k)$ matrix $\triangle_\alpha$ with $\det(\triangle_\alpha) \neq 0$.

We consider the function $\mathscr{F} : Cl(V) \to \mathbb{R}^k$ defined as

$$\mathscr{F}(a) = 0 = \xi\left(\int_0^T M_{z_\alpha}(t) F_1(t, x(t, z\alpha))\right). \tag{2.13}$$

If there exists $a \in V$ with $\mathscr{F}(a) = 0$ and $\det(d\mathscr{F}/d\alpha)(a)) \neq 0$, then there is a $T-$periodic solution $x(t,\varepsilon)$ of system (2.10) such that $x(0,\varepsilon) \to z_a$ as $\varepsilon \to 0$.

Proof 6 *It follows immediately from Theorem* 5 *taking* $k = n$.

We assume that there exists an open set V with $Cl(V) \subset \Omega$ such that for each $z \in Cl(V), x(t,z,0)$ is $T-$periodic, where $x(t,z,0)$ denotes the solution of the unperturbed system (2.11) with $x(0,z,0) = z$. The set $Cl(V)$ is isochronous for the system (2.10), i.e., it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of $T-$periodic solutions from the periodic solutions $x(t,z,0)$ contained in $Cl(V)$ is given in the following result.

Corollary 1 *(Perturbations of an isochronous set). We assume that there exists an open and bounded set* V *with* $Cl(V) \subset \Omega$ *and such that, for each* $z \in Cl(V)$, *the solution* $x(t,z)$ *is* $T-$*periodic; then we consider the function* $F : Cl(V) \mapsto \mathbb{R}^n$,

$$\mathscr{F}(z) = \int_0^T M_z^{-1}(t,z)F_1(t,x(t,z)). \tag{2.14}$$

If there exists $a \in V$ *with* $\mathscr{F}(a) = 0$ *and* $\det(d\mathscr{F}/dz)(a)) \neq 0$, *then there is a* $T-$*periodic solution* $x(t,\varepsilon)$ *of system* (2.10) *such that* $x(0,\varepsilon) \to a$ *as* $\varepsilon \to 0$.

Proof 7 *It follows immediately from Theorem* 5 *taking* $k = n$.

2.3 Another first order averaging method for periodic orbits

The next result extends the result of Theorem 5 to the case $n = 2m$ and when the matrix Δ_α is the zero matrix. Here, $\xi^\perp : \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is the projection of $\mathbb{R}^n$ onto its second set of m coordinates, i.e., $\xi^\perp(x_1,...,x_n) = (x_{m+1},...,x_n)$.

Theorem 6 *Let $V \subset \mathbb{R}^m$ be open and bounded, let $\beta_0 : C^1(V) \to \mathbb{R}^m$ be a Ck function and $Z = \{z_\alpha = (\alpha, \beta_0(\alpha)) | \alpha \in C^l(V)\} \subset \Omega$ its graphic in $\mathbb{R}^{2m}$. Assume that for each $z_\alpha \in Z$ the solution $x(t, z_\alpha)$ of* $(2.10)_{\varepsilon=0} = 0$ *is $T-$periodic and that there exists a fundamental matrix $M_{z_\alpha}(t)$ of* (2.3) *such that the matrix $M^{-1}_{z_\alpha}(0) - M^{-1}_{z_\alpha}(T)$ has in the upper right corner the $m \times m$ matrix Ω_α with $det(\Omega_\alpha) = 0$, and in the lower right corner the $m \times m$ zero matrix. Consider the function $G : C^l(V) \to \mathbb{R}^m$ defined by*

$$G(\alpha) = \xi^{\perp}\Big(\int_0^T M^{-1}_{z_\alpha}(t) F_1(t, x(t, z_\alpha)) dt\Big), \tag{2.15}$$

If there is $\alpha_0 \in V$ with $G(\alpha_0) = 0$ and $det((\partial G/\partial\alpha)(\alpha_0)) \neq 0$ then, for $\varepsilon \neq 0$ sufficiently small, there is a unique $T-$periodic solution $x(t, \varepsilon)$ of the system (2.10) *such that $x(t, \varepsilon) \to x(t, z_{\alpha_0})$ as $\varepsilon \to 0$.*

2.4 Proof of Theorem 4

Proof 8 (Proof of statement (i) of Theorem 4) *The assumptions guarantee the existence and uniqueness of the solutions of the initial valued problems* (2.3) *and* (2.4) *on the time-scale $1/\varepsilon$. We introduce*

$$u(t, x) = \int_0^t [F(s, x) - f^0(x)] ds. \tag{2.16}$$

Since we have subtracted the average of $f(s, x)$ in the integrand, the integral is bounded, i.e.,

$$||u(x, t)|| \leq 2MT, t \geq 0, x \in D.$$

We now introduce a transformation near the identity

$$x(t) = z(t) + \varepsilon u(t, z(t)). \tag{2.17}$$

This transformation will be used for simplifying equation (2.3). *Differentiation of* (2.17) *and substitution in* (2.3) *yields*

$$\dot{x} = \dot{z} + \varepsilon \frac{\partial}{\partial t} u(t, z) + \varepsilon \frac{\partial}{\partial z} u(t, z) \dot{z} = \varepsilon F(t, z + \varepsilon u(t, z)) + \varepsilon^2 R(t, z + \varepsilon u(t, z), \varepsilon). \tag{2.18}$$

Using (2.16), *we write this equation in the form*

$$\Big(I + \varepsilon \frac{\partial}{\partial t} u(t, z)\Big) \dot{z} = \varepsilon f^0(z) + S$$

with I the $n \times n$ identity matrix, and where

$$S = \varepsilon F(t,z+\varepsilon u(t,z)) - \varepsilon F(t,z) + \varepsilon^2 R(t,z+\varepsilon u(t,z),\varepsilon). \tag{2.19}$$

Since $\partial u/\partial z$ is uniformly bounded (as u) we can invert to obtain

$$\left(I+\varepsilon\frac{\partial}{\partial z}u(t,z)\right)^{-1} = I - \varepsilon\frac{\partial}{\partial z}u(t,z) + O(\varepsilon^2), \quad t \geq 0, \quad z \in D. \tag{2.20}$$

From the Lipschitz continuity of $F(t,z)$ we have

$$||F(t,z+\varepsilon u(t,z)) - F(t,z)|| \leq L\varepsilon||u(t,z)|| \leq L\varepsilon 2MT,$$

where L is the Lispchitz constant. Due to the boundedness of $\mathbb{R}$ it follows that, for some positive constant C independent from ε, we have

$$||S|| \leq \varepsilon^2 C, \quad t \geq 0, z \in D. \tag{2.21}$$

From (2.20) *and* (2.21) *we get that*

$$\dot{z} = \varepsilon f^0(z) + S - \varepsilon^2\frac{\partial u}{\partial z}f^0(z) + O(\varepsilon^3), \quad z(0) = x(0). \tag{2.22}$$

AsS $= O(\varepsilon^2)$ by introducing the time-like variable $\tau = \varepsilon t$, we obtain that the solution of

$$\frac{dy}{d\tau} = f^0(y), \quad y(0) = z(0), \tag{2.23}$$

approximates the solution of (2.17) *with error $O(\varepsilon)$ on the time-scale* 1 *in τ, i.e., on the time-scale $1/\varepsilon$ in t. Due to the near identity transformation* (2.17) *we obtain that*

$$x(t) - y(t) = O(\varepsilon) \tag{2.24}$$

in the time-scale $1/\varepsilon$.

Now we shall impose the periodicity condition after which we can apply the Implicit Function Theorem. We transform $x \to z$ with the near identity transformation (2.17), *then the equation for z becomes*

$$\dot{z} = \varepsilon f_0(z) + \varepsilon^2 S(t,z,\varepsilon). \tag{2.25}$$

Due to the choice of $u(t,z(t))$, a T-periodic solution $z(t)$ produces a T-periodic solution $x(t)$. For S we have the expression

$$S(t,z,\varepsilon) = \frac{\partial F}{\partial z}(t,z)u(t,z) - \frac{\partial u}{\partial z}(t,z)f^0(z) + R(t,z,0) + O(\varepsilon). \tag{2.26}$$

This expression is T-periodic in t and continuously differentiable with respect to z. Equation (2.25) *is equivalent to the integral equation*

$$z(t) = z(0) + \varepsilon \int_0^t f^0(z(s))ds + \varepsilon^2 \int_0^t S(s, z(s), \varepsilon)ds.$$

The solution $z(t)$ *is T-periodic if* $z(t+T) = z(t)$ *for all* $t \geq 0$*, which leads to the equation*

$$h(z(0), \varepsilon) = \int_0^T f^0(z(s))ds + \varepsilon \int_0^T S(s, z(s), \varepsilon)ds = 0. \tag{2.27}$$

Note that this is a short-hand notation. The right hand side of equation (2.27) *does not depend on* $z(0)$ *explicitly. But the solutions depend continuously on the initial values and so the dependence on* $z(0)$ *is implicitly by the bijection* $z(0) \to z(x)$.

It is clear that$h(p,0) = 0$. *If* ε *is in a neighborhood of* $\varepsilon = 0$*, then equation* (2.27) *has a unique solution* $x(t,\varepsilon) = z(t,\varepsilon)$. *If* $\varepsilon \to 0$ *then* $z(0,\varepsilon) \to p$. *This completes the proof of statement* (i).

For proving statement (ii) of Theorem 4 we need some preliminary results. The first result is Gronwall's inequality.

Lemma 4 *Let a be a positive constant. Assume that* $t \in [t_0, t_0 + a]$ *and*

$$\varphi(t) \leq \delta_1 \int_{t_0}^t \psi(s)\varphi(s)ds + \delta_2, \tag{2.28}$$

where $\psi(t) \leq 0$ *and* $\varphi(t) \leq 0$ *are continuous functions, and* $\delta_i > 0$ *for* $i = 1, 2$. *Then,*

$$\varphi(t) \leq \delta_2 e^{\delta_1 \int_{t_0}^t \psi(s)ds}.$$

Proof 9 *From* (2.28) *we get*

$$\frac{\varphi(t)}{\delta_1 \int_{t_0}^t \psi(s)\varphi(s)ds + \delta_2} \leq 1.$$

Multiplying by $\delta_1\psi(t)$ *and integrating we obtain*

$$\int_{t_0}^t \frac{\varphi(s)\psi(s)}{\delta_1 \int_{t_0}^t \psi(r)\varphi(r)dr + \delta_2} ds \leq \delta_1 \int_{t_0}^t \psi(s)ds,$$

therefore

$$\log(\delta_1 \int_{t_0}^t \psi(s)\varphi(s)ds + \delta_2) - \log(\delta_2) \leq \delta_1 \int_{t_0}^t \psi(s)ds.$$

Hence,

$$\delta_1 \int_{t_0}^{t} \psi(s)\varphi(s)ds + \delta_2 \leq \delta_2 e^{\delta_1 \int_{t_0}^{t} \psi(s)ds}.$$

From (2.28) *the lemma follows.*

We consider the linear differential system

$$\dot{x} = Ax, \tag{2.29}$$

where A is a constant $n \times n$ matrix. The eigenvalues $\lambda_1, ..., \lambda_n$ of system (2.29) are the zeros of the characteristic polynomial $det(A - \lambda Id)$.

If these eigenvalues λ_k are different, with eigenvectors e_k for $k = 1, ..., n$, then $e_k e^{\lambda_k t}$, for $k = 1, ..., n$, are n independent solutions of the system (2.29).

Assume now that not all eigenvalues are different, thus suppose that the eigenvalue λ has multiplicity $m > 1$. Then λ generates m independent solutions of the system (2.29) of the form

$$P_0 e^{\lambda t}, P_1(t) e^{\lambda t}, ..., P_{m-1}(t) e^{\lambda t}.$$

Where $P_i(t)$ for $i = 0, 1, ..., m-1$ are polynomial vectors of degree at most i.

With n independent solutions $x_1(t), ..., x_n(t)$ of system (2.29) we form a matrix

$$\Phi(t) = (x_1(t), ..., x_n(t)),$$

called a fundamental matrix of system (2.29) . Every solution $x(t)$ of system (2.29) can be written as $x(t) = \Phi(t)c$, where c is a constant vector. Moreover the solution $x(t)$ with $x(t_0) = x_0$ is

$$x(t) = \Phi(t)\Phi(t_0)^{-1} x_0.$$

Usually, we choose the fundamental matrix $\Phi(t)$ in such a way that $\Phi(t_0) = Id$. From (2.30) and the explicit form of the independent solutions of system (2.29), the next result follows easily.

Proposition 1 *We consider the linear differential system $\dot{x} = Ax$, where A is a constant $n \times n$ matrix with eigenvalues $\lambda_1, ..., \lambda_n$. Then the following statements hold:*

(*i*) *if $Re\lambda_k < 0$ for $k = 1, ..., n$ then, for each solution $x(t)$ with $x(t_0) = x_0$, there exist two positive constants C and μ satisfying*

$$||x(t)|| \leq C||x_0||e^{\mu t} \quad \textit{and} \quad \underset{t \to \infty}{Lim} \; x(t);$$

(ii) *if* $Re\lambda_k \leq 0$ *for* $k = 1, \ldots, n$ *and the eigenvalues with* $Re\lambda_k = 0$ *are different, then the solution* $x(t)$ *is bounded for* $t \geq t_0$*; more precisely,*

$$||x(t)|| \leq C||x_0|| \quad \text{and} \;\; C > 0;$$

(iii) *if there exists an eigenvalue* λ_k *with* $Re\lambda_k > 0$*, then in each neighborhood of* $x = 0$ *there are solutions* $x(t)$ *such that*

$$\lim_{t \to \infty} ||x(t)|| = \infty.$$

Under the assumptions of statement (i) *of Proposition 1, the solution* $x = 0$ *is called asymptotically stable. Under the assumptions of statement* (ii)*, the solution* $x = 0$ *is called Liapunov stable. Finally, under the assumptions of statement* (iii) *the solution* $x = 0$ *is called unstable.*

The next result is also known as the Poincaré–Liapunov Theorem.

Theorem 7 *Consider the differential system*

$$\dot{x} = Ax + B(t)x + f(t,x), \quad x(t_0) = x_0, \tag{2.30}$$

where $t \in \mathbb{R}$*,* A *is a constant* $n \times n$ *matrix having all its eigenvalues with negative real part, and* $B(t)$ *is a continuous* $n \times n$ *matrix such that* $\lim_{t \to \infty} ||B(t)|| = 0$*. The function* $f(t,x)$ *is continuous in* t *and* x*, and Lipschitz in* x *in a neighborhood of* $x = 0$*. If*

$$\lim_{||x|| \to 0} \frac{f(t,x)}{||x||} = 0, \quad \text{uniformly in } t,$$

then there exists positive constants C, t_0, δ *and* μ *such that* $||x_0|| \leq \delta$ *implies*

$$||x(t)|| \leq C||x_0||e^{\mu(t-t_0)} \quad for\, t \geq t_0.$$

The solution $x = 0$ *is asymptotically stable and the attraction is exponential in a* $\delta-$*neighborhood of* $x = 0$*.*

Proof 10 (Proof of Theorem 5) *We consider the function* $f : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$*, given by*

$$f(z,\varepsilon) = x(T,z,\varepsilon) - z. \tag{2.31}$$

Then, every $(z_\varepsilon, \varepsilon)$ *such that*

$$f(z_\varepsilon, \varepsilon) = 0. \tag{2.32}$$

provides the periodic solution $x(\cdot, z_\varepsilon, \varepsilon)$ *of* (2.10). *We need to study the zeros of the function* (2.31), *or, equivalently, of*

$$g(z,\varepsilon) = Y^{-1}(T,z) f(z,\varepsilon). \tag{2.33}$$

We have that $g(z_\alpha, 0) = 0$, *because* $x(\cdot, z_\alpha, 0)$ *is T-periodic, and we shall prove that*

$$G_\alpha = \frac{dg}{dz}(z_\alpha, 0) = Y_\alpha^{-1}(0) - Y_\alpha^{-1}(T) \tag{2.34}$$

For this, we need to know $(\partial x/\partial z)(\cdot, z, 0)$. *Since it is the matrix solution of* (2.12) *with* $(\partial x/\partial z)(0,z,0) = In$, *we have that* $(\partial x/\partial z)(t,z,0) = Y(t,z)Y^{-1}(0,z)$. *Moreover,*

$$\frac{df}{dz}(z,0) = \frac{\partial x}{\partial z}(T,z,0) - I_n = Y(T,z)Y^{-1}(T,z) - I_n$$

and

$$\frac{dg}{dz}(z,0) = Y^{-1}(0,z) - Y^{-1}(T,z) + \left(\frac{\partial Y^{-1}}{\partial z_1} f(z,0), \ldots, \frac{\partial Y^{-1}}{\partial z_n} f(z,0)\right),$$

which, for $z_\alpha \in Z$, *reduces to* (2.34).

We have

$$\frac{\partial g}{\partial \varepsilon}(z,0) = Y^{-1}(T,z)\frac{\partial x}{\partial \varepsilon}(T,z,0).$$

The function $(\partial x/\partial \varepsilon)(\cdot, z, 0)$ *is the unique solution of the initial value problem*

$$y' = D_x F_0(t, x(t,z,0))y + F_1(t, x(t,z,0)), \quad y(0) = 0.$$

Hence,

$$\frac{\partial x}{\partial \varepsilon}(t,z,0) = Y(t,z)\int_0^t Y^{-1}(s,z)F_1(s, x(s,z,0))\,ds.$$

Now we have

$$\frac{\partial g}{\partial \varepsilon}(z,0) = \int_0^T Y^{-1}(s,z)F_1(s, x(s,z,0))\,ds.$$

and hence

$$\frac{\partial(\pi g)}{\partial \varepsilon}(z_\alpha, 0) = f_1(\alpha),$$

where f_1 *is given by* (2.13). *There exists* $\alpha_{epsilon} \in V$ *such that* $g(z_{\alpha_\varepsilon}, \varepsilon) = 0$ *and, further,* $f(z_{\alpha_\varepsilon}, \varepsilon) = 0$, *which assures that* $\varphi(\cdot, \varepsilon) = x(\cdot, z_{\alpha_\varepsilon}, \varepsilon)$ *is a T-periodic solution of* (2.10).

For the proof of Theorem 6, since the result of Theorem 6 is analogous to the result of Theorem 5, their proofs are similar.

Proof 11 (Proof of Theorem 6) *Since Z is a compact set and $x(t,z_\alpha)$ is $T-$periodic for each $z_\alpha \in Z$, there is an open neighborhood D of Z in Ω, and $0<\varepsilon_1 \le \varepsilon_0$ such that any solution $x(t,z,\varepsilon)$ of* (2.10) *with initial conditions in $D_x(-\varepsilon_1,\varepsilon_1)$ is well defined in $(0,T)$. We consider the function $L: D_x(-\varepsilon_1,\varepsilon_1) \to \mathbb{R}^{2m}$, $(z,\varepsilon_1) \to x(T,z,\varepsilon)-z$. If $(\overline{z},\overline{\varepsilon}) \in D\times(-\varepsilon_1,\varepsilon_1)$ is such that $L(\overline{z},\overline{\varepsilon})=0$, then $x(t,\overline{z},\overline{\varepsilon})$ is a $T-$periodic solution of $(2.10)_\varepsilon=\overline{\varepsilon}$. Clearly, the converse is also true. Hence, the problem of finding $T-$periodic orbits of* (2.10) *close to the periodic orbits with initial conditions in Z is reduced to finding the zeros of $L(x,\varepsilon)$.*

The sets of zeros of $L(x,\varepsilon)$ and $\tilde{L}(z,\varepsilon)=M_z^{-1}(T)L(z,\varepsilon)$ coincide, since $M_z(T)$ is a fundamental matrix. Moreover, following the proof of Theorem 1.2.9, we can compute that

$$D_z\tilde{L}(z,\varepsilon)= (M_z^{-1}(0)-M_z^{-1}(T))+D_z\Big(\int_0^T M_z^{-1}(t)F_1(t,x(t,z,0))dt\Big)\varepsilon +O(\varepsilon_2). \tag{2.35}$$

We note that $\tilde{L}^{-1}(0)=(\xi^\perp\circ\tilde{L})^{-1}(0)\cap(\xi\circ\tilde{L})^{-1}(0)$. From (2.35) *we obtain $D_z\tilde{L}(z_\alpha,0)=M^{-1}z_\alpha(0)-M^{-1}z_\alpha(T)$. If we write $z\in\mathbb{R}^{2m}$ as $z=(u,v)$ with $u,v\in\mathbb{R}^m$, then $D_v(\xi\circ\tilde{L})(z_\alpha,0)$ is the upper right corner of $M^{-1}z(0)-M^{-1}z(T)$. Then, from (i), we can apply the Implicit Function Theorem, deducing the existence of an open neighborhood $U\times(-\varepsilon_2,\varepsilon_2)$ of $C^l(V)$ in $\xi(D)\times(-\varepsilon_2,\varepsilon_2)$, an open neighborhood O of $\beta_0(C^l(V))$ in $\mathbb{R}^m$ and a unique C^k function $\beta(\alpha,\varepsilon): U\times(-\varepsilon_2,\varepsilon_2)\to O$ such that $(\xi^\perp\circ\tilde{L})^{-1}(0)\cap(U\times O\times(-\varepsilon_2,\varepsilon_2))$ is exactly the graphic of $\beta(\alpha,\varepsilon)$. Now, if we define the function $\delta: U\times(-\varepsilon_2,\varepsilon_2)\to\mathbb{R}$ as $\delta(\alpha,\varepsilon)=(\xi^\perp\circ\tilde{L})(\alpha,\beta(\alpha,\varepsilon),\varepsilon)$, then δ is a function of class C^k and $\tilde{L}^{-1}(0)\cap(U\times O\times(-\varepsilon_2,\varepsilon_2))=\{(\alpha,\beta(\alpha,\varepsilon),\varepsilon)|(\alpha,\varepsilon)\in\delta^{-1}(0)\}$. Therefore, to describe the set $\tilde{L}^{-1}(0)$ in an open neighborhood of Z in $\mathbb{R}^n\times(-\varepsilon_0,\varepsilon_0)$, it suffices to describe $\delta^{-1}(0)$ in an open neighborhood of $C^l(V)$ in $\mathbb{R}\times(-\varepsilon_0,\varepsilon_0)$.*

Since $M^{-1}z_\alpha(0)-M^{-1}z_\alpha(T)$ has in the lower right corner the $m\times m$ zero matrix and $\delta(\alpha,0)=0$ in $V\times(-\varepsilon_2,\varepsilon_2)$, the function $\delta(\alpha,\varepsilon)$ can be written as $\delta(\alpha,\varepsilon)=\varepsilon G(\alpha)+\varepsilon^2 G(\alpha,\varepsilon)$ in V Ö$(-\varepsilon 2,\varepsilon 2)$, where $G(\alpha)$ is the function given in (2.15)*. In addition, if $\tilde{\delta}(\alpha,\varepsilon)=G(\alpha)+\varepsilon\tilde{G}(\alpha,\varepsilon)$ then $\delta^{-1}(0)=\tilde{\delta}^{-1}(0)$.*

If there is $\alpha_0\in V$ such that $\tilde{\delta}(\alpha_0,0)=G(\alpha_0)=0$ and $det((\partial G/\partial\alpha)(\alpha_0))\neq 0$ then, from the Implicit Function Theorem, there exist $\varepsilon_3>0$ small, an open neighborhood V_0 of α_0 in V and a unique function $\alpha(\varepsilon):(-\varepsilon_3,\varepsilon_3)\to V_0$ of class C^k such that $\tilde{\delta}^{-1}(0)\cap(V_0\times(-\varepsilon_3,\varepsilon_3))$ is the graphic of $\alpha(\varepsilon)$, which *also represents the set* $\delta^{-1}(0)\cap(V_0\times(-\varepsilon_3,\varepsilon_3))$. *This completes the proof of the theorem.*

2.5 Averaging theory of arbitrary order and dimension for finding periodic solutions

In this section we shall study periodic solutions of systems of the form

$$x'(t)=\sum_{i=0}^{k}\varepsilon^i F_i(t,x)+\varepsilon k+1R(t,x,\varepsilon), \tag{2.36}$$

where $F_i:\mathbb{R}\times D\to\mathbb{R}^n$ for $i=0,1,\dots,k$, and $R:\mathbb{R}\times D\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^n$ are locally Lipschitz functions, being T-periodic in the first variable, and where D is an open subset of $\mathbb{R}^n$; eventually F_0 can be the zero constant function.

The classical works using the averaging theory for studying the periodic solutions of a differential system (2.36) usually only provide this theory up to first ($k=1$) or second order ($k=2$) in the small parameter ε. Moreover, these theories assume differentiability of the functions F_i and R up to class C^2 or C^3, respectively.

Recently, in [10], this averaging theory for computing periodic solutions was developed up to second order in dimension n, and up to third order ($k=3$) in dimension 1, only using that the functions F_i and R are locally Lipschitz.

Also, in the recent work [19], the averaging theory for computing periodic solutions was developed to an arbitrary order k in ε for analytical differential equations in dimension 1. In this section we shall develop the averaging theory for studying the periodic solutions of a differential system (2.36) up to arbitrary order k in dimension n, with zero or non-zero F_0, and where the functions F_i and R are only locally Lipschitz.

An example that qualitative new phenomena can be found only when considering higher order analysis is the following. Consider arbitrary polynomial perturbations

$$\begin{aligned}\dot{x}&=-y+\sum_{j\geq1}\varepsilon^j f_j(x,y),\\ \dot{y}&=x+\sum_{j\geq1}\varepsilon^j g_j(x,y),\end{aligned} \tag{2.37}$$

of the harmonic oscillator, where ε is a small parameter. In this differential system the polynomials f_j and g_j are of degree n in the variables x

and y, and the system is analytic in the variables x, y and ε. Then in [19] it is proved that system (2.37) for $\varepsilon = 0$ sufficiently small has no more than $[s(n-1)/2]$ periodic solutions bifurcating from the periodic solutions of the linear center $\dot{x} = -y$, $\dot{y} = x$, using the averaging theory up to order s, and this bound can be reached. Here, [.] denotes the integer part function. So, higher order averaging theory can improve the results on the periodic solutions, both qualitatively and quantitatively. In short, the goal of this section is to extend the averaging theory for computing periodic solutions of the differential system in n variables (2.36) up to an arbitrary order k in ε for locally Lipschitz differential systems, using the Brouwer degree.

2.5.1 Statement of the main results

We are interested in studying the existence of periodic orbits of general differential systems expressed by

$$x'(t) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) + \varepsilon k + 1 R(t,x,\varepsilon), \tag{2.38}$$

where $F_i : \mathbb{R} \times D \to \mathbb{R}^n$ for $i = 0, 1, \ldots, k$, and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions, being T -periodic in the first variable, and where D is an open subset of $\mathbb{R}^n$. In order to state our main results we introduce some notation. Let L be a positive integer, let $x = (x_1, \dot{,} x_n) \in D$, $t \in \mathbb{R}$ and $y_j = (y_{j1}, \ldots, y_{jn}) \in \mathbb{R}^n$ for $j = 1, \ldots, L$. Given $F : \mathbb{R} \times D \to \mathbb{R}^n$ a sufficiently smooth function, for each $(t,x) \in R \times D$ we denote by $\partial^L F(t,x)$ a symmetric L-multilinear map which is applied to a "product" of L vectors of $\mathbb{R}^n$, which we denote as $L \odot_{j=1}^{L} y_j \in \mathbb{R}^{nL}$. The definition of this L-multilinear map is

$$\partial^L F(t,x) \odot_{j=1}^{L} y_j = \sum_{i_1,\ldots,iL=1}^{n} \frac{\partial^L F(t,x)}{\partial x_{i1} \ldots \partial x_{il}} \tag{2.39}$$

We define ∂^0 as the identity functional. Given a positive integer b and a vector $y \in \mathbb{R}^n$, we also write $y^b = \odot_{i=1}^{b} y \in \mathbb{R}^{nb}$.

Remark 3 *The L-multilinear map defined in* (2.39) *is the L^{th} Fréchet t derivative of the function $F(t,x)$ with respect to the variable x. Indeed, for every fixed $t \in \mathbb{R}$, if we consider the function $F_t : D \to \mathbb{R}^n$ such that $F_t(x) = F(t,x)$, then $\partial^L F(t,x) = F_t^{(L)}(x) = \partial^L/\partial x^L F(t,x)$.*

Example 5 *To illustrate the above notation* (2.39), *we consider a smooth function $F : \mathbb{R} \times \mathbb{R}^2 \to R^2$. So, for $x = (x_1, x_2)$ and $y^1 = (y_1^1, y_2^1)$, we have*

$$\partial F(t,x)y^1 = \frac{\partial F}{\partial x_1}(t,x)y_1^1 + \frac{\partial F}{\partial x_2}(t,x)y_2^1$$

Now, for $y^1 = (y_1^1, y_2^1)$ and $y^2 = (y_1^2, y_2^2)$, we have

$$\begin{aligned}\partial^2 F(t,x)(y^1,y^2) = \;& \frac{\partial^2 F}{\partial x_1 \partial x_1}(t,x)y_1^1 y_1^2 + \frac{\partial^2 F}{\partial x_1 \partial x_2}(t,x)y_1^1 y_2^2 + \\ & \frac{\partial^2 F}{\partial x_2 \partial x_1}(t,x)y_2^1 y_1^2 + \frac{\partial^2 F}{\partial x_2 \partial x_2}(t,x)y_2^1 y_2^2.\end{aligned}$$

Observe that, for each $(t,x) \in \mathbb{R} \times D$, $\partial F(t,x)$ is a linear map in $\mathbb{R}^2$ and $\partial^2 F(t,x)$ is a bilinear map in $\mathbb{R}^2 \times \mathbb{R}^2$. Let $\varphi(.,z) : [0,t_z] \to \mathbb{R}^n$ be the solution of the unperturbed system

$$\dot{x} = F_0(t,x) \tag{2.40}$$

such that $\varphi(.,z) = z$. For $i = 1,2,\ldots,k$, we define the averaged function of order i, $f_i : D \to \mathbb{R}^n$, as

$$f_i = \frac{y_i(T,z)}{i!}, \tag{2.41}$$

where $y_i : \mathbb{R} \times D \to \mathbb{R}^n$, for $i = 1,2,\ldots,k-1$, are defined recurrently by the integral equation

$$\begin{aligned} y_i(t,z) = \;& i! \int_0^t F_i(s,\varphi(s,z)) + \sum_{l=1}^{i} \sum_{s_l} \frac{1}{b_1! b_2! 2! b_2 \ldots b_l! l! b_l} \\ & \partial^L F_{i-l}(s,\varphi(s,z)) \odot_{j=1}^{l} y_j(s,z)^{b_j} ds, \end{aligned} \tag{2.42}$$

where S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, \ldots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$.

In Subsection 2.5.2 we compute the sets S_l for $l = 1,2,3,4,5$. Furthermore, we make the functions $f_k(z)$ explicit, up to $k = 5$ when $F_0 = 0$, and up to $k = 4$ when $F_0 = 0$. Related to the averaging functions (2.41) *there exist two cases of* (2.38)*, essentially different, that must be treated separately, namely, when $F_0 = 0$ and when $F_0 = 0$. It can be seen in the following remarks.*

Remark 4 *If $F_0 = 0$, then $\varphi((t,z) = z$ for each $t \in \mathbb{R}$. So,*

$$y_1(t,z) = \int_0^t F_1(t,z)ds, \; and \; f_1(t,z) = \int_0^T F_1(t,z)dt,$$

as usual in averaging theory; see, for instance [1].

Remark 5 *If $F_0 = 0$, then*

$$y_1(t,z) = \int_0^t F_1(s,\varphi(s,z)) + \partial F_0(s,\varphi(s,z))\,y_1(s,z)\,ds \tag{2.43}$$

The integral equation (2.43) *is equivalent to the following Cauchy problem:*

$$\dot{u} = F_1(s,\varphi(s,z)) + \partial F_0(s,\varphi(s,z))\,u\,ds \text{ and } u(0) = 0 \tag{2.44}$$

i.e., $y_1(t,z) = u(t)$. *If we write*

$$\eta(t,z) = \int_0^t \partial F_0(s,\varphi(s,z))\,ds, \tag{2.45}$$

we have

$$y_1(t,z) = e^{\eta(t,z)} \int_0^t e^{-\eta(t,z)} F_1(s,\varphi(t,z))\,dt, \tag{2.46}$$

and

$$f_1(z) = \int_0^T e^{-\eta(t,z)} F_1(s,\varphi(t,z))\,dt,$$

Moreover, each $y_i(t,z)$ is obtained similarly from a Cauchy problem. The formulae are given explicitly in Subsection 2.5.2.

In the following, we state our main results: Theorem 8 when $F_0 = 0$, and Theorem 9 when $F_0 = 0$.

Theorem 8 *Suppose that $F_0 = 0$. In addition, for the functions of* (2.38), *we assume the following conditions:*

(i) for each $t \in \mathbb{R}, F_i(t,.) \in C^{k-i}$ for $i = 1,2,\ldots,k$; $\partial^{k-i}F_i$ is locally Lipschitz in the second variable for $i = 1,2,\ldots,k$; and R is continuous and locally Lipschitz in the second variable;

(ii) $f_i = 0$ for $i = 1,2,\ldots,r-1$ and $f_r = 0$, where $r \in 1,2,\ldots,k$ (here, we are taking $f_0 = 0$). Moreover, suppose that for some $a \in D$ with $f_r(a) = 0$, there exists a neighborhood $V \subset D$ of a such that $f_r(z) = 0$ for all $z \in \bar{V}|\{a\}$, and that $d_B(f_r(z),V,a) \neq 0$.

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T-periodic solution $x(.,\varepsilon)$ of (2.38) *such that $x(0,\varepsilon) \to a$ when $\varepsilon \to 0$.*

Theorem 9 *Suppose that $F_0 = 0$. In addition, for the functions of (2.38), we assume the following conditions:*

(i) there exists an open subset W of D such that, for any $z \in \bar{W}$, $\varphi(t,z)$ is T-periodic in the variable t;

(ii) for each $t \in \mathbb{R}, F_i(t,.) \in C^{k-i}$ for $i = 0,1,2,\ldots,k$; $\partial^{k-i}F_i$ is locally Lipschitz in the second variable for $i = 0,1,2,\ldots,k$; and $\mathbb{R}$ is continuous and locally Lipschitz in the second variable;

(iii) $f_i = 0$ for $i = 1,2,\ldots,r-1$ and $f_r = 0$, where $r \in 1,2,\ldots,k$; moreover, suppose that for some $a \in W$ with $f_r(a) = 0$, there exists a neighborhood $V \subset W$ of a such that $f_r(z) = 0$ for all $z \in \bar{V}|\{a\}$, and that $d_B(f_r(z), V, a) = 0$.

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T-periodic solution $x(.,\varepsilon)$ of (2.38) such that $x(0,\varepsilon) \to a$ when $\varepsilon \to 0$.

Remark 6 *When the functions f_i defined in (2.41), for $i = 1,2,\ldots,k$, are C^1, the hypotheses (ii) from Theorem 8 and (iii) from Theorem 9 become: (iv) $f_i = 0$ for $i = 1,2,\ldots,r-1$ and $f_r = 0$, where $r \in 1,2,\ldots,k$; moreover, suppose that for some $a \in W$ with $f_r(a) = 0$ we have that $f_r(a) = 0$.*

2.5.2 Computing formulae

Now we shall illustrate how to compute the formulae from Theorems 8 and 9 for some $k \in \mathbb{N}$. In Subsection 2.5.3 we compute the formulae when $F_0 = 0$ for Theorem 8 up to k = 5. First of all, from (2.42) we should determine the sets S_l for $l = 1,2,3,4,5$:

$$S_1 = \{1\},$$

$$S_2 = \{(0,1),(2,0)\},$$

$$S_3 = \{(0,0,1),(1,1,0),(3,0,0)\},$$

$$S_4 = \{(0,0,0,1),(1,0,1,0),(2,1,0,0),(0,2,0,0),(4,0,0,0)\}.$$

To compute S_l it is convenient to exhibit a table of possibilities with the value b_i in the column i. We start from the last column.

Clearly, the last column can be filled only by zeroes and ones because $5b_5 > 5$ for $b_5 > 1$; the same happens with the fourth and the third column, because $3b_3$, $4b_4 > 5$, for $b_3, b_4 > 1$. Taking $b_5 = 1$, the unique possibility is $b_1 = b_2 = b_3 = b_4 = 0$, thus any other solution satisfies $b_5 = 0$. Taking $b_5 = 0$ and $b_4 = 1$, the unique possibility is $b_1 = 1$ and $b_2 = b_3 = 0$, thus any other solution must have $b_4 = b_5 = 0$. Finally, taking $b_5 = b_4 = 0$ and $b_3 = 1$, we have two possibilities either $b_1 = 2$ and $b_2 = 0$, or $b_1 = 0$ and $b_2 = 1$. Thus any other solution satisfies $b_3 = b_4 = b_5 = 0$.

Now we observe that the second column can be filled only by $0, 1$ or 2, since $2b_2 > 5$ for $b_2 > 2$; and taking $b_3 = b_4 = b_5 = 0$ and $b_2 = 1$ the unique possibility is $b_1 = 3$. Taking $b_3 = b_4 = b_5 = 0$ and $b_2 = 2$ the unique possibility is $b_1 = 1$, thus any other solution satisfies $b_2 = b_3 = b_4 = b_5 = 0$. Finally, taking $b_2 = b_3 = b_4 = b_5 = 0$ the unique possibility is $b_1 = 5$. Therefore the complete table of solutions is

$S_5=$

b_1	b_2	b_3	b_4	b_5
0	0	0	0	1
1	0	0	1	0
0	1	1	0	0
2	0	1	0	0
3	1	0	0	0
1	2	0	0	0
5	0	0	0	0

Now we can use (2.42) and (2.41) to compute the expressions of the y_i's and f_i's in each case.

2.5.3 Fifth order averaging of Theorem 8

Let us assume that $F_0 \equiv 0$. From (2.42) we obtain the functions $y_i(t,z)$ for $k = 1,2,3,4,5$

$$
\begin{aligned}
y_1(t,z) = & \int_0^t F_1(s,z)ds, \\
y_2(t,z) = & \int_0^t \left(2F_2(s,z) + 2\frac{\partial F_1}{\partial x}(s,z)y_1(s,z)\right)ds, \\
y_3(t,z) = & \int_0^t \left(6F_3(s,z) + 6\frac{\partial F_2}{\partial x}(s,z)y1(t,z)\right. \\
& \left. +3\frac{\partial^2 F_1}{\partial x^2}(s,z)y1(s,z)^2 + 3\frac{\partial F_1}{\partial x}(s,z)y_2(s,z)\right)ds, \\
y_4(t,z) = & \int_0^t \left(24F_4(s,z) + 24\frac{\partial F_3}{\partial x}(s,z)y_1(s,z) + 12\frac{\partial^2 F_2}{\partial x^2}(s,z)y_1(s,z)^2\right. \\
& +12\frac{\partial F_2}{\partial x}(s,z)y_2(s,z) + 12\frac{\partial^2 F_1}{\partial x^2}(s,z)y_1(s,z) \odot y2(s,z) \\
& \left. +4\frac{\partial^3 F_1}{\partial x^3}(s,z)y_1(s,z)^3 + 4\frac{\partial F_1}{\partial x}(s,z)y_3(s,z)\right)ds, \\
y_5(t,z) = & \int_0^t \left(120F_5(s,z) + 120\frac{\partial F_4}{\partial x}(s,z)y_1(s,z) + 60\frac{\partial^2 F_3}{\partial x^2}(s,z)y_1(s,z)^2\right. \\
& +60\frac{\partial F_3}{\partial x}(s,z)y_2(s,z) + 60\frac{\partial^2 F_2}{\partial x^2}(s,z)y_1(s,z) \odot y_2(s,z) \\
& +20\frac{\partial^3 F_2}{\partial x^3}(s,z)y_1(s,z)^3 + 20\frac{\partial F_2}{\partial x}(s,z)y_3(s,z) \\
& +20\frac{\partial^2 F_1}{\partial x^2}(s,z)y_1(s,z) \odot y_3(s,z) + 15\frac{\partial^2 F_1}{\partial x^2}(s,z)y_2(s,z)^2 \\
& +30\frac{\partial^3 F_1}{\partial x^3}(s,z)y_1(s,z)^2 \odot y_2(s,z) + 5\frac{\partial^4 F_1}{\partial x^4}(s,z)y_1(s,z)^4 \\
& \left. +5\frac{\partial F_1}{\partial x}(s,z)y_4(s,z)\right)ds,
\end{aligned}
$$

Therefore, from (2.41) we have that

$$
\begin{aligned}
f_0(z) =& \ 0,\\
f_1(z) =& \int_0^T F_1(t,z)dt,\\
f_2(z) =& \int_0^T \Big(F_2(t,z)ds + \frac{\partial F_1}{\partial x}(t,z)y_1(t,z)\Big)dt,\\
f_3(z) =& \int_0^T \Big(F_3(t,z) + \frac{\partial F_2}{\partial x}(t,z)y_1(t,z) + \frac{1}{2}\frac{\partial^2 F_1}{\partial x^2}(t,z)y_1(t,z)^2\\
&+\frac{1}{2}\frac{\partial F_1}{\partial x}(t,z)y_2(t,z)\Big)dt,\\
f_4(z) =& \int_0^T \Big(F_4(t,z) + \frac{\partial F_3}{\partial x}(t,z)y_1(t,z) + \frac{1}{2}\frac{\partial^2 F_2}{\partial x^2}(t,z)y_1(t,z)^2\\
&+\frac{1}{2}\frac{\partial F_2}{\partial x}(t,z)y_2(t,z) + \frac{1}{2}\frac{\partial^2 F_1}{\partial x^2}(t,z)y_1(t,z)\odot y2(t,z)\\
&+\frac{1}{6}\frac{\partial^3 F_1}{\partial x^3}(t,z)y_1(t,z)^3 + \frac{1}{6}\frac{\partial F_1}{\partial x}(t,z)y_3(t,z)\Big)dt,\\
f_5(z) =& \int_0^T \Big(F_5(t,z) + \frac{\partial F_4}{\partial x}(t,z)y_1(t,z) + \frac{1}{2}\frac{\partial^2 F_3}{\partial x^2}(t,z)y_1(t,z)^2\\
&+\frac{1}{2}\frac{\partial F_3}{\partial x}(t,z)y_2(t,z) + \frac{1}{2}\frac{\partial^2 F_2}{\partial x^2}(t,z)y_1(t,z)\odot y_2(t,z)\\
&+\frac{1}{6}\frac{\partial^3 F_2}{\partial x^3}(t,z)y_1(t,z)^3 + \frac{1}{6}\frac{\partial F_2}{\partial x}(t,z)y_3(t,z)\\
&+\frac{1}{6}\frac{\partial^2 F_1}{\partial x^2}(t,z)y_1(t,z)\odot y_3(t,z) + \frac{1}{8}\frac{\partial^2 F_1}{\partial x^2}(t,z)y_2(t,z)^2\\
&+\frac{1}{4}\frac{\partial^3 F_1}{\partial x^3}(t,z)y_1(t,z)^2\odot y_2(t,z) + \frac{1}{24}\frac{\partial^4 F_1}{\partial x^4}(t,z)y_1(t,z)^4\\
&+\frac{1}{24}\frac{\partial F_1}{\partial x}(t,z)y_4(t,z)\Big)dt.
\end{aligned}
$$

2.6 Three applications of Theorem 8

The first application studies the periodic solutions of the Hénon–Heiles Hamiltonian using the averaging theory of second order. The other two examples analyze the limit cycles of some classes of polynomial differential systems in the plane. These last two applications use the averaging theory of third order. More precisely, these three applications are based in Theorem 8.

In the next subsection we summarize the results of Theorem 8 up to third order, precisely the ones used in the applications here considered.

2.6.1 The averaging theory of first, second and third order

As far as we know, the averaging theory of third order for studying specifically periodic orbits was developed by first time in [10]. Now we summarize it here from Theorem 8 which is given at any order. Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 F_3(t,x) + \varepsilon^4 R(t,x,\varepsilon), \tag{2.47}$$

where $F_1, F_2, F_3 : \mathbb{R}\times D \to \mathbb{R}$ and R: $\mathbb{R}\times D\times(-\varepsilon_f,\varepsilon_f) \to \mathbb{R}$ are continuous functions, T -periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) and (ii) hold:
(i) $F_1(t,.) \in C^2(D)$, $F_2(t,.) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, F_3, R, $D_x^2 F_1, D_x F_2$ are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε. We define $F_{k0} : D \to \mathbb{R}$ for $k = 1,2,3$ as

$$\begin{aligned}
f_{10}(z) = &\ \frac{1}{T}\int_0^T F_1(s,z)ds,\\
f_{20}(z) = &\ \frac{1}{T}\int_0^T [D_z F_1(s,z).y_1(s,z) + F_2(s,z)]ds,\\
f_{30}(z) = &\ \frac{1}{T}\int_0^T \Big[\frac{1}{2}y_1(s,z)^T\frac{\partial^2 F_1}{\partial z^2}(s,z)y_1(s,z) + \frac{1}{2}\frac{\partial F_1}{\partial z}(s,z)y_2(s,z)+\\
&\ \frac{\partial F_2}{\partial z}(s,z)(y_1(s,z)) + F_3(s,z)\Big]ds,
\end{aligned}$$

Where

$$\begin{aligned}
y_1(s,z) = &\ \int_0^s F_1(t,z)dt,\\
y_2(s,z) = &\ \int_0^s \Big[\frac{\partial F_1}{\partial z}(t,z)\int_0^t F_1(r,z)dr + F_2(t,z)\Big]dt.
\end{aligned}$$

(ii) For $V \subset D$ an open and bounded set, and for each $\varepsilon \in (-\varepsilon_f,\varepsilon_f)$ $\{0\}$ there exists a $\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) + \varepsilon^2 F_{30}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20}$
$+\varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$.
Then for $|\varepsilon| > 0$ sufficiently small there exists a T-periodic solution $\varphi(.,\varepsilon)$ of the system such that $\varphi(0,\varepsilon) = a_\varepsilon$.

The expression $d_B(F_{10}+\varepsilon F_{20}+\varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10}+\varepsilon F_{20}+\varepsilon^2 F_{30} : V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10}+\varepsilon F_{20}+\varepsilon^2 F_{30}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10}+\varepsilon F_{20}+\varepsilon^2 F_{30}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case, the previous result provides the averaging theory of first order.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10}+\varepsilon F_{20}+\varepsilon^2 F_{30}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case, the previous result provides the averaging theory of second order.

If F_{10} and F_{20} are both identically zero and F_{30} is not identically zero, then the zeros of $F_{10}+\varepsilon F_{20}+\varepsilon^2 F_{30}$ are mainly the zeros of F_{30} for ε sufficiently small. In this case, the previous result provides the averaging theory of third order.

2.6.2 The Hénon–Heiles Hamiltonian

The results presented in this subsection have been proved by Jiménez–Llibre [24].

The classical Hénon–Heiles potential consists of a two dimensional harmonic potential plus two cubic terms. It was introduced in 1964, as a model for studying the existence of a third integral of motion of a star in a rotating meridian plane of a galaxy in the neighborhood of a circular orbit [23]. The classical Hénon–Heiles potential has been generalized by introducing two parameters to each cubic term,

$$\frac{1}{2}(p_x^2+p_y^2+x^2+y^2)+Bxy^2+\frac{1}{3}Ax^3, \tag{2.48}$$

such that $B \neq 0$, with $x, y, p_x, p_y \in \mathbb{R}$. Then the classical Hénon–Heiles Hamiltonian system corresponds to

$$\begin{aligned} \dot{x} &= p_x, \\ \dot{p}_x &= -x-(Ax^2+By^2), \\ \dot{y} &= p_y, \\ \dot{p}_y &= -y-2Bxy. \end{aligned} \tag{2.49}$$

As usual, the dot denotes derivative with respect to the independent

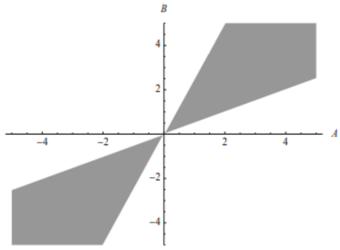

Figure 2.1: Open region $(2B-5A)(2B-A) < 0$ in the parameter space (A, B), where there is at least one periodic orbit with multipliers different from 1.

variable $t \in \mathbb{R}$, the time. We name (2.49) the Hénon–Heiles Hamiltonian systems with two parameters, or simply the Hénon–Heiles systems.

The periodic orbits in the Hénon–Heiles potential have been numerically studied and classified by Churchill–Pecelli–Rod [12], Davies–Huston–Barange [13] and others [9]. Maciejewski–Radzki–Rybicki did an analytical study of a more general Hénon–Heiles Hamiltonians including a third cubic term of the form Cx^2y, which can be removed by a proper rotation, and two more parameters associated with the quadratic part of the potential. They proved the existence of connected branches of non-stationary periodic orbits in the neighborhood of a given degenerate stationary point.

Theorem 10 *At every positive energy level the H´enon–Heiles Hamiltonian system* (2.49) *has at least*

(i) one periodic orbit if $(2B-5A)(2B-A) < 0$ *(see Figure 2.1),*

(ii) two periodic orbits if $A+B=0$ *and* $A \neq 0$ *(this case contains the classical Hénon–Heiles system), and*

(iii) three periodic orbits if $B(2B-5A) >$ *and* $A+B \neq 0$ *(see Figure 2.2).*

Proof 12 *For proving this theorem we shall apply Theorem 8 to the Hamiltonian system* (2.49)*. The periodic orbits of a Hamiltonian system with more than one degree of freedom are on cylinders fulfilled by periodic orbits. Then we must apply Theorem 8 to every Hamiltonian fixed level, where the periodic orbits are isolated.*

On the other hand, in order to apply Theorem 8 we need a small parameter ε. So in the Hamiltonian system (2.49) we change the variables

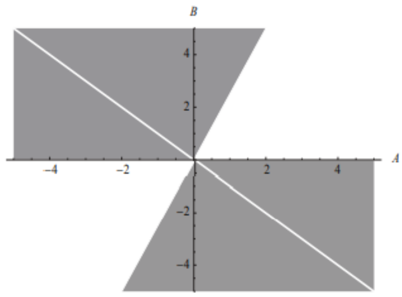

Figure 2.2: Open region $B(2B-5A)>0$ and $A+B\neq 0$ in the parameter space (A,B), where there are at least three periodic orbits with multipliers different from 1. When $A+B=0$, there are at least two periodic orbits with multipliers different from 1.

(x,y,p_x,p_y) to (X,Y,p_X,p_Y) where $x=\varepsilon X$, $y=\varepsilon Y$, $p_x=\varepsilon p_X$ and $p_y=\varepsilon p_Y$. In the new variables, the system (2.49) becomes

$$
\begin{aligned}
\dot{X} &= p_X,\\
\dot{p}_X &= -X-\varepsilon(AX^2+BY^2),\\
\dot{Y} &= p_Y,\\
\dot{p}_Y &= -Y-2\varepsilon BXY.
\end{aligned}
\tag{2.50}
$$

This system again is Hamiltonian with Hamiltonian

$$
\frac{1}{2}(p_X^2+p_Y^2+X^2+Y^2)+\varepsilon\left(BXY^2+\frac{1}{3}AX^3\right). \tag{2.51}
$$

As the change of variables is only a scale transformation, for all ε different from zero, the original and the transformed systems (2.49) and (2.50) have essentially the same phase portrait and, additionally, the system (2.50) for ε sufficiently small is close to an integrable one.

First we change the Hamiltonian (2.51) and the equations of motion (2.51) to polar coordinates for $\varepsilon=0$, which is an harmonic oscillator. Thus we have

$$
X=rcos\theta, p_X=rsin\theta, Y=\rho cos(\theta+\alpha), p_Y=\rho sin(\theta+\alpha).
$$

Recall that this is a change of variables when $r>0$ and $\rho>0$. Moreover, doing this change of variables, the angular variables θ and α appear in the system. Later on, the variable θ will be used for obtaining the periodicity necessary for applying the averaging theory.

The fixed value of the energy in polar coordinates is

$$
h=\frac{1}{2}(r^2+\rho^2)+\varepsilon\left(\frac{1}{3}Ar^3cos^3\theta+Br\rho^2cos\theta cos^2(\theta+\alpha)\right), \tag{2.52}
$$

and the equations of motion are given by

$$
\begin{aligned}
\dot{r} =&\quad -\varepsilon sin\theta\left(Ar^2cos^2\theta + Br\rho cos^2(\theta+\alpha)\right),\\
\dot{\theta} =&\quad -1-\varepsilon cos\theta\left(AXrcos^2\theta + \frac{\rho^2}{r}Bcos^2(\theta+\alpha)\right),\\
\dot{\rho} =&\quad -\varepsilon Brcos\theta sin(2(\theta+\alpha)),\\
\dot{\alpha} =&\quad \varepsilon\frac{cos\theta}{r}\left(Ar^2cos^2\theta + B(\rho^2-2r^2)cos^2(\theta+\alpha)\right).
\end{aligned}
\tag{2.53}
$$

However, the derivatives of the left hand side of these equations are with respect to the time variable t, which is not periodic. We change to the θ variable as the independent one, and we denote by a prime the derivative with respect to θ. The angular variable α cannot be used as the independent variable since the new differential system would not have the form (2.36) for applying Theorem 8 The system (2.53) goes over to

$$
\begin{aligned}
\dot{r} =&\quad \frac{\varepsilon r sin\theta\left(Ar^2cos^2\theta + B\rho^2cos^2(\theta+\alpha)\right)}{r+\varepsilon(Ar^2cos^3\theta + B\rho^2cos\theta cos^2(\theta+\alpha))},\\
\dot{\rho} =&\quad \frac{\varepsilon Br^2\rho cos\theta sin(2(\theta+\alpha))}{r+\varepsilon(Ar^2cos^3\theta + B\rho^2cos\theta cos^2(\theta+\alpha))},\\
\dot{\mathscr{A}} =-&\quad \frac{\varepsilon cos\theta\left(B\left(\rho^2-2r^2\right)cos^2(\theta+\alpha) + Ar^2cos^2\theta\right)}{r+\varepsilon(B\rho^2cos\theta cos^2(\theta+\alpha) + Ar^2cos^3\theta)}.
\end{aligned}
$$

Of course this system has now only three equations because we do not need the θ equation. If we write the previous system as a Taylor series in powers of ε, we have

$$
\begin{aligned}
\dot{r} =&\quad \varepsilon r sin\theta\left(Ar^2cos^2\theta + B\rho^2cos^2(\theta+\alpha)\right),\\
&-\varepsilon^2\frac{sin2\theta}{8r}\left(Ar^2(1+cos(2\theta)) + B\rho^2(1+cos(2(\theta+\alpha))\right)^2 + O(\varepsilon^3),\\
\dot{\rho} =&\quad \varepsilon Br\rho cos\theta sin(2(\theta+\alpha))\\
&\varepsilon^2B\rho cos^2\theta sin(2(\theta+\alpha))(Ar^2cos2\theta + B\rho^2cos^2(2(\theta+\alpha))) + O(\varepsilon^3),\\
\dot{\mathscr{A}} =-&\quad \varepsilon cos\theta r(Ar^2cos^2\theta + B(\rho^2-2r^2)cos^2(\theta+\alpha)).\\
&+\varepsilon^2cos^2\theta r^2(Ar^2cos^2\theta + B\rho^2cos^2(\theta+\alpha))\\
&(Ar^2cos^2\theta + B(\rho^2-2r^2)cos^2(\theta+\alpha)) + O(\varepsilon^3).
\end{aligned}
\tag{2.54}
$$

Now system (2.54) is 2π-periodic in the variable θ. In order to apply Theorem 8 we must fix the value of the first integral at $h>0$ and, by solving equation (2.52) for ρ, we obtain

$$\rho=\sqrt{\frac{h-r^2/2-\varepsilon Ar^3cos^3\theta/3}{1/2+\varepsilon Brcos\theta cos^2(\theta+\alpha)}} \tag{2.55}$$

$$\begin{aligned}\dot r=\;&\varepsilon sin\theta(Ar^2cos^2\theta+B(2h-r^2)cos^2(\theta+\alpha))\\&-\varepsilon^2\Big(\frac{sin2\theta}{8r}Ar^2(1+cos(2\theta))+B(2h-r^2)(1+cos(2(\theta+\alpha))))^2\\&+\frac{2}{3}ABr^3sin\theta cos^3\theta cos^2(\theta+\alpha)\\&+2B^2hrsin(2\theta)cos^4(\theta+\alpha)-B^2r^3sin(2\theta)cos^4(\theta+\alpha)+O(\varepsilon^3),\end{aligned} \tag{2.56}$$

and

$$\begin{aligned}\dot\alpha=\;&\varepsilon\Big(\frac{B}{r}(3r^2-2h)cos\theta cos^2(\theta+\alpha)-Arcos^3\theta\Big)\\&\varepsilon^2(A^2r^2cos^6\theta+\frac{2}{3}AB(6h-5r^2)cos^4\theta cos^2(\theta+\alpha)\\&B^2r^2(r^2-2h)2cos^2\theta cos^4(\theta+\alpha))+O(\varepsilon^3).\end{aligned} \tag{2.57}$$

Clearly, equations (2.56) and (2.56) satisfy the assumptions of Theorem 8, and it has the form of (2.36) with $F_1=(F_{11},F_{12})$ and $F_2=(F_{21},F_{22})$, where

$$F_{11}=sin\theta\left(Ar^2cos^2\theta+B(2h-r^2)cos^2(\theta+\alpha)\right),$$

$$F_{12}=\frac{B}{r}(3r^2-2h)cos\theta cos^2(\theta+\alpha)-Arcos^3\theta,$$

and

$$\begin{aligned}F_{21}=\;&-\frac{sin2\theta}{8r}\Big(Ar^2(1+cos(2\theta))+B(2h-r^2)(1+cos(2(\theta+\alpha)))\Big)^2,\\&-\frac{2}{3}ABr^3sin\theta cos^3\theta cos^2(\theta+\alpha)-2B^2hrsin(2\theta)cos^4(\theta+\alpha)\\&+B^2r^3sin(2\theta)cos^4(\theta+\alpha),\\F_{22}=\;&A^2r^2cos^6\theta+\frac{2}{3}AB(6h-5r^2)cos^4\theta cos^2(\theta+\alpha)\\&+\frac{B^2}{r^2}(r^2-2h)^2cos^2\theta cos^4(\theta+\alpha).\end{aligned}$$

As $r\neq0$ the functions F_1 and F_2 are analytical. Furthermore, they are $2pi$-periodic in the variable θ, the independent variable of the system

(2.56) -(2.57). However, the averaging theory of first order does not apply because the average functions of F_1 and F_2 in the period vanish,

$$f_1(r,A) = \int_0^{2\pi} (F_{11}, F_{12}) d\theta = (0,0). \tag{2.58}$$

As the function f_1 from Theorem 8 is zero, we procede to calculate the function f_2 by applying the second order averaging theory. We have that f_2 is defined by

$$f_2(r,A) = \int_0^{2\pi} [D_{rA}F_1(\theta,r,A).y_1(\theta,r,A) + F_2(\theta,r,A)] d\theta, \tag{2.59}$$

where

$$y_1(\theta,r,A) = \int_0^{\theta} F_1(t,r,A)dt.$$

The two components of the vector y_1 are

$$\begin{aligned} y_{11} = & \int_0^{\theta} F_{11}(t,r,A)dt \\ & = \frac{1}{3}\Big(B(2h-r^2)sin^2(\theta/2)\Big(cos(2(\theta+\alpha)) + 2cos(2\alpha+\theta)+3\Big) \\ & -Ar^2(cos^3\theta - 1)\Big), \end{aligned}$$

and

$$\begin{aligned} y_{12} = & \int_0^{\theta} F_{12}(t,r,A)dt \\ & = -\frac{Ar}{12}(9sin\theta + sin3\theta) - Bh6r(3sin(2\alpha+\theta) + sin(2\alpha+3\theta) \\ & -4sin2\alpha + 6sin\theta) \\ & +\frac{Br}{4}(3sin(2\alpha+\alpha) + sin(2\alpha+3\theta) - 4sin(2\alpha) + 6sin\theta). \end{aligned}$$

For the Jacobian matrix

$$D_{r\mathscr{A}}F_1(\theta,r,\mathscr{A}) = \begin{pmatrix} \dfrac{\partial F_{11}}{\partial r} & \dfrac{\partial F_{11}}{\partial \mathscr{A}} \\ \dfrac{\partial F_{12}}{\partial r} & \dfrac{\partial F_{12}}{\partial \mathscr{A}} \end{pmatrix},$$

we obtain

$$\begin{pmatrix} 2Arcos^2\theta - 2Brcos^2(\theta+\alpha))sin\theta & -2B(2h-r^2)cos(\theta+\alpha)sin\theta sin(\theta+\alpha) \\ -Acos3\theta + 6Bcos2(\theta+\alpha)cos\theta & \dfrac{2B}{r}(3r^2-2h)cos\theta cos(\theta+\alpha)sin(\theta+\alpha) \\ \dfrac{B}{r^2}(3r^2-2h)cos^2(\theta+\alpha)cos\theta & \end{pmatrix}$$

We can now calculate the function (2.59) from Theorem 8, and we obtain

$$f_2 = \Big(-\frac{Br}{12}(6B-A)(r^2-2h)sin2\mathscr{A},$$
$$= \frac{1}{12}\Big(r^2(5A^2-12AB-3B^2)-2B(A-6B)(h-r^2)cos(2\alpha)+$$
$$2Bh(6A-B)\Big).$$

We have to find the zeros $(r^*,\mathscr{A}^*)$ of $f_2(r,\mathscr{A})$, and to check that the Jacobian determinant

$$|D_{r,A}f_2(r^*,\mathscr{A}^*)| \neq 0. \tag{2.60}$$

Solving the equation $f_2(r,\mathscr{A}) = 0$, we obtain five solutions $(r^*,\mathscr{A}^*)$ with $r^* > 0$, namely

$$\left(\sqrt{2}h, \pm arcsec\frac{B(A-6B)}{4B^2+6AB-5A^2}\right), \left(\sqrt{\frac{2Bh}{3B-A}}.0\right),$$
$$\left(\sqrt{\frac{14Bh}{9B-5A}}, \pm\pi/2\right). \tag{2.61}$$

The first two solutions are not good, because for them we would get from (2.55) that $\rho = 0$ when $\varepsilon = 0$, and ρ must be positive. The third solution exists if $B(3B-A) > 0$. The last two solutions exist if $B(9B-5A) > 0$. The Jacobian (2.60) of the third solution is

$$-\frac{5B^2h^2(A-6B)(A-2B)(A+B)}{9(A-3B)} \tag{2.62}$$

and, for the last two solutions, the Jacobian coincides and is equal to

$$\frac{7B^2h^2(A-6B)(5A-2B)(A-B)}{9(5A-9B)}. \tag{2.63}$$

Summarizing, from Theorem 8 the third solution of $f_2(r,\mathscr{A}) = 0$ provides a periodic orbit for the system (2.56)-(2.57) (and consequently of the Hamiltonian system (2.50) on the Hamiltonian level $h > 0$) if $B(3B-A) > 0$, $(A-6B)(A-2B)(A+B) \neq 0$, and from (2.55) we get $\rho = \sqrt{2(A-2B)h/(A-3B)}$; we also need $(2B-A)(3B-A) > 0$. The conditions $B(3B-A) > 0$ and $(2B-A)(3B-A) > 0$ can be reduced to $B(2B-A) > 0$, where $(A-6B)(A-2B) \neq 0$ is included, but $A+B \neq 0$ is not. Then the third solution provides a periodic orbit when $B(2B-A) > 0$ and $A+B \neq 0$.

In a similar way the last two solutions of $f_2(r,A) = 0$ provide two periodic orbits for the system (2.56)-(2.57) if $B(9B-5A) > 0$, $(A-6B)(5A-2B)(A-B) \neq 0$, and from (2.55) we get $\rho = \sqrt{2(5A-2B)h/(5A-9B)}$; we also need $(2B-5A)(9B-5A) > 0$. The conditions $B(9B-5A) > 0$ and $(2B-5A)(9B-5A) > 0$ can be reduced to $B(2B-5A) > 0$, where the condition $(A-6B)(5A-2B)(A-B) \neq 0$ is included. Then the fourth and fifth solutions provide two periodic orbits whenever $B(2B-5A) > 0$.

There is one periodic orbit if the third solution exists, and the last two solutions do not. There are two periodic orbits if the two last solutions exist, and not the third one, i.e., when $A+B = 0$. Finally, there are three periodic orbits if the third, fourth and fifth solutions exist. Now the statements of Theorem (8) follow easily. The regions in the parameter space where periodic orbits exist are summarized in Figures 2.1 and 2.2.

Chapter 3

Applications

We recall that a limit cycle of a differential system is a periodic orbit isolated in the set of all periodic orbits of the system.

3.1 The van der Pol differential equation

Consider the van der Pol differential equation $\ddot{x}+x=\varepsilon(1-x^2)\dot{x}$, which can be written as the differential system

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=-x+\varepsilon(1-x^2).\end{aligned}\tag{3.1}$$

In polar coordinates (r,θ), where $x=r\cos\theta$, $y=r\sin\theta$, this system becomes

$$\begin{aligned}\dot{r}&=\varepsilon r(1-r^2\cos^2\theta)\sin^2\theta,\\ \dot{\theta}&=-1+\varepsilon\cos\theta(1-r^2\cos^2)\sin\theta.\end{aligned}$$

or, equivalently,

$$\frac{dr}{d\theta}=-\varepsilon r(1-r^2\cos^2\theta)\sin^2\theta+\bigcirc(\varepsilon^2)$$

Note that the previous differential system is in the normal form (2.3) for applying the averaging theory described in Theorem 4 if we take $x=r, t=\theta, T=2\pi$. From (2.5) we get that

$$f^0(r)=-\frac{1}{2\pi}\int_0^{2\pi} r(1-r^2\cos^2)\sin^2\theta\, d\theta=\frac{1}{8}r(r^2-4).$$

The unique positive root of $f^0(r)$ is $r = 2$. Since $(df^0/dr)(2) = 1$, by Theorem $4(i)$, it follows that system (3.1) has, for $\varepsilon = 0$ sufficiently small, a limit cycle bifurcating from the periodic orbit of radius 2 of the unperturbed system (3.1) with $\varepsilon = 0$. Moreover, since $(df^0/dr)(2) = 1 > 0$, by Theorem $4(ii)$, this limit cycle is unstable.

3.2 The Lienard differential system

The following result is due to Lins–de Melo–Pugh [25]. Here, we provide an easy and shorter proof with respect to the initial proof given by the mentioned authors

Proposition 2 *The Lienard differential systems of the form*

$$\dot{x} = y - \varepsilon(a_1 x + \cdots + a_n x^n),$$
$$\dot{y} = -x.$$

with ε sufficiently small and $a_n \neq 0$ have at most $[(n-1)/2]$ limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ and there are examples with exactly $[(n-1)/2]$ limit cycles; here $[.]$, denotes the integer part function

Proof 13 *We write the system*

$$\dot{x} = y - \varepsilon(a_1 x + \cdots + a_n x^n) \quad \dot{y} = -x$$

in polar coordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$, and we obtain

$$\dot{r} = -\varepsilon \Sigma_{k=1}^{n} a_k r^k \cos^{k+1}\theta,$$
$$\dot{\theta} = -1 + \varepsilon \sin\theta + \Sigma_{k=1}^{n} a_k r^{k-1} \cos^k\theta.$$

or, equivalently,

$$\frac{dr}{d\theta} = -\varepsilon \Sigma_{k=1}^{n} a_k r^k \cos^{k+1}\theta + \bigcirc(\varepsilon^2)$$

Again, taking $x = r$, $t = \theta$, $T = 2\pi$ and $F(t,x) = -\Sigma_{k=1}^{n} a_k r^k \cos^{k+1}\theta$, the previous he previous differential system is in the normal form (2.1) for applying the averaging theory described in Theorem 4

We have that

$$f^0(r) = -\frac{1}{2\pi}\Sigma_1^n a_k r^k \int_0^{2\pi} \cos^{k+1}\theta d\theta = -\frac{\varepsilon}{2\pi}\Sigma_{k=1,kodd}^{n} a_k b_k r^k = \rho(r)$$

where $b_k = \displaystyle\int_0^{2\pi} \cos^{k+1}\theta d\theta \neq 0$ if k is odd, and $b_k = 0$ if k is even. Now we apply Theorem 4 since the polynomial $p(r)$ has at most $[(n-1)/2]$ positive roots, and we can choose the coefficients a_k with k odd in such a way that $p(r)$ has exactly $[(n-1)/2]$ simple positive roots; the proposition follows.

3.3 Zero-Hopf bifurcation in $\mathbb{R}^n$

In this example we study a zero-Hopf bifurcation of C^3 differential systems in $\mathbb{R}^n$ with $n \geqslant 3$.

We assume that these systems have a singularity at the origin, whose linear part has eigenvalues $\varepsilon a \pm b_i$, with $b \neq 0$ and εc_k for $k = 3, \cdots, n$, where $\varepsilon = 0$ is a small parameter. Since the eigenvalues of the linearization at the origin when $\varepsilon = 0$ are $\pm b_i \neq 0$ and 0 with multiplicity $n-2$, if a_n infinitesimal periodic orbit bifurcates from the origin when $\varepsilon = 0$, we call such kind of bifurcation a zero-Hopf bifurcation. Such systems can be written into the form

$$\begin{aligned}
\dot{x} &= \varepsilon a x - b y + \Sigma_{i_1+\cdots+i_n=2} a_{i_1\cdots i_n} x^{i_1} y^{i_2} z^{i_3} \cdots z^{i_n} + \mathscr{A}, \\
\dot{y} &= b x + \varepsilon a y + \Sigma_{i_1+\cdots+i_n=2} b_{i_1\cdots i_n} x^{i_1} y^{i_2} z^{i_3} \cdots z^{i_n} + \mathscr{B}, \\
\dot{z} &= \varepsilon c_k z_k + \Sigma_{i_1+\cdots+i_n=2} c^{(k)}_{i_1\cdots i_n} x^{i_1} y^{i_2} z^{i_3} \cdots z^{i_n} + \mathscr{C}_k \quad 3, \cdots, n,
\end{aligned} \tag{3.2}$$

where $a_{i_1\ldots i_n}$, $b_{i_1\ldots i_n}$, $c^{(k)}_{i_1\ldots i_n}$, a, b and c^k are real parameters, $ab \neq 0$, and A, B and C^k are the Lagrange expression of the error function of third order in the expansion of the functions of the system in Taylor series.

Theorem 11 *There exist C^3 systems* (3.2) *for which $l \in \{0, 1, \cdots 2^{n-3}\}$ limit cycles bifurcate from the origin at $\varepsilon = 0$, i.e., for ε sufficiently small the system has exactly l limit cycles in a neighborhood of the origin, and these limit cycles tend to the origin when $\varepsilon \searrow 0$.*

As far as we know, Theorem 11 was the first result proving that the number of limit cycles that can bifurcate in a Hopf bifurcation increases exponentially with the dimension of the space. We recall that a Hopf bifurcation takes place when one or several limit cycles bifurcate from an equilibrium point. From the proof of Theorem 11 we get immediately the following result

Corollary 2 *There exist quadratic polynomial differential systems* (3.2) *(i.e., with $A = B = C^k = 0$) for which $l \in \{0, 1, \cdots 2^{n-3}\}$ limit cycles bifurcate from the origin at $\varepsilon = 0$, i.e., for ε sufficiently small the system has exactly l limit cycles in a neighborhood of the origin and these limit cycles tend to the origin when $\varepsilon \searrow 0$.*

Proof 14 *of Theorem $\mathring{D}$oing the cylindrical change of coordinates*

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z_i = z_i, \quad i = 3, \cdots, n, \tag{3.3}$$

in the region $r > 0$ the system (3.2) *becomes*

$$\begin{aligned}
\dot r = &\ \ \varepsilon ar + \Sigma_{i_1+\cdots+i_n=2}(a_{i_1\cdots i_n}\cos\theta + b_{i_1\cdots i_n}\sin\theta)(r\cos\theta)^{i_1} \\
&(r\sin\theta)^{i_2} z_3^{i_3}\cdots z_n^{i_n}) + \bigcirc(3), \\
\dot\theta = &\ \ \frac{1}{r}\Big[br + \Sigma_{i_1+\cdots+i_n=2}(b_{i_1\cdots i_n}\cos\theta - a_{i_1\cdots i_n}\sin\theta)(r\cos\theta)^{i_1} \\
&(r\sin\theta)^{i_2} z_3^{i_3}\cdots z^{i_n} + \bigcirc(3)\Big],
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\dot z = &\ \ \varepsilon c_k z_k + \Sigma_{i_1+\cdots+i_n=2} c^{(k)}_{i_1\cdots i_n}(r\cos\theta)^{i_1}(r\sin\theta)^{i_2} z_3^{i_3}\cdots z^{i_n} \\
&+\bigcirc(3) \quad 3, \cdots, n,
\end{aligned} \tag{3.5}$$

where $\bigcirc(3) = \bigcirc(3)(r, z_3, \cdots, z_n)$

As usual, $\mathbb{Z}_+$denotes the set of all non-negative integers. Taking $a_{00e_{ij}} = b_{00e_{ij}} = 0$ where $e_{ij} \in \mathbb{Z}_+^{n-2}$ has the sum of the entries equal to 2, it is easy to show that in a suitably small neighborhood of $(r, z_3, \cdots, z_n) = (0, 0, \cdots, 0)$ we have $\dot\theta \neq 0$. Then, choosing θ as the new independent variable system (3.4), in a neighborhood of $(r, z_3, \cdots, z_n) = (0, 0, \cdots, 0)$ it becomes

$$\begin{aligned}
\frac{\partial r}{\partial\theta} = &\ \ \frac{1}{M} r\Big(\varepsilon ar + \Sigma_{i_1+\cdots+i_n=2}(a_{i_1\cdots i_n}\cos\theta + b_{i_1\cdots i_n}\sin\theta) \\
&(r\cos\theta)^{i_1}(r\sin\theta)^{i_2} z_3^{i_3}\cdots z_n^{i_n} + \bigcirc(3)\Big), \\
\frac{\partial z_k}{\partial\theta} = &\ \ \frac{1}{M}\Big(1r\Big(\varepsilon c_k z_k + \Sigma_{i_1+\cdots+i_n=2} c^{(k)}_{i_1\cdots i_n}(r\cos\theta)^{i_1}(r\sin\theta)^{i_2} \\
&z_3^{i_3}\cdots z^{i_n} + \bigcirc(3)\Big)\Big).
\end{aligned} \tag{3.6}$$

Where

$$M_1 = \quad br + \Sigma_{i_1+\cdots+i_n=2}(b_{i_1\cdots i_n}\cos\theta - a_{i_1\cdots i_n}\sin\theta)(r\cos\theta)^{i_1}(r\sin\theta)^{i_2} \\ z_3^{i_3}\cdots z_n^{i_n} + \bigcirc(3)$$

for $k = 3, \cdots, n$. We note that this system is 2π periodic in the variable θ. In order to write system (3.6) in the normal form of the averaging theory we rescale the variables

$$(r, z_3, \cdots, z_n) = (\rho\varepsilon, \eta_3\varepsilon, \cdots, \eta_n\varepsilon). \tag{3.7}$$

Then the system (3.6) becomes

$$\begin{aligned} \frac{\partial\rho}{\partial\theta} &= \varepsilon f_1(\theta, \rho, \eta_3, \cdots, \eta_n) + \varepsilon^2 g(\theta, \rho, \eta_3, \cdots, \eta_n), \\ \frac{\partial\eta_k}{\partial\theta} &= \varepsilon f_k(\theta, \rho, \eta_3, \cdots, \eta_n) + \varepsilon^2 g_1(\theta, \rho, \eta_3, \cdots, \eta_n), \\ k &= 3, \cdots, n. \end{aligned} \tag{3.8}$$

where

$$f_1 = \frac{1}{b}\Big(a\rho + \Sigma_{i_1+\cdots+i_n=2}(a_{i_1\cdots i_n}\cos\theta + b_{i_1\cdots i_n}\sin\theta)(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2} \\ z_3^{i_3}\cdots z_n^{i_n})\Big)$$

$$f_1 = \frac{1}{b}\Big(c\eta_k + \Sigma_{i_1+\cdots+i_n=2}c^{(k)}_{i_1+\cdots+i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}z_3^{i_3}\cdots z_n^{i_n})\Big)$$

We note that the system (3.8) is in the normal form (2.1) of the averaging theory, with $x = (\rho, \eta_3, \cdots, \eta_n)$, $t = \theta$, $F(\theta, \rho, \eta_3, \cdots, \eta_n) = (f_1(\theta, \rho, \eta_3, \cdots, \eta_n), f_3(\theta, \rho, \eta_3, \cdots, \eta_n), \cdots,$ $f_n(\theta, \rho, \eta_3, \cdots, \eta_n)$, and $T = 2\pi$. The averaged system of (3.8) is

$$\dot{y} = \varepsilon f^0(y), \quad y = (\rho, \eta_3 \cdots, \eta_n) \in \Omega, \tag{3.9}$$

where Ω is a suitable neighborhood of the origin $(\rho, \eta_3, \cdots, \eta_n) = (0, 0, \cdots, 0)$, and

$$f^0(y) = (f_1^0(y), f_3^0(y), \cdots, f_n^0(y)),$$

with

$$f_i^0(y) = \frac{1}{2\pi}\int_0^{2\pi} f_i(\theta, \rho, \eta_3, \cdots, \eta_n)d\theta, \quad i = 1, 3, \cdots, n.$$

After some calculations we have that

$$f_1^0 = \frac{1}{2b}\rho\Big(2a + \Sigma_{j=3}^n(a_{10e_j} + b_{01e_j})\eta_j\Big),$$

$$f_k^0 = \frac{1}{2b}\rho\Big(2c_k\eta_k + \Big(c^{(k)}_{200_{n-2}} + c^{(k)}_{020_{n-2}}\rho^2\Big) + 2\Sigma_{3\leqslant i\leqslant j\leqslant n}c^{(k)}_{00e_{ij}}\eta_i\eta_j\Big), \quad k = 3, \cdots, n.$$

where $e_j \in \mathbb{Z}_+^{n-2}$ is the unit vector with the $j-$th entry equal to 1, and

$e_j \in \mathbb{Z}_+^{n-2}$ has the sum of the i-th and $j-$th entries equal to 2 and the other equal to 0.

Now we shall apply Theorem 4 for studying the limit cycles of system (3.8). Note that these limits, after the rescaling (3.7), will become infinitesimal limit cycles for system (3.6), which will tend to the origin when $\varepsilon \searrow 0$; consequently, they will be bifurcated limit cycles of the Hopf bifurcation of system (3.6) at the origin.

From Theorem 4 for studying the limit cycles of system (3.8) we only need to compute the non-degenerate singularities of system (3.9). Since the transformation from the cartesian coordinates $(r, z_3, \cdots, z_n)$ to the cylindrical ones $(\rho, \eta_3, \cdots, \eta_n)$ is not a diffeomorphism at $\rho = 0$, we deal with the zeros having the coordinate $\rho > 0$ of the averaged function f^0. So, we need to compute the roots of the algebraic equations

$$2a + \Sigma_{j=3}^{n}(a_{10e_j} + b_{01e_j})\eta_j = 0,$$

$$2c_k\eta_k + \left(c^{(k)}_{200_{n-2}} + c^{(k)}_{020_{n-2}}\right)\rho^2 + 2\Sigma_{3\leqslant i\leqslant j\leqslant n} c^{(k)}_{00e_{ij}}\eta_i\eta_j = 0, \qquad (3.10)$$

$$k = 3, \cdots, n.$$

Since the coefficients of system (3.10) are independent and arbitrary, in order to simplify the notation we write it as

$$a + \Sigma_{j=3}^{n} a_j\eta_j = 0, \quad c_0^k\rho^2 + c_k\eta_k + \Sigma_{3\leqslant i\leqslant j\leqslant n} c^{(k)}_{ij}\eta_i\eta_j = 0, \qquad (3.11)$$

$$k = 3, \cdots, n.$$

where a_j, $c_0^{(k)}$, c_k and $c^{(k)}_{ij}$ are arbitrary constants. Denote by $\mathscr{C}$ the set of algebraic systems of form (3.11). We claim that there is a system belonging to $\mathscr{C}$ which has exactly 2^{n-3} simple roots. The claim can beverified by the example

$$a + a_3\eta_3 = 0, \qquad (3.12)$$

$$c_0^{(3)}\rho^2 + c_3\eta_3 + \Sigma_{3\leqslant i\leqslant j\leqslant n} c^{(3)}_{ij}\eta_i\eta_j = 0, \qquad (3.13)$$

$$c_k\eta_k + \Sigma_{3\leqslant i\leqslant j\leqslant k} c^{(k)}_{ij}\eta_i\eta_j = 0, \quad k = 4, \cdots, n, \qquad (3.14)$$

with all the coefficients being non-zero. Equations (3.14) can be treated as quadratic algebraic equations in ea_k. Sbttuting the unque solution η_{30} of η_{30} . in (3.12) (3.14) with $k = 4$, this last equation has exactly two different solutions, namely η_{41} and η_{42} for η_4, choosing conveniently c_4. Introducing the two solutions (η_{30}, η_{4i}), $i = 1, 2$, into (1.22) with $k = 5$ and choosing conveniently the values of the coefficients of equation (3.14) with $k = 5$ and $(\eta_3, \eta_4) = (\eta_{30}, \eta_{4i})$, we get two different solutions η_{5i1} and η_{5i2} of η_5 for each i. Moreover, playing with the coefficients of the equations, the four solutions $(\eta_{30}, \eta_{4i}, \eta_{5_{ij}})$ for $i, j = 1, 2$, are distinct. By induction, we can prove that for suitable choice of the coefficients, equations (3.12) and (3.14) have 2^{n-3} different roots $(\eta_3, \cdots, \eta_n)$. Since $\eta_3 = \eta_{30}$ is fixed, for any given $c_{ij}^{(3)}$ there exist values of c_3 and $c_0^{(3)}$ such that equation (1.21) has a positive solution ρ for each of the 2^{n-3} solutions $(\eta_3, \cdots, \eta_n)$. of (3.12) and (3.14). Since the 2^{n-3} solutions are different, and the number of the solutions of (3.12)-(3.14) is the maximum that the equations can have (by the Bezout Theorem), it follows that every solution is simple, and consequently the determinant of the Jacobian of the system evaluated at it is not zero. This proves the claim. Using the same arguments which allowed us to prove the claim, we can also prove that we can choose the coefficients of the previous system in order to have $0, 1, \cdots, 2^{n-3} - 1$ simple real solutions.

Taking the averaged system (3.9) with f^0 having the convenient coefficients as in (3.12)-(3.14), the averaged system (3.9) has exactly $k \in \{0, 1, \cdots, 2^{n-1}\}$ singularities with the components $\eta > 0$. Moreover, the determinants of the Jacobian matrix $\partial f^0 / \partial y$ at these singularities do not vanish because all the singularities are simple. In short, by Theorem 4 we get that there are systems of the form (3.2) which have $k \in \{0, 1, \cdots, 2^{n-1}\}$ limit cycles. This proves the theorem.

3.4 The Hopf bifurcation of the Michelson system

The Michelson system

$$\dot{x} = y \quad \dot{y} = z \quad \dot{z} = c^2 y - \frac{x^2}{2}, \tag{3.15}$$

with $(x, y, z) \in \mathbb{R}^3$ and the parameter $c \geqslant 0$, was introduced by Michelson in the study of the travelling wave solutions of the Kuramoto–Sivashinsky equation. It is well known that system (3.15) is reversible with respect to the involution $R(x, y, z) = (-x, y, -z)$ and is volume-preserving under the flow of the system. It is easy to check that system (3.15) has two finite singularities $S_1 = (\sqrt{2c}, 0, 0) =$ and $S_2 = (\sqrt{2c}, 0, 0)$ for $c > 0$, which are both saddle-foci. The former has a two dimensional stable manifold and the latter has a two dimensional unstable manifold.

For $c > 0$ small numerical experiments and asymptotic expansions in sinus series and Webster–Elgin revealed the existence of a zero-Hopf bifurcation at the origin for $c = 0$. But their results do not provide an analytic proof on the existence of such zero-Hopf bifurcation. By a zero-Hopf bifurcation we mean that when $c = 0$ the Michelson system has the origin as a singularity having eigenvalues $0, \pm i$, and when $c > 0$ sufficiently small the Michelson system has a periodic orbit which tends to the origin when c tends to zero. The analytic proof of this zero-Hopf bifurcation has been provided by Llibre–Zang. Now we state this result and reproduce its proof.

Theorem 12 *For $c \geqslant 0$ sufficiently small the Michelson system* (3.15) *has a zero-Hopf bifurcation at the origin for $c = 0$. Moreover, the bifurcated periodic orbit satisfies $x(t) = -2c\cos t + \circ(c)$, $y(t) = 2c\sin t + \circ(c)$ and $z(t) = 2c\cot t + \circ(c)$, for $c > 0$ sufficiently small.*

Proof 15 *For any $\varepsilon \neq 0$ we apply the change of variables $x = \varepsilon x$, $y = \varepsilon y$, $z = \varepsilon z$ and $c = \varepsilon d$, and Michelson system* (3.15) *becomes*

$$\dot{x} = y \quad \dot{y} = z \quad \dot{z} = -y + \varepsilon d^2 - \varepsilon \frac{1}{2} x^2, \tag{3.16}$$

where we still use x, y, z instead of x, y, z. Now doing the change of variables $\dot{x} = x$, $y = r\cos\sin\theta$ and $z = r\cos\theta$, system (3.16) goes over to

$$\dot{x} = r\sin\theta \quad \dot{r} = \frac{\varepsilon}{2}(2d^2 - x^2)\cos\theta, \quad \dot{\theta} = 1 - \frac{\varepsilon}{2r}(2d^2 - x^2)\sin\theta. \tag{3.17}$$

This system can be written as

$$\begin{aligned} \frac{\partial x}{\partial \theta} &= r\sin\frac{\varepsilon}{2}(2d^2 - x^2)\sin^2\theta + \varepsilon^2 f_1(0, r, \varepsilon), \\ \frac{\partial r}{\partial \theta} &= \frac{\varepsilon}{2}(2d^2 - x^2)\cos\theta + \varepsilon^2 f_2(0, r, \varepsilon). \end{aligned} \tag{3.18}$$

where f_1 and f_2 are analytic functions in their variables.

For arbitrary $(x_0, r_0) \neq (0, 0)$, the system $(3.18)_\varepsilon = 0$ has the $2\pi-$periodic solution

$$x(\theta) = r_0 + x_0 - r_0\cos\theta, \quad r(\theta) = r_0, \tag{3.19}$$

such that $x(0) = x_0$ and $r(0) = r_0$. It is easy to see that the first variational equation of (3.17)$\varepsilon = 0$ along the solution (3.19) is

$$\begin{pmatrix} \dfrac{dy_1}{dt} \\ \dfrac{dy_2}{d\theta} \end{pmatrix} = \begin{pmatrix} 0 & \sin\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix}$$

It has the fundamental solution matrix

$$M = \begin{pmatrix} 1 & 1-\cos\theta \\ 0 & 1 \end{pmatrix} \tag{3.20}$$

which is independent from the initial condition (x_0, r_0). Applying Corollary 1 to the differential system (3.18) we have that

$$\mathscr{F} = \frac{1}{2}\int_0^{2\pi} M^{-1} \begin{pmatrix} (2d - x^2)\sin^2\theta \\ (2d - x^2)\cos\theta \end{pmatrix} |_{(1.43)}\, d\theta.$$

Then, $\mathscr{F}(x_0, r) = g_1(x_0, r_0), g_2(x_0, r_0)$ with

$$g_1(x_0, r_0) = \frac{1}{4}(4d^2 - 5r_0^2 - 6r_0x_0 - 2x_0^2), \quad g_2(x_0, r_0) = \frac{1}{2}r_0(x_0 + r_0).$$

We can check that $F = 0$ has a unique non-trivial solution $x_0 = -2d$ and $r_0 = 2d$, and that $\det DF(x_0, r_0)\ x_0 = -2d,\ r_0 = 2d = d^2$. Hence by Corollary 1 it follows that, for any given $d > 0$ and for $|\,\varepsilon\,| > 0$ sufficiently small, the system (3.18) has a periodic orbit $(x(\theta,\varepsilon), r(\theta,\varepsilon))$ of period 2π, such that $x(0,\varepsilon),\ r(0,\varepsilon) \to (-2d, 2d)$ as $\varepsilon \to 0$. We note that the eigenvalues of $DF(x_0, r_0)|_{x_0=-2d, r_0=2d}$ are $\pm di$. This shows that the periodic orbit is linearly stable.

Going back to system (3.15) we get that, for $c > 0$ sufficiently small, the Michelson system has a periodic orbit of period close to 2π given by $x(t) = -2c\cos t + \circ(c)$, $y(t) = 2c\sin t + \circ(c)$ and $z(t) = 2c\cot t + \circ(c)$. We think that this periodic orbit is symmetric with respect to the involution R, but we do not have a proof of it.

3.5 A third-order differential equation

Using Theorem 5 in the next result we present a third-order differential equation having as many limit cycles as we want.

Proposition 3 *Let us consider the third-order differential equation*

$$\dddot{x} - \ddot{x} + \dot{x} - x = \varepsilon\cos(x + t). \tag{3.21}$$

Then for all positive integer m there is $\varepsilon m > 0$ such that if $\varepsilon \in [-\varepsilon m, \varepsilon m] \setminus \{0\}$ the differential equation (3.21) has at least m limit cycles.

Proof 16 *If $y = \dot{x}$ and $z = \ddot{x}$, then* (2.10) *can be written as*

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \qquad\qquad (3.22)\\
\dot{z} &= x - y + z + \varepsilon \cos(x+t) = x - y + z + \varepsilon F(t, x, y, z)
\end{aligned}$$

The origin $(0,0,0)$ *is the unique singular point of* (3.22) *when* $\varepsilon = 0$. *The eigenvalues of the linearized system at this singular point are* $\pm i$ *and* 1. *By the linear invertible transformation* $(X, Y, Z)T = C(X, Y, Z)T$, *where*

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

we transform the differential system (3.22) *into another such that its linear part is the real Jordan normal form of the linear part of system* (3.22) *with* $\varepsilon = 0$, *i.e.,*

$$\begin{aligned}
\dot{X} &= -Y, \\
\dot{Y} &= X + \varepsilon F(X, Y, Z, t), \qquad\qquad (3.23)\\
\dot{Z} &= Z + \varepsilon F(X, Y, Z, t)
\end{aligned}$$

where

$$F(X, Y, Z, t) = F\Big(\frac{X - Y + Z}{2}, \frac{-X - Y + Z}{2} Y, \frac{-X + Y + Z}{2}\Big)$$

Using the notation introduced in (2.10) *we have that* $x(X, Y, Z)$, $F_0(x, t) = (-X, Y, Z)$, $F_1(x, t) = (0, F, F)$ *and* $F_2(x, t) = 0$ *Let* $x(t; X_0, Y_0, Z_0, \varepsilon)$ *be the solution to system* (3.23) *with* $x(0; X_0, Y_0, Z_0, \varepsilon) = (X_0, Y_0, Z_0)$. *Clearly the unperturbed system* (3.23) *with* $\varepsilon = 0$ *has a linear center at the origin in the* (X, Y) *plane, which is an invariant plane under the flow of the unperturbed system, and the periodic solution* $x(0; X_0, Y_0, 0, 0, \varepsilon) = x(X(t), Y(t), Z(t))$ *is*

$$X(t) = X_0 \cos t - Y_0 \sin t,\ Y(t) = Y_0 \cos t + X_0 \sin t,\ Z(t) = 0. \qquad (3.24)$$

Note that all these periodic orbits have period 2π.

For our system, V *and* α *from Theorem* (5) *are* $V = \{(X, Y, 0) : 0 < X_2 + Y_2 < \rho\}$, *for some arbitrary* $\rho > 0$ *and* $\alpha = (X_0, Y_0) \in V$.

The fundamental matrix solution $M(t)$ of the variational equation of the unperturbed system $(3.23)_\varepsilon = 0$ *with respect to the periodic orbits* (3.24) *satisfying that $M(t)$ is the identity matrix is*

$$M(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix},$$

We remark that it is independent from the initial condition $(X_0, Y_0, 0)$. *Moreover an easy computation shows that*

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-e^{2\pi} \end{pmatrix},$$

In short we have shown that all the assumptions of Theorem 5 hold. Hence we shall study the zeros $\alpha = (X_0, Y_0) \in V$ of the two components of the function $\mathscr{F}(\alpha)$ given in (1.26). *More precisely we have $\mathscr{F}(\alpha) = (\mathscr{F}_1(\alpha), \mathscr{F}_2(\alpha))$ where*

$$\begin{aligned}
\mathscr{F}_1(\alpha) &= \int_0^{2\pi} \sin t F(x(t; X_0, Y_0, 0, 0))\,dt, \\
&= \int_0^{2\pi} \sin t F\Big(t + \frac{X(t)-Y(t)}{2}, -\frac{X(t)+Y(t)}{2}, \frac{-X(t)+Y(t)}{2}, t\Big)\,dt, \\
\mathscr{F}_2(\alpha) &= \int_0^{2\pi} \cos t F(x(t; X_0, Y_0, 0, 0))\,dt, \\
&= \int_0^{2\pi} \cos t F\Big(t + \frac{X(t)-Y(t)}{2}, -\frac{X(t)+Y(t)}{2}, \frac{-X(t)+Y(t)}{2}, t\Big)\,dt.
\end{aligned}$$

where $X(t), Y(t)$ are given by (3.24).

First, we consider the third-order differential equation (3.21). *For this equation we have that*

$$\begin{aligned}
f_1(X_0, Y_0) &= \int_0^{2\pi} \sin t \cos\Big(t + \frac{(X_0-Y_0)\cos t - (X_0+Y_0)\sin t}{2}\Big)\,dt, \\
f_2(X_0, Y_0) &= \int_0^{2\pi} \cos t \cos\Big(t + \frac{(X_0-Y_0)\cos t - (X_0+Y_0)\sin t}{2}\Big)\,dt.
\end{aligned}$$

To simplify the computation of these two integrals we do the change of variables $(X_0, Y_0) \mapsto (r, s)$ *given by*

$$X_0 - Y_0 = 2r\cos s, \quad X_0 + Y_0 = -2r\sin s, \tag{3.25}$$

where $r > 0$ *and* $s \in [0, 2\pi]$. *From now on and until the end of the paper, we write* $f_1(r,s)$ *instead of*

$$f_1(X_0, Y_0) = f_1\Big(r(\cos s - \sin s), -r(\cos s + \sin s)\Big).$$

Similarly for $f_2(r,s)$.

We compute the two previous integrals and we get

$$\begin{aligned} f_1(r,s) &= -\pi J_2(r)\sin 2s, \\ f_2(r,s) &= 2\pi\Big(\frac{1}{r}J_1(r) - J_2(r)\cos^2 s\Big), \end{aligned} \tag{3.26}$$

where J_1 *and* J_2 *are the first and second Bessel functions of the first kind. For more details on Bessel functions. These computations become easier with the help of an algebraic manipulator such as Mathematica or Maple. Using the asymptotic expressions of the Bessel functions of first kind it follows that Bessel functions* $J_1(r)$ *and* J_2 *have different zeros. Hence,* $f_i(r,s) = 0$ *for* $i = 1,2$ *imply that* $s \in \{0, \pi/2, 3\pi/2\}$. *Therefore, we have to study the zeros of*

$$f_2(r,0) = f_2(r,\pi) = 2\pi\Big(\frac{1}{r}J_1(r) - J_2(r)\Big), \tag{3.27}$$

$$f_2(r,\pi/2) = f_2(r,3\pi/2) = \frac{2\pi}{r}J_1(r). \tag{3.28}$$

We claim that function (3.27) *has also infinite zeros for* $r \in (0,\infty)$. *Note that if* ρ *is sufficiently large, and we choose* $r < \rho$ *also sufficiently large, then*

$$J_n(r) \approx \sqrt{\frac{2}{\pi r}}\cos\Big(r - \frac{n\pi}{2} - \frac{\pi}{4}\Big), \quad for \;\; n = 1,2,$$

are asymptotic estimations. Considering (3.27)) *for r sufficiently large we obtain*

$$\begin{aligned} f_2(r,0) &\approx \frac{2}{r}\sqrt{\frac{2\pi}{r}}\Big(\cos\Big(r - \frac{3\pi}{4}\Big) + r\cos\Big(r - \frac{\pi}{4}\Big)\Big) \\ &= \frac{2}{r}\sqrt{\frac{\pi}{r}}((r-1)\cos r + (r+1)\sin r) \end{aligned}$$

The above function has infinite zeros because the equation

$$\tan r = \frac{1-r}{r+1}$$

has infinitely many solutions.

For every zero $r_0 > 0$ *of the function* (3.27) *we have two zeroes of system* (3.26), *namely* $(r,s) = (r_0, 0)$ *and* $(r,s) = (r_0, \pi)$. *We have from* (3.26) *that*

$$
\begin{aligned}
\left| \frac{\partial(f_1,f_2)}{\partial(r,s)} \right|_{(r,s)=(r_0,0)} = \; & \frac{1}{r_0^3}(4\pi^2(J_0(r_0)r_0 - 2J_1(r_0)) - (J_0(r_0)r_0 \\
& + (r^2-2)J_1(r_0))), \\
= \; & \frac{4\pi^2}{r_0} J_2(r_0)(J_1(r_0)r_0 - J_2(r_0)),
\end{aligned} \tag{3.29}
$$

where we have used several relations between the Bessel functions of the first kind. Clearly, it is impossible that (3.27) *and* (16) *are equal to zero at the same time. Therefore, by Theorem 4 there is a periodic orbit of the system* (3.21) *for each* $(r_0,)$, *that is, for each value of* $(X_0, Y_0) = (r_0, -r_0)$.

In an analogous way, there is a periodic orbit of the system (3.21) *for each* (r_0, π), *that is, for each value of* $(X_0, Y_0) = (-r_0, r_0)$. *In fact, the periodic orbit with these initial conditions and the previous one with initial conditions* $(X_0, Y_0) = (r_0, -r_0)$, *are the same.*

Similarly, since $J_1(r)$ *has infinitely many zeroes, the function* (16) *has infinitely many positive zeroes* r_1. *Every one of these zeroes provides two solutions to the system* (3.26) , *namely* $(r,s) = (r_1, \pi/2)$ *and* $(r,s) = (r_1, 3\pi/2)$

Moreover we have from (3.26) *that*

$$
\left| \frac{\partial(f_1,f_2)}{\partial(r,s)} \right|_{(r,s)=(r_1,\pi/2)} = \frac{4\pi^2}{r_1} J_2^2(r_1) \neq 0. \tag{3.30}
$$

Therefore, by Theorem 4 there is a periodic orbit of the system (3.21) *for each* $(r_1, \pi/2)$, *that is, for each value of* $(X_0, Y_0) = (-r_1, -r_1)$. *In an analogous way there is also a periodic orbit of the system* (3.21) *for each* $(r_1, 3\pi/2)$, *that is, for each value of* $(X_0, Y_0) = (r_1, r_1)$ *In fact, the periodic orbit with these initial conditions and the previous one with initial conditions* $(X_0, Y_0) = (-r_1, -r_1)$ *are the same. Taking the radius* ρ *of the disc* $V = \{(X_0, Y_0, 0) : 0 < X_2 + Y_2 < \rho\}$ *in the proof of Theorem 5 conveniently large, we include in it as many zeros of the system* $f_1(X_0, Y_0) = f_2(X_0, Y_0) = 0$ *as we want, so from Theorem 5, Proposition 3 follows.*

3.6 The Vallis system (El Nino phenomenon)

The Vallis system, introduced by Vallis in 1988, is a periodic nonautonomous three dimensional system modeling the atmosphere dynamics in the tropics over the Pacific Ocean, related to the yearly oscillations of precipitation, temperature and wind force. Denoting by x the wind force, by y the difference of near-surface water temperatures of the east and west parts of the Pacific Ocean, and by z the average near-surface water temperature, the Vallis system is

$$\begin{aligned} \frac{dx}{dt} &= -ax + by + au(t), \\ \frac{dy}{dt} &= -y + xz, \\ \frac{dz}{dt} &= -z - xy + 1. \end{aligned} \tag{3.31}$$

where $u(t)$ is some $C^1\, T-$periodic function describing the wind force under seasonal motions of air masses, and the parameters a and b are positive. Although this model neglects some effects like Earth's rotation, pressure field and wave phenomena, it provides a correct description of the observed processes and recovers many of the observed properties of El Nino. The properties of El Nino phenomena is shown that, taking $u \equiv 0$.

It is possible to observe the presence of chaos by considering $a = 3$ and $b = 102$. Later it is proved that there exists a chaotic attractor for the system (3.31) after a Hopf bifurcation. This chaotic motion can be easily understood if we observe the strong similarity between the system (3.31) and the Lorenz system, which becomes more clear under the replacement of z by $z+1$ in (3.31). Now we shall provide sufficient conditions in order that system (3.31) has periodic orbits and, additionally, we shall characterize the stability of these periodic orbits. As far as we know, the study of the periodic orbits in the non-autonomous Vallis system has not been considered in the literature, with the exception of the Hopf bifurcation studied.

We define

$$\int_0^T u(s)ds$$

Now we state our main result.

Theorem 13 *For $I \neq 0$ and $a \neq b$ the Vallis system* (3.31) *has a $T-$periodic solution* $(x(t), y(t), z(t))$ *such that*

$$(x(t), y(t), z(t)) \approx \left(\frac{aI}{T(a-b}), \frac{aI}{T(a-b)}, 1\right)$$

Moreover this periodic orbit is stable if $a > b$, and unstable if $a < b$.

We do not know the reliability of the Vallis model approximating the Nino phenomenon but it seems that, for the moment, this is one of the best existing models. Accepting this reliability we can say the following.

The stable periodic solution provided by Theorem 13 says that the Nino phenomenon exhibits a periodic behavior if the $T-$periodic function $u(t)$ and the parameters a and b of the system satisfy $I \neq 0$ and $a > b$. Moreover, Theorem 13 states that this periodic solution lives near the point

$$(x, y, z) \approx \left(\frac{aI}{T(a-b)}, \frac{aI}{T(a-b)}, 1\right)$$

Since the periodic solutions found in the following Theorems 15, 16 and17 are also stable, we can provide a similar physical interpretation for them as we have done for the periodic solution from Theorem 13

Theorem 14 *For $I \neq 0$ the Vallis system* (3.31) *has a $T-$periodic solution*
$(x(t), y(t), z(t))$ *such that*

$$(x(t), y(t), z(t)) \approx \left(-\frac{aI}{Tb}, -\frac{aI}{Tb}, 1\right)$$

Moreover this periodic orbit is always unstable

Theorem 15 *For $I \neq 0$ the Vallis system 3.31 has a $T-$periodic solution*
$(x(t), y(t), z(t))$ *such that*

$$(x(t), y(t), z(t)) \approx \left(\frac{I}{T}, \frac{I}{T}, 1\right)$$

Moreover this periodic orbit is always stable.

Theorem 16 *For $I \neq 0$ the Vallis system 3.31 has a $T-$periodic solution $(x(t), y(t), z(t)$ such that*

$$(x(t), y(t), z(t)) \approx \left(\frac{I}{T}, 0, 1\right)$$

Moreover this periodic orbit is always stable.

In what follows we consider the function

$$J(t) = \int_0^t u(s)ds$$

and note that $J(T) = I$. So, we have the following result.

Theorem 17 *Consider $I = 0$ and $J(t) \neq 0$ if $0 < t < T$. Then, the Vallis system 3.31 has a $T-$periodic solution $(x(t), y(t), z(t)$ such that*

$$(x(t), y(t), z(t)) \approx \left(-\frac{a}{T}\int_0^T J(s)ds, 0, 1\right)$$

Moreover this periodic orbit is always stable

The tool for proving our results will be the averaging theory. This theory applies to periodic non-autonomous differential systems depending on a small parameter ε. Since the Vallis system already is a $T-$periodic non-autonomous differential system, in order to apply to it the averaging theory we need to introduce in such system a small parameter. This is reached doing convenient rescalings in the variables (x, y, z), in the parameters (a, b), and in the function $u(t)$. Playing with different rescalings we shall obtain different results on the periodic solutions of the Vallis system. More precisely, in order to study the periodic solutions of the differential system 3.31, we start doing a rescaling of the variables (x, y, z), of the function $u(t)$, and of the parameters a and b, as follows:

$$\begin{aligned} &x = \varepsilon^{m_1} X, \quad y = \varepsilon^{m_2} Y, \quad z = \varepsilon^{m_3} Z, \\ &u(t) = \varepsilon^{n_1} U(t), \quad a = \varepsilon^{n_2} A, \quad b = \varepsilon^{n_3} Z. \end{aligned} \tag{3.32}$$

Where ε is always positive and sufficiently small, and where m_i and n_j are nonnegative integers, for all $i, j = 1, 2, 3$. Then, in the new variables

(x, y, z), the system 3.31 is written

$$\begin{aligned}
\frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{-m_1+m_2+n3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\
\frac{dY}{dt} &= -Y + \varepsilon^{m_1-m_2+m_3} XZ, \qquad (3.33) \\
\frac{dZ}{dt} &= -Z - \varepsilon^{m_1+m_2-m_3} XY + \varepsilon^{-m_3}.
\end{aligned}$$

Consequently, in order to have non-negative powers of ε we must impose the conditions

$$m_3 = 0 \quad and \quad 0 \leqslant m_2 \leqslant m_1 \leqslant L, \qquad (3.34)$$

where $L = \min\{m_2 + n_3, n_1 + n_2\}$. So, system 3.35 becomes

$$\begin{aligned}
\frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{-m_1+m_2+n3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\
\frac{dY}{dt} &= -Y + \varepsilon^{m_1-m_2} XZ, \qquad (3.35) \\
\frac{dZ}{dt} &= 1 - Z - \varepsilon^{m_1+m_2} XY.
\end{aligned}$$

Our aim is to find periodic solutions of the system (3.35) for some special values of $m_i, n_j, i, j = 1,2,3$, and after we go back through the rescaling 3.32 to guarantee the existence of periodic solutions in system 3.31. In what follows we consider the case where $n2$ and $n3$ are positive and $m_2 = m_1 < n_1 + n_2$. These conditions lead to the proofs of Theorems 13, 14 and 15. For this reason we present these proofs together in order to avoid repetitive arguments. Moreover, in what follows we consider

$$K = \int_0^T U(s)ds.$$

Proof 17 (Proofs of Theorems 13, 14 and 15) *We start considering system* (3.35) *with n_2 and n_3 positive and $m_2 = m_1 < n_1 + n_2$. So we have*

$$\begin{aligned}
\frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{n3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\
\frac{dY}{dt} &= -Y + XZ, \qquad (3.36) \\
\frac{dZ}{dt} &= 1 - Z - \varepsilon^{2m_1} XY.
\end{aligned}$$

Now we apply the averaging method to the differential system (3.36). *We have* $x = (X, Y, Z)^T$ *and*

$$F_0(t,x) = \begin{pmatrix} 0 \\ -Y + XZ \\ 1 - Z \end{pmatrix}, \tag{3.37}$$

We start considering the system

$$\dot{x} = F_0(t,x) \tag{3.38}$$

Its solution $x(t,z,0) = (X(t), Y(t), Z(t))$ *such that* $x(0,z,0) = z = (X_0, Y_0, Z_0)$ *is*

$$X(t) = X_0,$$

$$Y(t) = (1 - e^{-t}(1+t))X_0 + e^{-t}Y_0 + e^{-t}tX_0Z_0,$$

$$Z(t) = 1 - e^{-t} + e^{-t}Z_0.$$

In order that $x(t,z,0)$ *is a periodic solution we must choose* $Y0 = X0$ *and* $Z0 = 1$. *This implies that, through every point of the straight line* $X = Y$, $Z = 1$, *there passes a periodic orbit lying in the phase space* $(X, Y, Z, t) \in \mathbb{R}^3 \times S_1$. *Here and in what follows, S1 is the interval* $[0, T]$ *identifying* T *with* 0.

Observe that. We have $n = 3, k = 1, \alpha = X_0$ *and* $\beta(X_0) = (X_0, 1)$ *and, consequently,* $\mathscr{M}$ *is a one dimensional manifold given by* $\mathscr{M}(X_0, X_0, 1) \in \mathbb{R}^3 : X_0 \in \mathbb{R}$.

The fundamental matrix $M_z(t)$ *of* (3.38) *satisfying that* $M_z(0)$ *is the identity of* $\mathbb{R}^3$, *is*

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 - \cosh t + \sinh & e^{-t} & e^{-t}tX_0 \\ 0 & 0 & e^{-t} \end{pmatrix}$$

and its inverse matrix $M_z^{-1}(t)$ *is*

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 - e^{t} & e^{t} & -e^{t}tX_0 \\ 0 & 0 & e^{t} \end{pmatrix}$$

Since the matrix $M_z^{-1}(0) - M_z^{-1}(T)$ *has an* 1×2 *zero matrix in the upper right corner, and a* 2×2 *lower right corner matrix*

$$\Delta = \begin{pmatrix} 1 - e^{T} & e^{T}TX_0 \\ 0 & 1 - e^{T} \end{pmatrix}$$

with $\det(\Delta) = (1-e^T)^2 \neq 0$ *because* $T \neq 0$.

Let F *be the vector field of system* (3.36) *minus* F_0 *given in* (3.36). *Then the components of the function* $M_z^{-1}(t)F(t, x(t, z, 0))$ *are*

$$g_1(X_0, t) = -\varepsilon^{n_2} AX_0 + \varepsilon^{n_3} BX_0 + \varepsilon^{-m_1+n_1+n_2} AU(t),$$

$$g_2(X_0, t) = \varepsilon^{2m_1} e^t tX_0^3 + (1-e^t)g_1(X_0, t),$$

$$g_3(X_0, t) = -\varepsilon^{2m_1} e^t X_0^2.$$

In order to apply averaging theory of first order we need to consider only terms up to order ε. *Analysing the expressions of* g_1, g_2 *and* g_3 *we note that these terms depend on the values of* m_1 *and* n_j, *for each* $j = 1, 2, 3$. *In fact, we just need to study the integral of* $g1$ *because* $k = 1$. *Moreover, studying the function* g_1 *we observe that the only possibility to obtain an isolated zero of the function*

$$f_1(X_0) = \int_0^T g_1(X_0, t)\,dt$$

is assuming that $n_1 + n_2 - m_1$. *Otherwise, the only solution of* $f_1(X_0) = 0$ *is* X_0, *which corresponds to the equilibrium point* $(X_0, Y_0, Z_0) = (0, 0, 1)$ *of system* (3.38). *The same occurs if* n_2 *and* n_3 *are greater than* 1 *simultaneously. This analysis reduces the existence of possible periodic solutions to the following cases:*

(p_1) $n_2 = 1$ *and* $n_3 = 1$;
(p_2) $n_2 > 1$ *and* $n_3 = 1$;
(p_3) $n_2 = 1$ *and* $n_3 > 1$.

In the case (p_1) *we have* $M_z^{-1}(t)F_1(t, x(t, z, 0)) = -AX_0 + BX_0 + AU(t)$, *and then*

$$f_1(X_0) = (-A + B)TX_0 + AK.$$

Consequently, if $A \neq B$, *then* $f_1(X_0) = 0$ *implies* $X_0 = AK(T(A-B))$. *So, by Theorem 5, system* (3.36) *has a periodic solution* $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon))$ *such that*

$$(X(0,\varepsilon), Y(0,\varepsilon), Z(0,\varepsilon)) \mapsto (X_0, Y_0, Z_0) = \left(\frac{AK}{T(A-B)}, \frac{AK}{T(A-B)}, 1\right)$$

when $\varepsilon \mapsto 0$. *Note that the point* (X_0, Y_0, Z_0) *is an equilibrium point of the system* (3.32). *Then, taking* $n1 = n2 = n3 = 1$ *and going back through the rescaling* (3.36) *of the variables and parameters, we obtain that the periodic solution of system* (3.36) *becomes the periodic solution* $(x(t), y(t), z(t))$

of system (3.31) *satisfying*

$$(x(t), y(t), z(t)) \approx \left(\frac{aI}{T(a-b)}, \frac{aI}{T(a-b)}, 1\right).$$

Indeed, we observe that

$$x_0 = \varepsilon X_0 = \varepsilon \frac{(a\varepsilon^{-1})(I\varepsilon^{-1})}{T\varepsilon^{-1}(a-b)} = \frac{aI}{T(a-b)}$$

Moreover, we note that $f_1'(x_0) = \varepsilon f_1'(X_0) = -a + b \neq 0$ *so, the periodic orbit corresponding to* x_0 *is stable if* $a > b$*, and unstable otherwise. This completes the proof of Theorem 13*

Analogously the function f_1 *in the cases* (p_2) *and* (p_3) *is*

$$f_1(X_0) = TBX_0 + AK \;\; and \;\; f_1(X_0) = -TAX_0 + AK,$$

respectively. In the first case the condition $f_1(X_0) = 0$ *implies* $X_0 = -(AK)/(TB)$. *Now we observe that* $n_2 > 1$ *and* $n_3 = 1$. *So, going back through the rescaling, we obtain*

$$x_0 = \varepsilon X_0 = \varepsilon \frac{(-a\varepsilon^{-n_2})(I\varepsilon^{-n_1})}{Tb\varepsilon^{-1}} = -\frac{aI}{Tb\varepsilon^{n_1+n_2-2}}$$

and consequently, choosing $n_1 = 0$ *and* $n_2 = 2$*, we get* $x_0 = -aI/(Tb)$. *Note also that* $f_1'(x_0) = Tb > 0$*, then the periodic orbit corresponding to* x_0 *is always unstable. Thus, Theorem 14 is proved.*

Finally, in the case (p_3), $f_1(X_0) = 0$ *implies* $X_0 = K/T$. *So, taking* $n_1 = 1$ *and going back through the rescaling, we have* $x_0 = \varepsilon X_0 = \varepsilon I/(T\varepsilon) = I/T$. *Additionally,* $f_1'(x_0) = Ta < 0$. *Therefore, the periodic solution coming from* x_0 *is always stable. This proves Theorem 15.*

Proof 18 (Proof of Theorem 15) . *As in the proofs of Theorems 13, 14 and 15,*

we start by considering a more general case in the powers of ε *in* (3.35), *taking* $n_2 > 0$ *and* $m_2 < m_1 < L$. *In this case the function* $F_0(t, x)$ *of system* (2.10) *is*

$$F_0(t, x) = \begin{pmatrix} 0 \\ Y \\ 1 - Z \end{pmatrix}, \tag{3.39}$$

Then the solution $x(t, z, 0)$ *of system* (2.11) *satisfying* $x(0, z, 0) = z$ *is*

$$(X(t), Y(t), Z(t)) = (X_0, e^{-t}Y_0, 1 - e^{-t} + e^{-t}Z_0).$$

This solution is periodic if $Y_0 = 0$ *and* $Z_0 = 1$. *Then, through every point in the straight line* $Y = 0, Z = 1$ *there passes a periodic orbit lying in the phase space* $(X, Y, Z, t) \in \mathbb{R}^3 \times S^1$. *We have* $n = 3, k = 1, \alpha = X_0$ *and* $\beta(\alpha) = (0, 1)$. *Consequently,* M *is a one dimensional manifold given by* $M = \{(X_0, 0, 1) \in \mathbb{R}^3 : X_0 \in R\}$. *The fundamental matrix* $M_z(t)$ *from* (2.12) *and satisfying* $M_z(0) = Id_3$ *(with* F_0 *given by* (3.39). *Its inverse* $M_z^{-1}(t)$ *are, respectively*

$$M_z(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}, \ \textit{and} \ M_z^{-1}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix},$$

Since the matrix $M_z^{-1}(0) - M_z^{-1}(T)$ *has an* 1×2 *zero matrix in the upper right corner, and a* 2×2 *lower right corner matrix*

$$\Delta = \begin{pmatrix} 1 - e^T & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

with $\det(\Delta) = (1 - e^T)^2 \neq 0$, *we can apply the averaging theory Again, using the notations introduced in the proofs of Theorems 13, 14 and 15, since* $k = 1$ *we will look only to the integral of the first coordinate of* $F = (f_1, f_2, f_3)$. *In this case we have*

$$g_1(X_0, Y_0, Z_0, t) = -\varepsilon^{n_2} AX_0 + \varepsilon^{-m_1 + n_1 + n_2} AU(t).$$

Comparing this function g1 with the same function obtained in the proof of Theorems 13, 14 and 15, it is easy to see that this case corresponds to the case (p_3) *of the mentioned theorems. Then, in order to have periodic solutions, we need to choose* $n_2 = 1$ *and* $n_1 + n_2 - m_1 = 1$. *So, following the steps of the proof of case* (p_3) *by choosing* $n_1 = 1$ *and coming back through the rescaling 3.32 to system 3.31, Theorem 16 is proved.*

Proof 19 (Proof of Theorem 17) . *We start by considering the system 3.35 with* $n_3 = 2$, $n_2 > 0$, $m_1 = n_1 + n_2$ *and* $m_2 < m_1 < m_2 + n_3$. *With these conditions the system 3.35 becomes*

$$\begin{aligned} \frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{m_2 - n_1 - n_2 + n_3} BY + AU(t), \\ \frac{dY}{dt} &= -Y + \varepsilon^{-m_2 + n_1 + n_2} XZ, \\ \frac{dZ}{dt} &= 1 - Z - \varepsilon^{m_2 + n_1 + n_2} XY. \end{aligned} \tag{3.40}$$

Again, we will use the averaging theory So, considering $x = (X, Y, Z)T$ *we obtain*

$$F_0(t, x) = \begin{pmatrix} AU(t) \\ Y \\ 1 - Z \end{pmatrix}, \qquad (3.41)$$

Now we note that the solution $x(t, z, 0) = (X(t), Y(t), Z(t))$ *such that* $x(0, z, 0) = z = (X_0, Y_0, Z_0)$ *of the system*

$$\dot{x} = F_0(t, x) \qquad (3.42)$$

is

$$X(t) = X_0 + \int_0^t AU(s)ds, \quad Y(t) = e^{-t}Y_0, \quad Z(t) = 1 - e^{-t} + e^{-t}Z_0$$

Since $I = 0$ *and* $J(t) \neq 0$ *for* $0 < t < T$, *in order that* $x(t, z, 0)$ *is a periodic solution we need to fix* $Y_0 = 0$ *and* $Z_0 = 1$. *This implies that through every point in a neighbourhood of* X_0 *in the straight line* $Y = 0, Z = 1$ *there passes a periodic orbit lying in the phase space* $(X, Y, Z, t) \in\in \mathbb{R}^3 \times S_1$.

We have $n = 3, k = 1, \alpha = X_0$ *and* $\beta(X_0) = (0, 1)$. *Hence,* M *is a one dimensional manifold* $M = \{(X_0, 0, 1) \in \mathbb{R}^3 : X_0 \in \mathbb{R}\}$, *and the fundamental matrix* $M_z(t)$ *of* (3.42) *satisfying* $M_z(0) = Id_3$ *is*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix},$$

It is easy to see that the matrix $M_z^{-1}(0) - M^{-1}$ *has a* 1×2 *zero matrix in the upper right corner, and a* 2×2 *lower right corner matrix*

$$\Delta = \begin{pmatrix} 1 - e^T & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

with $\det(\Delta) = (1 - e^T)^2 \neq 0$. *Then, the hypotheses of Theorem 5 are satisfied. Now the components of the function* $M_z^{-1}(t)F(t, x(t, z, 0))$ *are*

$$g_1(X_0, t) = -\varepsilon^{n_2} A\left(X_0 + \int_0^t AU(s)ds\right) + AU(t),$$

$$g_2(X_0, t) = \varepsilon^{-m_2+n_1+n_2}\left(X_0 + \int_0^t AU(s)ds\right)e^t,$$

$$g_3(X_0, t) = 0.$$

Taking $n_1 = n_2 = 1$ and observing that $k = 1 and n = 3$, we are interested only in the first component of the function $F_1 = (F_{11}, F_{12}, F_3)$. Indeed, applying the averaging theory, we must study the zeros of the first component of the function

$$\mathscr{F}(X_0) = (f_1(X_0), f_2(X_0), f_3(X_0)) = \int_0^T M_z^{-1}(t,z)F_{11}(t,x(t,z))dt.$$

Since

$$F_{11} = -A\Big(X_0 + \int_0^t AU(s)ds\Big).$$

We deduce

$$\begin{aligned} f_1(X_0) &= \int_0^T -A\Big(X_0 + \int_0^t AU(s)ds\Big)dt \\ &= -ATX_0 - A^2\int_0^T \Big(\int_0^t U(s)ds\Big)ds. \end{aligned}$$

Therefore, from $f_1(X_0) = 0$ we obtain

$$X_0 = \frac{-A}{T}\int_0^T \Big(\int_0^t U(s)ds\Big)ds \neq 0$$

So, rescaling (3.32), *we get*

$$x_0 = \varepsilon^2 X_0 = -\varepsilon^2\frac{a\varepsilon^{-1}}{\varepsilon T}\int_0^T J(s)ds = -\frac{a}{T}\int_0^T J(s)ds$$

Moreover, since $f_1^{'}(x_0) = -a/T$, because a and ε are positive, the $T-$periodic orbit detected by the averaging theory is always stable. This ends the proof.

3.7 Periodic solutions of the Duffing differential equation revisited via the averaging theory

Hamel [11] in 1922 gaves the first general results for the existence of periodic solutions of the periodically forced pendulum equation

$$\ddot{y} + a\sin y = b\sin t, \tag{3.43}$$

where the dot denotes derivative with respect to the independent variable t, also called the time, and $y \in \mathbb{S}^1$ is an angle. Four years earlier this equation was the main subject of a monograph published by Duffing, who restricted his study to the periodic solutions of the following approximate equation

$$\ddot{y} + ay - cy^3 = b\sin t. \tag{3.44}$$

This equation is now known as the *Duffing differential equation*. The differential equation (3.44) describes the motion of a damped oscillator with a more complicated potential than in the harmonic motion (i.e. when $c = 0$). As usual the parameter a controls the size of stiffness, b controls the amplitude of the periodic driving force, and c controls the amount of nonlinearity in the restoring force. In particular, equation (3.44) models a spring pendulum such that its spring's stiffness only obey approximately the Hooke's law.

Many other different classes of Duffing differential equations have been investigated by several authors. They are mainly interested in the existence of periodic solutions, in their multiplicity, stability, bifurcation, ... See for instance the good survey of Mawhin [17].

In this work we shall study the periodic solutions of the Duffing differential equations (3.43) and (3.44), where a, b and c are real parameters, via the averaging theory.

Our main results on the periodic solutions of the Duffing differential equation (3.43) are the following.

Theorem 18 *Let ε be a small parameter. The Duffing differential equation* (3.43) *has*

(*a*) *four periodic solutions* $y_1(t) = -b\sin t + O(\varepsilon)$, $y_2(t) = \pi - b\sin t + O(\varepsilon)$, $y_3(t) = O(\varepsilon)$, $y_4(t) = \pi + O(\varepsilon)$ *if* $ab \neq 0$, $a = O(\varepsilon^2)$ *and* $b = O(\varepsilon)$;

(*b*) *two periodic solutions* $y_i(t)$ *for* $i = 3, 4$ *if* $b = 0$ *and* $a \neq 0$;

(*c*) *infinitely many periodic solutions* $y(t) = k - b\sin t$ *with* $k \in \mathbb{R}$ *if* $a = 0$ *and* $b \neq 0$;

(*d*) *no periodic solutions if* $a = b = 0$.

Theorem 18 will be proved in subsection 3.7.1 using the averaging theorems given in the Appendix.

Our main results on the periodic solutions of the differential system (3.44) are the following.

Theorem 19 *The Duffing differential equation* (3.44) *has*

(*a*) *one periodic solution* $y(t) = -\sqrt[3]{4b/(3c)}\sin t + O(\varepsilon)$, *if* $bc \neq 0$, $a = 1$ *and* $b = O(\varepsilon)$ *and* $c = O(\varepsilon)$;

(*b*) *one periodic solution* $y(t) = O(\varepsilon)$, *if* $b \neq 0$, $a = O(\varepsilon)$, $b = O(\varepsilon)$ *and* $c = O(\varepsilon^2)$;

(*c*) *two periodic solutions* $y_{\pm}(t) = \pm\sqrt{a/c}$, *if* $ac > 0$, $b \neq 0$, $a = O(\varepsilon^2)$, $b = O(\varepsilon)$ *and* $c = O(\varepsilon^2)$;

(*d*) *three periodic solutions* $y(t) = y_0$, *where* $y_0 \in \{0, \pm\sqrt{a/c}\}$ *if* $ac > 0$, $b \neq 0$, $a = O(\varepsilon^2)$, $b = O(\varepsilon^2)$ *and* $c = O(\varepsilon^2)$.

Theorem 19 will be proved in subsection 3.7.2 using three different averaging theorems.

3.7.1 Proof of Theorem 18

Instead of working with the Duffing differential equation (3.43) we shall work with the equivalent differential systems

$$\begin{aligned} \dot{x} &= -a\sin y + b\sin t, \\ \dot{y} &= x. \end{aligned} \tag{3.45}$$

In order to apply the theorems of the averaging theory of first order, given in the Appendix, for studying the periodic solutions of the differential system (3.45) we scale the variables and the parameters of this differential system.

We start doing a scaling of the variables (x, y) and of the parameter a and b as follows

$$x = \varepsilon^{m_1} X, \quad y = \varepsilon^{m_2} Y, \quad a = \varepsilon^{n_1} A, \quad b = \varepsilon^{n_2} B, \tag{3.46}$$

where m_1, m_2, n_1 and n_2 are integers such that the differential equation (3.45) becomes

$$\begin{aligned} \dot{X} &= -\varepsilon^{n_1-m_1} A\sin(\varepsilon^{m_2} Y) + \varepsilon^{n_2-m_1} B\sin t, \\ \dot{Y} &= \varepsilon^{m_1-m_2} X, \end{aligned} \tag{3.47}$$

where $m_1 - m_2$, $n_1 + m_2 - m_1$ (because $\sin(\varepsilon^{m_2} Y) = \mathcal{O}(\varepsilon^{m_2})$) and $n_2 - m_1$ must be non–negative integers such that

$$\{m_1 - m_2, n_1 + m_2 - m_1, n_2 - m_1\} \cap \{1\} \neq \emptyset,$$

because we want that the differential system (3.47) has some term of order one in ε in order to apply the averaging theory with respect to the small parameter ε of order one. Also we do not consider the case $n_2 - m_1 > 1$, otherwise in the averaging theory the term $b \sin t$ of system (3.45) would not contribute to the existence of periodic solutions, and we want to take it into account. Therefore, we distinguish the following seven cases

Case I: $m_1 - m_2 = 0$, $n_1 + m_2 - m_1 = 0$ and $n_2 - m_1 = 1$;

Case II: $m_1 - m_2 = 0$ and $n_1 + m_2 - m_1 = 1$;

Case III: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 = 0$;

Case IV: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 = 1$;

Case V: $m_1 - m_2 > 1$ and $n_1 + m_2 - m_1 = 1$;

Case VI: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 > 1$;

Case VII: $m_1 - m_2 > 1$, $n_1 + m_2 - m_1 > 1$ and $n_2 - m_1 = 1$;

and every case $\alpha \in \{II, III, \ldots, VI\}$ is separated into the following two subcases:

$$\begin{array}{ll} (\alpha.1) & n_2 - m_1 = 0, \\ (\alpha.2) & n_2 - m_1 = 1. \end{array}$$

We have applied the three theorems of averaging of the Appendix for studying the existence of periodic solutions in the 12 previous subcases of differential systems (3.47). As we shall see the proof of Theorem 18 when $ab \neq 0$ will come from the subcase (IV.1), and when $a \neq 0$ and $b = 0$ from the subcase (IV.2). All the other subcases, either do not satisfy the hypotheses of one of the three theorems of the averaging, or provide partial results of the ones stated in Theorem 18. Consequently, in what follows we only provide the details of the more positive results, i.e. we shall give the proofs of statements (a) and (b) of Theorem 18 only considering the subcases (IV.1) and (IV.2). The proofs of statements (c) and (d) are done without using the averaging theory.

Proof 20 (Proof of statement (a) of Theorem 18.) *For the case (IV.1), i.e. for*

$$m_1 - m_2 = 1, \quad n_1 + m_2 - m_1 = 1, \quad n_2 - m_1 = 0,$$

we take

$$m_1 = 1, \quad m_2 = 0, \quad n_1 = 2, \quad n_2 = 1. \tag{3.48}$$

Then system (3.47) *becomes*

$$\begin{aligned} \dot{X} &= -\varepsilon A \sin Y + B \sin t, \\ \dot{Y} &= \varepsilon X. \end{aligned} \tag{3.49}$$

Now we shall apply the averaging Theorem 20 to system (3.49). *In what follows we use the notation of Theorem 20. Thus* $\mathbf{x} = (X, Y)^T$ *and*

$$F_0(t, \mathbf{x}) = \begin{pmatrix} B \sin t \\ 0 \end{pmatrix}, \quad F_1(t, \mathbf{x}) = \begin{pmatrix} -A \sin Y \\ X \end{pmatrix}, \quad F_2(t, \mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In order to apply some of the three averaging theorems of the appendix for studying the periodic solutions of the differential system (3.49) *we must consider that the functions F_i for $i = 0, 1, 2$ are defined in $\mathbb{R} \times \Omega$, where Ω is a bounded open subset of $\mathbb{R}^2$, here we take Ω equal to the disc of center* $(0,0)$ *and radius $k+1$, being k the positive integer of the statement of Theorem 18.*

The unperturbed differential system (3.68) *(i.e. in our case system* (3.49) *with $\varepsilon = 0$) has the solution*

$$\mathbf{x}(t, \mathbf{z}, 0) = (X(t), Y(t))^T = (B + X_0 - B \cos t, Y_0)^T. \tag{3.50}$$

such that $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z} = \alpha = (X_0, Y_0)^T$. *All these solutions are $2\pi-$periodic if and only if $B \neq 0$, here we use the assumption that $b \neq 0$. Then, since $m = n = 2$ using the notation of Theorem 20, we have ξ = identity and the conditions (a) and (b) of Theorem 20 are satisfied trivially. So system* (3.49) *satisfies all the assumptions of Theorem 20, consequently in what follows we apply this theorem for studying the periodic solutions of system* (3.49). *We compute for our system* (3.49) *the fundamental matrix $M_{\mathbf{z}}(t)$ associated to the first variational system* (3.69) *such that $M_{\mathbf{z}}(0) = Id$ of $\mathbb{R}^2$, and we obtain*

$$M_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we must compute the function $\mathscr{F}(\alpha) = \mathscr{F}(X_0, Y_0)$ *given in* (3.70), *i.e.*

$$
\begin{aligned}
\mathscr{F}(X_0,Y_0) &= \begin{pmatrix} F_1(X_0,Y_0) \\ F_2(X_0,Y_0) \end{pmatrix} = \int_0^{2\pi} M_{\mathbf{z}}^{-1}(t) F_1(t,\mathbf{x}(t,\mathbf{z},0))\,dt, \\
&= \begin{pmatrix} -\displaystyle\int_0^{2\pi} A\sin Y_0\,ds \\ \displaystyle\int_0^{2\pi} (X_0 + B - B\cos s)\,ds \end{pmatrix} = 2\pi \begin{pmatrix} -A\sin Y_0 \\ B + X_0 \end{pmatrix}.
\end{aligned}
$$

Therefore, solving the system $\mathscr{F}(X_0,Y_0) = (0,0)$, *we obtain in* Ω *the solutions*

$$\alpha_j = (X_{0,j}, Y_{0,j}) = (-B, j\pi) \quad \text{for } j = -k,\dots,-1,0,1,\dots,k.$$

Moreover we have that the Jacobian

$$\det\left(\frac{\partial \mathscr{F}}{\partial \alpha}(\alpha_j)\right) = (-1)^k A \neq 0,$$

because by assumption $a \neq 0$. *Hence Theorem 20 says that if* $AB \neq 0$ *then for every solution* $(X_{0,j}, Y_{0,j}) = (-B, j\pi)$ *of the system* $\mathscr{F}(X_0,Y_0) = 0$, *the differential system* (3.49) *with* $\varepsilon = \varepsilon(k) \neq 0$ *sufficiently small has a* $2\pi-$ *periodic solution* $(X(t,\varepsilon), Y(t,\varepsilon))$ *such that* $(X(0,\varepsilon), Y(0,\varepsilon) \to (-B, j\pi)$ *when* $\varepsilon \to 0$. *So, from* (3.49) *and* (3.50) *the periodic solution* $(X(t,\varepsilon), Y(t,\varepsilon))$ *tends to the solution*

$$(X(t), Y(t)) \approx \big(-B\cos t, j\pi\big) \tag{3.51}$$

for ε *sufficiently small, i.e.*

$$(X(t,\varepsilon), Y(t,\varepsilon)) = \big(-B\cos t + O(\varepsilon), j\pi - \varepsilon B\sin t + O(\varepsilon^2)\big).$$

After the change of variables (3.46) *satisfying* (3.48), *i.e.*

$$x = \varepsilon X, \quad y = Y, \quad b = \varepsilon B, \quad a = \varepsilon^2 A,$$

we obtain that the 2π*–periodic solution* (3.51) *of system* (3.49) *becomes the* 2π*–periodic solution*

$$(x(t), y(t)) \approx \big(-b\cos t, j\pi - b\sin t\big)$$

of system (3.45). *Now taking into account that* y *is an angle, doing modulo* 2π *these* 2π*–periodic solutions of system* (3.45) *provide the following two* 2π*–periodic solutions* *(i)* $y(t) \approx -b \sin t$ *if* j *is even,*

(ii) $y(t) \approx \pi - b \sin t$ *if* j *is odd,*

of the differential equation (3.43). *This ends the proof of statement (a) of Theorem 18.*

Proof 21 (Proof of statement (b) **of Theorem 18.)** *For the case (IV.2), i.e. for*

$$m_1 - m_2 = 1, \quad n_1 + m_2 - m_1 = 1, \quad n_2 - m_1 = 1;$$

we take

$$m_1 = 1, \quad m_2 = 0, \quad n_1 = 2, \quad n_2 = 2. \tag{3.52}$$

Then system (3.47) *becomes*

$$\begin{aligned} \dot{X} &= -\varepsilon A \sin Y + \varepsilon B \sin t, \\ \dot{Y} &= \varepsilon X. \end{aligned} \tag{3.53}$$

We shall apply the averaging Theorem 22 to system (3.53). *In what follows we shall use the notation of system* (3.72) *and of Theorem 22. Thus* $\mathbf{x} = (X, Y)^T$ *and*

$$F_1(t, \mathbf{x}) = \begin{pmatrix} -A \sin Y + B \sin t \\ X \end{pmatrix}, \qquad F_2(t, \mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We must compute the function $g(\mathbf{y})$ *given in* (3.74), *i.e.*

$$g(\mathbf{y}) = \frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix} -A \sin Y_0 + B \sin s \\ X_0 \end{pmatrix} ds = \begin{pmatrix} -A \sin Y_0 \\ X_0 \end{pmatrix}.$$

Then, due to the fact that Y *is an angle we have two solutions for* (X_0, Y_0), *namely* $(X_1, Y_1) = (0, 0)$ *and* $(X_2, Y_2) = (0, \pi)$. *Since the Jacobian* (3.75) *in these two solutions is* A *and* $-A$ *respectively, and by assumptions* $a \neq 0$, *by Theorem 22 the system* (3.53) *has two periodic solutions* $(X_i(t, \varepsilon), Y_i(t, \varepsilon))$ *such that* $(X_i(0, \varepsilon), Y_i(0, \varepsilon))$ *tends to* (X_i, Y_i) *when* $\varepsilon \to 0$. *So by statement (a) of Theorem 22 we have*

$$(X_i(t, \varepsilon), Y_i(t, \varepsilon)) \approx \big(O(\varepsilon), Y_i + O(\varepsilon)\big) \tag{3.54}$$

for ε *sufficiently small.*

After the change of variables (3.46) *satisfying* (21), *i.e.*

$$x = \varepsilon X, \quad y = Y, \quad b = \varepsilon^2 B, \quad a = \varepsilon^2 A,$$

we obtain that the two periodic solutions (3.54) *of system* (3.53) *becomes the two periodic solutions*

$$(x_1(t), y_1(t)) \approx \big(O(\varepsilon), O(\varepsilon)\big), \quad (x_2(t), y_2(t)) \approx \big(O(\varepsilon), \pi + O(\varepsilon)\big),$$

of system (3.45). *This ends the proof of statement (b) of Theorem 18.*

Proof 22 (Proof of statement (c) **of Theorem 18.)** *Now the differential system* (3.45) *becomes*

$$\dot{x} = b\sin t, \qquad \dot{y} = x.$$

Its general solution is $x(t) = k_1 - b\cos t$ *and* $y(t) = k_1 t - b\sin t + k_2$, *where* k_1 *and* k_2 *are arbitrary constants. So, clearly the unique periodic solutions of the differential equation* (3.43) *are* $y(t) = -b\sin t + k_2$. *This proves the statement* (c) *Theorem 18.*

Proof 23 (Proof of statement (d) **of Theorem 18.)** *Under the assumptions of statement (d) the differential system* (3.45) *becomes*

$$\dot{x} = 0, \qquad \dot{y} = x.$$

Its general solution is $x(t) = k_1$ *and* $y(t) = k_1 t + k_2$, *where* k_1 *and* k_2 *are arbitrary constants. So the system has no periodic solutions.*

3.7.2 Proof of Theorem 19

Instead of working with the Duffing differential equation (3.44) we shall work with the equivalent differential system

$$\begin{aligned} \dot{x} &= -ay + cy^3 + b\sin t, \\ \dot{y} &= x. \end{aligned} \tag{3.55}$$

Again in order to apply the three theorems of the averaging theory of first order for studying the periodic solutions of the differential system (3.55) we scale the variables and the parameters of this differential system.

We start doing a rescaling of the variables (x, y) and of the parameter a, b and c as follows

$$x = \varepsilon^{m_1} X, \quad y = \varepsilon^{m_2} Y, \quad a = \varepsilon^{n_1} A, \quad b = \varepsilon^{n_2} B, \quad c = \varepsilon^{n_3} C, \tag{3.56}$$

where m_1, m_2, n_1, n_2 and n_3 are integers such that the differential equation (3.55) becomes

$$\begin{aligned} \dot{X} &= -\varepsilon^{n_1+m_2-m_1} AY + \varepsilon^{n_3+3m_2-m_1} CY^3 + \varepsilon^{n_2-m_1} B\sin t, \\ \dot{Y} &= \varepsilon^{m_1-m_2} X, \end{aligned} \tag{3.57}$$

where $m_1 - m_2$, $n_1 + m_2 - m_1$, $n_3 + 3m_2 - m_1$ and $n_2 - m_1$ must be non–negative integers such that

$$\{m_1 - m_2, n_1 + m_2 - m_1, n_3 + 3m_2 - m_1, n_2 - m_1\} \cap \{1\} \neq \emptyset.$$

We essentially distinguish the same seven cases of section 3.7.1, i.e. Therefore, we distinguish the following seven cases

Case I: $m_1 - m_2 = 0$ and $n_1 + m_2 - m_1 = 0$;

Case II: $m_1 - m_2 = 0$ and $n_1 + m_2 - m_1 = 1$;

Case III: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 = 0$;

Case IV: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 = 1$;

Case V: $m_1 - m_2 > 1$ and $n_1 + m_2 - m_1 = 1$;

Case VI: $m_1 - m_2 = 1$ and $n_1 + m_2 - m_1 > 1$;

Case VII: $m_1 - m_2 > 1$ and $n_1 + m_2 - m_1 > 1$.

Every case $\alpha \in \{II, III, \ldots, VI\}$ is divided into the following four sub-cases:

$$\begin{array}{llll} (\alpha.1) & n_3 + 3m_2 - m_1 = 0 & \text{and} & n_2 - m_1 = 0, \\ (\alpha.2) & n_3 + 3m_2 - m_1 = 0 & \text{and} & n_2 - m_1 = 1, \\ (\alpha.3) & n_3 + 3m_2 - m_1 = 1 & \text{and} & n_2 - m_1 = 0, \\ (\alpha.4) & n_3 + 3m_2 - m_1 = 1 & \text{and} & n_2 - m_1 = 1; \end{array}$$

and the cases I and VII are divided only into the following three sub-cases

$$\begin{array}{llll} (\alpha.2) & n_3 + 3m_2 - m_1 = 0 & \text{and} & n_2 - m_1 = 1, \\ (\alpha.3) & n_3 + 3m_2 - m_1 = 1 & \text{and} & n_2 - m_1 = 0, \\ (\alpha.4) & n_3 + 3m_2 - m_1 = 1 & \text{and} & n_2 - m_1 = 1. \end{array}$$

We have applied the three theorems of averaging (see the Appendix) for studying the existence of periodic solutions of the 26 previous sub-cases of differential systems (3.57). As we shall see in the proof of Theorem 19 statement (a) will come from the subcase (I.4), statement (b) from the subcase (III.3), statement (c) from the subcase (IV.3), statement (d) from the subcase (IV.4). All the other subcases, either do not satisfy the hypotheses of one of the three theorems of the averaging, or provide partial results of the ones stated in Theorem 19. As in the proof of Theorem 18 we only provide the details of the positive results in the proof of Theorem 19. We separate its proof into its four statements.

Proof 24 (Proof of statement (a) **of Theorem 19)** *For the case (I.4), i.e. for*

$$m_1 - m_2 = 0, \quad n_1 + m_2 - m_1 = 0, \quad n_3 + 3m_2 - m_1 = 1, \quad n_2 - m_1 = 1;$$

we take

$$m_1 = m_2 = n_1 = 0, \quad n_2 = n_3 = 1. \tag{3.58}$$

Then system (3.57) *becomes*

$$\begin{aligned} \dot{X} &= -AY + \varepsilon\big(CY^3 + B\sin t\big), \\ \dot{Y} &= X. \end{aligned} \tag{3.59}$$

We shall apply the averaging Theorem 20 to system (3.59) *and we shall obtain the statement* (a) *of Theorem 19. In what follows we shall use the notation of Theorem 20. Thus* $\mathbf{x} = (X, Y)^T$ *and*

$$F_0(t,\mathbf{x}) = \begin{pmatrix} -AY \\ X \end{pmatrix}, \quad F_1(t,\mathbf{x}) = \begin{pmatrix} CY^3 + B\sin t \\ 0 \end{pmatrix}, \quad F_2(t,\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The unperturbed differential system (3.68) *only has periodic solutions* $\mathbf{x}(t,\mathbf{z},0) = (X(t), Y(t))^T$ *such that* $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z} = (X_0, Y_0)^T$ *if* $A > 0$ *given by*

$$\begin{aligned} X(t) &= X_0 \cos\left(\sqrt{A}t\right) - Y_0\sqrt{A}\sin\left(\sqrt{A}t\right), \\ Y(t) &= \frac{X_0}{\sqrt{A}}\sin\left(\sqrt{A}t\right) + Y_0\cos\left(\sqrt{A}t\right). \end{aligned}$$

In order that $\mathbf{x}(t,\mathbf{z},0)$ *be a periodic solution of period* $T = 2\pi$*, the period of the function* $\sin t$ *and we can apply the averaging theory, we must choose* $A = 1$*. So we get*

$$(X(t), Y(t)) = (X_0\cos t - Y_0 \sin t, X_0 \sin t + Y_0 \cos t), \tag{3.60}$$

where $(X_0, Y_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

Now we compute the fundamental matrix $M_{\mathbf{z}}(t)$ *associated to the first variational system* (3.69) *such that* $M_{\mathbf{z}}(0) = Id$ *of* $\mathbb{R}^2$*, and we obtain*

$$M_{\mathbf{z}}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Again since $m = n = 2$ *in Theorem 20 we do not need to check any condition for applying the averaging theory described in Theorem 20 for studying the periodic solutions* (3.60)*, of system* (3.59) *with* $\varepsilon = 0$ *and* $A = 1$*, which can be prolonged to the perturbed system* (3.59) *with* $\varepsilon \neq 0$ *sufficiently small and* $A = 1$*. Note that here* $\mathbf{z} = \alpha$*. Moreover, since for our* *differential system we have* $\xi(X,Y)=(X,Y)$*, we must compute the function* (3.70)*, i.e.*

$$\mathscr{F}(X_0,Y_0)=\int_0^{2\pi} M_{\mathbf{z}}^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z},0))\,dt$$

$$=2\pi\begin{pmatrix}\frac{3}{8}CY_0\left(X_0^2+Y_0^2\right)\\ \frac{1}{8}\left(-4B-3CX_0\left(X_0^2+Y_0^2\right)\right)\end{pmatrix}.$$

The system $\mathscr{F}(X_0,Y_0)=(0,0)$ *has three solutions, but only one is real, namely*

$$(X_0^*,Y_0^*)=\left(-\sqrt[3]{\frac{4B}{3C}},0\right).$$

Since

$$\det\left(\frac{\partial\mathscr{F}(X_0,Y_0)}{\partial(X_0,Y_0)}\Big|_{(X_0,Y_0)=(X_0^*,Y_0^*)}\right)=\frac{3^{\frac{5}{3}}B^{\frac{4}{3}}C^{\frac{2}{3}}}{2^{\frac{10}{3}}}\neq 0,$$

Theorem 20 says that the periodic solution

$$(X(t),Y(t))=\left(-\sqrt[3]{\frac{4B}{3C}}\cos t,-\sqrt[3]{\frac{4B}{3C}}\sin t\right)$$

of the differential system (3.59) *with* $\varepsilon=0$ *can be prolonged to a periodic solution of system* (3.59) *with* $\varepsilon\neq 0$ *sufficiently small. Therefore, going back through the change of variables* (3.56) *satisfying* (3.58)*, i.e.*

$$x=X,\quad y=Y,\quad a=A,\quad b=\varepsilon B,\quad c=\varepsilon C,$$

we get that the differential system (3.55) *has the* 2π*–periodic solution given in the statement (a) of Theorem 19.*

Proof 25 (Proof of statement (b) **of Theorem 19)** *For the case (III.3), i.e. for*

$$m_1-m_2=1,\quad n_1+m_2-m_1=0,\quad n_3+3m_2-m_1=1,\quad n_2-m_1=0,$$

we take

$$m_1=n_1=n_2=1,\quad m_2=0,\quad n_3=2. \tag{3.61}$$

Then system (3.57) *becomes*

$$\begin{aligned} \dot{X} &= -AY + \varepsilon CY^3 + B\sin t, \\ \dot{Y} &= \varepsilon X, \end{aligned} \tag{3.62}$$

We shall apply the averaging Theorem 20 to system (3.62) *and we shall obtain the statement* (*b*) *of Theorem 19. In what follows we shall use the notation of Theorem 20. Thus* $\mathbf{x} = (X, Y)^T$ *and*

$$F_0(t,\mathbf{x}) = \begin{pmatrix} -AY + B\sin t \\ 0 \end{pmatrix}, \quad F_1(t,\mathbf{x}) = \begin{pmatrix} CY^3 \\ X \end{pmatrix},$$

$$F_2(t,\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The unperturbed differential system (3.68) *has the solution*

$$\mathbf{x}(t,\mathbf{z},0) = (X(t), Y(t))^T = (X_0 + B - B\cos t - AY_0 t, Y_0)\,.$$

such that $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z} = (X_0, Y_0)^T$. *In order that* $\mathbf{x}(t,\mathbf{z},0)$ *be a* $2\pi-$*periodic solution we must choose* $B \neq 0$ *and* $Y_0 = 0$. *Then we get*

$$\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) = (X_0 + B - B\cos t, 0)\,,$$

Therefore, using the notation of Theorem 20, we have $n = 2$ *and* $k = 1$ *for each one of these possible families of periodic solution.*

We compute the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ *associated to the first variational system* (3.69) *such that* $M_{\mathbf{z}_\alpha}(0) = Id$ *of* $\mathbb{R}^2$, *and we obtain*

$$M_{\mathbf{z}_\alpha}(t) = \begin{pmatrix} 1 & -At \\ 0 & 1 \end{pmatrix}.$$

Since the matrix

$$M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(2\pi) = \begin{pmatrix} 0 & -2A\pi \\ 0 & 0 \end{pmatrix}$$

has a non–zero 1×1 *matrix in the upper right corner, and a zero* 1×1 *matrix in its lower right corner. Therefore we can apply Theorem 22 of averaging if* $A \neq 0$, *then by applying this theorem we study the periodic solutions which can be prolonged from the unperturbed differential system to the perturbed one.*

Now we must compute the function $\mathscr{F}(\alpha) = \mathscr{F}(X_0, 0)$ *given in* (3.70), *i.e.*

$$\begin{aligned}
\mathscr{F}(X_0,0) &= \int_0^{2\pi} M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))\,dt \\
&= \begin{pmatrix} \displaystyle\int_0^{2\pi} A(X_0 + B - B\cos s)\, s\,ds \\ \displaystyle\int_0^{2\pi} (X_0 + B - B\cos s)\,ds \end{pmatrix} \\
&= \begin{pmatrix} 2\pi^2 A(B + X_0) \\ 2\pi(B + X_0) \end{pmatrix}.
\end{aligned}$$

The system $\mathscr{F}(X_0, Y_0) = (0,0)$ *has a solution*

$$(X_0, 0) = (-B, 0).$$

Hence Theorem 22 says that if $B \neq 0$ *and for every simple real root* $(X_0, 0)$ *of the system* $\mathscr{F}(X_0, 0) = (0,0)$, *the differential system* (3.62) *with* $\varepsilon \neq 0$ *sufficiently small has one* $2\pi-$*periodic solution*

$$(X(t), Y(t)) = (-B\cos t, 0)$$

which can be prolonged for to a periodic solution

$$(X(t,\varepsilon), Y(t,\varepsilon)) = (-B\cos t + O(\varepsilon), O(\varepsilon)).$$

Therefore, going back through the change of variables (3.56) *satisfying* (3.61), *i.e.*

$$\frac{x}{\varepsilon} = X, \quad y = Y, \quad C = \frac{c}{\varepsilon^2}, \quad B = \frac{b}{\varepsilon}, \quad A = \frac{a}{\varepsilon}.$$

we get that the differential system (3.55) *has the* $2\pi-$*periodic solution given in the statement (b) of Theorem 19.*

Proof 26 (Proof of statement (c) **of Theorem 19)** *For the case (IV.3), i.e. for*

$$m_1 - m_2 = 1, \quad n_1 + m_2 - m_1 = 1, \quad n_3 + 3m_2 - m_1 = 1, \quad n_2 - m_1 = 1$$

we take

$$n_2 = m_1 = 1, m_2 = 0, \qquad n_1 = n_3 = 2. \tag{3.63}$$

Then system (3.57) *becomes*

$$\begin{aligned} \dot{X} &= -\varepsilon AY + \varepsilon CY^3 + B\sin t, \\ \dot{Y} &= \varepsilon X, \end{aligned} \tag{3.64}$$

We shall apply the averaging Theorem 20 to system (3.64) *and we shall obtain the statement* (*c*) *of Theorem 19. In what follows we shall use the notation of Theorem 20. Thus* $\mathbf{x} = (X, Y)^T$ *and*

$$F_0(t,\mathbf{x}) = \begin{pmatrix} B\sin t \\ 0 \end{pmatrix}, \quad F_1(t,\mathbf{x}) = \begin{pmatrix} -AY + CY^3 \\ X \end{pmatrix}, \quad F_2(t,\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The unperturbed differential system (3.68) *only has periodic solutions* $\mathbf{x}(t,\mathbf{z},0) = (X(t), Y(t))^T$ *such that* $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z} = (X_0, Y_0)^T$ *given by*

$$X(t) = X_0 + B - B\cos t, \quad Y(t) = Y_0.$$

All these solutions are $2\pi-$*periodic if and only if* $B \neq 0$.

Since $k = n = 2$ *in the Theorem 20, we do not need to check any condition for applying the Theorem 20.*

We compute the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ *associated to the first variational system* (3.69) *associated to the vector field* $(\dot{Y}, \dot{X})$ *given by* (3.64) *with* $\varepsilon = 0$, *and such that* $M_{\mathbf{z}_\alpha}(0) = Id$ *of* $\mathbb{R}^2$.

We obtain

$$M_{\mathbf{z}_\alpha}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then by applying this theorem we study the periodic solutions which can be prolonged from the unperturbed differential system to the perturbed one, then we must compute the function $\mathscr{F}(\alpha) = \mathscr{F}(X_0, Y_0)$ *given in* (3.70), *i.e.*

$$\begin{aligned} \mathscr{F}(X_0, Y_0) &= \xi^{\perp}\left(\int_0^{2\pi} M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))\, dt\right) \\ &= \begin{pmatrix} -AY_0 + CY_0^3 \\ B + X_0 \end{pmatrix} \end{aligned}$$

The system $\mathscr{F}(X_0, Y_0) = (0,0)$ *for* $AC > 0$ *has three solutions* $(-B, Y_0)$ *with* $Y_0 \in \left\{0, \ \pm\sqrt{\dfrac{A}{C}}\right\}$, *we want to study the only solutions* $\left(-B, \pm\sqrt{\dfrac{A}{C}}\right)$ *because the solution* $(-B, 0)$ *appears in the statement* (*b*) *of Theorem 19.*

On the other hand we have the Jacobian matrix for the function $\mathscr{F}$ *on* (X_0, Y_0) *gives by*

$$\det\left(\left.\frac{\partial \mathscr{F}(X_0, Y_0)}{\partial (X_0, Y_0)}\right|_{(X_0, Y_0) = \left(-B, \pm\sqrt{\frac{A}{C}}\right)}\right) = -2A.$$

Theorem 22 says that if $A \neq 0$ and for two simple real roots $\left(-B, \pm\sqrt{\dfrac{A}{C}}\right)$ *of the function* $\mathscr{F}(X_0, Y_0)$, *the differential system* (3.64) *has two* $2\pi-$ *periodic solution* $(X(t), Y(t)) = (-B\cos t, \pm\sqrt{\dfrac{A}{C}})$ *with* $\varepsilon = 0$ *can be prolonged to a periodic solution*

$$(X(t,\varepsilon), Y(t,\varepsilon)) = \left(-B\cos t + O(\varepsilon), \pm\sqrt{\frac{A}{C}} - \varepsilon B\sin t + O(\varepsilon)\right).$$

of system (3.64) *with* $\varepsilon \neq 0$ *sufficiently small. Therefore, going back through the change of variables* (3.56) *satisfying* (3.63), *i.e.*

$$\frac{x}{\varepsilon} = X, \quad y = Y, \quad C = \frac{c}{\varepsilon^2}, \quad B = \frac{b}{\varepsilon}, \quad A = \frac{a}{\varepsilon^2}.$$

we get that the differential system (3.55) *has the tow* 2π*–periodic solution given in the statement (c) of Theorem 19.*

Proof 27 (Proof of statement (d) **of Theorem 19)** *For the case (IV.4)*

$$m_1 - m_2 = 1, \quad n_1 + m_2 - m_1 = 1, \quad n_3 + 3m_2 - m_1 = 1, \quad n_2 - m_1 = 1,$$

we take

$$m_1 = 1, \quad m_2 = 0, \qquad n_3 = n_2 = n_1 = 2. \tag{3.65}$$

Then system (3.57) *becomes*

$$\begin{aligned} \dot{X} &= -\varepsilon AY(t) + \varepsilon CY^3(t) + \varepsilon B\sin t \\ \dot{Y} &= \varepsilon X(t) \end{aligned} \tag{3.66}$$

We apply directly the Theorem 22 to system (3.66) *and we shall obtain the statement* (d) *of Theorem 19.*

In what follows we shall use the notation of Theorem 20 and Theorem 22. Thus $\mathbf{x} = (X, Y)^T$ *and*

$$F_0(t,\chi) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad F_1(t,\chi) = \begin{pmatrix} -AY(t) + CY^3(t) + B\sin t \\ X(t) \end{pmatrix}.$$

We must compute the function $g(\mathbf{y}) = \mathscr{G}(X_0, Y_0)$ *given in* (3.74), *i.e*

$$g(\mathbf{y}) = \begin{pmatrix} -AY_0 + CY_0^3 \\ X_0 \end{pmatrix}.$$

g is periodic of period $T = 2\pi$. If $AC > 0$, the system $\mathcal{G}(X_0, Y_0) = (0,0)$ has three solutions

$$(X_0, Y_0) = (0, Y_0) \text{ where } Y_0 \in \left\{0, \pm\sqrt{\frac{A}{C}}\right\}.$$

Since

$$\det\left(\frac{\partial\mathcal{G}(X_0, Y_0)}{\partial(X_0, Y_0)}\bigg|_{(X_0,Y_0)=(0,0)}\right) = A,$$

and

$$\det\left(\frac{\partial\mathcal{G}(X_0, Y_0)}{\partial(X_0, Y_0)}\bigg|_{(X_0,Y_0)=\left(0,\pm\sqrt{\frac{A}{C}}\right)}\right) = -2A,$$

Theorem 22 says that if $A \neq 0$ and for every simple real root $(0, Y_0)$ of the function $g(\mathbf{y})$,the differential system (3.66) has three $2\pi-$ periodic solutions

$$(X(t), Y(t)) = (0, Y_0) \qquad Y_0 \in \left\{0, \ \pm\sqrt{\frac{A}{C}}\right\}.$$

with $\varepsilon = 0$ can be prolonged to the three periodic solutions

$$(X(t,\varepsilon), Y(t,\varepsilon)) = (-\varepsilon B(\cos t - 1) + O(\varepsilon), O(\varepsilon)) \quad \text{for } (X_0, Y_0) = (0,0).$$

and

$$(X(t,\varepsilon), Y(t,\varepsilon)) = \left(-\varepsilon B(\cos t - 1) + O(\varepsilon), \pm\sqrt{\frac{A}{C}} + O(\varepsilon)\right),$$

for $(X_0, Y_0) = \left(0, \pm\sqrt{A/C}\right)$. Therefore, going back through the change of variables (3.56) satisfying (3.65), i.e.

$$\frac{x}{\varepsilon} = X, \quad y = Y, \quad C = \frac{c}{\varepsilon^2}, \quad B = \frac{b}{\varepsilon^2}, \quad A = \frac{a}{\varepsilon^2}.$$

We get that the differential system (3.55) has the three 2π–periodic solutions given in the statement (d) of Theorem 19.

3.7.3 Appendix: Periodic solutions via the averaging theory

We consider the problem of bifurcation of T–periodic solutions from the differential systems of the form

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{3.67}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. The functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathscr{C}^2$, T–periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}), \tag{3.68}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of system (3.68) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{3.69}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ a fundamental matrix of the linear differential system (3.69), by $\xi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ and $\xi^{\perp} : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ the projections of $\mathbb{R}^n$ onto its first m and $n-m$ coordinates respectively; i.e. $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_m)$, and $\xi^{\perp}(x_1, \ldots, x_n) = (x_{m+1}, \ldots, x_n)$

Theorem 20 *Let $V \subset \mathbb{R}^m$ be open and bounded, let $\beta_0 : Cl(V) \to \mathbb{R}^{n-m}$ be a $\mathscr{C}^k$ function and $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) \,|\, \alpha \in Cl(V)\} \subset \Omega$ its graphic in $\mathbb{R}^n$. Assume that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$ of (3.68) is T-periodic and that there exists a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (3.69) such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$*

(*a*) *has in the lower right corner the $(n-m)\times(n-m)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$, and*

(*b*) *has in the upper right corner the $m \times (n-m)$ zero matrix.*

Consider the function $\mathscr{F} : Cl(V) \to \mathbb{R}^m$ defined by

$$\mathscr{F}(\alpha) = \xi\left(\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))\,dt\right). \tag{3.70}$$

Suppose that there is $\alpha_0 \in V$ with $\mathscr{F}(\alpha_0) = 0$, then the following statements hold for $\varepsilon \neq 0$ sufficiently small.

(*i*) *If $\det((\partial\mathscr{F}/\partial\alpha)(\alpha_0)) \neq 0$, then there is a unique T-periodic solution $\mathbf{x}(t,\varepsilon)$ of system (3.67) such that $\mathbf{x}(t,\varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as $\varepsilon \to 0$.*

(*ii*) *If $m = 1$ and $\mathscr{F}'(\alpha_0) = \cdots = \mathscr{F}^{(s-1)}(\alpha_0) = 0$ and $\mathscr{F}^{(s)}(\alpha_0) \neq 0$ with $s \leq k$, then there are at most s T-periodic solutions $\mathbf{x}_1(t,\varepsilon), \ldots, \mathbf{x}_s(t,\varepsilon)$ of system (3.67) such that $\mathbf{x}_i(t,\varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as $\varepsilon \to 0$ for $i = 1, \ldots, s$.*

Theorem 20 is a classical result due to Malkin and Roseau.

As we shall see in this paper we have cases where Theorem 20 cannot be applied for studying the existence of periodic solutions, because its assumptions are not satisfied. Then in [15] the following result on averaging has been proved.

Theorem 21 *Let $V \subset \mathbb{R}^m$ be open and bounded, let $\beta_0 : Cl(V) \to \mathbb{R}^m$ be a $\mathscr{C}^k$ function and $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) \,|\, \alpha \in Cl(V)\} \subset \Omega$ its graphic in $\mathbb{R}^{2m}$. Assume that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$ of* (3.68) *is T-periodic and that there exists a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of* (3.69) *such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$*

(*a*) *has in the upper right corner the $m \times m$ matrix Δ_α with* $\det(\Delta_\alpha) \neq 0$, *and*

(*b*) *has in the lower right corner the $m \times m$ zero matrix.*

Consider the function $\mathcal{G} : Cl(V) \to \mathbb{R}^m$ defined by

$$\mathcal{G}(\alpha) = \xi^{\perp}\left(\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))\, dt\right). \tag{3.71}$$

Suppose that there is $\alpha_0 \in V$ with $\mathcal{G}(\alpha_0) = 0$, then the following statements hold for $\varepsilon \neq 0$ sufficiently small.

(*i*) *If* $\det((\partial \mathcal{G}/\partial \alpha)(\alpha_0)) \neq 0$, *then there is a unique T-periodic solution $\mathbf{x}(t, \varepsilon)$ of system* (3.67) *such that $\mathbf{x}(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as* $\varepsilon \to 0$.

(*ii*) *If $m = 1$ and $\mathcal{G}'(\alpha_0) = \cdots = \mathcal{G}^{(s-1)}(\alpha_0) = 0$ and $\mathcal{G}^{(s)}(\alpha_0) \neq 0$ with $s \leq k$, then there are at most s T-periodic solutions $\mathbf{x}_1(t, \varepsilon), \ldots, \mathbf{x}_s(t, \varepsilon)$ of system* (3.67) *such that $\mathbf{x}_i(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as $\varepsilon \to 0$ for $i = 1, \ldots, s$.*

In any case now we shall recall the more classical result on averaging theory for studying periodic solutions. We consider the initial value problems

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{3.72}$$

and

$$\dot{\mathbf{y}} = \varepsilon g(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{3.73}$$

with $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}_0$ in some open Ω of $\mathbb{R}^n$, $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$. We assume that $\mathbf{F_1}$ and $\mathbf{F_2}$ are periodic of period T in the variable t, and we set

$$g(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t,\mathbf{y})\,dt. \tag{3.74}$$

Theorem 22 *Assume that F_1, $D_{\mathbf{x}}F_1$,$D_{\mathbf{xx}}F_1$ and $D_{\mathbf{x}}F_2$ are continuous and bounded by a constant independent of ε in $[0,\infty) \times \Omega \times (0,\varepsilon_0]$, and that $y(t) \in \Omega$ for $t \in [0,1/\varepsilon]$. Then the following statements holds:*

(a) For $t \in [0,1/\varepsilon]$ we have $\mathbf{x}(t) - \mathbf{y}(t) = O(\varepsilon)$ as $\varepsilon \to 0$.

(b) If $p \neq 0$ is a singular point of system (3.73) and

$$\det D_{\mathbf{y}} g(p) \neq 0, \tag{3.75}$$

then there exists a periodic solution $\mathbf{x}(t,\varepsilon)$ of period T for system (3.72) such that $\mathbf{x}(0,\varepsilon) - p = O(\varepsilon)$ as $\varepsilon \to 0$.

(c) The stability of the periodic solution $\mathbf{x}(t,\varepsilon)$ is given by the stability of the singular point.

We have used the notation $D_{\mathbf{x}}g$ for all the first derivatives of g, and $D_{\mathbf{xx}}g$ for all the second derivatives of g.

3.8 Periodic solutions of a class of Duffing differential equations

3.8.1 Introduction and statement of the main result

Several classes of Duffing differential equations have been investigated by many authors. They are mainly interested in the existence of periodic solutions, in their multiplicity, stability, bifurcation, ... See the survey of J. Mawhin [17].

In this work we shall study the class of Duffing differential equations of the form

$$x'' + cx' + a(t)x + b(t)x^3 = h(t), \tag{3.76}$$

where $c > 0$ is a constant, and $a(t)$, $b(t)$ and $h(t)$ are continuous T–periodic functions. These differential equations were studied by Chen and Li in the papers. These authors studied the periodic solutions of equation (3.76) with the following additional conditions, either $b(t) > 0$, $h(t) > 0$ and $a(t)$ satisfies

$$a(t) \leq \frac{\pi^2}{T^2} + \frac{c^2}{4}, \quad \text{and} \quad a_0 = \frac{1}{T}\int_0^T a(t)\,dt > 0; \tag{3.77}$$

or $a(t) = a > 0$, $b(t) = 1$ and $c > 0$, a, c constants.

In [16] the authors studied the existence and the stability of periodic solutions of the Duffing differential equation (3.76) with $c = \varepsilon C > 0$, $a(t) = \varepsilon^2 A(t)$, $A_0 b_0 > 0$ where $A_0 = \frac{1}{T}\int_0^T A(t)\,dt$ and $b_0 = \frac{1}{T}\int_0^T b(t)\,dt$, and ε is sufficiently small.

Instead of working with the Duffing differential equation (3.76) we shall work with the equivalent differential system

$$\begin{aligned} x' &= y, \\ y' &= -cy - a(t)x - b(t)x^3 + h(t). \end{aligned} \tag{3.78}$$

We define the polynomial

$$\begin{aligned} p(x_0) = \; & -\left(\int_0^T e^{-ct}\int_0^t e^{cs} b(s)\,ds\,dt - \frac{e^{-cT}}{c}\int_0^T e^{cs} b(s)\,ds\right) x_0^3 \\ & -\left(\int_0^T e^{-ct}\int_0^t e^{cs} a(s)\,ds\,dt - \frac{e^{-cT}}{c}\int_0^T e^{cs} a(s)\,ds\right) x_0 \\ & + \frac{e^{-cT}}{c}\int_0^T e^{cs} h(s)\,ds + \int_0^T e^{-ct}\int_0^t e^{cs} h(s)\,ds\,dt. \end{aligned}$$

Our first result on the periodic solutions of the differential system (3.78) is the following.

Theorem 23 *For every simple real root of the polynomial $p(x_0)$ the differential system (3.78) has a periodic solution $(x(t), y(t))$ such that $(x(0), y(0)) = (x_0, 0)$.*

Theorem 20 will be proved in subsection 3.8.3 using Theorem 20 of the averaging theory.

Now we define the polynomial

$$q(x_0) = -\left(\int_0^T b(s)\,ds\right) x_0^3 - \left(\int_0^T a(s)\,ds\right) x_0 + \int_0^T h(s)\,ds.$$

Theorem 24 *For every simple real root of the polynomial $q(x_0)$ the differential system* (3.78) *has a periodic solution* $(x(t), y(t))$ *such that* $(x(0), y(0)) = (0, 0)$.

Theorem 26 will be proved in subsection 3.8.4 using Theorem 26 of the averaging theory.

As we shall see Theorem 27 of the averaging theory will provide results on the periodic solutions of system (3.78) which are already contained in Theorems 20 and 26.

In order to apply the three theorems of the averaging theory of first order for studying the periodic solutions of the differential system (3.78) in subsection 3.8.2 we rescale the variables, the parameters, and the functions of system (3.78).

The results of averaging theory that we use in this paper are described in subsection 3.8.4.

3.8.2 Preliminary results

We start doing a rescaling of the variables (x, y), of the functions $a(t)$, $b(t)$ and $h(t)$ and of the parameter c as follows

$$\begin{array}{ll} x = \varepsilon^{m_1} X, & y = \varepsilon^{m_2} Y, \\ c = \varepsilon^{m_3} C, & a(t) = \varepsilon^{n_1} A(t), \\ b(t) = \varepsilon^{n_2} B(t), & h(t) = \varepsilon^{n_3} H(t). \end{array} \tag{3.79}$$

In such a way that the differential equation (3.78) becomes

$$\begin{aligned} & X' = \varepsilon^{m_2 - m_1} Y, \\ & Y' = -\varepsilon^{m_3} C Y - \varepsilon^{n_1 + m_1 - m_2} A(t) X - \varepsilon^{n_2 + 3m_1 - m_2} B(t) X^3 + \varepsilon^{n_3 - m_2} H(t), \\ & \text{where} \quad 0 \le m_3, \quad 0 \le m_1 \le m_2 \le n_3, \quad m_2 \le n_1 + m_1, \quad m_2 \le n_2 + 3m_1, \\ & \text{and} \quad \{m_2 - m_1, m_3, n_1 + m_1 - m_2, n_2 + 3m_1 - m_2, n_3 - m_2\} \cap \{1\} \neq \emptyset. \end{aligned} \tag{3.80}$$

We distinguish the following seven cases with their corresponding subcases, recall that we want to apply the averaging theory of first order for studying the periodic solutions of the differential system (3.77), see a summary on this theory at the appendix.

Case I: $m_2 - m_1 = 0$ and $m_3 = 0$. Then we have the following subcases

$$
\begin{array}{llll}
(I.1) & n_1 + m_1 - m_2 = 0, & n_2 + 3m_1 - m_2 = 0, & n_3 - m_2 = 1; \\
(I.2) & n_1 + m_1 - m_2 = 0, & n_2 + 3m_1 - m_2 = 1, & n_3 - m_2 = 0; \\
(I.3) & n_1 + m_1 - m_2 = 1, & n_2 + 3m_1 - m_2 = 0, & n_3 - m_2 = 0; \\
(I.4) & n_1 + m_1 - m_2 = 0, & n_2 + 3m_1 - m_2 = 1, & n_3 - m_2 = 1; \\
(I.5) & n_1 + m_1 - m_2 = 1, & n_2 + 3m_1 - m_2 = 0, & n_3 - m_2 = 1; \\
(I.6) & n_1 + m_1 - m_2 = 1, & n_2 + 3m_1 - m_2 = 1, & n_3 - m_2 = 0; \\
(I.7) & n_1 + m_1 - m_2 = 1, & n_2 + 3m_1 - m_2 = 1, & n_3 - m_2 = 1.
\end{array}
$$

Case II: $m_2 - m_1 = 0$ and $m_3 \geq 1$.

Case III: $m_2 - m_1 = 1$ and $m_3 = 0$.

Case IV: $m_2 - m_1 = 1$ and $m_3 = 1$.

Case V: $m_2 - m_1 > 1$ and $m_3 = 1$.

Case VI: $m_2 - m_1 = 1$ and $m_3 > 1$.

Case VII: $m_2 - m_1 > 1$ and $m_3 > 1$.

Every case α from II to VII can be split into the following eight subcases:

$$
\begin{array}{llll}
(\alpha.1) & n_1 + m_1 - m_2 = 0, & n_2 + 3m_1 - m_2 = 0, & n_3 - m_2 = 0, \\
(\alpha.2) & n_1 + m_1 - m_2 = 0, & n_2 + 3m_1 - m_2 = 0, & n_3 - m_2 = 1, \\
(\alpha.3) & n_1 + m_1 - m_2 = 0, & n_2 + 3m_1 - m_2 = 1, & n_3 - m_2 = 0, \\
(\alpha.4) & n_1 + m_1 - m_2 = 1, & n_2 + 3m_1 - m_2 = 0, & n_3 - m_2 = 0, \\
(\alpha.5) & n_1 + m_1 - m_2 = 0, & n_2 + 3m_1 - m_2 = 1, & n_3 - m_2 = 1, \\
(\alpha.6) & n_1 + m_1 - m_2 = 1, & n_2 + 3m_1 - m_2 = 0, & n_3 - m_2 = 1, \\
(\alpha.7) & n_1 + m_1 - m_2 = 1, & n_2 + 3m_1 - m_2 = 1, & n_3 - m_2 = 0, \\
(\alpha.8) & n_1 + m_1 - m_2 = 1, & n_2 + 3m_1 - m_2 = 1, & n_3 - m_2 = 1.
\end{array}
$$

We have applied the three theorems of averaging (see section 3.8.4) for studying the existence of periodic solutions of the 55 previous subcases of differential systems (3.80). Theorem 20 comes from the subcase (III.1), and Theorem 26 follows from the subcase (IV.1).

All the subcases, different from (III.1) or (IV.1), either do not satisfy the hypotheses of one of the three theorems of averaging, or provide partial results of the ones stated in Theorems 20 and 26. So in what follows we shall consider only the subcases (III.1) or (IV.1).

Theorem 27 has been applied for studying the subcases $(\alpha, 8)$ for $\alpha = IV, \ldots, VII$, and either do not provide periodic solutions, or provide particular cases of the results given in Theorems 20 and 26.

In short, in what follows we only provide the details of the positive results, i.e. we shall give the proofs of Theorems 20 and 26.

3.8.3 Proof of Theorem 20

For case (III.1), i.e. for

$$m_2 = m_1 + 1, \quad m_2 = n_1 = n_2 = n_3 = 1 \text{ and } m_1 = m_3 = 0; \tag{3.81}$$

system 3.80 becomes

$$\begin{aligned} \dot{X} &= \varepsilon Y, \\ \dot{Y} &= -CY - A(t)X - B(t)X^3 + H(t), \end{aligned} \tag{3.82}$$

We shall apply the averaging Theorem 20 to system (3.82) and we shall obtain Theorem 20. In what follows we shall use the notation of Theorem 20, see

the appendix. Thus $\mathbf{x} = (X, Y)^T$ and

$$F_0(t,\mathbf{x}) = \begin{pmatrix} 0 \\ -CY - A(t)X - B(t)X^3 + H(t) \end{pmatrix},$$

$$F_1(t,\mathbf{x}) = \begin{pmatrix} Y \\ 0 \end{pmatrix},$$

$$F_2(t,\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The unperturbed differential system (3.68) has the solution $\mathbf{x}(t,\mathbf{z},0) = (X(t), Y(t))^T$ such that $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z} = (X_0, Y_0)^T$, where

$$\begin{aligned} X(t) &= X_0, \\ Y(t) &= e^{-Ct}\left(Y_0 + \int_0^t e^{Cs}\left(-B(s)X_0^3 - A(s)X_0 + H(s)\right) ds\right). \end{aligned}$$

In order that $\mathbf{x}(t,\mathbf{z},0)$ be a periodic solution we must choose

$$Y_0 = \frac{1}{e^{CT} - 1}\int_0^T e^{Cs}\left(-B(s)X_0^3 - A(s)X_0 + H(s)\right) ds.$$

So we get

$$\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) = \left(X_0, \frac{1}{e^{CT} - 1}\int_0^T e^{Cs}\left(-B(s)X_0^3 - A(s)X_0 + H(s)\right) ds\right).$$

Therefore, following the notation of Theorem 20, we have $n = 2$ and $k = 1$.

Now we compute the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ associated to the first variational system (3.69) such that $M_{\mathbf{z}_\alpha}(0) = \text{Id of } \mathbb{R}^2$, and we obtain

$$M_{\mathbf{z}_\alpha}(t) = \begin{pmatrix} 1 & 0 \\ -e^{-Ct}\left(\displaystyle\int_0^t e^{Cs}(3B(s)X_0 + A(s))\,ds\right) & e^{-Ct} \end{pmatrix}.$$

Its inverse matrix is

$$M_{\mathbf{z}_\alpha}^{-1}(t) = \begin{pmatrix} 1 & 0 \\ \displaystyle\int_0^t e^{Cs}\left(3B(s)X_0^2 + A(s)\right)ds & e^{Ct} \end{pmatrix}.$$

Since the matrix

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T) = \begin{pmatrix} 0 & 0 \\ -\displaystyle\int_0^t e^{Cs}\left(3B(s)X_0^2 + A(s)\right)ds & 1 - e^{CT} \end{pmatrix}$$

has a zero 1×1 matrix in the upper right corner and a non–zero 1×1 matrix in its lower right corner equal to $1-e^{CT}$, because $T \neq 0$. We can apply the averaging theory described in Theorem 20 for studying the periodic solutions which can be prolonged from the unperturbed differential system to the perturbed one. Therefore, since for our differential system we have $\xi(X,Y) = X$, then we must compute the function $\mathscr{F}(\alpha) = \mathscr{F}(X_0)$ given in (3.70), i.e.

$$\begin{aligned}
\mathscr{F}(X_0) = {} & \xi\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))\,dt\right) \\
= {} & \int_0^T \left[e^{-Ct}\left(\frac{1}{-1+e^{CT}} \int_0^T e^{Cs}\left(-B(s)X_0^3 - X_0A(s) + H(s)\right)ds \right.\right. \\
& \left.\left. + \int_0^t e^{Cs}\left(-B(s)X_0^3 - X_0A(s) + H(s)\right)ds \right)\right] dt \\
= {} & \int_0^T Y(t)\,dt.
\end{aligned}$$

Theorem 20 says that for every simple real root X_0 of the polynomial $\mathscr{F}(X_0)$ the differential system (3.82) with $\varepsilon \neq 0$ sufficiently small has a periodic solution $(X(t), Y(t))$ such that $(X(0), Y(0))$ tends to $(X_0, \beta_0(X_0))$ when $\varepsilon \to 0$.

Now it is to check that the function $\mathscr{F}(X_0)$ after the change of variables (3.79) satisfying (3.81), i.e.

$$X = x, \quad Y = \frac{y}{\varepsilon}, \quad H(t) = \frac{h(t)}{\varepsilon}, \quad B(s) = \frac{b(s)}{\varepsilon}, \quad A(s) = \frac{a(s)}{\varepsilon}.$$

becomes the polynomial $p(x_0)$ defined in subsection 3.8.1 just before the statement of Theorem 20. Hence Theorem 20 is proved.

3.8.4 Proof of Theorem 26

For case (IV.1), i.e.

$$m_2 = m_1 + 1, \quad m_3 = 1, \quad m_1 = 1, \quad m_2 = 2, \quad n_1 = -n_2 = 1 \text{ and } n_3 = 2; \tag{3.83}$$

system (3.80) becomes

$$\begin{aligned} \dot{X} &= \varepsilon Y, \\ \dot{Y} &= -\varepsilon C Y - A(t)X - B(t)X^3 + H(t), \end{aligned} \tag{3.84}$$

We shall apply the averaging Theorem 26 to system (3.84) and we shall obtain Theorem 26. In what follows we shall use the notation of Theorem 26.

Thus $\mathbf{x} = (X, Y)^T$ and

$$F_0(t, \mathbf{x}) = \begin{pmatrix} 0 \\ -A(t)X - B(t)X^3 + H(t) \end{pmatrix},$$

$$F_1(t, \mathbf{x}) = \begin{pmatrix} Y \\ -CY \end{pmatrix},$$

$$F_2(t, \mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The unperturbed differential system (3.68) has the solution $\mathbf{x}(t, \mathbf{z}, 0) = (X(t), Y(t))^T$ such that $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z} = (X_0, Y_0)^T$, where

$$\begin{aligned} X(t) &= X_0, \\ Y(t) &= Y_0 + \int_0^t \left(-B(s)X_0^3 - A(s)X_0 + H(s)\right) ds. \end{aligned}$$

In order that $\mathbf{x}(t, \mathbf{z}, 0)$ be a periodic solution X_0 must satisfy

$$\int_0^T \left(-B(s)X_0^3 - A(s)X_0 + H(s)\right) ds = 0, \tag{3.85}$$

Y_0 is arbitrary. Therefore we get

$$\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) = \left(Y_0, \bar{X}_0\right),$$

where $\bar{X}_0$ is a real root of the cubic polynomial (3.85). In short the unperturbed system (i.e. system (3.85) with $\varepsilon = 0$) has at most three families of periodic solutions because Y_0 is arbitrary and $\bar{X}_0$ is a real root of the cubic polynomial (3.85). Therefore, using the notation of Theorem 26, we have $n = 2$ and $k = 1$ for each one of these possible families of periodic solutions.

We compute the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ associated to the first variational system (3.69) associated to the vector field $(\dot{Y}, \dot{X})$ given by (3.84) with $\varepsilon = 0$, and such that $M_{\mathbf{z}_\alpha}(0) = \mathrm{Id}$ of $\mathbb{R}^2$, and we obtain

$$M_{\mathbf{z}_\alpha}(t) = \begin{pmatrix} 1 & -\displaystyle\int_0^t \left(3B(s)X_0^2 + A(s)\right) ds \\ 0 & 1 \end{pmatrix}.$$

Its inverse matrix is

$$M_{\mathbf{z}_\alpha}^{-1}(t) = \begin{pmatrix} 1 & \displaystyle\int_0^t \left(3B(s)X_0^2 + A(s)\right) ds \\ 0 & 1 \end{pmatrix}.$$

The matrix

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T) = \begin{pmatrix} 0 & -\displaystyle\int_0^T \left(3B(s)X_0^2 + A(s)\right) ds \\ 0 & 0 \end{pmatrix}$$

has a non–zero 1×1 matrix in the upper right corner if the real root $\bar{X}_0$ of the cubic polynomial (3.85) is simple, and a zero 1×1 matrix in its lower right corner. Therefore the assumptions of Theorem 26 hold, then by applying this theorem we study the periodic solutions which can be prolonged from the unperturbed differential system to the perturbed one. Since for our differential system we have $\xi^\perp(Y, X) = X$, then we must compute the function $\mathcal{G}(\alpha) = \mathcal{G}(Y_0)$ given in (3.70), i.e.

$$\mathcal{G}(Y_0) = \xi^\perp\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))\, dt\right) = -\int_0^T C Y_0 = -CTY_0.$$

Theorem 26 says that for every simple real root $Y_0 = 0$ of the polynomial $\mathcal{G}(Y_0)$ the differential system (3.84) with $\varepsilon \neq 0$ sufficiently small has a periodic solution $(X(t), Y(t))$ such that $(X(0), Y(0))$ tends to $(\bar{X}_0, 0)$

when $\varepsilon \to 0$, being $\bar{X}_0$ a simple real root of the cubic polynomial (3.85). Now it is easy to check that the cubic polynomial (3.85) after the change of variables (3.79) satisfying (3.83), i.e.

$$X = \frac{x}{\varepsilon}, \quad Y = \frac{y}{\varepsilon^2}, \quad H(t) = \frac{h(t)}{\varepsilon^2}, \quad B(s) = \frac{b(s)}{\varepsilon}, \quad A(s) = \frac{a(s)}{\varepsilon}.$$

becomes the polynomial $q(x_0)$ defined in subsection 3.8.1 just before the statement of Theorem 26. Hence Theorem 26 is proved.

The appendix: Periodic solutions via the averaging theory

In this subsection we present the basic results on the averaging theory of first order that we need for proving our results.

We consider the problem of bifurcation of T–periodic solutions from the differential systems of the form

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{3.86}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. The functions $F_0, F_1 : \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ are $\mathscr{C}^2$, T–periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}), \tag{3.87}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t,\mathbf{z},\varepsilon)$ be the solution of system (3.87) such that $\mathbf{x}(0,\mathbf{z},\varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t,\mathbf{z},0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}. \tag{3.88}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ a fundamental matrix of the linear differential system (3.88), by $\xi : \mathbb{R}^m\times\mathbb{R}^{n-m} \to \mathbb{R}^m$ and $\xi^\perp : \mathbb{R}^m\times\mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ the projections of $\mathbb{R}^n$ onto its first m and $n-m$ coordinates respectively; i.e. $\xi(x_1,\ldots,x_n) = (x_1,\ldots,x_m)$, and $\xi^\perp(x_1,\ldots,x_n) = (x_{m+1},\ldots,x_n)$

Theorem 25 *Let $V \subset \mathbb{R}^m$ be open and bounded, let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^{n-m}$ be a $\mathscr{C}^k$ function and $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) \mid \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ its graphic in $\mathbb{R}^n$. Assume that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$ of (3.87) is T-periodic and that there exists a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (3.88) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$*

(*a*) *has in the lower right corner the $(n-m) \times (n-m)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$, and*

(*b*) *has in the upper right corner the $m \times (n-m)$ zero matrix.*

Consider the function $\mathscr{F} : \mathrm{Cl}(V) \to \mathbb{R}^m$ defined by

$$\mathscr{F}(\alpha) = \xi\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))\, dt\right). \tag{3.89}$$

Suppose that there is $\alpha_0 \in V$ with $\mathscr{F}(\alpha_0) = 0$, then the following statements hold for $\varepsilon \neq 0$ sufficiently small.

(*i*) *If $\det((\partial \mathscr{F}/\partial \alpha)(\alpha_0)) \neq 0$, then there is a unique T-periodic solution $\varphi_1(t, \varepsilon)$ of system (3.86) such that $\varphi_1(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as $\varepsilon \to 0$.*

(*ii*) *If $m = 1$ and $\mathscr{F}'(\alpha_0) = \cdots = \mathscr{F}^{(s-1)}(\alpha_0) = 0$ and $\mathscr{F}^{(s)}(\alpha_0) \neq 0$ with $s \leq k$, then there are at most s T-periodic solutions $\varphi_1(t, \varepsilon), \ldots, \varphi_s(t, \varepsilon)$ of system (3.86) such that $\varphi_i(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as $\varepsilon \to 0$ for $i = 1, \ldots, s$.*

Theorem 25 is a classical result due to Malkin and Roseau .

As we shall see in this paper we have cases where Theorem 25 cannot be applied for studying the existence of periodic solutions, because its assumptions are not satisfied. Then in [15] the following result on averaging has been proved.

Theorem 26 *Let $V \subset \mathbb{R}^m$ be open and bounded, let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^m$ be a $\mathscr{C}^k$ function and $\mathscr{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) \,|\, \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ its graphic in $\mathbb{R}^{2m}$. Assume that for each $\mathbf{z}_\alpha \in \mathscr{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$ of (3.87) is T-periodic and that there exists a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (3.88) such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$*

(*a*) *has in the upper right corner the $m \times m$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$, and*

(*b*) *has in the lower right corner the $m \times m$ zero matrix.*

Consider the function $\mathscr{G} : \mathrm{Cl}(V) \to \mathbb{R}^m$ defined by

$$\mathscr{G}(\alpha) = \xi^\perp\left(\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))\,dt\right). \tag{3.90}$$

Suppose that there is $\alpha_0 \in V$ with $\mathscr{G}(\alpha_0) = 0$, then the following statements hold for $\varepsilon \neq 0$ sufficiently small.

(*i*) *If $\det((\partial\mathscr{G}/\partial\alpha)(\alpha_0)) \neq 0$, then there is a unique T-periodic solution $\varphi_1(t, \varepsilon)$ of system (3.86) such that $\varphi_1(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as $\varepsilon \to 0$.*

(*ii*) *If $m = 1$ and $\mathscr{G}'(\alpha_0) = \cdots = \mathscr{G}^{(s-1)}(\alpha_0) = 0$ and $\mathscr{G}^{(s)}(\alpha_0) \neq 0$ with $s \leq k$, then there are at most s T-periodic solutions $\varphi_1(t, \varepsilon), \ldots, \varphi_s(t, \varepsilon)$ of system (3.86) such that $\varphi_i(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0}, 0)$ as $\varepsilon \to 0$ for $i = 1, \ldots, s$.*

In any case now we shall recall the more classical result on averaging theory for studying periodic solutions. We consider the initial value problems

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{3.91}$$

and

$$\dot{\mathbf{y}} = \varepsilon g(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{3.92}$$

with $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}_0$ in some open Ω of $\mathbb{R}^n$, $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$. We assume that $\mathbf{F_1}$ and $\mathbf{F_2}$ are periodic of period T in the variable t, and we set

$$g(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t, \mathbf{y})\,dt.$$

Theorem 27 *Assume that* F_1, $D_\mathbf{x}F_1$,$D_{\mathbf{xx}}F_1$ *and* $D_\mathbf{x}F_2$ *are continuous and bounded by a constant independent of* ε *in* $[0,\infty)\times\ \Omega\times(0,\varepsilon_0]$, *and that* $y(t)\in\Omega$ *for* $t\in[0,1/\varepsilon]$. *Then the following statements hold:*

1. *For* $t\in[0,1/\varepsilon]$ *we have* $\mathbf{x}(t)-\mathbf{y}(t)=O(\varepsilon)$ *as* $\varepsilon\to 0$.

2. *If* $p\neq 0$ *is a singular point of system* (3.92) *and* $detD_\mathbf{y}g(p)\neq 0$, *then there exists a periodic solution* $\phi(t,\varepsilon)$ *of period* T *for system* (3.91) *which is close to* p *and such that* $\phi(0,\varepsilon)-p=O(\varepsilon)$ *as* $\varepsilon\to 0$.

3. *The stability of the periodic solution* $\phi(t,\varepsilon)$ *is given by the stability of the singular point.*

We have used the notation $D_\mathbf{x}g$ for all the first derivatives of g, and $D_{\mathbf{xx}}g$ for all the second derivatives of g.

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