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Study of the number of limit cycles in piecewise differential systems

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Table des matières

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Dedications

I dedicate the fruit of this humble work to those who have been my refuge and shelter, the secret of happiness and joy, my dear mother.

To the owner of the big heart and my first and last belonging, to the one who has given me so much, my dear father.

To my siblings : Antar, Abd Errahim, Lahcen, and Toumi, Ratiba, chafia, and Ibtisam. And my brother's wife (Abd Errahim's wife).

To my niblings : Sohaib, Is'haq, Adjoune, Loudjain, and the two little chicks, Kacem and Asem.

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To all those whom my pen writes about but have a special place in my heart.

Introduction

Ordinary differential equations (ODEs) are a fundamental tool for studying and understanding the behavior of a vast array of natural phenomena [1, 2]. They have become an essential part of the curriculum in many undergraduate science programs and are widely used in various fields, including engineering, technology, and the natural sciences. While ODEs can model a broad range of natural processes, only a select few allow for explicit solutions. For many other systems, qualitative theory and related methods offer alternative tools for surveying their behavior. Qualitative techniques can provide a more comprehensive understanding of significant subsets of solutions, see the book [5]. This approach reveals valuable information about the flow, parametric stability, and bifurcations of the ODEs. Therefore, qualitative techniques are a crucial complement to traditional methods, and their use should be encouraged to obtain a more complete understanding of the behavior of ODEs in diverse applications.

A dynamical system is any system that changes over time, and ODEs provide a concise and elegant way to capture this behavior. A dynamical system can be defined as a function that describes the time dependence of a point in an ambient space, such as a parametric curve. Examples of dynamical systems that can be modeled with ODEs include the oscillation of a clock pendulum, the flow of water through a pipe, the motion of particles in the air, and the population dynamics of a lake's fish species. By modeling these systems with ODEs, we gain insight into their behavior, which can inform important decisions in fields such as physics, engineering, and ecology. Therefore, ODEs play a crucial role in the study of dynamical systems and are essential tools for understanding the behavior of natural phenomena.

Piecewise dynamical systems are a particular type of dynamical system that is defined by different sets of differential equations in different regions of the state space. The boundary between regions, called the switching surface or switching manifold, defines the conditions under which the system switches from one set of equations to another, please refer to [6, 10, 18]. These systems are often used to model complex phenomena that exhibit different behaviors or dynamics under different conditions, such as ecological systems or mechanical systems subject to switching or control inputs.

The study of dynamical systems and piecewise dynamical systems involves the analysis of the system's behavior, stability, and bifurcations. The complexity of these systems often limits the effectiveness of analytical methods. Therefore, we utilize qualitative methods to study and gain a deeper understanding of these systems.

Dynamical systems and piecewise dynamical systems have numerous applications in science and engineering [3, 11]. Understanding the principles of these systems can help researchers and engineers gain insight into the behavior of real-world systems, leading to more effective control and optimization strategies.

In this thesis, we study planar dynamical systems, which are mathematical models that describe the motion of particles or objects in two-dimensional space, are incredibly important in many fields of science and engineering. One reason for their importance is that they can be used to study a wide range of physical phenomena, such as the motion of celestial bodies, the behavior of fluids, and the dynamics of electrical circuits. Furthermore, planar dynamical systems can help us gain insight into the behavior of more complex systems, by providing simplified models that capture the essential features of the system. This makes them useful for both theoretical analysis and practical applications, such as designing control systems for industrial processes or understanding the behavior of ecological systems. In addition, planar dynamical systems are often studied because they are mathematically interesting in their own right, and have deep connections to other areas of mathematics, such as topology and geometry. Overall, the study of planar dynamical systems is essential for advancing our understanding of the natural world and developing new technologies.

The thesis is structured into three chapters, each addressing different aspects of our study of the piecewise differential system. The first chapter provides the necessary background information and mathematical concepts that underpin our analysis, including definitions, lemmas, and theorems. These concepts include periodic orbits, Poincaré maps, averaging theory, first integrals, Hamiltonian systems, and Filippov systems. By establishing a solid foundation of these concepts, we aim to provide readers with a comprehensive understanding of the mathematical tools and techniques that will be employed throughout our analysis. In the subsequent chapters, we apply these concepts to our study of the piecewise differential system and draw conclusions based on our findings.

Chapter 2 of our thesis marks the beginning of our investigation into the behavior of the piecewise differential system. Specifically, we examine the limit cycles that arise from perturbations of the periodic orbits of the linear differential center $\dot{x} = -y, \quad \dot{y} = x$ by discontinuous piecewise differential systems. By studying these limit cycles, we aim to gain a deeper understanding of the dynamics of the piecewise differential system and draw conclusions about its long-term behavior.

Chapter 3 of our thesis focuses on the investigation of a specific type of piecewise differential system in the plane. This system is constructed by dividing the plane into four quadrants, each of which contains a linear Hamiltonian system. The system is formed by four pieces separated by the axes of coordinates, such that it is continuous on the x-axis but discontinuous on the y-axis. Our investigation of this particular sort of piecewise differential system intends to provide useful insights into its dynamics. We are particularly interested in determining the maximum number of limit cycles that can occur in these systems.

Chapitre 1

Preliminary

This chapter introduces the second and third chapters by introducing numerous basic concepts and theorems that will be used throughout the rest of the thesis. By definition, a polynomial system is a two-dimensional planar differential system expressed as follows

$$\frac{dx}{dt} = \dot{x} = P(x, y), \quad \frac{dx}{dt} = \dot{y} = Q(x, y). \tag{1.1}$$

The variables x and y are real, and t is the independent variable (time). The functions P and Q are polynomials with real coefficients in the variables x and y.

1.1 First Integrals

System (1.1) is said to be integrable on an open subset U of $\mathbb{R}^2$ if there exists a non-constant analytic function $H : U \to \mathbb{F}$, known as a first integral of the system on U, that remains constant along all solution curves $(x(t), y(t))$ of the system (1.1) contained within U. More specifically, $H(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ is defined and contained within U. The vector field χ associated with system (1.1) can be defined as follows

$$\chi = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

H is a first integral of system (1.1) on U if and only if it satisfies the following condition

$$\chi H = PH_x + QH_y \equiv 0.$$

1.2 Hamiltonian system

Let U be an open subset of $\mathbb{R}^2$ and let $H \in C^2(U)$ where $H = H(x, y)$ with $x, y \in \mathbb{R}^2$. A system of the form

$$\dot{x} = \frac{\partial H}{\partial y},$$
$$\dot{y} = -\frac{\partial H}{\partial x},$$

is called a Hamiltonian system with first integral H.

Example 1. *The Hamiltonian function*

$$H(x,y) = (x^2 + y^2)/2.$$

is the energy function for the pendulum

$$\begin{aligned} \dot{x} =& y, \\ \dot{y} =& -x, \end{aligned}$$

This system is equivalent to the harmonic oscillators

$$\ddot{x} + x = 0.$$

A fundamental property of Hamiltonian systems is that they are conservative, meaning that the Hamiltonian function or the total energy $H(x,y)$ is conserved along the system's trajectories . For more details of Hamiltonian system see [15].

1.3 Piecewise differential systems

A piecewise differential system on an open region $\Omega \subseteq \mathbb{R}^2$ is a set of vector fields

$$\dot{X} = f_i(X),$$

such that $X \in \Omega_i \subset \Omega$, f_i is a function defined from Ω_i to $\mathbb{R}^2$, and Ω_i is an open set in $\mathbb{R}^2$ satisfying that $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$ and $\cup_{i \in I}\overline{\Omega_i} = \Omega$. As usual $\overline{\Omega_i}$ denotes the closure of Ω_i, See [13, 14].

We see that the definition of a piecewise differential system contains no information regarding the flow's behavior at the boundaries between the regions, its classification is determined by how the vector field is extended to these borders. If the equations $f_i = f_j$ hold true for all points, the piecewise differential system is said to be continuous. If the equations do not hold, the system is classified as discontinuous

Example 2. *The differential equation*

$$\begin{cases} \dot{x} = x^2 + xy, \\ \dot{y} = e^x, \end{cases} \text{if } x > 0,$$

and

$$\begin{cases} \dot{x} = y^2 x + 1 \\ \dot{y} = \cos xy \end{cases} \text{if } x < 0,$$

is a piecewise differential system with

$$\Omega_1 = \{(x,y) \in \mathbb{R}^2 : x > 0\}, \;\; \Omega_2 = \{(x,y) \in \mathbb{R}^2 : x < 0\}, \;\; \Gamma = \{(x,y) \in \mathbb{R}^2 : x = 0\}.$$

1.4 Standard and sliding solutions of Filippov systems

Consider a discontinuous system

$$\dot{X} = \begin{cases} f_1(X), X \in \Omega_1 \\ f_2(X), X \in \Omega_2 \end{cases} \tag{1.2}$$

where $X \in \mathbb{R}^2$

$$\Omega_1 = \{X \in \mathbb{R}^2 : L(X) < 0\};\ \Omega_2 = \{X \in \mathbb{R}^2 : L(X) > 0\};$$

$L : \mathbb{R}^2 \to \mathbb{R}$ is smooth with $L_X(X) \neq 0$ on the **discontinuity boundary**

$$\sum = \{X \in \mathbb{R}^2 : L(X) = 0\};$$

L is a smooth scalar function with nonvanishing gradient $\nabla L = L_X(X) = (\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y})^T$, and $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ are smooth functions. Orbits of (1.2) are defined by concatenation of **standard** and **sliding** orbit segments.

For $X \in \sum$, define

$$\sigma(X) = \langle L_X(X), f_1(X)\rangle\langle L_X(X), f_2(X)\rangle$$

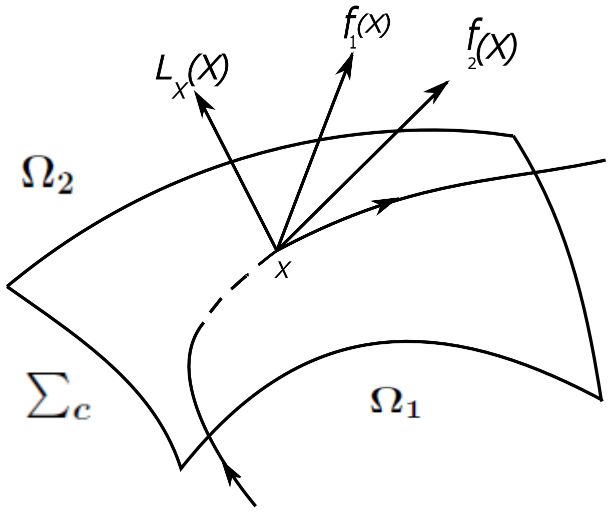
and introduce the sets of

a) crossing points : $\sum_c = \{X \in \sum : \sigma(X) > 0\}$,

b) sliding points : $\sum_s = \{X \in \sum : \sigma(X) \leq 0\}$,

c) regular sliding points : $\widehat{\sum}_s = \{X \in \sum_s : \langle L_X(X), f_2(X) - f_1(x)\rangle \neq 0\}$,

Crossing orbits : At $X \in \sum_c$, concatenate the standard orbit of f_1 reaching X from Ω_1 with the standard orbit of f_2 departing from X into Ω_2, or vice versa.



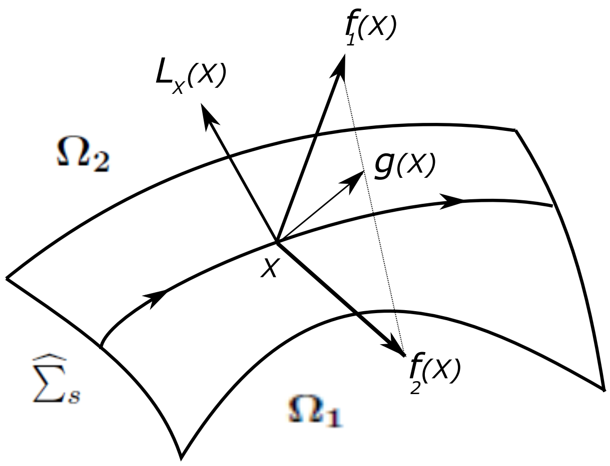
Sliding orbits : For $X \in \widehat{\sum}_s$ define the Filippov vector

$$g(X) = \lambda(X)f_1(X) + (1 - \lambda(X))f_2(X),$$

where

$$\lambda(X) = \frac{\langle L_X(X), f_2(X)\rangle}{\langle L_X(X), f_2(X) - f_1(X)\rangle}.$$

For more detailed information , please refer to [9].



1.5 Several models of Piecewise differential systems

1.5.1 Resonant LC power inverters under zero current switching strategy [3]

Resonant inverters are systems that include a switching network and a resonant $\boldsymbol{LC}$ tank circuit that converts a DC voltage into an $\boldsymbol{AC}$ voltage, thus providing $\boldsymbol{AC}$ power to a load. They have been around for a while, but their use has usually been limited to certain applications such as high-voltage power supplies or audio amplifiers. [16, 17]. Recently, they gained significant attention in various emerging applications, including wireless power transfer, battery charging in electric vehicles, induction heating, and powering high-intensity discharge lighting.

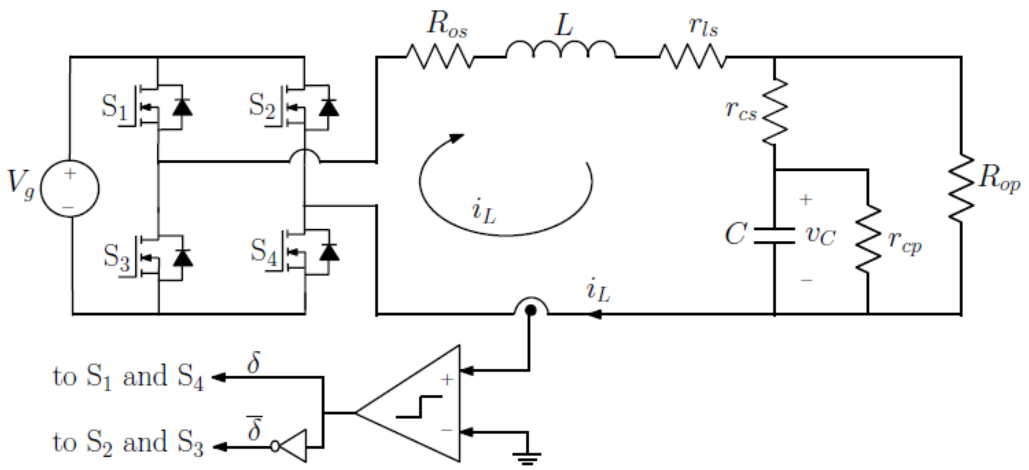


FIGURE 1.1 – Generalized schematic diagram of an $\boldsymbol{LC}$ resonant inverter.

By utilizing certain electronic laws and algebraic simplifications, it is possible to simplify and streamline the analysis and design processes in LC resonant inverter circuits.

$$\frac{d}{dt}\begin{pmatrix} v_C \\ i_L \end{pmatrix} = A\begin{pmatrix} v_C \\ i_L \end{pmatrix} + ub, \tag{1.3}$$

where

$$A = \begin{pmatrix} -\frac{G_p}{C} & -\frac{k}{C} \\ -\frac{k}{L} & -\frac{R_s}{L} \end{pmatrix}, b = \begin{pmatrix} 0 \\ \frac{V_g}{L} \end{pmatrix},$$

and the factor k is introduced to define the equivalent series resistance R_s and the equivalent parallel conductance G_p.

Zero Current Switching (ZCS) is a control approach that makes switching decisions based on the sign of the inductor current.The switching condition, which is generally a function of state variables and time, is reduced to relying just on inductor current. As a result, it can be represented utilizing the state variables as

$$h(v_C; i_L) = i_L$$

so that the ZCS control strategy leads to $u = \text{ sign } (h(v_C; i_L))$ and the system is autonomous. By change of variables system (1.3) becomes the following piecewise differential system

$$\frac{dX}{dt} = AX + ub,$$
$$h(X) = x_2, \tag{1.4}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -\dfrac{1}{Q} \end{pmatrix}, b = \begin{pmatrix} -\dfrac{\beta}{Q} \\ 1 \end{pmatrix}$$

and $u = 1$ if $h(X) > 0$ or $u = -1$ if $h(X) < 0$.

According to (1.4), the switching manifold $\sum$ is defined as $\sum = \{(x_1, x_2) : x_2 = 0\}$, and so the state space of the canonical form has two linearity regions, namely

$$\textstyle\sum^+ = \{(x_1, x_2) : x_2 > 0\}, \quad \sum^- = \{(x_1, x_2) : x_2 < 0\}.$$

1.5.2 System with purely elastic one-sided supports [11]

Consider the system represented in Figure 1.2, which is made up entirely of completely elastic one-sided supports. A mass (m), spring constants $(k$ and $k_f)$, and a force $(f_0(x)\cos(\omega t))$ acting on the mass in the direction of its acceleration comprise the system. A non-smooth continuous vector field can be used to represent this system.

$$m\ddot{x} + kx = f_0\cos(\omega t) - f(x),$$

such that

$$f(x) = \begin{cases} 0, & x \leq 0, \\ k_f x, & x > 0. \end{cases}$$

1.5.3 System with visco-elastic supports and dry friction [11]

Let's consider a system depicted in Figure 1.3, which consists of visco-elastic supports and dry friction. The system includes a mass (m), a spring constant (k), and a force $(f_0(x)cos(wt))$

acting on the mass in the direction of its acceleration. This system can be mathematically described by a differential equation with a discontinuous right-hand side, while maintaining a time-continuous state.

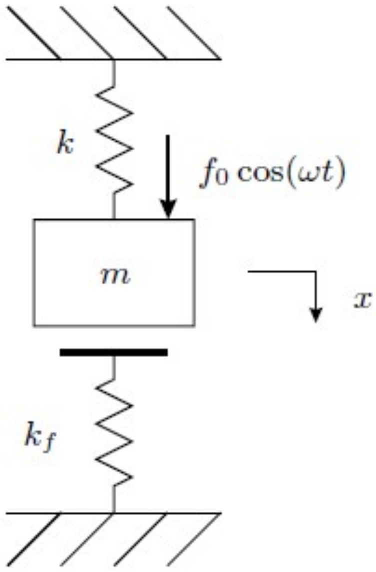


FIGURE 1.2 – elastic support model.

$$m\ddot{x} + kx = f_0 \cos(\omega t) - f(\dot{x}),$$

such that

$$f(x) = \begin{cases} -F_s, & \dot{x} < 0, \\ [-F_s, F_s], & \dot{x} = 0, \\ F_s, & \dot{x} > 0. \end{cases}$$

1.6 Periodic orbits : Poincare map

The Poincare map is the most important tool for studying flows near periodic orbits. Consider a differential equation

$$\dot{X} = f(X) \tag{1.5}$$

such that $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ and let $\phi(s, x)$ be the flow defined by (1.5). Let $\sum$ be a hypersurface in $\mathbb{R}^n$ and $p \in \sum \cap U$.

- If $f(p) \notin T_p \sum$ the flow ϕ is said to be transverse to $\sum$ at the point p.
- If $f(p) \in T_p \sum$ then p is called a contact point of the flow with $\sum$.

Remark 1.1. *Let V be an open subset of $\sum$. We say that the flow is transverse to $\sum$ at V if the flow is transverse to $\sum$ at every point in V.*

We will now look at two open hypersurfaces $\sum_1$, $\sum_2$ and $p_1 \in \sum_1 \cap U$, $p_2 \in \sum_2 \cap U$ such that $p_2 = \phi(s_1, p_1)$. There are neighbourhoods $V_1(p_1) \subset \sum_1 \cap U$, $V_2(p_2) \subset \sum_2 \cap U$, and a function $\tau : V_1 \to \mathbb{R}$ satisfying $\tau(p_1) = s_1$ and $\phi(\tau(q), q) \in V_2$ for every $q \in V_1$.

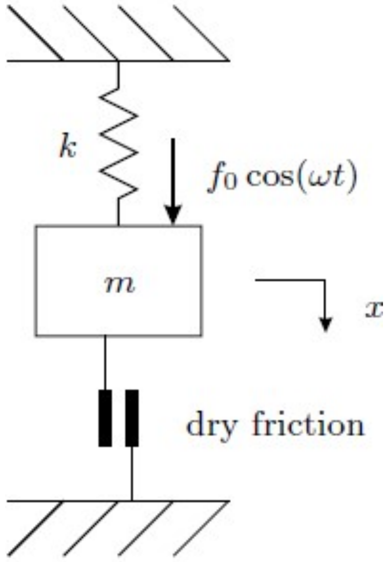


FIGURE 1.3 – elastic support model with dry friction.

In this case, we refer to the Poincare map as the map $\pi : V_1 \to V_2$ given by

$$\pi(q) = \phi(\tau(q), q),$$

for every $q \in V_1$, see Figure 1.4.

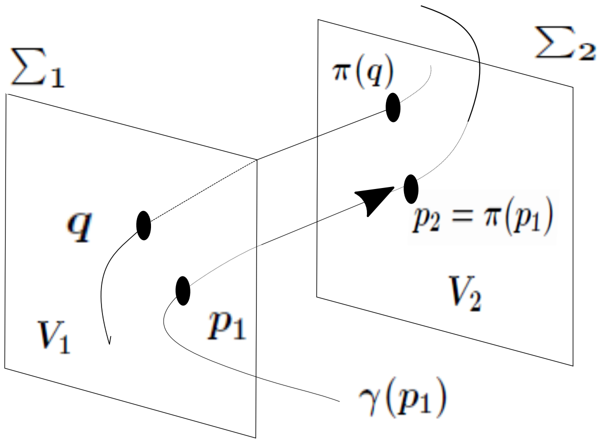


FIGURE 1.4 – Poincaré map π.

Remark 1.2. *When the vector field is globally Lipschitz, C^r with $r \geq 1$, or analytic, the Poincare map π is also continuous, C^r with $r \geq 1$, or analytic, respectively. By reversing the sense of the flow it is easy to conclude that the Poincare map is invertible and the inverse map π^{-1} is continuous, C^r with $r \geq 1$, or analytic,respectively. In the particular case when $\sum_1 = \sum_2$ the Poincare map π is called a return map.*

Consider $p \in \sum_1$ and let $\gamma(p)$ be a periodic orbit. The flow's continous dependence on the initial conditions implies that a return map π exists in a neighbourhood of p, and p is a fixed point of π. In contrast, if $p \in \sum_1$ is a fixed point of a return map π, then $\gamma(p)$ is a periodic orbit. As a result, limit cycles correspond to isolated fixed points of return maps. A hyperbolic limit cycle is one in which the absolute value of all the eigenvalues of the Jacobian matrix $D\pi(p)$ differs from one, instead, $\gamma(p)$ is referred to as a nonhyperbolic limit cycle. It should be noted that this definition is independent of the chosen point p or cross section $\sum_1$.

Theorem 1.1. *Let the differential equation $\dot{x} = f(x)$, f be a Lipschitz function in $U \subset \mathbb{R}^n$; $\gamma(p)$ be a hyperbolic limit cycle and π be a return map defined in a neighbourhood of $\gamma(p)$. Assume that π is differentiable in a neighbourhood of p. Then the following statements hold.*

(a) If every eigenvalue of $D_\pi(p)$ has an absolute value smaller than 1, then $\gamma(p)$ is a stable limit cycle.

(b) If the absolute value of at least one eigenvalue of $D_\pi(p)$ is greater than 1, then $\gamma(p)$ is an unstable limit cycle.

A proof of this result can be found in [5].

1.7 The Averaging theory for periodic orbit

The main idea behind averaging theory is to replace the original system with a simpler system that captures the essential features of the original system but is easier to analyze. This is done by averaging the behavior of the original system over one or more periods of the rapid oscillations, effectively smoothing out the fluctuations and simplifying the system's behavior. For some differential systems, averaging methods are valuable tools for determining the number of periodic orbits [4]. Numerous researchers have focused their efforts to studying the existence of periodic orbit via this method which has a long history. As we see in the work of Marsden and Mc. Cracken (Marsden,1976), chow and Hale (chow,1982), Samders , Buica and Llibre et al.

We must investigate the periodic solution of the following system

$$\dot{X} = \sum_{i=0}^{k} \epsilon^i F_i(t, X) + \epsilon^{k+1} R(t, X, \epsilon), \tag{1.6}$$

suth that

$$F_i : \mathbb{R} \times D \to \mathbb{R}^n \text{ for } i = 0, 1, 2, ..., k,$$

$$\mathbb{R} : \mathbb{R} \times D \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^n$$

are locally Lipschitz functions, and T–periodic in the first variable, being D an open subset of $\mathbb{R}^n$; eventually F_0 can be the zero constant function.

In classical works, the averaging theory is employed to investigate periodic solutions of a differential system (1.6) in the small parameter ϵ this theory is generally only provided up to first (k = 1) or second (k = 2) order. Moreover, these theories presuppose that the functions F_i and $\mathbb{R}$ are differentiable up to classes C^2 and C^3, respectively. The averaging theory for finding periodic solutions was recently developed up to second order in dimension n, and up to third order (k = 3) in dimension 1, only needing that the functions F_i and $\mathbb{R}$ be locally Lipschitz functions.The averaging theory for calculating periodic solutions to an arbitrary order k in ϵ for analytical differential equations in dimension 1 was developed in a recent paper [7]. Consider a harmonic oscillator that has been perturbed by an arbitrary polynomial

$$\begin{aligned} \dot{x} &= -y + \textstyle\sum_{j\geq 1} \epsilon^j f_j(x,y), \\ \dot{y} &= x + \textstyle\sum_{j\geq 1} \epsilon^j f_j(x,y), \end{aligned} \tag{1.7}$$

where ϵ is a small parameter, the polynomials f_j and g_j are of degree n in the variables x, y and analytic in the variables x, y and ϵ. Using the averaging theory up to order s in [7], the authors have provred no more than $[s(n-1)/2]$ periodic solutions can be bifurcating from linear center $\dot{x} = -y, \quad \dot{y} = x$, this bound can be reached when we perturbed this center by an arbitrary polynamial fo degree n.Here $[x]$ represents the integer part function of the real number x.

Remark 1.3. *Taking higher order averaging theory into consideration can improve the results on periodic solutions both qualitatively and quantitatively.*

Now, we study the existence of periodic orbits of general differential systems (1.6), for that we present some notation that will be required in the following part. Let L be a positive integer, let

$$X = (x_1, ..., x_n) \in D, \; t \in \mathbb{R} \text{ and } Y_j = (y_{j1}, ..., y_{jn}) \in \mathbb{R}^n$$

for $j = 1, ..., L$. Given

$$F_i : \mathbb{R} \times D \to \mathbb{R}^n$$

a sufficiently smooth function, for each $(t, X) \in \mathbb{R} \times D$ we denote by $\partial^L F(t, X)$ a symmetric L–multilinear map which is applied to a "product" of L vectors of $\mathbb{R}^n$, which we denote as $\bigodot_{j=1}^{L} y_j \in \mathbb{R}^{nL}$. The definition of this L–multilinear map is

$$\partial^L F(t,X) \bigodot_{j=1}^{L} Y_j = \sum_{i_1,...,i_L=1}^{n} \frac{\partial^L F(t,X)}{\partial x_i ... \partial x_{i_L}} y_{i_1} ... y_{L i_L} \tag{1.8}$$

We define ∂^0 as the identity functional. Given a positive integer b and a vector $Y \in \mathbb{R}^n$ we also denote $Y^b = \bigodot_{i=1}^{b} Y \in \mathbb{R}^{nb}$.

Remark 1.4. *The L-multilinear map defined in (1.8) is the L^{th} Frechet derivative of the function $F(t,x)$ with respect to the variable x. Indeed, fixed $t \in \mathbb{R}$, if we consider the function $F_t : D \to \mathbb{R}^n$ such that $F_t(x) = F(t,x)$, then $\partial^L F(t,x) = F_t^{(L)}(x) = \partial^L / \partial x^L F(t,x)$.*

Example 3. *To illustrate the above notation (1.8) we consider a smooth function $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$. So for $x = (x_1, x_2)$ and $y^1 = (y_1^1, y_2^1)$ we have*

$$\partial F(t,x) y^1 = \frac{\partial F(t,x)}{\partial x_1} y_1^1 + \frac{\partial F(t,x)}{\partial x_2} y_2^1.$$

Now, for $y^1 = (y_1^1, y_2^1)$ and $y^2 = (y_1^2, y_2^2)$ we have

$$\partial^2 F(t,x)(y^1, y^2) = \frac{\partial^2 F(t,x)}{\partial x_1 \partial x_1} y_1^1 y_1^2 + \frac{\partial^2 F(t,x)}{\partial x_1 \partial x_2} y_1^1 y_2^2 + \frac{\partial^2 F(t,x)}{\partial x_2 \partial x_1} y_2^1 y_1^2 + \frac{\partial^2 F(t,x)}{\partial x_2 \partial x_2} y_1^2 y_2^2.$$

Observe that for each $(t, x) \in \mathbb{R} \times D, \partial F(t, x)$ is a linear map in $\mathbb{R}^2$ and $\partial^2 F(t, x)$ is a bilinear map in $\mathbb{R}^2 \times \mathbb{R}^2$.

Let $\varphi(\Delta, z) : [0, tz] \to \mathbb{R}^n$ be the solution of the unperturbed system

$$\dot{X} = F_0(t, X) \tag{1.9}$$

such that $\varphi(0, z) = z$.

For $i = 1, 2, ..., k$, we define the Averaged Function $f_i : D \to \mathbb{R}^n$ of order i as

$$f_i = \frac{y_i(T, z)}{i!} \tag{1.10}$$

where $y_i : \mathbb{R} \times D \to \mathbb{R}$, for $i = 1, 2, ..., k - 1$ are defined recurrently by the following integral equation

$$\begin{aligned} y_i(t, z) = \;& i! \int_0^t (F_i(s, \varphi(s, z)) \\ & + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} ! ... b_l! l! 2!^{b_l}!} \partial^L (F_{i-l}(s, \varphi(s, z)) \bigodot_{j=1}^{l} y_j(s, z)^{b_j}) ds \end{aligned} \tag{1.11}$$

where S_l is the set of all l-tuples of non–negative integers $(b_1, b_2, ..., b_l)$ satisfying $b_1 + 2b_2 + ... + lb_l = l$, and $L = b_1 + b_2 + ... + b_l$.

In Section 2.1 we compute the sets S_l for $l = 1, 2, 3, 4.$ Furthermore, we make explicit the functions $f_k(z)$ up to $k = 4$ when $F_0 = 0.$ See [12]

Chapitre 2

On the limit cycles of the piecewise differential systems formed by linear systems in three zones

Our objective in this chapter is to study the limit cycles which bifurcate from the periodic orbits of the linear differential center $\dot{x} = -y, \ \dot{y} = x$, when we perturb this center by discontinuous piecewise differential systems, which becomes

$$\begin{cases} \dot{x}_A = \gamma_A + \alpha_A x + \beta_A y, \\ \dot{y}_A = \delta_A - \beta_A x + \alpha_A y, \end{cases} \tag{2.1}$$

$$\begin{cases} \dot{x}_M = \gamma_M + \alpha_M x + \beta_M y, \\ \dot{y}_M = \delta_M - \beta_M x + \alpha_M y, \end{cases} \tag{2.2}$$

$$\begin{cases} \dot{x}_B = \gamma_B + \alpha_B x + \beta_B y, \\ \dot{y}_B = \delta_B - \beta_B x + \alpha_B y, \end{cases} \tag{2.3}$$

in the following three zones

$$A = \{(x; y) \in \mathbb{R}^2 \ / \ \theta \in [\pi/4; 3\pi/4]\},$$
$$M = \{(x; y) \in \mathbb{R}^2 \ / \ \theta \in [0; \pi/4] \bigcup [3\pi/4; \pi]\},$$
$$B = \{(x; y) \in \mathbb{R}^2 \ / \ \theta \in [-\pi; 0]\},$$

respectively, where $x = r\cos\theta, \ \ y = r\sin\theta$;

$$\begin{aligned}
\alpha_A &= \alpha_{A_1}\epsilon + \alpha_{A_2}\epsilon^2 + \alpha_{A_3}\epsilon^3 + \alpha_{A_4}\epsilon^4, \\
\beta_A &= -1 + \beta_{A_1}\epsilon + \beta_{A_2}\epsilon^2 + \beta_{A_3}\epsilon^3 + \beta_{A_4}\epsilon^4, \\
\gamma_A &= \gamma_{A_1}\epsilon + \gamma_{A_2}\epsilon^2 + \gamma_{A_3}\epsilon^3 + \gamma_{A_4}\epsilon^4, \\
\delta_A &= \delta_{A_1}\epsilon + \delta_{A_2}\epsilon^2 + \delta_{A_3}\epsilon^3 + \delta_{A_4}\epsilon^4,
\end{aligned}$$

$$
\begin{aligned}
\alpha_M &= \alpha_{M_1}\epsilon + \alpha_{M_2}\epsilon^2 + \alpha_{M_3}\epsilon^3 + \alpha_{M_4}\epsilon^4, \\
\beta_M &= -1 + \beta_{M_1}\epsilon + \beta_{M_2}\epsilon^2 + \beta_{M_3}\epsilon^3 + \beta_{M_4}\epsilon^4, \\
\gamma_M &= \gamma_{M_1}\epsilon + \gamma_{M_2}\epsilon^2 + \gamma_{M_3}\epsilon^3 + \gamma_{M_4}\epsilon^4, \\
\delta_M &= \delta_{M_1}\epsilon + \delta_{M_2}\epsilon^2 + \delta_{M_3}\epsilon^3 + \delta_{M_4}\epsilon^4, \\
\alpha_B &= \alpha_{B_1}\epsilon + \alpha_{B_2}\epsilon^2 + \alpha_{B_3}\epsilon^3 + \alpha_{B_4}\epsilon^4, \\
\beta_B &= -1 + \beta_{B_1}\epsilon + \beta_{B_2}\epsilon^2 + \beta_{B_3}\epsilon^3 + \beta_{B_4}\epsilon^4, \\
\gamma_B &= \gamma_{B_1}\epsilon + \gamma_{B_2}\epsilon^2 + \gamma_{B_3}\epsilon^3 + \gamma_{B_4}\epsilon^4, \text{ and} \\
\delta_B &= \delta_{B_1}\epsilon + \delta_{B_2}\epsilon^2 + \delta_{B_3}\epsilon^3 + \delta_{B_4}\epsilon^4.
\end{aligned}
$$

Our main result is the following theorem.

Theorem 4. *For $\varepsilon \neq 0$ sufficiently small the maximum number of limit cycles of the piecewise differential systems obtained perturbing the linear differential center $\dot{x} = -y$, $\dot{y} = x$ by the discontinuous piecewise differential system formed by systems (2.1), (2.2) and (2.3) obtained using averaging theory up to fourth order is five.*

2.1 The averaging theory up to order 4 for computing limit cycles

In this section we present the basic results from the averaging theory for computing the periodic solutions of discontinuous piecewise differential systems that we shall need for proving the main results of this work. This improvement of the classical averaging theory for computing limit cycles of planar discontinuous piecewise differential systems was developed in [8], a summary of this theory is given in below. We consider discontinous differential systems of the form

$$
\dot{r}(\theta) = \begin{cases} F_A(\theta, r, \varepsilon) & \text{if } \pi/4 \le \theta \le 3\pi/4, \\ F_M(\theta, r, \varepsilon) & \text{if } 0 \le \theta \le \pi/4 \text{ or } 3\pi/4 \le \theta \le \pi, \\ F_B(\theta, r, \varepsilon) & \text{if } -\pi \le \theta \le 0, \end{cases} \tag{2.4}
$$

where $F_j(\theta, r, \varepsilon) = \sum_{i=1}^{4} \varepsilon^i F_i^{(j)}(\theta, r) + \varepsilon^5 R_j(\theta, r, \varepsilon)$, with $\theta \in \mathbb{S}^1$ and $r \in D$, where D is an open interval of $\mathbb{R}^+$, ε is a small real parameter,and j is A, M or B.

From [8] we define the following functions $y_i^{(j)}(t, r)$ for $i = 1, 2, 3$ related to system (2.4) :

$$
\begin{aligned}
y_1^{(j)}(s, r) &= \int_0^s F_1^{(j)}(t, r)dt, \\
y_2^{(j)}(s, r) &= \int_0^s [2F_2^{(j)}(t, r) + 2\partial F_1^{(j)}(t, r) y_1^{(j)}(t, r)]dt, \\
y_3^{(j)}(s, r) &= \int_0^s [6F_3^{(j)}(t, r) + 6\partial F_2^{(j)}(t, r) y_1^{(j)}(t, r) \\
&\qquad + 3\partial^2 F_1^{(j)}(t, r) y_1^{(j)}(t, r)^2 + 3\partial F_1^{(j)}(t, r) y_2^{(j)}(t, r)]dt,
\end{aligned}
$$

Here $\partial^k F_l(s, r)$ means the $k-th$ partial derivative of the function $F_l(s, r)$ with respect to the variable r. Also from [8] we have the functions

$$\begin{aligned}
f_1^{(j)}(r) &= \int_{j\in\Omega} F_1^{(j)}(t,r)dt,\\
f_2^{(j)}(r) &= \int_{j\in\Omega} [F_2^{(j)}(t,r) + \partial F_1^{(j)}(t,r)y_1^{(j)}(t,r)]dt,\\
f_3^{(j)}(r) &= \int_{j\in\Omega} [F_3^{(j)}(t,r) + \partial F_2^{(j)}(t,r)y_1^{(j)}(t,r)\\
&\qquad + \frac{1}{2}\partial^2 F_1^{(j)}(t,r)y_1^{(j)}(t,r)^2 + \frac{1}{2}\partial F_1^{(j)}(t,r)y_2^{(j)}(t,r)]dt,\\
f_4^{(j)}(r) &= \int_{j\in\Omega} [F_4^{(j)}(t,r) + \partial F_3^{(j)}(t,r)y_1^{(j)}(t,r) + \frac{1}{2}\partial^2 F_2^{(j)}(t,r)y_1^{(j)}(t,r)^2\\
&\qquad + \frac{1}{2}\partial F_2^{(j)}(t,r)y_2^{(j)}(t,r) + \frac{1}{2}\partial^2 F_1^{(j)}(t,r)y_1^{(j)}(t,r)y_2^{(j)}(t,r)\\
&\qquad + \frac{1}{6}\partial^3 F_1^{(j)}(t,r)y_1^{(j)}(t,r)^3 + \frac{1}{6}\partial F_1^{(j)}(t,r)y_3^{(j)}(t,r)]dt,
\end{aligned}$$

where $\Omega = \{A, M, B\}$. The function $f_k(r) = f_k^A(r) + f_k^M(r) - f_k^B(r)$ is called the averaged function of order k. If $f_k(r) \equiv 0$ for $k \in \{1, \ldots, 3\}$ but $f_{k+1}(r) \not\equiv 0$, then the simple positive real roots of the functions $f_{k+1}(r)$ provide limit cycles of the piecewise differential system (2.4).

2.2 Proof of Theorem 4

Consider the linear center we shall study which periodic orbits of this center become limit cycles when we perturb the center inside the discontinuous piecewise differential systems formed by systems

$$\begin{aligned}
\dot{x}_A = &\ -y + \epsilon(\gamma_{A_1} + \alpha_{A_1}x + \beta_{A_1}y) + \epsilon^2(\gamma_{A_2} + \alpha_{A_2}x + \beta_{A_2}y)\\
&+\epsilon^3(\gamma_{A_3} + \alpha_{A_3}x + \beta_{A_3}y) + \epsilon^4(\gamma_{A_4} + \alpha_{A_4}x + \beta_{A_4}y),\\
\dot{y}_A = &\ x + \epsilon(\delta_{A_1} - \beta_{A_1}x + \alpha_{A_1}y) + \epsilon^2(\delta_{A_2} - \beta A_2 x + \alpha_{A_2}y)\\
&+\epsilon^3(\delta_{A_3} - \beta_{A_3}x + \alpha_{A_3}y) + \epsilon^4(\delta_{A_4} - \beta_{A_4}x + \alpha_{A_4}y),
\end{aligned}$$

in the region A,

$$\begin{aligned}
\dot{x}_M = &\ -y + \epsilon(\gamma_{M_1} + \alpha_{M_1}x + \beta_{M_1}y) + \epsilon^2(\gamma_{M_2} + \alpha_{M_2}x + \beta_{M_2}y)\\
&+\epsilon^3(\gamma_{M_3} + \alpha_{M_3}x + \beta_{M_3}y) + \epsilon^4(\gamma_{M_4} + \alpha_{M_4}x + \beta_{M_4}y),\\
\dot{y}_M = &\ x + \epsilon(\delta_{M_1} - \beta_{M_1}x + \alpha_{M_1}y) + \epsilon^2(\delta_{M_2} - \beta_{M_2}x + \alpha_{M_2}y)\\
&+\epsilon^3(\delta_{M_3} - \beta_{M_3}x + \alpha_{M_3}y) + \epsilon^4(\delta_{M_4} - \beta_{M_4}x + \alpha_{M_4}y),
\end{aligned}$$

in the region M and

$$\begin{aligned}
\dot{x}_B = &\ -y + \epsilon(\gamma_{B_1} + \alpha_{B_1}x + \beta_{B_1}y) + \epsilon^2(\gamma_{B_2} + \alpha_{B_2}x + \beta_{B_2}y)\\
&+\epsilon^3(\gamma_{B_3} + \alpha B_3 x + \beta_{B_3}y) + \epsilon^4(\gamma_{B_4} + \alpha_{B_4}x + \beta_{B_4}y),\\
\dot{y}_B = &\ x + \epsilon(\delta_{B_1} - \beta_{B_1}x + \alpha_{B_1}y) + \epsilon^2(\delta_{B_2} - \beta_{B_2}x + \alpha_{B_2}y)\\
&+\epsilon^3(\delta_{B_3} - \beta_{B_3}x + \alpha_{B_3}y) + \epsilon^4(\delta_{B_4} - \beta_{B_4}x + \alpha_{B_4}y),
\end{aligned}$$

in the region B.

Now we write the discontinuous piecewise differential systems in polar coordinates $(\dot{r}, \dot{\theta})$, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$, which give the following systems

$$\begin{aligned}\dot{r}_A = \;& \epsilon(\gamma_{A_1}\cos(\theta) + \delta_{A_1}\sin(\theta) + \alpha_{A_1} r) + \epsilon^2(\gamma_{A_2}\cos(\theta) + \delta_{A_2}\sin(\theta) + \alpha_{A_2} r) \\ & + \epsilon^3(\gamma_{A_3}\cos(\theta) + \delta_{A_3}\sin(\theta) + \alpha_{A_3} r) + \epsilon^4(\gamma_{A_4}\cos(\theta) + \delta_{A_4}\sin(\theta) + \alpha_{A_4} r),\end{aligned}$$

$$\begin{aligned}\dot{\theta}_A = \;& 1 - \frac{\epsilon}{r}(\gamma_{A_1}\sin(\theta) - \delta_{A_1}\cos(\theta) + \beta_{A_1} r) - \frac{\epsilon^2}{r}(\gamma_{A_2}\sin(\theta) - \delta_{A_2}\cos(\theta) + \beta_{A_2} r) \\ & - \frac{\epsilon^3}{r}(\gamma_{A_3}\sin(\theta) - \delta_{A_3}\cos(\theta) + \beta_{A_3} r) - \frac{\epsilon^4}{r}(\gamma_{A_4}\sin(\theta) - \delta_{A_4}\cos(\theta) + \beta_{A_4} r),\end{aligned}$$

in the region A,

$$\begin{aligned}\dot{r_M} = \;& \epsilon(\gamma_{M_1}\cos(\theta) + \delta_{M_1}\sin(\theta) + \alpha_{M_1} r) + \epsilon^2(\gamma_{M_2}\cos(\theta) + \delta_{M_2}\sin(\theta) + \alpha_{M_2} r) \\ & + \epsilon^3(\gamma_{M_3}\cos(\theta) + \delta_{M_3}\sin(\theta) + \alpha_{M_3} r) + \epsilon^4(\gamma_{M_4}\cos(\theta) + \delta_{M_4}\sin(\theta) + \alpha_{M_4} r), \\ \dot{\theta_M} = \;& 1 - \frac{\epsilon}{r}(\gamma_{M_1}\sin(\theta) - \delta_{M_1}\cos(\theta) + \beta_{M_1} r) - \frac{\epsilon^2}{r}(\gamma_{M_2}\sin(\theta) - \delta_{M_2}\cos(\theta) + \beta_{M_2} r) \\ & - \frac{\epsilon^3}{r}(\gamma_{M_3}\sin(\theta) - \delta_{M_3}\cos(\theta) + \beta_{M_3} r) - \frac{\epsilon^4}{r}(\gamma_{M_4}\sin(\theta) - \delta_{M_4}\cos(\theta) + \beta_{M_4} r),\end{aligned}$$

in the region M, and

$$\begin{aligned}\dot{x}_B = \;& \epsilon(\gamma_{B_1}\cos(\theta) + \delta_{B_1}\sin(\theta) + \alpha_{B_1} r) + \epsilon^2(\gamma_{B_2}\cos(\theta) + \delta_{B_2}\sin(\theta) + \alpha_{B_2} r) \\ & + \epsilon^3(\gamma_{B_3}\cos(\theta) + \delta_{B_3}\sin(\theta) + \alpha_{B_3} r) + \epsilon^4(\gamma_{B_4}\cos(\theta) + \delta_{B_4}\sin(\theta) + \alpha_{B_4} r), \\ \dot{y}_B = \;& 1 - \frac{\epsilon}{r}(\gamma_{B_1}\sin(\theta) - \delta_{B_1}\cos(\theta) + \beta_{B_1} r) - \frac{\epsilon^2}{r}(\gamma_{B_2}\sin(\theta) - \delta_{B_2}\cos(\theta) + \beta_{B_2} r) \\ & - \frac{\epsilon^3}{r}(\gamma_{B_3}\sin(\theta) - \delta_{B_3}\cos(\theta) + \beta_{B_3} r) - \frac{\epsilon^4}{r}(\gamma_{B_4}\sin(\theta) - \delta_{B_4}\cos(\theta) + \beta_{B_4} r),\end{aligned}$$

in the region B.

Then we take as independent variable the angle θ, and the system $(\dot{r}, \dot{\theta})$ becomes the differential equation $dr/d\theta$. By doing a Taylor expansion truncated at 4-th order in ε we obtain an expression for $dr/d\theta$ written as the one of the differential system (2.4).

$$\begin{aligned}\frac{dr_A}{d\theta_A} =\;& \epsilon(\gamma_{A_1}\cos(\theta) + \delta_{A_1}\sin(\theta) + \alpha_{A_1} r) + \epsilon^2(\gamma_{A_2}\cos(\theta) + \delta_{A_2}\sin(\theta) + \alpha_{A_2} r \\ & + \frac{1}{r}((\gamma_{A_1}\sin(\theta) - \delta_{A_1}\cos(\theta) + \beta_{A_1} r)(\gamma_{A_1}\cos(\theta) + \delta_{A_1}\sin(\theta) + \alpha_{A_1} r)) \\ & + \epsilon^3(\gamma_{A_3}\cos(\theta) + \delta_{A_3}\sin(\theta) + \alpha_{A_3} r + \frac{1}{r}(\gamma_{A_2}\cos(\theta) + \delta_{A_2}\sin(\theta) + \alpha_{A_2} r) \\ & (\gamma_{A_1}\sin(\theta) - \delta_{A_1}\cos(\theta) + \beta_{A_1} r) + \frac{1}{r^2}((\gamma_{A_1}\cos(\theta) + \delta_{A_1}\sin(\theta) + \alpha_{A_1} r) \\ & (\gamma_{A_1}^2\sin^2(\theta) + \delta_{A_1}^2\cos^2(\theta) + r^2(\beta_{A_1}^2 + \beta_{A_2}) + r\sin(\theta)(2\beta_{A_1}\gamma_{A_1} + \gamma_{A_2})\end{aligned}$$

$$
\begin{aligned}
&- \cos(\theta)(2\gamma_{A_1}\delta_{A_1}\sin(\theta) + r(2\beta_{A_1}\delta_{A_1} + \delta_{A_2})) + \epsilon^4(\gamma_{A_4}\cos(\theta) + \delta_{A_4}\sin(\theta) \\
&+ \alpha_{A_4} r + \frac{1}{r}(\gamma_{A_3}\cos(\theta) + \delta_{A_3}\sin(\theta) + \alpha_{A_3} r)(\gamma_{A_1}\sin(\theta) - \delta_{A_1}\cos(\theta) + \beta_{A_1} r) \\
&+ \frac{1}{r^2}(\gamma_{A_2}\cos(\theta) + \delta_{A_2}\sin(\theta) + \alpha_{A_2} r)(\gamma_{A_1}^2\sin^2(\theta) + \delta_{A_1}^2\cos^2(\theta) \\
&+ r^2(\beta_{A_1}^2 + \beta_{A_2}) + r\sin(\theta)(2\beta_{A_1}\gamma_{A_1} + \gamma_{A_2}) - \cos(\theta)(2\gamma_{A_1}\delta_{A_1}\sin(\theta) \\
&+ r(2\beta A_1\delta_{A_1} + \delta_{A_2})) + \frac{1}{r^3}(\gamma_{A_1}\cos(\theta) + \delta_{A_1}\sin(\theta) + \alpha_{A_1} r) \\
&(r^3(\beta_{A_1}^3 + 2\beta_{A_1}\beta_{A_2} + \beta_{A_3}) - \delta_{A_1}^3\cos^3(\theta) + r^2\sin(\theta)(3\beta_{A_1}^2\gamma_{A_1} + 2\beta_{A_1}\gamma_{A_2} \\
&+ 2\beta_{A_2}\gamma_{A_1} + \gamma_{A_3}) + \gamma_{A_1} r(3\beta_{A_1}\gamma_{A_1} + 2\gamma_{A_2})\sin^2(\theta) + \gamma_{A_1}^3\sin^3(\theta) \\
&+ \delta_{A_1}\cos^2(\theta)(3\gamma_{A_1}\delta_{A_1}\sin(\theta) + 3\beta_{A_1}\delta_{A_1} r + 2\delta_{A_2} r) - \cos(\theta)(r^2(3\beta_{A_1}^2\delta_{A_1} \\
&+ 2\beta_{A_1}\delta_{A_2} + 2\beta_{A_2}\delta_{A_1} + \delta_{A_3}) + 2r\sin(\theta)(3\beta_{A_1}\gamma_{A_1}\delta_{A_1} + \gamma_{A_1}\delta_{A_2} + \gamma_{A_2}\delta_{A_1}) \\
&+ 3\gamma_{A_1}^2\delta_{A_1}\sin^2(\theta)),
\end{aligned}
$$

$$
\begin{aligned}
\frac{dr_M}{d\theta_M} =& \epsilon(\gamma_{M_1}\cos(\theta) + \delta_{M_1}\sin(\theta) + \alpha_{M_1} r) + \epsilon^2(\gamma_{M_2}\cos(\theta) + \delta_{M_2}\sin(\theta) + \alpha_{M_2} r \\
&+ \frac{1}{r}(\gamma_{M_1}\sin(\theta) - \delta_{M_1}\cos(\theta) + \beta_{M_1} r)(\gamma_{M_1}\cos(\theta) + \delta_{M_1}\sin(\theta) + \alpha_{M_1} r)) \\
&+ \epsilon^3(\gamma_{M_3}\cos(\theta) + \delta_{M_3}\sin(\theta) + \alpha_{M_3} r + \frac{1}{r}(\gamma_{M_2}\cos(\theta) + \delta_{M_2}\sin(\theta) + \alpha_{M_2} r) \\
&(\gamma_{M_1}\sin(\theta) - \delta_{M_1}\cos(\theta) + \beta_{M_1} r) + \frac{1}{r^2}(\gamma_{M_1}\cos(\theta) + \delta_{M_1}\sin(\theta) + \alpha_{M_1} r) \\
&(\gamma_{M_1}^2\sin^2(\theta) + \delta_{M_1}^2\cos^2(\theta) + r^2(\beta_{M_1}^2 + \beta_{M_2}) + r\sin(\theta)(2\beta_{M_1}\gamma_{M_1} + \gamma_{M_2}) \\
&- \cos(\theta)(2\gamma_{M_1}\delta_{M_1}\sin(\theta) + r(2\beta_{M_1}\delta_{M_1} + \delta_{M_2}))) + \epsilon^4(\gamma_{M_4}\cos(\theta) \\
&+ \delta_{M_4}\sin(\theta) + \alpha_{M_4} r + \frac{1}{r}(\gamma_{M_1}\sin(\theta) - \delta_{M_1}\cos(\theta) + \beta_{M_1} r)(\gamma_{M_3}\cos(\theta) \\
&+ \delta_{M_3}\sin(\theta) + \alpha_{M_3} r) + \frac{1}{r^2}(\gamma_{M_2}\cos(\theta) + \delta_{M_2}\sin(\theta) + \alpha_{M_2} r)(\gamma_{M_1}^2\sin^2(\theta) \\
&+ \delta_{M_1}^2\cos^2(\theta) + r^2(\beta_{M_1}^2 + \beta_{M_2}) + r\sin(\theta)(2\beta_{M_1}\gamma_{M_1} + \gamma_{M_2}) \\
&- \cos(\theta)(2\gamma_{M_1}\delta_{M_1}\sin(\theta) + r(2\beta_{M_1}\delta_{M_1} + \delta_{M_2})) + \frac{1}{r^3}(\gamma_{M_1}\cos(\theta) + \delta_{M_1}\sin(\theta) \\
&+ \alpha_{M_1} r)(\gamma_{M_1}^3\sin^3(\theta) - \delta_{M_1}^3\cos^3(\theta) + r^3(\beta_{M_1}^3 + 2\beta_{M_1}\beta_{M_2} + \beta_{M_3}) \\
&- \cos(\theta)(3\gamma_{M_1}^2\delta_{M_1}\sin^2(\theta) + r^2(3\beta_{M_1}^2\delta_{M_1} + 2\beta_{M_1}\delta_{M_2} + 2\beta_{M_2}\delta_{M_1} + \delta_{M_3}) \\
&+ 2r\sin(\theta)(3\beta_{M_1}\gamma_{M_1}\delta_{M_1} + \gamma_{M_1}\delta_{M_2} + \gamma_{M_2}\delta_{M_1}) + r^2\sin(\theta)(3\beta_{M_1}^2\gamma_{M_1} \\
&+ 2\beta_{M_1}\gamma_{M_2} + 2\beta_{M_2}\gamma_{M_1} + \gamma_{M_3}) + \gamma_{M_1} r\sin^2(\theta)(3\beta_{M_1}\gamma_{M_1} + 2\gamma_{M_2}) \\
&+ \delta_{M_1}\cos^2(\theta)(3\gamma_{M_1}\delta_{M_1}\sin(\theta) + 3\beta_{M_1}\delta_{M_1} r + 2\delta_{M_2} r))), \text{ and}
\end{aligned}
$$

$$\begin{aligned}
\frac{dr_B}{d\theta_B} =&\epsilon(\gamma_{B_1}\cos(\theta)+\delta_{B_1}\sin(\theta)+\alpha_{B_1}r)+\epsilon^2(\gamma_{B_2}\cos(\theta)+\delta_{B_2}\sin(\theta)+\alpha_{B_2}r)\\
&+\frac{1}{r}(\gamma_{B_1}\cos(\theta)+\delta_{B_1}\sin(\theta)+\alpha_{B_1}r)(\gamma_{B_1}\sin(\theta)-\delta_{B_1}\cos(\theta)+\beta_{B_1}r))\\
&+\epsilon^3(\gamma_{B_3}\cos(\theta)+\delta_{B_3}\sin(\theta)+\alpha_{B_3}r+\frac{1}{r}(\gamma_{B_2}\cos(\theta)+\delta_{B_2}\sin(\theta)+\alpha_{B_2}r)\\
&(\gamma_{B_1}\sin(\theta)-\delta_{B_1}\cos(\theta)+\beta_{B_1}r)+\frac{1}{r^2}(\gamma_{B_1}\cos(\theta)+\delta_{B_1}\sin(\theta)+\alpha_{B_1}r)\\
&(\gamma_{B_1}^2\sin^2(\theta)+\delta_{B_1}^2\cos^2(\theta)+r^2(\beta_{B_1}^2+\beta_{B_2})+r\sin(\theta)(2\beta_{B_1}\gamma_{B_1}+\gamma_{B_2})\\
&-\cos(\theta)(2\gamma_{B_1}\delta_{B_1}\sin(\theta)+r(2\beta_{B_1}\delta_{B_1}+\delta_{B_2})))+\epsilon^4(\gamma_{B_4}\cos(\theta)+\alpha_{B_4}r\\
&+\delta_{B_4}\sin(\theta)+\frac{1}{r}(\gamma_{B_3}\cos(\theta)+\delta_{B_3}\sin(\theta)+\alpha_{B_3}r)(\gamma_{B_1}\sin(\theta)-\delta_{B_1}\cos(\theta)\\
&+\beta_{B_1}r)\frac{1}{r^2}(\gamma_{B_2}\cos(\theta)+\delta_{B_2}\sin(\theta)+\alpha_{B_2}r)(\gamma_{B_1}^2\sin^2(\theta)+\delta_{B_1}^2\cos^2(\theta)\\
&+r^2(\beta_{B_1}^2+\beta_{B_2})+r\sin(\theta)(2\beta_{B_1}\gamma_{B_1}+\gamma_{B_2})-\cos(\theta)(2\gamma_{B_1}\delta_{B_1}\sin(\theta)\\
&+r(2\beta_{B_1}\delta_{B_1}+\delta_{B_2}))+\frac{1}{r^3}(\gamma_{B_1}\cos(\theta)+\delta_{B_1}\sin(\theta)+\alpha_{B_1}r)(\gamma_{B_1}^3\sin^3(\theta)\\
&-\delta_{B_1}^3\cos^3(\theta)+r^3(\beta_{B_1}^3+2\beta_{B_1}\beta_{B_2}+\beta_{B_3})-\cos(\theta)(3\gamma_{B_1}^2\delta_{B_1}\sin^2(\theta)\\
&+r^2(3\beta_{B_1}^2\delta_{B_1}+2\beta_{B_1}\delta_{B_2}+2\beta_{B_2}\delta_{B_1}+\delta_{B_3})+2r\sin(\theta)(3\beta_{B_1}\gamma_{B_1}\delta_{B_1}\\
&+\gamma_{B_1}\delta_{B_2}+\gamma_{B_2}\delta_{B1}))+r^2\sin(\theta)(3\beta_{B_1}^2\gamma_{B_1}+2\beta_{B_1}\gamma_{B_2}+2\beta_{B_2}\gamma_{B_1}+\gamma_{B_3})\\
&+\gamma_{B_1}r\sin^2(\theta)(3\beta_{B_1}\gamma_{B_1}+2\gamma_{B_2})+\delta_{B_1}\cos^2(\theta)(3\gamma_{B_1}\delta_{B_1}\sin(\theta)+3\beta_{B_1}\delta_{B_1}r\\
&+2\delta_{B_2}r)).
\end{aligned}$$

In short we have written our discontinuous piecewise differential system formed by systems (2.1), (2.2) and (2.3) in the normal form (2.4) for applying the averaging theory. We give only the expression of functions $F_i^{(j)}(r,\theta)$ for $i=1,2,3$. We remove the explicit expressions of $F_4^{(j)}(r,\theta)$ since they are extremely hyge, but they may be easily produced using algebraic manipulator such as Mathematica or Mapple.

$$\begin{aligned}
F_1^A(r,\theta) &= \gamma_{A_1}\cos(\theta)+\delta_{A_1}\sin(\theta)+\alpha_{R_1}r,\\
F_1^M(r,\theta) &= \gamma_{M_1}\cos(\theta)+\delta_{M_1}\sin(\theta)+\alpha_{M_1}r,\\
F_1^B(r,\theta) &= \gamma_{B_1}\cos(\theta)+\delta_{B_1}\sin(\theta)+\alpha_{B_1}r\\
F_2^A(r,\theta) &= \frac{1}{2r}(\gamma_{A_1}^2\sin(2\theta)-2\gamma_{A_1}\delta_{A_1}\cos(2\theta)-\delta_{A_1}^2\sin(2\theta)+2\alpha_{A_1}\beta_{A_1}r^2+2\alpha_{A_2}r^2\\
&\quad+2\alpha_{A_1}\gamma_{A_1}r\sin(\theta)-2\alpha_{A_1}\delta_{A_1}r\cos(\theta)+2\beta_{A_1}\gamma_{A_1}r\cos(\theta)\\
&\quad+2\beta_{A_1}\delta_{A_1}r\sin(\theta)+2\gamma_{A_2}r\cos(\theta)+2\delta_{A_2}r\sin(\theta)),\\
F_2^M(r,\theta) &= \frac{1}{2r}(\gamma_{M_1}^2\sin(2\theta)-2\gamma_{M_1}\delta_{M_1}\cos(2\theta)-\delta_{M_1}^2\sin(2\theta)-4\alpha_{B_1}\beta_{M_1}r^2+2\alpha_{M_2}r^2\\
&\quad-2\alpha_{A_1}\beta_{M_1}r^2-4\alpha_{B_1}\gamma_{M_1}r\sin(\theta)+4\alpha_{B_1}\delta_{M_1}r\cos(\theta)\\
&\quad-2\alpha_{A_1}\gamma_{M_1}r\sin(\theta)+2\alpha_{A_1}\delta_{M_1}r\cos(\theta)+2\beta_{M_1}\gamma_{M_1}r\cos(\theta)\\
&\quad+2\beta_{M_1}\delta_{M_1}r\sin(\theta)+2\gamma_{M_2}r\cos(\theta)+2\delta_{M_2}r\sin(\theta)),
\end{aligned}$$

$$\begin{aligned}
F_2^B(r,\theta) =& \frac{1}{4r}(2\gamma_{B_1}^2\sin(2\theta) - 4\gamma_{B_1}\delta_{M_1}\cos(2\theta) + 2\sqrt{2}\gamma_{B_1}\delta_{M_1}\cos(2\theta) \\
& - 2\sqrt{2}\gamma_{B_1}\delta_{A_1}\cos(2\theta) - 3\delta_{M_1}^2\sin(2\theta) + 2\sqrt{2}\delta_{M_1}^2\sin(2\theta) \\
& - 2\sqrt{2}\delta_{M_1}\delta_{A_1}\sin(2\theta) + 2\delta_{M_1}\delta_{A_1}\sin(2\theta) - \delta_{A_1}^2\sin(2\theta) + 4\alpha_{B_1}\beta_{B_1}r^2 \\
& + 4\alpha_{B_2}r^2 + 4\alpha_{B_1}\gamma_{B_1}r\sin(\theta) - 4\alpha_{B_1}\delta_{M_1}r\cos(\theta) + 2\sqrt{2}\alpha_{B_1}\delta_{M_1}r\cos(\theta) \\
& - 2\sqrt{2}\alpha_{B_1}\delta_{A_1}r\cos(\theta) + 4\beta_{B_1}\gamma_{B_1}r\cos(\theta) - 2\sqrt{2}\beta_{B_1}\delta_{M_1}r\sin(\theta) \\
& + 4\beta_{B_1}\delta_{M_1}r\sin(\theta) + 2\sqrt{2}\beta_{B_1}\delta_{A_1}r\sin(\theta) + 4\gamma_{B_2}r\cos(\theta) + 4\delta_{B_2}r\sin(\theta)),
\end{aligned}$$

$$\begin{aligned}
F_3^A(r,\theta) =& \gamma_{A_3}\cos(\theta) + \delta_{A_3}\sin(\theta) + \alpha_{A_3}r + \frac{1}{r}(\gamma_{A_2}\cos(\theta) + \delta_{A_2}\sin(\theta) + \alpha_{A_2}r) \\
& (\gamma_{A_1}\sin(\theta) - \delta_{A_1}\cos(\theta) + \beta_{A_1}r) + \frac{1}{r^2}(\gamma_{A_1}\cos(\theta) + \delta_{A_1}\sin(\theta) + \alpha_{A_1}r) \\
& (\gamma_{A_1}^2\sin^2(\theta) + \delta_{A_1}^2\cos^2(\theta) + r^2(\beta_{A_1}^2 + \beta_{A_2}) + r\sin(\theta)(2\beta_{A_1}\gamma_{A_1} + \gamma_{A_2}) \\
& - \cos(\theta)(2\gamma_{A_1}\delta_{A_1}\sin(\theta) + r(2\beta_{A_1}\delta_{A_1} + \delta_{A_2})))
\end{aligned}$$

$$\begin{aligned}
F_3^M(r,\theta) =& \gamma_{M_3}\cos(\theta) + \delta_{M_3}\sin(\theta) + \alpha_{M_3}r + \frac{1}{r}(\gamma_{M_2}\cos(\theta) + \delta_{M_2}\sin(\theta) \\
& + r(-\pi\alpha_{B_1}^2 - 2\pi\alpha_{B_1}\alpha_{A_1} - 2\alpha_{B_1}\beta_{B_1} + 2\alpha_{B_1}\beta_{M_1} - 2\alpha_{B_2} \\
& - \pi\alpha_{A_1}^2 + \alpha_{A_1}\beta_{M_1} - \alpha_{A_1}\beta_{A_1} - \alpha_{A_2}))(\beta_{M_1}r - \delta_{M_1}\cos(\theta) \\
& + \frac{1}{\delta_{M_1}}\gamma_{A_1}\delta_{A_1}\sin(\theta)) + \frac{1}{r^2}(r(-2\alpha_{B_1} - \alpha_{A_1}) + \frac{1}{\delta_{M_1}}\gamma_{A_1}\delta_{A_1}\cos(\theta)) \\
& (r^2(\beta_{M_1}^2 + \beta_{M_2}) + \delta_{M_1}^2\cos^2(\theta) + r\sin(\theta)(\gamma_{M_2} + \frac{1}{\delta_{M_1}}2\beta_{M_1}\gamma_{A_1}\delta_{A_1}) \\
& + \frac{1}{\delta_{M_1}^2}\gamma_{A_1}^2\delta_{A_1}^2\sin^2(\theta) - \cos(\theta)(2\gamma_{A_1}\delta_{A_1}\sin(\theta) + r(2\beta_{M_1}\delta_{M_1} + \delta_{M_2})),
\end{aligned}$$

$$\begin{aligned}
F_3^B(r,\theta) =& \gamma_{B_3}\cos(\theta) + \delta_{B_3}\sin(\theta) + \alpha_{B_3}r + \frac{1}{r}(\gamma_{B_1}\sin(\theta) - \frac{1}{2}(-\sqrt{2}\delta_{M_1} + 2\delta_{M_1} + \\
& \sqrt{2}\delta_{A_1})\cos(\theta) + \beta_{B_1}r)(\gamma_{B_2}\cos(\theta) + \alpha_{B_2}r + \frac{1}{4}(-8\alpha_{B_1}\gamma_{B_1} - \sqrt{2}\pi\alpha_{B_1}\delta_{M_1} \\
& + \pi\sqrt{2}\alpha_{B_1}\delta_{A_1} + 4\sqrt{2}\alpha_{A_1}\gamma_{A_1} - \pi\alpha_{A_1}\delta_{M_1} + \pi\alpha_{A_1}\delta_{A_1} - 4\beta_{B_1}\delta_{M_1} \\
& + 2\sqrt{2}\beta_{B_1}\delta_{M_1} - 2\sqrt{2}\beta_{B_1}\delta_{A_1} - 2\sqrt{2}\beta_{M_1}\delta_{M_1} + 4\beta_{M_1}\delta_{M_1} + 2\sqrt{2}\beta_{A_1}\delta_{A_1} \\
& - 2\sqrt{2}\delta_{M_2} + 4\delta_{M_2} + \frac{8\sqrt{2}\alpha_{B_1}\gamma_{A_1}\delta_{A_1}}{\delta_{M_1}} - \frac{16\alpha_{B_1}\gamma_{A_1}\delta_{A_1}}{\delta_{M_1}} + \frac{4\sqrt{2}\alpha_{A_1}\gamma_{A_1}\delta_{A_1}}{\delta_{M_1}} \\
& - \frac{8\alpha_{A_1}\gamma_{A_1}\delta_{A_1}}{\delta_{M_1}} + 2\sqrt{2}\delta_{A_2})\sin(\theta)) + \frac{1}{r^2}(\gamma_{B_1}\cos(\theta) + \frac{1}{2}(-\sqrt{2}\delta_{M_1} + 2\delta_{M_1} \\
& + \sqrt{2}\delta_{A_1})\sin(\theta) + \alpha_{B_1}r)(\gamma_{B_1}^2\sin^2(\theta) + \frac{1}{4}(-\sqrt{2}\delta_{M_1} + 2\delta_{M_1} \\
& + \sqrt{2}\delta_{A_1})^2\cos^2(\theta) + r^2(\beta_{B_1}^2 + \beta_{B_2}) + r\sin(\theta)(2\beta_{B_1}\gamma_{B_1} + \gamma_{B_2}) \\
& - \cos(\theta)(r(\beta_{B_1}(-\sqrt{2}\delta_{M_1} + 2\delta_{M_1} + \sqrt{2}\delta_{A_1}) + \frac{1}{4}(-8\alpha_{B_1}\gamma_{B_1} \\
& + \frac{8\sqrt{2}\alpha_{B_1}\gamma_{A_1}\delta_{A_1}}{\delta_{M_1}} - \frac{16\alpha_{B_1}\gamma_{A_1}\delta_{A_1}}{\delta_{M_1}} - \sqrt{2}\pi\alpha_{B_1}\delta_{M_1} + \pi\sqrt{2}\alpha_{B_1}\delta_{A_1}
\end{aligned}$$

$$+\frac{4\sqrt{2}\alpha_{A_1}\gamma_{A_1}\delta_{A_1}}{\delta_{M_1}}-\frac{8\alpha_{A_1}\gamma_{A_1}\delta_{A_1}}{\delta_{M_1}}+4\sqrt{2}\alpha_{A_1}\gamma_{A_1}-\pi\alpha_{A_1}\delta_{M_1}+\pi\alpha_{A_1}\delta_{A_1}$$
$$-4\beta_{B_1}\delta_{M_1}+2\sqrt{2}\beta_{B_1}\delta_{M_1}-2\sqrt{2}\beta_{B_1}\delta_{A_1}-2\sqrt{2}\beta_{M_1}\delta_{M_1}+4\beta_{M_1}\delta_{M_1}$$
$$+2\sqrt{2}\beta_{A_1}\delta_{A_1}-2\sqrt{2}\delta_{M_2}+4\delta_{M_2}+2\sqrt{2}\delta_{A_2}))+\gamma_{B_1}(-\sqrt{2}\delta_{M_1}+2\delta_{M_1}$$
$$+\sqrt{2}\delta_{A_1})\sin(\theta))).$$

Now we compute the averaged function $f_i(r)$ defined in section 2.1, and for $i=1$ we get

$$f_1(r)=-2\delta_{B_1}-\left(\sqrt{2}-2\right)\delta_{M_1}+\sqrt{2}\delta_{A_1}+\frac{1}{2}\pi r(2\alpha_{B_1}+\alpha_{M_1}+\alpha_{A_1}).$$

Then polynomial $f_1(r)$ can have at most one positive real roots r_1, which provide one limit cycle for the discontinuous piecewise differential system (2.1), (2.2) and (2.3) when ε is sufficiently small. This limit cycle tend to the circular periodic orbit of radius r_1 of the linear differential center

$$\dot{x}=-y,\dot{y}=x,\quad \text{when } \varepsilon\to 0. \tag{2.5}$$

In order to apply the averaging theory of second order we need that $f_1(r)\equiv 0$. In order to eliminate the coefficients of this polynomial we must take $\alpha_{M_1}=-2\alpha_{B_1}-\alpha_{A_1}$, $\delta_{B_1}=\frac{1}{2}\left(-\sqrt{2}\delta_{M_1}+2\delta_{M_1}+\sqrt{2}\delta_{A_1}\right)$. Computing the function $f_2(r)$ we obtain

$$\begin{aligned}f_2(r)=&\frac{1}{2}\pi r(\pi\alpha_{B_1}^2+2\pi\alpha_{B_1}\alpha_{A_1}+2\alpha_{B_1}\beta_{B_1}-2\alpha_{B_1}\beta_{M_1}+2\alpha_{B_2}+\alpha_{M_2}+\pi\alpha_{A_1}^2\\&-\alpha_{A_1}\beta_{M_1}+\alpha_{A_1}\beta_{A_1}+\alpha_{A_2})+\frac{1}{r}(\gamma_{A_1}\delta_{A_1}-\gamma_{M_1}\delta_{M_1})+\frac{1}{2}(-8\alpha_{B_1}\gamma_{B_1}\\&-16\alpha_{B_1}\gamma_{M_1}+8\sqrt{2}\alpha_{B_1}\gamma_{M_1}-\sqrt{2}\pi\alpha_{B_1}\delta_{M_1}+\pi\sqrt{2}\alpha_{B_1}\delta_{A_1}-8\alpha_{A_1}\gamma_{M_1}\\&+4\sqrt{2}\alpha_{A_1}\gamma_{M_1}+4\sqrt{2}\alpha_{A_1}\gamma_{A_1}-\pi\alpha_{A_1}\delta_{M_1}+\pi\alpha_{A_1}\delta_{A_1}-4\beta_{B_1}\delta_{M_1}\\&+2\sqrt{2}\beta_{B_1}\delta_{M_1}-2\sqrt{2}\beta_{B_1}\delta_{A_1}-2\sqrt{2}\beta_{M_1}\delta_{M_1}+4\beta_{M_1}\delta_{M_1}+2\sqrt{2}\beta_{A_1}\delta_{A_1}-4\delta_{B_2}\\&-2\sqrt{2}\delta_{M_2}+4\delta_{M_2}+2\sqrt{2}\delta_{A_2}).\end{aligned}$$

The polynomial $f_2(r)$ can have no more than two positive real roots r_1 and r_2, which provide two limit cycles for the discontinuous piecewise differential systems (2.1), (2.2) and (2.3) when ε is sufficiently small. These limit cycles tend to the circular periodic orbits of radius r_1 and r_2 of the linear differential center (2.5). To apply the averaging theory of third order, we must have $f_2(r)\equiv 0$. That we accept

$$\begin{aligned}\alpha_{M_2}=&-\pi\alpha_{B_1}^2-2\pi\alpha_{B_1}\alpha_{A_1}-2\alpha_{B_1}\beta_{B_1}+2\alpha_{B_1}\beta_{M_1}-2\alpha_{B_2}-\pi\alpha_{A_1}^2+\alpha_{A_1}\beta_{M_1}\\&-\alpha_{A_1}\beta_{A_1}-\alpha_{A_2},\\ \gamma_{M_1}=&\frac{1}{\delta_{M_1}}\gamma_{A_1}\delta_{A_1},\ \text{and}\\ \delta_{B_2}=&\frac{1}{4}(-8\alpha_{B_1}\gamma_{B_1}-16\alpha_{B_1}\gamma_{M_1}+8\sqrt{2}\alpha_{B_1}\gamma_{M_1}-\sqrt{2}\pi\alpha_{B_1}\delta_{M_1}+\pi\sqrt{2}\alpha_{B_1}\delta_{A_1}\\&-8\alpha_{A_1}\gamma_{M_1}+4\sqrt{2}\alpha_{A_1}\gamma_{M_1}+4\sqrt{2}\alpha_{A_1}\gamma_{A_1}-\pi\alpha_{A_1}\delta_{M_1}+\pi\alpha_{A_1}\delta_{A_1}-4\beta_{B_1}\delta_{M_1}\\&+2\sqrt{2}\beta_{B_1}\delta_{M_1}-2\sqrt{2}\beta_{B_1}\delta_{A_1}-2\sqrt{2}\beta_{M_1}\delta_{M_1}+4\beta_{M_1}\delta_{M_1}+2\sqrt{2}\beta_{A_1}\delta_{A_1}\\&-2\sqrt{2}\delta_{M_2}+4\delta_{M_2}+2\sqrt{2}\delta_{A_2}).\end{aligned}$$

Computing the function $f_3(r)$ we obtain

$$
\begin{aligned}
f_3(r) =& \frac{1}{32}\pi r(12\pi^2\alpha_{B_1}^3 + 42\pi^2\alpha_{B_1}^2\alpha_{A_1} + 32\pi\alpha_{B_1}^2\beta_{B_1} - 16\pi\alpha_{B_1}^2\beta_{M_1} + 32\pi\alpha_{B_1}\alpha_{B_2} \\
&+ 45\pi^2\alpha_{B_1}\alpha_{A_1}^2 + 32\pi\alpha_{B_1}\alpha_{A_1}\beta_{B_1} - 32\pi\alpha_{B_1}\alpha_{A_1}\beta_{M_1} + 32\pi\alpha_{B_1}\alpha_{A_1}\beta_{A_1} \\
&+ 32\pi\alpha_{B_1}\alpha_{A_2} + 32\alpha_{B_1}\beta_{B_1}^2 - 32\alpha_{B_1}\beta_{B_1}\beta_{M_1} + 32\alpha_{B_1}\beta_{B_2} - 32\alpha_{B_1}\beta_{M_2} \\
&+ 32\pi\alpha_{B_2}\alpha_{A_1} + 32\alpha_{B_2}\beta_{B_1} - 32\alpha_{B_2}\beta_{M_1} + 32\alpha_{B_3} + 16\alpha_{M_3} + 15\pi^2\alpha_{A_1}^3 \\
&- 16\pi\alpha_{A_1}^2\beta_{M_1} + 32\pi\alpha_{A_1}^2\beta_{A_1} + 32\pi\alpha_{A_1}\alpha_{A_2} - 16\alpha_{A_1}\beta_{M_1}\beta_{A_1} - 16\alpha_{A_1}\beta_{M_2} \\
&+ 16\alpha_{A_1}\beta_{A_1}^2 + 16\alpha_{A_1}\beta_{A_2} - 16\alpha_{A_2}\beta_{M_1} + 16\alpha_{A_2}\beta_{A_1} + 16\alpha_{A_3}) \\
&+ \frac{1}{r^2}\gamma_{A_1}\delta_{A_1}(\delta_{M_1} - \delta_{A_1}) + \frac{1}{8\delta_{M_1}^2 r}(8\pi\alpha_{B_1}\gamma_{B_1}^2\delta_{M_1}^2 - 10\pi\alpha_{B_1}\gamma_{A_1}^2\delta_{A_1}^2 \\
&+ 16\alpha_{B_1}\gamma_{A_1}^2\delta_{A_1}^2 - 8\pi\alpha_{B_1}\gamma_{A_1}\delta_{M_1}^2\delta_{A_1} + 2\pi\alpha_{B_1}\delta_{M_1}^4 - 16\alpha_{B_1}\delta_{M_1}^4 + 3\pi\alpha_{A_1}\gamma_{A_1}^2\delta_{M_1}^2 \\
&+ 8\alpha_{A_1}\gamma_{A_1}^2\delta_{M_1}^2 - 5\pi\alpha_{A_1}\gamma_{A_1}^2\delta_{A_1}^2 + 8\alpha_{A_1}\gamma_{A_1}^2\delta_{A_1}^2 - 8\pi\alpha_{A_1}\gamma_{A_1}\delta_{M_1}^2\delta_{A_1} + \pi\alpha_{A_1}\delta_{M_1}^4 \\
&- 8\alpha_{A_1}\delta_{M_1}^4 + \pi\alpha_{A_1}\delta_{M_1}^2\delta_{A_1}^2 - 8\alpha_{A_1}\delta_{M_1}^2\delta_{A_1}^2 - 16\beta_{M_1}\gamma_{A_1}\delta_{M_1}^2\delta_{A_1} \\
&+ 16\beta_{A_1}\gamma_{A_1}\delta_{M_1}^2\delta_{A_1} - 8\gamma_{M_2}\delta_{M_1}^3 + 8\gamma_{A_1}\delta_{M_1}^2\delta_{A_2} - 8\gamma_{A_1}\delta_{M_1}\delta_{M_2}\delta_{A_1} + 8\gamma_{A_2}\delta_{M_1}^2\delta_{A_1}) \\
&- \frac{1}{4\delta_{M_1}}(-16\sqrt{2}\pi\alpha_{B_1}^2\gamma_{A_1}\delta_{A_1} + 16\pi\alpha_{B_1}^2\gamma_{A_1}\delta_{A_1} - 24\sqrt{2}\alpha_{B_1}^2\delta_{M_1}^2 + 48\alpha_{B_1}^2\delta_{M_1}^2 \\
&- 8\sqrt{2}\alpha_{B_1}^2\delta_{M_1}\delta_{A_1} - 4\sqrt{2}\pi\alpha_{B_1}\alpha_{A_1}\gamma_{A_1}\delta_{M_1} - 20\sqrt{2}\pi\alpha_{B_1}\alpha_{A_1}\gamma_{A_1}\delta_{A_1} \\
&+ 24\pi\alpha_{B_1}\alpha_{A_1}\gamma_{A_1}\delta_{A_1} + \pi^2\alpha_{B_1}\alpha_{A_1}\delta_{M_1}^2 - 32\sqrt{2}\alpha_{B_1}\alpha_{A_1}\delta_{M_1}^2 + 64\alpha_{B_1}\alpha_{A_1}\delta_{M_1}^2 \\
&- \pi^2\alpha_{B_1}\alpha_{A_1}\delta_{M_1}\delta_{A_1} + 16\alpha_{B_1}\beta_{B_1}\gamma_{B_1}\delta_{M_1} - 16\sqrt{2}\alpha_{B_1}\beta_{M_1}\gamma_{A_1}\delta_{A_1} \\
&+ 3\alpha_{B_1}\beta_{M_1}\gamma_{A_1}\delta_{A_1} + 2\pi\sqrt{2}\alpha_{B_1}\beta_{M_1}\delta_{M_1}^2 - 2\sqrt{2}\pi\alpha_{B_1}\beta_{A_1}\delta_{M_1}\delta_{A_1} + 16\alpha_{B_1}\gamma_{B_2}\delta_{M_1} \\
&- 16\sqrt{2}\alpha_{B_1}\gamma_{M_2}\delta_{M_1} + 32\alpha_{B_1}\gamma_{M_2}\delta_{M_1} + 2\pi\sqrt{2}\alpha_{B_1}\delta_{M_1}\delta_{M_2} - 2\sqrt{2}\pi\alpha_{B_1}\delta_{M_1}\delta_{A_2} \\
&+ 16\alpha_{B_2}\gamma_{B_1}\delta_{M_1} - 16\sqrt{2}\alpha_{B_2}\gamma_{A_1}\delta_{A_1} + 32\alpha_{B_2}\gamma_{A_1}\delta_{A_1} + 2\pi\sqrt{2}\alpha_{B_2}\delta_{M_1}^2 \\
&- 2\sqrt{2}\pi\alpha_{B_2}\delta_{M_1}\delta_{A_1} - 4\pi\alpha_{A_1}^2\gamma_{A_1}\delta_{M_1} - 8\sqrt{2}\pi\alpha_{A_1}^2\gamma_{A_1}\delta_{A_1} + 12\pi\alpha_{A_1}^2\gamma_{A_1}\delta_{A_1} \\
&+ \pi^2\alpha_{A_1}^2\delta_{M_1}^2 - 8\sqrt{2}\alpha_{A_1}^2\delta_{M_1}^2 + 16\alpha_{A_1}^2\delta_{M_1}^2 - \pi^2\alpha_{A_1}^2\delta_{M_1}\delta_{A_1} + 8\sqrt{2}\alpha_{A_1}^2\delta_{M_1}\delta_{A_1} \\
&+ 8\sqrt{2}\alpha_{A_1}\beta_{B_1}\gamma_{A_1}\delta_{M_1} - 16\alpha_{A_1}\beta_{B_1}\gamma_{A_1}\delta_{A_1} + 8\sqrt{2}\alpha_{A_1}\beta_{B_1}\gamma_{A_1}\delta_{A_1} - 2\pi\alpha_{A_1}\beta_{B_1}\delta_{M_1}^2 \\
&+ 2\pi\alpha_{A_1}\beta_{B_1}\delta_{M_1}\delta_{A_1} - 8\sqrt{2}\alpha_{A_1}\beta_{M_1}\gamma_{A_1}\delta_{A_1} + 16\alpha_{A_1}\beta_{M_1}\gamma_{A_1}\delta_{A_1} + 2\pi\alpha_{A_1}\beta_{M_1}\delta_{M_1}^2 \\
&- 16\sqrt{2}\alpha_{A_1}\beta_{A_1}\gamma_{A_1}\delta_{M_1} - 8\sqrt{2}\alpha_{A_1}\beta_{A_1}\gamma_{A_1}\delta_{A_1} + 16\alpha_{A_1}\beta_{A_1}\gamma_{A_1}\delta_{A_1} + 2\pi\alpha_{A_1}\beta_{A_1}\delta_{M_1}^2 \\
&- 4\pi\alpha_{A_1}\beta_{A_1}\delta_{M_1}\delta_{A_1} - 8\sqrt{2}\alpha_{A_1}\gamma_{M_2}\delta_{M_1} + 16\alpha_{A_1}\gamma_{M_2}\delta_{M_1} - 8\sqrt{2}\alpha_{A_1}\gamma_{A_2}\delta_{M_1} \\
&+ 2\pi\alpha_{A_1}\delta_{M_1}\delta_{M_2} - 2\pi\alpha_{A_1}\delta_{M_1}\delta_{A_2} - 8\sqrt{2}\alpha_{A_2}\gamma_{A_1}\delta_{M_1} - 8\sqrt{2}\alpha_{A_2}\gamma_{A_1}\delta_{A_1} \\
&+ 16\alpha_{A_2}\gamma_{A_1}\delta_{A_1} + 2\pi\alpha_{A_2}\delta_{M_1}^2 - 2\pi\alpha_{A_2}\delta_{M_1}\delta_{A_1} - 4\sqrt{2}\beta_{B_1}\beta_{M_1}\delta_{M_1}^2 + 8\beta_{B_1}\beta_{M_1}\delta_{M_1}^2 \\
&+ 4\sqrt{2}\beta_{B_1}\beta_{A_1}\delta_{M_1}\delta_{A_1} - 4\sqrt{2}\beta_{B_1}\delta_{M_1}\delta_{M_2} + 8\beta_{B_1}\delta_{M_1}\delta_{M_2} + 4\sqrt{2}\beta_{B_1}\delta_{M_1}\delta_{A_2} \\
&- 4\sqrt{2}\beta_{B_2}\delta_{M_1}^2 + 8\beta_{B_2}\delta_{M_1}^2 + 4\sqrt{2}\beta_{B_2}\delta_{M_1}\delta_{A_1} - 8\beta_{M_1}^2\delta_{M_1}^2 + 4\sqrt{2}\beta_{M_1}^2\delta_{M_1}^2 \\
&- 8\beta_{M_1}\delta_{M_1}\delta_{M_2} + 4\sqrt{2}\beta_{M_1}\delta_{M_1}\delta_{M_2} - 8\beta_{M_2}\delta_{M_1}^2 + 4\sqrt{2}\beta_{M_2}\delta_{M_1}^2 - 4\sqrt{2}\beta_{A_1}^2\delta_{M_1}\delta_{A_1} \\
&- 4\sqrt{2}\beta_{A_1}\delta_{M_1}\delta_{A_2} - 4\sqrt{2}\beta_{A_2}\delta_{M_1}\delta_{A_1} + 8\delta_{B_3}\delta_{M_1} - 8\delta_{M_1}\delta_{M_3} + 4\sqrt{2}\delta_{M_1}\delta_{M_3} \\
&- 4\sqrt{2}\delta_{M1}\delta_{A_3}).
\end{aligned}
$$

Then the polynomial $f_3(r)$ can have at most three positive real roots, and therefore provide when ε is sufficiently small at most three limit cycles for the discontinuous piecewise differential systems

(2.1), (2.2) and (2.3).

In order to apply the averaging theory of fourth order we need that $f_3(r) \equiv 0$. So we must take

$$
\begin{aligned}
\delta_{M_1} =& \delta_{A_1},\ \delta_{A_2} = \frac{1}{8\gamma_{A_1}\delta_{M_1}^2}(-8\pi\alpha_{B_1}\gamma_{B_1}^2\delta_{M_1}^2 + 10\pi\alpha_{B_1}\gamma_{A_1}^2\delta_{A_1}^2 - 16\alpha_{B_1}\gamma_{A_1}^2\delta_{A_1}^2 \\
&+ 8\pi\alpha_{B_1}\gamma_{A_1}\delta_{M_1}^2\delta_{A_1} - 2\pi\alpha_{B_1}\delta_{M_1}^4 + 16\alpha_{B_1}\delta_{M_1}^4 - 3\pi\alpha_{A_1}\gamma_{A_1}^2\delta_{M_1}^2 \\
&- 8\alpha_{A_1}\gamma_{A_1}^2\delta_{M_1}^2 + 5\pi\alpha_{A_1}\gamma_{A_1}^2\delta_{A_1}^2 - 8\alpha_{A_1}\gamma_{A_1}^2\delta_{A_1}^2 + 8\pi\alpha_{A_1}\gamma_{A_1}\delta_{M_1}^2\delta_{A_1} \\
&- \pi\alpha_{A_1}\delta_{M_1}^4 + 8\alpha_{A_1}\delta_{M_1}^4 - \pi\alpha_{A_1}\delta_{M_1}^2\delta_{A_1}^2 + 8\alpha_{A_1}\delta_{M_1}^2\delta_{A_1}^2 + 16\beta_{M_1}\gamma_{A_1}\delta_{M_1}^2\delta_{A_1} \\
&- 16\beta_{A_1}\gamma_{A_1}\delta_{M_1}^2\delta_{A_1} + 8\gamma_{M_2}\delta_{M_1}^3 + 8\gamma_{A_1}\delta_{M_1}\delta_{M_2}\delta_{A_1} - 8\gamma_{A_2}\delta_{M_1}^2\delta_{A_1}), \\
\alpha_{l_2} =& \frac{1}{32(\pi\alpha_{B_1} + \pi\alpha_{A_1} + \beta_{B_1} - \beta_{M1})}(-12\pi^2\alpha_{B_1}^3 - 42\pi^2\alpha_{B_1}^2\alpha_{A_1} - 32\pi\alpha_{B_1}^2\beta_{B_1} \\
&+ 16\pi\alpha_{B_1}^2\beta_{M_1} - 45\pi^2\alpha_{B_1}\alpha_{A_1}^2 - 32\pi\alpha_{B_1}\alpha_{A_1}\beta_{B_1} + 32\pi\alpha_{B_1}\alpha_{A_1}\beta_{M_1} \\
&- 32\pi\alpha_{B_1}\alpha_{A_1}\beta_{A_1} - 32\pi\alpha_{B_1}\alpha_{A_2} - 32\alpha_{B_1}\beta_{B_1}^2 + 32\alpha_{B_1}\beta_{B_1}\beta_{M_1} - 32\alpha_{B_1}\beta_{B_2} \\
&+ 32\alpha_{B_1}\beta_{M_2} - 32\alpha_{B_3} - 16\alpha_{M_3} - 15\pi^2\alpha_{A_1}^3 + 16\pi\alpha_{A_1}^2\beta_{M_1} - 32\pi\alpha_{A_1}^2\beta_{A_1} \\
&- 32\pi\alpha_{A_1}\alpha_{A_2} + 16\alpha_{A_1}\beta_{M_1}\beta_{A_1} + 16\alpha_{A_1}\beta_{M_2} - 16\alpha_{A_1}\beta_{A_1}^2 - 16\alpha_{A_1}\beta_{A_2} \\
&+ 16\alpha_{A_2}\beta_{M_1} - 16\alpha_{A_2}\beta_{A_1} - 16\alpha_{A3}),
\end{aligned}
$$

$$
\begin{aligned}
\delta_{M_3} =& \frac{1}{4(\sqrt{2}-2)\delta_{M_1}}(-16\pi\alpha_{B_1}^2\gamma_{A_1}\delta_{A_1} + 16\pi\sqrt{2}\alpha_{B_1}^2\gamma_{A_1}\delta_{A_1} - 48\alpha_{B_1}^2\delta_{M_1}^2 + 24\sqrt{2}\alpha_{B_1}^2\delta_{M_1}^2 \\
&+ 8\sqrt{2}\alpha_{B_1}^2\delta_{M_1}\delta_{A_1} + 4\pi\sqrt{2}\alpha_{B_1}\alpha_{A_1}\gamma_{A_1}\delta_{M_1} - 24\pi\alpha_{B_1}\alpha_{A_1}\gamma_{A_1}\delta_{A_1} \\
&+ 20\pi\sqrt{2}\alpha_{B_1}\alpha_{A_1}\gamma_{A_1}\delta_{A_1} - \pi^2\alpha_{B_1}\alpha_{A_1}\delta_{M_1}^2 - 64\alpha_{B_1}\alpha_{A_1}\delta_{M_1}^2 \\
&+ 32\sqrt{2}\alpha_{B_1}\alpha_{A_1}\delta_{M_1}^2 + \pi^2\alpha_{B_1}\alpha_{A_1}\delta_{M_1}\delta_{A_1} - 16\alpha_{B_1}\beta_{B_1}\gamma_{B_1}\delta_{M_1} - 32\alpha_{B_1}\beta_{M_1}\gamma_{A_1}\delta_{A_1} \\
&+ 16\sqrt{2}\alpha_{B_1}\beta_{M_1}\gamma_{A_1}\delta_{A_1} - 2\sqrt{2}\pi\alpha_{B_1}\beta_{M_1}\delta_{M_1}^2 + 2\pi\sqrt{2}\alpha_{B_1}\beta_{A_1}\delta_{M_1}\delta_{A_1} - 16\alpha_{B_1}\gamma_{B_2}\delta_{M_1} \\
&- 32\alpha_{B_1}\gamma_{M_2}\delta_{M_1} + 16\sqrt{2}\alpha_{B_1}\gamma_{M_2}\delta_{M_1} - 2\sqrt{2}\pi\alpha_{B_1}\delta_{M_1}\delta_{M_2} + 2\pi\sqrt{2}\alpha_{B_1}\delta_{M_1}\delta_{A_2} \\
&- 16\alpha_{B_2}\gamma_{B_1}\delta_{M_1} - 32\alpha_{B_2}\gamma_{A_1}\delta_{A_1} + 16\sqrt{2}\alpha_{B_2}\gamma_{A_1}\delta_{A_1} - 2\sqrt{2}\pi\alpha_{B_2}\delta_{M_1}^2 \\
&+ 2\pi\sqrt{2}\alpha_{B_2}\delta_{M_1}\delta_{A_1} + 4\pi\alpha_{A_1}^2\gamma_{A_1}\delta_{M_1} - 12\pi\alpha_{A_1}^2\gamma_{A_1}\delta_{A_1} + 8\pi\sqrt{2}\alpha_{A_1}^2\gamma_{A_1}\delta_{A_1} \\
&- \pi^2\alpha_{A_1}^2\delta_{M_1}^2 - 16\alpha_{A_1}^2\delta_{M_1}^2 + 8\sqrt{2}\alpha_{A_1}^2\delta_{M_1}^2 + \pi^2\alpha_{A_1}^2\delta_{M_1}\delta_{A_1} - 8\sqrt{2}\alpha_{A_1}^2\delta_{M_1}\delta_{A_1} \\
&- 8\sqrt{2}\alpha_{A_1}\beta_{B_1}\gamma_{A_1}\delta_{M_1} - 8\sqrt{2}\alpha_{A_1}\beta_{B_1}\gamma_{A_1}\delta_{A_1} + 16\alpha_{A_1}\beta_{B_1}\gamma_{A_1}\delta_{A_1} + 2\pi\alpha_{A_1}\beta_{B_1}\delta_{M_1}^2 \\
&- 2\pi\alpha_{A_1}\beta_{B_1}\delta_{M_1}\delta_{A_1} - 16\alpha_{A_1}\beta_{M_1}\gamma_{A_1}\delta_{A_1} + 8\sqrt{2}\alpha_{A_1}\beta_{M_1}\gamma_{A_1}\delta_{A_1} \\
&- 2\pi\alpha_{A_1}\beta_{M_1}\delta_{M_1}^2 + 16\sqrt{2}\alpha_{A_1}\beta_{A_1}\gamma_{A_1}\delta_{M_1} - 16\alpha_{A_1}\beta_{A_1}\gamma_{A_1}\delta_{A_1} + 8\sqrt{2}\alpha_{A_1}\beta_{A_1}\gamma_{A_1}\delta_{A_1} \\
&- 2\pi\alpha_{A_1}\beta_{A_1}\delta_{M_1}^2 + 4\pi\alpha_{A_1}\beta_{A_1}\delta_{M_1}\delta_{A_1} - 16\alpha_{A_1}\gamma_{M_2}\delta_{M_1} + 8\sqrt{2}\alpha_{A_1}\gamma_{M_2}\delta_{M_1} \\
&+ 8\sqrt{2}\alpha_{A_1}\gamma_{A_2}\delta_{M_1} - 2\pi\alpha_{A_1}\delta_{M_1}\delta_{M_2} + 2\pi\alpha_{A_1}\delta_{M_1}\delta_{A_2} + 8\sqrt{2}\alpha_{A_2}\gamma_{A_1}\delta_{M_1} \\
&- 16\alpha_{A_2}\gamma_{A_1}\delta_{A_1} + 8\sqrt{2}\alpha_{A_2}\gamma_{A_1}\delta_{A_1} - 2\pi\alpha_{A_2}\delta_{M_1}^2 + 2\pi\alpha_{A_2}\delta_{M_1}\delta_{A_1} - 8\beta_{B_1}\beta_{M_1}\delta_{M_1}^2 \\
&+ 4\sqrt{2}\beta_{B_1}\beta_{M_1}\delta_{M_1}^2 - 4\sqrt{2}\beta_{B_1}\beta_{A_1}\delta_{M_1}\delta_{A_1} - 8\beta_{B_1}\delta_{M_1}\delta_{M_2} + 4\sqrt{2}\beta_{B_1}\delta_{M_1}\delta_{M_2} \\
&- 4\sqrt{2}\beta_{B_1}\delta_{M_1}\delta_{A_2} - 8\beta_{B_2}\delta_{M_1}^2 + 4\sqrt{2}\beta_{B_2}\delta_{M_1}^2 - 4\sqrt{2}\beta_{B_2}\delta_{M_1}\delta_{A_1} - 4\sqrt{2}\beta_{M_1}^2\delta_{M_1}^2 \\
&+ 8\beta_{M_1}^2\delta_{M_1}^2 - 4\sqrt{2}\beta_{M_1}\delta_{M_1}\delta_{M_2} + 8\beta_{M_1}\delta_{M_1}\delta_{M_2} - 4\sqrt{2}\beta_{M_2}\delta_{M_1}^2 + 8\beta_{M_2}\delta_{M_1}^2 \\
&+ 4\sqrt{2}\beta_{A_1}^2\delta_{M_1}\delta_{A_1} + 4\sqrt{2}\beta_{A_1}\delta_{M_1}\delta_{A_2} + 4\sqrt{2}\beta_{A_2}\delta_{M_1}\delta_{A_1} - 8\delta_{B_3}\delta_{M_1} + 4\sqrt{2}\delta_{M_1}\delta_{A_3}).
\end{aligned}
$$

Unluckily, the explicit expressions of $f_4^j,\ j \in \Omega$ are extremely large, for that we omit them.

However, as we can see in Mathematics, these polynomials can only have five roots. As a result, for the discontinuous piecewise differential systems (2.1), (2.2) and (2.3), the averaging theory up to 4 can only give five limit cycles.

Chapitre 3

Piecewise differential system in the plane with four zones

In this chapter, we will focus on a specific type of piecewise differential system in the plane by dividing the plane into four quadrants, each containing a linear Hamiltonian system, to determine the maximum number of limit cycles in these systems.

We consider a piecewise differential system in the plane formed by four pieces separated by the axes of coordinates having in each quadrant a linear Hamiltonian systems

$$\begin{cases} \dot{x} = -a_2 - a_4 x - 2a_5 y, \\ \dot{y} = a_1 + 2a_3 x + a_4 y, \end{cases} \quad \Omega_1 = \{(x;y) \in \mathbb{R}^2 / \ x > 0, \ \ y > 0\},$$

$$\begin{cases} \dot{x} = -b_2 - b_4 x - 2b_5 y, \\ \dot{y} = b_1 + 2b_3 x + b_4 y, \end{cases} \quad \Omega_2 = \{(x;y) \in \mathbb{R}^2 / \ x < 0, \ \ y > 0\},$$

$$\begin{cases} \dot{x} = -c_2 - c_4 x - 2c_5 y, \\ \dot{y} = c_1 + 2c_3 x + c_4 y, \end{cases} \quad \Omega_3 = \{(x;y) \in \mathbb{R}^2 / \ x < 0, \ \ y < 0\},$$

$$\begin{cases} \dot{x} = -d_2 - d_4 x - 2d_5 y, \\ \dot{y} = d_1 + 2d_3 x + d_4 y, \end{cases} \quad \Omega_4 = \{(x;y) \in \mathbb{R}^2 / \ x > 0, \ \ y < 0\},$$

such that the piecewise differential system is continuous on the x- axis and discontinuous on the y - axis . The Hamiltonian in the nth-quadrant for $n = 1, 2, 3, 4$ of the previous Hamiltonian systems are

$$\begin{aligned} H_1 &= a_1 x + a_2 y + a_3 x^2 + a_4 xy + a5 y^2, \\ H_2 &= b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2, \\ H_3 &= c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2, \\ H_4 &= d_1 x + d_2 y + d_3 x^2 + d_4 xy + d_5 y^2, \end{aligned}$$

respectively.

We impose that on the x - axis the piecewise differential system be continuous for that we must have

$$\dot{x}_{\Omega_1} - \dot{x}_{\Omega_4} = 0,$$
$$\dot{y}_{\Omega_1} - \dot{y}_{\Omega_4} = 0,$$
$$\dot{x}_{\Omega_2} - \dot{x}_{\Omega_3} = 0,$$
$$\dot{y}_{\Omega_2} - \dot{y}_{\Omega_3} = 0,$$

which give

$$-a_2 - a_4x + d_2 + d_4x = 0,$$
$$a_1 + 2a_3x - d_1 - 2d_3x = 0,$$
$$-b_2 - b_4x + c_2 + c_4x = 0,$$
$$b_1 + 2b_3x - c_1 - 2c_3x = 0,$$

to verify these equations we must take

$$c_1 = b_1,\ c_2 = b_2,\ c_3 = b_3,\ c_4 = b_4,\ d_1 = a_1,\ d_2 = a_2,\ d_3 = a_3,\ d_4 = a_4.$$

So the continuous - discontinuous piecewise differential system is formed by the four linear Hamiltonian systems

$$\dot{x} = -a_2 - a_4x - 2a_5y,\ \ \dot{y} = a_1 + 2a_3x + a_4y,$$

$$\dot{x} = -b_2 - b_4x - 2b_5y,\ \ \dot{y} = b_1 + 2b_3x + b_4y,$$

$$\dot{x} = -b_2 - b_4x - 2c_5y,\ \ \dot{y} = b_1 + 2b_3x + b_4y, \text{ and}$$

$$\dot{x} = -a_2 - a_4x - 2d_5y,\ \ \dot{y} = a_1 + 2a_3x + a_4y.$$

The Hamiltonians of these last Hamiltonian systems are

$$\begin{aligned}
H_1 =& a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2,\\
H_2 =& b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2,\\
H_3 =& b_1x + b_2y + b_3x^2 + b_4xy + c_5y^2,\\
H_4 =& a_1x + a_2y + a_3x^2 + a_4xy + d_5y^2.
\end{aligned}$$

Now we study the limit cycles of these continuous - discontinuous piecewise differential systems which intersect the positive and negative x and y axes in points of the form $(\alpha, 0)$, $(0, \beta)$, $(\delta, 0)$ and $(0, \gamma)$ with $\alpha, \beta > 0$ and $\delta, \gamma < 0$. If there exists a such limit cycle these four points must satisfy the following equations

$$\begin{cases}
H_1(\alpha, 0) - H_1(0, \beta) = 0,\\
H_2(0, \beta) - H_2(\delta, 0) = 0,\\
H_3(\delta, 0) - H_3(0, \gamma) = 0,\\
H_4(0, \gamma) - H_4(\alpha, 0) = 0
\end{cases}$$

From where

$$\begin{cases}
\alpha a_1 - a_2\beta + \alpha^2 a_3 - a_5\beta^2 = 0,\\
-b_1\delta + \beta b_2 - b_3\delta^2 + \beta^2 b_5 = 0,\\
b_1\delta - b_2\gamma + b_3\delta^2 - \gamma^2 c_5 = 0,\\
\alpha a_1 + a_2\gamma - \alpha^2 a_3 + \gamma^2 d_5 = 0.
\end{cases}$$

By solving this algebraic system we get the following sets of solutions

$$S_1 = \{a_5 = \frac{\alpha a_1 - a_2\beta + \alpha^2 a_3}{\beta^2},\ b_5 = \frac{b_1\delta - \beta b_2 + b_3\delta^2}{\beta^2},\ c_5 = \frac{b_1\delta - b_2\gamma + b_3\delta^2}{\gamma^2},$$
$$d_5 = \frac{\alpha a_1 - a_2\gamma + \alpha^2 a_3}{\gamma^2}\},$$
$$S_2 = \{a_1 = -\alpha a_3,\ b_1 = -b_3\delta\ , \beta = 0,\ \gamma = 0\},$$
$$S_3 = \{b_1 = -b_3\delta,\ \alpha = 0,\ \beta = 0,\ \gamma = 0\},$$
$$S_4 = \{a_1 = -\alpha a_3,\ \beta = 0,\ \gamma = 0, \delta = 0\},$$
$$S_5 = \{\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = 0\},$$
$$S_6 = \{a_5 = \frac{\alpha a_1 - a_2\beta + \alpha^2 a_3}{\beta^2},\ b_2 = \frac{\delta(\beta+\gamma)(b_1 + b_3\delta)}{\beta\gamma},\ b_5 = \frac{-b_1\delta - b_3\delta^2}{\beta\gamma},$$
$$c_5 = \frac{-b_1\delta - b_3\delta^2}{\beta\gamma}, d_5 = \frac{\alpha a_1 - a_2\gamma + \alpha^2 a_3}{\gamma^2}\},$$
$$S_7 = \{a_1 = -\alpha a_3,\ b_1 = -b_3\delta,\ c_5 = -\frac{b_2}{\gamma}, d_5 = -\frac{a_2}{\gamma},\ \beta = 0\},$$
$$S_8 = \{b1 = -b_3\delta,\ c_5 = -\frac{b_2}{\gamma},\ d_5 = -\frac{a_2}{\gamma},\ \alpha = 0,\ \beta = 0\},$$
$$S_9 = \{a_1 = -\alpha a_3,\ c_5 = -\frac{b_2}{\gamma},\ d_5 = -\frac{a_2}{\gamma},\ \beta = 0,\ \delta = 0\},$$
$$S_{10} = \{c_5 = -\frac{b_2}{\gamma},\ d_5 = -\frac{a_2}{\gamma},\ \alpha = 0,\ \beta = 0, \delta = 0\},$$
$$S_{11} = \{a_1 = -\alpha a_3,\ a_5 = -\frac{a_2}{\beta},\ b_1 = -b_3\delta,\ b_5 = -\frac{b_2}{\beta},\ \gamma = 0\},$$
$$S_{12} = \{a_5 = -\frac{a_2}{\beta},\ b_1 = -b_3\delta,\ b_5 = -\frac{b_2}{\beta},\ \alpha = 0,\ \gamma = 0\},$$
$$S_{13} = \{a_1 = -\alpha a_3,\ a_5 = -\frac{a_2}{\beta},\ b_5 = -\frac{b_2}{\beta},\ \gamma = 0,\ \delta = 0\},$$
$$S_{14} = \{a_5 = -\frac{a_2}{\beta},\ b_5 = -\frac{b_2}{\beta},\ \alpha = 0,\ \gamma = 0, \delta = 0\},$$
$$S_{15} = \{a_1 = -\alpha a_3,\ b_1 = -b_3\delta,\ b_2 = 0,\ c_5 = 0,\ d_5 = -\frac{a_2}{\gamma},\ \beta = 0\},$$
$$S_{16} = \left\{b_1 = -b_3\delta, b_2 = 0, c_5 = 0, d_5 = -\frac{a_2}{\gamma}, \alpha = 0, \beta = 0\right\},$$
$$S_{17} = \{a_1 = -\alpha a_3,\ b_2 = 0,\ c_5 = 0,\ d_5 = -\frac{a_2}{\gamma},\ \beta = 0,\ \delta = 0\},$$
$$S_{18} = \{b_2 = 0,\ c_5 = 0,\ d_5 = -\frac{a_2}{\gamma},\ \alpha = 0,\ \beta = 0,\ \delta = 0\},$$
$$S_{19} = \{a_5 = \frac{\alpha a_1 + a_2\gamma + \alpha^2 a_3}{\gamma^2},\ b_2 = 0,\ b_5 = \frac{\delta(b_1 + b_3\delta)}{\gamma^2},\ c_5 = \frac{\delta(b_1 + b_3\delta)}{\gamma^2},$$
$$d_5 = \frac{\alpha a_1 - a_2\gamma + \alpha^2 a_3}{\gamma^2},\ \beta = -\gamma\},$$
$$S_{20} = \{a_1 = -\alpha a_3,\ a_2 = 0,\ a_5 = 0,\ b_1 = -b_3\delta,\ b_5 = -\frac{b_2}{\beta},\ \gamma = 0\},$$

$$S_{21} =\{a_2 = 0,\ a_5 = 0,\ b_1 = -b_3\delta,\ b_5 = -\frac{b_2}{\beta},\ \alpha = 0,\ \gamma = 0\},$$

$$S_{22} =\{a_1 = -\alpha a_3,\ a_2 = 0,\ a_5 = 0,\ b_5 = -\frac{b_2}{\beta},\ \gamma = 0,\ \delta = 0\},$$

$$S_{23} =\{a_2 = 0,\ a_5 = 0,\ b_5 = -\frac{b_2}{\beta},\ \alpha = 0,\ \gamma = 0,\ \delta = 0\},$$

$$S_{24} =\{a_1 = -\alpha a_3,\ a_5 = \frac{a_2}{\gamma},\ b_2 = 0,\ b_5 = \frac{\delta(b_1 + b_3\delta)}{\gamma^2},\ c_5 = \frac{\delta(b_1 + b_3\delta)}{\gamma^2},\ d_5 = -\frac{a_2}{\gamma},\ \beta = -\gamma\},$$

$$S_{25} =\{a_5 = \frac{a_2}{\gamma},\ b_2 = 0,\ b_5 = \frac{\delta(b_1 + b_3\delta)}{\gamma^2},\ c_5 = \frac{\delta(b_1 + b_3\delta)}{\gamma^2},\ d_5 = -\frac{a_2}{\gamma},\ \alpha = 0,\ \beta = -\gamma\}.$$

From the previous sets of solutions the only which satisfy that $\alpha, \beta > 0$ and $\delta, \gamma < 0$ are the following four sets $S_1,\ S_6,\ S_{19},\ S_{24}$. From these sets of solutions a piecewise system with at most two limit cycles is possible. However, we have not found any examples of the case when we have two limit cycles. Here is an illustration of a piecewise system with one limit cycle.

Example 5. *The continuous - discontinuous piecewise differential systems formed by the Hamiltonian systems associated to the following for Hamiltonians have a unique limit cycle that intersects the axes at the points* $(1, 0), (0, 2), (-2, 0)$ *and* $(0, -3)$.

We take $a_1 = 12,\ a_2 = -2,\ a_3 = 11,\ a_4 = 0,\ a_5 = \frac{27}{4},\ b_1 = -2,\ b_2 = 11,\ b_3 = 5,\ b_4 = -3,\ b_5 = \frac{1}{2},\ c_5 = \frac{19}{3}$, and $d_5 = \frac{17}{9}$. Which give the continuous - discontinuous piecewise differential system formed by the four linear Hamiltonian systems

$$f_1(x, y) = \begin{cases} \dot{x} = 2 - \frac{27y}{2}, \\ \dot{y} = 22x + 12, \end{cases} \quad \Omega_1 = \{(x; y) \in \mathbb{R}^2 / \ x > 0,\ \ y > 0\},$$

$$f_2(x, y) = \begin{cases} \dot{x} = 3x - y - 11, \\ \dot{y} = 10x - 3y - 2, \end{cases} \quad \Omega_2 = \{(x; y) \in \mathbb{R}^2 / \ x < 0,\ \ y > 0\},$$

$$\sum\nolimits_{12} = \{X \in \mathbb{R}^2 : L(X) = 0\} = \{X \in \mathbb{R}^2 : x = 0\};$$

$$f_3(x, y) = \begin{cases} \dot{x} = 3x - \frac{38y}{3} - 11, \\ \dot{y} = 10x - 3y - 2, \end{cases} \quad \Omega_3 = \{(x; y) \in \mathbb{R}^2 / \ x < 0,\ \ y < 0\},$$

$$f_4(x, y) = \begin{cases} \dot{x} = 2 - \frac{34y}{9}, \\ \dot{y} = 22x + 12, \end{cases} \quad \Omega_4 = \{(x; y) \in \mathbb{R}^2 / \ x > 0,\ \ y < 0\},$$

$$\sum\nolimits_{34} = \{X \in \mathbb{R}^2 : L(X) = 0\} = \{X \in \mathbb{R}^2 : x = 0\};$$

we have

$$L_X(X) = (\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y})^T = (1\ \ 0)^T$$

The crossing set

$$\sum{}_c = \{X \in \sum{}_{12} : \langle L_X(X), f_1(X)\rangle \langle L_X(X), f_2(X)\rangle > 0\},$$

$$\langle L_X(X), f_1(X)\rangle = (1\ \ 0)\begin{pmatrix} 2 - \frac{27y}{2} \\ 12 \end{pmatrix} = (2 - \tfrac{27y}{2}),$$

$$\langle L_X(X), f_2(X)\rangle = (1\ \ 0)\begin{pmatrix} -y - 11 \\ -3y - 2 \end{pmatrix} = (-y - 11),$$

from it

$$\sum{}_c = \{X \in \sum{}_{12} : (2 - \frac{27y}{2})(-y - 11) > 0\},$$

$$\sum{}_c = \{X \in \sum{}_{12} : y \in]-\infty, -11[\bigcup]\frac{4}{24}, +\infty[\ \},$$

the point $(0, 2) \in \sum_c$

The crossing set

$$\sum{}_c = \{X \in \sum{}_{34} : \langle L_X(X), f_3(X)\rangle \langle L_X(X), f_4(X)\rangle > 0\},$$

$$\langle L_X(X), f_3(X)\rangle = (1\ \ 0)\begin{pmatrix} -\frac{38y}{3} - 11 \\ -3y - 2 \end{pmatrix} = (-\tfrac{38y}{3} - 11),$$

$$\langle L_X(X), f_4(X)\rangle = (1\ \ 0)\begin{pmatrix} 2 - \frac{34y}{9} \\ 12 \end{pmatrix} = (2 - \tfrac{34y}{9})$$

from it

$$\sum{}_c = \{X \in \sum{}_{34} : (-\frac{38y}{3} - 11)(2 - \frac{34y}{9}) > 0\},$$

$$\sum{}_c = \{X \in \sum{}_{12} : y \in]-\infty, -\frac{33}{31}[\bigcup]\frac{18}{34}, +\infty[\ \},$$

the point $(0, -3) \in \sum_c$

The Hamiltonians of these last Hamiltonian systems are

$$\begin{aligned} H_1 =& 11x^2 + 12x + \frac{27y^2}{4} - 2y, \\ H_2 =& 5x^2 - 3xy - 2x + \frac{y^2}{2} + 11y, \\ H_3 =& 5x^2 - 3xy - 2x + \frac{19y^2}{3} + 11y, \\ H_4 =& 11x^2 + 12x + \frac{17y^2}{9} - 2y. \end{aligned}$$

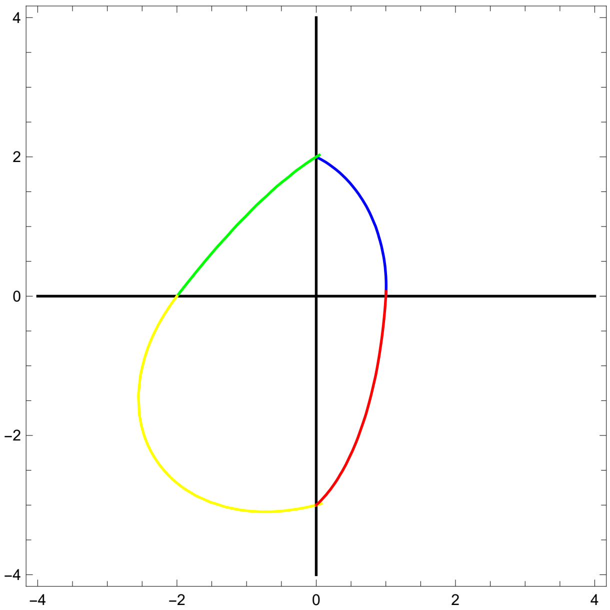


FIGURE 3.1 – continuous-discontinuous piecewise differential systems with one limit cycle.

Conclusion

The thesis is a study of a special type of piecewise differential systems. Which start by introduce an essential mathematical concepts for differential systems, including periodic orbits, averaging theory, first integrals, Hamiltonian systems, and Filippov systems. Second, we investigate the limit cycles that arise from perturbations of the periodic orbits of the linear differential center. Finally, we have focused on a specific type of piecewise differential system in the plane by dividing the plane into four quadrants, each of which contains a linear Hamiltonian system. Our investigation includes determining the maximum number of limit cycles that can occur in these systems.

Bibliographie

[1] A. A. Andronov, A. A. Vitt, and S. E. Khaikin. *Theory of Oscillators : International Series of Monographs in Physics*, volume 4. Oxford, 1966.

[2] S. Banerjee and G. C. Verghese. *Nonlinear phenomena in power electronics*. IEEE, 1999.

[3] L. Benadero García-Morato, E. Ponce, A. El Aroudi, F. Torres Peral, et al. Limit cycle bifurcations in resonant lc power inverters under zero current switching strategy. 2017.

[4] C. Casacuberta. Advanced courses in mathematics crm barcelona.

[5] F. Dumortier, J. Llibre, and J. C. Artés. *Qualitative theory of planar differential systems*. Springer, 2006.

[6] A. Filippov. *Differential Equation with Discontinuous Right-Hand Sides*. Kluwer Academic, Amsterdam, 1988.

[7] A. Gasull and J. Torregrosa. Small-amplitude limit cycles in liénard systems via multiplicity. *Journal of differential equations*, 159(1) :186–211, 1999.

[8] J. Itikawa, J. Llibre, and D. D. Novaes. A new result on averaging theory for a class of discontinuous planar differential systems with applications. *Rev. Mat. Iberoam.*, 33(4) :1247–1265, 2017.

[9] Y. A. Kuznetsov. Lecture 1. filippov systems : Sliding solutions and bifurcations. 2010.

[10] R. Leine and D. Van Campen. Bifurcation phenomena in non-smooth dynamical systems. *European Journal of Mechanics-A/Solids*, 25(4) :595–616, 2006.

[11] R. I. Leine and H. Nijmeijer. *Dynamics and bifurcations of non-smooth mechanical systems*, volume 18. Springer Science & Business Media, 2013.

[12] J. Llibre, R. Moeckel, C. Simó, and R. Moeckel. *Central Configurations*. Springer, 2015.

[13] J. Llibre and T. Salhi. On the limit cycles of the piecewise differential systems formed by a linear focus or center and a quadratic weak focus or center. *Chaos, Solitons & Fractals*, 160 :112256, 2022.

[14] J. Llibre and A. E. Teruel. *Introduction to the qualitative theory of differential systems*. Springer, 2014.

[15] L. Perko. *Differential equations and dynamical systems*, volume 7. Springer Science & Business Media, 2013.

[16] F. C. Schwarz. An improved method of resonant current pulse modulation for power converters. *IEEE Transactions on Industrial Electronics and Control Instrumentation*, (2) :133–141, 1976.

[17] H. Suryawanshi and S. Tarnekar. Resonant converter in high power factor, high voltage dc applications. *IEE Proceedings-Electric Power Applications*, 145(4) :307–314, 1998.

[18] Z. T. Zhusubaliyev and E. Mosekilde. *Bifurcations and chaos in piecewise-smooth dynamical systems*, volume 44. World Scientific, 2003.