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Limit cycles of discontinuous piecewise differential systems separated by a non–regular line and formed by an arbitrary linear center and an arbitrary quadratic center

By: Louiza Baymout

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In front of the jury composed of:

Dr. Rebiha Zeghdane	University of B.B.A	**President**
Pr. Rebiha Bernterki	University of B.B.A	**Supervisor**
Pr. Jaume Llibre	University of U.A.B	**Co-Supervisor**
Pr. Rachid Boukoucha	University of Bejaia	**Examiner**
Dr. Aziza Berbache	University of B.B.A	**Examiner**
Dr. Abdelkrim Kina	University of Ghardaia	**Examiner**
Dr. Bilal Ghermoul	University of B.B.A	**Examiner**

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List of publications

1. *Louiza Baymout, Rebiha Benterki and Jaume Llibre*: Limit cycles of the discontinuous piecewise differential systems separated by a non-regular line and formed by a linear center and a quadratic one. **International Journal of Bifurcation and Chaos**, **34** (5) (2024) 42 pp. Reference [10].

2. *Louiza Baymout and Rebiha Benterki*: Limit cycles of piecewise differential systems formed by linear center or focus and cubic uniform isochronous center. **Memoirs on Differential Equations and Mathematical Physics**, (2024). Reference [7].

3. *Louiza Baymout and Rebiha Benterki*: Four limit cycles of three-dimensional discontinuous piecewise differential systems having a sphere as switching manifold. **International Journal of Bifurcation and Chaos**, **34** (3) (2024) 13 pp. Reference [8].

4. *Louiza Baymout, Rebiha Benterki and Jaume Llibre*: Limit cycles of some families of discontinuous piecewise differential systems separated by a straight line. **International Journal of Bifurcation and Chaos**, **33** (14) (2023) 2350166. Reference [5].

5. *Louiza Baymout, Rebiha Benterki and Jaume Llibre*: The solution of the extended 16th Hilbert problem for some classes of piecewise differential systems. **Mathematics**, **12** (3) (2024) 464. Reference [6].

6. *Louiza Baymout, Rebiha Benterki and Jaume Llibre*: The limit cycles of a class of discontinuous piecewise differential systems. **International Journal of Dynamical Systems and Differential Equations**, **13** (4) (2023) 464. Reference [9].

7. *Rebiha Benterki, Loubna Damene and Louiza Baymout*: The Solution of the second part of the 16th Hilbert problem for a class of piecewise linear Hamiltonian saddles separated by conics. **Nonlinear Dynamics and Systems Theory**, **22** (3) (2022) 231–242. Reference [13].

Introduction

Over the past two centuries, the understanding and application of mathematics have expanded exponentially, with the rise of mathematical logic and algebra shedding new light on the structure of mathematical systems. Today, mathematics permeates nearly every aspect of science and modern life, providing fundamental tools for modeling, analyzing, and solving complex problems in various fields like computer science, physics, biology, and economics. Differential equations, in particular, have emerged as powerful instruments for modeling natural phenomena, facilitating the understanding of observable phenomena and enabling accurate predictions about the future. Their origin can be traced back to the evolution of infinitesimal calculus in the 17th century by Newton and Leibnitz, demonstrating their enduring importance in shaping our understanding of nature and advancing human knowledge and technology.

Periodic solutions or in particular isolated periodic solutions called *limit cycles* are one of the main remarkable and important solutions of differential equations. The notion of a limit cycle appeared first at the end of the 19th century with Poincaré [50]. Later on Hilbert stated a list of 23 problems for the advancement of mathematical science, and from then it started intensive research on these problems throughout the 20th century. From the 23 problems only the so-called 16th Hilbert's problem and the Riemann conjecture remain open until now. The second part of the 16th Hilbert problem has two parts and asks for an upper bound on the number of possible limit cycles and their positions for planar polynomial differential system of a given degree. We recall that a limit cycle for a differential system is an isolated periodic orbit in the set of all periodic orbits of this differential system.

Now moving from traditional differential equations to piecewise differential systems marks a significant transition in modeling dynamic systems. While differential equations provide a powerful framework for describing continuous behaviors, many real-world phenomena exhibit discontinuities or sudden changes in behavior. Piecewise differential systems address this limitation by allowing for the incorporation of discontinuities into the system dynamics. This transition enables a more accurate representation of complex systems with nonlinearities, switching behavior, or discontinuities, such as those encountered in control theory, circuit design, and biological systems, see for instance [21, 22, 34, 45]. By embracing discontinuities, piecewise differential systems offer a more comprehensive approach to modeling dynamical systems, bridging the gap between theoretical models and real-world observations. Thus the shift from differential equations to piecewise differential systems represents a crucial step toward enhancing our understanding and prediction of dynamical systems behaviors. In 1920 Andronov *et al.* [1] published their first research on piecewise linear discontinuous differential systems. Many studies on piecewise differential systems come from applications, for instance, control theory [35, 48] and electric circuit design [33, 46].

In our thesis we are interested in planar discontinuous piecewise differential systems (or simply PWS) of the form

$$(\dot{x},\dot{y})^T = \begin{cases} \mathcal{X}_1(x,y) = (X_1(x,y),\ Y_1(x,y))^T, & if \ \ (x,y) \in \Sigma_1^k; \\ \mathcal{X}_2(x,y) = (X_2(x,y),\ Y_2(x,y))^T, & if \ \ (x,y) \in \Sigma_2^k; \end{cases} \tag{1}$$

where Σ_j^k with $k = r, i$ and $j = 1,2$ represent the two regions Σ_j^k separated by the discontinuity curve Σ^k that can be either the regular line $\Sigma^r = \{(x,y) \in \mathbb{R}^2 :\ x = 0\}$, or the irregular line $\Sigma^i = \Gamma_1 \cup \Gamma_2$ formed by the to branches $\Gamma_1 = \{(x,y) : x = 0 \ \text{ and } \ y \geq 0\}$ or $\Gamma_2 = \{(x,y) : x \geq 0 \ \text{ and } \ y = 0\}$. In each region we consider a planar vecor field $\mathcal{X}_j = (X_j(x,y),\ Y_j(x,y))$ with $j = 1,2$.

For piecewise differential systems we distinguish between two kinds of limit cycles: sliding and crossing. A *sliding limit cycle* is a limit cycle that contains some arc of the curves of discontinuity that separate the different differential systems which form the PWS. The *crossing limit cycle* are the ones that contain only isolated points of the discontinuity curves. Here we focus only on the crossing limit cycles, for a more precise definition (see [49]). Rather than saying "crossing limit cycle", we will refer to it as "limit cycle".

Recently the second part of the 16th Hilbert's problem has become an interesting topic of research for many scientists because of the main role of limit cycles in understanding and explaining the dynamics of many natural phenomena, for example, the Sel'kov model of glycolysis [53], also some non-linear electrical circuits exhibit limit cycle oscillations, which inspired the original Van der Pol model [54, 55], or one of the Belousov Zhavotinskii model [11], etc.

Many publications concerning piecewise linear differential systems go back to the applications. The list of published articles devoted to these systems gives an idea of the main role of these systems in nature especially the easiest linear PWS systems in the plane separated by one straight line, that until now the maximum number of limit cycles is still unknown. The scientists provide only examples with at most three limit cycles, see [25, 29, 30, 37, 38, 40, 41, 42, 43] and the references therein. From here many questions arise, like how does the non-linearity of the systems affect the maximum number of limit cycles that may be created from a such class of PWS?. Regarding the separation curve, it's important to ask: If the separation curve is not a straight line what happens to the number of limit cycles? How do the kinds of the regions created by the separation curve affect the maximum number of limit cycles?

Nowadays many papers consider PWS whith a nonlinear differential system in some pieces. However keep the straight line as the separation curve and study the maximum number of limit cycles of such PWS. In [24] Esteban *et al.* solved the extension of the second part of the 16th Hilbert problem for PWS formed by isochronous polynomial centers of degrees one and two separated by a straight line. Next Benterki and Llibre [14] studied the same problem but for some classes of PWS formed by isochronous polynomial centers of degrees one and three. In [12] Benabdallah *et al.* studied the second part of the 16th Hilbert problem for a class of PWS separated by a straight line and formed by linear and quadratic centers where they proved that the maximum number of limit cycles of this class of systems is at most four. In [17] Buzzi *et al.* study the maximum number of limit cycles of some classes of planar PWS separated by a straight line and formed by combinations of linear centers (consequently isochronous) and cubic isochronous centers with homogeneous nonlinearities.

But if the discontinuity curve is not a straight line in 2013 [15, 16] the authors found more than three limit cycles for linear PWS with two zones, the same result was obtained in 2015 by Novaes *et al.* [47]. At the beginning of 2021 in [56] the authors separated two planar linear differential systems having centers by a irregular line and they proved that such systems have at most two limit cycles. In the same year in [39] Llibre solved the extension of the 16th Hilbert's problem for a family of planar continuous PWS formed by a linear center and a quadratic center where the separation curve is a parabola, he proved that the maximum number of limit cycles for this class of PWS is at most one.

Our contribution in this thesis is to study the limit cycles that can be created from four different non-linear families of PWS formed by five diffrent types of centers, firstly we study the problem of the existence and the maximum number of limit cycles for the PWS (1) separated by the regular line Σ^r and formed by either linear center or one of three types of cubic differential centers and we build examples that provides that this maximum is reached. Secondly by using the irregular line Σ^i we proved the maximum number of limit cycles of the PWS (1) formed by linear and quadratic centers.

This thesis comprises five chapters. The first chapter provides a concise overview of fundamental concepts, definitions, and results that are used to a better comprehend the other chapters.

In the second chapter we give the maximum number of limit cycles for some specific families of PWS generated by two differential systems and separated by the regular line Σ^r. We start by investigating the limit cycles that can be created from the family of PWS formed by arbitrary linear center and one of the six arbitrary nilpotent centers. Afterword we examine the limit cycles that can be created from the class of PWS formed by the combination of two arbitrary nilpotent centers. For reaching our results we give examples.

In the third chapter we solve an extension of the second part of the sixteenth Hilbert's problem for two families of PWS separated by the regular line Σ^r. The first family is formed by an arbitrary linear center and an arbitrary cubic isochronous center, and the second family is formed by arbitrary cubic isochronous centers. Addionlly we show that there are examples of each of these systems exhibiting one or three limit cycles.

In the fourth chapter we extend the second part of the 16th Hilbert's problem for the planar PWS separated by the regular line Σ^r and formed by an arbitrary linear center and an arbitrary cubic uniform isochronous center. We provide for this family of PWS an upper bound on its maximal number of limit cycles, and we prove that such an upper bound is reached.

Finally regarding the fifth chapter we are inspired to study the problem of Lum and Chua extended to the family of planar PWS with two regions, where in each region there is an arbitrary linear quadratic centers separated by the irregular line Σ^i instead of the regular line Σ^r.

CHAPTER 1

Preliminaries

Here we discussed some fundamental concepts, findings results, and some necessary tools from the qualitative theory of planar differential systems. The majority of those findings are presented without proof.

1.1 Autonomous polynomial differential systems

Definition 1 (Autonomous polynomial differential systems) *In the plane we called an autonomous polynomial differential system any system of the form*

$$\dot{x} = X(x,y), \quad \dot{y} = Y(x,y), \tag{1.1}$$

where X and Y are polynomials of the independent variables x and y where the of degree of the system is $max(deg(X), deg(Y))$.

1.2 Planar C^k-vector fields with their solutions

Before speaking on the solutions of a differential system, it is convenient to graphically describe the vector field since it provides us important information on the different forms of the possible solutions as well as their asymptotic behavior.

Definition 2 (Planar C^k-vector fields) *A planar C^k-vector field $\mathcal{X}$ is a region on the plane in which at any point p in the open subset $V \subseteq \mathbb{R}^2$, there exists a vector $\mathcal{X}(p)$, i.e. if there exist a C^k-application:*

$$\begin{aligned} \mathcal{X} : V \subseteq \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ p &\rightarrow \mathcal{X}(p) = (X(p), Y(p)). \end{aligned}$$

The corresponding vector field of system (1.1) can be expressed by the following way

$$\mathcal{X} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}. \tag{1.2}$$

The set of C^k-vector fields over $\mathbb{R}^2$ will be denoted by $\Omega^k(\mathbb{R}^2)$. For more information and definitions, see [36].

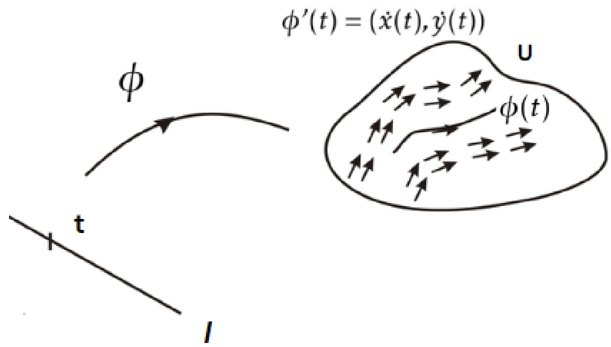


Figure 1.1: The representation of the planar vector field.

Definition 3 (Solution of the differential system (1.1)) *For all t in $I \subseteq \mathbb{R}$ an open subset, we called $\phi(t) = (x(t), y(t))$ a solution of differential system (1.1) the application*

$$\begin{aligned} \phi : I &\to U \subseteq \mathbb{R}^2 \\ t &\to \phi(t) = (x(t), y(t)). \end{aligned}$$

See [36] for more definitions.

Definition 4 (Periodic solution of the differential system (1.1)) *The solution $\phi(t) = (x(t), y(t))$ is called a periodic solution of system (1.1) if there exist a period $T > 0$ such that for all $t \in I$, $\phi(t + T) = \phi(t)$. The smallest number T that satisfies $\phi(t + T) = \phi(t)$ is called the period of the solution ϕ. For more information and definitions, see [36].*

1.3 Equilibrium points

Studying the equilibrium points, known also as singular points, is essential to understanding differential systems and their local qualitative behavior.

Definition 5 (Equilibrium points (or singular points) of system (1.1)) *A point $p_0 = (x_0, y_0)$ is called an equilibrium point of system (1.1) if $X(p_0) = Y(p_0) = 0$. An equilibrium point p_0 is called isolated if it has a neighborhood N where p_0 is the only equilibrium point in N.*

Proposition 1.1

All periodic solutions contain at least one equilibrium point.

Types of equilibrium points

Generaly the study of the local behavior of the trajectories near an equilibrium point p_0 of a planar $\mathcal{C}^k$-vector field (1.2) is very complicated. Already the linear systems show different classes, even for local topological equivalence.

Definition 6 (Jacobian Matrix) *The matrix*

$$D\mathcal{X}(p_0) = \begin{pmatrix} \dfrac{\partial X}{\partial x}(p_0) & \dfrac{\partial X}{\partial y}(p_0) \\ \dfrac{\partial Y}{\partial x}(p_0) & \dfrac{\partial Y}{\partial y}(p_0) \end{pmatrix},$$

associated to the $\mathcal{C}^k$-vector field (1.2) is called the Jacobian matrix of the vector field $\mathcal{X}$ at the equilibrium point p_0.

The equilibrium points of a planar differential (1.1) are classified into four different types according to the eigenvalues of the Jacobian matrix.

Definition 7

- *The hyperbolic equilibrium points are the ones where the Jacobian matrix yields eigenvalues with non-zero real parts.*
- *The semi-hyperbolic equilibrium points are the ones having a unique eigenvalue equal to zero.*
- *The nilpotent equilibrium points have both zero eigenvalues but their linear part is not identically zero.*
- *The linearly zero equilibrium points are the ones that have a linear part identically zero.*

For more information and for the classification of the local phase portraits of the hyperbolic, semi-hyperbolic, linearly zero equilibrium points that have been studied using the change of variables called Blow-ups, see [3, 23].

Definition 8 (Differential center) *The point p_0 is a center of the differential system (1.1) if p_0 is an equilibrium point that has a neighborhood N such that $N\backslash\{p_0\}$ is filled only with periodic orbits.*

Near a center, the linearization of the differential system yields an imaginary eigenvalue pair, indicating a spiral. This behavior encourages trajectories to spiral inward or outward, potentially leading to the formation of periodic orbits.

In the seventeenth century, Christiaan Huygens made significant contributions to the study of isochronous centers, which inspired further research in this area, see [28]. Huygens's efforts sparked widespread interest in the study of isochronous centers across various disciplines, including physics, mathematics, and engineering, see for example [18].

Definition 9 (Isochronous center) *For every $q \in N\backslash\{p_0\}$ let $T(q)$ denote the period of the periodic orbit of system (1.1) through the point q. If $T(q)$ is constant for all $q \in N\backslash\{p_0\}$, we say that the center p_0 is isochronous. If the equilibrium point p_0 is an isochronous center it does not imply that the angular velocity of the vector $\overrightarrow{p_0q}$ is the same for all periodic orbits in the set $N\backslash\{p_0\}$.*

In the following definition we give another type of center which is a particular case of isochronous center.

Definition 10 (Rigid center or uniform isochronous center) *We say that p_0 is a rigid center or a uniform isochronous center if the velocity at p_0 is constant.*

1.4 Five different classes of centers

Our results are based on the use of the following theorems and lemma concerning the linear center, quadratic center, cubic nilpotent center and cubic isochronous center.

Linear center

The general expression of linear planar differential centers is given in the following lemma.

Lemma 1.1

By doing a linear change of variables and a rescaling of the independent variables every planar linear center can be written

$$\dot{x} = d_1 - \beta x - \frac{(\omega^2 + 4\beta^2)}{4\alpha} y, \quad \dot{y} = c_1 + \alpha x + \beta y, \tag{1.3}$$

with $\alpha, \omega > 0$. The first integral of (1.3) is on the form

$$H(x, y) = y^2\omega^2 - 8(d_1 y - c_1 x)\alpha + 4(\beta y + \alpha x)^2. \tag{1.4}$$

Proof. *Any linear differential system in the plane can be written as*

$$\dot{x} = \quad d_1 + ax + by, \quad \dot{y} = c_1 + \alpha x + \beta y. \tag{1.5}$$

It is clear that $(a + \beta \pm \sqrt{4\alpha b + (\beta - a)^2})/2$ *represent the eigenvalues of system* (1.5)*. In order that this system has a center we have to take*

$$a = -\beta, \quad (a - \beta)^2 + 4\alpha b = -\omega^2, \quad \text{with} \ \ \omega > 0.$$

Consequently $b = \dfrac{-(\omega^2 + 4\beta^2)}{4\alpha}$ *and* $\alpha > 0$*. Under these conditions the linear system* (1.5) *becomes*

$$\dot{x} = \quad d_1 - \beta x - \frac{(\omega^2 + 4\beta^2)}{4\alpha} y, \quad \dot{y} = c_1 + \alpha x + \beta y.$$

Thus the proof of Lemma 1.1 is done. ■

Quadratic center in the classification of Kapteyn-Bautin

The normal form of the quadratic center in the classification of Kapteyn-Bautin is given in the following theorem.

Theorem 1.1 (Kapteyn-Bautin Theorem)

After performing an affine transformation and rescaling of the independent variable, any quadratic system candidate for a center can be expressed in the following manner.

$$\dot{x} = -bx^2 - dy^2 - y - Cxy, \quad \dot{y} = ax^2 - ay^2 + x + Axy. \tag{1.6}$$

This system has a center at the origin if and only if one of the next conditions holds

(i) $C = a = 0$,

(ii) $b + d = 0$,

(iii) $A - 2b = C + 2a = 0$,

(iv) $C + 2a = A + 3b + 5d = a^2 + bd + 2d^2 = 0$.

For the proof of Theorem 1.1 see [4, 23, 51, 52].

Cubic nilpotent center with linear plus cubic homogeneous terms

The following theorem gives the general form of Hamiltonian planar polynomial vector field with linear plus cubic homogeneous terms having a nilpotent center at the origin.

Theorem 1.2

A Hamiltonian planar polynomial vector field with linear plus cubic homogeneous terms having a nilpotent center at the origin if and only if, after a linear change of variables and a rescaling of its independent variable, it can be written as one of the following six classes:

$(\mathcal{C}_1)$ $\dot{x} = ax + by, \quad \dot{y} = \dfrac{-a^2}{b}x - ay + x^3$, with $b < 0$.

$(\mathcal{C}_2)$ $\dot{x} = ax + by - x^3, \quad \dot{y} = \dfrac{-a^2}{b}x - ay + 3x^2y$, with $a > 0$ and $b \neq 0$.

$(\mathcal{C}_3)$ $\dot{x} = ax + by - 3x^2y + y^3, \quad \dot{y} = (c - \dfrac{a^2}{b+c})x - ay + 3xy^2$, with either $a = b = 0$ and $c < 0$, or $c = 0$, $ab \neq 0$, and $a^2/b - 6b > 0$. *In this last case one can take* $a = 1$.

$(\mathcal{C}_4)$ $\dot{x} = ax + by - 3x^2y - y^3, \quad \dot{y} = (c - \dfrac{a^2}{b+c})x - ay + 3xy^2$, with either $a = b = 0$ and $c > 0$, or $c = 0$, $a \neq 0$, and $b < 0$. *In this last case one can take* $a = 1$.

$(\mathcal{C}_5)$ $\dot{x} = ax + by - 3\mu x^2y + y^3, \quad \dot{y} = (c - \dfrac{a^2}{b+c})x - ay + x^3 + 3\mu xy^2$, with either $a = b = 0$ and $c < 0$, or $c = 0$, $b \neq 0$, and $(a^4 - b^4 - 6\mu a^2b^2)/b > 0$. *In this last case and* *when* $a \neq 0$ one can take $a = 1$.

$(\mathcal{C}_6)$ $\dot{x} = ax + by - 3\mu x^2 y - y^3, \quad \dot{y} = (c - \dfrac{a^2}{b+c})x - ay + x^3 + 3\mu xy^2$, with either $a = b = 0$ and $c > 0$, or $c = 0$, $b \neq 0$, and $(a^4 + b^4 + 6\mu a^2 b^2)/b < 0$. *In this last case and when* $a \neq 0$ one can take $a = 1$. Where $a, b, c, \mu \in \mathbb{R}$.

For the proof see [19, 32].

In our work the cubic nilpotent center will be used to refer to Hamiltonian planar polynomial vector field with linear plus cubic homogeneous terms having a nilpotent center at the origin.

Rigid center

All polynomial rigid centers are classified in the following theorem.

Theorem 1.3

Any polynomial differential system has a rigid center at the origin can be written after an affine change of variables and a rescaling of the independent variable as the following form

$$\dot{x} = -y + xf(x,y), \quad \dot{y} = x + yf(x,y), \tag{1.7}$$

where $f(x,y) = a_1 x + a_2 y + a_4 xy$, and satisfies $a_1 a_2 = 0$ and $a_4 \neq 0$. System (1.7) has a rigid center at the origin if and only if one of the following conditions holds

Case 1. If $a_1^2 + a_2^2 = 0$.

Case 2. If $a_1^2 + a_2^2 \neq 0$ and

2.1. If $4a_4 - a_1^2 < 0$ and $a_2 = 0$, **2.2.** If $4a_4 + a_2^2 > 0$ and $a_1 = 0$,

2.3. If $4a_4 - a_1^2 > 0$ and $a_2 = 0$, **2.4.** If $4a_4 + a_2^2 < 0$ and $a_1 = 0$,

2.5. If $4a_4 - a_1^2 = 0$ and $a_2 = 0$, **2.6.** If $4a_4 + a_2^2 = 0$ and $a_1 = 0$.

For the proof of Theorem 1.3 see [20] or section 2 of [2].

In order to obtain the generalization of any center of (1.6), $(\mathcal{C}_i)$ with $i = 1, \ldots 6$ and (1.7) that satisfy the conditions of Theorem 1.1, 1.2 and 1.3, respectively, an arbitrary affine transformation applied to these systems

$$(x, y) = (\alpha_1 x + \beta_1 y + \gamma_1, \alpha_2 x + \beta_2 y + \gamma_2), \tag{1.8}$$

with $\alpha_2\beta_1 - \alpha_1\beta_2 \neq 0$, see Chapter 5, Chapter 2 and Chapter 4, respectively.

Cubic isochronous center of period 2π

According to the theorem of Manõsas and Villadelprat all analytic Hamiltonian functions with an isochronous center at the origin can be expressed as the sum of two squares.

Theorem 1.4

The Hamiltonian differential system

$$\dot{x} = -H_y(x, y), \quad \dot{y} = H_x(x, y),$$

has an isochronous center of period 2π at the origin if and only if

$$H(x, y) = \frac{f(x, y)^2 + g(x, y)^2}{2}, \tag{1.9}$$

where $(x, y) \to (f(x, y), g(x, y))$ is an analytic canonical mapping with $f(0, 0) = g(0, 0) = 0$, i.e. the Jacobian determinant of the map (f, g) is equal to one at any point.

For the proof see [44].

The primary methods for calculating the analytical limit cycles of differential systems are based on the Melnikov integral, the averaging theory, the Poincaré map, and the Poincaré map together with the Newton–Kantorovich theorem or the Poincaré–Miranda theorem. For the piecewise differential systems and in order to compute their limit cycles we add the method of first integrals. However it's important to note that this approach applies only to PWS where all their differential systems are integrable, meaning that for each of them, we know its first integral.

1.5 Integrability of differential systems

In the qualitative study of differential systems, the notion of integrability plays a main role but the determination of the first integral for a given differential system is not an easy task.

A differential system is said to be integrable if it admits a first integral.

Definition 11 (First integral) *A C^1-function $H : V \to \mathbb{R}$ on the open subset $V \subseteq \mathbb{R}^2$ is the first integral of the differential system (1.1) if the function H is constant on the trajectories of the differential system (1.1) contained in V, i.e, if $\dfrac{dH(x,y)}{dt} \equiv 0$ where*

$$\frac{d}{dt}H(x,y) = X\frac{\partial}{\partial x}H(x,y) + Y\frac{\partial}{\partial y}H(x,y). \tag{1.10}$$

Moreover $H = k$ where k constant is the general solution of $\dfrac{dH(x,y)}{dt} \equiv 0$. The differential system (1.1) is integrable in V if it has a C^1-first integral H in V.

As a particular case we give a definition of an interesting type of systems that arise in physical problems.

Definition 12 (Hamiltonian planar systems) *Let $H \in C^2(V)$ be a function of class C^2 on the open $V \subseteq \mathbb{R}^2$ where $H = H(x,y)$. We called a Hamiltonian system any system of the form*

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}, \tag{1.11}$$

with the hamiltonian function $H(x,y)$ =constant.

1.6 Planar piecewise differential systems

This section summarizes Filippov rules for representing planar piecewise differential systems given in [27]. Consider $h : V \subseteq \mathbb{R}^2 \to \mathbb{R}$ a C^1-function having 0 as a regular value and $\Sigma = h^{-1}(0)$ separation curve.

Definition 13 (Planar piecewise differential systems) *Let consider the arbitrary vector fields $\mathcal{X}_i \in \Omega^k(\mathbb{R}^2)$ with $i \in \{1,\dots,n\}$, where n is the number of connected components V_i of $\mathbb{R}^2/\Sigma$. The n-tuple $\mathcal{X} = (\mathcal{X}_1,\dots,\mathcal{X}_n)$ where $\mathcal{X}(x,y) = \mathcal{X}_i(x,y)$ if (x,y) in V_i is named by a planar piecewise vector field.*

We note that on the discontinuity curve Σ the piecewise vector field $\mathcal{X}$ is bi-valuated. When the neighbor vector fields $\mathcal{X}_i$ and $\mathcal{X}_j$ with $i,j \in \{1,\dots,n\}$ and $i \neq j$, coincide on the discontinuity curve Σ, we obtain a planar continuous piecewise differential system, that in general, will not be smooth on Σ.

In this thesis we are interested in discontinuous piecewise differential systems of the form (1) separated into two regions Σ_j^k with $j = 1,2$ and $k = r,i$ such that Σ^r represent the regular line and Σ^i the irregular line as we mentioned in the introduction. When the vector fields $\mathcal{X}_1$ and $\mathcal{X}_2$ coincide on the separation curve Σ^k with $k = r,i$, we obtain a continuous piecewise differential system on $\mathbb{R}^2$.

The vector field (1) is usually denoted by $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \Sigma^k)$ with $k = r,i$ or simply by $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2)$. Now in order to establish a definition for the trajectories of $\mathcal{X}$, we had to have a criterion for the transition of the trajectories between Σ_1^k and Σ_2^k across the curve of discontinuity Σ^k. The contact between the curve of discontinuity Σ^k with $k = r,i$ and the vector field $\mathcal{X}_1$ (or $\mathcal{X}_2$) is described by the directional derivative of h with respect to the vector field $\mathcal{X}_1$, i.e.,

$$\mathcal{X}_1 h(p_0) = \langle \nabla h(p_0), \mathcal{X}_1(p_0) \rangle.$$

Here $\langle , \rangle$ denotes the usual inner product of the plane $\mathbb{R}^2$. Filippov in [27] stated the main results of the discontinuous piecewise differential systems. The curve of discontinuity Σ^k with $k = r,i$ is divided into the three following sets:

- **Escaping region**: $R^e = \{p_0 \in \Sigma^k,\ \mathcal{X}_1 h(p_0) > 0 \text{ and } \mathcal{X}_2 h(p_0) < 0\}$ formed by escaping points.

- **Sliding region**: $R^s = \{p_0 \in \Sigma^k,\ \mathcal{X}_1 h(p_0) < 0 \text{ and } \mathcal{X}_2 h(p_0) > 0\}$ formed by sliding points.

- **Crossing region**: $R^c = \{p_0 \in \Sigma^k,\ (\mathcal{X}_1 h(p_0)) \cdot (\mathcal{X}_2 h(p_0)) > 0\}$ formed by crossing points.

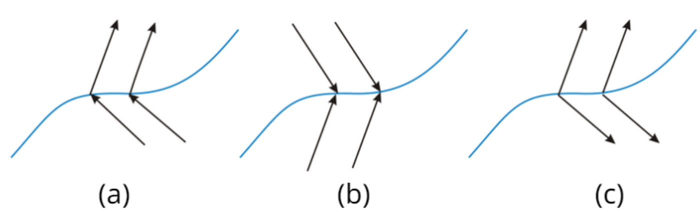


Figure 1.2: (a) Crossing, (b) sliding and (c) escaping regions.

Filippov's convention in [27] defines three types of limit cycles for piecewise vector fields, the escaping limit cycles, the sliding limit cycles, and finally the crossing limit cycles. Escaping limit cycles contain escaping points on the separation curve, sliding limit cycles contain sliding points on the separation curve, and crossing limit cycles contain only crossing points, see Figure 1.2. In this work we are interested only in the crossing limit cycles, or simply limit cycles.

CHAPTER 2

The Limit Cycles of Discontinuous Piecewise Differential System Formed by an Arbitrary Linear and Cubic Nilpotent Centers Separated by Σ^r

Chapter 2 is a result of our paper entitled "The solution of the extended 16th Hilbert problem for some classes of piecewise differential systems", published in Mathematics Journal.

The problem of determining the maximum number of limit cycles for PWS which are formed by linear differential systems separated by a regular line is particularly challenging. The difficulty of the problem increases when dealing with non-linear PWS. Furthermore, there are two important reasons that make this analysis difficult. First, while it is easy to integrate the solution of each linear differential system it is not as simple to directly calculate the amount of time an orbit spends in each region that each linear differential system governs. Second, in general, a large number of parameters are required to examine each possibility that could arise.

Lately numerous articles have focused on examining the problem of the existence and the maximum number of limit cycles for the simplest class of linear PWS separated by a regular line. In [14] Benterki and Llibre studied the same problem for a class of PWS having isochronous centers of degree one or three. Then from here we build the main objective of this chapter where we provide the upper bounds for the maximum number of limit cycles of certain classes of PWS separated by the regular line Σ^r and formed by either linear and cubic nilpotent centers or by only cubic nilpotent centers.

The family of the classes of PWS divided by the regular line Σ^r on two pieces, where in one piece there is a linear center and in the other piece there is a generalized cubic nilpotent center formed by a linear plus cubic homogeneous polynomial is denoted by $\mathcal{F}_1$.

The family of the classes of PWS divided by the regular line Σ^r on two pieces, where in each piece there is a generalized cubic nilpotent center formed by a linear plus cubic homogeneous polynomials after is denoted by $\mathcal{F}_2$.

Generalized cubic nilpotent center

In this part we make the general affine change of variables (1.8) to the planar cubic nilpotent center ($\mathcal{C}_i$) with $i = 1,\ldots 6$ and we get the six cubic nilpotent centers where the first one is given by the next differential system

$$\begin{aligned}
\dot{x} = \;& (b(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}(-a^2\beta_1(\alpha_1x+\beta_1y+\gamma_1)+b(3\beta_1^3y^2(\alpha_1x+\gamma_1)+3\beta_1^2y(\alpha_1x\\
&+\gamma_1)^2+\beta_1(\alpha_1x+\gamma_1)^3+\beta_1^4y^3-b\beta_2(\gamma_2+\alpha_2x+\beta_2y))-ab(\beta_2(\alpha_1x+\gamma_1)+\beta_1\\
&(\gamma_2+\alpha_2x+2\beta_2y))),\\
\dot{y} = \;& (b(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}(a^2\alpha_1(\alpha_1x+\beta_1y+\gamma_1)+b(-\alpha_1^4x^3-3\alpha_1^3x^2(\beta_1y+\gamma_1)-3\alpha_1^2x\\
&(\beta_1y+\gamma_1)^2-\alpha_1(\beta_1y+\gamma_1)^3+\alpha_2b(\gamma_2+\alpha_2x+\beta_2y))+ab(\alpha_1(\gamma_2+2\alpha_2x+\beta_2y)\\
&+\alpha_2\beta_1y+\alpha_2\gamma_1)),
\end{aligned}\tag{2.1}$$

that has the first integral

$$\begin{aligned}
H_1(x,y) \;=\;& a^2b^{-1}(\alpha_1x+\beta_1y+\gamma_1)^2+a(\alpha_1x+\beta_1y+\gamma_1)(\gamma_2+\alpha_2x+\beta_2y)-\frac{1}{2}(\alpha_1x+\beta_1y\\
&+\gamma_1)^4+b(\gamma_2+\alpha_2x+\beta_2y)^2.
\end{aligned}$$

The second differential cubic nilpotent is

$$\begin{aligned}
\dot{x} = \;& (b(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}(b\beta_1^3y^2(3\gamma_2+3\alpha_2x+4\beta_2y)-b\beta_2(a\alpha_1x-\alpha_1^3x^3-3\alpha_1\gamma_1^2x\\
&-\gamma_1^3+\gamma_1(a-3\alpha_1^2x^2)+b(\gamma_2+\alpha_2x+\beta_2y))+\beta_1^2y\Big(3b(\alpha_1x+\gamma_1)(2\gamma_2+2\alpha_2x+3\\
&\beta_2y)-a^2\Big)+\beta_1(3b\gamma_1^2(\gamma_2+\alpha_2x+2\beta_2y)-a^2\alpha_1x+3\alpha_1^2bx^2(\gamma_2+\alpha_2x+2\beta_2y)-a\\
&b(\gamma_2+\alpha_2x+2\beta_2y)+\gamma_16\alpha_1bx(\gamma_2+\alpha_2x+2\beta_2y)-a^2)),\\
\dot{y} = \;& (b(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}((\alpha_2b-\alpha_1^3bx^2(4\alpha_2x+3(\gamma_2+\beta_2y))(a\beta_1y-\beta_1^3y^3-3\beta_1\gamma_1^2y\\
&-\gamma_1^3+\gamma_1(a-3\beta_1^2y^2)+b(\gamma_2+\alpha_2x+\beta_2y))-\alpha_1^2x(3b(\beta_1y+\gamma_1)(3\alpha_2x+2(\gamma_2\\
&+\beta_2y))-a^2)+\alpha_1(-3b\gamma_1^2(\gamma_2+2\alpha_2x+\beta_2y)+a^2\beta_1y+\gamma_1(a^2-6b\beta_1y(\gamma_2+2\alpha_2\\
&x+\beta_2y))+ab(\gamma_2+2\alpha_2x+\beta_2y)-3b\beta_1^2y^2(\gamma_2+2\alpha_2x+\beta_2y))),
\end{aligned}\tag{2.2}$$

its first integral is

$$\begin{aligned}
H_2(x,y) \;=\;& \frac{a^2}{2b}(\alpha_1x+\beta_1y+\gamma_1)^2+a(\alpha_1x+\beta_1y+\gamma_1)(\gamma_2+\alpha_2x+\beta_2y)+\frac{1}{2}b(\gamma_2+\alpha_2x\\
&+\beta_2y)^2-(\alpha_1x+\beta_1y+\gamma_1)^3(\gamma_2+\alpha_2x+\beta_2y).
\end{aligned}$$

The third differential cubic nilpotent has the form

$$
\begin{aligned}
\dot{x} = \;& -((b+c)(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}(a^2\beta_1(\alpha_1x+\beta_1y+\gamma_1)+a(b+c)(\beta_2(\alpha_1x+\gamma_1)+\beta_1\\
& (\gamma_2+\alpha_2x+2\beta_2y))+(b+c)(\beta_2(\gamma_2+\alpha_2x+\beta_2y)(-3\alpha_1^2x^2-6\alpha_1\gamma_1x+b-3\gamma_1^2\\
& +(\gamma_2+\alpha_2x+\beta_2y)^2)-\beta_1(\alpha_1x+\gamma_1)(c+3(\gamma_2+\alpha_2x+\beta_2y)(\gamma_2+\alpha_2x+3\beta_2y))\\
& -\beta_1^2y(c+3(\gamma_2+\alpha_2x+\beta_2y)(\gamma_2+\alpha_2x+2\beta_2y)))),\\
\dot{y} = \;& ((b+c)(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}(a^2\alpha_1(\alpha_1x+\beta_1y+\gamma_1)+a(b+c)(\alpha_1(\gamma_2+2\alpha_2x+\beta_2y)\\
& +\alpha_2\beta_1y+\alpha_2\gamma_1)-(b+c)(-\alpha_2(\gamma_2+\alpha_2x+\beta_2y)(b-3\beta_1^2y^2-6\beta_1\gamma_1y-3\gamma_1^2+(\gamma_2\\
& +\alpha_2x+\beta_2+y)^2)\alpha_1^2x(c+3(\gamma_2+\alpha_2x+\beta_2y)(\gamma_2+2\alpha_2x+\beta_2y))(\beta_1y+\alpha_1+\gamma_1)(c\\
& +3(\gamma_2+\alpha_2x+\beta_2y)(\gamma_2+3\alpha_2x+\beta_2y)))),
\end{aligned}
\tag{2.3}
$$

its first integral is

$$
\begin{aligned}
H_3(x,y) \;=\;& \frac{1}{2}\left(\frac{a^2}{b+c}-c\right)(\alpha_1x+\beta_1y+\gamma_1)^2+a(\alpha_1x+\beta_1y+\gamma_1)(\gamma_2+\alpha_2x+\beta_2y)+\frac{1}{2}b(\gamma_2\\
& +\alpha_2x+\beta_2y)^2-\frac{3}{2}(\alpha_1x+\beta_1y+\gamma_1)^2(\gamma_2+\alpha_2x+\beta_2y)^2+\frac{1}{4}(\gamma_2+\alpha_2x+\beta_2y)^4.
\end{aligned}
$$

The fourth one is

$$
\begin{aligned}
\dot{x} = \;& -((b+c)(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}(a^2\beta_1(\alpha_1x+\beta_1y+\gamma_1)+a(b+c)(\beta_2(\alpha_1x+\gamma_1)+\beta_1\\
& (\gamma_2+\alpha_2x+2\beta_2y))+(-\beta_2(\gamma_2+\alpha_2x+\beta_2y)(3\alpha_1^2x^2+6\alpha_1\gamma_1x-b+3\gamma_1^2+(\gamma_2\\
& +\alpha_2x+\beta_2y)^2)+\beta_1^2y(c+3(\gamma_2+\alpha_2x+\beta_2y)(\gamma_2+\alpha_2x+2\beta_2y))-\beta_1(\alpha_1x+\gamma_1)\\
& (c+3(\gamma_2+\alpha_2x+\beta_2y)(\gamma_2+\alpha_2x+3\beta_2y))),\\
\dot{y} = \;& ((b+c)(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}(a^2\alpha_1(\alpha_1x+\beta_1y+\gamma_1)+a(b+c)(\alpha_1(\gamma_2+2\alpha_2x+\beta_2y)\\
& +\alpha_2)\beta_1y+\alpha_2\gamma_1-(b+c)(\alpha_2(\gamma_2+\alpha_2x+\beta_2y)(-b+3\beta_1^2y^2+6\beta_1\gamma_1y+3\gamma_1^2+(\gamma_2\\
& +\alpha_2x+\beta_2y)^2)+\alpha_1^2x(c+3(\gamma_2+\alpha_2x+\beta_2y)(\gamma_2+2\alpha_2x+\beta_2y))+\alpha_1(\beta_1y+\gamma_1)(c\\
& +3)(\gamma_2+\alpha_2x+\beta_2y)(\gamma_2+3\alpha_2x+\beta_2y))),
\end{aligned}
\tag{2.4}
$$

its first integral is

$$
\begin{aligned}
H_4(x,y) \;=\;& -2\left(c-\frac{a^2}{b+c}\right)(\beta_1y+\gamma_1+\alpha_1x)^2+a(\beta_1y+\gamma_1+\alpha_1x)(\gamma_2+\alpha_2x+\beta_2y)+2b(\gamma_2\\
& +\alpha_2x+\beta_2y)^2-6(\alpha_1x+\beta_1y+\gamma_1)^2(\gamma_2+\alpha_2x+\beta_2y)^2-(\gamma_2+\alpha_2x+\beta_2y)^4.
\end{aligned}
$$

The fifth differential cubic nilpotent is

$$
\begin{aligned}
\dot{x} = \;& -((b+c)(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}(a^2\beta_1(\alpha_1x+\beta_1y+\gamma_1)+a(b+c)(\beta_2(\alpha_1x+\gamma_1)+\beta_1\\
& (\gamma_2+\alpha_2x+2\beta_2y))+(b+c)(-3\beta_1^3y^2(\alpha_1x+\gamma_1)+\beta_2(\gamma_2+\alpha_2x+\beta_2y)(-3\mu(\alpha_1x
\end{aligned}
$$

$$
\begin{aligned}
& +\gamma_1)^2+b+(\gamma_2+\alpha_2 x+\beta_2 y)^2)-\beta_1^4 y-\beta_1(\alpha_1 x+\gamma_1)((\alpha_1 x+\gamma_1)^2+c+3\mu(\gamma_2 \\
& +\alpha_2 x+\beta_2 y)(\gamma_2+\alpha_2 x+3\beta_2 y))-\beta_1^2 y(3((\alpha_1 x+\gamma_1)^2+\mu(\gamma_2+\alpha_2 x+\beta_2 y)(\gamma_2 \\
& +\alpha_2 x+2\beta_2 y))+c))), \\
\dot{y}= \; & ((b+c)(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}(a^2\alpha_1(\alpha_1 x+\beta_1 y+\gamma_1)+a(b+c)(\alpha_1(\gamma_2+2\alpha_2 x+\beta_2 \\
& y)+\alpha_2\beta_1 y+\alpha_2\gamma_1)-(b+c)(+3\alpha_1^3 x^2(\beta_1 y+\gamma_1)+\alpha_2(\gamma_2+\alpha_2 x+\beta_2 y)(-b+3 \\
& \mu(\beta_1 y+\gamma_1)^2-(\gamma_2+\alpha_2 x+\beta_2 y)^2)+\alpha_1^4 x^3+\alpha_1(\beta_1 y+\gamma_1)((\beta_1 y+\gamma_1)^2+c+3\mu \\
& (\gamma_2+\alpha_2 x+\beta_2 y)(\gamma_2+3\alpha_2 x+\beta_2 y))+\alpha_1^2 x(3((\beta_1 y+\gamma_1)^2+\mu(\gamma_2+\alpha_2 x+\beta_2 y) \\
& (\gamma_2+2\alpha_2 x+\beta_2 y))+c))),
\end{aligned}
\tag{2.5}
$$

its first integral is

$$
\begin{aligned}
H_5(x,y) \;=\; & 2\left(\frac{a^2}{b+c}-c\right)(\alpha_1 x+\gamma_1+\beta_1 y)^2+a(\alpha_1 x+\gamma_1+\beta_1 y)(\gamma_2+\alpha_2 x+\beta_2 y)-(\alpha_1 x \\
& +\beta_1 y+\gamma_1)^4-6\mu(\alpha_1 x+\beta_1 y+\gamma_1)^2(\gamma_2+\alpha_2 x+\beta_2 y)^2+2b(\gamma_2+\alpha_2 x+\beta_2 y)^2 \\
& +(\gamma_2+\alpha_2 x+\beta_2 y)^4.
\end{aligned}
$$

Finally the sixth one is

$$
\begin{aligned}
\dot{x}= \; & -((b+c)(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}(a^2\beta_1(\alpha_1 x+\beta_1 y+\gamma_1)+a(b+c)(\beta_2(\alpha_1 x+\gamma_1)+\beta_1 \\
& (\gamma_2+\alpha_2 x+2\beta_2 y))+(b+c)(-\beta_2\gamma_2+\alpha_2 x+\beta_2 y(3\mu(\alpha_1 x+\gamma_1)^2-b+(\gamma_2+\alpha_2 x \\
& +\beta_2 y)^2-3\beta_1^3 y^2(\alpha_1 x+\gamma_1)-\beta_1^4 y^3-\beta_1(\alpha_1 x+\gamma_1)((\alpha_1 x+\gamma_1)^2+c+3\mu(\gamma_2+\alpha_2 \\
& x+\beta_2 y)(\alpha_2 x+\beta_2 y))-\beta_1^2 y(3((\alpha_1 x+\gamma_1)^2+\mu(\gamma_2+\alpha_2 x+\beta_2 y)(\gamma_2+\alpha_2 x))+c))), \\
\dot{y}= \; & ((b+c)(\alpha_2\beta_1-\alpha_1\beta_2))^{-1}(a^2\alpha_1(\alpha_1 x+\beta_1 y+\gamma_1)+a(b+c)(\alpha_1(\gamma_2+2\alpha_2 x+\beta_2 y) \\
& +\alpha_2\beta_1 y+\alpha_2\gamma_1)-(b+c)(3\alpha_1^3 x^2(\beta_1 y+\gamma_1)+\alpha_2(\gamma_2+\alpha_2 x+\beta_2 y)(-b+3\mu(\beta_1 y \\
& +\gamma_1)^2+(\gamma_2+\alpha_2 x+\beta_2 y)^2)+\alpha_1^4 x^3+\alpha_1(\beta_1 y+\gamma_1)((\beta_1 y+\gamma_1)^2+c+3\mu(\gamma_2+\alpha_2 \\
& x+\beta_2 y)(\gamma_2+3\alpha_2 x)+\beta_2 y)+\alpha_1^2 x(3((\beta_1 y+\gamma_1)^2+\mu(\gamma_2+\alpha_2 x+\beta_2 y)(\gamma_2+2\alpha_2 x \\
& +\beta_2 y))+c))),
\end{aligned}
\tag{2.6}
$$

its first integral is

$$
\begin{aligned}
H_6(x,y)= \; & 2\left(\frac{a^2}{b+c}-c\right)(\alpha_1 x+\gamma_1+\beta_1 y)^2+a(\alpha_1 x+\gamma_1+\beta_1 y)(\gamma_2+\alpha_2 x+\beta_2 y)-(\alpha_1 x+\beta_1 y+\gamma_1)^4 \\
& -6\mu(\alpha_1 x+\beta_1 y+\gamma_1)^2(\gamma_2+\alpha_2 x+\beta_2 y)^2+2b(\gamma_2+\alpha_2 x+\beta_2 y)^2-(\gamma_2+\alpha_2 x+\beta_2 y)^4.
\end{aligned}
$$

2.1 The limit cycles of the first family of PWS $\mathcal{F}_1$

The next theorem gives the results of the maximum number of limit cycles for the first family of PWS $\mathcal{F}_1$.

Theorem 2.1

For the first family of PWS $\mathcal{F}_1$, the maximum number of limit cycles is at most one. Additionally all the classes attain this maximum; see Figures 2.1 and 2.2.

Proof of Theorem 2.1

Proof. *Now we need to establish Theorem 2.1 for the family of PWS partitioned by the regular line Σ^r, and composed of the pair of differential systems (1.3)–(2.i) for $i \in 1,\dots,6$.*

In Σ_1^r we take the linear differential center (1.3) with its corresponding first integral $H(x,y)$ given by (1.4). In Σ_2^r we consider the nilpotent center (2.i) with its own first integral $H_i(x,y)$. If there is a limit cycle for the PWS (1.3)–(2.i), this limit cycles must cross the discontinuity line Σ^r in $p_1 = (0,y_1)$ and $p_2(0,y_2)$ a pair of distinct points where $y < Y$. Furthermore the points p_1 and p_2 need to satisfy the next system

$$H(p_1) - H(p_2) = (y - Y)h(y,Y) = 0, \quad H_i(p_1) - H_i(p_2) = h_i(y,Y) = 0, \tag{2.7}$$

where $h(y,Y) = 2d_1\alpha - \beta^2 y - \beta^2 Y - \omega^2(y - Y)$, and $h_i(y,Y)$ is a quadratic polynomial for all $i \in \{1,\dots,6\}$. Since $y < Y$, we get the function $Y = g(y)$ that has y as the unique variable by solving the equation $h(y,Y) = 0$. Then after substituting the finding Y in the equation $h_i(y,Y) = 0$, we get a new quadratic equation having y as the unique unknown variable. It is evident that since this equation is quadratic it can has at most a maximum of two real solutions named by $(y_1, f(y_1))$ and $(y_2, f(y_2))$. We know that these two solutions reflect the same solution of (2.7) because of the symmetry $(y_1, f(y_1)) = (f(y_2), y_2)$. Then both solutions provide the same limit cycle for the PWS (1.3)–(2.i). It has been shown that there is a maximum of one limit cycle for the family of PWS (1.3)–(2.i). ∎

Illustrative examples for the PWS (1.3)–(2.i) for $i \in 1,\ldots,6$

Example of one limit cycle for the PWS (1.3)-(2.1)**.** In what follows we give a PWS for the PWS (1.3)–(2.1) with exactly one limit cycle. In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} = {} & (10^{-2}/15001)(100(-x^3-30x^2(-1+y)+x(-300y(y-2)-13714)-1000\\ & (y-3)y^2)-1302001y+1050950),\\ \dot{y} = {} & (10^{-2}/1500.1)(x^3+30x^2(-1+y)+100x(1963+3(y-2)y)+20(y(50(y\\ & -3)y-6857)+6600)),\end{aligned} \tag{2.8}$$

of the form (2.1), this system has the first integral

$$\begin{aligned}H_1(x,y) = {} & -0.25\,(0.1x-1+y)^4-5\,(0.1x-1+y)^2-(0.01y+50)-15x)\,(0.1x+y-1)\\ & -0.5\;10^{-5}(50-1500x+y)^2.\end{aligned}$$

In Σ_1^r we consider the linear differential system

$$\dot{x} = -0.1x-0.101y+0.1, \quad \dot{y} = 0.1x+0.1y+0.5, \tag{2.9}$$

that has the first integral $H(x,y) = 2(-1+x)y+1x(10+x)+1.01y^2$. Now for the solutions of system (2.7) with $i=1$ satisfying $y<Y$, the unique solution of (2.7) is $(y,Y) \approx (-5.45516, 7.43536)$, and it provides the unique limit cycle of the PWS (2.8)–(2.9) that seen in Figure 2.1(a).

Example of one limit cycle for the PWS (1.3)-(2.2)**.** Here we consider the cubic nilpotent center (2.2) in the region Σ_2^r

$$\begin{aligned}\dot{x} = {} & (10^{-4}/29)(91x^3+60x^2(-31y+455)+300x(y(-950+33y)+7722)-10^3\\ & (y(4(y-30)y+981)+585)),\\ \dot{y} = {} & (10^{-4}/29)(12x^3+4035x^2+60(31x+2375)y^2-39(7x(x+200)+59400)\\ & y+1294510x-3300y^3+7844000,\end{aligned} \tag{2.10}$$

that has the first integral

$$\begin{aligned}H_2(x,y) \;\; = \;\; & 0.1\,(-0.01x+0.1y-1.5)\,(-3x+y+5)-0.05\,(-0.01x+0.1y-1.5)^2\\ & -0.05(-3x+y+5)^2-(-0.01x+0.1y-1.5)^3\,(5+y-3x).\end{aligned}$$

In Σ_1^r we consider the linear center

$$\dot{x} = -0.1x - 0.101y - 0.1, \quad \dot{y} = x + 0.1y - 0.3, \tag{2.11}$$

that has the first integral $H(x,y) = y^2 + (x+0.1y)^2 + 2(0.1y - 0.3x)$. For the PWS (2.10)–(2.11), the unique solution of system (2.7) with $i = 2$ such that $y < Y$ is $(y, Y) \approx (-3.57003, 3.37201)$. This proves the uniqueness of the limit cycle of the PWS (2.10)–(2.11), see Figure 2.1(b).

Example of one limit cycle for the PWS (1.3)-(2.3). In the region Σ_1^r we take the cubic nilpotent center

$$\begin{aligned}\dot{x} = {} & (10^{-4}/699)(-1472099x^3 - 210x^2(-80207 + 704000y) - 10^2x(6\times 10^2y \\ & (1051y - 3650) + 212897) - 10^3\left(10^2y\left(50y^2 - 456y + 809\right) - 7149\right)), \\ \dot{y} = {} & (10^{-6}/699)(2939999x^3 + 30x^2(14720990y - 1574999) + 700x(60y(352 \\ & y) - 42899) + 10^2(10y(2\times 10^2y(1051y - 5475) + 21297) + 237501)),\end{aligned} \tag{2.12}$$

of the form (2.3) having the first integral

$$H_3(x,y) \;=\; 5(0.01(x-9)+y)^4 + (7x-5+y)^2 - 0.003(7x-5+y)^2(x-10+100y)^2.$$

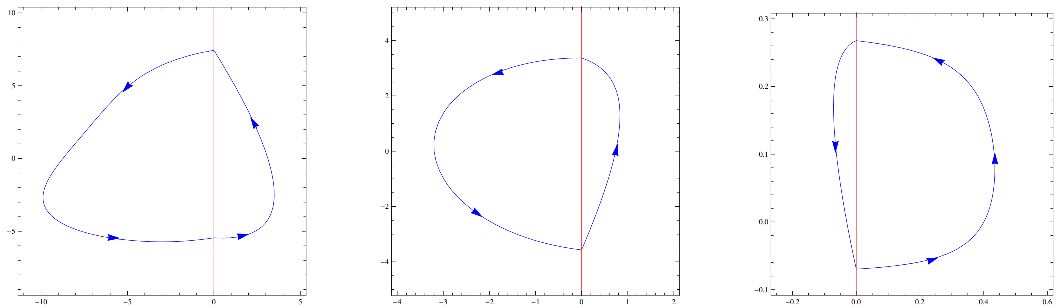
In the region Σ_2^r we place the linear differential center

$$\dot{x} = 0.01 - 0.101y - 0.1x, \quad \dot{y} = -0.03 + 0.1x + 0.1y, \tag{2.13}$$

with the first integral $H(x,y) = (x+y)^2 + (-3x-y)/5 + 10^{-2}y^2$. Theis a unique limit cycles for the PWS (2.12)–(2.13), because when $i = 3$, system (2.7) has exactly one real solution namely $(y, Y) \approx (-0.0697386,$
$0.267758)$ and satisfying $y < Y$, which provides the unique limit cycle seen in Figure 2.1(c).

Example of one limit cycle for the PWS (1.3)-(2.4). In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} = {} & (1/980)(6(3094 + 737x)y^2 - (33292 + 33x(364 + 31x))y - 10(x(x(191x \\ & +1785) + 4732) + 2892) - 2344y^3), \\ \dot{y} = {} & (1/980)(10\left(573x^2 + 4732 + 3570x\right)y + 33(182 + 31x)y^2 + 5(x(5x(798 \\ & 97x) + 14042) + 20924) - 1474y^3),\end{aligned} \tag{2.14}$$



(*a*) (*b*) (*c*)

Figure 2.1: (a) Represent the unique limit cycle of the PWS (2.8)–(2.9), (b) is the unique limit cycle of the PWS (2.10)–(2.11), and (c) is the unique limit cycle of the PWS (2.12)–(2.13).

of type (2.4), with the first integral

$$\begin{aligned} H_4(x,y) &= -5(2x-0.9y+7)^2-0.0125(x+2(y+1))^2-0.05(20x-9y+70)(x+2y+2) \\ &\quad -0.00375(20x+70-9y)^2(2(y+1)+x)^2-1/64(x+1+2y)^4. \end{aligned}$$

In Σ_2^r we consider

$$\dot{x}=-0.1x-0.3-2.26y, \quad \dot{y}=x+0.1y-0.5, \tag{2.15}$$

the linear center with $H(x,y)=(x+0.1y)^2+(0.6-x)+2.25y^24$, as the first integral. Since when $i=4$, system (2.7) has exactly one real solution $(y,Y)\approx(-2.06113,1.79564)$ then the PWS (2.14)–(2.15) has exactly one limit cycle drawn in Figure 2.2(a).

Example of one limit cycle for the PWS (1.3)-(2.5). In Σ_2^r we consider the following cubic nilpotent center

$$\begin{aligned} \dot{x} &= \frac{10^{-4}}{1064}(230728x^3-3x^2(160269699y+41725553)+3x(5y(22466369014y \\ &\quad +81962755532)+793469623268)-5(y(25y(6388790068751y \\ &\quad +1159455046638)+499170228203556)-446074006985264)), \\ \dot{y} &= \frac{10^{-4}}{5320}(x^2(572720190-34609275y)-15425x^3+x(15y(1602696995y \\ &\quad +834511076)-3519832393532)+7135887059545552-15y(5y \\ &\quad (74887896715y+40981377766)+793469623268)), \end{aligned} \tag{2.16}$$

that has the following first integral

$$
\begin{aligned}
H_5(x,y) \;=\;& 0.4\,10^{-3}((-3x+5y+18118)^4)-(10^{-5}/5)(3(-5x+3555y+212)^2(-3x+5y\\
&+18118)^2)+(-0.5x+711y/2+106/5)^2-0.25\,10^{-4}((-5x+3555y+212)^4).
\end{aligned}
$$

In Σ_1^r we consider

$$
\dot{x}=-0.1x-0.2y+0.1, \quad \dot{y}=0.1x+0.1y-0.1, \tag{2.17}
$$

the linear differential center with the first integral $H(x,y)=(0.1x+0.1y)^2+0.2(-0.1x-0.1y)+0.01y^2$. For the PWS (2.16)–(2.17), the unique solution of system (2.7) when $i=5$ is $(y,Y)\approx(-0.875101,1.8751)$. This proves the uniqueness of the limit cycle of the PWS (2.16)–(2.17) shown in Figure 2.2(b).

Example of one limit cycle for the PWS (1.3)-(2.6). We consider the following cubic nilpotent center in Σ_1^r

$$
\begin{aligned}
\dot{x} =\;& \frac{10^{-4}}{49}(10(-765(343+4932x)y^2+54x(-15395x+11711)y+x(1020\\
&(-65x+147)x-436303)+879650-6251501y^3-18870103y),\\
\dot{y} =\;& \frac{10^{-3}}{49}(3900x^2(-49+51y)+10x(9y(-3332+9237y)+82033)+y(27y\\
&(-11711+46580y)+436303)-1247640+26000x^3),
\end{aligned} \tag{2.18}
$$

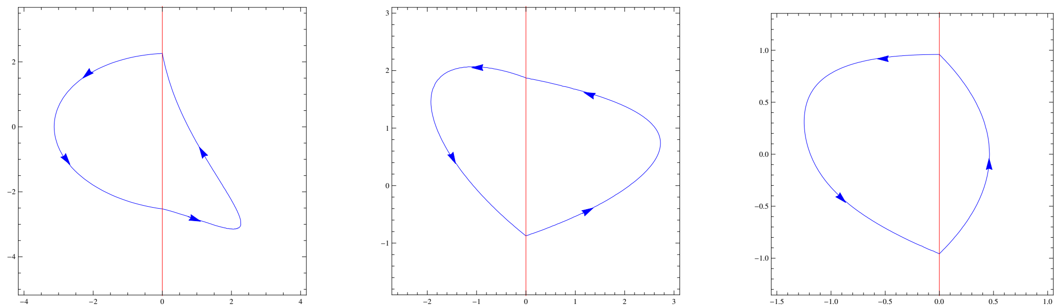
with the first integral

$$
\begin{aligned}
H_6(x,y) \;=\;& -5\,(x-5+0.1y)^4-3\,(x+0.1+5y)^2\,(x-5+0.1y)^2-(x-5+0.1y)^2\\
&-5(x+0.1+5y)^4.
\end{aligned}
$$

In Σ_2^r we consider the linear differential center

$$
\dot{x}=0.01-0.1x-1.01y, \quad \dot{y}=10+0.1x+0.1y, \tag{2.19}
$$

with the first integral $H(x,y)=(0.1x+0.1y)^2+0.2\,(-0.1x-0.1y)+0.01y^2$. Because of system (2.7) when $i=6$ has exactly one real solution namely $(y,Y)\approx(-0.95781,0.95979)$, the PWS (2.18)–(2.19) has one limit cycle drawn in Figure 2.2(c).



(a) (b) (c)

Figure 2.2: (a) Represent the unique limit cycle of the PWS (2.14)–(2.15), (b) is the unique limit cycle of the PWS (2.16)–(2.17), and (c) is the unique limit cycle of the PWS (2.18)–(2.19).

2.2 The limit cycles of the second family of PWS $\mathcal{F}_2$

Here our results devides on two partes concerning the kind of classes formed the second family of PWS $\mathcal{F}_2$.

The limit cycles of the class of PWS $(2.i)$–$(\widetilde{2.i})$ for $i \in 1,\ldots,6$

The following result gives the maximum number of limit cycles for the second family of PWS $\mathcal{F}_2$ formed by the same class of differential system in each region.

Theorem 2.2

For the second family of PWS $\mathcal{F}_2$, the maximum number of limit cycles for the class of PWS $(2.i)$–$(\widetilde{2.i})$ with $i \in 1,\ldots,6$ is at most four. Moreover all the classes attain this maximum; see Figures 2.3 and 2.4.

Proof of Theorem 2.2

Proof. *In this subsection we prove Theorems 2.2 for the class of PWS* $(2.i)$–$(\widetilde{2.i})$ *where* $i \in \{1,\ldots,6\}$.

In Σ_1^r *we consider the general cubic nilpotent center* $(2.i)$ *where* $i \in \{1,\ldots,6\}$ *with its first integral* $H_i(x,y)$. *To get the second cubic nilpotent center* $(\widetilde{2.i})$ *with its first intefral* $\widetilde{H}_i\,(x,y)$

that we consider in Σ_2^r we replace the parameters $(\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\gamma_2)$ with the parameters $(\widetilde{\alpha}_1,\widetilde{\beta}_1,\widetilde{\gamma}_1,\widetilde{\alpha}_2,\widetilde{\beta}_2,\widetilde{\gamma}_2)$ in system (2.i) and in the first integral $H_i(x,y)$. Now if there is a limit cycle of the PWS (2.i)–($\widetilde{2.i}$), it must interesects the discontinuity line Σ^r at two distinct points $p_1=(0,y)$ and $p_2(0,Y)$ where $y<Y$. Moreover the points p_1 and p_2 must satisfy the next system

$$H_i(p_1)-H_i(p_2)=h_i(y,Y)=0,\quad \widetilde{H}_i\,(p_1)-\widetilde{H}_i\,(p_2)=\widetilde{h}_i\,(y,Y)=0, \tag{2.20}$$

where $h_i(y,Y)$ and $\widetilde{h}_i\,(y,Y)$ are cubic polynomials for all $i\in\{1,\ldots,6\}$. By to Bézout Theorem (see for instance [31]), the maximum number of the solutions of system (2.20) is at most nine. Because of the symmetry of the solutions of this system we know that the maximum number of the solutions of system (2.20) satisfying $y<Y$ is at most four. Then there are at most four limit cycles for the PWS (2.i)–($\widetilde{2.i}$) with $i\in\{1,\ldots,6\}$. ■

Illustrative examples for the class of PWS (2.i)–($\widetilde{2.i}$) for $i\in 1,\ldots,6$

Example of four limit cycles for the PWS (2.1)–($\widetilde{2.1}$). In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x}\approx\;& 0.14862x^3+x^2(-0.20045y-0.03758)+y(-(0.0075963+0.013504y)y\\&-0.755137)+x(0.214281+(0.033794+0.0901174y)y)+0.725322,\\ \dot{y}\approx\;& 0.33059x^3+x^2(-0.4458y-0.083602)+y((-0.030039y-0.016897)y\\&-0.214281)+x((0.07517+0.200453y+0.0661793x)y)+0.202988,\end{aligned}\tag{2.21}$$

with its first integral

$$\begin{aligned}H_1(x,y)\;\approx\;& -x^3(0.0345268+0.184143y)+0.1024x^4+x^2((0.0465664+0.124177y)y\\&+0.0409968)+x(y(-0.265486-(0.0209348+0.0372174y)y)+0.43251\\&+0.251494)+y(0.467793-0.898647x+y((0.0031372+0.00418293y)y)).\end{aligned}$$

In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x}=\;&(1/1883)(5(0.7(442x+91-224y)+0.16(70x-3-16y)-7(0.4(-7x+0.4y\\&+0.3)^3+49(3x-7+7y)))),\\ \dot{y}=\;&(5/26)(-7(-7x+0.3+0.4y)^3+21(3x-7+7y)-35x+0.2+49(1-y)+3y),\end{aligned}\tag{2.22}$$

that has the first integral

$$\begin{aligned}\widetilde{H_1}(x,y) &= (7x-0.4y)^4+0.54\,(7x-0.4y)^2+1.2\,(0.4y-7x)^3+0.108\,(1.6y-7x)+8.1\;10^{-3}\\ &\quad +14(3x+7y-7)^2-0.4(70x-3-16y)(3x-7+7y)+(1/350)(-70x+3+16y)^2.\end{aligned}$$

For system (2.20) with $i=1$, the four real solutions of satisfying $y<Y$ which provide four limit cycles for the PWS (2.21)–(2.22) shown in Figure 2.3(a) are collected in the set $S_{1,1}$ given by

$$\begin{aligned}S_{1,1}\approx\ &\{(0.420187,1.44392),(0.496882,1.3699),(0.589761,1.27971),\\ &(0.718353,1.15381)\}.\end{aligned}$$

Example of four limit cycles for the PWS (2.2)–($\widetilde{2.2}$). In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} =\ & (1/0.0204)(-44800x^3+960x^2(9887+3190y)-60x(220y(3965+1199y)\\ &-5617)+35727973+2y(4609389+2420y(14907+4400y))),\\ \dot{y} =\ & (1/0.0510)(256000x^3+4800x^2(1793+70y)-480x(10y(9887+1595y)\\ &+1331)+50y(110y(11895+2398y)-16851)-9154111),\end{aligned}\tag{2.23}$$

with its first integral

$$\begin{aligned}H_2(x,y) &= -(0.1x+0.449+y)(-0.4x+0.01+1.1y)^3-(0.1x+0.449+y)^2-0.25\times10^{-8}\\ &\quad (-40x+1+110y)^2-10^{-6}(40x-1-110y)(10x+449+100y).\end{aligned}$$

In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x}\approx\ & 0.276953x^3+x^2(5.21482+1.68257y)+y((0.635998+0.187723y)y\\ &+0.0812631)+x(y(15.8565+2.61772y)+23.5608)+0.31494,\\ \dot{y}\approx\ & -0.122318x^3+x^2(-2.59563-0.830858y)+x((-10.4296-1.68257y)\\ &y-16.2432)+y(-23.5608+(-7.92823-0.872574y)y)-22.7062,\end{aligned}\tag{2.24}$$

which has the first integral

$$\begin{aligned}\widetilde{H_2}(x,y) &\approx -0.0135x^4-x^3(0.381968+0.122267y)+x^2(-(2.30221+0.371406y)\\ &\quad y-3.58548)-x(10.4015+y((3.50011+0.385219y)y)-10.0242)\end{aligned}$$

$$-1.17785 + y(y(-(0.0935921 + 0.0207187y)y - 0.0179378) - 0.139038).$$

The four real solutions of system (2.20) with $i = 2$ satisfying $y < Y$, which provide four limit cycles for the PWS (2.23)–(2.24) drawn in Figure 2.3(b) are collected in the set $S_{2,2}$ where

$$\begin{aligned} S_{2,2} \approx \ & \{(-4.27681, -2.06171), (-4.16402, -2.32352), (-4.02224, -2.59599), \\ & (-3.81634, -2.91886)\}. \end{aligned}$$

Example of four limit cycles for the PWS (2.3)-($\widetilde{2.3}$)**.** In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} = \ & \frac{10^{-2}}{23}(-x(154871 + 240y(1634 + 765y)) - 10(y(1350y(173 + 94y) + 69677) \\ & -21159) - x^2(5493 - 600y) + 231x^3), \\ \dot{y} = \ & \frac{10^{-3}}{23}(174083 + x(-60y(100y - 1831)) + 10(y(240y(817 + 255y) + 154871) \\ & +268) - 42x^2(307 + 165y) + 215x^3), \end{aligned} \tag{2.25}$$

with its first integral

$$\begin{aligned} H_3(x,y) \ = \ & (-0.1x + 3y + 4)^4 - 4(-0.1x + 4 + 3y)^2 + 0.04(30y - x + 40)(6x + 50y \\ & -1) + 0.01(-6x + 1 - 50y)^2 - 0.6 \times 10^{-3}(x - 40 - 30y)^2(6x - 1 + 50y)^2. \end{aligned}$$

In Σ_2^r we consider the cubic nilpotent center

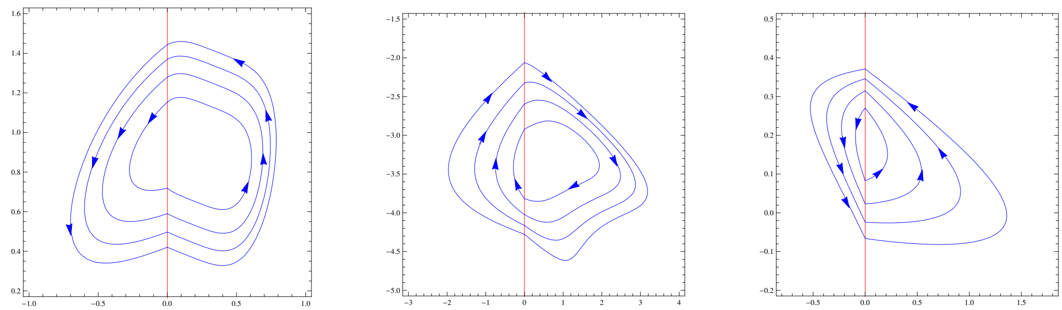
$$\begin{aligned} \dot{x} \approx \ & -0.00706076x^3 + x^2(0.347189 + 0.366337y) + y(-(31.7738 + 17.2644y) \\ & y - 9.47936) + x(-(4.23999 + 0.320759y)y - 3.06582) + 2.87862, \\ \dot{y} \approx \ & 0.0028674x^3 - x^2(0.031237 - 0.0211823y) + x((-0.694378 - 0.366337y) \\ & y + 0.40340) - 1.70916 + y(3.06582 + (2.11999 + 0.10692y)y), \end{aligned} \tag{2.26}$$

which has the first integral

$$\begin{aligned} \widetilde{H}_3(x,y) \ \approx \ & -x^3(0.00123121y - 0.00181564) + x^2((0.0605407 + 0.0319398y) \\ & y - 0.0351717) - 0.000125x^4 - 0.894789 + x(y(-(0.369671 + 0.018644y)y) \\ & +0.298033) - y(0.82647y + ((1.84684 + 0.752614y)y6) + 0.501956). \end{aligned}$$

The four real solutions of system (2.20) with $i = 3$ satisfying $y < Y$, which provide four limit cycles for the PWS (2.25)–(2.26) seen in Figure 2.3(c) are the set $S_{3,3}$ given by

$$\begin{aligned} S_{3,3} \approx \; & \{(-0.066374, 0.37132), (-0.024393, 0.34618), (0.022844, 0.31494), \\ & (0.082568, 0.27046)\}. \end{aligned}$$



(a) (b) (c)

Figure 2.3: (a) Represent the four limit cycles of the PWS (2.21)–(2.22), (b) is the four limit cycles of the PWS (2.23)–(2.24), and (c) is the four limit cycles of the PWS (2.25)–(2.26).

Example of four limit cycles for the PWS (2.4)–($\widetilde{2.4}$). In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} = \; & (10^{-2}/81)(100x(-368 - 39y(2 - 17y)) - 2000(y(y(39 + 22y) + 44) + 12) \\ & -4744x^3 - 15x^2(6574 + 6097y)), \\ \dot{y} = \; & (10^{-2}/405)(25x(3y(13148 + 6097y) + 53180) - 500(y(13y(-3 + 17y) - 368) \\ & -1308) + 60x^2(1186y + 1231) + 2401x^3), \end{aligned} \tag{2.27}$$

with its first integral

$$H_4(x, y) \;=\; -0.25\,(0.1x + 2 + 2y)^4 - 2.5(-4x - 2 + y)^2 - 0.00375(-4x - 2 + y)^2(20(1 + y) + x)^2.$$

In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} \approx \; & x^2(-9.24004y - 2.27664) - 4.13647x^3 + x((-5.43808 - 7.40128y) \\ & y - 3.06562) + y(-4.59542 + (-2.29771y - 4.07322)y) - 1.2533, \\ \dot{y} \approx \; & +x^2(2.746 + 12.4094y)5.63647x^3 + y(y(2.71904 + 2.46709y) \\ & +3.06562) - 4.77918 + x(y(4.55327 + 9.24004y) + 6.59598), \end{aligned} \tag{2.28}$$

which has the first integral

$$\widetilde{H}_4(x,y) \approx -0.375(x+y+0.685221)^2(-x+1.05753-0.11292y)^2-1.25 \\ (-x+1.05753-0.11292y)^2-0.25(x+0.685221+y)^4.$$

The four real solutions of system (2.20) with $i=4$ satisfying $y<Y$, which provide four limit cycles for the PWS (2.27)–(2.28) shown in Figure 2.4(a) are the set $S_{4,4}$ given by

$$S_{4,4} \approx \{(-2.82875, 1.71996), (-2.59836, 1.50325), (-2.26931, 1.20032), \\ (-1.60171, 0.62322)\}.$$

Example of four limit cycles for the PWS (2.5)–($\widetilde{2.5}$). In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} = {} & (10^{-3}/15)(-135x^2(431+5885y)+9x(-64985y(8606+30775y))-75y^2 \\ & (4819+10375y)+4974y+128925x^3+28448), \\ \dot{y} = {} & (10^{-3}/5)(135(321-955y)x^2+3x(15y(862+5885y)-18014)+y(19494 \\ & -5y(12909+30775y))-2025x+16528), \end{aligned} \tag{2.29}$$

with its first integral

$$\begin{aligned} H_5(x,y) = {} & (10^{-3}/5)(6075x^4+540x^3(-321+955y)-18x^2(15y(5885y+862)-18014) \\ & +12x(y(5y(30775y+12909)-19494)-16528)+y(y(9948-25y(31125y \\ & +19276))+113792)). \end{aligned}$$

In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} \approx {} & x^2(-1.32335y+0.267014)+0.0626179x^3+y(y(-17.692-38.0898y) \\ & +0.243481)+1.39255+x(y(-0.669421+11.0206y)-0.110061), \\ \dot{y} \approx {} & y(y(-3.67353y+0.334711)+0.110061)+0.00478179x^3+x^2(-0.187854y \\ & -0.0586411)-2.13677+x(y(-0.534028+1.32335y)-0.892372), \end{aligned} \tag{2.30}$$

which has the first integral

$$\widetilde{H}_5(x,y) \approx x^2(y(0.112573-0.278961y)+0.188112)-0.000504x^4+x^3(0.0263997$$

$$y+0.00824102)+x(y(y(-0.141114+1.54876y)-0.0464015)+1.9306$$
$$+0.900861)+y(y(y(-2.48632-4.01466y)+0.0513258)+0.587099).$$

The four real solutions of system (2.20) with $i=5$ satisfying $y<Y$, which provide four limit cycles for the PWS (2.29)–(2.30) shown in Figure 2.4(b) are the set $S_{5,5}$ given by

$$\begin{aligned} S_{5,5} \approx\ & \{(-0.31212, 0.45996), (-0.14074, 0.43160), (-0.020105, 0.39546), \\ & (0.092971, 0.34164)\}. \end{aligned}$$

Example of four limit cycles for the PWS (2.6)–($\widetilde{2.6}$). In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} =\ & x(77y^2+(1381y/25)+(1026.81/55))+1/49500(y(220y(6072+6905y) \\ & +945963)+156374)+24x^3+3/25x^2(590y+257), \\ \dot{y} =\ & (1/0.033)(-660(1381+3540x)y^2-7920x(257+300x)y-847000y^3 \\ & -864x(932+55x(29+20x))-(148411+616086y), \end{aligned} \tag{2.31}$$

with its first integral

$$\begin{aligned} H_6(x,y) =\ & (10^{-4}/44)(-12509920y-2471451+40(-54x(220x(200x+257)+34227)y \\ & -4840(525x+184)y^3-3x(288x(55x(15x+29)+1398)+148411)-9 \\ & (220x(1770x+1381)+105107)y^2-759550y^4)). \end{aligned}$$

In Σ_2^r we consider the cubic nilpotent center

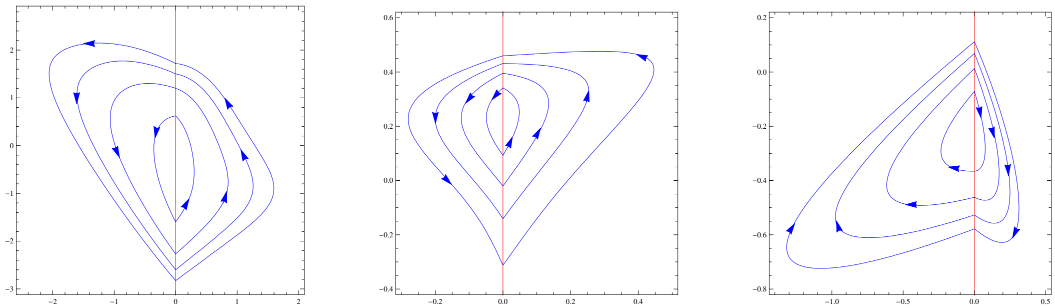
$$\begin{aligned} \dot{x} \approx\ & -7.88317x^3+x^2(23.9206+71.1577y)+x((-135.413-214.918y)y \\ & -46.5053)+22.3614+y(y(191.024+217.23y)+135.272), \\ \dot{y} \approx\ & -2.63012x^3+x^2(8.4588+23.6495y)+x((-47.8411-71.1577y)y \\ & -16.5638)+11.4369+y(y(67.7066+71.6392y)+46.5053), \end{aligned} \tag{2.32}$$

which has the first integral

$$\begin{aligned} \widetilde{H_6}(x,y) \approx\ & -0.250036x^4+x^3(1.07219+2.99769y)+x^2((-9.09615-13.5294y)y \\ & -3.14931)-8.26225+x(y(y(25.7465+27.2419y)+17.6843)) \\ & +4.34905+y(y((-24.2133-20.6513y)y-25.7196)-8.50325). \end{aligned}$$

The four real solutions of system (2.20) with $i = 6$ satisfying $y < Y$, which provide four limit cycles for the PWS (2.31)–(2.32) shown in Figure 2.4(c) are the set $S_{6,6}$ given by

$$S_{6,6} \approx \{(-0.57830, 0.11190), (-0.52717, 0.068607), (-0.46202, 0.01295), (-0.3659, -0.07126)\}.$$



(a) (b) (c)

Figure 2.4: (a) Represent the four limit cycles of the PWS (2.27)–(2.28), (b) is the four limit cycles of the PWS (2.29)–(2.30), (c) is the four limit cycles of the PWS (2.31)–(2.32).

The limit cycles of the class of PWS $(2.i)$–$(2.j)$, with $i, j \in \{1, ..., 6\}$ and $i \neq j$

The next result gives the maximum number of limit cycles for the second family of PWS $\mathcal{F}_2$ formed by two different cupic nilpetent centers.

Theorem 2.3

For the second family of PWS $\mathcal{F}_2$, the maximum number of limit cycles for the class of PWS $(2.i)$–$(2.j)$ with $i, j = 1, \ldots, 6$ and $i \neq j$ is at most four. Moreover all the classes attain this maximum; see Figures 2.5, 2.6, 2.7, 2.8 and 2.9.

Proof of Theorem 2.3

Proof. *In this part we prove Theorem 2.3 for the class of PWS $(2.i)$–$(2.j)$ where $i, j \in \{1, ..., 6\}$ and $i \neq j$.*

For the class $(2.i)$–$(2.j)$ with $i, j \in \{1, ..., 6\}$ and $i \neq j$, we consider in the first region the cubic nilpotent center $(2.i)$ with its first integral $H_i(x, y)$. In the second region we consider the cubic *nilpotent center* (2.j) *with its first integral* $H_j(x,y)$. *If the PWS* (2.i)–(2.j) *has a limit cycle, this limit cycle must cross the discontinuity line* Σ^r *at a pair of distinct points* $p_1 = (0,y)$ *and* $p_2(0,Y)$ *where* $y < Y$. *Moreover the points* p_1 *and* p_2 *must satisfy the next system*

$$H_i(p_1) - H_i(p_2) = h_i(y,Y) = 0, \quad H_j(p_1) - H_j(p_2) = h_j(y,Y) = 0, \tag{2.33}$$

where $h_i(y,Y)$ *and* $h_j(y,Y)$ *are cubic polynomials. Based on the symmetry of that system's solutions and the Bézout Theorem, the maximum number of solutions of system* (2.33) *that satisfy* $y < Y$ *is at most four. Thus there are a maximum of four limit cycles for the PWS* (2.i)–(2.j) *with* $i,j \in \{1,\ldots,6\}$ *and* $i \neq j$. ■

Illustrative examples for the PWS (2.i)–(2.j) with $i,j \in \{1,\ldots,6\}$ and $i \neq j$

Example of four limit cycles for the PWS (2.1)-(2.2)**.** In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} = {} & (1/8676)(25(\frac{10^{-3}}{2}(31(600x-155y-34)) - \frac{36}{5}(\frac{3.1}{2}\left(-6x+\frac{3.1y}{2}+\frac{1.7}{5}\right)^3 \\ & +\frac{126}{25}(40x-71+70y)) + \frac{9}{50}(1432x+345-868y))), \\ \dot{y} = {} & \frac{1}{1446}(5(\frac{-6}{25}(1200x+895y-1099) - 6(6\left(-6x+\frac{3.1y}{2}+\frac{1.7}{5}\right)^3 \\ & -\frac{72}{25}(40x+70y-71)) + 30x - \frac{31y}{4} - 1.7,\end{aligned} \tag{2.34}$$

of type (2.1) with its first integral

$$\begin{aligned}H_1(x,y) = {} & \left(6x-\frac{3.1y}{2}\right)^4 + \frac{867}{125}\left(6x-\frac{31y}{20}\right)^2 + \frac{34}{25}\left(\frac{31y}{20}-6x\right)^3 + \frac{491.3}{3125}\left(\frac{31y}{20}-6x\right) \\ & +\frac{8.3521}{625} + \frac{5}{18}\left(-6x+\frac{31y}{20}+\frac{17}{50}\right)^2 + \frac{18}{125}(40x+70y-71)^2 - \frac{1}{250}(600x \\ & -155y-34)(40x+70y-71).\end{aligned}$$

In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} \approx {} & 0.0300114x^3 + x^2(-1.26886y+1.19984) + y((-0.0705721-0.107242y) \\ & y-6.97976) + x(y(13.4167y-25.3561)+10.6326) + 6.80631, \\ \dot{y} \approx {} & x^2(-0.0900341y+0.0851568) + y(y(-4.47222y+12.678)-10.6326)\end{aligned} \tag{2.35}$$

$$+0.00189308x^3+2.48977+x(y(-2.39968+1.26886y)+1.3799),$$

of type (2.2) with the first integral

$$\begin{aligned} H_2(x,y) \approx\ & x^2(y(-67.0262y+126.761)-72.8918)-0.05x^4+x^3(-2.99889+3.17064y) \\ & -345.933+x(y(y(-1339.41+472.482y)+1123.31)-263.039)+y(y \\ & ((-2.48527-2.83248y)y-368.699)+719.073). \end{aligned}$$

The four real solutions of system (2.33) with $i=1$ and $j=2$ satisfying $y<Y$, which provide four limit cycles for the PWS (2.34)–(2.35) shown in Figure 2.5(a) are the set $S_{1,2}$ given by

$$\begin{aligned} S_{1,2} \approx\ & \{(0.432123, 1.46424), (0.500649, 1.39791), (0.581365, 1.31939), \\ & (0.684976, 1.21799)\}. \end{aligned}$$

Example of four limit cycles for the PWS (2.1)-(2.3). In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} \approx\ & 0.0000231195x^3+x^2(-0.00667458-0.00693586y)+y((-23.1195y \\ & -66.7458)y-95.7292)-1.47427+x((1.33492+0.693586y)y-0.693397), \\ \dot{y} \approx\ & 2.3119546737227093-7x^3+x^2(-0.0000667458-0.0000693586y) \\ & +y(0.693397+(-0.667458-0.231195y)y)-1.01726+x((0.00693586y \\ & +0.0133492)y+0.063066), \end{aligned} \tag{2.36}$$

of type (2.1) with the first integral

$$\begin{aligned} H_1(x,y) \approx\ & -2.5-9(-x+96.2329+100y)^4-0.000625(-x+100y+96.2329)^2)-0.005 \\ & (x-96.2329-100y)(x+190.456+332.534y-0.1(x+190.456+332.534y)^2. \end{aligned}$$

In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} =\ & \frac{1}{280}(3x^2(983-468y)+3x\left(6200y^2-3425+5812y\right)-2(y(50y(843+292y) \\ & +60453)+931)-298x^3), \\ \dot{y} =\ & \frac{1}{560}(-46x^3+3x^2(283+596y)+x(12y(-983+234y)-4171)-4y^2(3100y \\ & +4359)+6594+20550y), \end{aligned} \tag{2.37}$$

of type (2.3) with the first integral

$$
\begin{aligned}
H_3(x,y) \;=\;& (-(3x-1+50y)^2+4(-2(2+y)+x)(3x-1+50y)-16(-2(2+y)+x)^2\\
& 4(-2(2+y)+x)^4-(12/50)(3x-1+50y)^2(-2(2+y)+x)^2).
\end{aligned}
$$

The four real solutions of system (2.33) with $i=1$ and $j=3$ satisfying $y<Y$, which provide four limit cycles for the PWS (2.36)–(2.37) seen in Figure 2.5(b) are the set $S_{1,3}$ given by

$$
\begin{aligned}
S_{1,3} \approx\;& \{(-0.18047, 0.13754), (-0.15244, 0.11308), (-0.11784, 0.08204),\\
& (-0.064769, 0.03252)\}.
\end{aligned}
$$

Example of four limit cycles for the PWS (2.1)-(2.4). In Σ_1^r we consider the cubic nilpotent center

$$
\begin{aligned}
\dot{x} =\;& (10^{-5}/4890)(32574287591-34036948070y+800(49432500x^3-228150\\
& x^2(37+170y)+13x(9180y(85y+37)-1786537)-5202y^2(111+170y))),\\
\dot{y} =\;& (10^{-4}/7524)(-298134377+303711290y+50(49432500x^3-228150x^2\\
& (37+170y)+13x(9180y(37+85y)+320183)-5202y^2(111+170y))),
\end{aligned}
\tag{2.38}
$$

of type (2.1) with the first integral

$$
\begin{aligned}
H_1(x,y) \;=\;& \frac{10^{-8}}{36}(-1302971503640y+625767092449+200(8032781250x^4-49432500\\
& x^3(37+170y)+4225x^2(9180y(37+85y)+320183)+y^2(442170y(74+85y)\\
& +3403694807)-13x(170y(1530y(111+170y)-1786537)+298134377))).
\end{aligned}
$$

In Σ_2^r we consider the cubic nilpotent center

$$
\begin{aligned}
\dot{x} \approx\;& x^2(-6.08187y+6.84222)+x(y(-0.416004y+0.193773)-0.804278)\\
& +0.27556-0.787102x^3+y((-0.00415197-0.00635887y)y-0.287934),\\
\dot{y} \approx\;& x^2(2.36131y-1.18342)+8.13871x^3+y((-0.0968865+0.138668y)y\\
& +0.804278)-0.740904+x(y(-13.6844+6.08187y)+12.0021),
\end{aligned}
\tag{2.39}
$$

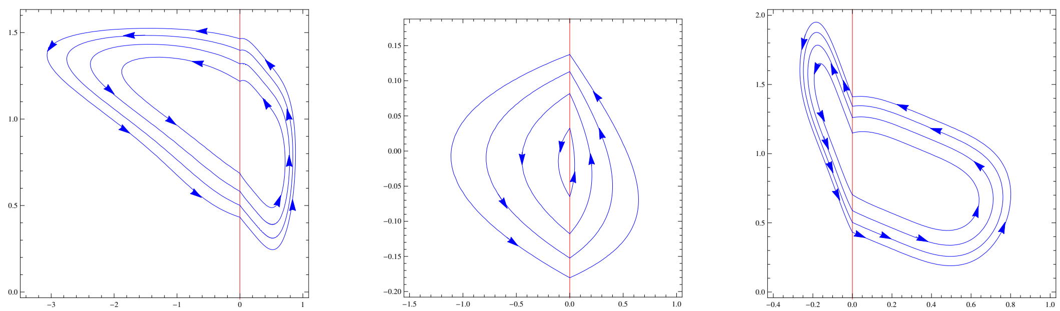
of type (2.4) with the first integral

$$
H_4(x,y) \;\approx\; -4.06(x^4+x^3(-0.193876+0.386844y)+x^2(y(-3.3628+1.49455y)+2.9494)
$$

$+0.06387 + x(y((-0.0476176 + 0.0681523y)y + 0.395285) - 0.364138) + y(y$
$((0.000680201 + 0.000781312y)y + 0.0707566) - 0.135432).$

The four real solutions of system (2.33) with $i = 1$ and $j = 4$ satisfying $y < Y$, which provide four limit cycles for the PWS (2.38)–(2.39) drawn in Figure 2.5(c) are the set $S_{1,4}$ given by

$$\begin{aligned} S_{1,4} \approx \quad & \{(0.43188, 1.41107), (0.502868, 1.34301), (0.588021, 1.26079), \\ & (0.702739, 1.14902)\}. \end{aligned}$$



(b) (b) (a)

Figure 2.5: (a) Represent the four limit cycles of the PWS (2.34)–(2.35), (b) is the four limit cycles of the PWS (2.36)–(2.37), and (c) the four limit cycles of the PWS (2.38)–(2.39).

Example of four limit cycles for the PWS (2.1)-(2.5)**.** In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} = \quad & (1/8151)(2(0.17(650x - 37 - 170y) - 75(17 \times 10^{-7}(-650x + 37 + 170y)^3 \\ & +(21/2)(26x - 25 + 35y)) + (3/4)(3666x + 591 - 2380y))), \\ \dot{y} = \quad & (10^{-4}/2508)(50(-76050x^2 16477500x^3(37 + 170y) + 13x(3060y(37 + 85y) \\ & +112321) - 1734y^2(111 + 170y)) - 74411959 + 106305430y), \end{aligned} \tag{2.40}$$

of type (2.1) with the first integral

$$\begin{aligned} H_1(x,y) \quad = \quad & (10^{-8}/12)(-321888027880y + 110296574483 + 200(2677593750x^4 - 164775 \\ & x^3(37 + 170y) + 4225x^2(3060y(37 + 85y) + 112321) + y^2(147390y(74 + 85y) \end{aligned}$$

$$+1178616769) - 13x(10y(8670y(111+170y) - 10630543) + 74411959))).$$

In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} \approx\ & x^2(-11.9089y + 7.93667) + x(y(-5.95413y + 7.98447) - 0.58277) \\ & -0.696039 - 6.24628x^3 + y((0.0142604 + 0.021644y)y + 1.01937), \\ \dot{y} \approx\ & 8.60663x^3 + x^2(-12.4536 + 18.7389y) + y(y(-3.99223 + 1.98471y) \\ & +0.58277) + 0.805031 + x(y(-15.8733 + 11.9089y) + 1.17967), \end{aligned} \tag{2.41}$$

of type (2.5) with the first integral

$$\begin{aligned} H_5(x,y) \approx\ & 3.98198x^4 + x^3(-7.68244 + 11.5597y) + x^2(y(-14.688 + 11.0197y) \\ & +1.09158) - 0.4349 + x(y(y(-7.38825 + 3.67301y) + 1.07851) \\ & +1.48983) + y(y((-0.008797 - 0.0100139y)y - 0.943252) + 1.28813). \end{aligned}$$

The four real solutions of system (2.33) with $i = 1$ and $j = 5$ satisfying $y < Y$, which provide four limit cycles for the PWS (2.40)–(2.41) drawn in Figure 2.6(a) are the set $S_{1,5}$ given by

$$\begin{aligned} S_{1,5} \approx\ & \{(0.239326, 1.09438), (0.321936, 1.01403), (0.321936, 1.01403), \\ & (0.431104, 0.907126)\}. \end{aligned}$$

Example of four limit cycles for the PWS (2.1)-(2.6)**.** In Σ_1^r we consider the cubic nilpotent centerr

$$\begin{aligned} \dot{x} =\ & (1/11.32)(-80802x^2(4+15y) + 67x(-37 + 270y(8+15y)) - 5y(810y \\ & (45y) + 113089) + 662916 + 1804578x^3), \\ \dot{y} =\ & (1/56.60)(-1804578(4+15y)x^2 + x(435169 + 404010(8+15y)y) \\ & -335y(270y(4+5y) - 37) - 137276 + 40302242x^3), \end{aligned} \tag{2.42}$$

of type (2.1) with the first integral

$$\begin{aligned} H_1(x,y) =\ & (1/0.0004)(-1203052(4+15y)x^3 + 20151121x^4 + x^2(404010y(8+15y) \\ & +435169) - 2x(335y(-37 + 270y(4+5y)) + 137276) + 3920656 + 5y(5y \\ & -1325832(135y(16+15y) + 113089)). \end{aligned}$$

In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} \approx\ & x^2(-45.3121-56.2053y)+21.3589x^3+y((-16.9101-21.1376y)y\\ & -590.23)+691.974+x(y(66.1925+54.2197y)+64.3626),\\ \dot{y} \approx\ & x^2(-72.6467-64.0768y)+28.0462x^3+y((-33.0963-18.0732y)y\\ & -64.3626)+x(y(90.6242+56.2053y)+77.2486)+0.0391091,\end{aligned} \tag{2.43}$$

of type (2.6) with the first integral

$$\begin{aligned}H_6(x,y) \approx\ & -64.0648(x^3(-3.45366-3.04625y)+x^4+x^2(y(6.462484.00804y)\\ & +5.50866)+x(y((-4.72024-2.57763y)y-9.1795)+0.0055778)\\ & +65.0845+y(y((0.803915+0.75367y)y+42.0898)-98.6904).\end{aligned}$$

The four real solutions of system (2.33) with $i=1$ and $j=6$ satisfying $y<Y$, which provide four limit cycles for the PWS (2.42)–(2.43) seen in Figure 2.6(b) are the set $S_{1,6}$ given by

$$\begin{aligned}S_{1,6} \approx\ & \{(0.322324,1.81602),(0.439432,1.71116),(0.580949,1.582),\\ & (0.777225,1.39821)\}.\end{aligned}$$

Example of four limit cycles for the PWS (2.2)-(2.3). In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} =\ & (1/0.318)(-8820(2446+187x)y^2+1260x(856+67x)y+x(45x(734\\ & -27x)+5560472)-41906248y+129395280+9878400y^3),\\ \dot{y} =\ & (1/0.318)(-1260(428+67x)y^2+45x(-1468+81x)y+2x(238679\\ & -15(-249+x)x)-5560472y+549780y^3+2421420),\end{aligned} \tag{2.44}$$

of type (2.2) with the first integral

$$\begin{aligned}H_2(x,y) =\ & (10^{-6}/12)(-72962(x+30-14y)^2+22920(x-422-120y)(x+30-14y)\\ & -1800(x-422-120y)^2-15(x-422-120y)(x+30-14y)^3).\end{aligned}$$

In Σ_1^r we consider the cubic nilpotent center

$$\dot{x} \approx\ 0.0920409x^3+x^2(1.19586y+2.68903)+y((0.0174702y-0.0381536)y$$

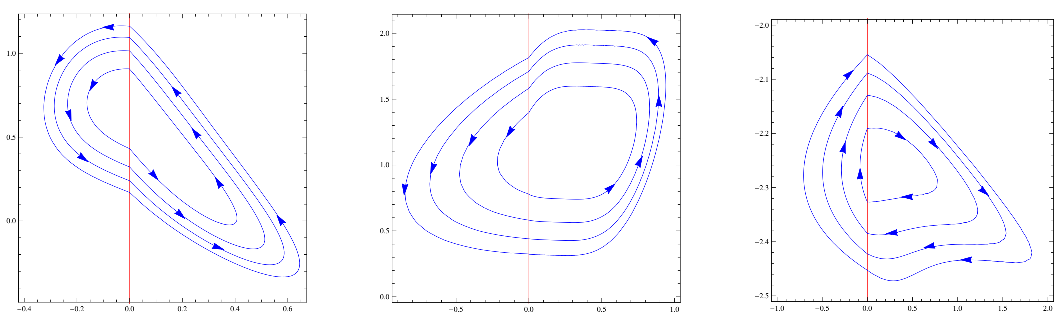
$$
\begin{aligned}
& -0.0741121)+x(y(16.5284+3.68178y)+18.5409)+0.228838, \\
\dot{y} \approx\ & +x^2(-0.621409-0.276123y)-0.0172082x^3+y((-8.26419-1.22726y) \qquad (2.45)\\
& y-18.5409)+x(-6.04679+(-5.37806-1.19586y)y)-13.8587,
\end{aligned}
$$

of type (2.2) with the first integral

$$
\begin{aligned}
H_3(x,y) \approx\ & -6(-0.01x+0.921687+0.407922y)^2(0.2x+2.24993+y)^2 \\
& +(0.2x+2.24993+y)^4+0.00004(-x+92.1687+40.7922y)^2.
\end{aligned}
$$

The four real solutions of system (2.33) with $i=2$ and $j=3$ satisfying $y<Y$, which provide four limit cycles for the PWS (2.44)–(2.45) shown in Figure 2.6(c) are the set $S_{2,3}$ given by

$$
\begin{aligned}
S_{2,3} \approx\ & \{(-2.45273,-2.05487),(-2.42253,-2.0884),(-2.38468,-2.12957), \\
& (-2.32697,-2.19059)\}.
\end{aligned}
$$



(a) (b) (c)

Figure 2.6: (a) Represent the four limit cycles of the PWS (2.40)–(2.41), (b) is the four limit cycles of the PWS (2.42)–(2.43), and (c) is the four limit cycles of the PWS (2.44)–(2.45).

Example of four limit cycles for the PWS (2.2)-(2.4). In Σ_1^r we consider the cubic nilpotent center

$$
\begin{aligned}
\dot{x} =\ & (10^{-2}/1695)(2205(-367x+2446)y^2+630x(-127x+751)y+x(2865001 \\
& -45x(94447x+1))+10476562y-32348820-2469600y^3), \qquad (2.46)\\
\dot{y} =\ & (10^{-2}/1695)(45(377+141y)x^2+x(-487498+90(944+889y)y)+y
\end{aligned}
$$

$$(105y(-2253+2569y)-2865001)+985485+60x^3),$$

of type (2.2) with the first integral

$$\begin{aligned} H_2(x,y) &= (10^{-5}/15)(-15(x+422+120y)(x-15+7y)^3-36481(x-15+7y)^2 \\ &\quad +5730(x+422+120y)(x-15+7y)-225(x+422+120y)^2). \end{aligned}$$

In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} \approx{}& x^2(-20.3414y+4.49553)+x(y(-28.7946y+30.4253)+76.3734) \\ & -174.323-4.63556x^3+56.4567+y(y(-13.3083y+29.0644)), \\ \dot{y} \approx{}& x^2(1.03313+13.9067y)+3.06778x^3+y(y(-15.2126+9.59821y) \\ & -76.3734)+x(y(-8.99106+20.3414y)-53.9556)+8.93679, \end{aligned} \tag{2.47}$$

of type (2.4) with the first integral

$$\begin{aligned} H_4(x,y) \approx{}& -1.5625x^4+x^2(54.962+y(-20.7208y+9.15875))+x^3(1-9.44405y) \\ & -786.27+x(155.596+y(y(-19.5545y+30.9927))-18.2069) \\ & -355.149+y(y(y(19.7377-6.77827y)+57.5097)). \end{aligned}$$

The four real solutions of system (2.33) with $i=2$ and $j=4$ satisfying $y<Y$, which provide four limit cycles for the PWS (2.46)–(2.47) seen in Figure 2.7(a) are the set $S_{2,4}$ given by

$$\begin{aligned} S_{2,4} \approx{}& \{(-2.45273,-2.05487),(-2.42253,-2.0884),(-2.38468,-2.12957), \\ & (-2.32697,-2.19059)\}. \end{aligned}$$

Example of four limit cycles for the PWS (2.2)-(2.5). In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} ={}& (10^{-4}/6)(9(-245(1223+680x)y^2+210x(1023+320x)y-3x(15x(200x \\ & +823)-101087)+137200y^3-5238281y+16174410)), \\ \dot{y} ={}& (1/0.02)(-315(1023+640x)y^2+270x(823+300x)y-27x(5x(80x \\ & +267)-16641)+166600y^3+1881630-909783y), \end{aligned} \tag{2.48}$$

of type (2.2) with the first integral

$$H_2(x,y) = (10^{-5}/15)((-30(20x-211-60y)(3x+15-7y)^3-36481(3x+15-7y)^2 +11460(20x-211-60y)(3x+15-7y)-900(-20x+211+60y)^2).$$

In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} \approx\ & x^2(-2.68368y+13.6637)+x(-5181.26+y(-277.103y+2198.16)) \\ & +549164-0.0123584x^3+y(-177854+y(-91560.6+41924.7y)), \\ \dot{y} \approx\ & x^2(0.176651+0.0370751y)-0.000276416x^3+x(y(2.68368y-27.3275) \\ & -29.9689)+y(y(92.3677y-1099.08)+5181.26)+427.069,\end{aligned} \tag{2.49}$$

of type (2.5) with the first integral

$$\begin{aligned}H_5(x,y) \approx\ & -0.0006x^4+x^3(0.511262+0.107302y)+x^2(y(-118.636+11.6506y) \\ & -130.103)+1.3308\times 10^6+y(y(772111+y(-91003.4y+264993)) \\ & -4.76815\times 10^6)+x(y(44986.6+y(-9542.81+801.987y))+3708.05.\end{aligned}$$

The four real solutions of system (2.33) with $i=2$ and $j=5$ satisfying $y<Y$, which provide four limit cycles for the PWS (2.48)–(2.49) seen in Figure 2.7(b) are the set $S_{2,5}$ given by

$$\begin{aligned}S_{2,5} \approx\ & \{(-2.45273,-2.05487),(-2.42253,-2.0884),(-2.38468,-2.12957), \\ & (-2.32697,-2.19059)\}.\end{aligned}$$

Example of four limit cycles for the PWS (2.2)-(2.6). In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} =\ & (1/365.25)(-360x^2(2413+3311y)-22032x^3+x(1575y(-10031y+4958) \\ & +59419145)-50(y(16174410+2205y(-1223+560y)-5238281))), \\ \dot{y} =\ & (1/1826.25)(720x^2(459y+1733)-768x^3+10x(180y(4826+3311y) \\ & -3991259)+25(2815815+y(-11883829+105y(10031y-7437)))),\end{aligned} \tag{2.50}$$

of type (2.2) with the first integral

$$H_2(x,y) = (1/0.01875)(-182405(4x-75+35y)^2-28650(x-2110-600y)(4x-75$$

$$+35y) - 1125(x - 2110 - 600y)^2 + 3(x - 2110 - 600y)(4x - 75 + 35y)^3.$$

In Σ_2^r we consider the cubic nilpotent center

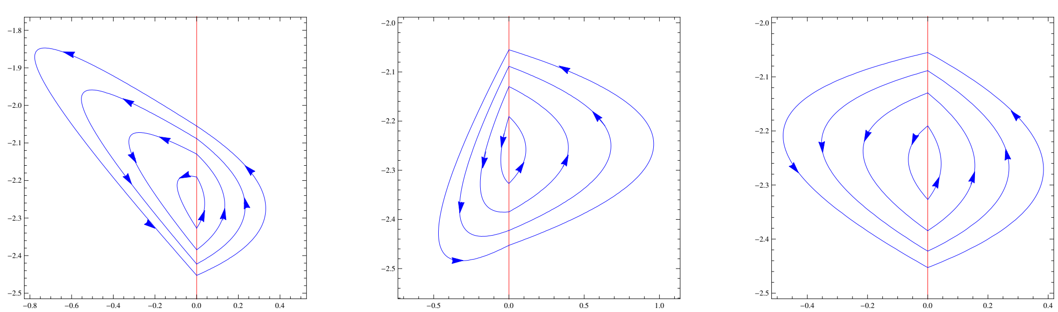
$$\begin{aligned}
\dot{x} \approx\ & -178.362x^3 + x(15822.9 + y(-3239.16y + 1093.52)) + x^2(-2024.74y \\
& -3067.91) - 18986.8 + y(6149.1 + y(-1449.5y + 3165.61)), \\
\dot{y} \approx\ & x^2(835.977 + 535.086y) + 45.3362x^3 + y(y(-546.759 + 1079.72y) \\
& -15822.9) + x(4536.54 + y(6135.83 + 2024.74y)) - 21574.7,
\end{aligned} \tag{2.51}$$

of type (2.6) with the first integral

$$\begin{aligned}
H_6(x,y) \approx\ & -531.5(x^3(24.586 + 15.7368y) + x^4 + x^2(y(270.681 + 89.3211y) \\
& +200.129) + x(y(-1396.05 + y(-48.2404 + 95.2634y)) - 1903.53) \\
& +y(1675.2 + y(-271.267 + y(-93.1004 + 31.9723y)))).
\end{aligned}$$

The four real solutions of system (2.33) with $i = 2$ and $j = 6$ satisfying $y < Y$, which provide four limit cycles for the PWS (2.50)–(2.51) seen in Figure 2.7(c) are the set $S_{2,6}$ given by

$$\begin{aligned}
S_{2,6} \approx\ & \{(-2.45273, -2.05487), (-2.42253, -2.0884), (-2.38468, -2.12957), \\
& (-2.32697, -2.19059)\}.
\end{aligned}$$



(a) (b) (c)

Figure 2.7: (a) Represent the four limit cycles of the PWS (2.46)–(2.47), (b) is the four limit cycles of the PWS (2.48)–(2.49), and (c) is the four limit cycles of the PWS (2.50)–(2.51).

Example of four limit cycles for the PWS (2.3)-(2.4). In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} = &\ (1/1.575)(43875x^2(145+188y)17820x^3 - x(5400y(2741+435y)\\ &+12496) - 20(y(23084027 - 3511546 + 4500y(8511+2920y)))),\\ \dot{y} = &\ (1/0.00315)(-267300x^2(353+40y) + 4787100x^3 + x(426674243 - 1755\\ &\ y(145+94y)) + 20(y(124969919 + 27000y(2741+290y)) + 3839138)),\end{aligned} \tag{2.52}$$

of type (2.3) with the first integral

$$\begin{aligned}H_3(x,y) \ = \ &\ 0.25(-0.3x+4+2y)^4 - 0.9(-0.3x+4+2y)^2 + 2(x-0.01+5y)(-0.3x+4+2y)\\ &-1.5(-0.3x+4+2y)^2(x-0.001+5y)^2 - 0.9(x-0.01+5y)^2.\end{aligned}$$

In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} \approx &\ x^2(1.29997+0.385691y) - 0.0202932x^3 + y((-266.672 - 91.4913y)y\\ &-160.73) + x(y(-6.95693+4.25192y) - 20.8016) + 24.4502,\\ \dot{y} \approx &\ y(20.8016 + y(-1.41731y+3.47847)) + 0.00535288x^3 + x^2(0.0608796y\\ &-0.227359) + x((-2.59993 - 0.385691y)y + 3.33596) - 20.3172,\end{aligned} \tag{2.53}$$

of type (2.4) with the first integral

$$\begin{aligned}H_4(x,y) \ \approx \ &\ (-0.0001(-x+6.19385-15y)^4 - 0.4(-x+22.7859+8.35192y)^2\\ &-0.0024(-x+6.19385-15y)^2(-x+22.7859+8.35192y)^2).\end{aligned}$$

The four real solutions of system (2.33) with $i=3$ and $j=4$ satisfying $y<Y$, which provide four limit cycles for the PWS (2.52)–(2.53) shown in Figure 2.8(a) are the set $S_{3,4}$ given by

$$\begin{aligned}S_{3,4} \approx \ &\{(-0.10950, 0.31947), (-0.071226, 0.29253), (-0.026675, 0.25905),\\ &(0.032266, 0.21093)\}.\end{aligned}$$

Example of four limit cycles for the PWS (2.3)-(2.5). In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} = &\ (1/0.065)(15x^2(-7217+3000y) + 5150x^3 - 3x(1600y(2749+1675y)\\ &+1332199) - 20(-323519 + 2y(200y(13773+10000y) + 570303))),\\ \dot{y} = &\ (10^{-5}/26)(-150x^2(593+4120y) + 14900x^3 + x(1200y(-1500y+7217)\end{aligned} \tag{2.54}$$

$$+1129403)+40(y(3996597+800y(8247+3350y))+154122)),$$

of type (2.3) with the first integral

$$H_3(x,y) \quad = (-0.1x+4+4y)^4 - 6\left(0.5x-0.01+6y\right)^2\left(-0.1x+4+4y\right)^2 + 2(0.5x-0.01+6y)^2.$$

In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} \approx & \ x^2(-2560.18-4227.15y)130.423x^3+y((-195146-141687y)y \\ & -40402.3)+x(7549.92+y(46247.9+43422.8y))+11459.6, \\ \dot{y} \approx & \ x^2(-270.532-391.268y)+11.3396x^3+y((-23124-14474.3y)y \\ & -7549.92)+x(1034.18+y(5120.37+4227.15y))+202.374, \end{aligned} \tag{2.55}$$

of type (2.5) with the first integral

$$\begin{aligned} H_5(x,y) \quad \approx \quad & x^3(4091.62+5917.67y)-128.627x^4+x^2(-23461.8+(-116163-95899.2 \\ & y)y)+229911+x(y(342563+y(1.0492\times 10^6+656741y))-9182.31)+y(y \\ & (-916588+(-2.9514\times 10^6-1.60719\times 10^6 y)y)+519958). \end{aligned}$$

The four real solutions of system (2.33) with $i=3$ and $j=5$ satisfying $y<Y$, which provide four limit cycles for the PWS (2.54)–(2.55) seen in Figure 2.8(b) are the set $S_{3,5}$ given by

$$\begin{aligned} S_{3,5} \approx \quad & \{(-0.1288, 0.34684), (-0.070429, 0.31926), (-0.0076186, 0.28364), \\ & (0.073805, 0.22667)\}. \end{aligned}$$

Example of four limit cycles for the PWS (2.3)-(2.6). In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} = \quad & (10^2/43)(-6x(35837+5y(19508+9675y))-20(y(135y(719+430y) \\ & +8137)-20774)+516x^3-15x^2(1021+301y)), \\ \dot{y} = \quad & (10^2/215)(3x(25y(2042+301y)+54561)+10(3y(5y(9754+3225y) \\ & +35837)+3628)-6x^2(2149+1290y)+215x^3), \end{aligned} \tag{2.56}$$

of type (2.3) with the first integral

$$
\begin{aligned}
H_3(x,y) \quad &= \quad 0.25\,(-0.1x+4+3y)^4 - 1.5\,(1.5x-0.1+3.5y)^2\,(-0.1x+4+3y)^2 \\
&\quad\quad + (0.6x-0.1+3.5y)^2 .
\end{aligned}
$$

In Σ_2^r we consider the cubic nilpotent center

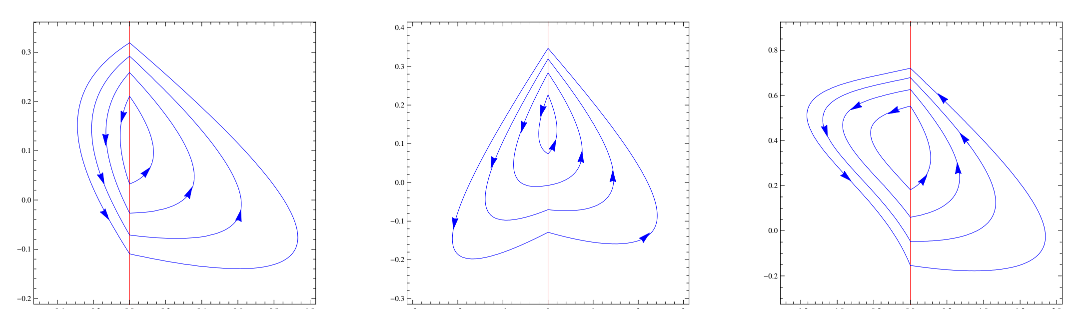
$$
\begin{aligned}
\dot{x} \approx \quad & -108.667x^3 + x^2(-828.334y + 815.182) + y((-5631.92 - 3368.19y)y \\
& -472.126) + x(y(1656.62 + 3908.52y) - 1781.38) + 1205.35, \\
\dot{y} \approx \quad & x^2(-294.748326y) + 38.3667x^3 + y(y(-828.31 - 1302.84y) \\
& +1781.38) + x(y(-1630.36 + 828.334y) + 750.82) - 617.22,
\end{aligned}
\tag{2.57}
$$

of type (2.6) with the first integral

$$
\begin{aligned}
H_6(x,y) \quad \approx \quad & -156.25(x - 1.75156 - 1.08253y)^4 - 2.5(x - 1.75156 - 1.08253y)^2 - 375(x \\
& -1.75156 - 1.08253y)^2(x - 4.04242 + 10y)^2 - 0.25(x - 4.04242 + 10y)^4.
\end{aligned}
$$

The four real solutions of system (2.33) with $i = 3$ and $j = 6$ satisfying $y < Y$, which provide four limit cycles for the PWS (2.56)–(2.57) seen in Figure 2.8(c) are the set $S_{3,6}$ given by

$$
\begin{aligned}
S_{3,6} \approx \quad & \{(-0.153565, 0.720851), (-0.0467576, 0.678862), (0.0599189, 0.626584), \\
& (0.18123, 0.552656)\}.
\end{aligned}
$$



(a) (b) (c)

Figure 2.8: (a) Represent the four limit cycle of the PWS (2.52)–(2.53), (b) is the four limit cycles of the PWS (2.54)–(2.55), and (c) is the four limit cycle of the PWS (2.56)–(2.57).

Example of four limit cycles for the PWS (2.4)-(2.5). In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} = &\ (1/1.86)(x(343573+225y(236+1017y))-5(y(41881+2025y(2+11y))\\ &-36946)-10449x^3-45x^2(2482+3133y)),\\ \dot{y} = &\ (1/0.093)(x(225y(4964+3133y)+4421809)-5(y(225y(118+339y)\\ &+343573)-425818)+45x^2(2522+3483y)+7203x^3),\end{aligned} \tag{2.58}$$

of type (2.4) with the first integral

$$\begin{aligned}H_4(x,y) \ = \ & -3.3(-2x-1+y)^2+0.1(2x+1-y)(x+10+15y)-1.5\times 10^{-4}(x+15y\\ &+10)^2-0.25\times 10^{-4}(x+10+15y)^4-0.015(-2x-1+y)^2(x+10+15y)^2.\end{aligned}$$

In Σ_1^r we consider the cubic nilpotent center

$$\begin{aligned}\dot{x} \approx &\ x^2(-88.2943-1306.36y)+87.0946x^3+y((-1979.26-10886y)y\\ &-20467.6)+x(4025.15+y(837.747+6531.61y))+18055.8,\\ \dot{y} \approx &\ x^2(-18.3513-261.284y)+17.42x^3+y((-418.874-2177.2y)y\\ &-4025.15)+x(y(176.589+1306.36y)+628.569)+49010.8,\end{aligned} \tag{2.59}$$

of type (2.5) with the first integral

$$\begin{aligned}H_5(x,y) \ \approx \ & x^3(-5.61926-80.0064y)+4.00058x^4+x^2(y(81.1085+600.02y)\\ &+288.706)-8.723\times 10^6+x(y((-384.784-2000.01y)y-3697.57)\\ &+45022.1)+y(y(9400.91+y(606.061+2500y))-16586.3).\end{aligned}$$

The four real solutions of system (2.33) with $i=4$ and $j=5$ satisfying $y<Y$, which provide four limit cycles for the PWS (2.58)–(2.59) seen in Figure 2.9(a) are the set $S_{4,5}$ given by

$$\begin{aligned}S_{4,5} \approx &\ \{(-2.02489, 2.41933), (-1.7375, 2.21959), (-1.32018, 1.9543),\\ &(-0.552275, 1.52342)\}.\end{aligned}$$

Example of four limit cycles for the PWS (2.4)-(2.6). In Σ_2^r we consider the cubic nilpotent center

$$\dot{x} = \ (10^{-5}/37)(-2100x^2(22148+26267y)-3691300x^3-y(9100y(7074$$

$$
\begin{aligned}
& +4771y)) + 99840207 - 3x(3100y(12554 + 10387y) + 47561113)),\\
\dot{y} = \; & (10^{-5}/37)(540100x^3 + 300x^2(3242 + 36913y) + y(310y(18831 + 10387y)\\
& +142683339) - 2692564 + x(2100y(44296 + 26267y) + 218366903)),
\end{aligned} \tag{2.60}
$$

of type (2.4) with the first integral

$$
\begin{aligned}
H_4(x,y) \;\; = \;\; & 0.5 \times 10^{-6}(-50(x + 12 + 13y)^4 - 500(x + 12 + 13y)^2 + 2000(200y + 300x\\
& -3(x + 12 + 13y)^2(300x + 1 + 200y)^2 - 2000(300x + 200y + 1)^2.
\end{aligned}
$$

In Σ_1^r we consider the cubic nilpotent center

$$
\begin{aligned}
\dot{x} \approx \; & x^2(0.430192 + 1.21879y) + 0.0759897x^3 + y((-2.42318 - 1.6343y)y\\
& -3.75825) - 0.0149919 + x(y(1.18861 + 2.19373y) - 5.21479),\\
\dot{y} \approx \; & 0.010401x^3 + x^2(-0.0969069 - 0.227969y) + x(y(-0.860385 - 1.21879y)\\
& +10.5872) - 0.284961 + y((-0.594303 - 0.731243y)y + 5.21479),
\end{aligned} \tag{2.61}
$$

of type (2.6) with the first integral

$$
\begin{aligned}
H_6(x,y) \;\; \approx \;\; & x^3(0.0204567y + 0.00869591) - 0.0007x^4 + x^2((0.11581 + 0.164052y)\\
& y - 1.42505) - 0.00515174 + x(y((0.159989 + 0.196854y)y - 1.40384)\\
& +0.0767126) + y(y(-0.505868 + (-0.217444 - 0.10999y)y) - 0.00403587).
\end{aligned}
$$

The four real solutions of system (2.33) with $i = 4$ and $j = 6$ satisfying $y < Y$, which provide four limit cycles for the PWS (2.60)–(2.61) seen in Figure 2.9(b) are the set $S_{4,6}$ given by

$$
\begin{aligned}
S_{4,6} \approx \;\; & \{(-2.06178, 1.45127), (-1.84186, 1.28924), (-1.5337, 1.07579),\\
& (-1.00683, 0.739019)\}.
\end{aligned}
$$

Example of four limit cycles for the PWS (2.5)-(2.6). In Σ_1^r, we consider the cubic nilpotent center

$$
\begin{aligned}
\dot{x} = \; & (10^{-5}/9)(-540x^2(783 + 74835y) + 8059500x^3 - y(120y(13833 + 212420y)\\
& +159917) + 1963206 + 3x(-1238443 + 60y(123439 + 314700y))),\\
\dot{y} = \; & (10^{-5}/3)(270x^2(-29850y + 11087) - 16200x^3 + 3x(60y(15766 + 74835y)
\end{aligned} \tag{2.62}
$$

$$-1360603) + y(-30y(123439 + 209800y)1238443) + 1308666),$$

of type (2.5) with the first integral

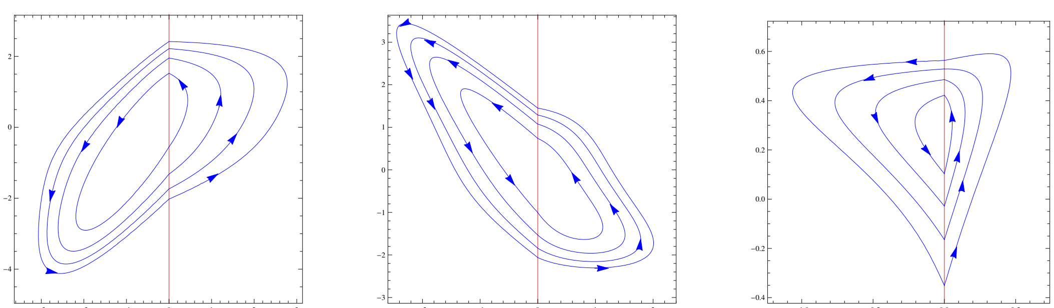
$$\begin{aligned} H_5(x,y) &= 0.5 \times 10^{-5}(540x^3(-1107 + 2950y) + 24300x^4 - 9x^2(60y(1566 + 7485y) \\ &\quad -1360603) + 1609167 - 6x(y(30y(123439 + 209800y) - 1238443) \\ &\quad -1308666) + y(y(159917 + 80y(138733 + 159315y)) - 3926412). \end{aligned}$$

In Σ_2^r we consider the cubic nilpotent center

$$\begin{aligned} \dot{x} \approx &\ -3.56089x^3 + x^2(85.2029 + 116.091y) + y((-7456.68 - 11417.2y)y \\ &-71.6274) + 879.327 + x(-442.536 + y(-2367.7 + 1210.81y)), \\ \dot{y} \approx &\ y(442.536 + y(-403.605y + 1183.85)) + 0.699665x^3 + x^2(10.6827y \\ &-20.2353) + x(197.017 + y(-170.406 - 116.091y)) - 622.065, \end{aligned} \tag{2.63}$$

of type (2.6) with the first integral

$$\begin{aligned} H_6(x,y) \approx &\ x^3(-363.433y + 688.421) - 17.8524x^4 + x^2(y(8696.03 + 5924.27y) \\ &-10054) + x(y(-45166.4 + y(-120827 + 41193y)) + 63489.6) - 14006 \\ &+y(y(-3655.24 + (-253683 - 291318y)y) + 89746.4). \end{aligned}$$



(*a*) (*b*) (*c*)

Figure 2.9: (*a*) Represent the four limit cycle of the PWS (2.58)–(2.59), (*b*) is the four limit cycle of the PWS (2.60)–(2.61), and (*c*) is the four limit cycle of the PWS (2.62)–(2.63).

The four real solutions of system (2.33) with $i = 5$ and $j = 6$ satisfying $y < Y$, which provide four limit cycles for the PWS (2.62)–(2.63) shown in Figure 2.9(*c*) are the set $S_{5,6}$

given by

$$S_{5,6} \approx \{(-0.3509, 0.563625), (-0.164808, 0.529183), (-0.0283422, 0.485655), (0.102906, 0.422227)\}.$$

CHAPTER 3

The Limit Cycles of Discontinuous Piecewise Differential System Formed by an Arbitrary Linear and Cubic Isochronous Centers of Period 2π Separated by Σ^r

This chapter is a result of our paper entitled "limit cycles of some families of discontinuous piecewise differential systems separated by a straight line", published in International Journal of Bifurcation and Chaos.

Recently some papers have studied the maximum number of limit cycles for the PWS formed by isochronous quadratic or cubic centers separated by a regular line instead of linear centers in order to control the variation of the number of limit cycles according to the considered systems, see [14, 24].

In this chapter we solve an extension of the second part of the sixteenth Hilbert's problem for two families of PWS separated by the regular line Σ^r. The first family is formed by a linear center and a cubic Hamiltonian isochronous center, and the second family is formed by cubic Hamiltonian isochronous centers.

Generalized cubic isochronous center of period 2π

In the next proposition we will classify all the cubic Hamiltonian isochronous centers generated by the Hamiltonian function (1.9) being f and g quadratic polynomials.

Proposition 3.1

Any planar cubic isochronous center can be written as one of the following three systems:

The first system $(\mathcal{C}_1)$ has the form

$$\begin{aligned}\dot{x} = \; & -(4b_3b_4(b_1b_4-2b_2b_3))^{-2}((a_4(2b_2b_3-b_1b_4)\left(2b_3(b_2+b_4x)+b_4^2y\right)+2\\ & b_3b_4^2)(a_4(2b_2b_3-b_1b_4)(4b_3x(b_1+b_3x)+4b_3y(b_2+b_4x)+b_4^2y^2)+4b_3\end{aligned}$$

$$
\begin{aligned}
& b_4(2b_3x + b_4y)) - 1/(8b_3^2)(2b_3(b_2 + b_4x) + b_4^2y)(4b_3x(b_1 + b_3x) + 4b_3y \\
& (b_2 + b_4x) + b_4^2y^2), \\
\dot{y} = \ & (2b_3b_4(b_1b_4 - 2b_2b_3))^{-2}((a_4(b_1b_4 - 2b_2b_3)(b_1 + 2b_3x + b_4y) - 2b_3b_4) \\
& (-a_4(2b_2b_3 - b_1b_4)(4b_3x(b_1 + b_3x) + 4b_3y(b_2 + b_4x) + b_4^2y^2) - 4b_3b_4 \\
& (2b_3x + b_4y))) + (2b_3x + b_1 + b_4y)\Big(b_1x + b_2y + \frac{1}{4b_3}(2b_3x + b_4y)^2\Big),
\end{aligned} \tag{3.1}
$$

with the first integral

$$
\begin{aligned}
H_1(x,y) = \ & \Big(\frac{a_4}{8b_4}\Big(4b_1x + 4b_2y + \frac{1}{b_3}(2b_3x + b_4y)^2\Big) + (2b_2b_3 - b_1b_4)^{-1}(2b_3x + b_4y)\Big)^2 \\
& + \frac{1}{2}(b_1x + b_2y + \frac{1}{4b_3}(2b_3x + b_4y)^2)^2.
\end{aligned}
$$

The second system $(\mathcal{C}_2)$ written as

$$
\begin{aligned}
\dot{x} = \ & -\frac{a_3^2b_2}{b_3^2}\Big(x(b_1 + b_3x) + b_2y\Big) - \frac{a_3}{b_3}x - b_2\Big(x(b_1 + b_3x) + b_2y\Big), \\
\dot{y} = \ & \frac{1}{b_2^2b_3^2}\Big(a_3b_2(b_1 + 2b_3x) + b_3\Big)\Big(a_3b_2(x(b_1 + b_3x) + b_2y) + b_3x\Big) \\
& + (b_1 + 2b_3x)\Big(x(b_1 + b_3x) + b_2y\Big),
\end{aligned} \tag{3.2}
$$

with the first integral

$$
H_2(x,y) = \ \Big((b_2b_3)^{-1}(a_3b_1b_2 + b_3)x + a_3b_2b_3^{-1}y + a_3x^2\Big)^2 + (b_1x + b_2y + b_3x^2)^2.
$$

The third system $(\mathcal{C}_3)$ is

$$
\begin{aligned}
\dot{x} &= -(b_1b_5)^{-2}\Big(a_5b_1(b_2 + 2b_5y) - b_5\Big)\Big(a_5b_1(b_1x + y(b_2 + b_5y)) - b_5y\Big) \\
& \quad (b_1x + y(b_2 + b_5y))\Big), \\
\dot{y} &= -((b_2 + 2b_5y)a_5b_5^{-2}((a_5b_1b_2 - b_5)y + a_5b_5^{-1}x + a_5y^2) + b_1(b_1x \\
& \quad + b_2y + b_5y^2),
\end{aligned} \tag{3.3}
$$

with the first integral

$$H_3(x,y) = \left(\frac{1}{b_1b_5}(a_5b_1b_2-b_5)y + \frac{a_5b_1}{b_5}x + a_5y^2\right)^2 + \left(b_1x+b_2y+b_5y^2\right)^2.$$

Proof of Proposition 3.1. *In this part we will classify all the cubic Hamiltonian isochronous centers generated by the Hamiltonian functions (1.9) where f and g are the quadratic polynomials*

$$f(x,y) = a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2, \quad g(x,y) = b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2.$$

It is clear that $f(0,0) = g(0,0) = 0$, and the Jacobian matrix $\mathbb{J}(f,g)$ of the map (f,g) is

$$\mathbb{J}(f,g) = \begin{bmatrix} a_1 + 2a_3x + a_4y & a_2 + a_4x + 2a_5y \\ b_1 + 2b_3x + b_4y & b_2 + b_4x + 2b_5y \end{bmatrix}.$$

We denote by

$$\begin{aligned}\det(\mathbb{J}(f,g)) = \;& a_1b_2 - a_2b_1 + (a_1b_4 - 2a_2b_3 + 2a_3b_2 - a_4b_1)x + (2a_1b_5 - a_2b_4 + a_4b_2 - 2a_5b_1)y \\ & +(4a_3b_5 - 4a_5b_3)xy + (2a_3b_4 - 2a_4b_3)x^2 + (2a_4b_5 - 2a_5b_4)y^2,\end{aligned}$$

the determinant of the Jacobian $\mathbb{J}(f,g)$. Then (f,g) is a canonical mapping if and only if the equation $\det(\mathbb{J}(f,g)) = 1$ is verified for all (x,y). So it is necessary that all the coefficients of the polynomial $\det(\mathbb{J}(f,g))-1$ be zero. Then we find three pairs of maps (f_i,g_i) with $i = 1,2,3$, where

$$\begin{aligned}f_1(x,y) = \;& \frac{\left(2a_4b_1b_2b_3 - a_4b_1^2b_4 + 2b_3b_4\right)}{b_4(2b_2b_3 - b_1b_4)}x + \frac{\left(a_4b_1b_2b_4 - 2a_4b_2^2b_3 - b_4^2\right)}{b_4(b_1b_4 - 2b_2b_3)}y + \frac{a_4b_3}{b_4}x^2 \\ & + \frac{a_4b_4}{4b_3}y^2 + a_4xy, \\ g_1(x,y) = \;& b_1x + b_2y + \frac{b_4^2}{4b_3}y^2 + b_3x^2 + b_4xy, \\ f_2(x,y) = \;& \frac{(a_3b_1b_2 + b_3)}{b_2b_3}x + \frac{a_3b_2}{b_3}x + a_3x^2, \quad g_2(x,y) = b_1x + b_2y + b_3x^2, \\ f_3(x,y) = \;& \frac{(a_5b_1b_2 - b_5)}{b_1b_5}y + \frac{a_5b_1}{b_5}x + a_5y^2, \quad g_3(x,y) = b_1x + b_2y + b_5y^2.\end{aligned}$$

By using Theorem 1.2 we find system $(\mathcal{C}_1)$ with its first integral $H_1(x,y)$, or $(\mathcal{C}_2)$ with its first *integral $H_2(x,y)$, or $(\mathcal{C}_3)$ with its first integral $H_3(x,y)$, by considering the map (f_1,g_1), or (f_2,g_2), or (f_3,g_3), respectively. This completes the proof of Proposition 3.1 .* ■

3.1 The limit cycles of the family of PWS separated by Σ^r and formed by a linear center and one of the three classes $(\mathcal{C}_i)$ with $i = 1,2,3$

The following theorem summarize the results concerning the maximum number of limit cycles for the family of PWS separated by the regular line Σ^r and formed by a linear center and one of the three systems $(\mathcal{C}_i)$ with $i = 1,2,3$.

Theorem 3.1

The maximum number of limit cycles for the family of PWS separated by the regular line Σ^r and formed by a linear center (1.3) and

(a) system $(\mathcal{C}_1)$ is at most one. This upper limit is reached, see Figure 3.1(a);

(b) system $(\mathcal{C}_2)$ is zero;

(c) system $(\mathcal{C}_3)$ is at most one. This upper limit is reached, see Figure 3.1(b).

Proof of Theorem 3.1

Proof. *Theorem 3.1 studies the maximum number of limit cycles of the PWS formed by a linear center and one of the three systems of cubic centers listed in Proposition 3.1 and separated by the regular line Σ^r.*

If there is a limit cycle of a such piecewise differential system, such that this limit cycle intersects in two points $p_1 = (0,y)$ and $p_2 = (0,Y)$ the discontinuity line Σ^r with $y < Y$. These two points must satisfy the following system

$$e_1 = H(p_1) - H(p_2) = P(y,Y) = 0, \quad e_2 = H_i(p_1) - H_i(p_2) = P_i(y,Y) = 0, \tag{3.4}$$

where $P(y,Y) = -8\alpha d_1 + 4\beta^2 y + 4\beta^2 Y + \omega^2 y + \omega^2 Y$ and $P_i(y,Y)$ with $i = 1,2,3$ are polynomial functions in the variables y and Y. Now by solving $e_1 = 0$, we get $Y = \dfrac{8\alpha d_1}{4\beta^2 + \omega^2} - y$ and by substituting it in $e_2 = 0$ we obtain an equation $F_i(y) = 0$ in the variable y that varies with respect to the first integrals H_1, H_2 and H_3.

Proof of statement (a) of Theorem 3.1. *For $i=1$, the function $F_1(y)$ is given as*

$$\begin{aligned}F_1(y)=\ & (2\alpha d_1)/(b_4^2\left(4\beta^2+\omega^2\right)(b_1b_4-2b_2b_3)^2)(a_4^2(b_1b_4-2b_2b_3)^2(8b_2^2b_3^2-4b_2b_3b_4^2y+b_4^4\\ & y^2)+4a_4b_3b_4^2(2b_2b_3-b_1b_4)(4b_2b_3-b_4^2y)+8b_3^2b_4^2(b_1^2b_2^2b_4^2-4b_1b_2^3b_3b_4+4b_2^4b_3^2+b_4^2)\\ & +b_4^6y^2(b_1b_4-2b_2b_3)^2-4b_2b_3b_4^4y(b_1b_4-2b_2b_3)^2)+(16\alpha^2d_1^2)/((4\beta^2+\omega^2)^2(2b_2b_3\\ & -b_1b_4))(a_4^2(2b_2b_3-b_1b_4)(4b_2b_3-b_4^2y)+4a_4b_3b_4^2+b_4^2(b_1b_4-2b_2b_3)(b_4^2y-4b_2b_3))\\ & +b_3y^2\left(a_4^2b_2+(a_4b_4^2)/(2b_2b_3-b_1b_4)+b_2b_4^2\right)+(64(a_4^2+b_4^2)/(4\beta^2+\omega^2)^3)\alpha^3b_4^2d_1^3.\end{aligned}$$

Due to the fact $F_1(y)=0$ is a quadratic equation in the variable y which has at most two real solutions y_1 and y_2, system (3.4) can have at most two real solutions $(y_1,F_1(y_1))$ and $(y_2,F_1(y_2))$. Since $(y_1,F_1(y_1))=(F_1(y_2),y_2)$ which means that these solutions are symmetric and provide the same limit cycle for the PWS (1.3)–$(\mathcal{C}_1)$.

To complete the proof of this statement we will give an example of one limit cycle. We consider the linear differential center in the half-plane Σ_1^r

$$\dot{x}\approx -x-1.03639y+0.2,\quad \dot{y}\approx 1.92978x+y-0.5, \tag{3.5}$$

with its first integral $H(x,y)\approx 14.8963x^2+x(15.4383y-7.71913)+y(8y-3.08765)$. In the second half-plane Σ_2^r we consider the cubic isochronous center

$$\begin{aligned}\dot{x}=\ & (116x+45)y^2+0.5x(5-87x)y+2.5x\Big(25x(116x-155)-19\Big)-58y^3-5.2y,\\ \dot{y}=\ & \Big(725x^3-75x^2(58y+25)+x\Big(25y(34y+155)+2\Big)+y\Big(961-50y(116y+5)\Big)\Big),\end{aligned} \tag{3.6}$$

taking the form of system $(\mathcal{C}_1)$, which has the first integral

$$H_1(x,y)\ =\ (-1.25x^2+x(5y+2.1)-5y^2+2.05y)^2+0.25(x^2-2x(2y+1)+y(4y-1))^2.$$

The unique real solution of system (3.4) is $(-0.385939,0.771896)$ that guarantee that the PWS (3.5)–(3.6) has exactly one limit cycle shown in Figure 3.1(a). ∎

Proof of statement (b) of Theorem 3.1. *For $i=2$, $F_2(y)$ is given by $F_2(y)=8\alpha b_2^2d_1b_3^{-2}(a_3^2+b_3^2)(4\beta^2y-4\alpha d_1+y\omega^2)(4\beta^2+\omega^2)^{-2}$. We have $y=4\alpha d_1\left(4\beta^2+\omega^2\right)^{-1}$ is the unique solution of the equation $F_2(y)=0$, then we obtain $Y=4\alpha d_1\left(4\beta^2+\omega^2\right)^{-1}=y$, and this is a contradiction with the assumption of the proof of Theorem 3.1 , where we assumed that $y<Y$. Consequently there are no limit cycles for the PWS (1.3)–$(\mathcal{C}_2)$.* ∎

Proof of statement (c) of Theorem 3.1. *In this case the equation $F_3(y)=0$ is a cubic equation where*

$$F_3(y)= \quad -4\alpha d_1+4\beta^2y+y\omega^2\Big(a_5^2b_1^2\left(b_2\left(4\beta^2+\omega^2\right)+8\alpha b_5d_1\right)(16(\alpha d_1\beta^2(b_2-2b_5y)+2\alpha^2b_5 d_1^2+b_5\beta^4y^2)+4\omega^2\left(\alpha b_2d_1+2b_5y\left(\beta^2y-\alpha d_1\right)\right)+b_5y^2\omega^4)-a_5b_1b_5\left(4\beta^2+\omega^2\right)(16 (2\alpha d_1\beta^2(b_2-b_5y)+4\alpha^2b_5d_1^2+b_5\beta^4y^2)+8\omega^2(\alpha b_2d_1+b_5y\left(\beta^2y-\alpha d_1\right))+b_5y^2\omega^4) +b_5^2(b_1^2\left(b_2\left(4\beta^2+\omega^2\right)+8\alpha b_5d_1\right)(16\left(\alpha d_1\beta^2(b_2-2b_5y)+2\alpha^2b_5d_1^2+b_5\beta^4y^2\right)+4\omega^2 (\alpha b_2d_1+2b_5y\left(\beta^2y-\alpha d_1\right))+b_5y^2\omega^4)+4\alpha d_1\left(4\beta^2+\omega^2\right)^2\Big)\Big).$$

It is clear that $F_3(y)=0$ has at most three real solutions, and due to the symmetry as in the proof of statement (a) of Theorem 3.1 , we know that system (3.4) has at most one real solution, which provides at most one limit cycle for the PWS (1.3)–$(\mathcal{C}_3)$.

We require an example with a unique limit cycle to complete the proof of this statement. We consider the linear differential center in the half-plane Σ_1^r

$$\dot{x}\approx-2(x+26.9922y+0.4),\quad \dot{y}\approx 0.115774x+2y+0.9, \tag{3.7}$$

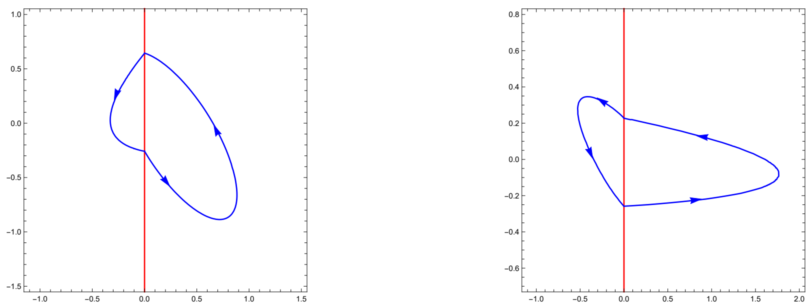
and its first integral $H(x,y)\approx 0.0536145x^2+x(1.85238y+0.833573)+y(25y+0.740954)$. In the half-plane Σ_2^r we consider the cubic isochronous center

$$\begin{aligned}\dot{x}= &\quad (10^{-3}/5)\Big(-2x(2501y+2551)-y\big(y(2501y+7653)+10202\big)\Big),\\ \dot{y}= &\quad (10^{-3}/5)\big(5002x+y(2501y+5102)\big),\end{aligned}\tag{3.8}$$

taking the form of system $(\mathcal{C}_3)$, which has the first integral $H_2(x,y)=\big(x+0.5y(y+2)\big)^2+10^{-4}\big(2x+y(y+102)\big)^2$. Then the pair $(-0.25869, 0.229052)$ is the unique real solution of system (3.4) which guarantee that the PWS (3.7)–(3.8) has exactly one limit cycle shown in Figure 3.1(b). The proof of Theorem 3.1 is done. ∎

3.2 The limit cycles of the family of PWS separated by Σ^r and formed by one of the three classes $(\mathcal{C}_i)$ with $i=1,2,3$ in each region

The next theorem summarize the results that concern the maximum number of limit cycles for the family of PWS separated by the regular line Σ^r and formed by one of the



(a) (b)

Figure 3.1: The unique limit cycle of the PWS, (a) for (3.5)–(3.6) and (b) for (3.7)–(3.8).

three systems $(\mathcal{C}_i)$ with $i = 1, 2, 3$ in each region.

Theorem 3.2

For the PWS separated by the regular line Σ^r and formed by

(a) system $(\mathcal{C}_1)$ in each region, there are at most three limit cycles. This maximum is reached, see Figure 3.2(a);

(b) system $(\mathcal{C}_2)$ in each region, there are no limit cycles;

(c) system $(\mathcal{C}_3)$ in each region, there are at most three limit cycles. This maximum is reached, see Figure 3.2(b).

Proof of Theorem 3.2

Proof. *In this section we provide the maximum number of limit cycles of the PWS separated by the regular line Σ^r, and formed in each half-plane by one of the three cubic Hamiltonian isochronous centers mentioned in Proposition 3.1.*

In the half-plane Σ_1^r we consider the cubic Hamiltonian isochronous center $(\mathcal{C}_i)$ with its corresponding first integral $H_i(x,y)$, where $i = 1, 2, 3$. By changing the parameters $(a_1, a_2, a_3, a_4, a_5)$ and $(b_1, b_2, b_3, b_4, b_5)$ with the parameters $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5)$ and $(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4, \tilde{b}_5)$, respectively in system $(\mathcal{C}_i)$ and in its first integral, we get the second differential system $(\tilde{\mathcal{C}}_i)$ with cubic Hamiltonian isochronous center and its corresponding first integral $\widetilde{H}_i(x,y)$ in the half-plane Σ_2^r. We know that the existence of a limit cycle of the PWS $(\mathcal{C}_i)$–$(\tilde{\mathcal{C}}_i)$ forces the existence of the solutions $p_1 = (0, y)$ and $p_2 = (0, Y)$ with $y < Y$ satisfying the system

$$e_1 = H_i(p_1) - H_i(p_2) = P_i(y, Y) = 0, \quad e_2 = \widetilde{H}_i(p_1) - \widetilde{H}_i(p_2) = \widetilde{P}_i(y, Y) = 0, \tag{3.9}$$

where $P_i(y,Y)$ and $\widetilde{P}_i\,(y,Y)$ are two polynomials in the variables y and Y for all $i \in \{1,2,3\}$.

Proof of statement (a) of Theorem 3.2. *The quartic polynomials $P_1(y,Y)$ and $\widetilde{P}_1\,(y,Y)$ has the form*

$$
\begin{aligned}
P_1(y,Y) = \;& a_4^2(b_1b_4-2b_2b_3)^2\Big(4b_2b_3+b_4^2(y+Y)\Big)\Big(4b_2b_3(y+Y)+b_4^2\Big(y^2+Y^2\Big)\Big)+8a_4\\
& b_3b_4^2(2b_2b_3-b_1b_4)\Big(4b_2b_3(y+Y)+b_4^2\Big(y^2+yY+Y^2\Big)\Big)+b_4^2\Big(4b_2^2b_3^2b_4^2\Big(4b_1^2\\
& (y+Y)-8b_1b_4\Big(y^2+yY+Y^2\Big)+b_4^2(y+Y)\Big(y^2+Y^2\Big)\Big)+b_4^2(y+Y)\Big(b_1^2b_4^4\Big(y^2\\
& +Y^2\Big)+16b_3^2\Big)+32b_2^3b_3^3b_4\Big(b_4\Big(y^2+yY+Y^2\Big)-2b_1(y+Y)\Big)+4b_1b_2b_3b_4^4\\
& \Big(2b_1\Big(y^2+yY+Y^2\Big)-b_4(y+Y)\Big(y^2+Y^2\Big)\Big)+64b_2^4b_3^4(y+Y)\Big),\\
\widetilde{P}_1\,(y,Y) = \;& \tilde{a}_4^2(\tilde{b}_1\tilde{b}_4-2\tilde{b}_2\tilde{b}_3)^2\Big(4\tilde{b}_2\tilde{b}_3+\tilde{b}_4^2(y+Y)\Big)\Big(4\tilde{b}_2\tilde{b}_3(y+Y)+\tilde{b}_4^2\Big(y^2+Y^2\Big)\Big)+8\tilde{a}_4\\
& \tilde{b}_3\tilde{b}_4^2(2\tilde{b}_2\tilde{b}_3-\tilde{b}_1\tilde{b}_4)\Big(4\tilde{b}_2\tilde{b}_3(y+Y)+\tilde{b}_4^2\Big(y^2+yY+Y^2\Big)\Big)+\tilde{b}_4^2\Big(4\tilde{b}_2^2\tilde{b}_3^2\tilde{b}_4^2\Big(4\tilde{b}_1^2\\
& (y+Y)-8\tilde{b}_1\tilde{b}_4\Big(y^2+yY+Y^2\Big)+\tilde{b}_4^2(y+Y)\Big(y^2+Y^2\Big)\Big)+\tilde{b}_4^2(y+Y)\Big(\tilde{b}_1^2\tilde{b}_4^4\Big(y^2\\
& +Y^2\Big)+16\tilde{b}_3^2\Big)+32\tilde{b}_2^3\tilde{b}_3^3\tilde{b}_4\Big(\tilde{b}_4\Big(y^2+yY+Y^2\Big)-2\tilde{b}_1(y+Y)\Big)+4\tilde{b}_1\tilde{b}_2\tilde{b}_3\tilde{b}_4^4\\
& \Big(2\tilde{b}_1\Big(y^2+yY+Y^2\Big)-\tilde{b}_4(y+Y)\Big(y^2+Y^2\Big)\Big)+64\tilde{b}_2^4\tilde{b}_3^4(y+Y)\Big).
\end{aligned}
$$

As we said at the beginning of this proof, the existence and the number of limit cycles of the PWS $(\mathcal{C}_1)$–$(\tilde{\mathcal{C}_1})$ entirely depend on the number of real solutions of system (3.9) with $i=1$. Therefore we compute the resultants, $Resultant[e_1,e_2,y]$ and $Resultant[e_1,e_2,Y]$, of e_1 and e_2 with regard to y and Y, respectively, in order to determine the common zeros (y,Y) of e_1 and e_2.

Here we know that $Resultant[e_1,e_2,y]$ and $Resultant[e_1,e_2,Y]$, or simply R_y and R_Y have the same expression because of the symmetry of the solutions of $e_1=0$ and $e_2=0$ with respect to y and Y. So it is sufficient to compute only one of them, and in this case, we find that $Resultant[e_1,e_2,Y]$ is a polynomial of degree six, and due to the big expression of this equation we omit it. Therefore we have at most six pairs (y,Y) of solutions of system (3.9), that can provide the intersection points of the limit cycles with the separation line Σ^r, but if (y,Y) is one of these pairs then (Y,y) is the other one. Thus these two pairs define the same limit cycle. Consequently the maximum number of limit cycles of the PWS $(\mathcal{C}_1)$–$(\tilde{\mathcal{C}_1})$ is at most three.

Now we give an example with three limit cycles for the system formed by $(\mathcal{C}_1)$–$(\tilde{\mathcal{C}_1})$ to complete the proof of statement (a) of Theorem 3.1 .

In the half-plane Σ_1^r we consider the cubic isochronous center

$$\begin{aligned}\dot{x} \approx\ & x^2(16.2544-13.4305y)+y\Big(y(56.36-111.92y)-6.35263\Big)\\ & +0.895364x^3+x\Big(y(67.1523y-92.544)+13.0205\Big),\\ \dot{y} \approx\ & x^2(4.65088-2.68609y)+y\Big(y(46.272-22.3841y)-13.0205\Big)\\ & +0.179073x^3+x\Big(y(13.4305y-32.5088)+26.8446\Big),\end{aligned} \tag{3.10}$$

with the first integral

$$\begin{aligned}H_1(x,y) \ \approx\ & \frac{1}{2}\Big(\big(y(2.29554-7.05055y)-0.282022x^2+x(2.82022y-4.86883)\big)^2\\ & +0.01\Big(x(17.7176-10.y)+x^2+y(25y-10.4073)\big)^2\Big).\end{aligned}$$

In the half-plane Σ_2^r we consider the Hamiltonian differential cubic isochronous center

$$\begin{aligned}\dot{x} =\ & \frac{1}{98}\Big(980x^3-7x^2(840y+349)+x(112y(105y+26)-929)+y\Big(28(141\\ & -280y)y-445\Big)\Big),\\ \dot{y} =\ & 5x^3-\frac{3}{14}x^2(140y+99)+x\Big(60y^2+\frac{349y}{7}+\frac{1961}{98}\Big)+\frac{1}{98}y(929-112y\\ & (35y+13)),\end{aligned} \tag{3.11}$$

having the first integral

$$\widetilde{H}_1(x,y) \ =\ 49\Big(x^2-x(4y+3)+y(4y-1)\Big)^2+\Big(-21x^2+x(84y+59)+y(29-84y)\Big)^2.$$

The PWS separated by the regular line Σ^r and formed by the Hamiltonian cubic isochronous centers (3.10)–(3.11) has exactly three limit cycles because the system of equations (3.9) with $i=1$ has exactly the three real solutions (y_i, Y_i) given as follows $(y_1, Y_1) \approx (-0.364040, 0.699217)$, $(y_2, Y_2) \approx (-0.437999, 0.773309)$ and $(y_3, Y_3) \approx (-0.508934, 0.84433)$, see Figure 3.2(a). ■

Proof of statement (b) of Theorem 3.2. *For $i=2$ the solution of system (3.9) is $y=-Y$, which means that the PWS $(\mathcal{C}_2)$–$(\tilde{\mathcal{C}}_2)$ has a continuum of periodic solutions. Consequently no limit cycles.* ■

Proof of statement (c) of Theorem 3.2. *The cubic polynomials $P_3(y,Y)$ and $\widetilde{P}_3(y,Y)$ has* *the form*

$$\begin{aligned} P_3(y,Y) = \;& a_5^2 b_1^2(b_2+b_5(y+Y))\Big(b_2(y+Y)+b_5\Big(y^2+Y^2\Big)\Big)-2a_5b_1b_5\Big(b_2(y+Y)+b_5\Big(y^2+y\\ & Y+Y^2\Big)\Big)+b_5^2\Big(b_1^2b_5y^2(2b_2+b_5Y)+y\Big(b_1^2(b_2+b_5Y)^2+1\Big)+b_1^2Y(b_2+b_5Y)^2+b_1^2\\ & b_5^2y^3+Y\Big),\\ \widetilde{P}_3\,(y,Y) = \;& \tilde{a}_5^2\tilde{b}_1^2(\tilde{b}_2+\tilde{b}_5(y+Y))\Big(\tilde{b}_2(y+Y)+\tilde{b}_5\Big(y^2+Y^2\Big)\Big)-2\tilde{a}_5\tilde{b}_1\tilde{b}_5\Big(\tilde{b}_2(y+Y)+\tilde{b}_5\Big(y^2+y\\ & Y+Y^2\Big)\Big)+\tilde{b}_5^2\Big(\tilde{b}_1^2\tilde{b}_5y^2(2\tilde{b}_2+\tilde{b}_5Y)+y\Big(\tilde{b}_1^2(\tilde{b}_2+\tilde{b}_5Y)^2+1\Big)+\tilde{b}_1^2Y(\tilde{b}_2+\tilde{b}_5Y)^2+\tilde{b}_1^2\\ & \tilde{b}_5^2y^3+Y\Big). \end{aligned}$$

As in the proof of statement (a) of Theorem 3.2 , and by computing the resultant R_y of system (3.9) with $i=3$, we get a polynomial of degree six in the variable y. Due to the symmetry of the solutions of system (3.9) we notice that the maximum number of limit cycles is at most three.

Now we provide an example of three limit cycles for the system formed by $(\mathcal{C}_3)$–$(\tilde{\mathcal{C}}_3)$ to justify the proof of statement (c) of Theorem 3.2 .

In the half-plane Σ_1^r we consider the cubic isochronous center

$$\begin{aligned} \dot{x} = \;& \frac{1}{100}\Big(-8x(29y+34)-y\Big(58y^2+204y+181\Big)\Big),\\ \dot{y} = \;& \frac{1}{25}\Big(116x+y(29y+68)\Big), \end{aligned} \tag{3.12}$$

taking the form of system $(\mathcal{C}_3)$, with its first integral $H_3(x,y)=(2x+0.5y(y+2))^2+10^{-2}(8x+y(2y+9))^2$. In the half-plane Σ_2^r we consider the second cubic Hamiltonian isochronous center of system $(\mathcal{C}_3)$

$$\begin{aligned} \dot{x} \approx \;& x(-2.32y-2.72)+y\Big((-1.10327y-3.88048)y-3.44297\Big),\\ \dot{y} \approx \;& 2.43929x+y(1.16y+2.72), \end{aligned} \tag{3.13}$$

which has the first integral

$$\widetilde{H}_3\,(x,y) \approx 1.21964x^2+xy(1.16y+2.72)+y^2((0.275818y+1.29349)y+1.72149).$$

The three real solutions (y_i,Y_i) with $i=1,2,3$ of system (3.9) with $i=3$, that provide the three limit cycles of the PWS (3.12)–(3.13) shown in Figure 3.2(b) are given as follows $(y_1,Y_1)\approx(-0.288497,0.236378)$, $(y_2,Y_2)\approx(-0.411453,0.311843)$ and $(y_3,Y_3)\approx(-0.52063,0.368918)$.

Hence the proof of Theorem 3.2 is done. ■

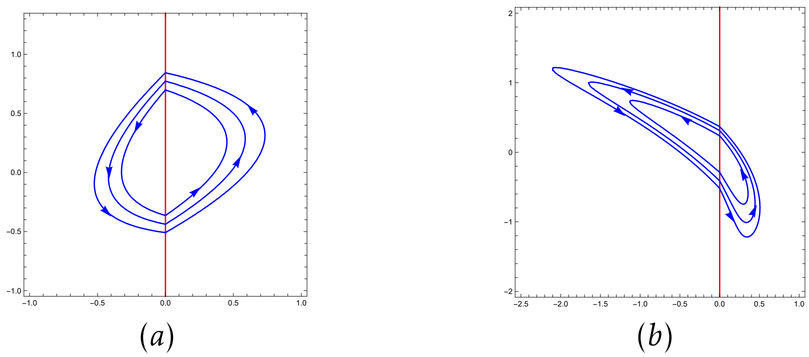


Figure 3.2: The three limit cycles of the PWS, (a) for (3.10)–(3.11) and (b) for (3.12)–(3.13).

3.3 The limit cycles of the family of PWS $(\mathcal{C}_i) - (\mathcal{C}_j)$ with $i, j \in \{1, 2, 3\}$ and $i \neq j$ separated by Σ^r

In what follow we summarize the results that concern the maximum number of limit cycles for the family of PWS $(\mathcal{C}_i)$–$(\mathcal{C}_j)$ with $i, j \in \{1, 2, 3\}$ and $i \neq j$ separated by the regular line Σ^r.

Theorem 3.3

For the PWS separated by the regular line Σ^r and formed by system

(a) $(\mathcal{C}_1)$ in one region and system $(\mathcal{C}_2)$ in the other region, there are no limit cycles;

(b) $(\mathcal{C}_1)$ in one region and system $(\mathcal{C}_3)$ in the other region, there are at most three limit cycles. This maximum is reached, see Figure 3.3;

(c) $(\mathcal{C}_2)$ in one region and system $(\mathcal{C}_3)$ in the other region, there are no limit cycles.

Proof of Theorem 3.3

Proof. *This subsection is devoted to provide the maximum number of limit cycles of the PWS separated by the regular line Σ^r, and formed system $(\mathcal{C}_i) - (\mathcal{C}_j)$ with $i, j \in \{1, 2, 3\}$ and $i \neq j$.*

For the PWS $(\mathcal{C}_i)-(\mathcal{C}_j)$, with $i,j \in \{1,2,3\}$ and $i \neq j$, we consider in Σ_1^r the Hamiltonian cubic isochronous center $(\mathcal{C}_i)$ with its first integral $H_i(x,y)$. In Σ_2^r we consider the Hamiltonian cubic isochronous center $(\mathcal{C}_j)$ with its first integral $H_j(x,y)$. If there exists a limit cycle of the PWS $(\mathcal{C}_i)$–$(\mathcal{C}_j)$, it must intersects the separation line Σ^r in two distinct points $p_1=(0,y)$ and $p_2=(0,Y)$ with $y<Y$. These two points must satisfy $e_1=0$ and $e_2=0$ such that

$$e_1 = H_i(p_1) - H_i(p_2) = P_i(y,Y) = 0, \quad e_2 = H_j(p_1) - H_j(p_2) = P_j(y,Y) = 0, \tag{3.14}$$

where $P_i(y,Y)$ and $P_j(y,Y)$ for all $i,j \in \{1,2,3\}$ and $i \neq j$ are polynomial functions.

Proof of statement (a) of Theorem 3.3. *In this case $i=1$ and $j=2$, where $P_1(y,Y)$ is the polynomial given in the proof of statement (a) of Theorem 3.2 , and $P_2(y,Y)$ is $P_2(y,Y) = b_2^2 b_3^{-2}\left(a_3^2+b_3^2\right)(y+Y)$. Here $Y=-y$ is the solution of $P_2(y,Y)=0$, and by substituting Y in $P_2(y,Y)$, we get $y=0$. Consequently the PWS $(\mathcal{C}_1)$–$(\mathcal{C}_2)$ has no limit cycles.* ∎

Proof of statement (b) of Theorem 3.3. *For $i=1$ and $j=3$, we have the two polynomials $P_1(y,Y)$ and $P_3(y,Y)$ given in the proof of Theorem 3.2 . As in the proof of statement (a) of Theorem 3.2 to study the maximum number of solutions of system (3.14) with $i=1$ and $j=3$ it is necessary to compute the resultant R_y which is a polynomial of degree six. Then we know that the maximum number of limit cycles is at most three.*

To complete the proof of this statement we establish an example of three limit cycles for the system $(\mathcal{C}_1)$-$(\mathcal{C}_3)$.

In the half-plane Σ_2^r we consider the cubic isochronous center

$$\begin{aligned} \dot{x} = &\ -\frac{6}{5}(100x+13)y^2 - \frac{1}{50}(40x(75x+82)+37)y - \frac{1}{50}x(5x(100x+289) \\ &+181) - 80y^3, \\ \dot{y} = &\ 5x^3 + x^2\left(30y + \frac{207}{10}\right) + x\left(60y^2 + \frac{289y}{5} + \frac{953}{50}\right) + \frac{1}{50}y(40y(50y+41)+181), \end{aligned} \tag{3.15}$$

of the form of system $(\mathcal{C}_1)$ with the first integral

$$H_1(x,y) = \left(15x^2 + x(60y+41) + y(60y+7)\right)^2 + 25\left(4xy + x(x+3) + 4y^2 + y\right)^2.$$

In the half-plane Σ_1^r we consider the cubic isochronous center

$$\begin{aligned} \dot{x} &\approx x(50y+3.25) + y((-255.525y - 49.8275)y - 2.36361), \\ \dot{y} &\approx 4.89188x + y(-25y - 3.25), \end{aligned} \tag{3.16}$$

of the form of system $(\mathcal{C}_3)$ *with the first integral*

$$H_3(x,y) \approx 2.44594x^2 + xy(-25y - 3.25) + y^2(y(63.8814y + 16.6092) + 1.18181).$$

The PWS formed by the cubic Hamiltonian isochronous center (3.15)–(3.16) *has exactly three limit cycles because the system of equations* (3.14) *with* $i = 1$ *and* $j = 3$ *has the three real solutions* (y_i, Y_i) *for* $i \in \{1,2,3\}$, *where* $(y_1, Y_1) \approx (-0.5525470, 0.422990)$, $(y_2, Y_2) \approx (-0.650101, 0.520408)$ *and* $(y_3, Y_3) \approx (-0.714291, 0.584539)$. *These limit cycles are shown in Figure* 3.3.■

Proof of statement (c) **of Theorem 3.3.** *In this case* $i = 2$ *and* $j = 3$, *and the solution of system* (3.14) *produces the unique real solution* $(0,0)$, *which means that there are no limit cycles for the PWS* $(\mathcal{C}_2)$-$(\mathcal{C}_3)$. *So the proof of Theorem* 3.3 *is done.* ■

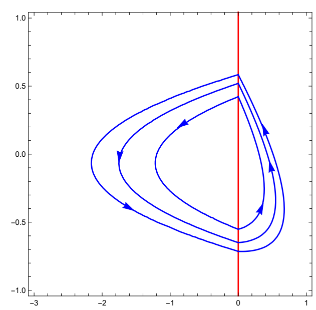


Figure 3.3: The three limit cycles of the PWS (3.15)–(3.16).

CHAPTER 4

The Limit Cycles of Discontinuous Piecewise Differential System Formed by an Arbitrary Linear and Rigid Centers Separated by Σ^r

This chapter is a result of our paper entitled "The limit cycles of a class of discontinuous piecewise differential systems", published in International Journal of Dynamical Systems and Differential Equations.

Recently in [12] the authors studied the second part of the 16th Hilbert problem for a class of PWS separated by a regular line and formed by linear and quadratic centers. In [14, 24] the authors solved the extension of the second part of the 16th Hilbert problem for two classes of PWS formed by quadratic and cubic isochronous centers separated by a regular line.

In this chapter we extend the second part of the 16th Hilbert's problem to the planar PWSs separated by a regular line Σ^r and formed by an arbitrary linear center and an arbitrary rigid center. We provide for this class of piecewise differential systems an upper bound on its maximal number of limit cycles, and we prove that such an upper bound is reached.

Generalized rigid centers

Now we give the expression of the rigid centers (1.7) with their corresponding first integrals after doing the arbitrary affine change of variables (1.8). In this way we obtain the expression of all rigid centers.

System (1.7) becomes

$$\begin{aligned}\dot{x} = \ & (\alpha_2\beta_1-\alpha_1\beta_2)^{-1}\Big(\gamma_2(a_1\beta_1\gamma_1-\beta_2\gamma_1(a_2+a_4\gamma_1)+\beta_2)+x^2(a_1\alpha_1(\alpha_2\beta_1-\alpha_1\beta_2)\\ & +a_2\alpha_2(\alpha_2\beta_1-\alpha_1\beta_2)+a_4(\alpha_2\gamma_1(\alpha_2\beta_1-2\alpha_1\beta_2)+\alpha_1\gamma_2(2\alpha_2\beta_1-\alpha_1\beta_2)+y\\ & (\alpha_1\beta_2+\alpha_2\beta_1)(\alpha_2\beta_1-\alpha_1\beta_2)))+x(\alpha_2(a_1\beta_1(\gamma_1+\beta_1 y)+(a_2+a_4(\gamma_1+\beta_1 y))\end{aligned}$$

$$
\begin{aligned}
&(2\beta_1\gamma_2-\beta_2\gamma_1+\beta_1\beta_2y)+\beta_2)-\alpha_1(\beta_1(\beta_2y-\gamma_2)(a_1+a_4(\gamma_2+\beta_2y))+2a_1\beta_2\\
&\gamma_1)(\gamma_2+\beta_2y)-\beta_1))+y(\beta_1\beta_2(a_2\gamma_2-a_1\gamma_1)+\beta_1^2\gamma_2(a_1+a_4\gamma_2)-\beta_2^2\gamma_1(a_2+a_4\\
&\gamma_1)+\beta_1^2+\beta_2^2)+\gamma_1(\beta_1-a_1\beta_2\gamma_1)+\beta_1\gamma_2^2(a_2+a_4\gamma_1)+\alpha_1\alpha_2a_4x^3(\alpha_2\beta_1-\alpha_1\beta_2)\\
&+a_4\beta_1\beta_2y^2(\beta_1\gamma_2-\beta_2\gamma_1)\Big),\\
\dot{y}=\ &(\alpha_2\beta_1-\alpha_1\beta_2)^{-1}\Big(\gamma_1(a_1\alpha_2\gamma_1-\alpha_1)-a_1\alpha_1\gamma_1\gamma_2-x(\alpha_1^2+\alpha_1\alpha_2(a_2(\gamma_2+\beta_2y)-a_1\\
&(\gamma_1+\beta_1y))+\alpha_1^2(\gamma_2+\beta_2y)(a_1+a_4(\gamma_2+\beta_2y))+\alpha_2^2-\alpha_2^2(\gamma_1+\beta_1y)(a_2+a_4(\gamma_1\\
&+\beta_1y)))+y^2(a_1\beta_1(\alpha_2\beta_1-\alpha_1\beta_2)+\beta_2(-\alpha_1a_2\beta_2+a_2\alpha_2\beta_1-\alpha_1a_4\beta_2\gamma_1+2\alpha_2a_4\\
&\beta_1\gamma_1)+a_4\beta_1\gamma_2(\alpha_2\beta_1-2\alpha_1\beta_2))-\alpha_1y(\beta_1\gamma_2(a_1+a_4\gamma_2)+a_1\beta_2\gamma_1+2\beta_2\gamma_2(a_2\\
&+a_4\gamma_1)+\beta_1)+\alpha_2y(2\beta_1\gamma_1(a_1+a_4\gamma_2)+\beta_2\gamma_1(a_2+a_4\gamma_1)+a_2\beta_1\gamma_2-\beta_2)-\alpha_1\gamma_2^2\\
&(a_2+a_4\gamma_1)\alpha_2\gamma_2(\gamma_1(a_2+a_4\gamma_1)-1)+\alpha_1\alpha_2a_4x^2(-\alpha_1\gamma_2+\alpha_2\gamma_1-\alpha_1\beta_2y+\alpha_2\beta_1\\
&y)+a_4\beta_1\beta_2y^3(\alpha_2\beta_1-\alpha_1\beta_2)\Big),
\end{aligned}
\tag{4.1}
$$

its corresponding first integrals is given as follows.

Case 1. $a_1^2+a_2^2=0$, the corresponding first integral of system (4.1) becomes

$$
H_1(x,y)=\ ((\gamma_1+\alpha_1x+\beta_1y)^2+(\gamma_2+\alpha_2x+\beta_2y)^2)/(1-a_4(\gamma_1+\alpha_1x+\beta_1y)^2). \tag{4.2}
$$

Case 2. $a_1^2+a_2^2\neq 0$, if $4a_4-a_1^2<0$, $a_2=0$ and $S=\sqrt{a_1^2-4a_4}$, the first integral of system (4.1) writes as

$$
\begin{aligned}
H_2^{(1)}(x,y)=\ &(-a_1-2a_4(\gamma_2+\alpha_2x+\beta_2y)+S)^{-(S+a_1)/a_1}(a_1+2a_4(\gamma_2+\alpha_2x\\
&+\beta_2y)+S)^{(a_1-S)/a_1}\left((\gamma_1+\alpha_1x+\beta_1y)^2+(\gamma_2+\alpha_2x+\beta_2y)^2\right)^{S/a_1}.
\end{aligned}
\tag{4.3}
$$

If $4a_4+a_2^2>0$, $a_1=0$ and $S=\sqrt{4a_4+a_1^2}$, the first integral of system (4.1) becomes

$$
\begin{aligned}
H_2^{(2)}(x,y)=\ &(-a_2-2a_4(\gamma_1+\alpha_1x+\beta_1y)+S)^{-(S+a_2)/a_2}(a_2+2a_4(\gamma_1+\alpha_1x+\beta_1y)+S)^{(a_2-S)/a_2}\\
&\left((\gamma_1+\alpha_1x+\beta_1y)^2+(\gamma_2+\alpha_2x+\beta_2y)^2\right)^{S/a_2}.
\end{aligned}
\tag{4.4}
$$

If $4a_4-a_1^2>0$, $a_2=0$, $R_1(x,y)=\dfrac{1}{S}(a_1+2a_4(\gamma_2+\alpha_2x+\beta_2y))$ and $S=\sqrt{4a_4-a_1^2}$, the first integral of system (4.1) writes

$$H_2^{(3)}(x,y) = \quad e^{-2\arctan(R_1(x,y))}\left(\frac{(\gamma_1+\alpha_1 x+\beta_1 y)^2+(\gamma_2+\alpha_2 x+\beta_2 y)^2}{a_1(\gamma_2+\alpha_2 x+\beta_2 y)+a_4(\gamma_2+\alpha_2 x+\beta_2 y)^2+1}\right)^{S/a_1}. \tag{4.5}$$

If $4a_4+a_2^2<0$, $a_1=0$, $R_2(x,y)=\dfrac{1}{S}(a_2+2a_4(\gamma_1+\alpha_1 x+\beta_1 y))$ and $S=\sqrt{-a_2^2-4a_4}$, the first integral of system (4.1) becomes

$$H_2^{(4)}(x,y) = \quad e^{-2\arctan(R_2(x,y))}\left(\frac{(\gamma_1+\alpha_1 x+\beta_1 y)^2+(\gamma_2+\alpha_2 x+\beta_2 y)^2}{a_2(\gamma_1+\alpha_1 x+\beta_1 y)+a_4(\gamma_1+\alpha_1 x+\beta_1 y)^2-1}\right)^{S/a_2}. \tag{4.6}$$

If $4a_4-a_1^2=0$, $a_2=0$ and $R_3(x,y)=4/(a_1(\gamma_2+\alpha_2 x+\beta_2 y)+2)$, the first integral of system (4.1) becomes

$$H_2^{(5)}(x,y) = \frac{(\gamma_1+\alpha_1 x+\beta_1 y)^2+(\gamma_2+\alpha_2 x+\beta_2 y)^2}{(a_1(\gamma_2+\alpha_2 x+\beta_2 y)+2)^2}e^{R_3(x,y)}. \tag{4.7}$$

If $4a_4+a_2^2=0$, $a_1=0$ and $R_4(x,y)=4/(2-a_2(\gamma_1+\alpha_1 x+\beta_1 y))$, the first integral of system (4.1) becomes

$$H_2^{(6)}(x,y) = \frac{(\gamma_1+\alpha_1 x+\beta_1 y)^2+(\gamma_2+\alpha_2 x+\beta_2 y)^2}{(2-a_2(\gamma_1+\alpha_1 x+\beta_1 y))^2}e^{R_4(x,y)}. \tag{4.8}$$

4.1 The limit cycles of the family of PWS separated by Σ^r and formed by linear center and rigid center

The main result is given in the following theorem.

Theorem 4.1

For a piecewise smooth differential system with two zones separated by the regular line Σ^r, and formed by an arbitrary linear center and an arbitrary rigid center the maximum number of limit cycles is at most

(I) one if $a_1^2+a_2^2=0$. There are systems satisfying this condition having exactly one limit cycle, see Figure 4.2(a);

(II) three if $4a_4-a_1^2<0$ and $a_2=0$, or $4a_4+a_2^2>0$ and $a_1=0$; four if $4a_4-a_1^2>0$ and $a_2=0$, or $4a_4+a_2^2<0$ and $a_1=0$; two if $4a_4-a_1^2=0$ and $a_2=0$, or

$4a_4 + a_2^2 = 0$ and $a_1 = 0$. There are systems of these types having exactly three limit cycles shown in Figure 4.2(b), three limit cycles shown in Figure 4.5(a), and two limit cycles shown in Figure 4.5(b),

We get the upper bound on the maximum number of limit cycles by studying the intersections of the graphics of many functions.

Remark *In the proofs of Chapters 4 and 5 we only give the graphics of the functions, when the first derivative's sign of all the functions started when $y \to -\infty$ with a positive sign and also with a negative sign when $y \to -\infty$. The cases that we omit to consider explicitly will be called the symmetric cases of the ones that we considered.*

Proof of Theorem 4.1

Proof. *Here we are going to show the upper bound number of limit cycles for the PWS with an arbitrary linear and rigid centers separated by Σ^r.*

In the right half-plane Σ_1^r we consider the linear differential center (1.3) with the first integral $H(x,y)$ of the form (1.4). In the left half-plane Σ_2^r we consider system (4.1), with its first integrals $H_j^{(k)}(x,y)$ with $k = 1,\ldots,6$ and $j = 1,2$, where $H_1^{(k)}(x,y) = H_1(x,y)$.

The next system of equations must be verified if the PWS (1.3)–(4.1) have a limit cycle that intersects the line Σ^r in the two points $p_1 = (0,y)$ and $p_2 = (0,Y)$, with $y \neq Y$

$$E_1 = H(p_1) - H(p_2) = (y-Y)h(y,Y) = 0, \quad E_2 = H_j^{(k)}(p_1) - H_j^{(k)}(p_2) = h_j^{(k)}(y,Y) = 0. \quad (4.9)$$

where $h(y,Y) = 8\alpha d_1 - 4\beta^2 y - y\omega^2 - 4\beta^2 Y - \omega^2 Y$. By solving $E_1 = 0$, we get $Y = \dfrac{8\alpha d_1}{4\beta^2+\omega^2} - y$ and by replacing it in $E_2 = 0$ we get an equation $F(y) = 0$ with the variable y, that changes depending on the first integrals $H_j^{(k)}(x,y)$ of system (4.1).

Proof of statement (I) of Theorem 4.1. *We start the proof of this statement for the PWS separated by Σ^r and formed by the arbitrary linear differential center (1.3) and the arbitrary rigid center (4.1) satisfying $a_1^2 + a_2^2 = 0$, where*

$$\begin{aligned} F(y) = \; & \left(4\beta^2+\omega^2\right)\left(\beta_2\gamma_2\left(a_4\gamma_1^2-1\right) + a_4\beta_1^2\beta_2\gamma_2 y^2 - \beta_1\gamma_1\left(a_4\gamma_2^2 + a_4\beta_2^2 y^2 + 1\right)\right) \\ & -4\alpha d_1\left(\beta_2^2\left(1-a_4\gamma_1^2\right) + \beta_1^2(a_4\gamma_2(\gamma_2+2\beta_2 y)+1) - 2a_4\beta_1\beta_2^2\gamma_1 y\right). \end{aligned}$$

The quadratic equation $F(y)=0$ has at most two real solutions. Consequently system (4.9) can have at most two real solutions $(y_1,F(y_1))$ and $(y_2,F(y_2))$. Since $(y_1,F(y_1))=(F(y_2),y_2)$ these solutions provide the same limit cycle for the PWS (1.3)–(4.1). Consequently the planar PWS (1.3)–(4.1) can have at most one limit cycle under the condition $a_1^2+a_2^2=0$.

To complete the proof of this statement we present a PWS that has only one limit cycle and satisfies $a_1^2+a_2^2=0$. We take the linear differential center in the half-plane Σ_1^r

$$\dot{x}=-x+(13/8)y+0.1,\quad \dot{y}=-2x+y+0.3, \tag{4.10}$$

with the first integral $H(x,y)=80x^2-8x(10y+3)+y(65y+8)$. In the half-plane Σ_2^r we consider the rigid center

$$\begin{aligned}\dot{x}&=\frac{10^{-2}}{9}\Big(x\Big(45x^2-84x-1036\Big)+24(3x+4)y^2-6x(33x+56)y+904y+208\Big),\\ \dot{y}&=\frac{10^{-2}}{9}\Big(15x^2(3y+4)-2x(3y(33y+68)+746)+4y(6y(3y+1)+115)-992\Big),\end{aligned} \tag{4.11}$$

with the first integral $H_1(x,y)=(13x^2+2x(8-7y)+2y(5y+4)+16)/((5x-2y+14)(5x-2(y+3)))$. The unique real solution of system (4.9) is $(y_1,y_2)\approx(-1.32287,1.1998)$ which produces the unique limit cycle for the PWS (4.10)–(4.11), see Figure 4.2(a). ■

Proof of statement (II) of Theorem 4.1. *Here we demonstrate the statement for the PWS separated by Σ^r and formed by the arbitrary linear differential center (1.3) and the arbitrary rigid centers (4.1) satisfying $a_1^2+a_2^2\neq 0$, and we distinguish the following subcases.*

Subcase 2.1. *If $4a_4-a_1^2<0$ and $a_2=0$, then $k=1$ and $j=2$ in system (4.9), then the first integral $H_2^{(1)}(x,y)$ of system (4.1) is given by (4.3). In this case finding the solution of the equation $F(y)=0$ is equivalent to solving the equation $f_1(y)=g_1(y)=0$ such that*

$$f_1(y)=\left(\frac{k_0+k_1\,y+k_2\,y^2}{G_0+G_1\,y+k_2\,y^2}\right)^r\quad \textit{and}\quad g_1(y)=\left(\frac{m_1+m_2\,y}{m_3-m_2\,y}\right)^p\left(\frac{n_1+m_2\,y}{n_3-m_2\,y}\right)^q,$$

where

$$m_1=S-a_1-2a_4\gamma_2,\;\; m_2=-2a_4\beta_2,\;\; m_3=S-a_1-2a_4\gamma_2-16\alpha a_4\beta_2d_1k^{-1},\;\; p=-(S+a_1)a_1^{-1},$$

$$n_1=-S-a_1-2a_4\gamma_2,\;\; n_3=-S-a_1-2a_4\gamma_2-(16\alpha a_4\beta_2d_1)k^{-1},\;\; q=(a_1-S)a_1$$

$$k_0=\gamma_1^2+\gamma_2^2+64\alpha^2d_1^2k^{-2}k_2+16\alpha d_1k^{-1}(\beta_1\gamma_1+\beta_2\gamma_2),\;\; k_1=-2(\beta_1\gamma_1+\beta_2\gamma_2)-16\alpha d_1k^{-1}k_2,$$

$$k_2=\beta_1^2+\beta_2^2,\;\; G_0=\gamma_1^2+\gamma_2^2,\;\; G_1=2(\beta_1\gamma_1+\beta_2\gamma_2),\;\; S=\sqrt{a_1^2-4a_4},\;\; r=Sa_1^{-1},\quad k=4\beta^2+\omega^2.$$

The maximum number of the real solutions of system (4.9) is equivalent to the maximum number of the intersection points of the graphics of the function $f_1(y)$ with the ones of $g_1(y)$.

We denote by $g_1'(y)$ the first derivative of the function $g_1(y)$, given by

$$g_1'(y) = (m_1 + m_2\, y)^{p-1}(n_1 + m_2\, y)^{q-1}(m_3 - m_2\, y)^{-(p+1)}(n_3 - m_2\, y)^{-(q+1)}\; P_1(y),$$

where

$$\begin{aligned} P_1(y) = \quad & m_2 n_1 n_3 p(m_1 + m_3) + m_1 m_2 m_3 q(n_1 + n_3) + (m_2^2(q(m_3 - m_1)(n_1 + n_3) \\ & -p(m_1 + m_3)(n_1 - n_3)))\; y + (-m_2^3(p(m_1 + m_3) + q(n_1 + n_3)))\; y^2. \end{aligned}$$

In all the graphics of the functions $f_i(y)$ and $g_i(y)$, with $i = 1, 2, 3$, the dashed lines represent the vertical asymptote straight lines, and the horizontal straight line is the $y-$axis.

Since $p \neq 0 \neq q$, for $p, q > 0$, and from the geometric study, the function $g_1(y)$ has two distinct vertical asymptote straight lines $y_1 = m_3/m_2$ and $y_2 = n_3/m_2$, and its variation depends on the sign of its first derivative, the nature of the parameters p and q, the roots of the quadratic polynomial $P_1(y)$ with their possible positions with respect to y_1 and y_2, and with respect to the two roots $r_1 = -m_1/m_2$ and $r_2 = -n_1/m_2$ of $g_1'(y)$.

So if we suppose that $p > q > 0$ and $y_1 < y_2$, the possible positions of the two real roots r_1 and r_2 with respect to the vertical asymptote y_1 and y_2 can be as follows.

1. *$r_1 < r_2 < y_1 < y_2$ with its symmetric $y_1 < y_2 < r_1 < r_2$;*

2. *$r_1 < y_1 < r_2 < y_2$ with its symmetric $y_1 < r_1 < y_2 < r_2$;*

3. *$r_1 < y_1 < y_2 < r_2$;*

4. *$y_1 < r_1 < r_2 < y_2$.*

Similarly we find the same possible positions if $y_2 < y_1$.

Now we will analyze the possible positions of the real roots for the quadratic polynomial $P_1(y)$ with respect to y_1, y_2, r_1 and r_2. We denote by

$$\begin{aligned} \Delta = \quad & m_2^4(p(m_1 + m_3)(n_1 - n_3) + q(m_1 - m_3)(n_1 + n_3))^2 + 4m_2^2(m_2 p(m_1 + m_3) \\ & + m_2 q(n_1 + n_3))(m_2 n_1 n_3 p(m_1 + m_3) + m_1 m_2 m_3 q(n_1 + n_3)), \end{aligned}$$

the discriminant of $P_1(y)$, and by using the expressions of y_1, y_2, r_1 and r_2 we can write Δ in *the form*

$$\begin{aligned}\Delta = \;& m_2^8(r_1-y_1)(r_2-y_2)\Big(p^2(r_1-y_1)(r_2-y_2)+2pq(r_1(r_2-2y_1+y_2)+r_2y_1\\ &+q^2(r_1-y_1)(r_2-y_2)-2r_2y_2+y_1y_2)\Big).\end{aligned}$$

If the polynomial $P_1(y)$ *has a pair of distinct real roots* r_3 *and* r_4*, i.e.,* $\Delta > 0$*, their expressions are given by*

$$\begin{aligned}r_3 = \;& \frac{p(r_1-y_1)(r_2+y_2)+q(r_1+y_1)(r_2-y_2)+\sqrt{\Delta/m_2^8}}{2(p(r_1-y_1)+q(r_2-y_2))},\\ r_4 = \;& \frac{p(r_1-y_1)(r_2+y_2)+q(r_1+y_1)(r_2-y_2)-\sqrt{\Delta/m_2^8}}{2(p(r_1-y_1)+q(r_2-y_2))}.\end{aligned}$$

If we suppose that $r_1 < r_2 < y_1 < y_2$ *it results that* $r_3 < r_4$ *because* $p(r_1-y_1)+q(r_2-y_2) < 0$*. By fixing the position of* r_3 *between* r_1 *and* r_2 *with* $r_1 < r_3 < r_2$*, then we obtain the position of* r_4*.*

The first inequality $r_1 < r_3$ *is equivalent to*

$$(r_1-y_1)(-y_2(p+q)+2pr_1-pr_2+qr_2)-\sqrt{\Delta/m_2^8} > 0. \tag{4.12}$$

The second inequality $r_3 < r_2$ *is equivalent to*

$$-(r_2-y_2)(p(r_1-y_1)-q(r_1-2r_2+y_1))+\sqrt{\Delta/m_2^8} > 0, \tag{4.13}$$

by summing the two inequalities (4.12) and (4.13) we get $(r_1-r_2)(p(r_1-y_1)+q(r_2-y_2)) > 0$*, which is satisfied for all* $r_1 < r_2 < y_1 < y_2$*. So to find the position of* r_4*, we assume that* $r_3 < r_4 < r_2$*, it is clear that the first inequality* $r_3 < r_4$ *holds.*

For $r_4 < r_2$ *we get*

$$-(r_2-y_2)(p(r_1-y_1)-q(r_1-2r_2+y_1))-\sqrt{\Delta/m_2^8} > 0. \tag{4.14}$$

A necessary condition in order that this last inequality holds is $p(r_1-y_1)-q(r_1-2r_2+y_1) > 0$*, i.e.,* $p(r_1-y_1) > q(r_1-2r_2+y_1) = q(r_1-y_1)+2q(y_1-r_2)$*. We know that* $p \geq q > 0$ *and* $r_1-y_1 < 0$*, then* $p(r_1-y_1) \leq q(r_1-y_1) < 0$*. Since* $2q(y_1-r_2) > 0$ *then* $p(r_1-y_1) \leq q(r_1-y_1) < q(r_1-y_1)+2q(y_1-r_2) = q(r_1-2r_2+y_1)$*, which is a contradiction. Then the position* $r_1 < r_3 < r_4 < r_2 < y_1 < y_2$ *is not possible.*

Now we assume that $r_2 < r_4 < y_1$*, the inequality* $r_2 < r_4$ *is equivalent to*

$$(r_2-y_2)(p(r_1-y_1)-q(r_1-2r_2+y_1))+\sqrt{\Delta/m_2^8} > 0, \tag{4.15}$$

and $r_4 < y_1$ *is equivalent to*

$$(r_1 - y_1)(p(r_2 - 2y_1 + y_2) + q(r_2 - y_2)) - \sqrt{\Delta/m_2^8} > 0. \tag{4.16}$$

So (4.15)+(4.16) *implies* $(y_1 - r_2)(p(r_1 - y_1) + q(r_2 - y_2)) > 0$ *which is a contradiction, because* $y_1 - r_2 > 0$, $r_1 - y_1 < 0$ *and* $r_2 - y_2 < 0$. *Then the position* $r_1 < r_3 < r_2 < r_4 < y_1 < y_2$ *is not possible.*

Now we assume that $y_1 < r_4 < y_2$, *the inequality* $y_1 < r_4$ *is equivalent to*

$$(r_1 - y_1)(p(r_2 - 2y_1 + y_2) + q(r_2 - y_2)) - \sqrt{\Delta/m_2^8} < 0, \tag{4.17}$$

and $r_4 < y_2$ *equivalent to*

$$-(r_2 - y_2)(p(r_1 - y_1) + q(r_1 + y_1 - 2y_2)) + \sqrt{\Delta/m_2^8} < 0. \tag{4.18}$$

So (4.17)+(4.18) *gives* $(y_1 - y_2)(p(y_1 - r_1) + q(y_2 - r_2)) < 0$. *Since* $y_1 - y_2 < 0$, $y_1 - r_1 > 0$ *and* $y_2 - r_2 > 0$, *this inequality holds. In a similar way we provide all the positions of the real roots of* $P_1(y)$ *with respect to the vertical asymptote* y_1 *and* y_2 *and to* r_1 *and* r_2 *that are given in what follows:*

(1) $r_1 < r_3 < r_2 < y_1 < r_4 < y_2$ *with its symmetric* $y_1 < r_3 < y_2 < r_1 < r_4 < r_2$;

(2) $y_1 < r_3 < r_4 < r_1 < y_2 < r_2$ *with its symmetric* $r_1 < y_1 < r_2 < r_3 < r_4 < y_2$;

(3) $r_1 < r_3 < r_4 < y_1 < r_2 < y_2$ *with its symmetric* $y_1 < r_1 < y_2 < r_3 < r_4 < r_2$;

(4) $r_3 < r_1 < y_1 < r_4 < y_2 < r_2$ *with its symmetric* $r_1 < y_1 < r_3 < y_2 < r_2 < r_4$;

(5) $y_1 < r_1 < r_3 < r_2 < y_2 < r_4$ *with its symmetric* $r_3 < y_1 < r_1 < r_4 < r_2 < y_2$;

(6) $r_3 < r_4 < y_1 < r_1 < y_2 < r_2$ *with its symmetric* $r_1 < y_1 < r_2 < y_2 < r_3 < r_4$.

If $P_1(y)$ *has a pair of complex roots, we have only the position* $r_1 < y_1 < r_2 < y_2$ *together with its symmetric* $y_1 < r_1 < y_2 < r_2$.

If $P_1(y)$ *has a double real root* r_0, *the possible positions of this double root with respect to* y_1, y_2, r_1 *and* r_2 *are*

(1) $r_1 < r_0 < y_1 < r_2 < y_2$ *together with its symmetric* $y_1 < r_1 < y_2 < r_0 < r_2$;

(2) $r_1 < y_1 < r_2 < r_0 < y_2$ *together with its symmetric* $y_1 < r_0 < r_1 < y_2 < r_2$.

Figures 4.7, 4.9 and 4.11 are the possible graphics for the function $g_1(y)$. Indeed if p and q are even integers or if p is an even integer and $q = 2l_1/(2l_2+1)$ with $l_1, l_2 \in \mathbb{N}$, or if $p = 2l_1/(2l_2+1)$ and $q = 2l'_1/(2l'_2+1)$ with $l_1, l_2, l'_1, l'_2 \in \mathbb{N}$, we give all the graphics of $g_1(y)$ in Figure 4.7. If $P_1(y)$ has a pair of distinct real roots r_3 and r_4 which can take the position (1) where the graphic of $g_1(y)$ is given in Figure 4.7(a), or either the position (2), or (3), or (4), or (5) or (6), where graphics of $g_1(y)$ are given either in (b), or (c), or (d), or (e) or (f) of Figure 4.7, respectively. If $P_1(y)$ has a pair of complex roots, the graphic of $g_1(y)$ is illustrated in Figure 4.7(g). If $P_1(y)$ has a double real root taking either the position (1) or (2), then the graphics of $g_1(y)$ are given by Figure 4.7(h) or Figure 4.7(i).

If p and q are odd integers, or if p is an odd integer and $q = (2l_1+1)/(2l_2+1)$ with $l_1, l_2 \in \mathbb{N}$, or if $p = (2l_1+1)/(2l_2+1)$ and $q = (2l'_1+1)/(2l'_2+1)$ with $l_1, l_2, l'_1, l'_2 \in \mathbb{N}$, we give all the graphics of $g_1(y)$ in Figure 4.9. If $P_1(y)$ has a pair of distinct real roots r_3 and r_4 either in the position (1), or (2), or (3), or (4), or (5) or (6), then the graphics of $g_1(y)$ are given either in (a) and (b), or (c) and (d), or (e) and (f), or (g) and (h), or (i) and (j), or (k) and (l) of Figure 4.9, respectively. If $P_1(y)$ has a pair of complex roots, then the graphics of $g_1(y)$ are shown in Figures 4.9(m) and 4.9(n). If $P_1(y)$ has a double real root r_0 either in the position (1) or (2), then the graphics of $g_1(y)$ are given either in (o) and (p), or (q) and (r) of Figure 4.9, respectively.

In a similar way we obtain that if p is an odd integer and q is an even integer, or if $p = (2l_1+1)/(2l_2+1)$ and $q = (2l'_1)/(2l'_2+1)$ with $l_1, l_2, l'_1, l'_2 \in \mathbb{N}$, or if p is an even integer and $q = (2l_1+1)/(2l_2+1)$ with $l_1, l_2 \in \mathbb{N}$, or if p is an odd integer and $q = 2l_1/(2l_2+1)$ with $l_1, l_2 \in \mathbb{N}$, we give all the graphics of $g_1(y)$ in Figure 4.11.

If p is an odd integer and q is either irrational or $q = l_1/(2l_2)$ with $l_1, l_2 \in \mathbb{N}$ and $l_2 \neq 0$, the function $g_1(y)$ is well defined on $D_{g_1} = [r_2, y_2)$ and the sign of $g'_1(y)$ depends on the sign of the polynomial $P_1(y)$, therefore the graphics of $g_1(y)$ are the parts drawn on D_{g_1} when both p and q are odd integers.

If p is an even integer and q is either irrational or $q = l_1/(2l_2)$ with $l_1, l_2 \in \mathbb{N}$ and $l_2 \neq 0$, the function $g_1(y)$ is well defined on $D_{g_1} = [r_2, y_2)$ and the sign of $g'_1(y)$ is determined by the sign of $P_1(y)$ and on the sign of the product $(n_1 + n_2 y_2)(n_1 + n_3 y_2)$, therefore the graphics of $g_1(y)$ are the parts drawn on D_{g_1} in which one of the integers p and q is odd and the other is even.

If p is either irrational or $p = l_1/(2l_2)$ and q is either irrational or $q = l'_1/(2l'_2)$ with $l_1, l_2, l'_1, l'_2 \in \mathbb{N}$ and $l_2 \neq 0 \neq l'_2$, or if p is either irrational or $p = l_1/(2l_2)$ and $q = (2l'_1+1)/(2l'_2+1)$ with $l_1, l_2, l'_1, l'_2 \in \mathbb{N}$ and $l_2 \neq 0$, the function $g_1(y)$ is well defined on $D_{g_1} = [r_1, y_1) \cap [r_2, y_2)$ and the sign of $g'_1(y)$ is determined on sign of $P_1(y)$, therefore the graphics of $g_1(y)$ are the *parts drawn on D_{g_1} when both p and q are odd integers.*

If p is either irrational or $p = l_1/(2l_2)$ and $q = 2l'_1/(2l'_2+1)$ with $l_1, l_2, l'_1, l'_2 \in \mathbb{N}$ and $l_2 \neq 0$, the function $g_1(y)$ is well defined on $D_{g_1} = [r_1, y_1) \cap [r_2, y_2)$ and the sign of $g'_1(y)$ is determined by the sign of $P_1(y)$ and $(n_1 + m_2\ y)(n_3 - m_2\ y)$, therefore the graphics of $g_1(y)$ are the parts drawn on D_{g_1} in which one of the integers p and q is odd and the other is even.

For the case $p, q < 0$ or $pq < 0$, we found the same graphics by the same way.

Now for the function $f_1(y)$ we denote by Δ_1 and Δ_2 the discriminant of the quadratic polynomials $G_0 + G_1\ y + k_2\ y^2$ and $k_0 + k_1\ y + k_2\ y^2$, respectively. We have $\Delta = \Delta_1 = \Delta_2 = -4(\beta_2\gamma_1 - \beta_1\gamma_2)^2 \leq 0$.
If $\Delta = 0$, the function $f_1(y)$ with its first derivative have the form

$$f_1(y) = \left(\frac{G_1 + 2k_2\ y}{k_1 + 2k_2\ y}\right)^{2r}, \quad f'_1(y) = \frac{\eta(G_1 + 2k_2\ y)^{2r-1}}{(k_1 + 2k_2\ y)^{2r+1}},$$

with $\eta = 4rk_2(k_1 - G_1)$. Then to draw all the graphics of the function $f_1(y)$ with $\Delta = 0$ we have to study the sign of its derivative which depends on η and on the nature of the parameter $2r$.

For $r > 0$ it is clear that $f'_1(y)$ vanish at $z_1 = -G_1/(2k_2)$ that can have only one possible position with respect to the vertical asymptote straight line $z_2 = -k_1/(2k_2)$. Thus in this case we only draw the graphics of the function $f_1(y)$ when $z_1 < z_2$, and we omit the case when $z_2 < z_1$ as we did in the graphics of $g_1(y)$.

If r is a natural number or $r = l_1/(2l_2 + 1)$ with $l_1, l_2 \in \mathbb{N}$, the sign of $f'_1(y)$ depends on the sign of the product $\eta(G_1 + 2k_2\ y)(k_1 + 2k_2\ y)$. So the only possible graphic of this function is shown in Figure 4.6(a).

If $r = (2l_1 + 1)/(4l_2 + 2)$ with $l_1, l_2 \in \mathbb{N}$, the sign of $f'_1(y)$ is related only on the parameter η. Here the graphics of this function are shown in Figure 4.6(b) if $\eta < 0$ and in Figure 4.6(c) if $\eta > 0$.

If either r is irrational or $r = (2l_1 + 1)/(4l_2)$ with $l_1, l_2 \in \mathbb{N}$ and $k_2 \neq 0$, the function $f_1(y)$ is well defined on $D_{f_1} = [z_1, z_2)$, and the sign of $f'_1(y)$ is related only on the nature of η. Thus the graphics of $f_1(y)$ are the parts drawn on D_{f_1} in Figure 4.6(b) if $\eta < 0$ and in Figure 4.6(c) if $\eta > 0$.

Similarly when $r < 0$, we obtain the identical graphics as $r > 0$.

If $\Delta < 0$ the function $f_1(y)$ has two extremums. Thus Figures 4.6(d) and 4.6(e) are the only possible graphics for the function $f_1(y)$.

For the function $g_1(y)$ when both integers p and q are even, we notice that the derivative's *sign changes at most seven times, but in the other cases it changes at most five times, which guarantees that the even case is the one that gives the maximum number of the intersection points of the function $f_1(y)$ with $g_1(y)$, so to provide this maximum it is sufficient to obtain the upper bound number of the intersection points of $f_1(y)$ with $g_1(y)$ when p and q are even integers. Then we are only interested in the graphics of the function $g_1(y)$ drawn in Figure 4.7. In this case since $y = 1$ is the common horizontal asymptote straight line for these two functions it ensures that no intersection points exist between these graphics at infinity. Then the graphics of $f_1(y)$ and $g_1(y)$ can intersect in at most seven points. Consequently system (4.9) can have at most seven real solutions. It is simple to demonstrate that if (y, Y) is a solution of system (4.9), then the symmetry (Y, y) is also a solution of that system. Therefore the maximum number of limit cycles of the PWS (1.3)–(4.1) for $4a_4 - a_1^2 < 0$ and $a_2 = 0$ is at most three.*

By considering $(m_1, m_2, m_3, n_1, n_3, p, q, k_0, k_1, k_2, G_0, G_1, r) \approx (0.3, 1, 0.01, 2, 2, 2, 2, 0.33, 1, 0.75, 0.75, -1.5, 3)$ *we build an example in which the graphics of $f_1(y)$ and $g_1(y)$ interesect in seven points, see Figure 4.1.*

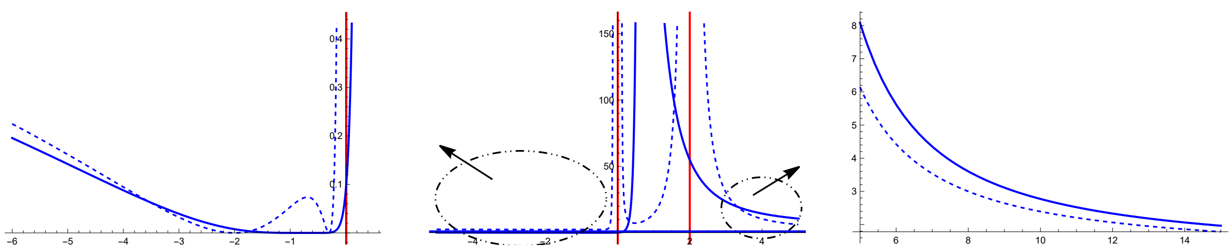


Figure 4.1: The seven intersection points between the graphics of $f_1(y)$ presented in a continuous line and $g_1(y)$ presented in a dashed line. The vertical lines represent the asymptote's straight lines.

Subcase 2.2. *If $4a_4 + a_2^2 > 0$ and $a_1 = 0$, then $k = 2$ and $j = 2$ in system (4.9), and $H_2^{(2)}(x, y)$ given by (4.4) represent the first integral of system (4.1). Thus the solutions of $F(y) = 0$ are identical to the solutions of $f_1(y) = g_1(y)$ given in subcase 2.1, with*

$$\begin{aligned}
m_1 = &\quad S - a_2 - 2a_4\gamma_1, \;\; m_2 = -2a_4\beta_1, \;\; m_3 = S - a_2 - 2a_4\gamma_1 - 16\alpha a_4\beta_1 d_1(4\beta^2 + \omega^2)^{-1},\\
n_1 = &\quad -S - a_2 + 2a_4\gamma_1, \;\; n_3 = -S - a_2 - 2a_4\gamma_1 - 16\alpha a_4\beta_1 d_1(4\beta^2 + \omega^2)^{-1},\\
p = &\quad -(S + a_2)/a_2, \;\; q = (a_2 - S)/a_2, \;\; r = S/a_2, \;\; S = \sqrt{a_2^2 + 4a_4}.
\end{aligned}$$

Therefore the maximum number of the intersection points between the graphics of $f_1(y)$ and $g_1(y)$ is at most five. Then as in subcase 2.1 the maximum number of limit cycles of the PWS (1.3)–(4.1) with $4a_4 + a_2^2 > 0$ and $a_1 = 0$ is at most three.

To prove that our results are reached we will give an example of three limit cycles for the class formed by a linear center and a rigid center with $4a_4 + a_2^2 > 0$ and $a_1 = 0$. In the region Σ_1^r we consider the linear differential center

$$\dot{x} = x - \frac{41y}{32} + \frac{9}{10}, \quad \dot{y} = 2x - y + 1, \tag{4.19}$$

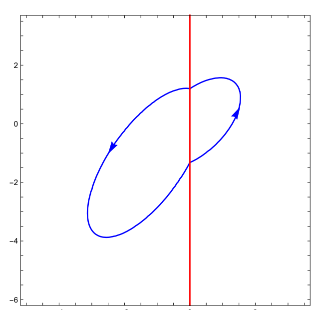
with the first integral $H(x,y) = -\dfrac{8}{5}(10x+9)y + 16x(x+1) + \dfrac{41y^2}{4}$. *In the region Σ_2^r we consider the rigid center*

$$\begin{aligned}
\dot{x} \approx\ & x(y(0.0880169 - 0.000688837y) - 1.99445) + 0.000375x^3 + x^2 \\
& (0.00336918y + 0.0960476) + (-0.0179787y - 17.9585)y + 12.6236, \\
\dot{y} \approx\ & y(y(0.00207603 - 0.0000688837y) + 1.94435) + x^2(0.0000375y \\
& -0.0108607) + x((0.00336918y - 0.0994057)y + 0.27169) + 13.1497,
\end{aligned} \tag{4.20}$$

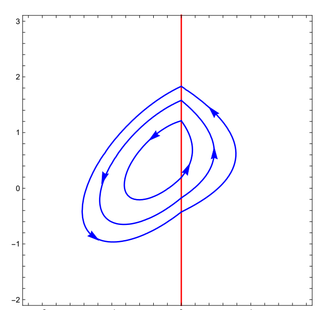
with the first integral

$$\begin{aligned}
H_2^{(2)}(x,y) \approx\ & (0.0015625(x^2 + x(14.615y + 98.717) + y(67.491y - 94.819 + 14255.7)^2)) \\
& /((0.0075x - 0.0015y + 0.30095)^3(-0.0075x + 0.00015y + 0.0999052)).
\end{aligned}$$

For the PWS (4.19)–(4.20), system (4.9) has the three solutions $(y_1, y_2) \approx (-0.4262, 1.831)$, $(y_3, y_4) \approx (-0.173113, 1.57799)$ and $(y_5, y_6) \approx (0.193249, 1.21163)$ which provide the three limit cycles intersecting the separation regular line Σ^r in the six points $(0, y_j)$ with $j = 1, \ldots, 6$, see Figure 4.2(b).



(a)



(b)

Figure 4.2: (a) The unique limit cycle of the PWS (4.10)–(4.11), (b) the three limit cycles of the PWS (4.19)–(4.20).

Subcase 2.3. *If $4a_4 - a_1^2 > 0$ and $a_2 = 0$, then $k = 3$ and $j = 2$ in system (4.9), and the first integral $H_2^{(3)}(x,y)$ of (4.1) is given by (4.5). The solutions of the equation $F(y) = 0$ are* *identically equivalent to the ones of the equation* $f_2(y) = g_2(y)$ *such that*

$$f_2(y) = \left(\frac{k_0 + k_1\ y + k_2\ y^2}{G_0 + G_1\ y + G_2\ y^2} \right)^r \quad and \quad g_2(y) = e^{2(\arctan(m_1 + m_2\ y) - \arctan(m_3 - m_2\ y))},$$

where

$$\begin{aligned}
k_0 = \;& (\gamma_1^2 + \gamma_2^2)k^{-2}(16\beta^4(\gamma_2(a_1 + a_4\gamma_2) + 1) + 8\beta^2\Big(\omega^2(\gamma_2(a_1 + a_4\gamma_2) + 1) + 4\alpha\beta_2 d_1(a_1 \\
& +2a_4\gamma_2)\Big) + a_1\gamma_2\omega^4 + 8\alpha a_1\beta_2 d_1\omega^2 + a_4\left(\gamma_2\omega^2 + 8\alpha\beta_2 d_1\right)^2 + \omega^4), \\
k_1 = \;& 16\alpha\beta_2 d_1 k^{-1}\left(a_1\beta_1\gamma_1 + a_1\beta_2\gamma_2 + 2a_4\beta_1\gamma_1\gamma_2 - a_4\beta_2\gamma_1^2 + a_4\beta_2\gamma_2^2\right) + 2a_1\beta_1\gamma_1\gamma_2 - a_1 \\
& \beta_2\gamma_1^2 + a_1\beta_2\gamma_2^2 + 2a_4\beta_1\gamma_1\gamma_2^2 - 2a_4\beta_2\gamma_1^2\gamma_2 + 128\alpha^2 a_4\beta_2^2 d_1^2 k^{-2}(\beta_1\gamma_1 + \beta_2\gamma_2) + 2\beta_1 \\
& \gamma_1 + 2\beta_2\gamma_2, \\
k_2 = \;& 8\alpha\beta_2 d_1 k^{-1}\left(a_1\beta_1^2 + a_1\beta_2^2 - 4a_4\beta_1\beta_2\gamma_1 + 2a_4\gamma_2(\beta_1 - \beta_2)(\beta_1 + \beta_2)\right) + a_1\beta_1^2\gamma_2 - 2a_1 \\
& \beta_1\beta_2\gamma_1 - a_1\beta_2^2\gamma_2 + a_4\beta_1^2\gamma_2^2 - 4a_4\beta_1\beta_2\gamma_1\gamma_2 + a_4\beta_2^2\gamma_1^2 - 2a_4\beta_2^2\gamma_2^2 + 64\alpha^2 a_4\beta_2^2 d_1^2 k^{-2} \\
& \left(\beta_1^2 + \beta_2^2\right) + \beta_1^2 + \beta_2^2, \\
G_0 = \;& (\gamma_2(a_1 + a_4\gamma_2) + 1)k^{-2}(\left(4\beta^2 + \omega^2\right)^2\left(\gamma_1^2 + \gamma_2^2\right) + 64\alpha^2 d_1^2\left(\beta_1^2 + \beta_2^2\right) + 16\alpha d_1\left(4\beta^2 + \omega^2\right) \\
& (\beta_1\gamma_1 + \beta_2\gamma_2)), \\
G_1 = \;& -2\beta_1\gamma_1(\gamma_2(a_1 + a_4\gamma_2) + 1) + \beta_2\left(a_1(\gamma_1 - \gamma_2)(\gamma_1 + \gamma_2) + 2\gamma_2\left(a_4\gamma_1^2 - 1\right)\right) + 64\alpha^2\beta_2 \\
& d_1^2 k^{-2}\left(\beta_1^2 + \beta_2^2\right)(a_1 + 2a_4\gamma_2) - 16\alpha d_1 k^{-1}(\beta_1^2(\gamma_2(a_1 + a_4\gamma_2) + 1) - \beta_1\beta_2\gamma_1(a_1 + 2 \\
& a_4\gamma_2) + \beta_2^2(1 - a_4\gamma_2^2)), \\
G_2 = \;& -16\alpha\beta_2 d_1 k^{-1}\left(a_1\left(\beta_1^2 + \beta_2^2\right) + a_4\left(2\beta_1^2\gamma_2 - \beta_1\beta_2\gamma_1 + \beta_2^2\gamma_2\right)\right) + a_1\beta_1^2\gamma_2 - 2a_1\beta_1\beta_2\gamma_1 \\
& -a_1\beta_2^2\gamma_2 + a_4\beta_1^2\gamma_2^2 - 4a_4\beta_1\beta_2\gamma_1\gamma_2 + a_4\beta_2^2\gamma_1^2 - 2a_4\beta_2^2\gamma_2^2 + 64\alpha^2 a_4\beta_2^2 d_1^2 k^{-2}\left(\beta_1^2 + \beta_2^2\right) \\
& +\beta_1^2 + \beta_2^2, \;\; r = \frac{S}{a_1}, \;\; S = \sqrt{4a_4 - a_1^2}, \quad k = 4\beta^2 + \omega^2, \\
m_1 = \;& S^{-1}\left(a_1 + 2a_4\gamma_2 + 16\alpha a_4\beta_2 d_1 k^{-1}\right), \;\; m_2 = 2a_4\beta_2 S^{-1}, \;\; m_3 = (a_1 + 2a_4\gamma_2)S^{-1}.
\end{aligned}$$

The first derivative of the function $f_2(y)$ *is*

$$f_2'(y) = \left(k_0 + k_1\ y + k_2\ y^2\right)^{r-1}\left(G_0 + G_1\ y + G_2\ y^2\right)^{-(r+1)}\ P_2(y),$$

where $P_2(y) = r(G_0k_1 - G_1k_0) + r(2G_0k_2 - 2G_2k_0)\ \ y + r(G_1k_2 - G_2k_1)\ \ y^2$. *Now to draw all the possible graphics of the function* $f_2(y)$, *we denote by* Δ, Δ_1 *and* Δ_2 *the discriminants of the quadratic equations* $G_0 + G_1\ y + G_2\ y^2 = 0$, $k_0 + k_1\ y + k_2\ y^2 = 0$ *and* $P_2(y) = 0$, *respectively.*

If $\Delta, \Delta_1 > 0$ the function $f_2(y)$ becomes a particular case of $g_1(y)$, i.e., $p = q = r$. Then the graphics of $f_2(y)$ are equivalently the same graphics as the ones of $g_1(y)$ when both p, q are odd or even which provide the graphics shown in Figures 4.7 and 4.9.

If $\Delta, \Delta_1 \leq 0$ the function $f_2(y)$ have the same graphics as the function $f_1(y)$. Then the graphics of $f_2(y)$ are illustrated in Figure 4.6.

For $r > 0$ and according to the sign of the derivative $f_2'(y)$ which is related to the nature of the parameter r and on the sign of discriminates Δ, Δ_1 and Δ_2, we study all the possible graphics of the function $f_2(y)$ in what follows.

If either r is an even integer or $r = k_1/(2k_2+1)$ with k_1 and k_2 in $\mathbb{N}$, the possible graphics of the function $f_2(y)$ are shown in Figure 4.10(a) and Figure 4.10(b) if $\Delta < 0$, $\Delta_1 = 0$ and $\Delta_2 > 0$; or in Figure 4.10(c) if $\Delta < 0$ and $\Delta_1, \Delta_2 > 0$; or in Figure 4.10(d) and Figure 4.10(e) if Δ, $\Delta_2 > 0$ and $\Delta_1 < 0$; or in Figure 4.10(f) and Figure 4.10(g) if $\Delta = 0$, $\Delta_1 < 0$ and $\Delta_2 > 0$.

If the integer r is odd, the possible graphics of the function $f_2(y)$ are illustrated in Figures 4.10(a) and 4.10(b) if $\Delta < 0$, $\Delta_1 = 0$ and $\Delta_2 > 0$; or in Figures 4.10(h) and 4.10(i) if $\Delta < 0$ and $\Delta_1, \Delta_2 > 0$; or in Figures 4.10(j) and 4.10(k) if $\Delta, \Delta_2 > 0$ and $\Delta_1 < 0$; or in Figures 4.10(l) and 4.10(m) if $\Delta = 0$, $\Delta_1 < 0$ and $\Delta_2 > 0$.

If $r = k_1/(2k_2)$ with k_1, k_2 in $\mathbb{N}$ and $k_1 \neq 0 \neq k_2$, the function $f_2(y)$ is well defined on D_{f_2} when $(k_0 + k_1\, y + k_2\, y^2)(G_0 + G_1\, y + G_2\, y^2) \geq 0$ and $G_0 + G_1\, y + G_2\, y^2 \neq 0$, then the graphics of $f_2(y)$ are the parts drawn on D_{f_2} when r is an odd integer.

Similarly if $r < 0$ we obtain the identical graphics as $r > 0$.

According to the sign of the derivative $g_2'(y)$ given by

$$g_2'(y) = 2m_2\left(\frac{(m_1+m_2\, y)^2 + (m_3 - m_2\, y)^2 + 2}{((m_1+m_2\, y)^2+1)((m_3-m_2\, y)^2+1)}\right)e^{2\left(\arctan(m_1+m_2\, y) - \arctan(m_3 - m_2\, y)\right)},$$

which depends only on the parameter m_2, we get the two different possible graphics (a) of Figure 4.8 if $m_2 > 0$, and (b) of Figure 4.8 if $m_2 < 0$.

In this case the upper bound number of the intersection points between $f_2(y)$ and $g_2(y)$ is reached when r is an even integer, $\Delta > 0$ and $\Delta_1 > 0$, and the graphics of $f_2(y)$ are the ones drawn in Figure 4.7 because in this case the function $f_2(y)$ has the form of the function $g_1(y)$ and we proved in the previous subcases the reason in order that r must be an even integer. According to the graphics of the function $g_2(y)$ shown in Figure 4.8 and due to the fact there are no intersecting points at infinity, it results that these graphics can intersect at most in nine points. Consequently the maximum number of limit cycles of the PWS (1.3)–(4.1) for

$4a_4 - a_1^2 > 0$ *and* $a_2 = 0$ *is at most four.*

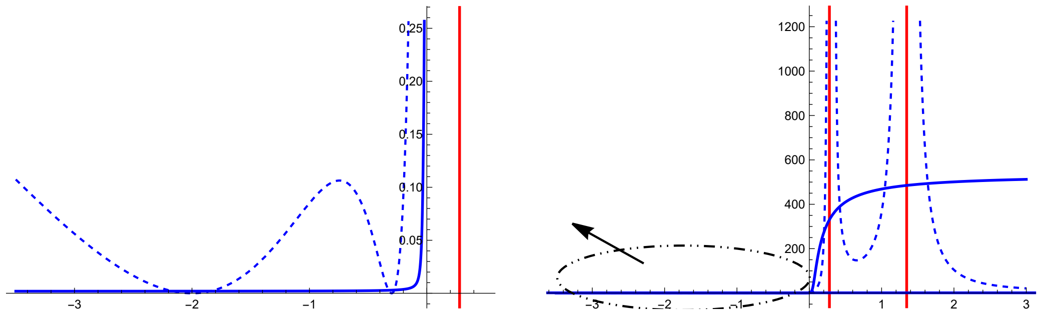


Figure 4.3: The nine points intersection between the graphics of $f_2(y)$ presented in a continuous line and $g_2(y)$ presented in a dashed line. The vertical lines represent the asymptote's straight lines.

By taking $\{k_0, k_1, k_2, G_0, G_1, G_2, r, s_1, s_2, s_3\} \longrightarrow \{0.3, 1.15, 0.5, 0.15, -0.65, 0.4, 2, 1, 1, -30\}$ *we build an example in which the graphics of the two functions* $f_2(y)$ *and* $g_2(y)$ *intersect in nine points. These points are shown in Figure 4.4.*

Subcase 2.4. *If* $4a_4 + a_2^2 < 0$ *and* $a_1 = 0$*, so* $k = 4$ *and* $j = 2$ *in system (4.9), and the first integral of system (4.1) is* $H_2^{(4)}(x,y)$ *given by (4.6). Then to solve* $F(y) = 0$ *it is equivalent to solve* $f_2(y) = g_2(y)$ *given in the previous subcase 2.3, with*

$$
\begin{aligned}
k_0 = {} & (\gamma_1^2+\gamma_2^2)k^{-2}(16\beta^4(\gamma_1(a_2+a_4\gamma_1)-1)+8\beta^2(\omega^2(\gamma_1(a_2+a_4\gamma_1)-1)+4\alpha\beta_1 d_1(a_2\\
& +2a_4\gamma_1))+\omega^4(a_2\gamma_1-1)+8\alpha a_2\beta_1 d_1\omega^2+a_4(\gamma_1\omega^2+8\alpha\beta_1 d_1)^2),\\
k_1 = {} & 16\alpha\beta_1 d_1 k^{-1}(\beta_1\gamma_1(a_2+a_4\gamma_1)+\beta_2\gamma_2(a_2+2a_4\gamma_1)-a_4\beta_1\gamma_2^2)+a_2\beta_1\gamma_1^2-a_2\beta_1\gamma_2^2\\
& +2a_2\beta_2\gamma_1\gamma_2-2a_4\beta_1\gamma_1\gamma_2^2+2a_4\beta_2\gamma_1^2\gamma_2+128\alpha^2 a_4\beta_1^2 d_1^2 k^{-2}(\beta_1\gamma_1+\beta_2\gamma_2)-2\beta_1\\
& \gamma_1-2\beta_2\gamma_2,\\
k_2 = {} & 8\alpha\beta_1 d_1 k^{-1}\left(a_2\left(\beta_1^2+\beta_2^2\right)-2a_4\left(\beta_1^2\gamma_1+2\beta_1\beta_2\gamma_2-\beta_2^2\gamma_1\right)\right)-a_2\beta_1^2\gamma_1-2a_2\beta_1\beta_2\gamma_2\\
& +a_2\beta_2^2\gamma_1-2a_4\beta_1^2\gamma_1^2+a_4\beta_1^2\gamma_2^2-4a_4\beta_1\beta_2\gamma_1\gamma_2+a_4\beta_2^2\gamma_1^2+64\alpha^2 a_4\beta_1^2 d_1^2 k^{-2}(\beta_1^2+\beta_2^2)\\
& -\beta_1^2-\beta_2^2,\\
G_1 = {} & \beta_1\gamma_2^2(a_2+2a_4\gamma_1)-2\beta_2\gamma_2(\gamma_1(a_2+a_4\gamma_1)-1)+64k^{-2}\alpha^2\beta_1 d_1^2(a_2+2a_4\gamma_1)\left(\beta_1^2+\beta_2^2\right)\\
& +16\alpha d_1/(4\beta^2+\omega^2)\left(\beta_1\beta_2\gamma_2(a_2+2a_4\gamma_1)-\beta_2^2\gamma_1(a_2+a_4\gamma_1)+\beta_1^2\left(a_4\gamma_1^2+1\right)+\beta_2^2\right)\\
& +\beta_1\gamma_1(2-a_2\gamma_1),\\
G_2 = {} & -16\alpha\beta_1 d_1 k^{-1}(a_2(\beta_1^2+\beta_2^2)+a_4(\beta_1^2\gamma_1-\beta_1\beta_2\gamma_2+2\beta_2^2\gamma_1))-a_2\beta_1^2\gamma_1-2a_2\beta_1\beta_2\gamma_2+a_2\beta_2^2\\
& \gamma_1-2a_4\beta_1^2\gamma_1^2+a_4\beta_1^2\gamma_2^2-4a_4\beta_1\beta_2\gamma_1\gamma_2+a_4\beta_2^2\gamma_1^2+64\alpha^2 a_4\beta_1^2 d_1^2 k^{-2}\left(\beta_1^2+\beta_2^2\right)-\beta_1^2-\beta_2^2,\\
m_1 = {} & (a_2+2a_4\gamma_1)/S, \;\; m_2 = (2a_4\beta_1)/S, \;\; m_3 = S^{-1}\left(a_2+2a_4\gamma_1+16\alpha a_4\beta_1 d_1 k^{-1}\right),\\
r = {} & S/a_2, \;\; S = \sqrt{-a_2^2-4a_4}, \quad k = 4\beta^2+\omega^2.
\end{aligned}
$$

Consequently the maximum number of limit cycles of the PWS (1.3)–(4.1) for $4a_4 - a_2^2 < 0$

and $a_1 = 0$ is at most four.

The maximum number of limit cycles in subcase 2.3 and 2.4 is at most four, but we can only build an example having three limit cycles for the class formed by a linear center and a rigid center with $4a_4 - a_1^2 > 0$ and $a_2 = 0$. In the region Σ_1^r we consider the linear differential center

$$\dot{x} = \frac{4x}{5} - \frac{881y}{800} + \frac{9}{10}, \quad \dot{y} = 2x - \frac{4y}{5} + 1, \tag{4.21}$$

with the first integral $H(x,y) = 4\left(2x - \frac{4y}{5}\right)^2 + 16\left(x - \frac{9y}{10}\right) + \frac{25y^2}{4}$. In the region Σ_2^r we consider the rigid center

$$\begin{aligned}
\dot{x} \approx\ & x^2(1.33839 - 0.126705y) + x(y(5.14213 - 0.155116y) - 8.99951) \\
& -0.025x^3 + y(4.56894y - 16.6475) + 11.9886, \\
\dot{y} \approx\ & x^2(-0.025y - 0.270459) + x((-0.126705y - 0.76873)y + 0.93212) \\
& +y((-0.155116y - 0.268073)y + 0.502979) + 4.79534,
\end{aligned} \tag{4.22}$$

with the first integral

$$\begin{aligned}
H_2^{(3)}(x,y) \approx\ & (52x^2 + x(218.821y - 696.045) + y(231.874y - 1428.38) + 2524.64)/(x^2 \\
& +x(6y - 14) + y(9y - 42) + 149)e^{2\arctan(0.1x + 0.3y - 0.7)},
\end{aligned}$$

For the PWS (4.21)–(4.22), system (4.9) has the three solutions $(y_1, y_2) \approx (-0.842999, 2.47751)$, $(y_3, y_4) \approx (-0.654549, 2.28905)$ and $(y_5, y_6) \approx (-0.43812, 2.07263)$ which provide the three limit cycles intersecting the separation regular line Σ^r in the six points $(0, y_j)$ with $j = 1, \ldots, 6$, see Figure 4.5(a).

Subcase 2.5. *If $4a_4 - a_1^2 = 0$ and $a_2 = 0$, then $k = 5$ and $j = 2$ in system (4.9), and $H_2^{(5)}(x,y)$ given by (4.7) is the first integral of the rigid center (4.1). Now to solve $F(y) = 0$ it is sufficient to solve $f_3(y) = g_3(y)$ with*

$$f_3(y) = \frac{k_0 + k_1\, y + k_2\, y^2}{G_0 + G_1\, y + k_2\, y^2} \quad \text{and} \quad g_3(y) = \left(\frac{m_1 + m_2\, y}{m_3 - m_2\, y}\right)^2 e^{(m_1 + m_2\, y)^{-1} - (m_3 - m_2\, y)^{-1}},$$

and

$$m_1 = -(a_1\gamma_2 + 2)/4, \;\; m_2 = -(a_1\beta_2)/4, \;\; m_3 = -\left(a_1\gamma_2 + (8\alpha a_1\beta_2 d_1)/(4\beta^2 + 2 + \omega^2)\right)/4,$$

and k_0, k_1, k_2, G_0, G_1 are the same with the subcase 2.1.

It is clear that the function $f_3(y)$ is a particular case of $f_1(y)$ where $r = 1$ and $\Delta = -4(\beta_2\gamma_1 - \beta_1\gamma_2)^2$, then the corresponding graphics of $f_3(y)$ are Figure 4.6(a) if $\Delta = 0$, and Figures 4.6(d) and 4.6(e) if $\Delta < 0$.

For the function $g_3(y)$, the first derivative of this function is

$$g_3'(y) = (m_3 - m_2\, y)^{-4} P_3(y) e^{(m_1+m_2\, y)^{-1}-(m_3-m_2\, y)^{-1}},$$

where

$$P_3(y) = m_2\left(2m_1^2 m_3 - m_1^2 + 2m_1 m_3^2 - m_3^2\right) - 2m_2^2(m_1 - m_3)(m_1 + m_3 + 1)\, y - 2m_2^3(m_1 + m_3 + 1)\, y^2.$$

Since $m_2 \neq 0$, and the variation of $g_3(y)$ depends on the ones of the quadratic polynomial $P_3(y)$. In Figure 4.12 we show all the possible graphics of $g_3(y)$, where (a) and (b) correspond to the case in which the polynomial $P_3(y)$ has two distinct real roots, (c) and (d) when $P_3(y)$ has one double real root for $P_3(y)$ and (e) and (f) if $P_3(y)$ has two complex roots for $P_3(y)$.

From the graphics of the function $g_3(y)$ shown in Figure 4.12, and due to the variation of this function it is obvious that we get the maximum number of the intersection points by intersecting the graphics (a) and (b) of Figure 4.12 with the graphics of the function $f_3(y)$. We guarantee that at infinity there are no intersection points, because the two functions $f_3(y)$ and $g_3(y)$ share the same horizontal asymptote straight line $y = 1$. Then we remark that these graphics can intersect at most in five points. Consequently the upper bound of the number of limit cycles in this case for the PWS (1.3)–(4.1) for $4a_4 - a_1^2 = 0$ and $a_2 = 0$ is at most two.

By taking $\{m_1, m_2, m_3, k_0, k_1, k_2, G_0, G_1\} \longrightarrow \{-2, 2, 6, 1/4, 1, 1, 25/4, 5\}$ we build an example in which the functions $f_3(y)$ and $g_3(y)$ intersects in five points shown in Figure 4.4.

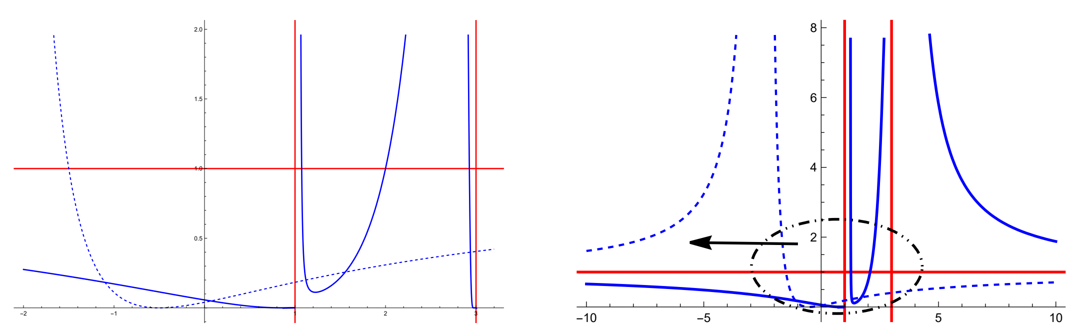


Figure 4.4: The five intersection points between the graphics of $f_3(y)$ presented in a continuous line and $g_3(y)$ presented in a dashed line. The straight lines represent the asymptotes straight lines.

Subcase 2.6. *If $4a_4+a_2^2=0$ and $a_1=0$, then $k=6$ and $j=2$ in system (4.9), and $H_2^{(6)}(x,y)$ given by (4.8) is the first integral of system (4.1). To solve $F(y)=0$ it is sufficient to solve the equation $f_3(y)=g_3(y)$ mentioned in the subcase 2.5, with*

$$m_1 = \ 0.25(a_2\gamma_1-2),\ \ m_2=0.25a_2\beta_1,\ \ m_3=0.25(a_2\gamma_1+8\alpha a_2\beta_1 d_1(4\beta^2+\omega^2)^{-1}-2),$$

and the expressions of k_0,k_1,k_2,G_0,G_1 are the same as the ones given in subcase 2.1.

Working in a similar way to the previous subcase the maximum number of limit cycles of the PWS (1.3)–(4.1) under the present conditions is at most two.

Finally we construct an example with two limit cycles of the PWS (1.3)–(4.1) satisfying $4a_4+a_2^2=0$ and $a_1=0$ to reach the result of statement (II). In the right half-plane Σ_1^r we consider the linear differential center

$$\dot{x}=0.5x-461/580y+0.9,\quad \dot{y}=0.05(29x-10y+30), \tag{4.23}$$

with the first integral $H(x,y)=4(2.9x/2-0.5y)^2+58/5(1.5x-0.9y)+3.61y^2$. In the left half-plane Σ_2^r we consider the rigid center

$$\begin{aligned}\dot{x}\approx\ & 59.062-2.5\times10^{-9}x^3+x^2(0.00162997+1.25625\times10^{-6}y)+x((0.006187y\\&-0.089255)y+2.33048)+y(4.41817y-57.1156),\\ \dot{y}\approx\ & -81.4419+x^2(5.70054\times10^{-8}-2.5\times10^{-9}y)+x(-0.00260418+(6.00594\\&+1.25626y)10^{-6}y)+y((0.000601877y-0.112201)y+5.23317),\end{aligned} \tag{4.24}$$

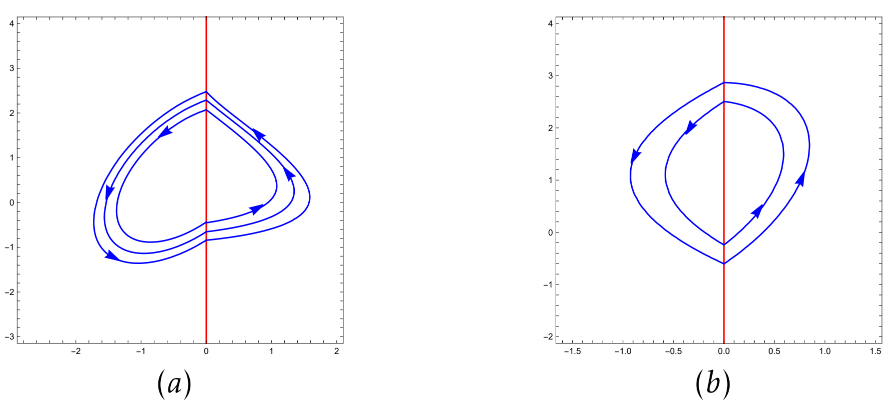


Figure 4.5: (a) The three limit cycles of the PWS (4.21)–(4.22), (b) the two limit cycles of the PWS (4.23)–(4.24).

with the first integral

$$H_2^{(6)}(x,y) \approx \quad (x(26139.5 - 502.503y) + x^2 + y(367006y - 2.04258 \times 10^7) + 3.28816 \times 10^8)/((x + 300y - 19500)^2)e^{-R(x,y)},$$

with $R(x,y) = (4 \times 10^4)/(x + 300(y - 65))$. *In this case system* (4.9) *has the two solutions* $(y_1, y_2) \approx (-0.604777, 2.86942)$ *and* $(y_3, y_4) \approx (-0.242279,\ 2.50692)$ *which produce the two limit cycles for the PWS* (4.23)–(4.24), *see Figure* 4.5(b). *Then statement* (II) *is held.* ■

All the graphics of the functions $f_k^{(j)}(y)$ and $g_k^{(k)}(y)$ with $j = i, ii$ and $k = 1, 2, 3$

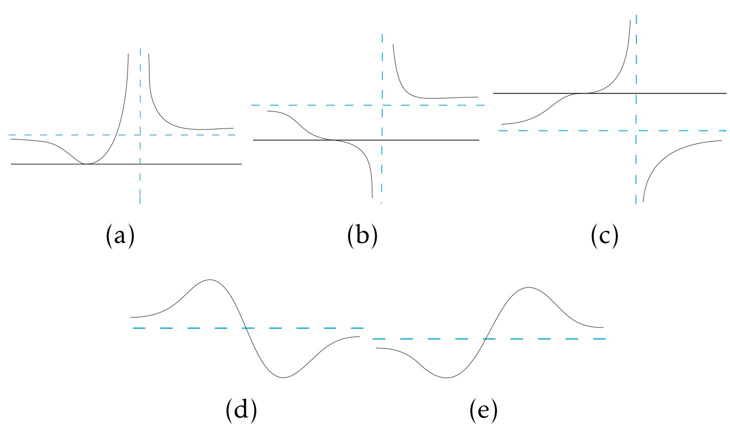


Figure 4.6: The graphics of the function $f_1(y)$.

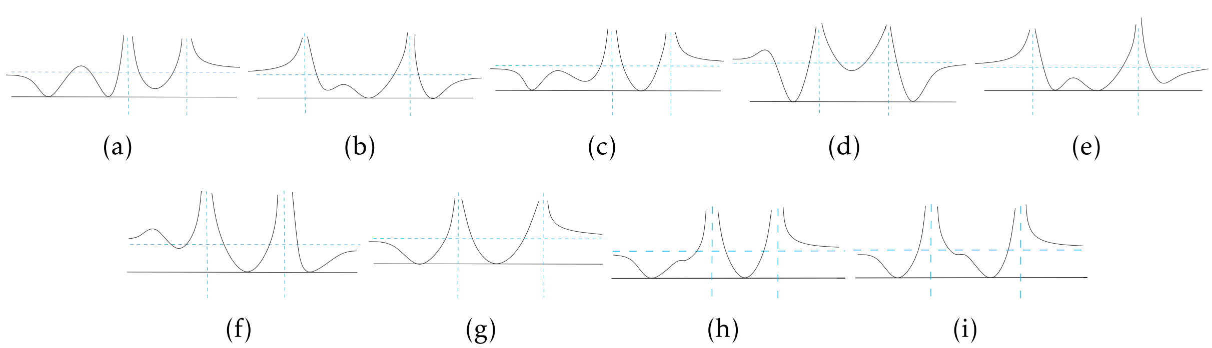


Figure 4.7: The graphics of the function $g_1(y)$ if p and q are even, or if p even and $q = k_1/(2k_2 + 1)$ with $k_1, k_2 \in \mathbb{N}$.

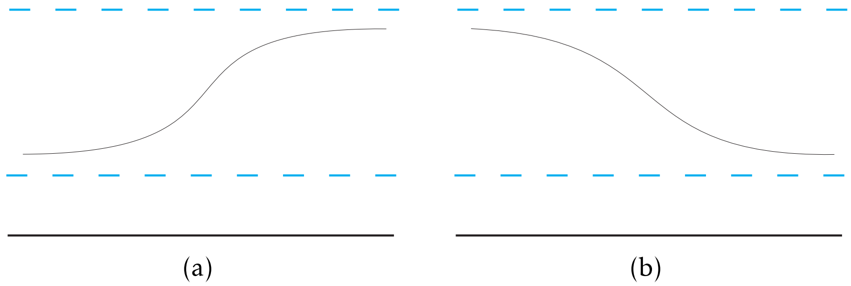


Figure 4.8: The two possible graphics of the function $g_2(y)$.

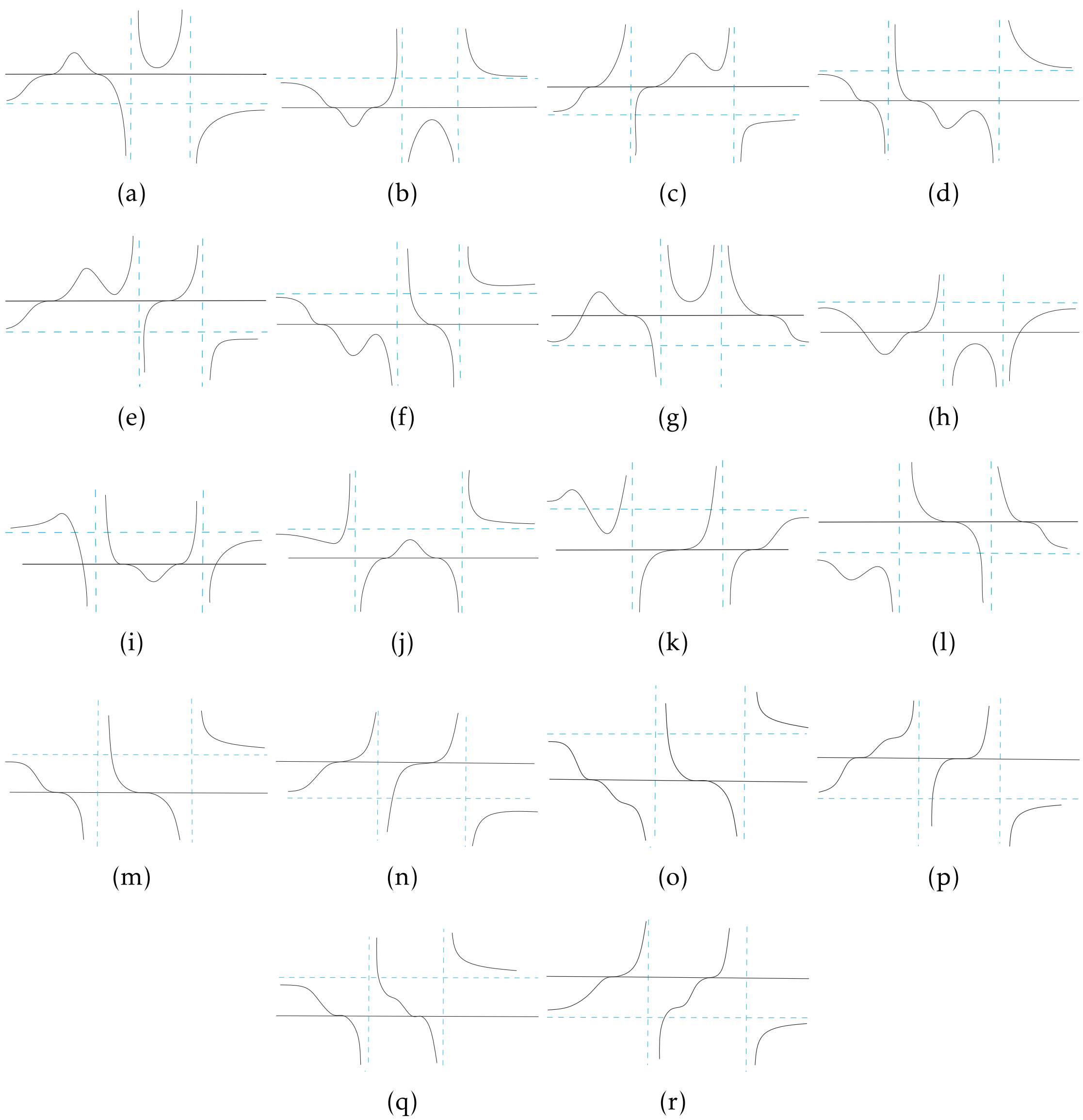


Figure 4.9: The graphics of the function $g_1(y)$ if p and q are odd.

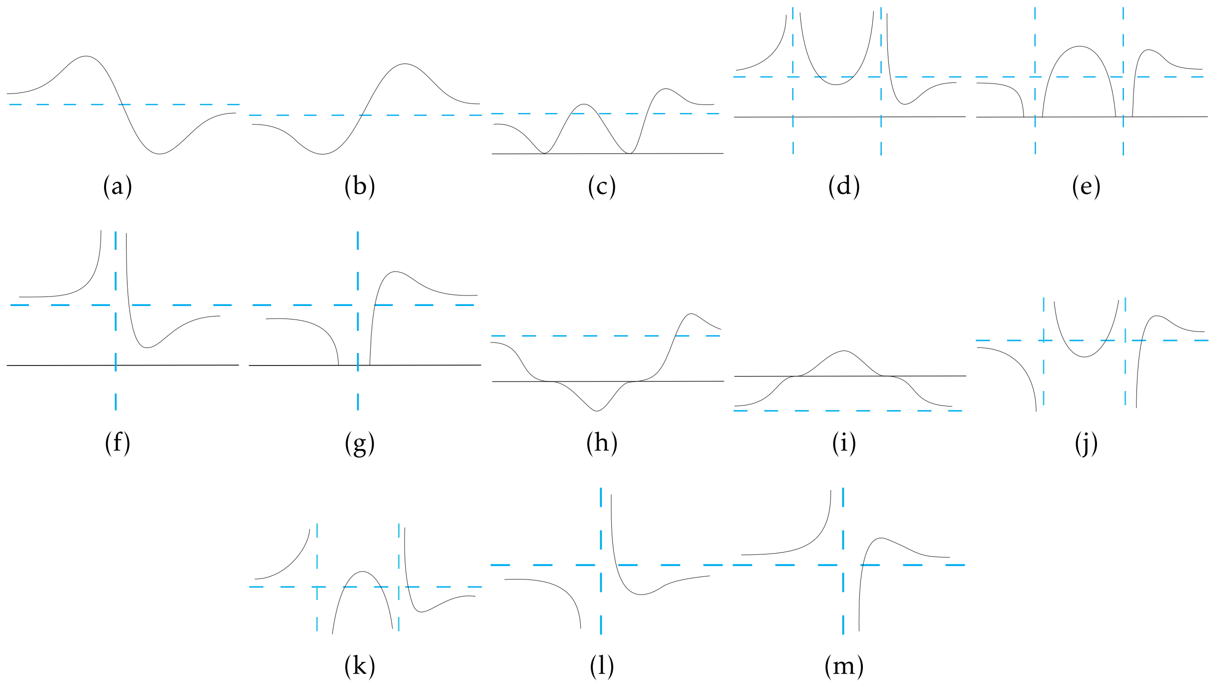


Figure 4.10: The graphics of the function $f_2(y)$.

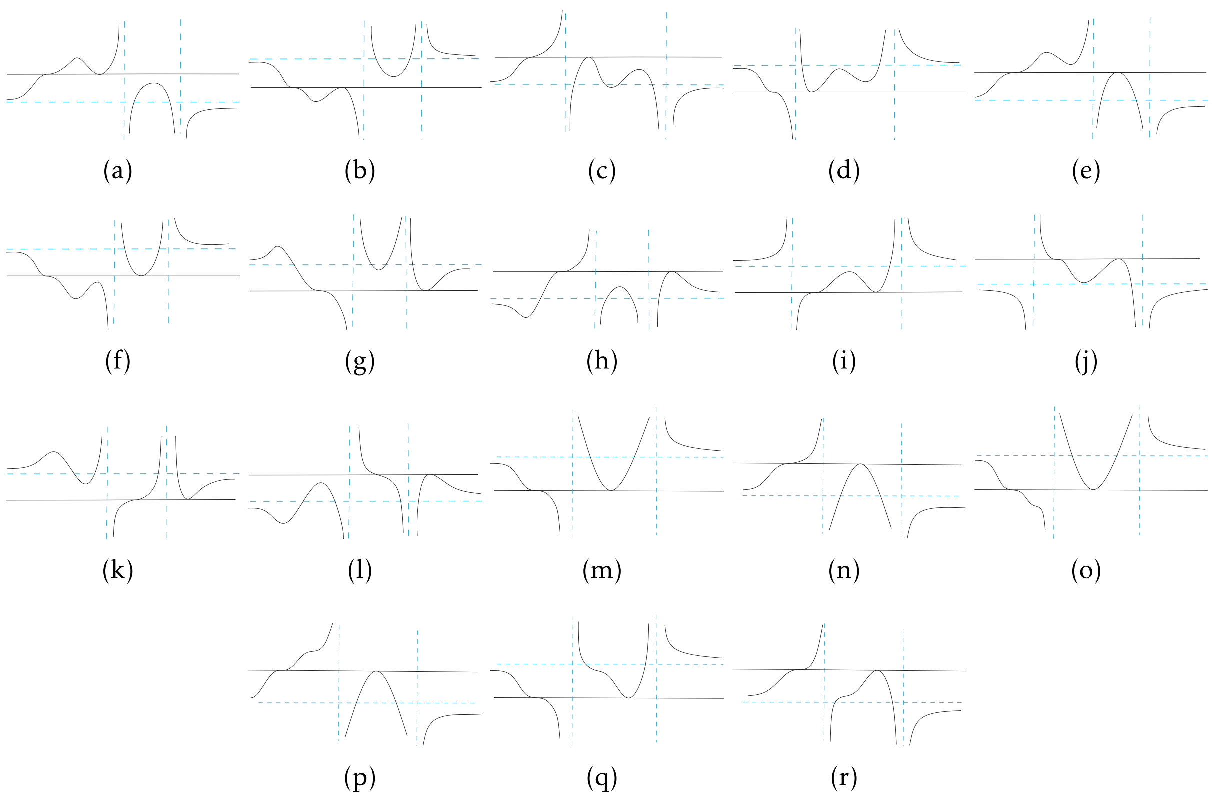


Figure 4.11: The graphics of the function $g_1(y)$ if p odd and q even, or p odd and $q = l_1/(2l_2+1)$ with $l_1, l_2 \in \mathbb{N}$.

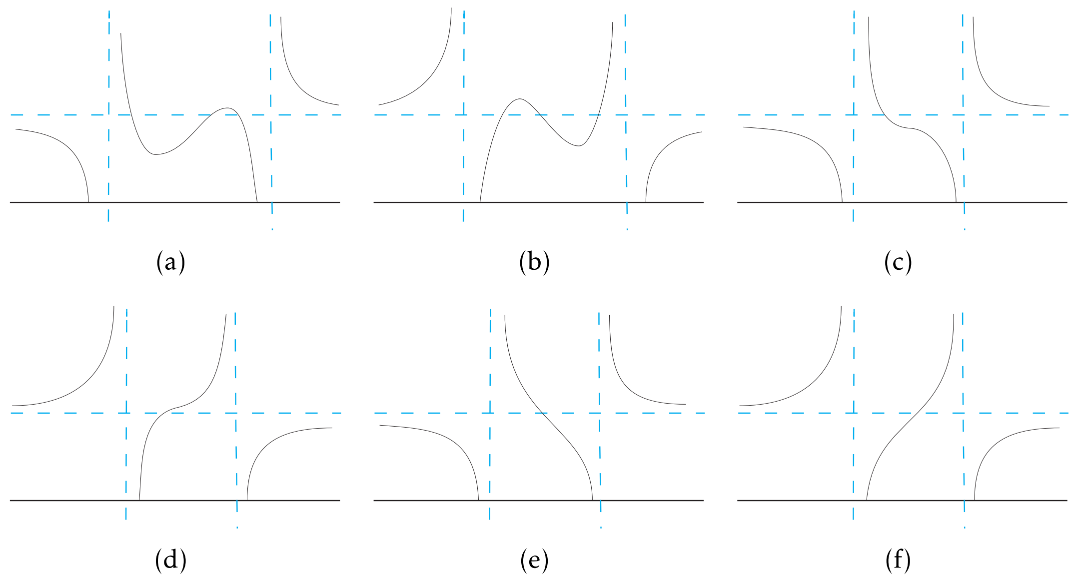


Figure 4.12: The graphics of the function $g_3(y)$.

CHAPTER 5

The Limit Cycles of Discontinuous Piecewise Differential System Formed by an Arbitrary Linear and Quadratic Centers Separated by Σ^i

This chapter is a result of our paper entitled "Limit cycles of the discontinuous piecewise differential systems separated by a non-regular line and formed by a linear center and a quadratic one", published in International Journal of Bifurcation and Chaos.

Up to now most of the published papers interested in planar PWS separated by a regular line. But when the discontinuity curve is not a regular line in 2013 [15, 16, 47] the authors found more than three limit cycles for PWS formed by linear systems with two zones. In 2021 [56] the authors found two limit cycles for a class of PWS formed by linear differential centers and separated by the irregular line Σ^i.

From [12] Benabdellah *et al.* have proved that there are no more than four limit cycles for the class of PWS separated by a regular line and formed by linear and a quadratic centers, because of that we inspired to study the maximum number of limit cycles for the class of PWS formed by linear center and quadratic centers and separated by the irregular line Σ^i instead of Σ^r.

We denote by $\mathcal{C}_k$ with $k = i, ii, iii, iv$ the four classes of PWS separated by the irregular line Σ^i, and formed by two pieces, in one piece there is an arbitrary linear differential center, and in the other piece there is a quadratic center in the classification of Kapteyn-Bautin Theorem after an arbitrary affine change of variables.

Generalized quadratic center

In this part we will make the general affine change of variables (1.8) for the quadratic system and its first integral. Thus the quadratic differential system (1.6) becomes:

$$\dot{x} = \quad (\alpha_2\beta_1 - \alpha_1\beta_2)^{-1}\Big(x^2\Big(a\beta_1(\alpha_1-\alpha_2)(\alpha_1+\alpha_2) + A\alpha_1\alpha_2\beta_1 + \beta_2\Big(\alpha_1^2 b + \alpha_1\alpha_2 C + \alpha_2^2 d\Big)\Big)$$

$$
\begin{aligned}
&+y^2(a\beta_1^3+\beta_1^2\beta_2(A+b)\beta_1\beta_2^2(C-a)+\beta_2^3 d)+\gamma_1(a\beta_1\gamma_1+b\beta_2\gamma_1+\beta_1)+\gamma_2(A\beta_1\\
&\gamma_1+\beta_2+\beta_2 C\gamma_1)+\gamma_2^2(\beta_2 d-a\beta_1)y(\beta_1\beta_2(-2a\gamma_2+A\gamma_1+2b\gamma_1+C\gamma_2)+\beta_1^2\\
&(2a\gamma_1+A\gamma_2+1)+\beta_2^2(C\gamma_1+2d\gamma_2+1))+x(\alpha_1(2a\beta_1^2 y+\beta_1+2a\beta_1\gamma_1+\beta_1\\
&\beta_2 y+(A+2b)+A\beta_1\gamma_2+2b\beta_2\gamma_1+\beta_2 C(\gamma_2+\beta_2 y))+\alpha_2(A\beta_1(\gamma_1\beta_1 y)+\beta_2\\
&-2a\beta_1(\gamma_2+\beta_2 y)+\beta_2+C(\gamma_1+\beta_1 y)+2\beta_2 d(\gamma_2+\beta_2 y)))\Big),\\
\dot{y}=\;&(\alpha_1\beta_2-\alpha_2\beta_1)^{-1}\Big(x^2\Big(a\alpha_1^3+\alpha_1\alpha_2^2(C-a)+\alpha_1^2\alpha_2(A+b)+\alpha_2^3 d\Big)+y^2(a\alpha_1(\beta_1\\
&-\beta_2)(\beta_1+\beta_2)+\beta_2(A\alpha_1\beta_1+\alpha_2\beta_1 C+\alpha_2\beta_2 d)+\alpha_2 b\beta_1^2)+\gamma_1(a\alpha_1\gamma_1+\alpha_1+\alpha_2\\
&b\gamma_1)+\gamma_2^2(\alpha_2 d-a\alpha_1)+\gamma_2(A\alpha_1\gamma_1+\alpha_2+\alpha_2 C\gamma_1)+y(\alpha_1(2a\beta_1\gamma_1-2a\beta_2\gamma_2\\
&+A\beta_1\gamma_2+A\beta_2\gamma_1+\beta_1)+\alpha_2(2b\beta_1\gamma_1+\beta_2+\beta_1 C\gamma_2+\beta_2 C\gamma_1+2\beta_2 d\gamma_2))+x\\
&(\alpha_1\alpha_2(-(2a-C)(\gamma_2+\beta_2 y)+A(\gamma_1+\beta_1 y)+2b(\gamma_1+\beta_1 y))+\alpha_1^2(2a(\gamma_1+\beta_1\\
&y)+A(\gamma_2+\beta_2 y)+1)+\alpha_2^2(C(\gamma_1+\beta_1 y)+2d(\gamma_2+\beta_2 y)+1))\Big).
\end{aligned}
\tag{5.1}
$$

I. If the quadratic system (5.1) satisfies (i) of Theorem 1.1. The first integral becomes If $A+b=0\neq A$

$$
H_1^{(i)}(x,y)=\Big(A(\gamma_2+\alpha_2 x+\beta_2 y)+1\Big)^{2d}e^{(1+A(\gamma_2+\alpha_2 x+\beta_2 y))^{-2}Z_1(x,y)}, \tag{5.2}
$$

where

$$
Z_1(x,y)=A(A^2(\gamma_1+\alpha_1 x+\beta_1 y)^2-2A(\gamma_2+\alpha_2 x+\beta_2 y)+4d(\gamma_2+\alpha_2 x+\beta_2 y)-1)+3d.
$$

If $A=0\neq b$, then

$$
\begin{aligned}
H_2^{(i)}(x,y)=\;&\Big(2b^3(\gamma_1+\alpha_1 x+\beta_1 y)^2+2b^2 d(\gamma_2+\alpha_2 x+\beta_2 y)^2+2b(b-d)\\
&(\gamma_2+\alpha_2 x+\beta_2 y)+d-b\Big)e^{2b(\gamma_2+\alpha_2 x+\beta_2 y)}.
\end{aligned}
\tag{5.3}
$$

If $b=0\neq A$

$$
H_3^{(i)}(x,y)=\Big(1+A(\gamma_2+\alpha_2 x+\beta_2 y)\Big)^{2(d-A)}e^{Z_2(x,y)}, \tag{5.4}
$$

where

$$
Z_2(x,y)=A\Big(A^2(\gamma_1+\beta_1 y+\alpha_1 x)^2+Ad(\gamma_2+\beta_2 y+\alpha_2 x)^2+2(A-d)(\gamma_2+\alpha_2 x+\beta_2 y)\Big).
$$

If $A = b = 0$

$$H_4^{(i)}(x,y) = \quad 2d\Big(\gamma_2+\alpha_2 x+\beta_2 y\Big)^3 + 3\Big((\gamma_1+\alpha_1 x+\beta_1 y)^2 + (\gamma_2+\alpha_2 x+\beta_2 y)^2\Big). \tag{5.5}$$

II. If the quadratic system (5.1) satisfies the second condition (ii) of Theorem 1.1 and $\Delta = C^2 + 4b(A+b)$. The first integral becomes

If $A + b = 0$ and $a = 0 \neq Cb$

$$\begin{aligned} H_1^{(ii)}(x,y) = \quad & e^{Z_3(x,y)}\Big(-b(\gamma_2+\alpha_2 x+\beta_2 y) + C(\gamma_1+\alpha_1 x+\beta_1 y)+1\Big)^{b^2} \\ & \Big(1-b(\gamma_2+\alpha_2 x+\beta_2 y)\Big)^{-b^2-C^2}, \end{aligned} \tag{5.6}$$

where $Z_3(x,y) = bC(b(\gamma_1+\alpha_1 x+\beta_1 y) + C(\gamma_2+\alpha_2 x+\beta_2 y))(b(\gamma_2+\alpha_2 x+\beta_2 y)-1)^{-1}$.

If $AbC(A+b)\Delta \neq 0 = a$ and $\Delta = -L^2 < 0$

$$\begin{aligned} H_2^{(ii)}(x,y) = \quad & \Big(1-\frac{1}{4b}(4b^2+C^2+L^2)(\gamma_2+\alpha_2 x+\beta_2 y)\Big)^r\Big(b^2(\gamma_2+\alpha_2 x+\beta_2 y)^2 \\ & -b(\gamma_2+\alpha_2 x+\beta_2 y)(C(\gamma_1+\alpha_1 x+\beta_1 y)+2) + \frac{1}{4}\Big(C^2+L^2\Big) \\ & (\gamma_1+\alpha_1 x+\beta_1 y)^2 + C(\gamma_1+\alpha_1 x+\beta_1 y)+1\Big)^{\frac{1}{b}} e^{Z_4(x,y)}, \end{aligned} \tag{5.7}$$

where $r = -8b(4b^2+C^2+L^2)^{-1}$ and

$$Z_4(x,y) = \frac{-2C}{bL}\cot^{-1}\left(\frac{L(\gamma_1+\alpha_1 x+\beta_1 y)}{2b(\gamma_2+\alpha_2 x+\beta_2 y)-C(\gamma_1+\alpha_1 x+\beta_1 y)-2}\right).$$

If $C = b = 0$

$$\begin{aligned} H_3^{(ii)}(x,y) = \quad & \Big(1+((\gamma_2+\alpha_2 x+\beta_2 y)(-(a^2(\gamma_2+\alpha_2 x+\beta_2 y))+aA(\gamma_1+\alpha_1 x+\beta_1 y) \\ & +A)\Big)/(a(\gamma_1+\alpha_1 x+\beta_1 y)+1)^2)^{-r}(a(\gamma_1+\alpha_1 x+\beta_1 y)+1)^{-2r} \\ & e^{2ar(\gamma_1+\alpha_1 x+\beta_1 y)-2A\coth^{-1}(Z_5(x,y))}, \end{aligned} \tag{5.8}$$

where $r = \sqrt{4a^2+A^2}$ and $Z_5(x,y) = \dfrac{-2a^2(\gamma_2+\alpha_2 x+\beta_2 y)+aA(\gamma_1+\alpha_1 x+\beta_1 y)+A}{r(a(\gamma_1+\alpha_1 x+\beta_1 y)+1)}$.

If $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$

$$H_4^{(ii)}(x,y) = \quad \left(\frac{1}{2}\Big(C-\sqrt{\Delta}\Big)(\gamma_1+\alpha_1 x+\beta_1 y) - b(\gamma_2+\alpha_2 x+\beta_2 y)+1\right)^{r_-}$$

$$\begin{aligned}&\left(\frac{1}{2}\left(C+\sqrt{\Delta}\right)(\gamma_1+\alpha_1 x+\beta_1 y)-b(\gamma_2+\alpha_2 x+\beta_2 y)+1\right)^{r_+}\\&\left(A(\gamma_2+\alpha_2 x+\beta_2 y)+1\right)^{\frac{1}{A}},\end{aligned} \tag{5.9}$$

where $r_\pm = \dfrac{\sqrt{\Delta}\pm C}{2b\sqrt{\Delta}}$.
If $b=a=0$ and $AC\neq 0$

$$H_5^{(ii)}(x,y) = \left(C(\alpha_1 x+\gamma_1+\beta_1 y)+1\right)^{2A^2}\left(A(\alpha_2 x+\gamma_2+\beta_2 y)+1\right)^{2C^2} e^{Z_5(x,y)}, \tag{5.10}$$

where $Z_5(x,y) = -2AC\Big(A(\alpha_1 x+\gamma_1+\beta_1 y)+C(\alpha_2 x+\gamma_2+\beta_2 y)\Big)$.

If $A=a=0$ and $Cb\neq 0$

$$\begin{aligned}H_6^{(ii)}(x,y) = \; & e^{\gamma_2+\alpha_2 x+\beta_2 y}\left(\frac{1}{2}(C-\sqrt{\Delta})(\gamma_1+\beta_1 y+\alpha_1 x)-b(\gamma_2+\beta_2 y+\alpha_2 x)+1\right)^{r_+}\\&\left(\frac{1}{2}(C+\sqrt{\Delta})(\alpha_1 x+\gamma_1+\beta_1 y)-b(\alpha_2 x+\gamma_2+\beta_2 y)+1\right)^{r_-},\end{aligned} \tag{5.11}$$

where $r_\pm = \dfrac{\sqrt{\Delta}\pm C}{2b\sqrt{\Delta}}$.
If $\Delta=a=0$ and $C\neq 0$

$$\begin{aligned}H_7^{(ii)}(x,y) = \; & \frac{1}{2}\left(-\frac{C^2}{4b}(\gamma_2+\alpha_2 x+\beta_2 y)-b(\gamma_2+\alpha_2 x+\beta_2 y)+1\right)^{-\frac{4b^2}{4b^2+C^2}}\\&\Big(-2b(\gamma_2+\alpha_2 x+\beta_2 y)+C(\gamma_1+\alpha_1 x+\beta_1 y)+2\Big)e^{Z_6(x,y)},\end{aligned} \tag{5.12}$$

where $Z_6(x,y) = 1+C(\alpha_1 x+\gamma_1+\beta_1 y)/(2b(\alpha_2 x+\gamma_2+\beta_2 y)-C(\alpha_1 x+\gamma_1+\beta_1 y)-2)$.

If $A=b=0$

$$H_8^{(ii)}(x,y) = (1+C(\alpha_1 x+\gamma_1+\beta_1 y))^2 e^{-C^2(\alpha_2 x+\gamma_2+\beta_2 y)^2-2C(\alpha_1 x+\gamma_1+\beta_1 y)}. \tag{5.13}$$

III. If the quadratic system (5.1) satisfies the third condition (iii) of Theorem 1.1. The first integral is

$$\begin{aligned}H^{(iii)}(x,y) = \; & \frac{1}{6}(2a(\gamma_1+\alpha_1 x+\beta_1 y)^3+6b(\gamma_1+\alpha_1 x+\beta_1 y)^2(\gamma_2+\alpha_2 x+\beta_2 y)\\&+3(\gamma_1+\alpha_1 x+\beta_1 y)^2-6a(\gamma_1+\alpha_1 x+\beta_1 y)(\gamma_2+\alpha_2 x+\beta_2 y)^2+2\\&d(\gamma_2+\alpha_2 x+\beta_2 y)^3+3(\gamma_2+\alpha_2 x+\beta_2 y)^2).\end{aligned} \tag{5.14}$$

IV. If the quadratic system (5.1) satisfies the fourth condition (iv) of Theorem 1.1. The first integral is

$$\begin{aligned} H^{(iv)}(x,y) = \ & \Big((a^2+d^2)(d(\gamma_2+\alpha_2 x+\beta_2 y)-a(\gamma_1+\alpha_1 x+\beta_1 y))^3-3ad(a^2 \\ & +d^2)(\gamma_1+\alpha_1 x+\beta_1 y)(\gamma_2+\alpha_2 x+\beta_2 y)+3d^2(a^2+d^2)(\gamma_2+\alpha_2 \\ & x+\beta_2 y)^2+3d(a^2+d^2)(\gamma_2+\alpha_2 x+\beta_2 y)+d^2\Big)^2/\Big((a^2+d^2)(a \\ & (\gamma_1+\alpha_1 x+\beta_1 y)-d(\gamma_2+\alpha_2 x+\beta_2 y))^2+2d(a^2+d^2)(\gamma_2+\alpha_2 \\ & x+\beta_2 y)+d^2\Big)^3. \end{aligned} \tag{5.15}$$

Here we interested to study the limit cycles of the four classes $\mathcal{C}_k$ with $k = i, ii, iii, iv$ of PWS that intersect with the discontinuity line Σ^i in two points, where we find two possible configurations of limit cycles.

The first configuration that we will denote by **Cnf 1** is when the limit cycles have two intersection points with $\Gamma_1 = \{(x,y) : x = 0 \text{ and } y \geq 0\}$ or $\Gamma_2 = \{(x,y) : x \geq 0 \text{ and } y = 0\}$ where $\Sigma^i = \Gamma_1 \cup \Gamma_2$. However the study of the limit cycles of the four classes of PWS $\mathcal{C}_k$ with $k = i, ii, iii, iv$ that intersect the rays Γ_1 or Γ_2 in two points is equivalent to the study of the same classes of systems separated by a regular line, done by Benabdallah *et al.* see [12].

The second configuration is denoted by **Cnf 2**, where the limit cycles have two intersection points with the separation curve Σ^i, the first point in Γ_1 and the second one in Γ_2, i.e., the first point of intersection is $(x_1, 0) \in \Gamma_1$ and the second point is $(0, y_2) \in \Gamma_2$. We notice that when we combine the two configurations **Cnf 1** and **Cnf 2** we obtain another configuration **Cnf 3** that has a combination between the two kinds of limit cycles. The case where we have two arbitrary linear differential centers (1.3) in each region has been solved in Theorem 3 of [41] and Theorem 4 of [26].

In the following theorem we will give the maximum number of limit cycles of the four classes $\mathcal{C}_k$ with $k = i, ii, iii, iv$ of PWS separated by the regular line Σ^r.

Theorem 5.1

The maximum number of limit cycles satisfying **Cnf 1** of the PWS separated by the regular line Σ^r and formed

(*a*) the class $\mathcal{C}_i$ is at most three if $A + b = 0 \neq A$; and one limit cycle if either $A = 0 \neq b$, $b = 0 \neq A$ or $A = b = 0$;

(b) the class $\mathcal{C}_{ii}$ is at most three if either $A+b=0$ and $a=0\neq Cb$, or $AbC(A+b)(4b(A+b)+C^2)\neq 0$, or $b=C=0$, or $A=a=0\neq Cb$ or $b=a=0\neq AC$; two if $\Delta=a=0$ and $C\neq 0$; and one if $A=b=0$;

(c) the class $\mathcal{C}_{iii}$ is at most one;

(d) the class $\mathcal{C}_{iv}$ is at most four.

Theorem 5.1 is proved in [12].

5.1 The limit cycles of the four classes of PWS $\mathcal{C}_k$ with $k=i,ii,iii,iv$ separated by Σ^i satisfying Cnf 2

The first main result concerning the upper bound of the number of limit cycles for the four classes $\mathcal{C}_k$ with $k=i,ii,iii,iv$ of PWS separated by the irregular line Σ^i satisfying **Cnf 2** is given in the following theorem.

Theorem 5.2

The maximum number of limit cycles satisfying **Cnf 2** for

(a) the class $\mathcal{C}_i$ is at most three if $A+b=0\neq A$; two if either $A=0\neq b$ or $b=0\neq A$; and one if $A=b=0$. There are systems of this class with three limit cycles see Figure 5.4(a), two limit cycles in Figure 5.4(b); and one limit cycle in Figure 5.4(c), respectively;

(b) the class $\mathcal{C}_{ii}$ is at most four either if $AbC(A+b)\Delta\neq 0=a$ and $\Delta\neq 0$ or $C=b=0$, or $A+b=0$ and $a=0\neq Cb$; three if either $b=a=0\neq AC$ or $A=a=0\neq Cb$; and two if $\Delta=a=0$, or $A=b=0$. There are systems of this class with four limit cycles in Figure 5.11(a), three limit cycles in Figure 5.11(b), and two limit cycles in Figure 5.11(c), respectively;

(c) the class $\mathcal{C}_{iii}$ is at most one. There are systems of this class with one limit cycle, see Figure 5.12(a);

(d) the class $\mathcal{C}_{iv}$ is at most five. There are systems of this class with five limit cycles, see Figure 5.12(b).

As in the pervious chapter we get the upper bound on the maximum number of limit cycles by studying the intersection of the graphics of many functions. Because of that we only give the graphics of the functions starting with a positive sign.

Proof of Theorem 5.2

Proof. *In one region we consider the linear differential center at the origin with its first integral $H(x,y)$. In the second region we consider the quadratic center (5.1) satisfying one of the four conditions of Theorem 1.1, with its first integral $H_k^{(j)}(x,y)$ with $k = 1,\dots,4$ for $j = i$, and $k = 1,\dots,8$ for $j = ii$ and $H^{(j)}(x,y)$ in case of $j = iii, iv$. If the PWS (1.3)–(5.1) has a limit cycle, this limit cycle must intersect the separation line Σ^i in two distinct points $p_1 = (x_1,0) \in \Gamma_2$ and $p_2 = (0,y_2) \in \Gamma_1$ where $x_1 > 0$ and $y_2 > 0$. These two points must satisfy the system of equations*

$$H(p_1) - H(p_2) = h(x_1,y_2) = 0, \quad H_k^{(j)}(p_1) - H_k^{(j)}(p_2) = h^{(j)}(x_1,y_2) = 0, \tag{5.16}$$

by solving $h(x_1,y_2) = 0$, we get $x_1 = \lambda y_2$ with $\lambda = \sqrt{\alpha^2+\omega^2}$ which is a function of the variable y_2. Substituting x_1 in $h^{(j)}(x_1,y_2) = 0$ we obtain $F_k^{(j)}(y_2) = 0$ when $j = i, ii$ and $F^{(j)}(y_2) = 0$ when $j = iii, iv$ that are equations in the variable y_2.

Proof of statement (a) of Theorem 5.2. *First we prove the statement for the first class $\mathcal{C}_i$ when $C = a = 0$. If $A + b = 0 \neq A$ corresponding to $k = 1$ and $j = i$ in system (5.16), the first integral of (5.1) is $H_1^{(i)}(x,y)$ given in (5.2), and the solutions of $F_1^{(i)}(y_2) = 0$ are equivalent to the solutions of the equation $f_1^{(i)}(y_2) = g_1^{(i)}(y_2)$ where*

$$f_1^{(i)}(y_2) = \left(\frac{s_1 + s_2\, y_2}{s_1 + s_3\, y_2}\right)^r \quad \text{and} \quad g_1^{(i)}(y_2) = e^{k(y_2)},$$

and

$$\begin{aligned}
k(y_2) = &\quad (k_1\ y_2 + k_2\ y_2^2 + k_3\ y_2^3 + k_4\ y_2^4)/\left[(s_1 + s_2\ y_2)(s_1 + s_3\ y_2)\right]^2,\\
k_1 = &\quad 2A(A\gamma_2+1)(A^2(\lambda(\alpha_1\gamma_1+\alpha_2\gamma_2)+A\gamma_1(\lambda(\alpha_1\gamma_2-\alpha_2\gamma_1)-\beta_1\gamma_2+\beta_2\gamma_1)-\beta_2\gamma_2)\\
&\quad -A^2\beta_1\gamma_1+d(2A\gamma_2+1)(\beta_2-\alpha_2\lambda)),\\
k_2 = &\quad A^2(A^3(\alpha^2(\alpha_1^2\gamma_2^2-\alpha_2^2\gamma_1^2)+\gamma_2(-4\alpha_2\beta_1\gamma_1\lambda+\alpha_1^2\gamma_2\omega^2-\beta_1^2\gamma_2)+4\alpha_1\beta_2\gamma_1\gamma_2\lambda-\alpha_2^2\\
&\quad \gamma_1^2\omega^2+\beta_2^2\gamma_1^2)+2A^2(\alpha^2\gamma_2(\alpha_1^2+\alpha_2^2)+2\alpha_1\beta_2\gamma_1\lambda-2\alpha_2\beta_1\gamma_1\lambda+\gamma_2\omega^2(\alpha_1^2+\alpha_2^2)-\\
&\quad \gamma_2(\beta_1^2+\beta_2^2))+A(\alpha^2(\alpha_1^2+\alpha_2^2(1-4d\gamma_2)-\beta_1^2)+\omega^2(\alpha_1^2+\alpha_2^2-4\alpha_2^2d\gamma_2)+(4d\gamma_2-1))
\end{aligned}$$

$$\begin{aligned}
&+3d(\beta_2^2-\alpha_2^2\lambda^2)),\\
k_3 = &\quad 2A^3(A^2\alpha_1\beta_2^2\gamma_1\lambda-A^2\alpha_2\beta_1^2\gamma_2\lambda+A\lambda^2(\alpha_1^2(A\beta_2\gamma_2+\beta_2)+\alpha_2^2(\beta_2-A\beta_1\gamma_1))-A\alpha_2\beta_1^2\lambda\\
&\quad -A\alpha_2\beta_2^2\lambda-2\alpha_2^2\beta_2d\lambda^2+2\alpha_2\beta_2^2d\lambda),\\
k_4 = &\quad -A^5\lambda^2(\alpha_2\beta_1-\alpha_1\beta_2)(\alpha_1\beta_2+\alpha_2\beta_1),\ \ s_1=A\gamma_2+1,\ \ s_2=A\beta_2,\ \ s_3=A\alpha_2\lambda,\ \ r=2d.
\end{aligned}$$

These solutions represent the intersection points between these two functions $f_1^{(i)}(y_2)$ and $g_1^{(i)}(y_2)$, for that we will draw all the possible graphics of these functions.

For $r > 0$, the function $f_1^{(i)}(y_2)$ has the horizontal asymptote straight line $h_1 = (s_2/s_3)^r$ and the vertical asymptote straight line $y_{22} = -s_1/s_3$. We denote by

$$\left(f_1^{(i)}\right)'(y_2) = \eta(s_1 + s_2\ y_2)^{r-1}(s_1 + s_3\ y_2)^{-(r+1)},$$

the first derivative of the function $f_1^{(i)}(y_2)$ with $\eta = rs_1(s_2 - s_3)$. Then to draw all the possible graphics for the function $f_1^{(i)}(y_2)$ we have to study the sign of its derivative which depends on the sign of the parameter η and r. It is clear that the first derivative of the function $f_1^{(i)}(y_2)$ vanish at $y_{21} = -s_1/s_2$ that can have the only possible position $y_{21} < y_{22}$ with the vertical asymptote straight line y_{22}, but we have $y_{22} < y_{21}$ when $r > 1$.

Here we draw only the graphics of the function $f_1^{(i)}(y_2)$ corresponding to the case when $y_{21} < y_{22}$, and by the symmetry with the vertical asymptote straight line we get the graphics corresponding to the case $y_{22} < y_{21}$. Then

1- *if either r is even or $r = 2k_1/(2k_2+1)$ with $k_1, k_2 \in \mathbb{N}$, the sign of $\left(f_1^{(i)}\right)'(y_2)$ depends on the sign of the product $\eta(s_1+s_2y_2)(s_1+s_3y_2)$. So the only possible graphic of this function is shown in Figure 5.16(a);*

2- *if r is odd or $r = (2k_1+1)/(2k_2+1)$ with $k_1, k_2 \in \mathbb{N}$, the sign of $\left(f_1^{(i)}\right)'(y_2)$ depends only on the sign of the parameter η. Here the graphics of this function are shown in Figure 5.16(b) if $\eta < 0$ and Figure 5.16(c) if $\eta > 0$;*

3- *if r is irrational, or $r = k_1/(2k_2)$ with k_1 an odd integer, then the sign of $\left(f_1^{(i)}\right)'(y_2)$ depends only on the sign of η. Consequently the graphics of $f_1^{(i)}(y_2)$ are given in Figure 5.16(b) and (c) restricted on its definition domain.*

For $r < 0$ and in a similar way we found the same graphics than the case $r > 0$.

Now to study all the possible graphics for the function $g_1^{(i)}(y_2)$, we need to study the sign of its derivative $\left(g_1^{(i)}\right)'(y_2) = [(s_1 + s_2\, y_2)(s_1 + s_3\, y_2)]^{-3}\, P(y_2).e^{k(y_2)}$, where

$$\begin{aligned} P(y_2) = \;& k_1 s_1^2 + s_1(2k_2 s_1 - k_1(s_2+s_3))\, y_2 + 3(k_3 s_1^2 - k_1 s_2 s_3)\, y_2^2 + (-2k_2 s_2 s_3 \\ & + k_3 s_1(s_2+s_3) + 4k_4 s_1^2)\, y_2^3 + (2k_4 s_1(s_2+s_3) - k_3 s_2 s_3)\, y_2^4. \end{aligned}$$

Here $y_{21} = -s_1/s_2$ and $y_{22} = -s_1/s_3$ represent the two vertical asymptote straight lines for the function $g_1^{(i)}(y_2)$, and $h_2 = e^{k_4(s_2 s_3)^{-2}}$ is the horizontal asymptote straight line, and if $s_2 \neq s_3$ we know that all the possible graphics of the function $g_1^{(i)}(y_2)$ are given as follows:

1- *If $P(y_2)$ has four simple real roots r_i for $i = 1,2,3,4$, and according to the possible positions of these roots with respect to the two vertical asymptotes, the graphics of $g_1^{(i)}(y_2)$ are given in Figure 5.20(a) and Figure 5.20(b) if $r_1 < r_2 < r_3 < y_{21} < r_4 < y_{22}$, or Figure 5.20(c) and Figure 5.20(d) if $r_1 < r_2 < y_{21} < r_3 < y_{22} < r_4$, or Figure 5.20(e) and Figure 5.20(f) if $r_1 < r_2 < y_{21} < r_3 < r_4 < y_{22}$, or Figure 5.20(g) and Figure 5.20(h) if $r_1 < y_{21} < r_2 < r_3 < y_{22} < r_4$, or Figure 5.20(i) and Figure 5.20(j) if $r_1 < y_{21} < r_2 < r_3 < r_4 < y_{22}$.*

2- *If $P(y_2)$ has one simple real root and one triple real root, or two complex roots and two simple real roots named by r_1 and r_2, then there are two possible positions for these roots with respect to the two vertical asymptotes. Then in this case the graphics of $g_1^{(i)}(y_2)$ are given in Figure 5.20(k) and Figure 5.20(l) if $r_1 < y_{21} < r_2 < y_{22}$, or Figure 5.20(m) and Figure 5.20(n) if $y_{21} < r_1 < r_2 < y_{22}$.*

3- *If $P(y_2)$ has two double real roots, the graphics are shown in Figure 5.20(o) and Figure 5.20(p).*

4- *If $P(y_2)$ has one double real and two complex roots, the graphics are shown in Figure 5.20(q) and Figure 5.20(r).*

5- *If $P(y_2)$ has four complex roots, the graphics are shown in Figure 5.20(s) and Figure 5.20(t).*

6- *If $P(y_2)$ has one real root of order 4, the graphics are shown in Figure 5.20(u) and Figure 5.20(v).*

7- *If $P(y_2)$ has one double real root r_0 and two simple real roots r_1 and r_2, the graphics of the function $g_1^{(i)}(y_2)$ with the different possible positions of these roots with the vertical asymptote are shown in Figure 5.23(a) and Figure 5.23(b) if $y_{21} < r_0 < y_{22} < r_1 < r_2$, or Figure 5.23$(c)$ and Figure 5.23(d) if $r_0 < y_{21} < r_1 < r_2 < y_{22}$, or Figure 5.23$(e)$ and Figure 5.23(f) if $r_0 < y_{21} < r_1 < y_{22} < r_2$, or Figure 5.23$(g)$ and Figure 5.23(h) if $y_{21} < r_1 < y_{22} < r_0 < r_2$, or Figure 5.23$(i)$ and Figure 5.23(j) if $r_1 < y_{21} < r_0 < r_2 < y_{22}$, or Figure 5.23$(k)$ and Figure 5.23(l) if $r_1 < y_{21} < r_0 < y_{22} < r_2$.*

For $s_2 = s_3$ the function $f_1^{(i)}(y_2)$ becomes a constant function $f_1^{(i)}(y_2) \equiv 1$, and $P(y_2)$ becomes a cubic polynomial which can have at most three real roots, and we omit the graphics of this case.

From the function $f_1^{(i)}(y_2)$, it is clear that the sign of the derivative $\left(f_1^{(i)}\right)'(y_2)$ can change at most three times when r is an even integer or $r = k_1/(2k_2+1)$ such that $k_1, k_2 \in \mathbb{N}$, see Figure 5.16(a), and this change of sign guarantees that this case gives the maximum number of the intersection points with the other function. Since the derivative of $g_1^{(i)}(y_2)$ can change its sign at most seven times and this appears in the first ten graphics of Figure 5.20. By considering the dependence between the horizontal and the vertical asymptotes of the two functions, the maximum number of intersection points will be reduced. Therefore if $y_{21} < y_{22}$ is fixed we have to study three distinct cases depending on the position of the horizontal asymptote straight line h_1 of the function $f_1^{(i)}(y_2)$ with the horizontal asymptote straight line h_2 of the function $g_1^{(i)}(y_2)$.

Now assuming that $h_1 < h_2$, we can locate four, or three, or two, or one intersection point on the left side of the vertical asymptote y_{21} between the functions $f_1^{(i)}(y_2)$ and $g_1^{(i)}(y_2)$. The four intersection points between $f_1^{(i)}(y_2)$ and $g_1^{(i)}(y_2)$ are resulting from the intersection between Figure 5.16(a) and Figure 5.20(a), the three intersection points are resulting from Figure 5.16(a) and Figure 5.20(b), or Figure 5.20(c) or Figure 5.20(f), the two intersection points are resulting from Figure 5.16(a) and Figure 5.20(d), or Figure 5.20(e), or Figure 5.20(h) or Figure 5.20(i), and finally the unique intersection point comming from Figure 5.16(a) and Figure 5.20(g) or Figure 5.20(j). In a similar way and between y_{21} and y_{22} we can find four intersection points between Figure 5.16(a) and Figure 5.20(i) or Figure 5.20(j), *three intersection points from Figure 5.16(a) and Figure 5.20(e)-(h), two intersection points between Figure 5.16(a) and Figure 5.20(a)-(d). On the right side of y_{22} we can find only one intersection points between Figure 5.16(a) and all the graphics of Figure 5.20. Then for $h_1 < h_2$ we know that the maximum number of intersection point between $f_1^{(i)}(y_2)$ and $g_1^{(i)}(y_2)$ is at most seven. See for example the intersetion of Figure 5.16(a) and Figure 5.20(a).*

Similarly we find that the maximum of intersection points of the graphics of Figure 5.16 and Figure 5.20 is at most seven for $h_1 \geq h_2$. We know that if (y_1, y_2) is a solution of (5.16) then (y_2, y_1) is also a solution of this system. Consequently the maximum number of limit cycles is at most three.

By taking the values $\{r, s_1, s_2, s_3, k_1, k_2, k_3, k_4\} = \{2, -2, 2, 1, -0.45, 1.38, -1.32, 0.38\}$ we construct an example with exactly six intersection points between the graphics of the two functions $f_1^{(i)}(y_2)$ and $g_1^{(i)}(y_2)$, these points are shown in Figure 5.1. Then we have three limit cycles for the class of PWS $\mathcal{C}_i$ with $A \neq 0$.

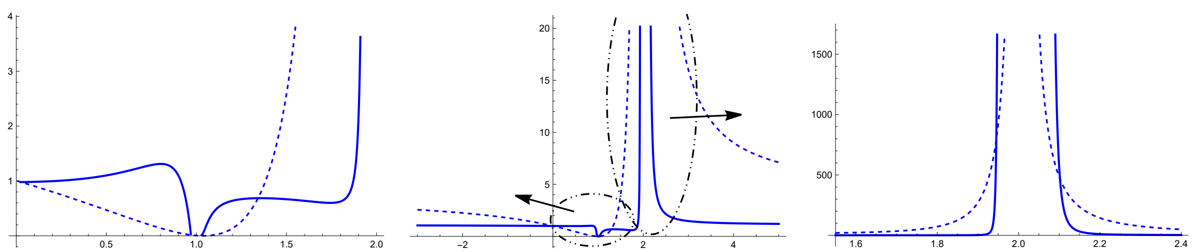


Figure 5.1: Example with six intersection points between the graphics of the two functions $f_1^{(i)}(y_2)$ drawn in dashed line and $g_1^{(i)}(y_2)$ drawn in a continuous line.

In what follows we give a PWS of the class $\mathcal{C}_i$ where $A \neq 0$ with exactly three limit cycles. In the region Σ_1^i we consider the quadratic center

$$\begin{aligned} \dot{x} \approx & -0.564609x^2 + x(-0.669869y - 30.2403) + (-0.203527y - 15.8386)y \\ & -430.571, \\ \dot{y} \approx & \ 0.203527x^2 + x(0.0459715y + 8.26475) + (-0.0592883y - 3.18892)y \\ & +96.2791, \end{aligned} \tag{5.17}$$

its first integral is $H_1^{(i)}(x,y) = Exp(k(x,y))/(0.1x + 0.1y + 3)^4$ where

$$k(x,y) \approx (13.038x^2 + x(18.979y + 704.713) + y(6.9071y + 485.714) + 9716.81)/(x + y + 30)^2.$$

In the region Σ_2^i *we consider the linear differential center*

$$\dot{x} = -2x - 20y, \quad \dot{y} = x + 2y, \tag{5.18}$$

with the first integral $H(x,y) = (x+2y)^2 + 16y^2$. *For the PWS* (5.17)–(5.18), *system* (5.16) *has the three solutions* $(x_1, y_1) \approx (2.96648, 0.663325)$, $(x_2, y_2) \approx (2.14476, 0.479583)$ *and* $(x_3, y_3) \approx (0.632456, 0.141421)$ *which provide the three limit cycles intersecting the rays* Γ_1 *and* Γ_2 *in the six points* $(x_j, 0)$ *and* $(0, y_j)$ *with* $j = 1, 2, 3$, *see Figure* 5.4(*a*).

If $A = 0 \neq b$ *corresponding to* $k = 2$ *and* $j = i$ *in system* (5.16), *the first integral is* $H_2^{(i)}(x,y)$ *given in* (5.3), *the study of the solutions* y_2 *satisfying* $F_2^{(i)}(y_2) = 0$, *is equivalent to study the solutions* y_2 *of the equation* $f_2^{(i)}(y_2) = g_2^{(i)}(y_2)$ *such that*

$$f_2^{(i)}(y_2) = e^{L_0 + L_1\ y_2 + L_2\ y_2^2} \quad \textit{and} \quad g_2^{(i)}(y_2) = (K_0 + K_1\ y_2 + K_2\ y_2^2)/(K_0 + G_1\ y_2 + G_2\ y_2^2),$$

where

$$\begin{aligned}
K_0 = &\ \ 2b^3\gamma_1^2 + 2b^2 d\gamma_2^2 + 2b^2\gamma_2 - 2bd\gamma_2 - b + d,\\
K_1 = &\ \ (4\alpha_1 b^3\gamma_1 + 2\alpha_2 b^2 + 4\alpha_2 b^2 d\gamma_2 - 2\alpha_2 bd)\lambda, \quad K_2 = (2\alpha_1^2 b^3 + 2\alpha_2^2 b^2 d)\lambda^2,\\
G_1 = &\ \ 4b^3\beta_1\gamma_1 + 2b^2\beta_2 + 4b^2\beta_2 d\gamma_2 - 2b\beta_2 d, \ \ G_2 = 2b^3\beta_1^2 + 2b^2\beta_2^2 d,\\
L_0 = &\ \ L_2 = 0, \ \ L_1 = 2b\beta_2 - 2\alpha_2 b\lambda.
\end{aligned}$$

The function $g_2^{(i)}(y_2)$ *has a horizontal asymptote straight line* $h = K_2/G_2$, *and its first derivative is* $(g_2^{(i)})'(y_2) = P_1(y_2)/P_2(y_2)^2$, *with* $P_1(y_2) = K_0K_1 - G_1K_0 + (2K_0K_2 - 2G_2K_0)\ y_2 + (G_1K_2 - G_2K_1)\ y_2^2$ *and* $P_2(y_2) = K_0 + G_1\ y_2 + G_2\ y_2^2$. *We denoted by* $\Delta = (2K_0K_2 - 2G_2K_0)^2 - 4(G_1K_2 - G_2K_1)(K_0K_1 - G_1K_0)$ *the discriminant of the quadratic equation* $P_1(y_2)$. *Then according to the different kinds of solutions of the quadratic equation* $P_2(y_2) = 0$ *and on the sign of* Δ, *we know that the possible graphics of* $g_2^{(i)}(y_2)$ *are given as follows:*

1- *If the quadratic equation* $P_2(y_2) = 0$ *has two real solutions named by* y_{21}, y_{22} *and* $\Delta > 0$, *the possible graphics of* $g_2^{(i)}(y_2)$ *are* (*a*) *and* (*b*) *of Figure* 5.19. *If* $\Delta \leq 0$ *the possible graphics of* $g_2^{(i)}(y_2)$ *are* (*c*) *and* (*d*) *of Figure* 5.19.

2- *If the equation* $P_2(y_2) = 0$ *has one double real solution* y_2 *and* $\Delta > 0$, *then the possible graphics of* $g_2^{(i)}(y_2)$ *are* (*e*) *and* (*f*) *of Figure* 5.19. *If* $\Delta \leq 0$ *the possible graphics of* $g_2^{(i)}(y_2)$ *are* (*g*) *and* (*h*) *of Figure* 5.19.

3- *If* $P_2(y_2) = 0$ *has two complex solutions and* $\Delta > 0$ *the possible graphics of* $g_2^{(i)}(y_2)$ *are* (*i*) *and* (j) *of Figure* 5.19.

Since the possible graphics of $g_2^{(i)}(y_2)$ *are given in Figure* 5.19, *and all the possible graphics for* $f_2^{(i)}(y_2)$ *are given by* (c) *and* (d) *of Figure* 5.19 *because* $L_2 = 0$. *Then using the same arguments for studying the intersection points between the graphics of* $f_1^{(i)}(y_2)$ *and* $g_1^{(i)}(y_2)$, *we obtain that the maximum number of intersection points between these two functions is at most four. So by the symmetry as in the first case when* $k = 1$, *the maximum number of limit cycles is at most two.*

By taking the values $\{L_0, L_1, L_2, K_0, K_1, K_2, G_1, G_2\} = \{0, -3, 0, -0.1, 0.9, -0.8, 0.6, 0.4\}$. *We construct an example with precisely four intersection points between the graphics of the two functions* $f_2^{(i)}(y_2)$ *and* $g_2^{(i)}(y_2)$. *These intersection points are presented in Figure* 5.2.

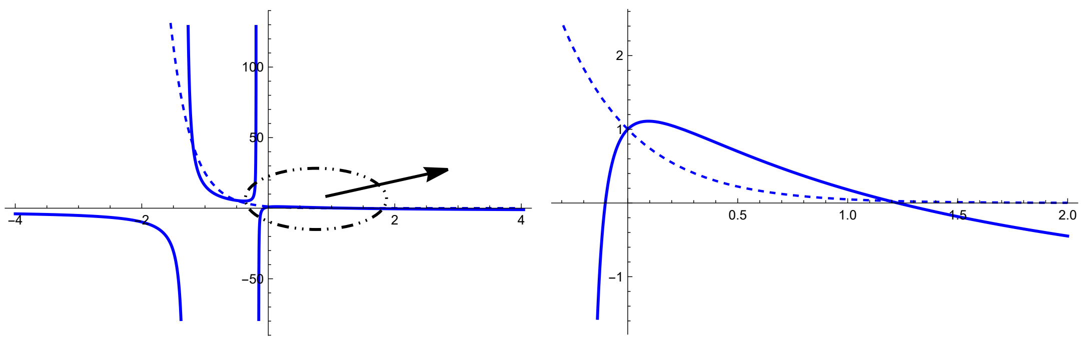


Figure 5.2: The four intersection points between the graphics of the two functions $f_2^{(i)}(y_2)$ drawn in dashed line and $g_2^{(i)}(y_2)$ drawn in a continuous line.

If $b = 0 \neq A$ *corresponding to* $k = 3$ *and* $j = i$ *in system* (5.16), *the first integral of the quadratic center satisfying the first condition of Theorem* 1.1 *is* $H_3^{(i)}(x,y)$ *given by* (5.4), *and the solutions of* $F_3^{(i)}(y_2) = 0$ *are equivalent to the solutions of the equation* $f_3^{(i)}(y_2) = g_3^{(i)}(y_2)$ *such that*

$$f_3^{(i)}(y_2) = f_1^{(i)}(y_2) \ \text{ with } \ r = 2(d-A) \ \text{ and } \ g_3^{(i)}(y_2) = f_2^{(i)}(y_2),$$

where

$$\begin{aligned} L_0 = &\ \ 0, \quad L_1 = 2A\Big(A^2\gamma_1(\alpha_1\lambda - \beta_1) - A(d\gamma_2 + 1)(\beta_2 - \alpha_2\lambda) + d(\beta_2 - \alpha_2\lambda)\Big), \\ L_2 = &\ \ A^2\Big(A\alpha_1^2\lambda^2 - A\beta_1^2 + \alpha_2^2 d\lambda^2 - \beta_2^2 d\Big). \end{aligned}$$

As in the precedent case we give all the possible graphics of $g_3^{(i)}(y_2)$ *shown in Figure* 5.26, *and the possible ones for* $f_3^{(i)}(y_2)$ *are shown in Figure* 5.16. *Hence the maximum number* *of intersection points between these two functions is at most four. Then by symmetry, the maximum number of limit cycles is at most two.*

By taking $\{L_0, L_1, L_2, r, s_1, s_2, s_3\} = \{0, 0.9, -0.09, 2, 2, -2, -1\}$ *we produce an example with precisely four intersection points between the graphics of the function* $f_3^{(i)}(y_2)$ *and* $g_3^{(i)}(y_2)$. *These intersection points appear in Figure 5.3. Then we have two limit cycles for the class of PWS* $\mathcal{C}_i$ *when* $b = 0 \neq A$.

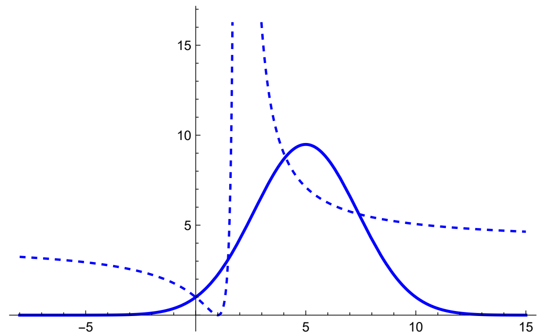


Figure 5.3: The four intersection points between the graphics of the two functions $g_3^{(i)}(y_2)$ drawn in a continuous line and $f_3^{(i)}(y_2)$ drawn in dashed line.

In what follows we give a PWS of this class $\mathcal{C}_i$ *when* $b = 0 \neq A$ *with two limit cycles. In the region* Σ_1^i *we consider the quadratic center*

$$\begin{aligned} \dot{x} \approx \; & x(0.547606 - 0.794014y) + y(2.73803 - 6.69883y) + 0.10915x^2 \\ & -1.82025, \\ \dot{y} \approx \; & 0.0126244x^2 + x(-0.0460281y - 0.0295211) + (-0.545752y \\ & -0.147606)y + 0.364049, \end{aligned} \tag{5.19}$$

its corresponding first integral is $H_3^{(i)}(x,y) \approx (x+5y)e^{k(x,y)}$, *where* $k(x,y) = 0.0173389x^2 + x(-0.299823y - 0.081091) + y(1.84009y - 1.50421)$. *In the region* Σ_2^i *we consider the linear differential center*

$$\dot{x} = -x - 17y, \quad \dot{y} = x + y, \tag{5.20}$$

with the first integral $H(x,y) = (x+y)^2 + 16y^2$. *For the PWS (5.19)–(5.20), system (5.16) has the two solutions* $(x_1, y_1) \approx (2.12132, 0.514496)$ *and* $(x_2, y_2) \approx (1, 0.242536)$ *which provide the two limit cycles intersecting the rays* Γ_1 *and* Γ_2 *in the four points* $(x_j, 0)$ *and* $(0, y_j)$ *with* $j = 1, 2$, *see Figure 5.4(b).*

If $A = b = 0$ *corresponding to* $k = 4$ *and* $j = i$ *in system (5.16), the first integral in this case* *is $H_4^{(i)}(x,y)$ given in (5.5), and $F_4^{(i)}(y_2)$ is a polynomial in the variable y_2 of degree three. The maximum number of the real solutions of $F_4^{(i)}(y_2) = 0$ is three. Then in this case the maximum number of limit cycles for the class of PWS $\mathcal{C}_i$ when $A = b = 0$ is at most one.*

In what follows we give a PWS of the class $\mathcal{C}_i$ when $A = b = 0$ with one limit cycle. In the region Σ_1^i we consider the quadratic center

$$\begin{aligned}\dot{x} \approx\ & x(0.275591 - 0.0393701y) - 0.00393701x^2 + (-0.0984252y\\ & -0.622047)y - 1.25984,\\ \dot{y} \approx\ & 0.000787402x^2 + x(0.00787402y + 4.94488) + (0.019685y\\ & -0.275591)y - 5.74803,\end{aligned} \tag{5.21}$$

this system has the first integral

$$H_4^{(i)}(x,y) \approx 2(0.1x + 0.5y + 1)^3 + 3\Big((-2.5x + 0.2y + 3)^2 + (0.1x + 0.5y + 1)^2\Big).$$

In the region Σ_2^i we consider the linear differential center

$$\dot{x} = -0.4x - 1.1571y, \quad \dot{y} = x + 0.4y, \tag{5.22}$$

with the first integral $H(x,y) = (x + 0.4y)^2 + 0.997096y^2$. For the PWS (5.21)–(5.22), system (5.16) has the unique solution $(x_1, y_1) \approx (3.2662, 3.0364)$ that provides the unique limit cycle intersecting the rays Γ_1 and Γ_2 in the two points $(x_1, 0)$ and $(0, y_1)$, see Figure 5.4(c). This example completes the proof of statement (a).

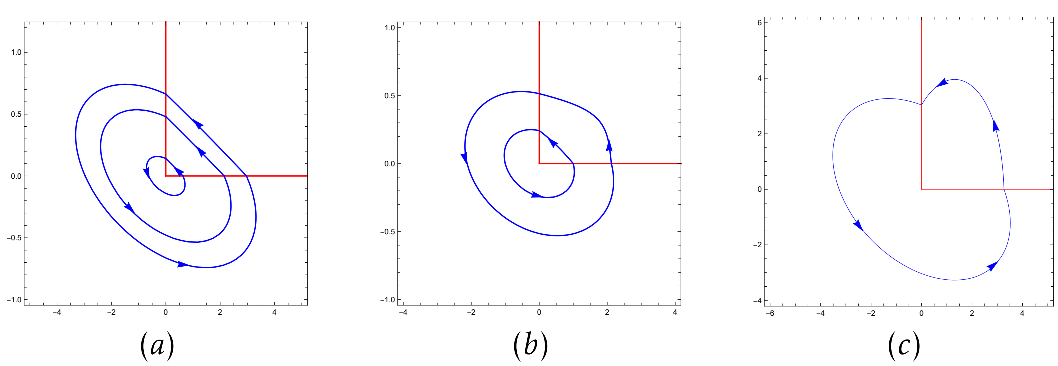


Figure 5.4: (a) The three limit cycles of the PWS (5.17)–(5.18), (b) the two limit cycles of the PWS (5.19)–(5.20), and (c) the unique limit cycle of the PWS (5.21)–(5.22).

Proof of statement (b) of Theorem 5.2. *Now we will prove the statement for the second class $\mathcal{C}_{ii}$ when $b+d=0$. If $A+b=0$ and $a=0\neq Cb$ corresponding to $k=1$ and $j=ii$ in system (5.16), the first integral of the quadratic center is $H_1^{(ii)}(x,y)$ given in (5.6). The study of the solutions y_2 satisfying $F_1^{(ii)}(y_2)=0$ is equivalent to study the solutions y_2 of the equation $f_1^{(ii)}(y_2)=g_1^{(ii)}(y_2)$ such that*

$$f_1^{(ii)}(y_2)=\left(\frac{m_1+m_2\,y_2}{m_1+m_3\,y_2}\right)^p\left(\frac{n_1+n_2\,y_2}{n_1+n_3\,y_2}\right)^q \quad \text{and} \quad g_1^{(ii)}(y_2)=e^{k(y_2)},$$

where

$$\begin{aligned}
k(y_2)= &\quad (K_1\,y_2+K_2\,y_2^2)/[(m_1+m_2\,y_2)(m_1+m_3\,y_2)],\\
m_1= &\quad 1-b\gamma_2,\ \ m_2=-b\beta_2,\ \ m_3=-\alpha_2 b\lambda,\ \ p=b^2+C^2,\\
n_1= &\quad 1-b\gamma_2+C\gamma_1,\ \ n_2=(\alpha_1 C-\alpha_2 b)\lambda,\ \ n_3=\beta_1 C-b\beta_2,\ \ q=b^2,\\
K_1= &\quad bC\Big(b^2\,(\lambda(\alpha_2\gamma_1-\alpha_1\gamma_2)+\beta_1\gamma_2-\beta_2\gamma_1)+b\,(\alpha_1\lambda-\beta_1)+\alpha_2 C\lambda-\beta_2 C\Big),\\
K_2= &\quad b^3 C\lambda(\alpha_2\beta_1-\alpha_1\beta_2).
\end{aligned}$$

For $p,q>0$ and from the geometric study, we can divide the study of the function $f_1^{(ii)}(y_2)$ on two parts according to the number of the vertical asymptotes straight lines.

The function $f_1^{(ii)}(y_2)$ has one vertical asymptote straight line if either $p=0$ and $q\neq 0$, or $q=0$ and $p\neq 0$, or $m_2=m_3$, or $n_2=n_3$ in these cases the graphics of the function $f_1^{(ii)}(y_2)$ are the same as the ones of the function $f_1^{(i)}(y_2)$ shown in Figure 5.16.

To draw all the possible graphics of the function $f_1^{(ii)}(y_2)$ which has two vertical asymptotes straight lines $y_{21}=-m_1/m_3$ and $y_{22}=-n_1/n_3$, we need to study the sign of its first derivative that depend on the nature of the parameters p and q, the roots of the quadratic polynomial $P(y_2)=M_0+M_1\,y_2+M_2\,y_2^2$ and of the possible position of these two roots with the positions of y_{21} and y_{22} and on the two roots $y_{01}=-m_1/m_2$ and $y_{02}=-n_1/n_2$ of $\left(f_1^{(ii)}\right)'(y_2)$, where

$$\left(f_1^{(ii)}\right)'(y_2)=(m_1+m_2\,y_2)^{p-1}(n_1+n_2\,y_2)^{q-1}(m_1+m_3\,y_2)^{-(p+1)}(n_1+n_3\,y_2)^{-(q+1)}P(y_2),$$

where

$$\begin{aligned}
M_0= &\quad m_1^2 n_1 q(n_2-n_3)+m_1 n_1^2 p(m_2-m_3),\\
M_1= &\quad m_1 n_1\Big(p(m_2-m_3)(n_2+n_3)+q(m_2+m_3)(n_2-n_3)\Big),\\
M_2= &\quad m_1 n_2 n_3 p(m_2-m_3)+m_2 m_3 n_1 q(n_2-n_3).
\end{aligned}$$

Now we will give the possible positions of the real roots of the quadratic polynomial with respect to y_{21}, y_{22}, y_{01} and y_{02}.

I-If $P(y_2)$ has two distinct real roots r_1 and r_2, the possible positions of these two roots with respect to y_{21}, y_{22}, y_{01} and y_{02} are

(1) $y_{01} < r_1 < y_{02} < y_{21} < r_2 < y_{22}$; (2) $y_{01} < y_{21} < y_{02} < r_1 < r_2 < y_{22}$;

(3) $y_{01} < r_1 < r_2 < y_{21} < y_{02} < y_{22}$; (4) $r_1 < y_{01} < y_{21} < r_2 < y_{22} < y_{02}$;

(5) $y_{21} < y_{01} < r_1 < y_{02} < y_{22} < r_2$; (6) $r_1 < r_2 < y_{21} < y_{01} < y_{22} < y_{02}$.

II - If $P(y_2)$ has one double real root r_0, the possible positions of this double root with respect to y_{21}, y_{22}, y_{01} and y_{02} are

(1) $r_0 < y_{01} < y_{02} < y_{21} < y_{22}$; (2) $y_{01} < r_0 < y_{21} < y_{02} < y_{22}$; (3) $y_{01} < y_{21} < r_0 < y_{02} < y_{22}$.

III - If $P(y_2)$ has two complex roots, we have $y_{01} < y_{21} < y_{02} < y_{22}$ as the only possible position.

Now all the possible graphics of the function $f_1^{(ii)}(y_2)$ are shown in Figures 5.18, 5.17 and 5.22 and in what follows we explain how they have been obtained.

1- *If p and q are even integers, or if p is even and $q = k_1/(2k_2+1)$ with $k_1, k_2 \in \mathbb{N}$, we give all the graphics of $f_1^{(ii)}(y_2)$ in Figure 5.18. If $P(y_2)$ has two distinct real roots taking either the position (1), or (2), or (3), or (4), or (5) or (6), then the graphics of $f_1^{(ii)}(y_2)$ are given by (a), or (b), or (c), or (d), or (e) or (f) of Figure 5.18, respectively. If $P(y_2)$ has two complex roots, the graphic of $f_1^{(ii)}(y_2)$ is given by (g) of Figure 5.18. If $P(y_2)$ has one double real root taking either the position (1), or (2) or (3), then the graphics of $f_1^{(ii)}(y_2)$ are given by (h), or (i) or (j) of Figure 5.18, respectively.*

2- *If p and q are odd integers we give all the graphics of $f_1^{(ii)}(y_2)$ in Figure 5.17. If $P(y_2)$ has two distinct real roots then the graphics of $f_1^{(ii)}(y_2)$ are given by (a) and (b) of Figure 5.17 when these roots taking position (1), (c) and (d) of Figure 5.17 when these roots taking position (2), (e) and (f) of Figure 5.17 when these roots taking position (3), (g) and (h) of Figure 5.17 when these roots taking position (4), (i) and (j) of Figure 5.17 when these roots taking position (5), (k) and (l) of Figure 5.17 when these roots taking position (6). If $P(y_2)$ has two complex roots, the graphics of $f_1^{(ii)}(y_2)$ is given in (m) and (n) of Figure 5.17. If $P(y_2)$ has one double real root then the graphics of $f_1^{(ii)}(y_2)$ are given by (o) and (p) of Figure 5.17 when this root taking position (1), (q) and (r) of*

Figure 5.17 when this root taking position (2), (*s*) *and* (*t*) *of Figure 5.17 when this root taking position* (3).

3- *In a similar way if p odd and q even, or if p odd and $q = k_1/(2k_2+1)$ with $k_1, k_2 \in \mathbb{N}$, we give all the graphics of $f_1^{(ii)}(y_2)$ in Figure 5.22.*

4- *If p is odd and $q = k_1/(2k_2)$ and k_1 is an odd integer and $k_2 \in \mathbb{N}$, then the sign of the derivative depends on the sign of the quadratic polynomial $P(y_2)$, therefore the graphics of $f_1^{(ii)}(y_2)$ are the same than in the case that both p, q are odd, but in its domain of definition.*

5- *If p is even and $q = k_1/(2k_2)$ and $k_1, k_2 \in \mathbb{N}$, then the sign of the derivative depends on the sign of the quadratic polynomial $P(y_2)$ and on the sign of the product $(n_1+n_2y_2)(n_1+n_3y_2)$. Therefore the graphics of $f_1^{(ii)}(y_2)$ are the same than in the case q is odd and p is even, but in its domain of definition.*

6- *If $p = k_1/(2k_2)$ and $q = k_1'/(2k_2')$ and k_1, k_1' are odd integers, then the sign of the derivative depends on the sign of the quadratic polynomial $P(y_2)$, therefore the graphics of $f_1^{(ii)}(y_2)$ are the same than the case p, q are odd, but in its domain of definition.*

7- *If $p = k_1/(2k_2+1)$ and $p = k_1'/(2k_2'+1)$ with $k_1, k_2, k_1', k_2' \in \mathbb{N}$, then the sign of the derivative depends on the sign of the quadratic polynomial $P(y_2)$ and on the sign of the product $(n_1+n_2y_2)(n_1+n_3y_2)(m_1+m_2y_2)(m_1+m_3y_2)$, therefore the graphics of $f_1^{(ii)}(y_2)$ are topologically equivalent to graphics of case of both p, q are even but in its domain of definition.*

8- *If $p = k_1/(2k_2)$ and $p = k_1'/(2k_2'+1)$ and k_1 is an odd integer with $k_2, k_1', k_2' \in \mathbb{N}$, then the sign of the derivative depends on the sign of the quadratic polynomial $P(y_2)$ and on the sign of the product $(n_1+n_2y_2)(n_1+n_3y_2)$, therefore the graphics of $f_1^{(ii)}(y_2)$ are the same than in the case p odd and q even, but in its domain of definition.*

In a similar way we find the same graphics than in the case $p, q > 0$ if both p and q are negative, or one of them is negative and the other one positive.

Now for the function $g_1^{(ii)}(y_2)$, the horizontal asymptote straight line is $h = K_2/(m_2m_3)$, and the first derivative of $g_1^{(ii)}(y_2)$ is $\left(g_1^{(ii)}\right)'(y_2) = [(m_1+m_2\,y_2)(m_1+m_3\,y_2)]^{-1}P(y_2)e^{k(x,y)}$, where $P(y_2) = K_1m_1^2 + 2K_2m_1^2\,y_2 + (K_2m_1(m_2+m_3) - K_1m_2m_3)\,y_2^2$. For $m_2 \neq m_3$ and according to the different possible kind of roots of the quadratic polynomial $P(y_2)$ and their possible positions *with the two vertical asymptote straight lines y_{01} and y_{21} of the function $f_1^{(ii)}(y_2)$, all the graphics of the function $g_1^{(ii)}(y_2)$ are given in what follows.*

1- *If $P(y_2)$ has two distinct real roots, the graphics of the function $g_1^{(ii)}(y_2)$ are shown in (a) and (b) of Figure 5.21.*

2- *If $P(y_2)$ has two complex roots, the graphics of the function $g_1^{(ii)}(y_2)$ are shown in Figures 5.21(c) and 5.21(d).*

3- *If $P(y_2)$ has one double real root this root is either y_{01} or y_{21}. Here the graphics of the function $g_1^{(ii)}(y_2)$ are shown in Figures 5.21(e) and 5.21(f).*

As in the precedent case and for the same reason we only have drawn the graphics of the function $g_1^{(ii)}(y_2)$ for $m_2 \neq m_3$.

The function $g_1^{(ii)}(y_2)$ has at most three changes in the sign of its derivative which appears in (a) and (b) of Figure 5.21. We also know that $g_1^{(ii)}(y_2)$ is a positive function, so to get the maximum number of intersection points between the graphics (a) and (b) of Figure 5.21 and the graphics of $f_1^{(ii)}(y_2)$ it is sufficient to solve the problem of the intersection points between the graphics (a) and (b) of Figure 5.21 with the graphics of Figure 5.18. As we mentioned in the precedent cases the change of sign of the derivative of $f_1^{(ii)}(y_2)$ plays a main role, here we see that seven is the maximum number of the changes of $\left(f_1^{(ii)}\right)'(y_2)$ and this is shown in $(a), \cdots, (f)$ of Figure 5.18. In this case we remark that y_{01} and y_{21} represent a root and a vertical asymptote, respectively, of the function $f_1^{(ii)}(y_2)$ and also the vertical asymptotes for the function $g_1^{(ii)}(y_2)$. Then if we chose the horizontal asymptote of the function $g_1^{(ii)}(y_2)$ less than the one of $f_1^{(ii)}(y_2)$, we get at most four intersection point between Figure 5.21(a) and (a), (b), (e) and (f) of Figure 5.18. Similarly we study the case when the horizontal asymptote of the function $g_1^{(ii)}(y_2)$ is greater than the one of $f_1^{(ii)}(y_2)$. These points are less than y_{01}, and at most four intersection points between y_{01} and y_{21} by intersecting Figure 5.21(a) with Figure 5.18(a) and no intersection points after y_{21}. So we remark that these graphics can intersect at most in eight points. Then the upper bound number of limit cycles for the class of PWS $\mathcal{C}_{ii}$ when $A + b = 0$ and $a = 0 \neq Cb$ is four.

By taking $\{K_1, K_2, m_1, m_2, m_3, p, n_1, n_2, n_3, q\} = \{-6, 0.1, -2, 2.4, 0.9, 4, -2, 5, 2, 4\}$ we construct an example with exactly eight intersection points between the graphics of $f_1^{(ii)}(y_2)$ and $g_1^{(ii)}(y_2)$, these points are highlighted in Figure 5.5.

If $AbC(A + b)\Delta \neq 0 = a$ and $\Delta < 0$ corresponding to $k = 2$ and $j = ii$ in system (5.16), the first integral of the quadratic center (5.1) is $H_2^{(ii)}(x, y)$ given in (5.9). Then to study the

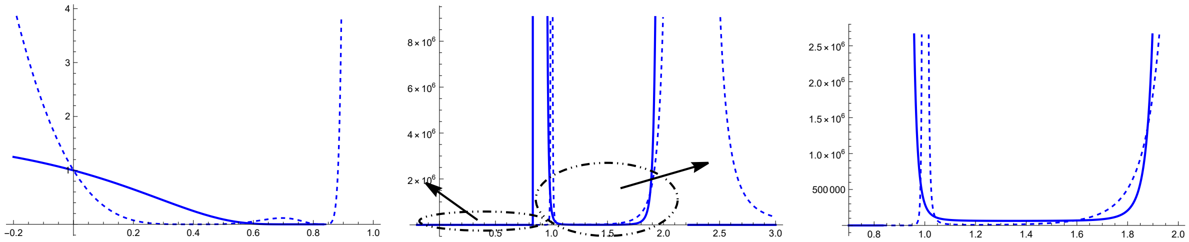


Figure 5.5: The eight intersection points between the graphics of the two functions $f_1^{(ii)}(y_2)$ drawn in dashed line and $g_1^{(ii)}(y_2)$ drawn in continuous line.

solutions y_2 satisfying $F_2^{(ii)}(y_2) = 0$ is equivalent to study the solutions y_2 of $f_2^{(ii)}(y_2) = g_2^{(ii)}(y_2)$ such that

$$f_2^{(ii)}(y_2) = \left(\frac{s_1 + s_2\ y_2}{s_1 + s_3\ y_2}\right)^r \left(\frac{K_0 + K_1\ y_2 + K_2\ y_2^2}{K_0 + G_1\ y_2 + G_2\ y_2^2}\right)^{r'} \quad \text{and} \quad g_2^{(ii)}(y_2) = e^{M\ k(y_2)},$$

where

$$\begin{aligned}
k(y_2) = \quad & \cot^{-1}[(M_0 + M_1 y_2 + M_2 y_2^2)/(N_0 + N_1 y_2 + N_2 y_2^2)],\\
M = \quad & \frac{2C}{bL}, \quad M_0 = L(2b\gamma_2 - C\gamma_1 + \gamma_1 L - 2),\\
M_1 = \quad & L(\gamma_1(2b\beta_2 - \beta_1 C + \beta_1 L) + \alpha_1\lambda(2b\gamma_2 - C\gamma_1 + \gamma_1 L - 2)),\\
M_2 = \quad & \alpha_1 L\lambda(2b\beta_2 - \beta_1 C + \beta_1 L), \quad N_0 = 2\gamma_1 L(-2b\gamma_2 + C\gamma_1 + 2),\\
N_1 = \quad & 2L(-b\beta_2\gamma_1 - b\gamma_2(\beta_1 + \alpha_1\lambda) - \alpha_2 b\gamma_1\lambda + \beta_1 + \beta_1 C\gamma_1 + \alpha_1 C\gamma_1\lambda + \alpha_1\lambda),\\
N_2 = \quad & -2L\lambda(\alpha_1 b\beta_2 + \alpha_2 b\beta_1 - \alpha_1\beta_1 C),\\
K_0 = \quad & b^2\gamma_2^2 - bC\gamma_1\gamma_2 - 2b\gamma_2 + \frac{1}{4}\gamma_1^2\left(C^2 + L^2\right) + C\gamma_1 + 1,\\
K_1 = \quad & \frac{1}{2}\lambda\left(4\alpha_2 b(b\gamma_2 - 1) - 2C(\alpha_1(b\gamma_2 - 1) + \alpha_2 b\gamma_1) + \alpha_1 C^2\gamma_1 + \alpha_1\gamma_1 L^2\right),\\
K_2 = \quad & \frac{1}{4}\lambda^2\left(4\alpha_2^2 b^2 - 4\alpha_1\alpha_2 bC + \alpha_1^2\left(C^2 + L^2\right)\right),\\
G_1 = \quad & 2b^2\beta_2\gamma_2 - 2b\beta_2 - b\beta_1 C\gamma_2 - b\beta_2 C\gamma_1 + \frac{1}{2}\beta_1\gamma_1\left(C^2 + L^2\right) + \beta_1 C,\\
G_2 = \quad & b^2\beta_2^2 - b\beta_1\beta_2 C + \frac{1}{4}\beta_1^2\left(C^2 + L^2\right), \quad r' = \frac{1}{b}, \quad r = 8b/(4b^2 + C^2 + L^2),\\
s_1 = \quad & 1 - 2\gamma_2 r^{-1}, \quad s_2 = -2\beta_2 r^{-1}, \quad s_3 = -2\alpha_2\lambda r^{-1}.
\end{aligned}$$

The solutions of $f_2^{(ii)}(y_2) = g_2^{(ii)}(y_2)$ represent the intersection points between the graphics of *the two functions* $f_2^{(ii)}(y_2)$ *and* $g_2^{(ii)}(y_2)$*. We have that*

$$\left(f_2^{(ii)}\right)'(y_2) = \frac{(s_1+s_2\ y_2)^{r-1}\left(K_0+K_1\ y_2+K_2\ y_2^2\right)^{r'-1}}{(s_1+s_3\ y_2)^{r+1}\left(K_0+G_1\ y_2+G_2\ y_2^2\right)^{r'+1}}\ P_1(y_2),$$

and

$$\left(g_2^{(ii)}\right)'(y_2) = \frac{P_2(y_2)\ e^{M\ k(y_2)}}{(M_0+M_1y_2+M_2y_2^2)^2+(N_0+N_1y_2+N_2y_2^2)^2},$$

with

$$\begin{aligned} P_1(y_2) = \ & K_0s_1(r's_1(K_1-G_1)+K_0r(s_2-s_3))+(K_0s_1(-s_3(G_1(r'+r)+K_1(r-r'))-G_1r's_2+\\ & G_1rs_2-2G_2r's_1+K_1r's_2+K_1rs_2+2K_2r's_1))\ y_2+\Big(G_1(-K_0r's_2s_3+K_1rs_1(s_2-s_3)\\ & +K_2r's_1^2)-G_2s_1(2K_0r'(s_2+s_3)+K_0r(s_3-s_2)+K_1r's_1)+K_0K_1r's_2s_3+K_0K_2s_1(2r'\\ & (s_2+s_3)+r(s_2-s_3))\Big)\ y_2^2+(G_1K_2s_1(s_2(r'+r)+s_3(r'-r))-G_2(2K_0r's_2s_3+K_1s_1\\ & (s_2(r'-r)+s_3(r'+r)))+2K_0K_2r's_2s_3)\ y_2^3+(G_1K_2r's_2s_3-G_2K_1r's_2s_3+G_2K_2rs_1\\ & (s_2-s_3))\ y_2^4, \end{aligned}$$

and

$$P_2(y_2) = \ -M(M_1N_0-M_0N_1)-M(2M_2N_0-2M_0N_2)\ y_2-M(M_2N_1-M_1N_2)\ y_2^2.$$

Let $\Delta_1 = K_1^2-4K_0K_2 = -L^2(\alpha^2+\omega^2)(\alpha_1-\alpha_1b\gamma_2+\alpha_2b\gamma_1)^2$ *and* $\Delta_2 = G_1^2-4K_0G_2 = -L^2(-b\beta_1\gamma_2+b\beta_2\gamma_1+\beta_1)^2$ *be the discriminant of the quadratic equations* $K_0+K_1\ y_2+K_2\ y_2^2=0$ *and* $K_0+G_1\ y_2+G_2\ y_2^2=0$*, respectively. It is clear that* $\Delta_i \leq 0$ *with* $i=1,2$*.*

1^{st} **case.** *If either* $\Delta_i = 0$ *with* $i=1,2$*, or if* $\Delta_1 = 0$ *and the exponent* $r'<0$*, or if* $\Delta_2=0$ *and* $r'>0$*, then the graphics of the function* $f_2^{(ii)}(y_2)$ *are equivalent to the graphics of the function* $f_1^{(ii)}(y_2)$ *that are shown in Figures 5.18, 5.17 and 5.22.*

2^{nd} **case.** *If either* $\Delta_i<0$ *with* $i=1,2$*, or if* $\Delta_1<0$ *and* $r'<0$ *or if* $\Delta_2<0$ *and* $r'>0$*. Here for* $r>0$ *the only vertical asymptote straight line for the function* $f_2^{(ii)}(y_2)$ *is* $y_{21}=-s_1/s_3$*, and* $h=(s_2/s_3)^r(K_2/G_2)^{r'}$ *is the horizontal asymptote straight line, then according to the sign of* $\left(f_2^{(ii)}\right)'(y_2)$ *which depends on the parameter* r *and on the different possible kind of the roots of the quartic polynomial* $P_1(y_2)$ *and their possible positions with respect to the vertical asymptote* y_{21}*, and denoted by* $r_0 = \dfrac{-s_1}{s_2} < y_{21}$ *we will obtain all the graphics of the function* $f_2^{(ii)}(y_2)$ *as follows.*

1- If either r is even, or $r = k_1/(2k_2+1)$ with $k_1, k_2 \in \mathbb{N}$, and If

(1.a) the polynomial $P_1(y_2)$ has four real roots r_i with $i = 1,2,3,4$, then the graphics of $f_2^{(ii)}(y_2)$ are shown in Figures 5.25(a) and 5.25(b) when $r_j < y_{21}$ with $j \in \{0,1,2,3,4\}$, or in Figures 5.25(c) and 5.25(d) when $r_j < y_{21}$ with $j \in \{0,1,2,3\}$ and $r_4 > y_{21}$, or in Figures 5.25(e) and 5.25(f) when $r_j < y_{21}$ with $j \in \{0,1,2\}$ and $r_4 > r_3 > y_{21}$.

(1.b) the polynomial $P_1(y_2)$ has four complex roots, the graphics of this function are shown in (g) and (h) of Figure 5.25.

(1.c) the polynomial $P_1(y_2)$ has one double real root r_1 and two complex roots, the graphics of $f_2^{(ii)}(y_2)$ are shown in (i) and (j) of Figure 5.25 when $r_1 < r_0 < y_{21}$, or in (k) and (l) of Figure 5.25 when $r_0 < r_1 < y_{21}$, or in (m) and (n) of Figure 5.25 when $r_0 < y_{21} < r_1$.

(1.d) the polynomial $P_1(y_2)$ has one triple and one simple real root, or two simple real roots r_1 and r_2 and two complex roots, the graphics of $f_2^{(ii)}(y_2)$ are given in (o) and (p) of Figure 5.25 when $r_j < y_{21}$ with $j = 0,1,2$, or (q) and (r) of Figure 5.25 when $r_j < y_{21}$ with $j = 0,1$ and $r_2 > y_{21}$.

(1.e) the polynomial $P_1(y_2)$ has two double real roots r_1 and r_2, the graphics are given in (s) and (t) of Figure 5.25 when $r_1 < r_2 < r_0 < y_{21}$, or in (u) and (v) of Figure 5.25 when $r_1 < r_0 < r_2 < y_{21}$, or in (w) and (x) of Figure 5.25 when $r_1 < r_0 < y_{21} < r_2$.

2- If r is odd we have the same graphics as the case when r is even where now r_0 represents an inflection point of the function $f_2^{(ii)}(y_2)$.

3- If $r = k_1/(2k_2)$ with $k_1, k_2 \in \mathbb{N}$, the sign of the derivative depends only on the sign of $(K_0 + K_1\ y_2 + K_2\ y_2^2)(K_0 + G_1\ y_2 + G_2\ y_2^2)P_1(y_2)$, then the possible graphics of the function $f_2^{(ii)}(y_2)$ are the same than the ones of the case where r is an odd integer drawn on its definition domain.

For $r < 0$ and in a similar way we find the same graphics than in the case $r > 0$.

Now for the function $g_2^{(ii)}(y_2)$ it is clear that the sign of its derivative $\left(g_2^{(ii)}\right)'(y_2)$ depends only on the sign of the quadratic polynomial $P_2(y_2)$. So to study the variation of the function $g_2^{(ii)}(y_2)$ we distinguish three different cases.

1- When the function $g_2^{(ii)}(y_2)$ has two vertical asymptotes straight lines y_{21} and y_{22}. i.e, when $N_1^2 - 4N_0N_2 > 0$. According to the different possible kinds of roots of $P_2(y_2)$ and of their possible position with respect to the vertical asymptote y_{21} and y_{22}, all the graphics of the function $g_2^{(ii)}(y_2)$ are given in Figure 5.24. Indeed if $P_2(y_2)$ has two distinct real *roots r_1 and r_2, the graphics are given in (a) and (b) of Figure 5.24 when $r_1 < r_2 < y_{21} < y_{22}$, or in (c) and (d) of Figure 5.24 when $r_1 < y_{21} < r_2 < y_{22}$, or in (e) and (f) of Figure 5.24 when $y_{21} < r_1 < r_2 < y_{22}$, or in (g) and (h) of Figure 5.24 when $r_1 < y_{21} < y_{22} < r_2$. If $P_2(y_2)$ has two complex roots or one double real root, then the graphics of $g_2^{(ii)}(y_2)$ are in (i) and (j) of Figure 5.24 are the two possible graphics.*

2- *When the function $g_2^{(ii)}(y_2)$ has only one vertical asymptote straight line y_{21} if $P_2(y_2)$ has two distinct real solutions r_1 and r_2, then the graphic of $g_2^{(ii)}(y_2)$ are in (k) and (l) of Figure 5.24 when $r_1 < r_2 < y_{21}$ or (m) and (n) of Figure 5.24 when $r_1 < y_{21} < r_2$. If $P_2(y_2)$ has two complex roots or one double real root, then the graphics of $g_2^{(ii)}(y_2)$ are in (o) and (p) of Figure 5.24 are the two possible graphics.*

3- *When the function $g_2^{(ii)}(y_2)$ has no vertical asymptote straight line, the only possible graphics of $g_2^{(ii)}(y_2)$ are given in (q) and (r) of Figure 5.24.*

In order to find the maximum number of intersection points between the graphics of the two functions $f_2^{(ii)}(y_2)$ and $g_2^{(ii)}(y_2)$, we begin with the function $g_2^{(ii)}(y_2)$, and since Figure 5.24 illustrates all the possible graphics of the function $g_2^{(ii)}(y_2)$, then the maximum number of changes in the sign of its first derivative is at most three, as shown in $(a),\ldots,(h)$ of Figure 5.24. We know that seven is the maximum number of changes in the sign the first derivative of $f_2^{(ii)}(y_2)$ shown in $(a),\ldots,(f)$ of Figures 5.18 and 5.25. If we fixe (a) of Figure 5.18, we remark that eight is the maximum number of the intersection points between this figure and Figure 5.24(a). Similarly we check out the maximum number of intersection points between the remaining graphics of the functions $f_2^{(ii)}(y_2)$ and $g_2^{(ii)}(y_2)$. Thus these two functions can clearly intersect in a maximum of eight points. Hence $F_2^{(ii)}(y_2) = 0$ can have at most eight real solutions. Consequently the maximum number of limit cycles for the class of the PWS $\mathcal{C}_{ii}$ under the present conditions is at most four.

By taking $\{r, s_1, s_2, s_3, r', K_0, K_1, K_2, G_1, G_2, M, M_0, M_1, M_2, N_0, N_1, N_2\} = \{2, 2, -1.5, 2.2, 1.1, -3.5, 2.8, -1.6, 4.2, 4.2, 1.5, 3.5, 15, 3.3, -0.01, 1.8, -0.8\}$ we construct an example with exactly eight intersection points between the graphics of $f_2^{(ii)}(y_2)$ and $g_2^{(ii)}(y_2)$, these points are shown in Figure 5.6.

If $C = b = 0$ corresponding to $k = 3$ and $j = ii$ in system (5.16), the first integral of the quadratic center is $H_3^{(ii)}(x, y)$ given in (5.8), the study of the solutions y_2 satisfying $F_3^{(ii)}(y_2) =$

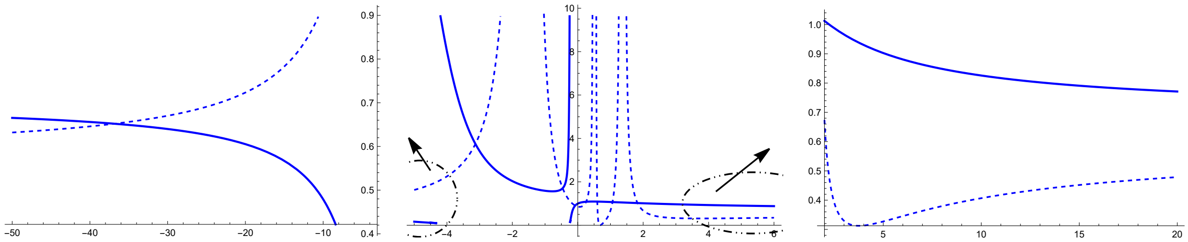


Figure 5.6: The eight intersection points between the graphics of the two functions $f_2^{(ii)}(y_2)$ drawn in dashed line and $g_2^{(ii)}(y_2)$ drawn in continuous line.

0 is equivalent to study the solutions y_2 of the equation $f_3^{(ii)}(y_2) = g_3^{(ii)}(y_2)$ such that

$$f_3^{(ii)}(y_2) = \left(\frac{k_0 + k_1\ y_2 + k_2\ y_2^2}{k_0 + g_1\ y_2 + g_2\ y_2^2}\right)^A e^{M\ y_2} \quad \textit{and} \quad g_3^{(ii)}(y_2) = (g_2^{(i)}(y_2))^r,$$

where

$$\begin{aligned}
K_0 = &\ -a^2\gamma_2^2 + A\gamma_2(a\gamma_1+1) + (a\gamma_1+1)^2,\ \ K_1 = \lambda((a\gamma_1+1)(2a\alpha_1 + A\alpha_2) + a\gamma_2(A\alpha_1 - 2a\alpha_2)),\\
K_2 = &\ a\lambda^2(a(\alpha_1-\alpha_2)(\alpha_1+\alpha_2) + A\alpha_1\alpha_2),\ \ G_1 = (a\gamma_1+1)(2a\beta_1 + A\beta_2) + a\gamma_2(A\beta_1 - 2a\beta_2),\\
G_2 = &\ a(a(\beta_1-\beta_2)(\beta_1+\beta_2) + A\beta_1\beta_2),\ \ r = \sqrt{4a^2+A^2},\ \ M = 2ar(\beta_1 - \alpha_1\lambda),\\
k_0 = &\ -4a^4\gamma_2^2 + 4a^2A\gamma_2(a\gamma_1+1) - (aA\gamma_1 + A)^2 + (a\gamma_1 r + r)^2,\\
k_1 = &\ -a\lambda(-2a\alpha_2 + A\alpha_1 - \alpha_1 r)\Big((a\gamma_1+1)(A+r) - 2a^2\gamma_2\Big) - a(\beta_1(A+r) - 2a\beta_2)((a\gamma_1+1)\\
&\ (A-r) - 2a^2\gamma_2),\\
k_2 = &\ -a^2\lambda(-2a\alpha_2 + A\alpha_1 - \alpha_1 r)(\beta_1(A+r) - 2a\beta_2),\\
g_1 = &\ -a\lambda(\alpha_1(A+r) - 2a\alpha_2)\Big((a\gamma_1+1)(A-r) - 2a^2\gamma_2\Big) - a(-2a\beta_2 + A\beta_1 - \beta_1 r)((a\gamma_1+1)\\
&\ (A+r) - 2a^2\gamma_2),\\
g_2 = &\ -a^2\lambda(\alpha_1(A+r) - 2a\alpha_2)(-2a\beta_2 + A\beta_1 - \beta_1 r).
\end{aligned}$$

Since $\Delta_1 = (r\lambda)^2(-a\alpha_1\gamma_2 + a\alpha_2\gamma_1 + \alpha_2)^2$ and $\Delta_2 = r^2(-a\beta_1\gamma_2 + a\beta_2\gamma_1 + \beta_2)^2$ are the discriminants of the numerator and the denominator of the function $g_3^{(ii)}(y_2)$, respectively, and they are positive, the possible graphics of the function $g_3^{(ii)}(y_2)$ are the ones drawn in Figure 5.16 if $\Delta_i = 0$ with $i = 1, 2$, Figures 5.18 and 5.17 if either $\Delta_i > 0$ with $i = 1, 2$, or $\Delta_1 > 0$ and $\Delta_2 = 0$, or $\Delta_1 = 0$ and $\Delta_2 > 0$.

For the function $f_3^{(ii)}(y_2)$ we have that

$$\begin{aligned}\Delta_1 = \;& a^2(\lambda(\alpha_1(A-m)-2a\alpha_2)\Big((a\gamma_1+1)(A+m)-2a^2\gamma_2\Big)-(\beta_1(A+m)-2a\beta_2)((a\gamma_1+1)\\ &(A-m)-2a^2\gamma_2))^2,\\ \Delta_2 = \;& a^2(\lambda(\alpha_1(A+m)-2a\alpha_2)\Big((a\gamma_1+1)(A-m)-2a^2\gamma_2\Big)-(\beta_1(A-m)-2a\beta_2)((a\gamma_1+1)\\ &(A+m)-2a^2\gamma_2))^2,\end{aligned}$$

are the discriminants of the numerator and the denominator of the function $f_3^{(ii)}(y_2)$, respectively. It is clear that since $\Delta_i \geq 0$, and

$$\left(f_3^{(ii)}\right)'(y_2) = \left(k_0 + k_1\ y_2 + k_2\ y_2^2\right)^{A-1}\left(k_0 + g_1\ y_2 + g_2\ y_2^2\right)^{-(A+1)} P(y_2)e^{M\ y_2},$$

where $P(y_2) = k_0(A(k_1-g_1)+k_0M)+k_0(2A(k_2-g_2)+M(g_1+k_1))\,y_2+(A(g_1k_2-g_2k_1)+g_1k_1M+k_0M(g_2+k_2))\,y_2^2+M(g_1k_2+g_2k_1)\,y_2^3+g_2k_2M\,y_2^4$, the function $f_3^{(ii)}(y_2)$ has the same variation than the function $f_2^{(ii)}(y_2)$, i.e, they have the same graphics, the only difference is at infinity. If $M>0$, $f_3^{(ii)}(y_2)$ has a parabolic branch at $+\infty$ but if $M<0$, the parabolic branch is at $-\infty$.

1- *If $\Delta_i = 0$ with $i = 1,2$ the graphics of the function $f_3^{(ii)}(y_2)$ are equivalent to the graphics of the function $f_2^{(ii)}(y_2)$ when both discriminants of the function $f_3^{(ii)}(y_2)$ are strictly negative, see Figure 5.25. But there is a parabolic branch at infinity instead of a horizontal asymptote.*

2- *If either $\Delta_i > 0$ with $i = 1,2$, or if $\Delta_1 = 0$ and $\Delta_2 > 0$, or if $\Delta_1 > 0$ and $\Delta_2 = 0$, then the graphics of the function $f_3^{(ii)}(y_2)$ are equivalent to the graphics of the function $f_2^{(ii)}(y_2)$ that are shown in Figures 5.18, 5.17 and 5.22.*

Since $f_2^{(ii)}(y_2)$ and $f_3^{(ii)}(y_2)$ have the same behavior we conclude that $(a),\ldots,(f)$ of Figures 5.18 and 5.25 with the graphics of the function $g_3^{(ii)}(y_2)$ are the ones that are going to give the maximum number of intersection points between the graphics of these functions. For the function $g_3^{(ii)}(y_2)$, Figure 5.18 is one that will give the maximum number of the intersection points between $g_3^{(ii)}(y_2)$ and $f_3^{(ii)}(y_2)$ because of the positive sign of $f_3^{(ii)}(y_2)$. Then it is clear that these two functions can intersect at most in eight points. Hence the maximum number of limit cycles for the class of PWS $\mathcal{C}_{ii}$ when $C = b = 0$ is at most four.

By taking $\{r, K_0, K_1, K_2, G_1, G_2, A, M, k_1, k_2, k_3, k_4, k_5, k_6, k_7\} = \{2, -2, 2.4, 0.9, -2, 5, 2, 0.5, 1.5, 4, 1, 6, 0.8\}$ *we build an example with exactly eight intersection points between the graphics of the two functions $f_3^{(ii)}(y_2)$ and $g_3^{(ii)}(y_2)$, these points are shown in Figure 5.7.*

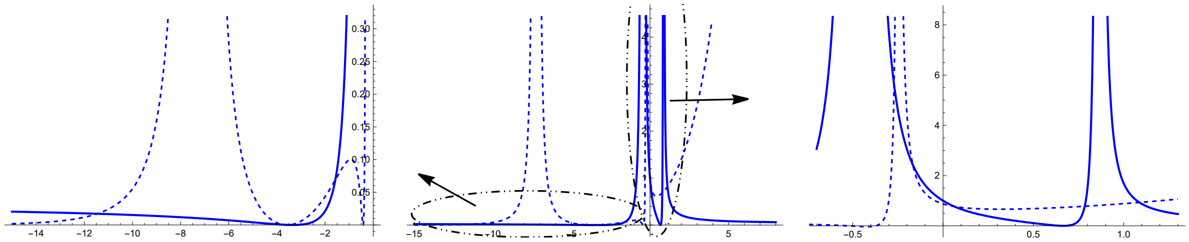


Figure 5.7: The eight intersection points between the graphics of the two functions $f_3^{(ii)}(y_2)$ drawn in dashed line and $g_3^{(ii)}(y_2)$ drawn in a continuous line.

If $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$ corresponding to $k=4$ and $j=ii$ in system (5.16), the first integral of the quadratic center (5.1) is $H_4^{(ii)}(x,y)$ given in (5.9). Then to study the solutions y_2 satisfying $F_4^{(ii)}(y_2)=0$ is equivalent to study the solutions y_2 of $f_4^{(ii)}(y_2) = g_4^{(ii)}(y_2)$ such that

$$f_4^{(ii)}(y_2) = f_1^{(i)}(y_2) \ \ with \ \ r = 1/A \ \ and \ \ g_4^{(ii)}(y_2) = \left(\frac{m_1^+ + m_2^+ \, y_2}{m_1^+ + m_3^+ \, y_2}\right)^{p^+} \left(\frac{m_1^- + m_2^- \, y_2}{m_1^- + m_3^- \, y_2}\right)^{p^-},$$

where

$$\begin{aligned} m_1^\pm = & \ \frac{1}{2}\gamma_1(C \pm \sqrt{\Delta}) - b\gamma_2 + 1, \ \ m_2^\pm = \frac{1}{2}\beta_1(C \pm \sqrt{\Delta}) - b\beta_2, \\ m_3^\pm = & \ (\frac{1}{2}\alpha_1(C \pm \sqrt{\Delta}) - b\alpha_2)\lambda, \ \ p^\pm = \frac{1}{2b}\left(1 \pm C/\sqrt{\Delta}\right). \end{aligned}$$

Figures 5.18, 5.17 and 5.22 represent all the possible graphics of the function $g_4^{(ii)}(y_2)$. The function $f_4^{(ii)}(y_2)$ is drawn in the previous cases and all its graphics are shown in Figure 5.16. As in the proof of statement (a) of Theorem 4 for the function $f_4^{(ii)}(y_2)$ we chose Figure 5.16(a) to get the maximum number of the intersections points between this figure and the graphics of $g_4^{(ii)}(y_2)$. Since Figure 5.16(a) of $f_4^{(ii)}(y_2)$ is up to the y_2-axis, the number of the changes of the sign of $\left(g_4^{(ii)}\right)'(y_2)$ and the position of the graphics $g_4^{(ii)}(y_2)$ play a main role in finding the maximum number of the intersection points. Hence the graphic of the function $g_4^{(ii)}(y_2)$ is up to the y_2-axis when both p and q are even and $\left(g_4^{(ii)}\right)'(y_2)$ has seven changes of the sing in $(a), \cdots, (f)$ of Figure 5.18. Then the maximum number of the intersection points between these two functions is at most eight. By symmetry we obtain that the maximum number of limit cycles for the class of PWS $\mathcal{C}_{ii}$ when $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$ is at most four.

By taking $\{r, s_1, s_2, s_3, p^+, m_1^+, m_2^+, m_3^+, p^-, m_1^-, m_2^-, m_3^-\} = \{8, -2, 2, 0.8, 2, -2, 2.4, 0.9, 6, -2, 5, 2\}$ we build an example with exactly eight intersection points between the graphics of the two functions $f_4^{(ii)}(y_2)$ and $g_4^{(ii)}(y_2)$, these points are shown in Figure 5.9. Then we have four limit cycles for the class of PWS $\mathcal{C}_{ii}$ when $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$.

In what follows we give a PWS of the class $\mathcal{C}_{ii}$ when $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$ with *four limit cycles. In the region Σ_1^i we consider the quadratic center*

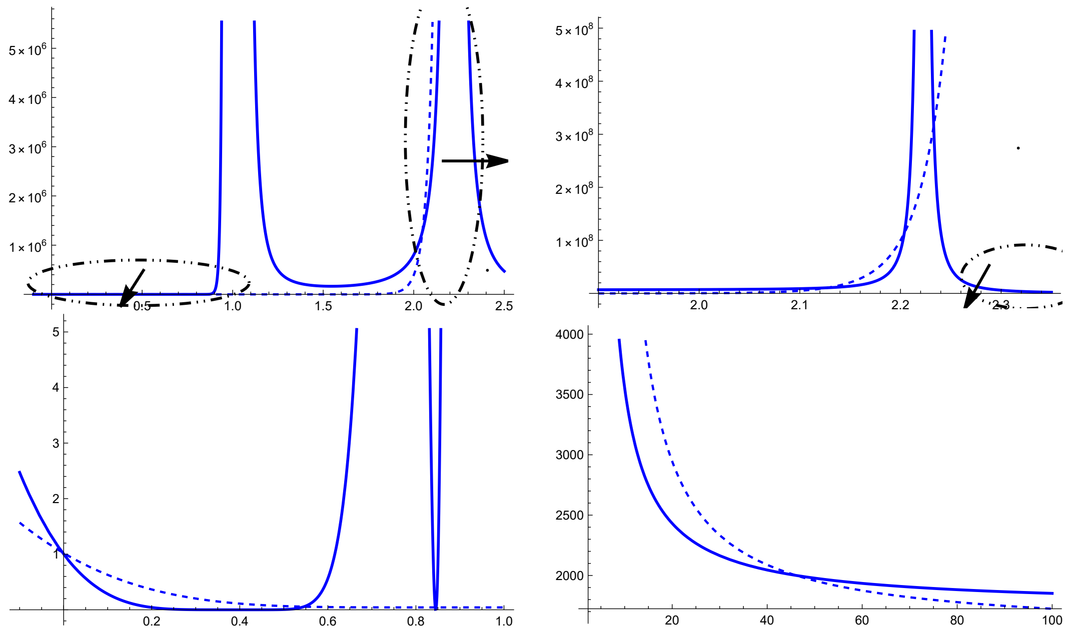


Figure 5.8: The eight intersection points between the graphics of the two functions $f_4^{(ii)}(y_2)$ drawn in dashed line and $g_4^{(ii)}(y_2)$ drawn in a continuous line.

$$\begin{aligned}\dot{x} \approx & \ -73350.3x^2 + x(0.224264y + 758819) + (3.62214591491`*^{\wedge}\text{-}7y \\ & -1.16003)y - 1.96252\,10^6, \\ \dot{y} \approx & \ -0.0239117x^2 + x(73350.3y - 151764) + (-0.0747547y - 379409)y \\ & +785009,\end{aligned} \tag{5.23}$$

this system has the first integral

$$\begin{aligned}H_4^{(ii)}(x,y) \approx & \ (103733x - 0.3y - 536564)(-103733x - 0.0885618y + 536568) \\ & (-3.167224349`*^{\wedge}\text{-}8x + 0.194281y - 0.401973)^2.\end{aligned}$$

In the region Σ_2^i we consider the linear differential center

$$\dot{x} = -\frac{3x}{2} - \frac{25y}{4}, \quad \dot{y} = x + \frac{3y}{2}, \tag{5.24}$$

with the first integral $H(x,y) = \left(x + \dfrac{3y}{2}\right)^2 + 4y^2$. For the PWS (5.23)–(5.24), system (5.16) has the four solutions $(x_1,y_1) \approx (4.5812588, 1.83303)$, $(x_2,y_2) \approx (3.87298, 1.54919)$, $(x_3,y_3) \approx (3, 1.2)$ and $(x_4,y_4) \approx (1.7320, 0.6928)$ which provides the four limit cycles intersecting the rays Γ_1 and Γ_2 in the eight points $(x_j, 0)$ and $(0, y_j)$ with $j = 1,2,3,4$, see Figure 5.11(a).

If $b = a = 0 \neq AC$ corresponding to $k = 5$ and $j = ii$ in system (5.16), the first integral of the quadratic center is $H_5^{(ii)}(x, y)$ given in (5.10). The equation $F_5^{(ii)}(y_2) = 0$ is equivalent to the equation $f_5^{(ii)}(y_2) = g_5^{(ii)}(y_2)$ such that

$$f_5^{(ii)}(y_2) = f_1^{(ii)}(y_2) \quad \textit{and} \quad g_5^{(ii)}(y_2) = f_2^{(i)}(y_2),$$

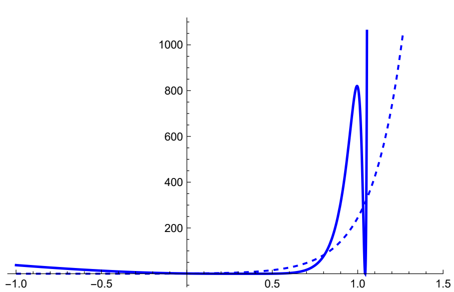
where

$$\begin{aligned} L_0 = &\ \ L_2 = 0,\ \ L_1 = 2AC\left((A\alpha_1 + \alpha_2 C)\lambda - A\beta_1 - \beta_2 C\right), \\ m_1 = &\ \ A\gamma_2 + 1,\ \ m_2 = A\alpha_2\lambda,\ \ m_3 = A\beta_2,\ \ p = 2C^2, \\ n_1 = &\ \ C\gamma_1 + 1,\ \ n_2 = \alpha_1 C\lambda,\ \ n_3 = \beta_1 C,\ \ q = 2A^2. \end{aligned}$$

Due to $L_2 = 0$ we show the graphics of the function $g_5^{(ii)}(y_2)$ in Figures 5.26(c) and 5.26(d). For the function $f_5^{(ii)}(y_2)$ the possible graphics are shown in Figures 5.18, 5.17 and 5.22.

Since the function $g_5^{(ii)}(y_2) = e^{L_1 y_2}$ is a positive function and it has the x-axis as a horizontal asymptote straight line and as we mentioned in the precedent cases the number and the location of the extrema according to the x-axis is important for the maximum number of the intersection points between the graphics of $g_5^{(ii)}(y_2)$ and $f_5^{(ii)}(y_2)$. For that reason we guarantee that the maximum number of intersection points between the graphics of these two functions can be precisely between the graphics of (c), (d) of Figure 5.26 and the graphics of Figure 5.18. Then we remark that the graphics of these functions can intersect at most in seven points. By symmetry the maximum number of limit cycles for the class of PWS $\mathcal{C}_{ii}$ when $b = a = 0 \neq AC$ is at most three.

By taking $\{p, q, m_1, m_2, m_3, n_1, n_2, n_3, L_0, L_1, L_2\} = \{2, 4, -2.4, 2.3, 1, -2, 6, 1.8, 0, 5.5, 0\}$ we build an example with exactly seven intersection points between the graphics of the two functions $f_5^{(ii)}(y_2)$ and $g_5^{(ii)}(y_2)$, these points are shown in Figure 5.9.



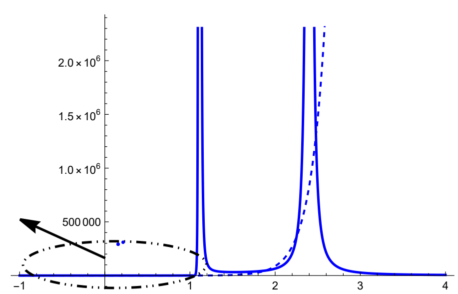


Figure 5.9: The seven intersection points between the graphics of the two functions $f_5^{(ii)}(y_2)$ drawn in a continuous line and $g_5^{(ii)}(y_2)$ drawn in dashed line.

If $A = a = 0 \neq Cb$ corresponding to $k = 6$ and $j = ii$ in system (5.16), the first integral *in this case is* $H_6^{(ii)}(x,y)$ *given in* (5.11)*, the solutions of* $F_6^{(ii)}(y_2) = 0$ *are equivalent to the solutions of the equation* $f_6^{(ii)}(y_2) = g_6^{(ii)}(y_2)$ *such that*

$$f_6^{(ii)}(y_2) = g_4^{(ii)}(y_2) \quad \text{and} \quad g_6^{(ii)}(y_2) = f_2^{(i)}(y_2),$$

where

$$\begin{aligned} m_1^{\pm} = & \ \frac{1}{2}\gamma_1(C \pm \sqrt{\Delta}) - b\gamma_2 + 1, \ \ m_2^{\pm} = \frac{1}{2}\beta_1(C \pm \sqrt{\Delta}) - b\beta_2, \\ m_3^{\pm} = & \ \frac{1}{2}(\alpha_1(C \pm \sqrt{\Delta}) - 2\alpha_2 b)\lambda, \ \ p^{\pm} = \frac{1}{2b}\Big(1 \pm C/\sqrt{\Delta}\Big), \\ L_0 = & \ 2\gamma_2, \ \ L_1 = \alpha_2\lambda - \beta_2, \ \ L_2 = 0. \end{aligned}$$

Here we remark that this case is similar to the precedent one where $b = 0 \neq A$*. In the precedent case we have proved that three is the maximum number of limit cycles for the class of PWS* $\mathcal{C}_{ii}$ *when* $b = a = 0 \neq AC$*. Hence the maximum number of limit cycles for the class of PWS* $\mathcal{C}_{ii}$ *when* $A = 0 \neq b$ *is at most three.*

In what follows we give a PWS of the class of PWS $\mathcal{C}_{ii}$ *when* $A = 0 \neq b$ *with three limit cycles. In the region* Σ_1^i *we consider the quadratic center*

$$\begin{aligned} \dot{x} \approx & \ -0.198885x^2 + x(-0.109869y - 0.912492) + (0.387068y - 1.53815)y \\ & -1.22818, \\ \dot{y} \approx & \ y(0.599866 - 0.096767y) + 0.0497213x^2 + x(0.0274673y + 0.834053) \\ & +0.672247, \end{aligned} \tag{5.25}$$

this system has the first integral

$$H_6^{(ii)}(x,y) \approx \ (-x - 1.69835y + 4.22541)^2(1.x - 1.14593y + 5.99997)e^{0.2x+0.8y}.$$

In the region Σ_2^i *we consider the linear differential center*

$$\dot{x} = \frac{4x}{5} - \frac{289y}{100}, \quad \dot{y} = x - \frac{4y}{5}, \tag{5.26}$$

with the first integral $H(x,y) = \left(x - \dfrac{4y}{5}\right)^2 + \dfrac{9y^2}{4}$*. For the PWS* (5.25)–(5.26)*, system* (5.16) *has the three solutions* $(x_1,y_1) \approx (3.87298, 2.27823)$*,* $(x_2,y_2) \approx (3, 1.76471)$ *and* $(x_3,y_3) \approx (1.73205, 1.01885)$ *that provide the three limit cycles intersecting the rays* Γ_1 *and* Γ_2 *in the six points* $(x_j,0)$ *and* $(0,y_j)$ *with* $j = 1,2,3$*, see Figure* 5.11*(b).*

If $\Delta = a = 0$ *corresponding to* $k = 7$ *and* $j = ii$ *in system* (5.16)*, the first integral of the* *quadratic center* (5.1) *is* $H_7^{(ii)}(x,y)$ *given in* (5.12)*. Then to study* $F_7^{(ii)}(y_2)=0$ *it is enough to study the solutions* y_2 *of the equation* $f_7^{(ii)}(y_2)=g_7^{(ii)}(y_2)$ *such that*

$$f_7^{(ii)}(y_2)=f_1^{(ii)}(y_2) \quad \textit{and} \quad g_7^{(ii)}(y_2)=g_1^{(ii)}(y_2),$$

where

$$\begin{aligned}
m_1 &= -2b\gamma_2+C\gamma_1+2, \;\; m_2=\beta_1C-2b\beta_2, \;\; m_3=(\alpha_1C-2\alpha_2b)\lambda, \;\; p=1,\\
n_1 &= (4b^2+C^2)\gamma_2-4b, \;\; n_2=\alpha_2\left(4b^2+C^2\right)\lambda, \;\; n_3=\beta_2\left(4b^2+C^2\right), \;\; q=\frac{4b^2}{4b^2+C^2},\\
K_1 &= -bC\Big(b^2(\lambda(\alpha_2\gamma_2-\alpha_1\gamma_2)+\beta_1\gamma_2-\beta_2\gamma_2)+b(\alpha_1\lambda-\beta_1)+\alpha_2C\lambda-\beta_2C\Big),\\
K_2 &= -b^3C(\alpha_2\beta_1-\alpha_1\beta_2)\lambda.
\end{aligned}$$

It is clear that since $q=1$ *all the possible graphics of the function* $f_7^{(ii)}(y_2)$ *are shown in Figure* 5.17 *if* q *is an odd number and in Figure* 5.22 *if* q *is an even number. For the function* $g_7^{(ii)}(y_2)$ *we show all its possible graphics in Figure* 5.21.

Since $g_7^{(ii)}(y_2)$ *is a positive function and the maximum number of changes of the sign of the derivative of this function is when* $\left(g_7^{(ii)}\right)'(y_2)$ *has two extrema. Figures* 5.19(a) *and* 5.19(b) *are the ones that give the maximum number of the intersection points with the graphics of* $f_7^{(ii)}(y_2)$*. Because the first vertical asymptote straight line of* $f_7^{(ii)}(y_2)$ *is also a vertical asymptote straight line for* $g_7^{(ii)}(y_2)$ *and the second vertical asymptote of* $f_7^{(ii)}(y_2)$ *is an extremum of* $g_7^{(ii)}(y_2)$*, the maximum number of intersection points between these two functions is at most four. Then by symmetry the maximum number of limit cycles for the class of PWS* $\mathcal{C}_{ii}$ *when* $\Delta=a=0$ *is two.*

By taking $\{K_1,K_2,m_1,m_2,m_3,p,n_1,n_2,n_3,q\}=\{-0.5,2,-2,2,1,1,1,2,0.5,2\}$ *we build an example with exactly four intersection points between the graphics of the two functions* $f_7^{(ii)}(y_2)$ *and* $g_7^{(ii)}(y_2)$*, these points are shown in Figure* 5.10.

If $A=b=0$ *corresponding to* $k=8$ *and* $j=ii$ *in system* (5.16)*, the first integral in this case is* $H_8^{(ii)}(x,y)$ *given in* (5.13)*, the study of the solutions* y_2 *satisfying* $F_8^{(ii)}(y_2)=0$ *is equivalent to study the solutions* y_2 *of the equation* $f_8^{(ii)}(y_2)=g_8^{(ii)}(y_2)$ *such that*

$$f_8^{(ii)}(y_2)=f_1^{(i)}(y_2) \;\; \textit{with} \;\; r=2 \;\; \textit{and} \;\; g_8^{(ii)}(y_2)=f_2^{(i)}(y_2),$$

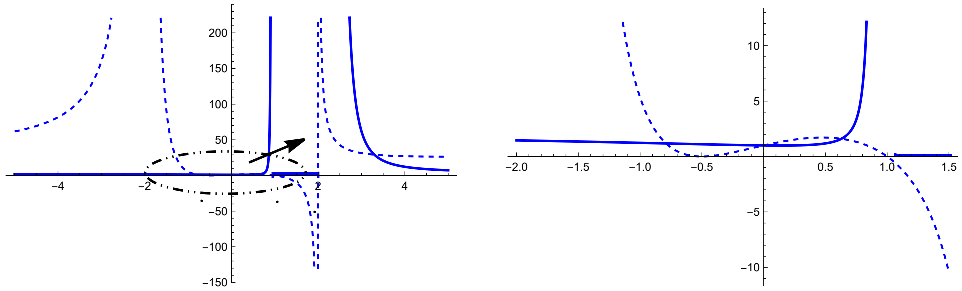


Figure 5.10: The four intersection points between the graphics of the two functions $f_7^{(ii)}(y_2)$ drawn in a continuous line and $g_7^{(ii)}(y_2)$ drawn in dashed line.

where

$$
\begin{aligned}
L_0 = & \; 0, \;\; L_1 = -2C(\beta_1 + \beta_2 C\gamma_2 - (\alpha_1 + \alpha_2 C\gamma_2)\lambda), \;\; L_2 = C^2(\alpha^2\alpha_2^2 + \alpha_2^2\omega^2 - \beta_2^2), \\
s_1 = & \; C\gamma_1 + 1, \;\; s_2 = \alpha_1 C\lambda, \;\; s_3 = \beta_1 C.
\end{aligned}
$$

Due to the fact that the graphics of $f_8^{(ii)}(y_2)$ are equivalent to the graphics of $f_3^{(i)}(y_2)$ with $r = 2$, and to the fact that the graphics of $g_8^{(ii)}(y_2)$ are equivalent to the graphics of $g_3^{(i)}(y_2)$, it follows from the proof of the case $b = 0$ of statement (a) of Theorem 4, that the maximum number of intersection points between the graphics of the two functions $f_8^{(ii)}(y_2)$ and $g_8^{(ii)}(y_2)$ is at most four. Then the maximum number of limit cycles for the class of PWS C_{ii} when $A = b = 0$ is at most two.

Now we give an example having two limit cycles for the class of PWS C_{ii} when $A = b = 0$. In the region Σ_1^i we consider the quadratic center

$$
\begin{aligned}
\dot{x} \approx & \; x(0.741461 - 0.529638y) + 0.0173273x^2 + (0.294564y + 2.21484)y \\
& -5.42593, \\
\dot{y} \approx & \; x(1.50454 - 0.934655y) + 0.0305776x^2 + (0.519819y - 1.97381)y \\
& +1.78917,
\end{aligned}
\tag{5.27}
$$

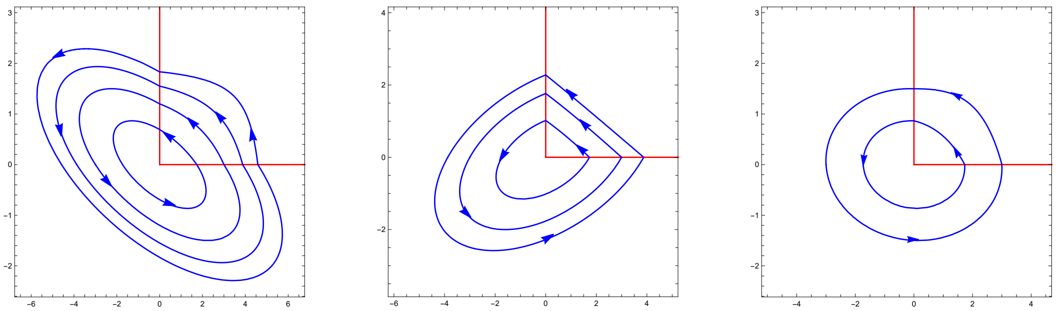
this system has the first integral

$$
H_8^{(ii)}(x,y) \approx \; (x - 30y + 47.9582)^2 e^{-0.81(x-0.566667y+1.36927)^2+0.2x-6y}.
$$

In the region Σ_2^i we consider the linear differential center

$$
\dot{x} = -0.1x - 4.01y, \quad \dot{y} = x + 0.1y, \tag{5.28}
$$

with the first integral $H(x,y)=(x+0.1y)^2+4y^2$. *For the PWS* (5.27)–(5.28), *system* (5.16) *has the two solutions* $(x_1,y_1)\approx(1,1.49813)$ *and* $(x_2,y_2)\approx(1.73205,0.864945)$ *that provide the two limit cycles intersecting the rays* Γ_1 *and* Γ_2 *in the four points* $(x_j,0)$ *and* $(0,y_j)$ *with* $j=1,2$, *see Figure* 5.11(c). *This example completes the proof of statement* (b).



(a) (b) (c)

Figure 5.11: (a) The four limit cycles of the PWS (5.23)–(5.24), (b) the three limit cycles of the PWS (5.25)–(5.26), and (c) the two limit cycles of the PWS (5.27)–(5.28).

Proof of statement (c) of Theorem 5.2. *Now we will prove this statement for the third class* $\mathcal{C}_{iii}$ *when* $A-2b=C+2a=0$. *In this case* $j=iii$ *in system* (5.16), *then the equation* $F^{(iii)}(y_2)=0$ *where*

$$
\begin{aligned}
F^{(iii)}(y_2)= \; & \Big(\alpha_1\gamma_1\lambda+\alpha_2\gamma_2\lambda+a\alpha_1\gamma_1^2\lambda-\beta_1(a\gamma_1^2-a\gamma_2^2+2b\gamma_1\gamma_2+\gamma_1)-\beta_2\gamma_2(-2a\gamma_1+d\gamma_2\\
& +1)-a\alpha_1\gamma_2^2\lambda+2a\alpha_2\gamma_1\gamma_2\lambda+b\gamma_1(\lambda(2\alpha_1\gamma_2+\alpha_2\gamma_1)-\beta_2\gamma_1)+\alpha_2 d\gamma_2^2\lambda\Big)\,y_2+\frac{1}{2}\\
& \Big(4\beta_1\beta_2(a\gamma_2-b\gamma_1)-\beta_1^2(2a\gamma_1+2b\gamma_2+1)+\alpha^2(\alpha_1^2(2a\gamma_1+2b\gamma_2+1)+4\alpha_1\alpha_2(b\\
& \gamma_1-a\gamma_2)+\alpha_2^2(-2a\gamma_1+2d\gamma_2+1))+\beta_2^2(2a\gamma_1-2d\gamma_2-1)+\omega^2(\alpha_1^2(2a\gamma_1+2b\gamma_2\\
& +1)+4\alpha_1\alpha_2(b\gamma_1-a\gamma_2)+\alpha_2^2(-2a\gamma_1+2d\gamma_2+1))\Big)\,y_2^2+\frac{1}{3}\Big(a(\alpha_1\lambda^3(\alpha_1^2-3\alpha_2^2)-\\
& \beta_1^3+3\beta_1\beta_2^2)+3\alpha_1^2\alpha_2 b\lambda^3-3b\beta_1^2\beta_2+\alpha_2^3 d\lambda^3-\beta_2^3 d\Big)\,y_2^3,
\end{aligned}
$$

is a cubic equation in the variable y_2. *This equation has at most three real solutions. Then by the symmetry the class of PWS* $\mathcal{C}_{iii}$ *when* $A-2b=C+2a=0$ *can have at most one limit cycle. So we have proved that there is at most one limit cycle for the class of PWS* $\mathcal{C}_{iii}$ *when* $A-2b=C+2a=0$.

To complete the proof of this case we build an example of the class of PWS $\mathcal{C}_{iii}$ *when* $A-2b=$

$C+2a=0$ *with one limit cycle. In the region* Σ_1^i *we consider the quadratic center*

$$\begin{aligned}\dot{x} &= \frac{1}{1300}\Big(-169x^2 + x(988y+6720) - 2y(247y+3760) - 27900\Big),\\ \dot{y} &= \frac{1}{650}\Big(2678x^2 + x(169y-3290) - y(247y+3360) - 9700\Big),\end{aligned} \tag{5.29}$$

this system has the first integral

$$\begin{aligned}H^{(iii)}(x,y) = \;& 5356x^3 + x^2(507y-9870) - 6x(y(247y+3360)+9700) + 2y(y(247y\\ &+5640)+41850) + 201000.\end{aligned}$$

In the region Σ_2^i *we consider the linear differential center*

$$\dot{x} \approx -2x - 10.8228y, \quad \dot{y} \approx x + 2y, \tag{5.30}$$

with the first integral $H(x,y) \approx (x+2y)^2 + 6.82276y^2$. *For the PWS* (5.29)–(5.30), *system* (5.16) *has only the solution* $(x_1, y_1) \approx (0.567633, 2.66406)$ *which provides the unique limit cycle intersecting the rays* Γ_1 *and* Γ_2 *in the two points* $(x_1, 0)$ *and* $(0, y_1)$, *see Figure 5.12*(*a*). *This example completes the proof of statement* (*c*). ■

Proof of statement (d) **of Theorem 5.2.** *Here we prove this statement for the 4th class* $\mathcal{C}_{iv}$ *when* $C+2a = A+4b+5d = a^2+bd+2d^2 = 0$ *and* $j = iv$ *in system* (5.16), *then the equation* $F^{(iv)}(y_2) = 0$ *is a polynomial equation of degree nine and because of the large expression of this equation we omit it. This equation has at most nine real solutions. In fact these nine solutions represent four real solutions of* (5.16) *because of the symmetry. Then all these ten solutions provide the four limit cycles for the class of PWS* $\mathcal{C}_{iv}$ *when* $C+2a = A+4b+5d = a^2+bd+2d^2 = 0$. *So we have proved that there are at most four limit cycles for the class of PWS* $\mathcal{C}_{iv}$ *when* $C+2a = A+4b+5d = a^2+bd+2d^2 = 0$.

To complete the proof of this case we introduce an example with four limit cycles for the class of PWS $\mathcal{C}_{iv}$ *when* $C+2a = A+4b+5d = a^2+bd+2d^2 = 0$. *In the region* Σ_2^i *we consider the quadratic center*

$$\begin{aligned}\dot{x} \approx\;& x(0.898898 - 0.154472y) + 0.0752303x^2 + (0.00129847y\\ &-1.58169)y - 1.51959,\\ \dot{y} \approx\;& x(1.93316 - 0.196016y) + 0.150808x^2 + (-0.047621y\\ &-1.81001)y + 1.01708,\end{aligned} \tag{5.31}$$

this system has the first integral

$$H^{(IV)}(x,y) \approx \Big(x^2(36.2311 - 1.67779y) + y(y(12.5561 - 0.174923y) - 269.803) + x^3 + x \\ ((0.938325y - 42.7139)y + 434.695) + 1710.3\Big)^2 \Big/ \Big(x(24.1541 - 1.11853y) \\ + x^2 + (0.312775y - 14.9674)y + 142.483\Big)^3.$$

In the region Σ_1^i *we consider the linear differential center*

$$\dot{x} = x - 2y, \quad \dot{y} = x - y, \tag{5.32}$$

with the first integral $H(x,y) = (x-y)^2 + y^2$. *For the PWS* (5.31)–(5.32), *system* (5.16) *has the four solutions* $(x_1,y_1) \approx (0.836666, 0.591608)$, $(x_2,y_2) \approx (0.707107, 0.5)$, $(x_3,y_3) \approx (0.547723, 0.387298)$ *and* $(x_4,y_4) \approx (0.316228, 0.223607)$ *that provide the four limit cycles intersecting the rays* Γ_1 *and* Γ_2 *in the eight points* $(x_i,0)$ *and* $(0,y_i)$ *with* $i = 1,2,3,4$, *see Figure 5.12(b). This example completes the proof of Theorem 5.2.*

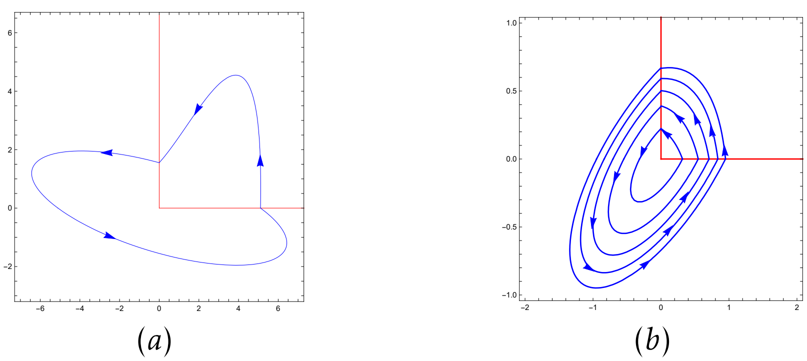


Figure 5.12: (a) The unique limit cycle of the PWS (5.29)–(5.30), and (b) the five limit cycles of the PWS (5.31)–(5.32).

5.2 The limit cycles satisfying of the four classes of PWS $\mathcal{C}_k$ with $k = i, ii, iii, iv$ separated by Σ^i Cnf 3

In the second main result we provide the maximum number of limit cycles of the four classes $\mathcal{C}_k$, $k = i, ii, iii, iv$, of PWS having limit cycles satisfying **Cnf 3**.

Theorem 5.3

The maximum number of limit cycles satisfying **Cnf 3** for

(a) the class $\mathcal{C}_i$ is at most six if $A+b=0 \neq A$; three either if $A=0\neq b$ or $b=0\neq A$; and two if $A=b=0$. There are systems of this class with five limit cycles see Figure 5.13(a), three limit cycles in Figure 5.13(b), and two limit cycles in Figure 5.13(c), respectively;

(b) the class $\mathcal{C}_{ii}$ is at most seven either if $AbC(A+b)\Delta \neq 0 = a$ and $\Delta \neq 0$ or $C=b=0$, or $A+b=0$ and $a=0\neq Cb$; six either if $b=a=0\neq AC$, or $A=a=0\neq Cb$; four if $\Delta=a=0$; and three if $A=b=0$. There are systems of this class with five limit cycles in Figure 5.14(a), six limit cycles in Figure 5.14(b), four limit cycles in Figure 5.14(c), and three limit cycles in Figure 5.14(d), respectively;

(c) the class $\mathcal{C}_{iii}$ is at most two. There are systems of this class with two limit cycles, see Figure 5.15(a);

(d) the class $\mathcal{C}_{iv}$ is at most nine. There are systems of this class with five limit cycles, see Figure 5.15(b);

Proof of Theorem 5.3

Proof. *We simultaneously study the existence of limit cycles of configuration* ***Cnf 1*** *and* ***Cnf 2****. The limit cycles of configuration* ***Cnf 1*** *intersect the separation line* Σ_2 *in two distinct points* $p_1=(0,y_1)$ *and* $p_2=(0,y_2)$ *with* $0\leq y_1<y_2$*. These two points must satisfy the system of equations*

$$H(p_1)-H(p_2)=(y_1-y_2)h(y_1,y_2)=0, \quad H_k^{(j)}(p_1)-H_k^{(j)}(p_2)=h_k^{(j)}(y_1,y_2)=0, \tag{5.33}$$

with $k=1,\ldots,4$ *for* $j=i$*, and* $k=1,\ldots,8$ *for* $j=ii$*. For* $j=iii,iv$ *we have* $H_k^{(j)}(x,y)=H^{(j)}(x,y)$ *and* $h_k^{(j)}(y_1,y_2)=h^{(j)}(y_1,y_2)$*. On the other hand, the two intersection points of limit cycles of configuration* ***Cnf 2*** *with the separation line* Σ^i *must satisfy (5.16). In Theorems 5.1 and 5.2 the maximum number of limit cycles is already provided for each one of the configurations* ***Cnf 1*** *and* ***Cnf 2****, respectively. Then we have the following results.*

Proof of statement (a) of Theorem 5.3. *We give an example with five limit cycles for the class of PWS $\mathcal{C}_i$ when $A+b=0\neq A$. In what follows we give a PWS of the class* (1.3)–(5.1) *when $A+b=0\neq A$ with five limit cycles two satisfying **Cnf 1** and three satisfying **Cnf 2**. In the region Σ_1^i we consider the quadratic center*

$$\begin{aligned}\dot{x} \approx\ & y(2.05262-0.0538376y)+0.00875921x^2+x(0.00952768y\\ & +0.72735)-3.62673,\\ \dot{y} \approx\ & -4.884707923516701\text{*\^{}-}7x^2+x(0.00882947y-0.907179)\\ & +(0.00615901y-0.719153)y-0.362589,\end{aligned} \tag{5.34}$$

with the first integral $H_1^{(i)}(x,y)\approx e^{k(x,y)}(0.000204571x-0.00980828y+1.03)$, where

$$k(x,y)=\frac{184.31x^2+x(305.64y+8913.59)+y(126.70y-4217)+3.474\times 10^7}{(1x-47.945y+5034.93)^2}.$$

In the region Σ_2^i we consider the linear differential center

$$\dot{x}=-\frac{11x}{5}+\frac{1346y}{125}-20,\quad \dot{y}=-5x+\frac{11y}{5}-2, \tag{5.35}$$

with the first integral $H(x,y)=-8(11x+100)y+20x(5x+4)+5384y^2/25$. For the PWS (5.34)–(5.35)*, system* (5.33) *has the two solutions $(y_1,y_2)=((2500-25\sqrt{5962})/1346,(25\sqrt{5962}+2500)/1346)$ and $(y_3,y_4)=((2500-25\sqrt{8654})/1346,(25\sqrt{8654}+2500)/1346)$, which provide two limit cycles intersecting Γ_1 in the four points $(0,y_i)$ with $i=1,2,3,4$, and system* (5.16) *has the three solutions $(x_5,y_5)=((1/5)\left(\sqrt{29}-2\right),(25\sqrt{11346}+2500)/1346))$, $(x_6,y_6)=((1/5)\left(\sqrt{79}-2\right),(25\sqrt{1038}+250)/1346)$ and $(x_7,y_7)=((1/5)(\sqrt{129}-2),(25\sqrt{1673}+250)/1346)$, which provide the six intersecting points $(x_j,0)$ and $(0,y_j)$ with $j=5,6,7$ of the three limit cycles with the separation line Σ^i. Then the PWS* (5.34)–(5.35) *has exactly five limit cycles, see Figure* 5.13*(a).*

We give an example with three limit cycles for the class of PWS $\mathcal{C}_i$ when $b=0\neq A$. In what follows we give a PWS of the class (1.3)–(5.1) *when $b=0\neq A$ with three limit cycles one satisfying **Cnf 1** and two satisfying **Cnf 2**. In the region Σ_1^i we consider the quadratic center*

$$\begin{aligned}\dot{x} \approx\ & y(15.0335-0.254667y)+0.739273x^2+x(2.47274y+1.86447)\\ & -14.1078,\\ \dot{y} \approx\ & -0.607274x^2+x(-0.678545y-4.62004)+(0.0739273y-4.18426)y\\ & -6.3485,\end{aligned} \tag{5.36}$$

this system has the first integral $H_3^{(i)}(x,y) \approx e^{(x-0.1y+4.80783)^2+(-0.8x-2.62059y+2.31979)^2}$. *In the region* Σ_2^i *we consider the linear differential center*

$$\dot{x} = -2.2x + 20.968..y - 20, \quad \dot{y} = -5x + 2.2y - 9, \tag{5.37}$$

with the first integral $H(x,y) = x(360-88y)+100.x^2+y(419.36y-800)$. *For the PWS* (5.36)–(5.37), *system* (5.33) *has the unique solution* $(y_1,y_2) \approx (0.29589, 1.61177)$, *which provides a unique limit cycle intersecting* Γ_1 *in the two points* $(0,y_i)$ *with* $i=1,2$, *and system* (5.16) *has the two solutions* $(x_3,y_3) \approx (0.69799, 2.2286)$ *and* $(x_4,y_4) \approx (1.5526, 2.6323)$, *which provide the four intersecting points* $(x_j,0)$, $(0,y_j)$ *with* $j=3,4$ *of the two limit cycles with the separation line* Σ^i. *Then the PWS* (5.36)–(5.37) *has exactly three limit cycles, see Figure* 5.13(*b*).

We give an example with two limit cycles for the class of PWS $\mathcal{C}_i$ *when* $A=b=0$. *In what follows we give a PWS of the class* (1.3)–(5.1) *when* $A=b=0$ *with two limit cycles one satisfying* ***Cnf 1*** *and the other one satisfying* ***Cnf 2***. *In the region* Σ_1^i *we consider the quadratic center*

$$\begin{aligned}
\dot{x} \approx\ & y(0.810541-0.00095672y)-0.095672x^2+x(0.0191344y \\
& +0.056492)-0.789731, \\
\dot{y} \approx\ & -0.95672x^2+x(0.191344y-1.43508)+(-0.0095672y \\
& -0.056492)y+0.0611942,
\end{aligned} \tag{5.38}$$

this system has the first integral

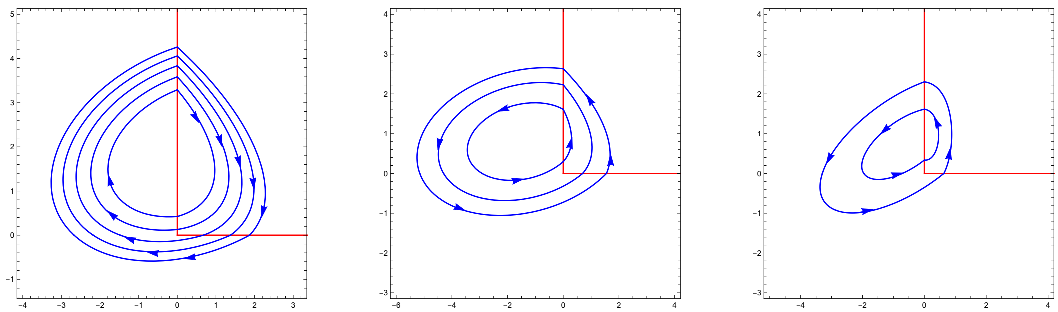
$$\begin{aligned}
H_4^{(i)}(x,y) \approx\ & 3.12x^2+1.6(x-0.1y+0.1)^3+x(0.379428y-0.35502)+2.0285y^2 \\
& -3.95739y+1.93013.
\end{aligned}$$

In the region Σ_2^i *we consider the linear differential center*

$$\dot{x} = -6x + \frac{123y}{8} - 15, \quad \dot{y} = -6x + 6y - 8, \tag{5.39}$$

with the first integral $H(x,y) = -144(2x+5)y + 48x(3x+8) + 369y^2$. *For the PWS* (5.38)–(5.39), *system* (5.33) *has the unique solution* $(y_1,y_2) \approx (0.335447, 1.6157)$ *which provides one limit cycle intersecting* Γ_1 *in the two points* $(0,y_1)$ *and* $(0,y_2)$, *and system* (5.16) *has the unique solution* $(x_3,y_3) \approx (0.63163, 2.3040)$, *which provides the two intersecting points* $(x_3,0)$, $(0,y_3)$ *of the limit cycle with the separation line* Σ^i. *Then the PWS* (5.38)–(5.39) *has exactly two limit cycles, see Figure* 5.13(*c*). *This example completes the proof of statement* (*a*).

Proof of statement (*b*) of Theorem 5.3. *We give an example with five limit cycles for the*



(a) (b) (c)

Figure 5.13: (a) The five limit cycles satisfying **Cnf 3** of the PWS (5.34)–(5.35), (b) the three limit cycles satisfying **Cnf 3** of the PWS (5.36)–(5.37), and (c) the two limit cycles **Cnf 3** of the PWS (5.38)–(5.39).

class of PWS $\mathcal{C}_{ii}$ when $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$. In what follows we give a PWS of the class (1.3)–(5.1) when $AbC(A+b)\Delta \neq 0 = a$ and $\Delta > 0$ with five limit cycles one limit cycle from the first configuration and four limit cycles from the second configuration. In the region Σ_1^i we consider the quadratic center

$$\begin{aligned} \dot{x} \approx & \ 0.514763x^2 + x(-0.224056y - 5.81653) + (-0.439253y - 3.53537)y \\ & +4.46104, \\ \dot{y} \approx & \ x(8.61407 - 0.796302y) - 1.4867x^2 + (0.365704y + 8.05432)y \\ & +5.53532, \end{aligned} \tag{5.40}$$

this system has the first integral

$$\begin{aligned} H_4^{(ii)}(x,y) \approx & \ (-0.829829x - 0.417505y + 2.68699)^2(0.5x - 0.3y - 3.85171) \\ & (1.15966x + 1.13501y + 2.47773). \end{aligned}$$

In the region Σ_2^i we consider the linear differential center

$$\dot{x} = \frac{x}{2} - \frac{461y}{580} + \frac{9}{10}, \quad \dot{y} = \frac{1}{20}(29x - 10y + 30), \tag{5.41}$$

with the first integral $H(x,y) = -116(5x+9)y + 29x(29x+60) + 461y^2$. For the PWS (5.40)–(5.41), system (5.33) has the unique solution $(y_1, y_2) \approx (0.25965, 2.0049)$ which provides the unique limit cycle intersecting Γ_1 in the two points $(0, y_1)$ and $(0, y_2)$, and system (5.16) has the four solutions $(x_3, y_3) \approx (0.77051, 3.42873)$, $(x_4, y_4) \approx (0.59023, 3.1683)$, $(x_5, y_5) \approx (0.387278, 2.86942)$ and $(x_6, y_6) \approx (0.150039, 2.50692)$, which provide the eight intersecting points $(x_k, 0)$, $(0, y_k)$ with $k = 3, 4, 5, 6$ of the four limit cycles with the separation line Σ^i.

Then the PWS (5.40)–(5.41) has exactly five limit cycles, see Figure 5.15(a).

We give an example with six limit cycles for the class of PWS $\mathcal{C}_{ii}$ when $A = a = 0 \neq Cb$. In what follows we give a PWS of the class (1.3)–(5.1) when $A = a = 0 \neq Cb$ with three limit cycles from each configuration. In the region Σ_2^i we consider the quadratic center

$$\begin{aligned}\dot{x} \approx\ & 0.0208423x^2 + x(-0.294257y - 0.687356) + (0.498832y + 2.99943)y \\ & -7.00792, \\ \dot{y} \approx\ & y(1.7807 - 0.166277y) - 0.00694744x^2 + x(0.0980857y - 0.587584) \\ & -3.24858,\end{aligned} \tag{5.42}$$

this system has the first integral

$$H_6^{(ii)}(x,y) \approx \ (1.x - 12.1481y + 55.7642)^2(1.x - 1.97015y - 1.18853)e^{0.1x+0.3y}.$$

In the region Σ_1^i we consider the linear differential center

$$\dot{x} = -\frac{3x}{2} + \frac{53y}{8} - \frac{59}{5}, \quad \dot{y} = -10x + \frac{3y}{2} - 5, \tag{5.43}$$

with the first integral $H(x,y) = -8(15x+118)y + 400x(x+1) + 265y^2$. For the PWS (5.42)–(5.43), system (5.33) has the three solutions $(y_1,y_2) \approx (1.05249, 2.509)$, $(y_3,y_4) \approx (0.49155,$ $3.0707)$ and $(y_5,y_6) \approx (0.109285, 3.45298)$ which provide three limit cycles intersecting Γ_1 in the six points $(0,y_i)$ with $i = 1,2,3,4,5,6$, and system (5.16) has the three solutions $(x_7,y_7) \approx$ $(0.366025, 3.76284)$, $(x_8,y_8) \approx (0.724745, 4.0304)$ and $(x_9,y_9) \approx (1, 4.26936)$, which provide the six intersecting points $(x_j,0)$, $(0,y_j)$ with $j = 7,8,9$ of the three limit cycles with the separation line Σ^i. Then the PWS (5.42)–(5.43) has exactly six limit cycles, see Figure 5.15(b).

We give an example with four limit cycles for the class of PWS $\mathcal{C}_{ii}$ when $\Delta = a = 0$. In what follows we give a PWS of the class (1.3)–(5.1) when $\Delta = a = 0$ with three limit cycles from each configuration. In the region Σ_2^i we consider the quadratic center

$$\begin{aligned}\dot{x} \approx\ & x(0.225773 - 0.133734y) - 0.00957539x^2 + (0.00407421y \\ & -0.593254)y + 0.94234, \\ \dot{y} \approx\ & y(0.532632 - 0.173414y) + 0.749123x^2 + x(0.0191508y \\ & +3.16459) + 1.55252,\end{aligned} \tag{5.44}$$

this system has the first integral $H_7^{(ii)}(x,y) \approx \dfrac{x + 0.146272y + 3.84598}{\sqrt{x - 0.254197y + 3.64465}} e^{k(x,y)}$, where $k(x,y) =$

$\dfrac{0.333333(-x-0.947209y+0.751366)}{x+0.146272y+3.84598}$. *In the region* Σ_1^i *we consider the linear differential center*

$$\dot{x} = \frac{x}{2} - \frac{65y}{116} + \frac{9}{10}, \quad \dot{y} = \frac{1}{20}(29x - 10y + 30), \tag{5.45}$$

with the first integral $H(x,y) = -116(5x+9)y + 29x(29x+60) + 325y^2$. *For the PWS* (5.42)–(5.43), *system* (5.33) *has the two solutions* $(y_1, y_2) = ((-2/25)(\sqrt{6371} - 261), (2/325)(\sqrt{6371} + 261))$ *and* $(y_3, y_4) = ((-2/325)(\sqrt{48621} - 261), (2/325)(261 + \sqrt{48621}))$ *which provide two limit cycles intersecting* Γ_1 *in the four points* $(0, y_i)$ *with* $i = 1,2,3,4$, *and system* (5.16) *has the two solutions* $(x_5, y_5) = ((2/29)\left(\sqrt{295} - 15\right), (2/325)(11\sqrt{751} + 261))$ *and* $(x_6, y_6) = ((10/29)\left(\sqrt{17} - 3\right), (2/325)(261 + \sqrt{133121}))$, *which provide the six intersecting points* $(x_j, 0)$, $(0, y_j)$ *with* $j = 5,6$ *of the two limit cycles with the separation line* Σ^i. *Then the PWS* (5.44)–(5.45) *has exactly four limit cycles, see Figure* 5.15(*c*).

We give an example with three limit cycles for the class of PWS $\mathcal{C}_{ii}$ *when* $A = b = 0$. *In what follows we give a PWS of the class* (1.3)–(5.1) *when* $A = b = 0$ *with three limit cycles one satisfying* ***Cnf 1*** *and two satisfying* ***Cnf 2***. *In the region* Σ_2^i *we consider the quadratic center*

$$\begin{aligned} \dot{x} &\approx -0.046451x^2 + x(0.1887y + 1.8225) + (-0.14806y - 2.3677)y + 10.612, \\ \dot{y} &\approx -0.015483x^2 + x(0.062903y + 3.2741) + (-0.049354y - 3.6225)y + 1.8709, \end{aligned} \tag{5.46}$$

this system has the first integral $H_8^{(ii)}(x,y) \approx (-x + 1.0625y + 4.79167)^2 e^{k(x,y)}$, *where* $k(x,y) = -0.0009(x - 3y + 60)^2 + 0.48x - 0.51y$. *In the region* Σ_1^i *we consider the linear differential center*

$$\dot{x} = -0.2x - 0.202784y + 0.806423, \quad \dot{y} = x + 0.2y + 0.117971, \tag{5.47}$$

with the first integral $H(x,y) = 4(x+0.2y)^2 + 8(0.117971x - 0.806423y) + 0.651134y^2$. *For the PWS* (5.46)–(5.47), *system* (5.33) *has the unique solution* $(y_1, y_2) \approx (0.493476, 7.4601)$ *which provides one limit cycle intersecting* Γ_1 *in the two points* $(0, y_1)$ *and* $(0, y_2)$, *and system* (5.16) *has the two solutions* $(x_3, y_3) \approx (1.4422, 9.2446)$ *and* $(x_4, y_4) \approx (2.59897, 11.1981)$, *which provide the four intersecting points* $(x_j, 0)$, $(0, y_j)$ *with* $j = 3,4$ *of the two limit cycles with the separation line* Σ^i. *Then the PWS* (5.46)–(5.47) *has exactly three limit cycles, see Figure* 5.15(*d*). *The proof of statement* (*b*) *is completed with this example.*

Proof of statement (*c*) of Theorem 5.3. *Here we give an example with two limit cycles for the class of PWS* $\mathcal{C}_{iii}$ *when* $A - 2b = C + 2a = 0$, *one satisfying* ***Cnf 1*** *and the other satisfying*

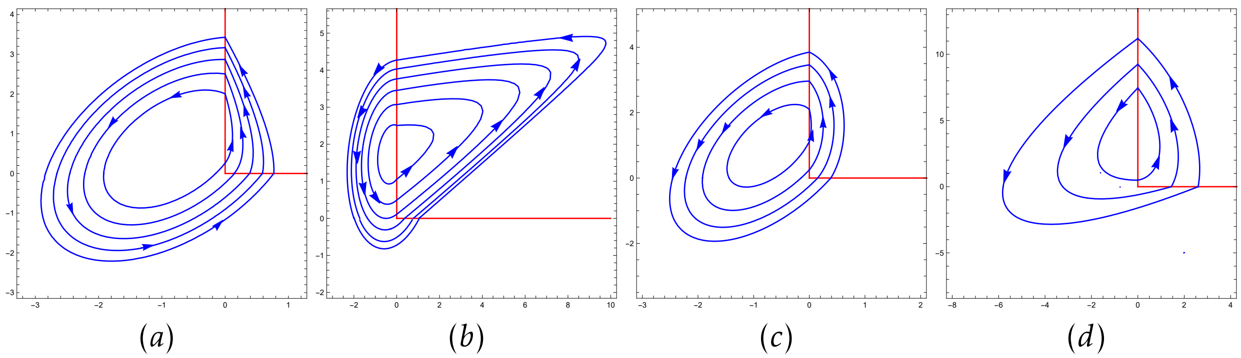


Figure 5.14: (a) The five limit cycles satisfying **Cnf 3** of the PWS (5.40)–(5.41), (b) the six limit cycles satisfying **Cnf 3** of the PWS (5.42)–(5.43), (c) the four limit cycles satisfying **Cnf 3** of the PWS (5.44)–(5.45), and (d) the three limit cycles satisfying **Cnf 3** of the PWS (5.46)–(5.47).

Cnf 2. *In the region* Σ_1^i *we consider the quadratic center*

$$\begin{aligned}\dot{x} \approx\ & -0.108378x^2 + x(6.81021y + 15.0732) + y(3.33019y + 2.60493)\\ & -10.2841,\\ \dot{y} \approx\ & 0.268378x^2 + x(0.216755y - 1.01461) + y(-3.40511y - 15.0732)\\ & -14.6995,\end{aligned} \tag{5.48}$$

this system has the first integral

$$\begin{aligned}H^{(iii)}(x,y) \approx\ & x^2(1.89027 - 0.403825y) - 0.333333x^3 + x(y(12.6877y + 56.164)\\ & +54.7718) + y(y(4.1362y + 4.85311) - 38.3196) - 57.3826.\end{aligned}$$

In the region Σ_2^i *we consider the linear differential center*

$$\dot{x} = -6x + 9.83871y - 13, \quad \dot{y} = -6.2x + 6y - 5, \tag{5.49}$$

with the first integral $H(x,y) = 4(6y - 6.2x)^2 - 49.6(13y - 5x) + 100y^2$. *For the PWS* (5.48)–(5.49), *system* (5.33) *has the unique solution* $(y_1, y_2) \approx (0.35892, 2.283)$ *which provides one limit cycle intersecting* Γ_1 *in the two points* $(0, y_1)$ *and* $(0, y_2)$, *and system* (5.16) *has the unique solution* $(x_3, y_3) \approx (0.80645, 3.0462)$, *which provides the two intersecting points* $(x_3, 0)$, $(0, y_3)$ *of the limit cycle with the separation line* Σ^i. *Then the PWS* (5.48)–(5.49) *has exactly two limit cycles, see Figure* 5.15(a). *With this example, statement* (c) *proof is complete.* ■

Proof of statement (d) of Theorem 5.3. *Finally we give an example with five limit cycles for the class of PWS* $\mathcal{C}_{iv}$ *when* $C + 2a = A + 4b + 5d = a^2 + bd + 2d^2 = 0$, *two satisfying* ***Cnf 1***

and three satisfying **Cnf 2**. *In the region* Σ_1^i *we consider the quadratic center*

$$
\begin{aligned}
\dot{x} \approx\ & x(4.38066 - 1.97521y) + 0.766728x^2 + (0.992492y - 5.78735)y \\
& + 5.07476, \\
\dot{y} \approx\ & x(2.05083 - 0.597499y) + 0.27823x^2 + (0.135742y - 2.38066)y \\
& + 1.84875,
\end{aligned}
\tag{5.50}
$$

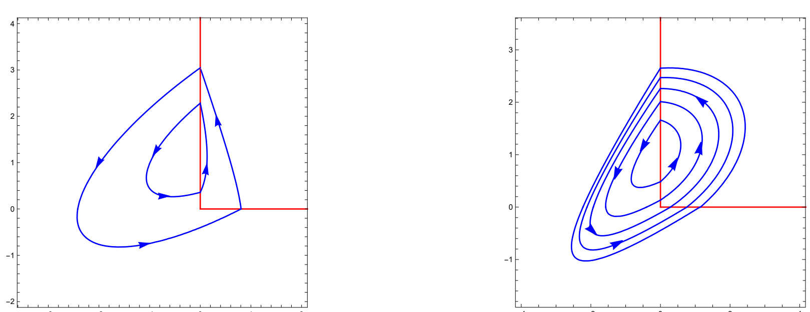
this system has the first integral

$$
\begin{aligned}
H^{(IV)}(x,y) \approx\ & \Big(x^2(15.8595 - 6.24882y) + y(y(40.0457 - 9.03711y) + 1.28941) \\
& + x^3 + x(y(13.0159y - 52.26) - 81.759) - 343.77\Big)^2 \Big/ \Big(x(10.573 \\
& - 4.16588y) + x^2 + y(4.33863y - 12.8171) + 37.4499\Big)^3.
\end{aligned}
$$

In the region Σ_2^i *we consider the linear differential center*

$$
\dot{x} = \frac{7x}{2} - \frac{373y}{20} + 20, \quad \dot{y} = 5x - \frac{7y}{2} + 8, \tag{5.51}
$$

with the first integral $H(x,y) = -20(7x+40)y + 20x(5x+16) + 373y^2$. *For the PWS* (5.50)–(5.51), *system* (5.33) *has the two solutions* $(y_1,y_2) \approx (0.4844042, 1.66037)$ *and* $(y_3,y_4) \approx (0.133283, 2.01149)$ *that provide two limit cycle intersecting* Γ_1 *in the four points* $(0,y_i)$ *with* $i = 1,2,3,4$, *and system* (5.16) *has the three solutions* $(x_5,y_5) \approx (0.286796, 2.26323)$, $(x_6,y_6) \approx (0.75796, 2.4703)$ *and* $(x_7,y_7) \approx (1.1495533, 2.65052)$, *which provide the six intersecting points* $(x_j,0)$, $(0,y_j)$ *with* $j = 5,6,7$ *of the three limit cycles with the separation line* Σ^i. *Then the PWS* (5.50)–(5.51) *has exactly five limit cycles, see Figure* 5.15(*b*). *With this example the proof of Theorem* 5.3 *is complete.*



(*a*) (*b*)

Figure 5.15: (*a*) The two limit cycles satisfying **Cnf 3** of the PWS (5.48)–(5.49), (*b*) the five limit cycles satisfying **Cnf 3** of the PWS (5.50)–(5.51).

All the graphics of the functions $f_k^{(j)}(y_2)$ and $g_k^{(k)}(y_2)$ with $j = i, ii$ and $k = 1, 2$

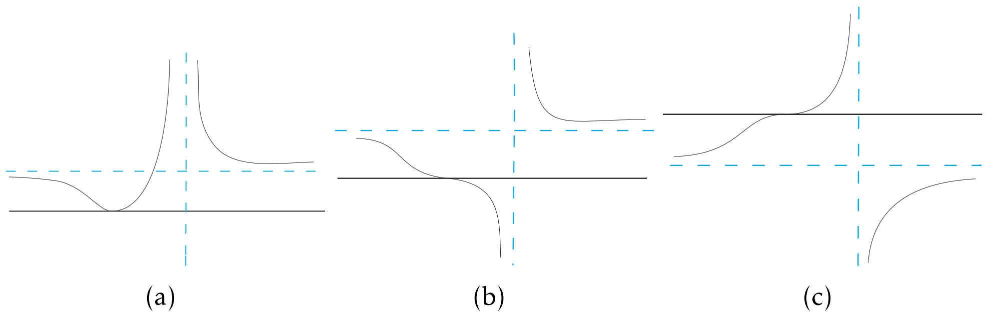


Figure 5.16: The graphics of the function $f_1^{(i)}(y_2)$. The dashed lines represent the asymptote straight lines, and the horizontal straight line is the y_2−axis.

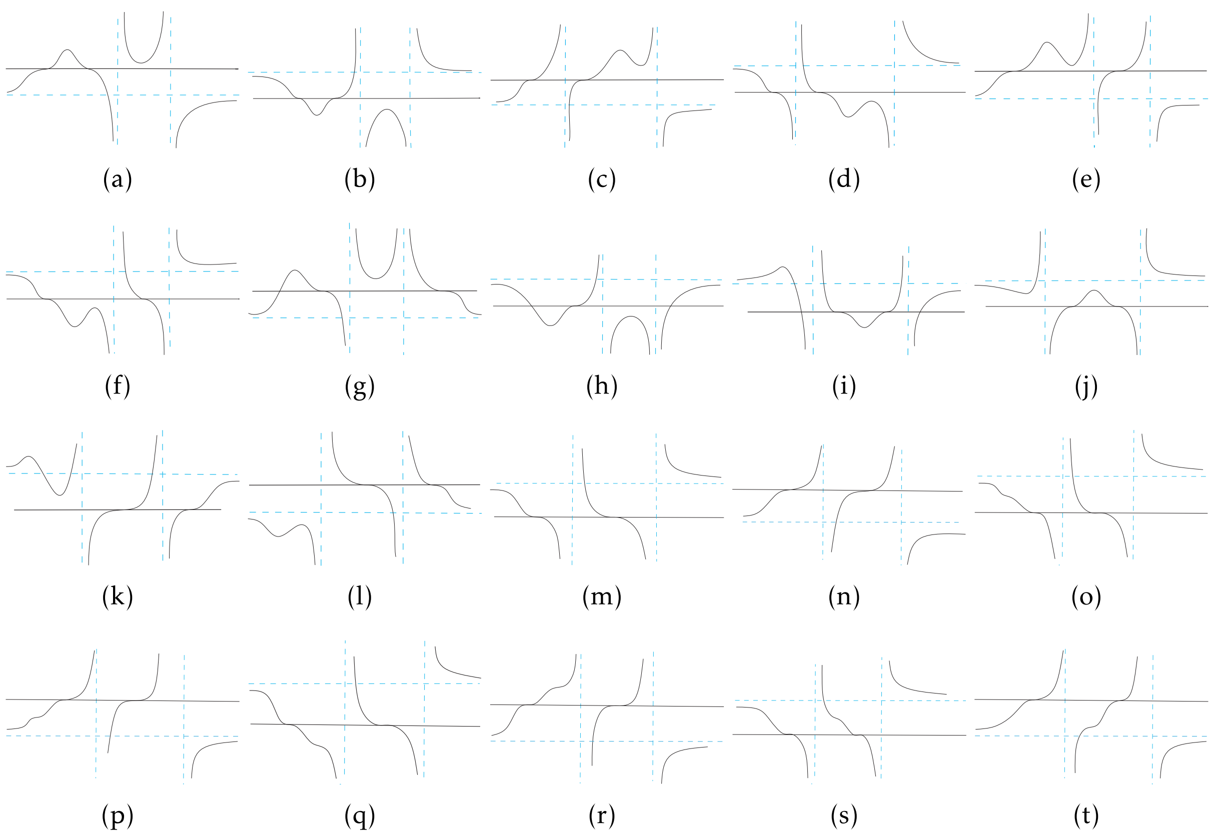


Figure 5.17: The graphics of the function $f_1^{(ii)}(y_2)$ if p and q are odd. The dashed lines represent the vertical asymptotes straight lines, and the horizontal straight line is the y_2−axis.

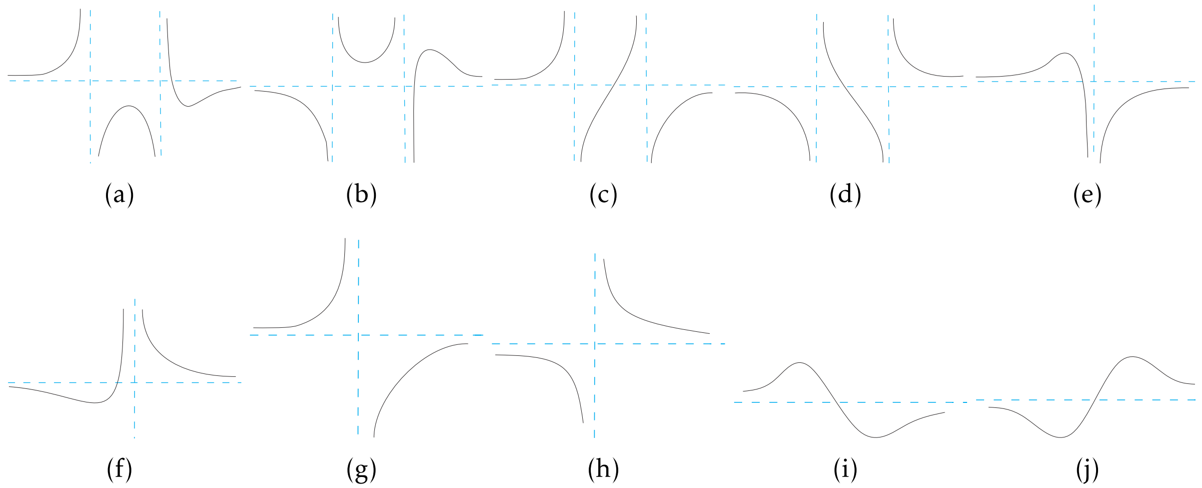


Figure 5.19: The graphics of the function $g_2^{(i)}(y_2)$. The dashed lines represent the asymptotes straight lines.

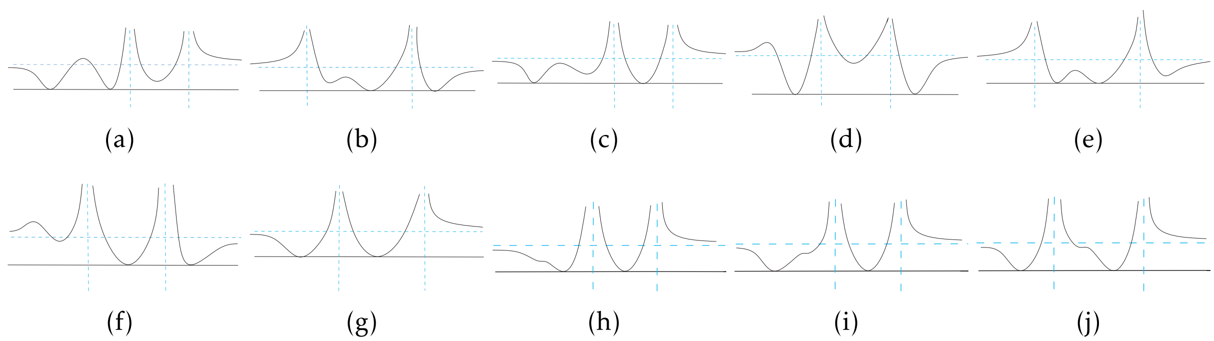


Figure 5.18: The graphics of the function $f_1^{(ii)}(y_2)$ if p and q are even or if p is even and $q = k_1/(2k_2+1)$ with $k_1, k_2 \in \mathbb{N}$. The dashed lines represent the vertical asymptote straight lines, and the horizontal straight line is the y_2–axis.

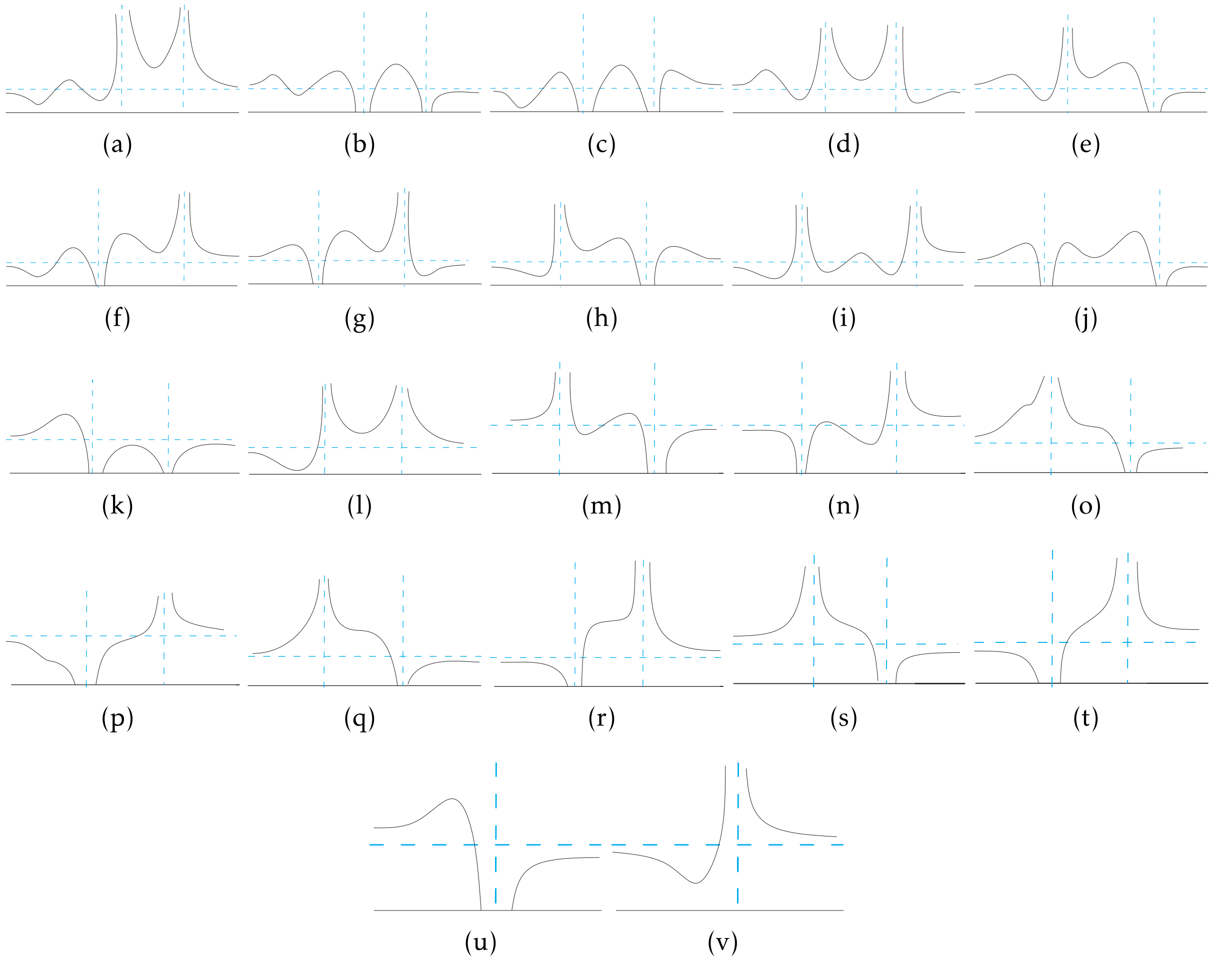


Figure 5.20: Continuous of the graphics of the function $g_1^{(i)}(y_2)$ with $s_2 \neq s_3$. The dashed lines represent the vertical asymptotes straight lines, and the horizontal straight line is the y_2–axis.

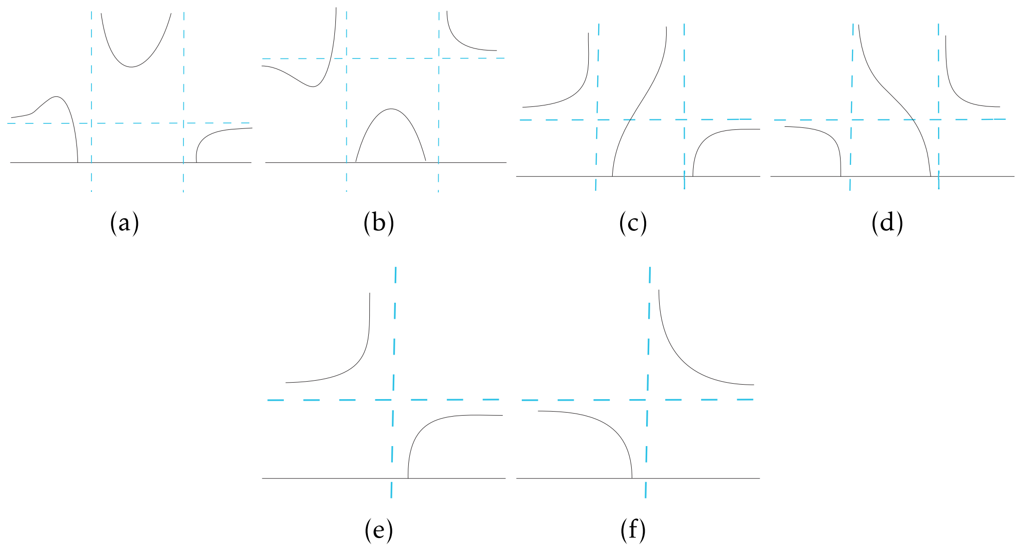


Figure 5.21: The graphics of the function $g_1^{(ii)}(y_2)$. The dashed lines represent the vertical asymptotes straight lines, and the horizontal straight line is the y_2–axis.

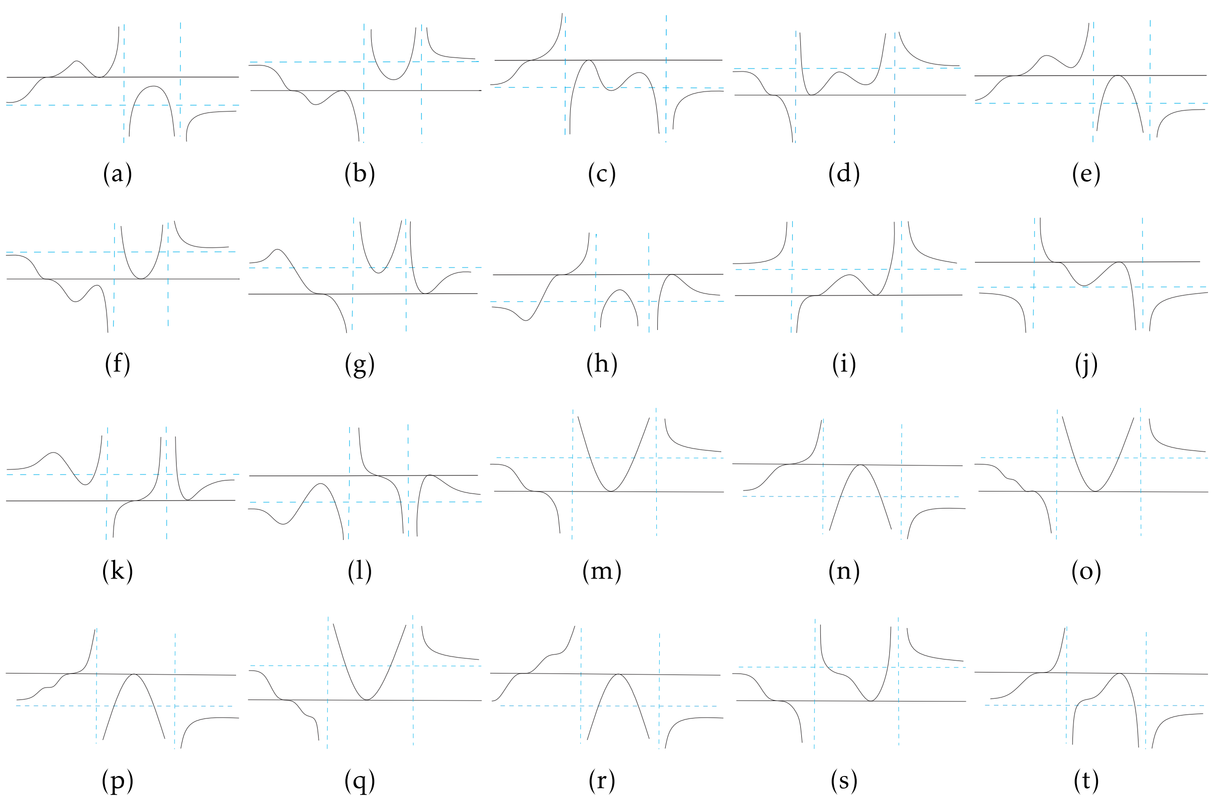


Figure 5.22: The graphics of the function $f_1^{(ii)}(y_2)$ if p odd and q even, or if p odd and $q = k_1/(2k_2+1)$ with $k_1, k_2 \in \mathbb{N}$. The dashed lines represent the vertical asymptotes straight lines, and the horizontal straight line is the y_2–axis.

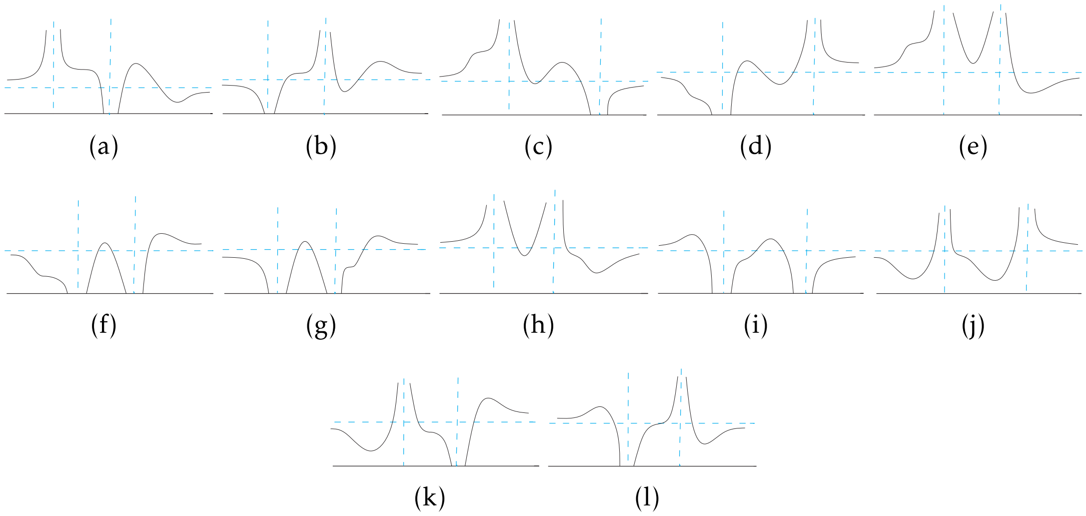


Figure 5.23: The graphics of the function $g_1^{(i)}(y_2)$ with $s_2 \neq s_3$. The dashed lines represent the vertical asymptotes straight lines, and the horizontal straight line is the y_2–axis.

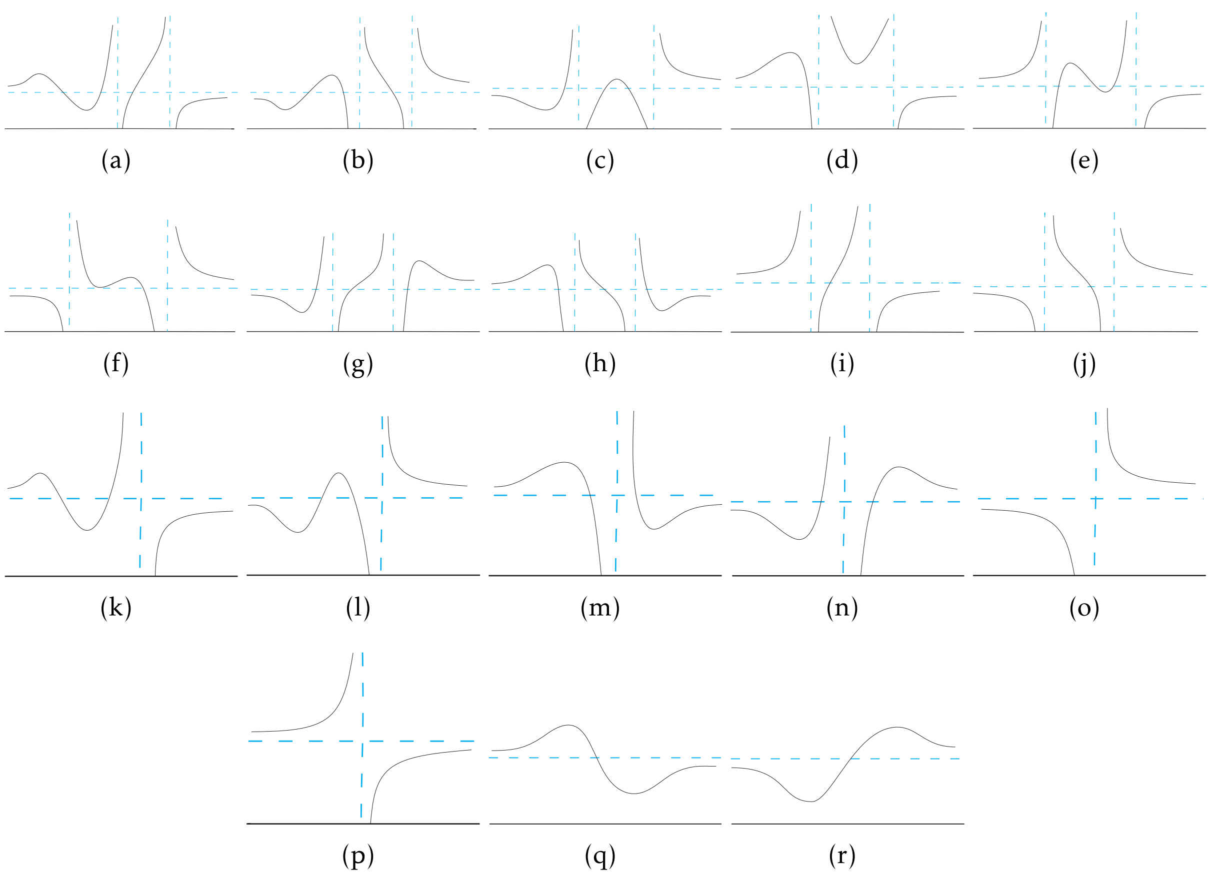


Figure 5.24: The graphics of the function $g_2^{(ii)}(y_2)$. The dashed lines represent the asymptotes straight lines, and the horizontal straight line is the y_2–axis.

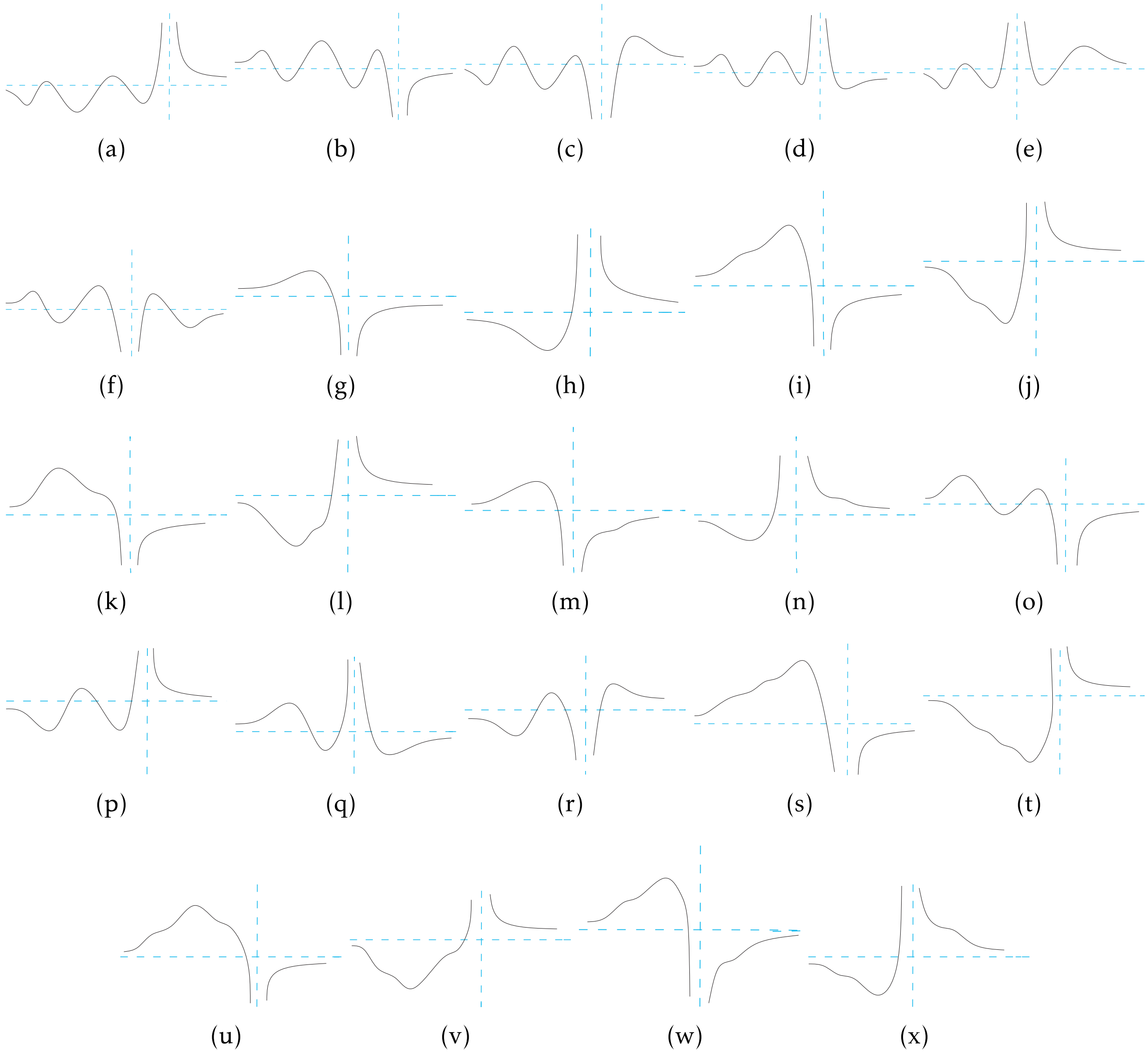


Figure 5.25: The graphics of the function $f_2^{(ii)}(y_2)$ with r is an even number, or $r = k_1/(2k_2+1)$ with $k_1, k_2 \in \mathbb{N}$. The dashed lines represent the asymptotes straight lines.

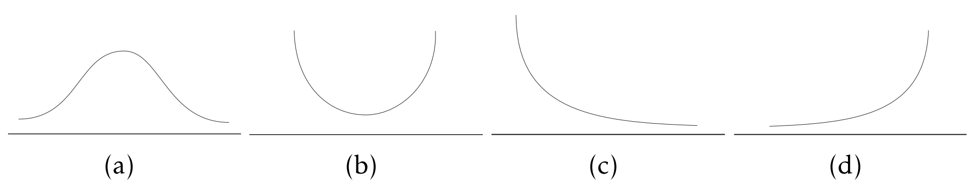


Figure 5.26: The graphics of the function $f_2^{(i)}(y_2)$, the horizontal straight line is the y_2–axis.

Conclusion

This thesis devoted to solve the second part of the sixteenth Hilbert problem for two types of planar PWS depending on the nature of the separation curve, the first type separated by the regular line Σ^r and the second type separated by the irregular line Σ^i.

Firstly when the separation curve is the regular line Σ^r we study the limit cycles for PWS formed either by linear and cubic differential centers or only cubic differential centers in each region. We analyze the maximum number of limit cycles that can be bifurcate from the five families of PWS by using standard techniques such as Bézout theorem, Resultant theory, or the maximum number of the intersection points between the graphics of non-algebraic functions. Consequently we demonstrate that when the separation curve is a regular line, there exist families of PWS with at most four limit cycles. We also provide examples which confirm that this upper bound is reached.

Secondly when we consider the irregular separation curve Σ^i we examine the limit cycles for the class of PWS formed by linear and quadratic differential centers. We analyze the maximum number of limit cycles that can be created from this class of PWS by using the maximum number of intersection points between the graphics of non-algebraic functions. We prove that eight is the maximum number of limit cycles for this class of systems. Then we demonstrate how the nature of the separation curve affects this maximum number for planar PWS. Additionally we provide examples with exactly six limit cycles.

While this study has provided valuable insights into the behavior of some non-linear PWS, there remains a vast and promising frontier awaiting exploration in the realm of non-linear PWS. Additionally this work couragously gives light to the uses of separation curves beyond regular lines that affect the dynamics within PWS.

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Abstract

This thesis consists of two important parts, the first one is devoted to the study of the upper bound on the number of limit cycles that can be created from three different non-linear families of discontinuous piecewise differential systems separated by a regular line.
The second part focuses on the study of the existence and the maximum number of limit cycles of a class of non-linear discontinuous piecewise differential systems but in this case we use an irregular line as the separation curve instead of regular line.

Keywords : Discontinuous piecewise differential system, limit cycle, linear center, (regular/ irregular) line, quadratic center, isochronous center, nilpotent center.

ملخص

تتكون هذه الاطروحة من جزأين مهمين، الجزء الأول مخصص لدراسة الحد الاعلى لعدد الحلول الدورية المعزولة ووضعيتها على المستوى و التى يمكن أن تنشأ من ثلاث عائلات مختلفة غير خطية من الأنظمة التفاضلية المتقطعة غير المتصلة و المفصولة بمستقيم.
يتمحور الجزء الثانى على دراسة وجود الحد الأقصى لعدد الحلول الدورية المعزولة لفئة من الأنظمة التفاضلية المتقطعة غير المتصلة و غير الخطية وفى هذا الجزء استخدمنا خط غير منتظم للفصل بين هذه الانظمة التفاضلية بدل خط مستقيم.

الكلمات المفتاحية: نظام تفاضلى متقطع غير متصل، حل دورى معزول، خط (منتظم/ غير منتظم)، مركز خطى، مركز تربيعى، مركز متزامن (ثابت الزاوية)، مركز عديم القوى.

Résumé

Cette thèse se compose de deux parties importantes, la première est consacrée à l'étude de la limite supérieure du nombre de cycles limites qui peuvent être obtenus à partir de trois familles différentes de systèmes différentiels non linéaires discontinus par morceaux séparés par une ligne régulière.
La deuxième partie se focalise sur l'étude de l'existence et du nombre maximal de cycles limites d'une classe de systèmes différentiels non linéaires discontinus, mais dans cette partie la courbe de séparation est une ligne irrégulière au lieu d'une ligne droite.

Mots clés : **Système différentiel discontinu par morceaux, cycle limite, ligne (régulière/ irrégulière), centre linéaire, centre quadratique, centre isochrone, centre nilpotent.**