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THEME

Limit cycles of continuous and discontinuous piecewise differential systems separated by straight line and formed by two arbitrary quadratic centers

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Dedication

I dedicate this work to **Allah** Almighty my creator, my strong pillar, my source of inspiration wisedom and knowledge.

Also I dedicate to :

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List of Conferences

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- **Limit cycle of the discontinuous piecewise system formed by cubic Hamiltonian system separated by a straight line**, in the conference intilted: RECENT DEVELOPMENT IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS. University of Abderrahmane Mira, Bejaia, Algeria. May 22-26/ 2022.

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Introduction

It is frequently stated that mathematics is the language of science and nature. Ever since mankind became aware of the world around it, there has been a pressing need to comprehend the laws that govern natural phenomena, and no body can ignore the central role played by mathematics in this game. They enable us to use a strict, global language to explain what we see and predict what's going to happen. More particularly, differential equations have shown to be among the most effective methods to model the relationships between events and things from our everyday reality. not just for characterizing the natural laws but also, for explaining the behavioral characteristics of certain social processes for example.

In the 17th century, I. Newton (1642-1727) and G. W. Leibniz (1646-1716) built the theory of ordinary differential equations [52], which laid the foundations for the next 350 years. After that in 18th century, the French mathematicians J. L. Lagrange (1736–1813) and P. S. Laplace (1749–1827) introduced the concept of partial differential equations, see [52].

Solving a nonlinear differential system directly (explicitly) is generally difficult, if not impossible. Despite the importance of numerical methods, they only allow us to calculate an approximate solution over a finite time interval, corresponding to given initial conditions, by discretizing the interval.

Due to these difficulties, Henri Poincaré (1854-1912) proposed a new approach to the study of ordinary differential equations at the end of the 19th century, establishing what is known as the qualitative theory of differential equations [51]. In his series of publications Mémoire sur les courbes définies par une équation différentielle, published between 1881 and 1886 [46]. Where the qualitative study of differential systems and consists of examining the properties and characteristics, and provides information on the behavior of the solutions of a differential system without the need for explicit solution.

One of the most important and remarkable solutions of differential equations is the limit cycle, which was first defined by Pioncaré at the end of the 19th century. In the qualitative theory of differential systems in the plane, we recall that any periodic solutions which is isolated in the set of all periodic solution of the system is called a limit cycle. The significance of these isolated solutions lies in their key role in understanding the dynamics of a certain differential system.

Despite the fact that the study of the existence and maximum number of isolated periodic solutions of differential systems is one of the most challenging problems in the qualitative theory of such systems, it is of great interest due to its numerous applications in social science or in natural phenomena where we can observe periodic behaviour. Well-known classical examples include the periodicity of heartbeats, bridge vibrations or airplane wings, oscillations in RLC circuits, Van der Pol oscillations [49, 50] or the Belousov Zhavotinskii model [3]. Furthermore, several research projects have focused on modeling limit cycles in physics and, more recently in biology, economics and engineering, see for example [14, 19, 37, 44].

In 1900, D. Hilbert introduced a series of problems which were to have a significant impact on mathematics throughout the twentieth century. Ten of these problems were exhibited at the Paris International Congress of Mathematicians. One of them is known as Hilbert's sixteenth problem which consists in studying the existence and determination of the upper bound on the maximum number, noted by $H(n)$ of limit cycles of the planar polynomial differential systems $\dot{x} = P_n(x,y),\ \dot{y} = Q_n(x,y)$, where P_n and Q_n are polynomial functions of degree n.

It has been proved by Ilyashenko and Yakovenko in 1991 [29] and by Ecalle in 1992 [22] that $H(n) < \infty$ for any given planar polynomial nonlinear differential system. It is still unknown if there is a finite uniform upper bound $H(n)$ on the number of limit cycles of planar polynomial differential system of degree n. Many papers were published for the quadratic systems, and in what follows it was obtained that $H(2) \geq 4$ [18, 48]. Recently this result was also proved for near-integrable quadratic systems [9]. For cubic polynomial systems, many results was obtained that $H(3) \geq 13$ [35, 36]. See [33, 28] for more on the research of the 16th Hilbert problem.

Since the 1930s, the study of limit cycles has also gained significance in piecewise differential systems (PWDS) separated by a straight line. Due to their widely applications in various scientific domains of studies such as electronics, mechanics, engineering, and physics, see for instance [1, 14, 24, 32, 43].

The simplest kind of discontinuous piecewise differential systems are formed by linear ones, in which a straight line serves as the separation curve.

The planar piecewise differential systems, which are composed by two pieces and possessing in each piece a linear differential system, in which a straight line serves as the separation curve, are the most simple kind of piecewise differential systems. Since these systems can create a significant number of crossing limit cycles, several researchers have attempted to identify them to solve the extension of sixteenth Hilbert problem.

In 2001 Giannakopoulos and Pliete [25] established the existence of discontinuous piecewise linear differential systems having two crossing limit cycles. Later in 2010 Han and Zhang [27] discovered other discontinuous piecewise linear differential systems with two crossing limit cycles, and they hypothesized that the maximum number of crossing limit cycles for discontinuous piecewise linear differential systems with two pieces separated by a straight line is two. However, in 2012 Huan and Yang [26] presented numerical proof for the existence of three crossing limit cycles in this type of discontinuous systems. In the same year, Llibre and Ponce [39], motivated by Huan and Yang's numerical example, showed for the first time that discontinuous piecewise linear differential systems with two zones separated by a straight line can exhibit three crossing limit cycles. Later on, numerous studies have answered the second part of sixteenth Hilbert problem for particular classes of linear discontinuous piecewise differential systems separated by a straight line, see for example [2, 16, 34, 38], or classes separated by cubic or conic curves, see [4, 8, 10, 11, 12, 30, 31]. We can therefore observe that the majority of publications focus on piecewise linear differential systems separated by a straight line, while nonlinear systems are rarely treated. For example, in [13, 23], the authors have solved the extension of Hilbert's 16th problem for certain nonlinear discontinuous differential systems of degree two or three.

A great deal of research has attempted to provide an upper bound for the maximum number of limit cycles for a given piecewise differential system in the plane, but in general this study is a very challenging problem. Various tools are used by mathematicians to compute analytically the limit cycles of piecewise differential systems, such as averaging theory, Melnikov functions and the Poincaré map. The main objective of this thesis is to provide the maximum number of limit cycles of such non linear piecewise differential systems separated by a straight line, based on the first integral.

Now we present the structure of our thesis, which is divided into four chapters. In the first one, we introduce the background findings required for performing our study.

Next, in the second chapter, we use a new method to study the maximum number of limit cycle of new class of discontinuous piecewise differential systems having the straight line $x = 0$ as a separation curve, and formed by an arbitrary linear center and an arbitrary quadratic center.

The third chapter devoted to solve the second part of sixteenth Hilbert problem for two famlies of discontinuous piecewise differentail systems with cubic center and separated by a straight line $x = 0$, where the first is created by a linear center and one of three classes of isochronous cubic center having a rational first integral. And the second family is formed by two isochronous cubic centers with a rational first integral in each half-plane.

Finally, we solve the previous chapter's problem for discontinuous piecewise differential system with nilpotent Hamiltonian saddles of linear and cubic homogeneous polynomials and separated by the straight line $x = 0$. More precisely, we proved that the discontinuous piecewise differential systems created by a linear differential center in one half-plane, and by one of the six classes of the Hamiltonian nilpotent saddles can have at most one limit cycle. In the other hand we show that four is the maximum number of limit cycles of the discontinuous piecewise differential systems formed by two Hamiltonian nilpotent saddles in each half-plane.

Chapter 1

Some Preliminary Concepts on Discontinuous Piecewise Differential Systems

This chapter gives some of the fundamental concepts and findings from the qualitative theory of ordinary differential equations, which provide the basis for the development of this thesis. First, we go over the basic concepts of a discontinuous piecewise differential system. then, we end the chapter with a technique that describes how to use the first integral to construct the limit cycle of a piecewise differential system.

Section 1.1 Planar differential systems

Definition 1.1

Let consider the following differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1.1}$$

where P and Q are the relatively prime functions in the dependent variables x and y, and the time t is the real independent variable. Recall that the derivative with respect to t is denoted by the dot " $\cdot$ ". System (1.1) is called a polynomial differential system if P and Q are polynomials with real coefficients in the variables x and y over $\mathbb{R}^n$, where $m = \max\{degP; degQ\}$ is the degree of the polynomial differential system, [21].

Section 1.2 Vector fields

Definition 1.2 (Vector fields)
Let $\mathcal{D} \subset \mathbb{R}^n$ is an open subset. A vector field of class C^r on $\mathcal{D}$, with $0 \leq r \leq +\infty$ is defined as a map $X : \mathcal{D} \longrightarrow \mathbb{R}^n$, of class C^r. Where the free part of a vector associated at the point $x \in \mathcal{D}$ is denoted by $X(x)$. See [21].

Drawing carefully selected vectors $(x, X(x))$ as shown in Figure 1.1 is how a vector field is graphically represented on a plane.

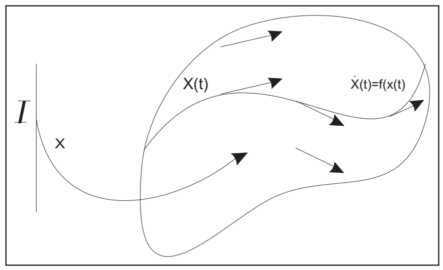


Figure 1.1: Vector field.

Remark 1 *When we integrate a vector field, we search for curves $x(t)$, where t belongs to $I \subset \mathbb{R}$, that satisfy the differential equation*

$$\dot{x} = X(x). \tag{1.2}$$

Such that $\dot{x} = dx/dt$ with $x \in \mathcal{D}$ is the dependent variable and t is independent variable of the equation (1.2).

Section 1.3 Filippov systems

Definition 1.3 (Discontinuous piecewise differential systems (PWS).)
Let $(X(x,y), Y(x,y))$ be a pair of C^r differential systems on $\mathbb{R}^2$, with $r \geq 1$. And let Σ be a smooth codimension one manifold (switching manifold) which separate the plane in two regions .

Any planar differential system of the form

$$Z(x,y) = \begin{cases} X(x,y) & if \quad h(x,y) \geq 0, \\ Y(x,y) & if \quad h(x,y) \leq 0, \end{cases} \tag{1.3}$$

is called a discontinuous piecewise planar differential system, which has $\Sigma = h^{-1}(0)$ *as a separation curve, where* $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is a differentiable function has the regular value* 0.

It is important to note that the separation curve Σ *separate the plane in two regions* $\Sigma^- = \{(x,y) : h(x,y) \leq 0\}$, *and* $\Sigma^+ = \{(x,y) : h(x,y) \geq 0\}$.

As usual, system (1.3) *is defined by* $Z = (X, Y, \Sigma)$ *or just by* $Z = (X, Y)$, *in cases when the separation line* Σ *is well known, [15].*

Our area of interest is the study of limit cycles of polynomial differential systems.

Section 1.4 Limit cycles

Definition 1.4 (Solution) *A differentiable map* $X(t) = (x(t), y(t))$ *for* $t \in I \subset \mathbb{R}$ *which satisfies* $\dot{X}(t) = \dfrac{dX(t)}{dt} = F(x,y)$ *is called a solution of system* (1.1), *[21].*

Definition 1.5 (Periodic solutions) *Let* $X(t)$ *be a solution of system* (1.1). *If there is a minimal finite time* $T > 0$ *such that* $X(t+T) = X(t)$ *for all* $t \in \mathbb{R}$ *then the solution* $X(t)$ *is periodic of period* T, *[21].*

Definition 1.6 (Limit cycles) *An isolated periodic trajectory of a planar vector field given by* (1.2) *is called a limit cycle. In another way, if a vector field's periodic trajectory has an annular neighborhood that is free of other periodic trajectories, then it is a limit cycle, [21].*

We refer to the limit cycles as LC and to the discontinuous piecewise differential system as PWS.

the derivative of h in the direction of the vector field X characterizes the contact between the curve of discontinuity Σ and the vector field X (or Y).i.e

$$Xh(p) =< \nabla h(p), X(p) >,$$

where we denote by $< .,. >$ the usual inner product in $\mathbb{R}^2$. The discontinuity curve (switching manifold) Σ can be classified into the following sets, see Figure 1.2.

(I) Crossing set:

$$\Sigma^c = \{p \in \Sigma : Xh(x).Yh(x) > 0\}.$$

(II) Sliding set:

$$\Sigma^s = \{p \in \Sigma : Xh(x) < 0 \text{ and } Yh(x) > 0\}.$$

(III) Escaping set:

$$\Sigma^e = \{p \in \Sigma : Xh(x) > 0 \text{ and } Yh(x) < 0\}.$$

If vector fields $X(\mathbf{x})$ and $Y(\mathbf{x})$ point in the same direction respect to Σ where $\mathbf{x} \in \Sigma$, the point $\mathbf{x}$ is called a crossing point. And if the vector fields $X(\mathbf{x})$ and $Y(\mathbf{x})$ points inward, (resp outward) Σ we said that the point $\mathbf{x}$ is of sliding, (resp escaping) type. We assume that the trajectories of X and Y are transverse to Σ in each situation. [41].

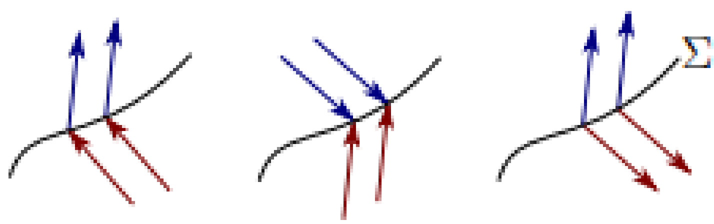


Figure 1.2: (a)Crossing, (b) sliding and (c) escaping sets.

Section 1.5 First integrals

Definition 1.7 *Let X be the vector field associated to the differential system (1.1) which is defined by*

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

Let U be an open subset of $\mathbb{R}^n$, we recall that the polynomial differential system (1.1) is integrable on U if a non-constant analytic function $H : U \to \mathbb{R}$ exists, and it is constant *on all solution curves $(x(t), y(t))$ of system (1.1) that are contained on U, i.e*

$$\frac{dH}{dt} = H_x P + H_y Q = XH = 0 \ \ on \ \ U.$$

Then H is called a first integral of system (1.1) on U. See [42]

Section 1.6 Linear centers

We recall that a singular point p of planar differential system, with a neighborhood U such that $U/\{p\}$ is filled with periodic orbits is called a center.

Now we will define the following lemma that provides a normal form for an arbitrary linear differential center.

LEMMA 1.1 *After doing a linear change of variables and a rescaling of the independent variable every linear center can be written as*

$$\dot{x} = -\beta x - \frac{\left(4\beta^2 + \omega^2\right)}{4\alpha} y + \sigma_1, \quad \dot{y} = \alpha x + \beta y + \delta_1, \ \ with \ \ \omega > 0, \ \ \alpha > 0, \tag{1.4}$$

and its first integral is

$$H(x,y) = 8\alpha(\delta_1 x - \sigma_1 y) + 4(\alpha x + \beta y)^2 + y^2\omega^2. \tag{1.5}$$

For a proof of Lemma 1.1 see [40].

By doing the following change $\{\beta, \alpha, \omega, \delta_1, \sigma_1\} \rightarrow \{A, 1, 2\omega, B, C\}$ the linear center (1.4) becomes

$$\dot{x} = -Ax - \left(A^2 + \omega^2\right) y + B, \quad \dot{y} = Ay + C + x, \tag{1.6}$$

this system has the first integral

$$H(x,y) = (Ay + x)^2 + 2(Cx - By) + \omega^2 y^2. \tag{1.7}$$

Since our research focuses on the number of common zeros of a system of polynomial equations, we sometimes have to use Bézout's theorem, which is a statement in algebraic geometry concerning how many common zeros n polynomials have in n indeterminates.

Theorem 1.1 (Bézout Theorem) *Consider two algebraic curves F and G of degree p, q respectively. Assume that is no common component between the curves. Then there are a maximum of pq intersections points of F and G, [47].*

Section 1.7 The study of limit cycle by using the first integral

We will now provide an example that shows how we use the first integral tool to study the limit cycles of a discontinuous piecewise differential system having the straight line $x = 0$ as a switching curve.

In the following, we consider the discontinuous piecewise differential system composed of a linear system and a quadratic differential system.

In the half-plane $x > 0$, we consider the following linear differential system

$$\dot{x} \simeq -1 + 1.5x + 0.615065y, \qquad \dot{y} \simeq 1 - 4.06461x - 1.5y, \tag{1.8}$$

with the first integral $\quad H_1(x, y) \simeq 4(-4.06461x - 1.5y)^2 + y^2 - 32.5169(x + y).$

In the half-plane $x < 0$ we consider the quadratic polynomial differential system

$$\begin{aligned} \dot{x} &\simeq -6.0926 + 5.0438x^2 + 15.7359x - 13.1746xy - 8.43438y + 6.59599y^2, \\ \dot{y} &\simeq 12.8296 + 4.7784x^2 + 8.44355x - 10.0876xy - 15.7359y + 6.5873y^2, \end{aligned} \tag{1.9}$$

its first integral is

$$\begin{aligned} H_2(x, y) \simeq \; & 84.5119 - 8.50555x^3 - 32.5346y - 22.5198y^2 + 11.7409y^3 - 22.5443x^2 \\ & + 26.934x^2y - 68.5098x + 84.0298xy - 35.1762xy^2. \end{aligned}$$

So the piecewise differential system separated by the straight line $x = 0$ and formed by systems (1.8) and (1.9) has a limit cycle which crosses the line $x = 0$ in exactly two points $(0, 0.587624)$ and $(0, 2.66407)$. This limit cycle is shown in Figure 1.3.

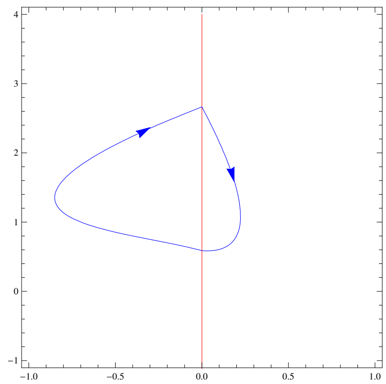


Figure 1.3: The unique crossing limit cycle of the discontinuous piecewise differential system (1.8) – (1.9).

Chapter 2

LC of the PWS Separated by a Straight Line and Formed by a Linear Center and Quadratic One

In this century, the study of the extinction Hilbert's sixteenth problem for planar discontinuous piecewise differential systems (or simply PWS) has developed strongly, due to its wide range of uses in modeling various natural phenomena. The literature has published numerous papers on the maximum number of limit cycles of piecewise linear differential systems in the plane separated by a straight line, and few papers on the existence or non-existence of isolated periodic solutions for piecewise nonlinear differential systems.

This chapter is devoted to solve the second part of Hilbert's 16th problem for a new class of PWS separated by the straight line $\Sigma = \{(x, y) : x = 0\}$ and created by an arbitrary linear center in one half-plane and a quadratic center in the other half-plane, i.e we provide the upper bound of the maximum number of isolated periodic solutions of this classe.

Using a new method that focuses on the intersections points between the graphics of various functions, and by basing on the first integrals of quadratic and linear centers, we provide the maximum number of limit cycles of this class of PWS.

Section 2.1 Quadratic differential centers

In this section we define the quadratic centers in the classification of Kapteyn Bautin, and we give their expressions after a general affine change of variables.

The following theorem defines a normal form of the quadratic centers.

Theorem 2.1 (Kapteyn-Bautin Theorem) *Any quadratic system candidate to have a center can be written after an affine transformation and a rescaling of the independent variable in the form*

$$\dot{x} = -y - bx^2 - Cxy - dy^2, \quad \dot{y} = x + ax^2 + Axy - ay^2. \tag{2.1}$$

This system has a center at the origin if and only if one of the following conditions holds

(i) $C = a = 0$,

(ii) $b + d = 0$,

(iii) $C + 2a = A - 2b = 0$,

(iv) $C + 2a = A + 3b + 5d = a^2 + bd + 2d^2 = 0$.

For more details see for instance Theorem 8.15 of [21].

Quadratic centers after an affine change of variables

Now we give the expression of an arbitrary quadratic differential center with its corresponding first integral obtained after the general affine change of variables $\{x \to \alpha_1 x + \gamma_1 y + \delta_1, y \to \alpha_2 x + \gamma_2 y + \delta_2\}$ with $\alpha_1\gamma_2 - \alpha_2\gamma_1 \neq 0$.

Thus system (2.1) becomes

$$\begin{aligned}\dot{x} = \quad & \frac{1}{\alpha_2\gamma_1 - \alpha_1\gamma_2}\Big(x^2\Big(a\gamma_1(\alpha_1-\alpha_2)(\alpha_1+\alpha_2) + A\alpha_1\alpha_2\gamma_1 + \gamma_2\Big(\alpha_1^2 b + \alpha_1\alpha_2 C + \alpha_2^2 d\Big)\Big) \\ & + y^2(a\gamma_1^3 + \gamma_1^2\gamma_2(A+b)\gamma_1\gamma_2^2(C-a) + \gamma_2^3 d) + \delta_1(a\gamma_1\delta_1 + b\gamma_2\delta_1 + \gamma_1) + \delta_2(A\gamma_1\delta_1 \\ & + \gamma_1) + \delta_2(A\gamma_1\delta_1 + \gamma_2 + \gamma_2 C\delta_1) + \delta_2^2(\gamma_2 d - a\gamma_1)y(\gamma_1\gamma_2(-2a\delta_2 + A\delta_1 + 2b\delta_1 + C \\ & \delta_2) + \gamma_1^2(2a\delta_1 + A\delta_2 + 1) + \gamma_2^2(C\delta_1 + 2d\delta_2 + 1)) + x(\alpha_1(2a\gamma_1^2 y + \gamma_1 + 2a\gamma_1\delta_1 \\ & + A + 2b) + \gamma_1\gamma_2 y + 2b\gamma_2\delta_1 + A\gamma_1\delta_2 + \gamma_2 C(\delta_2 + \gamma_2 y)) + \alpha_2(A\gamma_1(\delta_1\gamma_1 y) + \gamma_2\end{aligned}$$

$$-2a\gamma_1(\delta_2+\gamma_2 y)+\gamma_2+C(\delta_1+\gamma_1 y)+2\gamma_2 d(\delta_2+\gamma_2 y)))\Big),$$

$$\begin{aligned}\dot{y} = \;& \frac{1}{\alpha_1\gamma_2-\alpha_2\gamma_1}\Big(x^2\left(a\alpha_1^3+\alpha_1\alpha_2^2(C-a)+\alpha_1^2\alpha_2(A+b)+\alpha_2^3 d\right)+y^2(a\alpha_1(\gamma_1-\gamma_2)\\ &(\gamma_1+\gamma_2)+\gamma_2(A\alpha_1\gamma_1+\alpha_2\gamma_1 C+\alpha_2\gamma_2 d)+\alpha_2 b\gamma_1^2)+\delta_1(a\alpha_1\delta_1+\alpha_1+\alpha_2 b\delta_1)\\ &+\delta_2^2(\alpha_2 d-a\alpha_1)+\delta_2(A\alpha_1\delta_1+\alpha_2+\alpha_2 C\delta_1)+y(\alpha_1(2a\gamma_1\delta_1-2a\gamma_2\delta_2+A\gamma_1\delta_2\\ &+A\gamma_2\delta_1+\gamma_1)+\alpha_2(2b\gamma_1\delta_1+\gamma_2+\gamma_1 C\delta_2+\gamma_2 C\delta_1+2\gamma_2 d\delta_2))+x(\alpha_1\alpha_2(-(2a\\ &-C)(\delta_2+\gamma_2 y)+A(\delta_1+\gamma_1 y)+2b(\delta_1+\gamma_1 y))+\alpha_1^2(2a(\delta_1+\gamma_1 y)+A(\delta_2+\gamma_2 y)\\ &+1)+\alpha_2^2(C(\delta_1+\gamma_1 y)+2d(\delta_2+\gamma_2 y)+1))\Big).\end{aligned} \tag{2.2}$$

For its corresponding first integral we distinguish the following cases.

I. The quadratic system (2.2) satisfying condition (i) of Theorem 2.1. In this case the corresponding first integral of the differential system (2.2) for $A=-b\neq 0$ becomes

$$H_1^{(1)}(x,y) = \;(A(\delta_2+\alpha_2 x+\gamma_2 y)+1)^{2d}e^{Z_1(x,y)}, \tag{2.3}$$

where

$$\begin{aligned}Z_1(x,y) = \frac{1}{(A(\delta_2+\alpha_2 x+\gamma_2 y)+1)^2}\;&\Big(A\Big(A^2(\delta_1+\alpha_1 x+\gamma_1 y)^2-2A(\delta_2+\alpha_2 x+\gamma_2 y)\\ &+4d(\delta_2+\alpha_2 x+\gamma_2 y)-1\Big)+3d\Big).\end{aligned}$$

If $A=0\neq b$ it is given by

$$\begin{aligned}H_2^{(1)}(x,y) = \;& e^{2b(\delta_2+\alpha_2 x+\gamma_2 y)}\Big(2b^3(\delta_1+\alpha_1 x+\gamma_1 y)^2+2b^2 d(\delta_2+\alpha_2 x+\gamma_2 y)^2\\ &+2b(b-d)(\delta_2+\alpha_2 x+\gamma_2 y)-b+d\Big).\end{aligned} \tag{2.4}$$

If $b=0\neq A$ it becomes

$$\begin{aligned}H_3^{(1)}(x,y) = \;& e^{A\left(A^2(\delta_1+\alpha_1 x+\gamma_1 y)^2+Ad(\delta_2+\alpha_2 x+\gamma_2 y)^2+2(A-d)(\delta_2+\alpha_2 x+\gamma_2 y)\right)}(A(\delta_2+\alpha_2 x\\ &+\gamma_2 y)+1)^{2d-2A}.\end{aligned} \tag{2.5}$$

If $A=b=0$ the corresponding first integral is

$$H_4^{(1)}(x,y) = \;2d(\delta_2+\alpha_2 x+\gamma_2 y)^3+3\Big((\delta_1+\alpha_1 x+\gamma_1 y)^2+(\delta_2+\alpha_2 x+\gamma_2 y)^2\Big). \tag{2.6}$$

II. The quadratic system (2.2) satisfying condition (ii) of Theorem 2.1. The first integral of the differential system (2.2) if $A=-b\neq 0$ and $a=0\neq C$ becomes

$$H_1^{(2)}(x,y)= e^{Z(x,y)}(1-b(\delta_2+\alpha_2x+\gamma_2y))^{-b^2-C^2}(-b(\delta_2+\alpha_2x+\gamma_2y)+C(\delta_1 +\alpha_1x+\gamma_1y)+1)^{b^2}, \tag{2.7}$$

where

$$Z(x,y)=\frac{bC}{b(\delta_2+\alpha_2x+\gamma_2y)-1}\Big(b(\delta_1+\alpha_1x+\gamma_1y)+C(\delta_2+\alpha_2x+\gamma_2y)\Big).$$

If $AbC(A+b)\Delta\neq 0$ and $a=0$ with $\Delta=4b(A+b)+C^2<0$ and $L=\sqrt{-\Delta}$ it is given by

$$H_2^{(2)}(x,y)= \left(-\frac{\left(C^2+L^2\right)(\delta_2+\alpha_2x+\gamma_2y)}{4b}-b(\delta_2+\alpha_2x+\gamma_2y)+1\right)^{-\frac{8b}{4b^2+C^2+L^2}} e^{\frac{-2CM}{bL}}, \tag{2.8}$$

where $M=\arctan\left(\dfrac{2b(\delta_2+\alpha_2x+\gamma_2y)-C(\delta_1+\alpha_1x+\gamma_1y)-2}{L(\delta_1+\alpha_1x+\gamma_1y)}\right)$.

If $AbC(A+b)\Delta\neq 0$ and $a=0$ with $\Delta=4b(A+b)+C^2>0$ and $r=\sqrt{\Delta}$ it becomes

$$H_3^{(2)}(x,y)= (A(\delta_2+\alpha_2x+\gamma_2y)+1)^{1/A}\left(\frac{1}{2}(C-r)(\delta_1+\alpha_1x+\gamma_1y)-b(\delta_2+\alpha_2x +\gamma_2y)+1\right)^{\frac{r-C}{2rb}}\left(\frac{1}{2}(C+r)(\delta_1+\alpha_1x+\gamma_1y)-b(\delta_2+\alpha_2x+\gamma_2y)+1\right)^{\frac{r+C}{2rb}}. \tag{2.9}$$

If $b=C=0$ the corresponding first integral is

$$H_4^{(2)}(x,y)= e^{Z(x,y)}(a(\delta_1+\alpha_1x+\gamma_1y)+1)^{-2\sqrt{4a^2+A^2}}\Big(\frac{1}{(a(\delta_1+\alpha_1x+\gamma_1y)+1)^2} ((\delta_2+\alpha_2x+\gamma_2y)\Big(a^2(-(\delta_2+\alpha_2x+\gamma_2y))+aA(\delta_1+\alpha_1x+\gamma_1y)+A\Big) +\Big(a(\delta_1+\alpha_1x+\gamma_1y)+1)^2\Big)\Big)^{-\sqrt{4a^2+A^2}}, \tag{2.10}$$

where

$$Z(x,y)=2a\sqrt{4a^2+A^2}(\delta_1+\alpha_1x+\gamma_1y)-2A\tanh^{-1}\left(\frac{-2a^2(\delta_2+\alpha_2x+\gamma_2y)+aA(\delta_1+\alpha_1x+\gamma_1y)+A}{\sqrt{4a^2+A^2}(a(\delta_1+\alpha_1x+\gamma_1y)+1)}\right).$$

If $A=a=0$, $C\neq 0$ and $b\neq 0$ it is given as follows

$$H_5^{(2)}(x,y)= e^{\delta_2+\alpha_2x+\gamma_2y}\left(\frac{1}{2}(C-r)(\delta_1+\alpha_1x+\gamma_1y)-b(\delta_2+\alpha_2x+\gamma_2y)+1\right)^{\frac{r+C}{2rb}} \left((\frac{1}{2})(C+r)(\delta_1+\alpha_1x+\gamma_1y)-b(\delta_2+\alpha_2x+\gamma_2y)+1\right)^{\frac{r-C}{2rb}}, \tag{2.11}$$

where $r=\sqrt{4b^2+C^2}$.

If $b=a=0$, $C\neq 0$ and $A\neq 0$ the corresponding first integral is

$$H_6^{(2)}(x,y)= e^{-2AC(A(\delta_1+\alpha_1x+\gamma_1y)+C(\delta_2+\alpha_2x+\gamma_2y))}(C(\delta_1+\alpha_1x+\gamma_1y)+1)^{2A^2}(A(\delta_2 +\alpha_2x+\gamma_2y)+1)^{2C^2}. \tag{2.12}$$

If $\Delta=4b(A+b)+C^2=0$ and $a=0\neq C$ it is

$$H_7^{(2)}(x,y)=\frac{1}{2}\; e^{Z(x,y)+1}\left(-\frac{C^2(\delta_2+\alpha_2x+\gamma_2y)}{4b}-b(\delta_2+\alpha_2x+\gamma_2y)+1\right)^{-\frac{4b^2}{4b^2+C^2}} (-2b(\delta_2+\alpha_2x+\gamma_2y)+C(\delta_1+\alpha_1x+\gamma_1y)+2), \tag{2.13}$$

where $Z(x,y)=C(\delta_1+\alpha_1x+\gamma_1y)/(2b(\delta_2+\alpha_2x+\gamma_2y)-C(\delta_1+\alpha_1x+\gamma_1y)-2)$.

If $A=b=0$ and $a=0\neq C$ it becomes

$$H_8^{(2)}(x,y)= (C(\delta_1+\alpha_1x+\gamma_1y)+1)^2e^{-C\left(C(\delta_2+\alpha_2x+\gamma_2y)^2+2(\delta_1+\alpha_1x+\gamma_1y)\right)}. \tag{2.14}$$

III. The quadratic system (2.2) satisfying condition (iii) of Theorem 2.1. Has the first integral

$$H_1^{(3)}(x,y)=\frac{1}{6}\; (2a(\delta_1+\alpha_1x+\gamma_1y)^3+6b(\delta_1+\alpha_1x+\gamma_1y)^2(\delta_2+\alpha_2x+\gamma_2y)+3(\delta_1 +\alpha_1x+\gamma_1y)^2-6a(\delta_1+\alpha_1x+\gamma_1y)(\delta_2+\alpha_2x+\gamma_2y)^2+2d(\delta_2+\alpha_2x +\gamma_2y)^3+3(\delta_2+\alpha_2x+\gamma_2y)^2). \tag{2.15}$$

IV. The quadratic system (2.2) satisfying condition (iv) of Theorem 2.1. Has the first integral

$$H_1^{(4)}(x,y)= \Big((a^2+d^2)(d(\delta_2+\alpha_2x+\gamma_2y)-a(\delta_1+\alpha_1x+\gamma_1y))^3-3ad(a^2+d^2) (\delta_1+\alpha_1x+\gamma_1y)(\delta_2+\alpha_2x+\gamma_2y)+3d^2(a^2+d^2)(\delta_2+\alpha_2x+\gamma_2y)^2 +3d(a^2+d^2)(\delta_2+\alpha_2x+\gamma_2y)+d^2\Big)^2\Big/\Big((a^2+d^2)(a(\delta_1+\alpha_1x+\gamma_1y) -d(\delta_2+\alpha_2x+\gamma_2y))^2+2d(a^2+d^2)(\delta_2+\alpha_2x+\gamma_2y)+d^2\Big)^3. \tag{2.16}$$

Section 2.2 LC of PWS formed by a linear center and quadratic one

This section devoted to provide the maximum number of limit cycles of PWS separated by the straight line $x=0$, and formed by an arbitrary linear center and an arbitrary quadratic center.

2.2.1 The main results

The following theorem defines our main results.

Note that the results stated in the following theorem do not depend on which half-plane $\Sigma^+ = \{(x,y) : x \geq 0\}$ or $\Sigma^- = \{(x,y) : x \leq 0\}$ are located the linear and the quadratic centers.

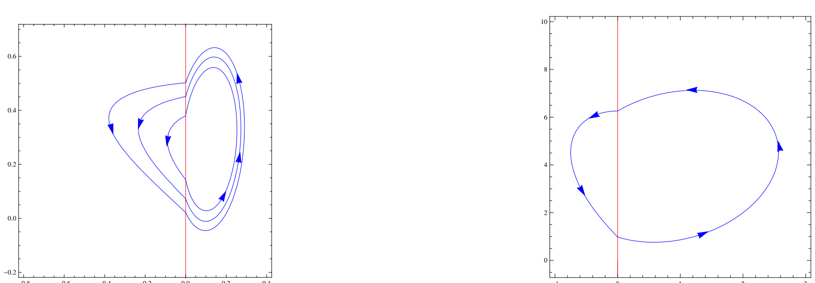
Theorem 2.2 *The maximum number of limit cycles of the discontinuous piecewise differential systems separated by the straight line Σ and formed by an arbitrary linear center and an arbitrary quadratic center satisfying the condition of Kapteyn-Bautin Theorem of*

(I) type $C = a = 0$ is three if $A = -b \neq 0$, and one if eiher $A = 0 \neq b$, or $b = 0 \neq A$, or $A = b = 0$. There are discontinuous piecewise differential systems of these types with three limit cycles see Figure 2.1(a) and with one limit cycle see Figure 2.1(b).

(II) type $b+d = 0$ is three if either $A+b = 0$ and $b \neq 0$, or $AbC(A+b)(4b(A+b)+C^2) \neq 0$, or $b = C = 0$, or $A = 0$ and $b \neq 0$, or $b = 0$ and $A \neq 0$; two if $(4b(A+b)+C^2) = 0$; and one if $A = b = 0$. There are discontinuous piecewise differential systems of this type with three limit cycles see Figure 2.2(a) and with one limit cycle see Figure 2.2(b).

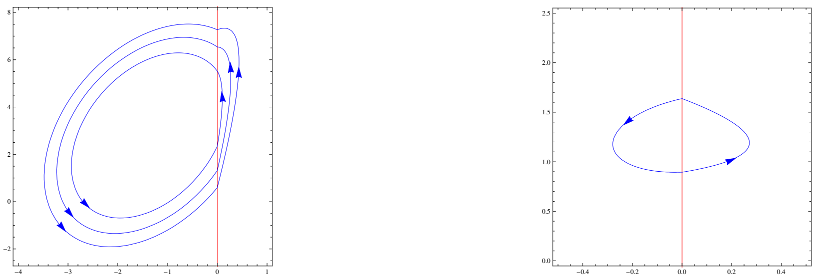
(III) type $C + 2a = A - 2b = 0$ is one. There are discontinuous piecewise differential systems of this type with one limit cycle, see Figure 2.3(a).

(IV) type $C + 2a = A + 3b + 5d = a^2 + bd + 2d^2 = 0$ is four. There are discontinuous piecewise differential systems of this type with four limit cycles, see Figure 2.3(b).



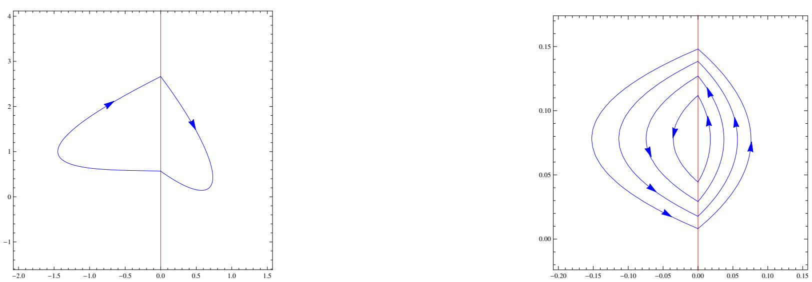
(a) (b)

Figure 2.1: (a) The three limit cycles of the discontinuous piecewise differential system (2.18)–(2.19), and (b) the unique limit cycle of the discontinuous piecewise differential system (2.20)–(2.21).



(*a*) (*b*)

Figure 2.2: (*a*) The three limit cycles of the discontinuous piecewise differential system (2.22)–(2.23), and (*b*) the unique limit cycle of the discontinuous piecewise differential system (2.24)–(2.25).



(*a*) (*b*)

Figure 2.3: (*a*) The unique limit cycle of the discontinuous piecewise differential system (2.26)–(2.27), and (*b*) the four limit cycles of the discontinuous piecewise differential system (2.28)–(2.29).

2.2.2 Proof of Theorem 2.2

Now we should give the proof of Theorem 2.2, where we provide the maximum number of limit cycles of the discontinuous piecewise differential system separated by the straight line Σ, and formed by an arbitrary linear center and an arbitrary quadratic center.

Proof.

In one half-plane we consider the linear differential center (1.4) *with its first integral* $H(x,y)$ *given in* (1.5)*. In the other half-plane we consider system* (2.2) *satisfying one of the four condition of Theorem* 2.1*, with its corresponding first integral* $H_k^{(j)}(x,y)$ *with* $k = 1,\ldots,8$ *and* $j = 1,\ldots,4$.

In order that the discontinuous piecewise differential system (1.4)–(2.2) *has a limit cycle that intersects the straight line* Σ *at the points* $(0,y_1)$ *and* $(0,y_2)$ *with* $y_1 < y_2$*, these points* *must satisfy the following system*

$$\begin{aligned} e_1 &= H(0,y_1) - H(0,y_2) = (y_1 - y_2)\Big(\left(4\beta^2 + \omega^2\right)(y_1 + y_2) - 8\alpha\sigma_1\Big) = 0, \\ e_2 &= H_k^{(j)}(0,y_1) - H_k^{(j)}(0,y_2) = h_k^{(j)}(y_1,y_2) = 0. \end{aligned} \tag{2.17}$$

From $e_1 = 0$, *we obtain* $y_1 = \dfrac{8\alpha\sigma_1}{4\beta^2 + \omega^2} - y_2$ *and by substituting it in* $e_2 = 0$ *we obtain the equation* $F(y_2) = 0$ *in the variable* y_2*, which differs according with the first integrals of system* (2.2).

Proof of statement (I) ***of Theorem*** 2.2. *Now we prove the statement* (I) *for the discontinuous piecewise differential system formed by the linear differential center* (1.4) *and the quadratic differential center* (2.2) *of type* $C = a = 0$*, and we distinguish the following cases:*

Case 1. If $A = -b \neq 0$ *then* $k = 1$ *and* $j = 1$ *in system* (2.17)*, the first integral of* (2.2) *is* $H_1^{(1)}(x,y)$ *given in* (2.3)*, so the solutions of* $F(y_2) = 0$ *are equivalent to the solutions of the non-algebraic equation* $f_1(y_2) = g_1(y_2)$ *where*

$$f_1(y_2) = \left(\frac{L_1 + L_2\, y_2}{L_3 - L_2\, y_2}\right)^r, \quad g_1(y_2) = e^{\dfrac{k_0 + k_1\, y_2 + k_2\, y_2^2 + k_3\, y_2^3}{(L_1 + L_2\, y_2)^2 (L_3 - L_2\, y_2)^2}},$$

and

$$\begin{aligned} k_0 = {} & \frac{1}{(4\beta^2+\omega^2)^2} 16A\sigma_1\alpha(A^3(\beta^2(8\gamma_1\delta_1\delta_2 - 4\gamma_2\delta_1^2 + 4\gamma_2\delta_2^2) + \omega^2(2\gamma_1\delta_1\delta_2 - \gamma_2\delta_1^2 + \gamma_2\delta_2^2) \\ & + 8\alpha\sigma_1\delta_2(\gamma_1^2 + \gamma_2^2)) - \gamma_2 d(4\beta^2 + \omega^2) - 3A\gamma_2 d\left(\delta_2\left(4\beta^2+\omega^2\right) + 4\alpha\gamma_2\sigma_1\right) - A^4(\gamma_2\delta_1 \\ & - \gamma_1\delta_2)\Big(\delta_1\delta_2\left(4\beta^2+\omega^2\right) + 4\alpha\sigma_1(\gamma_1\delta_2 + \gamma_2\delta_1)\Big) + A^2\Big(\left(4\beta^2+\omega^2\right)(\gamma_1\delta_1 + \gamma_2\delta_2(1 \\ & - 2d\delta_2)) + 4\alpha\sigma_1\left(\gamma_1^2 + \gamma_2^2 - 4\gamma_2^2 d\delta_2\right)\Big)\Big), \\ k_1 = {} & 4A\Big((A\delta_2+1)(d(2A\gamma_2\delta_2 + \gamma_2) - A^2(\delta_2(A\gamma_1\delta_1 + \gamma_2) + \delta_1(\gamma_1 - A\gamma_2\delta_1))) + 32\alpha^2A^2\gamma_2\sigma_1^2 \\ & (4\beta^2+\omega^2)^{-2}(A^2\gamma_1(\gamma_1\delta_2 - \gamma_2\delta_1) + A(\gamma_1^2 + \gamma_2^2) - 2\gamma_2^2 d) - 4\alpha A\sigma_1/(4\beta^2+\omega^2)(A^3(\gamma_1^2\delta_2^2 \\ & - \gamma_2^2\delta_1^2) + 2A^2\delta_2(\gamma_1^2 + \gamma_2^2) + A(\gamma_1^2 + \gamma_2^2(1 - 4d\delta_2)) - 3d\gamma_2^2)\Big), \\ k_2 = {} & -\frac{48\alpha A^3\gamma_2\sigma_1}{4\beta^2+\omega^2}\left(A^2\gamma_1(\gamma_1\delta_2 - \gamma_2\delta_1) + A\left(\gamma_1^2 + \gamma_2^2\right) - 2\gamma_2^2 d\right), \\ k_3 = {} & 4A^3\gamma_2\left(A^2\gamma_1(\gamma_1\delta_2 - \gamma_2\delta_1) + A\left(\gamma_1^2 + \gamma_2^2\right) - 2\gamma_2^2 d\right), \quad r = 2d, L_1 = A\delta_2 + 1, \quad L_2 = A\gamma_2, \\ L_3 = {} & \frac{8\alpha A\gamma_2\sigma_1}{4\beta^2+\omega^2} + A\delta_2 + 1. \end{aligned}$$

We note that $\left(f_1\right)'(y_2)$ *and* $\left(g_1\right)'(y_2)$ *are the derivatives of the functions* $f_1(y_2)$ *and* $g_1(y_2)$*,* *respectively. Where*

$$\left(f_1\right)'(y_2) = \frac{\eta (L_1 + L_2\ y_2)^{r-1}}{(L_3 - L_2\ y_2)^{r+1}},$$

and

$$\left(g_1\right)'(y_2) = e^{\dfrac{k_0 + k_1\ y_2 + k_2\ y_2^2 + k_3\ y_2^3}{(L_1 + L_2\ y_2)^2 (L_3 - L_2\ y_2)^2}} \frac{P(y_2)}{(L_1 + L_2\ y_2)^3 (L_1 - L_2\ y_2)^3},$$

with $\eta = rL_2(L_3 + L_1)$*, and*

$$\begin{aligned} P(y_2) = \ & k_3 L_2^2\ y_2^4 + L_2\Big(2k_2 L_2 + k_3(L_3 - L_1)\Big)\ y_2^3 + 3\Big(k_1 L_2^2 + k_3 L_1 L_3\Big)\ y_2^2 + \Big(4k_0 L_2^2 + k_1 L_2 (L_1 \\ & -L_3) + 2k_2 L_1 L_3\Big)\ y_2 + 2k_0 L_1 L_2 - 2k_0 L_2 L_3 + k_1 L_1 L_3. \end{aligned}$$

We denoted by (C_{f_1}) *and* (C_{g_1}) *the graphics of the functions* $f_1(y_2)$ *and* $g_1(y_2)$*, respectively. According with the sign of* $\left(f_1\right)'(y_2)$ *which depends on* r *and with the sign of the parameter* $\eta \in \mathbb{R}$*, we obtain that all the possible graphics* (C_{f_1}) *of the function* $f_1(y_2)$ *are as follows.*

If r *is an even integer or the rational* $r = p/(2q+1)$ *with* p *is an even integer and* q *is an arbitrary integer, then the sign of* $\left(f_1\right)'(y_2)$ *depends on the sign of* $\eta\ (L_1 + L_2\ y_2)\ (L_3 - L_2\ y_2)$*. Therefore the graphic* (C_{f_1}) *is given in Figure 2.14(a) if* $\eta > 0$*, or 2.14(b) if* $\eta < 0$*.*

If r *is an odd integer or the rational* $r = p/(2q+1)$ *with* p *is an odd integer and* q *is an arbitrary integer, then the sign of* $\left(f_1\right)'(y_2)$ *depends only on the sign of* η*. Therefore the graphic* (C_{f_1}) *is given in Figure 2.14(c) if* $\eta < 0$*, or Figure 2.14(d) if* $\eta > 0$*.*

If r *is irrational or the rational* $r = p/(2q)$ *where* p *is an odd integer and* q *is an arbitrary integer, then the sign of* $\left(f_1\right)'(y_2)$ *depends on the sign of* η*. Consequently the graphics* (C_{f_1}) *are the same than in the case that* r *is an odd integer but in the domain of definition of* $f_1(y_2)$*.*

According with the sign of $\left(g_1\right)'(y_2)$ *and with the different kind of the roots* r_i *with* $i \in \{1,\ldots,4\}$ *of the polynomial* $P(y_2)$*, and by considering the case when* $L_3 \neq -L_1$*, we shall obtain the different possible topologically distinct graphics* (C_{g_1})*.*

If $P(y_2)$ *has four simple real roots, then the positions of these roots with respect to the two vertical asymptotes straight lines* $y_{21} = -\dfrac{L_1}{L_2}$ *and* $y_{22} = -\dfrac{L_3}{L_2}$ *play a main role in the variation of the graphics* (C_{g_1})*. So all the possible topologically distinct graphics* (C_{g_1}) *are given in Figure 2.16(a) if* $y_{21} < r_1 < y_{22} < r_2 < r_3 < r_4$*, or Figure 2.16(b) if* $r_1 < r_2 < r_3 < y_{21} < r_4 < y_{22}$*, or Figure 2.16(c) if* $r_1 < y_{21} < r_2 < y_{22} < r_3 < r_4$*, or Figure 2.16(d) if* $r_1 < r_2 < y_{21} < r_3 < y_{22} < r_4$*, or Figure 2.16(e) if* $y_{21} < r_1 < r_2 < y_{22} < r_3 < r_4$*, or Figure 2.16(f) if* $r_1 < r_2 < y_{21} < r_3 < r_4 < y_{22}$*, or Figure 2.16(g) if* $r_1 < y_{21} < r_2 < r_3 < y_{22} < r_4$*, or Figure 2.16(h) if* $r_1 < y_{21} < r_2 < r_3 < r_4 < y_{22}$*, or Figure 2.16(i) if* $y_{21} < r_1 < r_2 < r_3 < y_{22} < r_4$*.*

If $P(y_2)$ *has one triple and one simple real root, or two complex and two simple real roots, the graphics* (C_{g_1}) *are given in Figure 2.16(j) if* $y_{21} < r_1 < r_2 < y_{22}$*, or Figure 2.16(k) if* $r_1 < y_{21} < r_2 < y_{22}$*, or Figure 2.16(l) if* $y_{21} < r_1 < y_{22} < r_2$*.*

If $P(y_2) = 0$ *has one double real and two complex roots, the graphics* (C_{g_1}) *are given in Figure 2.17(a).*

If $P(y_2) = 0$ *has two double real roots, the graphics* (C_{g_1}) *are given in Figure 2.17(b) if* $r_1 < y_{21} < r_2 < y_{22}$*, or Figure 2.17(c) if* $y_{21} < r_1 < y_{22} < r_2$*.*

If $P(y_2) = 0$ *has four complex roots, see Figure 2.17(d).*

If $P(y_2) = 0$ *has one double real* r_0 *and two simple real roots* r_1 *and* r_2*, then if* $r_0 < r_1 < y_{21} < r_2 < y_{22}$ *see Figure 2.17(e), or if* $y_{21} < r_1 < y_{22} < r_0 < r_2$ *see Figure 2.17(f), or if* $r_1 < y_{21} < r_0 < r_2 < y_{22}$ *see Figure 2.17(g), or if* $y_{21} < r_1 < r_0 < y_{22} < r_2$ *see Figure 2.17(h), or if* $r_1 < y_{21} < r_0 < y_{22} < r_2$ *see Figure 2.17(i), or if* $y_{21} < r_1 < r_2 < y_{22} < r_0$ *see Figure 2.17(j), or if* $r_0 < y_{21} < r_1 < r_2 < y_{22}$ *see Figure 2.17(k), or if* $y_{21} < r_0 < y_{22} < r_1 < r_2$ *see Figure 2.17(l), or if* $r_1 < r_2 < y_{21} < r_0 < y_{22}$ *see Figure 2.18(a), or if* $r_0 < y_{21} < r_1 < y_{22} < r_2$ *see Figure 2.18(b).*

If $P(y_2) = 0$ *has one real root of order four this root must equal one of the two asymtotes* y_{21} *or* y_{22}*, then the graphics* (C_{g_1}) *are given in Figure 2.18(c), or Figure 2.18(d).*

Now if $L_3 = -L_1$ *we obtain that* $P(y_2) = 0$ *is a cubic equation, therefore the graphics* (C_{g_1}) *are as follows.*

If $P(y_2) = 0$ *has one triple real root or one simple and two complex roots, the graphics* (C_{g_1}) *are given in Figure 2.18(c) or 2.18(d).*

If $P(y_2) = 0$ *has one double real and one simple real root, the graphics* (C_{g_1}) *are given in Figure 2.18(e) or 2.18(f) if* $r_1 = r_2 < y_{21} < r_3$*, or* $r_1 < y_{21} < r_2 = r_3$*, respectively.*

If $P(y_2) = 0$ *has three real roots, the graphics* (C_{g_1}) *are given in Figure 2.18(g) if* $y_{21} < r_1 < r_2 < r_3$*, or Figure 2.18(h) if* $r_1 < r_2 < y_{21} < r_3$*, or Figure 2.18(i) if* $r_1 < y_{21} < r_2 < r_3$*, or Figure 2.18(j) if* $r_1 < r_2 < r_3 < y_{21}$*.*

We will only give the graphics (C_{g_1}) *when the derivative* $\left(g_1\right)'(y_2)$ *starts with a negative sign because when the derivative start with a positive sign their graphics are topologically equivalent to the previous ones.*

For the function $f_1(y_2)$ *we remark that the sign of the derivative changes at most three times when* r *an even integer or* $r = p/(2q+1)$ *with* p *an even integer and* q *is an arbitrary integer which guarantees that* (C_{f_1}) *can have at most one local extrem in (a) or (b) of Figure 2.14, and on the other hand it is obvious that the graphics* (C_{g_1}) *can have at most four local extremes in (a), or (b), or (c), or (d), or (e), or (f), or (g), or (h), or (i) or (j) of Figure 2.16, and since the* *function $g_1(y_2)$ is positive and it has the horizontal asymptote straight line $g_1(y_2) = 1$, then we guarantee that the maximum number of intersection points between the graphics (C_{f_1}) and (C_{g_1}) can be precisely between (a) or (b) of Figure 2.14 and (a), or (b), or (c), or (d), or (e), or (f), or (g), or (h), or (i) or (j) of Figure 2.16. It is clear that the graphics (C_{f_1}) and (C_{g_1}) intersect at most in seven points, see for example Figure 2.4. Hence, $F(y_2) = 0$ has at most seven real solutions. We can show easily that if (y_1, y_2) is a solution of (2.17), then (y_2, y_1) is also a solution of this system. Consequently, the maximum number of limit cycles of the discontinuous piecewise differential system (1.4)–(2.2) in this case is at most three.*

In what follows we construct an example with exactly seven intersection points between the graphics (C_{f_1}) and (C_{g_1}) by considering $\{L_1, L_2, L_3, k_0, k_1, k_2, k_3, r\} \rightarrow \{-2, 1, -3.5, 2, 2.5, -1, -6.9, 2\}$, these points are shown in Figure 2.4.

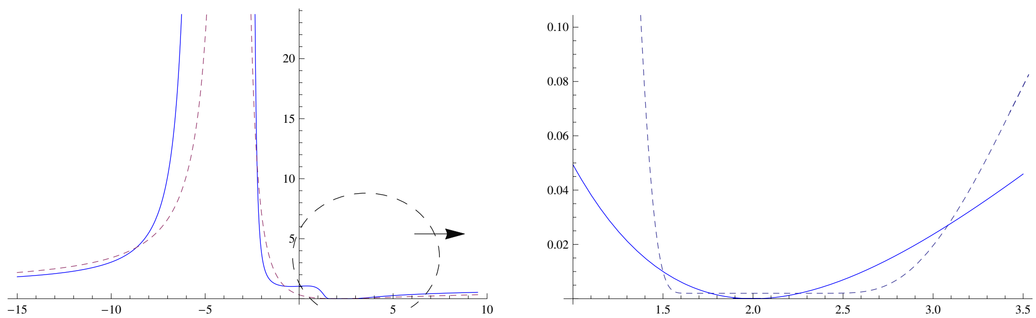


Figure 2.4: The seven intersection poitns between the two functions $f_1(y_2)$ drawn in continuous line and $g_1(y_2)$ drawn in dashed line.

To complete the proof of this case we provide an example with three limit cycles.

Three limit cycles for a discontinuous piecewise differential system (1.4)–(2.2) of type $C = a = 0$ with $A = -b \neq 0$.

In the half-plane Σ^- we consider the quadratic center

$$\begin{aligned} \dot{x} \simeq & -0.41137x^2 + x(-4.7083y - 0.8342) + y(-12.0159y - 0.866409) \\ & +1.05188, \\ \dot{y} \simeq & \ 0.053171x^2 + x(0.67274y + 0.0188591) + y(1.9488y + 0.11029) \\ & -0.720841, \end{aligned} \tag{2.18}$$

its corresponding first integral is

$$\begin{aligned} H_2^{(1)}(x,y) \simeq & -0.001233(-x - 10.5578y + 9.06)^2 \Big(x^2 + (9.8313y + 6.51292)x \\ & +(21.4047y + 14.3032)y + 2.38945 \Big). \end{aligned}$$

In the half-plane Σ^+ we consider the linear differential center

$$\dot{x} = \frac{1}{5}x - \frac{229}{300}y + \frac{1}{5}, \quad \dot{y} = 3x - \frac{1}{5}y - \frac{3}{10}, \tag{2.19}$$

with the first integral $\quad H(x,y) = 4\Big(3x - \frac{2}{10}y\Big)^2 + 24\Big(-\frac{3}{10}x - \frac{2}{10}y\Big) + 9y^2.$

In this case system (2.17) *has the three solutions* $(y_1, y_2) \simeq \{(0.0217348, 0.502283),$ $(0.0725424, 0.451475), (0.143419, 0.380598)\}$ *which provide the three limit cycles for the discontinuous piecewise differential system* (2.18)–(2.19) *shown in Figure* 2.1(*a*).

Case 2. If $A = 0 \neq b$ then $k = 2$ and $j = 1$ in system (2.17)*, then the first integral of system* (2.2) *is $H_2^{(1)}(x,y)$ given in* (2.4)*, and the solutions y_2 of $F(y_2) = 0$ are the ones of the non-algebraic equation $f_2(y_2) = g_2(y_2)$, where*

$$f_2(y_2) = e^{(l_0 + l_1 y_2)} \ \text{ and } \ g_2(y_2) = \frac{k_0 + k_1\, y_2 + k_2\, y_2^2}{z_0 + z_1\, y_2 + k_2\, y_2^2},$$

with

$$\begin{aligned}
l_0 = & \quad -16\alpha b \gamma_2 \sigma_1/(4\beta^2 + \omega^2), \ \ l_1 = 4b\gamma_2, \ \ k_0 = b\Big(2b^2\delta_1^2 + 2b\delta_2(d\delta_2 + 1) - 2d\delta_2 - 1\Big) + d, \\
k_1 = & \quad 2b\Big(2b^2\gamma_1\delta_1 + b(\gamma_2 + 2\gamma_2 d\delta_2) - \gamma_2 d\Big), \ \ k_2 = 2b^2\Big(b\gamma_1^2 + \gamma_2^2 d\Big), \\
z_0 = & \quad 2b(8\alpha\sigma_1\Big(2b^2\gamma_1\delta_1 + b(\gamma_2 + 2\gamma_2 d\delta_2) - \gamma_2 d\Big)/(4\beta^2 + \omega^2) + b^2\delta_1^2 + 64\alpha^2 b\sigma_1^2\Big(b\gamma_1^2 + \gamma_2^2 d\Big) \\
& \quad (4\beta^2 + \omega^2)^{-2} + bd\delta_2^2 + b\delta_2 - d\delta_2) - b + d, \\
z_1 = & \quad 2b\Big(-2b^2\gamma_1\delta_1 - b\gamma_2 - 2b\gamma_2 d\delta_2 - 16\alpha b\sigma_1/(4\beta^2 + \omega^2)\Big(b\gamma_1^2 + \gamma_2^2 d\Big) + \gamma_2 d\Big).
\end{aligned}$$

We denoted by (C_{f_2}) and (C_{g_2}) the graphics of $f_2(y_2)$ and $g_2(y_2)$, respectively.

The possible graphics of $f_2(y_2)$ are shown either in Figure 2.15(*a*) *if $l_1 > 0$, or in Figure* 2.15(*b*) *if $l_1 < 0$.*

For the function $g_2(y_2)$ its derivative is $\Big(g_2\Big)'(y_2) = P_1(y_2)\Big(P_2(y_2)\Big)^{-2}$, with

$$P_1(y_2) = \quad (k_2 z_1 - k_1 k_2)y_2^2 + (2k_2 z_0 - 2k_0 k_2)y_2 - k_0 z_1 + k_1 z_0 \ \text{ and } \ P_2(y_2) = k_2 y_2^2 + z_1 y_2 + z_0.$$

We see that the discriminant of the numerator of $g_2(y_2) = 0$ is equal to the discriminant of $P_2(y_2) = 0$ which is $\Delta_0 = k_1^2 - 4k_0k_2 = z_1^2 - 4z_0k_2$, and $\Delta = (2k_2z_0 - 2k_0k_2)^2 - 4(k_2z_1 - k_1k_2)(-k_0z_1 + k_1z_0)$ is the discriminant of $P_1(y_2) = 0$, then according to the sign of the determinants Δ_0 and Δ the graphics of $g_1(y_2)$ are given in Figure 2.19(*a*) *or Figure* 2.19(*b*) *if $\Delta_0 > 0$ and $\Delta < 0$, Figure* 2.19(*c*) *or Figure* 2.19(*d*) *if $\Delta_0 > 0$ and $\Delta > 0$, Figure* 2.19(*e*) *or Figure* 2.19(*f*) *if $\Delta_0 > 0$ and $\Delta = 0$ or $\Delta_0 = 0$ and $\Delta > 0$, and Figure* 2.19(*g*) *or Figure* 2.19(*h*) *if* $\Delta_0 < 0$ *and* $\Delta < 0$.

It is clear that the graphics (C_{g_2}) *can have the maximum number of local extremes in* (c), *or* (d), *or* (g) *or* (h) *of Figure 2.19, then we know that the maximum number of intersection points between the graphics* (C_{f_2}) *and* (C_{g_2}) *can be precisely between* (a) *or* (b) *of Figure 2.15 and* (c), *or* (d), *or* (g) *or* (h) *of Figure 2.19. It is obvious that the graphics* (C_{f_2}) *and* (C_{g_2}) *intersect at most in three points see for example Figure 2.5. Due to the symmetry of the solutions of system* (2.17) *we know that the maximum number of limit cycles in this case is at most one.*

In the following we build an example with exactly three intersection points between the graphics (C_{f_2}) *and* (C_{g_2}) *by taking* $\{l_0, l_1, k_0, k_1, k_2, z_0, z_1\} \to \{0.35, 0.2, 3.9, -2, -1, 4, -2\}$, *these points are shown in Figure 2.5.*

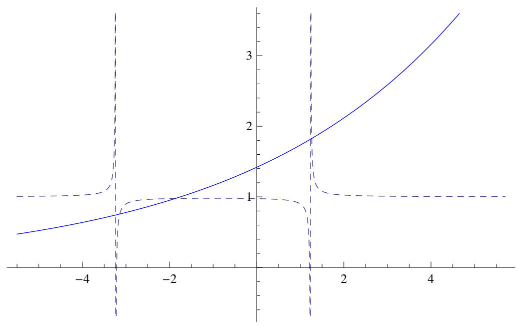


Figure 2.5: The three intersection poitns between the two functions $f_2(y_2)$ drawn in continous line and $g_2(y_2)$ drawn in dashed line.

Case 3. If $b = 0 \neq A$ *then* $k = 3$ *and* $j = 1$ *in system* (2.17), *the first integral of system* (2.2) *is* $H_3^{(1)}(x,y)$ *given in* (2.5), *and the solutions of* $F(y_2) = 0$ *are equivalent to the ones of the equation* $f_3(y_2) = g_3(y_2)$.

We have $f_3(y_2) = f_1(y_2)$, *with* $r = 2d - 2A$, *therefore the graphics of* $f_3(y_2)$ *are given in Figure 2.14.*

We have also $g_3(y_2) = f_2(y_2)$, *then we know that the graphics of* $g_3(y_2)$ *are given in Figure 2.15. The parameters of the function* $g_3(y_2)$ *are:*

$$
\begin{aligned}
l_0 = {} & \frac{1}{(4\beta^2+\omega^2)^2}\Big(16\alpha A\sigma_1(A^2\gamma_1(\delta_1(4\beta^2+\omega^2)+4\alpha\gamma_1\sigma_1)+A\gamma_2(4\beta^2(d\delta_2+1)+4\alpha\gamma_2\sigma_1 d \\
& +d\delta_2\omega^2+\omega^2)-\gamma_2 d(4\beta^2+\omega^2))\Big),\\
l_1 = {} & -\frac{1}{4\beta^2+\omega^2}\Big(4A(A^2\gamma_1(\delta_1(4\beta^2+\omega^2)+4\alpha\gamma_1\sigma_1)+A\gamma_2(4\beta^2(d\delta_2+1)+4\alpha\gamma_2\sigma_1 d+d\delta_2\omega^2 \\
& +\omega^2)-\gamma_2 d(4\beta^2+\omega^2))\Big).
\end{aligned}
$$

Due to the fact that $f_3(y_2) = f_1(y_2)$ *there is at most one local extrem at zero in* (a) *or* (b) *of Figure 2.14. We know also that* $g_3(y_2) = f_2(y_2)$, *then it is clear that the maximum number of* *intersections points between the graphics* (C_{g_3}) *and* (C_{f_3}) *can be precisely between* (a) *or* (b) *of Figure 2.14 and Figure 2.15. Consequently the graphics of the functions* $f_3(y_2)$ *and* $g_3(y_2)$ *intersect at most in three points see for example Figure 2.6. Due to the symmetry of the solutions of system* (2.17) *we know that the maximum number of limit cycles of the discontinuous piecewise differential system* (1.4)–(2.2) *is at most one.*

By considering $\{l_0, l_1, L_1, L_2, L_3, r\} \to \{-0.2, 1.75, 4, -7, -0.4, -2\}$ *we constuct an example with exactly three intersection points between the graphics of the two functions* $f_3(y_2)$ *and* $g_3(y_2)$*, these points are illustrated in Figure 2.6.*

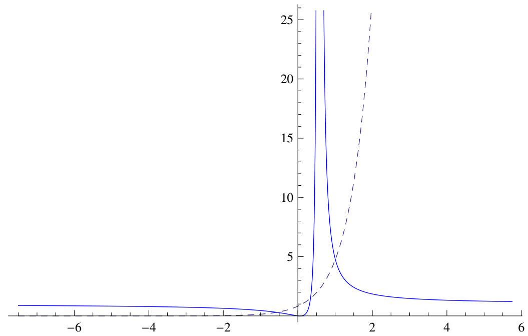


Figure 2.6: The three intersection poitns between the two functions $f_3(y_2)$ drawn in continuous line and $g_3(y_2)$ drawn in dashed line.

Case 4. If $A = b = 0$ *then* $k = 4$ *and* $j = 1$ *in system* (2.17)*, the first integral in this case is* $H_4^{(1)}(x, y)$ *given in* (2.6)*, and*

$$\begin{aligned} F(y_2) = \quad & 2d\Big(\frac{8\alpha\gamma_2\sigma_1}{4\beta^2+\omega^2} + \delta_2 - \gamma_2 y_2\Big)^3 - 2d(\delta_2 + \gamma_2 y_2)^3 + 3\Big(\Big(\frac{8\alpha\gamma_1\sigma_1}{4\beta^2+\omega^2} + \delta_1 - \gamma_1 y_2\Big)^2 \\ & + \Big(\frac{8\alpha\gamma_2\sigma_1}{4\beta^2+\omega^2} + \delta_2 - \gamma_2 y_2\Big)^2\Big) - 3\Big((\delta_1 + \gamma_1 y_2)^2 + (\delta_2 + \gamma_2 y_2)^2\Big). \end{aligned}$$

Since $F(y_2) = 0$ *is a cubic equation in the variable* y_2 *the maximum number of real solutions of system* (2.17) *is at most three. Eventually, the upper bound of the maximum number of limit cycles for this case is at most one.*

To complete the proof of this case we build an example with one limit cycle of the discontinuous piecewise differential system (1.4)–(2.2) *of type* $C = a = 0$ *with* $A = b = 0$.

One limit cycle of the discontinuous piecewise differential system (1.4)–(2.2) ***of type*** $C = a = 0$ ***with*** $A = b = 0$.

We consider the quadratic center

$$\dot{x} \simeq \; -1.06383\Big(x\Big(\frac{11}{5} - \frac{1}{10}\Big(1 - 4\Big(-0.2y - 0.5\Big)\Big)\Big) + \frac{1}{10}x^2 + \frac{8}{500}y^2 + \frac{28}{25}y - \frac{43}{10}\Big), \qquad (2.20)$$

$$\dot{y} \simeq \quad 1.06383\Big(-\frac{1}{4}x^2 + x\Big(\frac{1}{4}\Big(1-4\Big(-\frac{1}{5}y-\frac{1}{2}\Big)\Big)+\frac{121}{25}\Big)-\frac{1}{25}y^2+\frac{19}{10}y-\frac{52}{5}\Big),$$

in the half-plane Σ^-, with its first integral

$$H_2^{(1)}(x,y) = 3\Big(\Big(\frac{1}{2}x-\frac{1}{5}y-\frac{1}{2}\Big)^2+\Big(-\frac{11}{5}x-y+\frac{9}{2}\Big)^2\Big)-4\Big(\frac{1}{2}x-\frac{1}{5}y-\frac{1}{2}\Big)^3.$$

In the half-plane Σ^+ we consider the linear differential center

$$\dot{x} \simeq \frac{1}{10}x - 0.27632y + 1, \quad \dot{y} = x - \frac{1}{10}y - \frac{1}{2}, \tag{2.21}$$

with the first integral $H(x,y) \simeq 4\Big(x-\frac{1}{10}y\Big)^2 + 8\Big(-\frac{1}{2}x-y\Big) + 1.06528y^2.$

In this case system (2.17) has the unique solution $(y_1, y_2) \simeq (0.978592, 6.25938)$ which provides the unique limit cycle for the discontinuous piecewise differential system (2.20)–(2.21), see Figure 2.1(b). This example completes the proof of statement (I).

Proof of statement (II) ***of Theorem 2.2****. Now we must prove the statement for the discontinuous piecewise differential system formed by the linear center (1.4) and the quadratic center (2.2) of type $b+d=0$, and we distinguish the following cases:*

Case 1. If $A+b=0$ and $a=0\neq C$, then $k=1$ and $j=2$ in system (2.17), the first integral of the quadratic center is $H_1^{(2)}(x,y)$ given in (2.7), the solutions of $F(y_2)=0$ are the same as the solutions of the equation $\tilde{f}_1(y_2)=\tilde{g}_1(y_2)$ where

$$\tilde{f}_1(y_2) = \left(\frac{m_1+m_2\,y_2}{m_3-m_2\,y_2}\right)^{r_1}\left(\frac{n_1+n_2\,y_2}{n_3-n_2\,y_2}\right)^{r_2} \quad \text{and} \quad \tilde{g}_1(y_2) = e^{\frac{k_0+k_1\,y_2}{(m_1+m_2\,y_2)(m_3-m_2\,y_2)}},$$

with

$$\begin{aligned}
m_1 = &\ -\frac{8\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2} - b\delta_2 + 1,\ \ m_2 = b\gamma_2,\ \ m_3 = 1-b\delta_2,\ \ r_1 = b^2+C^2,\ r_2 = b^2,\\
n_1 = &\ -b\delta_2 + C\delta_1 + 1,\ \ n_2 = \gamma_1 C - b\gamma_2,\ \ n_3 = \frac{8\alpha\sigma_1}{4\beta^2+\omega^2}(\gamma_1 C - b\gamma_2) - b\delta_2 + C\delta_1 + 1,\\
k_0 = &\ -\frac{8\alpha bC\sigma_1}{4\beta^2+\omega^2}(b(-b\gamma_1\delta_2 + b\gamma_2\delta_1 + \gamma_1) + \gamma_2 C),\ \ k_1 = 2bC(b(-b\gamma_1\delta_2 + b\gamma_2\delta_1 + \gamma_1) + \gamma_2 C).
\end{aligned}$$

The derivative of the function $\tilde{f}_1(y_2)$ and $\tilde{g}_1(y_2)$ are

$$\left(\tilde{f}_1\right)'(y_2) = \left(M_0 + M_1\,y_2 + M_2\,y_2^2\right)\frac{(m_1+m_2\,y_2)^{r_1-1}(n_1+n_2\,y_2)^{r_2-1}}{(m_3-m_2\,y_2)^{r_1+1}(n_3-n_2\,y_2)^{r_1+1}},$$

and

$$\left(\tilde{g}_1\right)'(y_2) = \frac{\left(N_0 + N_1 y_2 + N_2 y_2^2\right)}{(m_1 + m_2 y_2)^2 (m_3 - m_2 y_2)^2} e^{\dfrac{k_0 + k_1\, y_2}{(m_1 + m_2\, y_2)(m_3 - m_2\, y_2)}}.$$

with

$$\begin{aligned} M_0 =&\ \ m_2 n_1 n_3 r_1 (m_1 + m_3) + m_1 m_3 n_2 r_2 (n_1 + n_3), \\ M_1 =&\ \ m_2 n_2 (r_1 (m_1 + m_3)(n_3 - n_1) - r_2 (m_1 - m_3)(n_1 + n_3)), \\ M_2 =&\ \ -m_2 n_2 (n_2 r_1 (m_1 + m_3) + m_2 r_2 (n_1 + n_3)), \\ N_0 =&\ \ -m_3 m_2 k_0 + m_1 m_2 k_0 + m_1 m_3 k_1,\ N_1 = 2 k_0 m_2^2,\ N_2 = k_1 m_2^2. \end{aligned}$$

According to the number of the vertical asymptotes of the function $\tilde{f}_1(y_2)$ we can divide the study of this function into two parts.

If $m_1 = n_1$, $m_2 = n_2$, $m_3 = n_3$, or $r_1 = 0$ and $r_2 \neq 0$, or $r_1 \neq 0$ and $r_2 = 0$, then the function $\tilde{f}_1(y_2)$ has one vertical asymptote and the graphics $(C_{\tilde{f}_1})$ of the function $\tilde{f}_1(y_2)$ are the same as the ones of the function $f_1(y_2)$ shown in Figure 2.14.

If $m_1 \neq n_1$, or $m_2 \neq n_2$, or $m_3 = n_3$, or $r_1 \neq 0$ and $r_2 \neq 0$, then the function $\tilde{f}_1(y_2)$ has two vertical asymptotes. Therefore according with the derivative $\left(\tilde{f}_1\right)'(y_2)$ which depends on the parameters r_1, r_2 and with the sign of the discriminant $\Delta_1 = M_1^2 - 4M_0M_2$ also according with the positions of the roots of the numerator of $\left(\tilde{f}_1\right)'(y_2)$ with respect to the two vertical asymptotes, we know that all the possible topologically distinct graphics $(C_{\tilde{f}_1})$ of the function $\tilde{f}_1(y_2)$ are given as follows.

If r_1 and r_2 are even integers, or r_1 is an even integer and r_2 is rational such that $r_2 = 2p/(2q+1)$ with $p,q \in \mathbb{Z}$, all the graphics $(C_{\tilde{f}_1})$ are given in Figure 2.20 if $\Delta_1 > 0$. If $\Delta_1 = 0$ the graphics $(C_{\tilde{f}_1})$ are given in (a), or (b), or (c), or (d), or (e) of Figure 2.21. If $\Delta_1 < 0$ the graphics $(C_{\tilde{f}_1})$ are given in (f), or (g) of Figure 2.21.

If r_1 and r_2 are odd integers, or r_1 is an odd integer and r_2 is rational such that $r_2 = (2p+1)/(2q+1)$ with $p,q \in \mathbb{N}$, therefore if $\Delta_1 > 0$ the graphics $(C_{\tilde{f}_1})$ are given in (h), or (i), or (j), or (k), or (l) of Figure 2.21 and in (a), or (b), or (c), or (d), or (e), or (f), or (g) of Figure 2.22. If $\Delta_1 = 0$ the graphics $(C_{\tilde{f}_1})$ are given in (h), or (i), or (j), or (k), or (l) of Figure 2.22. If $\Delta_1 < 0$ the graphics $(C_{\tilde{f}_1})$ are given in (a), or (b) of Figure 2.23.

If r_1 is an odd integer and r_2 is an even integer, or r_2 is an even integer and $r_1 = (2p+1)/(2q+1)$ with $p,q \in \mathbb{Z}$, or r_1 is an odd integer and $r_2 = (2p)/(2q+1)$ with $p,q \in \mathbb{Z}$, then the sign of the derivative $\left(\tilde{f}_1\right)'(y_2)$ depends on the sign of $\left(M_0 + M_1\, y_2 + M_2\, y_2^2\right)(n_1 + n_2\, y_2)(n_3 - n_2\, y_2)$, therefore if $\Delta_1 > 0$ the graphics $(C_{\tilde{f}_1})$ are given in (c), or (d), or (e), or (f), or (g), or (h), or

(i), or (j), or (k), or (l) of Figure 2.23 and in (a), or (b) of Figure 2.24. If $\Delta_1 = 0$ the graphics $(C_{\tilde{f}_1})$ are given in (c), or (d), or (e), or (f), or (g) of Figure 2.24. If $\Delta_1 < 0$ the graphics $(C_{\tilde{f}_1})$ are given in (h), or (i) of Figure 2.24.

If r_1 is an odd integer and r_2 is irrational or $r_2 = p/2q$ with $p, q \in \mathbb{Z}$, then the sign of the derivative $\left(\tilde{f}_1\right)'(y_2)$ depends on the sign of the quadratic polynomial $\left(M_0 + M_1\, y_2 + M_2\, y_2^2\right)$, therefore the graphics $(C_{\tilde{f}_1})$ are the same as the case in which r_1 and r_2 are odd integers but in their domain of definition.

If r_2 is an even integer and r_1 is irrational or $r_1 = p/2q$ with $p, q \in \mathbb{Z}$, then the sign of the derivative $\left(\tilde{f}_1\right)'(y_2)$ depends on the sign of the products $\left(M_0 + M_1\, y_2 + M_2\, y_2^2\right)(n_1 + n_2\, y_2)(n_3 - n_2\, y_2)$, therefore the graphics $(C_{\tilde{f}_1})$ are the same as in the case that r_1 is an odd integer and r_2 is an even integer but in their domain of definition.

If r_1 is irrational or $r_1 = p_0/2q_0$ and r_2 is irrational or rational with $r_2 = p/2q$ and p_0, p are odd integers, then the sign of the derivative $\left(\tilde{f}_1\right)'(y_2)$ depends on the sign of the quadratic polynomial $\left(M_0 + M_1\, y_2 + M_2\, y_2^2\right)$, therefore the graphics of $\tilde{f}_1(y_2)$ are the same as in the case where r_1 and r_2 are odd integers, but in their domain of definition.

If both r_1, r_2 are rational with $r_1 = (2p_0)/(2q_0+1)$ and $r_2 = (2p)/(2q+1)$ such that $p, q, p_0, q_0 \in \mathbb{Z}$, then the sign of the derivative $\left(\tilde{f}_1\right)'(y_2)$ depends on the sign of $\left(M_0 + M_1\, y_2 + M_2\, y_2^2\right)(m_1 + m_2\, y_2)(m_3 - m_2\, y_2)(n_1 + n_2\, y_2)(n_3 - n_2\, y_2)$, therefore the graphics of $\tilde{f}_1(y_2)$ are the same as in the case that r_1 and r_2 are even integers.

If both r_1, r_2 are rational with $r_1 = (2p_0 + 1)/(2q_0 + 1)$ and $r_2 = (2p + 1)/(2q + 1)$ such that $p, q, p_0, q_0 \in \mathbb{Z}$, then the sign of the derivative $\left(\tilde{f}_1\right)'(y_2)$ depends on the sign of $\left(M_0 + M_1\, y_2 + M_2\, y_2^2\right)$, therefore the graphics of $\tilde{f}_1(y_2)$ are the same as in the case where r_1 and r_2 are odd integers.

If both r_1, r_2 are rational with $r_1 = (2p_0 + 1)/(2q_0 + 1)$ and $r_2 = (2p)/(2q + 1)$ such that $p, q, p_0, q_0 \in \mathbb{Z}$, then the sign of the derivative $\left(\tilde{f}_1\right)'(y_2)$ depends on the sign of $\left(M_0 + M_1\, y_2 + M_2 y_2^2\right)$, therefore the graphics of $\tilde{f}_1(y_2)$ are the same as in the case of r_1 is an odd integer and r_2 is an even integer.

If r_1 is irrational or rational and r_2 rational with r_1 irrational or $r_1 = p_0/(2q_0)$ and $r_2 = (2p)/(2q + 1)$ and p_0 is an odd integer and q_0, p and q are integers, therefore the graphics of $\tilde{f}_1(y_2)$ are the same in the case that r_1 is an odd integer and r_2 is an even integer but in their domain of definition.

If r_1 is irrational or rational and r_2 rational with r_1 irrational or $r_1 = p_0/(2q_0)$ and $r_2 = (2p + 1)/(2q + 1)$ and p_0 is an odd integer and q_0, p and q are integers, therefore the graphics *of $\tilde{f}_1(y_2)$ are the same in the case where r_1 and r_2 are odd integers but in their domain of definition.*

According to the sign of the derivative of the function $\tilde{g}_1(y_2)$ which depends on the sign of the quadratic polynomial $P(y_2) = N_0 + N_1 y_2 + N_2 y_2^2$, then the topologically distinct graphics of $\tilde{g}_1(y_2)$ are shown in (a) and (b) of Figure 2.25 if $m_3 \neq -m_1$ and $P(y_2)$ has two distinct real roots, or (c) of Figure 2.25 if $m_3 \neq -m_1$ and $P(y_2)$ has two complex roots, or (d) of Figure 2.25 if $m_3 \neq -m_1$ and $P(y_2)$ has one double real root, or (e) and (f) of Figure 2.25(e) if $m_3 = -m_1$. The possible graphics of $\tilde{f}_1(y_2)$ are given in Figures 2.14, 2.20, 2.21, 2.22, 2.23 and 2.24, and the graphics of $\tilde{g}_1(y_2)$ are given in Figure 2.25.

Since the graphics of $\tilde{f}_1(y_2)$ can have the maximum number of local extremes in (a)–(l) of Figure 2.20 and due to the fact that the function $\tilde{g}_1(y_2)$ is positive and its graphics can have at most two extremes in (a) or (b) of Figure 2.25. We know that the maximum number of intersection points between the graphics of $\tilde{f}_1(y_2)$ and $\tilde{g}_1(y_2)$ can be precisely between (a)–(l) of Figure 2.20 and (a) or (b) of Figure 2.25. In this case the two functions $\tilde{f}_1(y_2)$ and $\tilde{g}_1(y_2)$ have $\tilde{f}_1(y_2) = \tilde{g}_1(y_2) = 1$ as an horizontal asymptote which ensures that at infinity there are no intersection points between their graphics. Therefore the graphics $(C_{\tilde{f}_1})$ and $(C_{\tilde{g}_1})$ of the two functions $\tilde{f}_1(y_2)$ and $\tilde{g}_1(y_2)$ intersect at most in six points see for example Figure 2.7. Then the upper bound of the maximum number of limit cycles in this case is at most three.

By taking $\{k_0, k_1, m_1, m_2, m_3, r_1, r_2, n_1, n_2, n_3\} \to \{-0.21, -1.7, -1.7, -5.8, -2.5, 4, 2, -1.28, 2.75, -3.9\}$, *we build an example with exaclty six intersection poitns between graphics of the functions $\tilde{f}_1(y_2)$ and $\tilde{g}_1(y_2)$. These points are shown in Figure 2.7.*

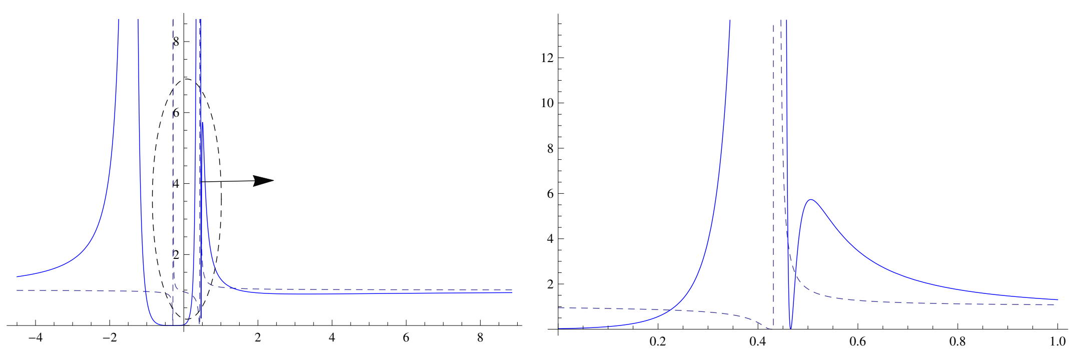


Figure 2.7: The six intersection poitns between the two functions $\tilde{f}_1(y_2)$ drawn in continuous line and $\tilde{g}_1(y_2)$ drawn in dashed line.

Case 2. If $AbC\Delta \neq 0$ and $a = 0$ with $\Delta = 4b(A + b) + C^2 < 0$, then $k = 2$ and $j = 2$ in system (2.17), the first integral of (2.2) is $H_2^{(2)}(x, y)$ given in (2.8). The equation $F(y_2) = 0$ is *equivalent to the equation* $\tilde{f}_2(y_2) = \tilde{g}_2(y_2)$ *where*

$$\tilde{f}_2(y_2) = e^{m_1\left(\arctan\left(\frac{s_3y_2+s_4}{s_1y_2+s_2}\right)+\arctan\left(\frac{s_3y_2+s_6}{s_5-s_1y_2}\right)\right)}, \text{ and } \tilde{g}_2(y_2) = \left(\frac{t_1y_2+t_2}{t_3-t_1y_2}\right)^{r_2}\left(\frac{K_1y_2^2+K_2y_2+K_3}{K_1y_2^2+K_4y_2+K_5}\right)^{r_1},$$

with

$$\begin{aligned}
m_1 = {} & \frac{2C}{bL},\ s_1 = \gamma_1 L,\ s_2 = \delta_1,\ s_3 = \gamma_1 C - 2b\gamma_2,\ s_4 = -2b\delta_2 + C\delta_1 + 2,\ s_5 = \frac{8\alpha\gamma_1\sigma_1 L}{4\beta^2+\omega^2} + \delta_1 L,\\
s_6 = {} & \frac{16\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2} + 2b\delta_2 - \frac{8\alpha\gamma_1 C\sigma_1}{4\beta^2+\omega^2} - C\delta_1 - 2,\ K_1 = \frac{1}{4}\left(4b^2\gamma_2^2 - 4b\gamma_1\gamma_2 C + \gamma_1^2C^2 + \gamma_1^2L^2\right),\\
K_2 = {} & -\frac{1}{2(4\beta^2+\omega^2)}\Big(8\alpha\sigma_1\left(4b^2\gamma_2^2+\gamma_1^2L^2\right) + \left(4\beta^2+\omega^2\right)\left(4b^2\gamma_2\delta_2 - 4b\gamma_2 + \gamma_1\delta_1L^2\right) - 2C\\
& (4\alpha b\gamma_1\gamma_2\sigma_1 + (4\beta^2+\omega^2)(b\gamma_1\delta_2 + b\gamma_2\delta_1 - \gamma_1)) + \gamma_1C^2(8\alpha\gamma_1\sigma_1 + \delta_1(4\beta^2+\omega^2))\Big),\\
K_3 = {} & \frac{1}{4\left(4\beta^2+\omega^2\right)^2}\Big(-4C\left(\delta_1\left(4\beta^2+\omega^2\right) + 8\alpha\gamma_1\sigma_1\right)(4\beta^2(b\delta_2-1) + 8\alpha b\gamma_2\sigma_1 + b\delta_2\omega^2\\
& -\omega^2) + 4\left(4\beta^2(b\delta_2-1) + 8\alpha b\gamma_2\sigma_1 + b\delta_2\omega^2 - \omega^2\right)^2 + C^2\left(\delta_1\left(4\beta^2+\omega^2\right) + 8\alpha\gamma_1\sigma_1\right)^2\\
& +L^2\left(\delta_1(4\beta^2+\omega^2) + 8\alpha\gamma_1\sigma_1\right)^2\Big),\\
K_4 = {} & \frac{1}{2}\left(4b^2\gamma_2\delta_2 - 4b\gamma_2 - 2C(\gamma_1(b\delta_2-1) + b\gamma_2\delta_1) + \gamma_1C^2\delta_1 + \gamma_1\delta_1L^2\right),\\
K_5 = {} & \frac{1}{4}\left(-4C\delta_1(b\delta_2-1) + 4(b\delta_2-1)^2 + C^2\delta_1^2 + \delta_1^2L^2\right),\ r_1 = \frac{1}{b},\ r_2 = -\frac{8b}{4b^2+C^2+L^2},\\
t_1 = {} & \frac{\gamma_2\left(4b^2+C^2+L^2\right)}{4b},\ t_2 = -\frac{1}{4b}\left(\frac{8\alpha\gamma_2\sigma_1\left(4b^2+C^2+L^2\right)}{4\beta^2+\omega^2} + 4b^2\delta_2 - 4b + C^2\delta_2 + \delta_2L^2\right),\\
t_3 = {} & -\frac{C^2\delta_2}{4b} - b\delta_2 - \frac{\delta_2L^2}{4b} + 1.
\end{aligned}$$

We note that $\left(\tilde{f}_2\right)'(y_2)$ *and* $\left(\tilde{g}_2\right)'(y_2)$ *are the derivatives of the functions* $\tilde{f}_2(y_2)$ *and* $\tilde{g}_2(y_2)$*, respectively, where*

$$\left(\tilde{f}_2\right)'(y_2) = m_1\tilde{f}_2(y_2)\frac{P_1(y_2)}{((s_1y_2+s_2)^2+(s_3y_2+s_4)^2)((s_5-s_1y_2)^2+(s_3y_2+s_6)^2)},$$

and

$$\begin{aligned}
\left(\tilde{g}_2\right)'(y_2) = {} & -\frac{P_2(y_2)}{(t_1y_2+t_2)(-t_1y_2+t_3)\left(K_1y_2^2+K_2y_2+K_3\right)\left(K_1y_2^2+K_4y_2+K_5\right)}\left(\frac{t_1y_2+t_2}{-t_1y_2+t_3}\right)^{r_2}\\
& \left(\frac{K_1y_2^2+K_2y_2+K_3}{K_1y_2^2+K_4y_2+K_5}\right)^{r_1},
\end{aligned}$$

with

$$
\begin{aligned}
P_1(y_1) = \; & y_2^2\left(s_1^2+s_3^2\right)(s_1(s_4-s_6)-s_2s_3-s_3s_5)-2y_2\left(s_1^2+s_3^2\right)(s_2s_6+s_4s_5)-s_1s_2^2s_6-s_1s_4^2s_6+s_1 \\
& s_4s_5^2+s_1s_4s_6^2-s_2^2s_3s_5-s_2s_3s_5^2-s_2s_3s_6^2-s_3s_4^2s_5, \\
P_2(y_2) = \; & y_2^4\left(K_1^2r_2t_1t_2+K_1^2r_2t_1t_3+K_1K_2r_1t_1^2-K_1K_4r_1t_1^2\right)+y_2^3\Big(K_1K_2r_1t_1t_2-K_1K_2r_1t_1t_3+K_1 \\
& K_2r_2t_1t_2+K_1K_2r_2t_1t_3+2K_1K_3r_1t_1^2-K_1K_4r_1t_1t_2+K_1K_4r_1t_1t_3+K_1K_4r_2t_1t_2+K_1K_4 \\
& r_2t_1t_3-2K_1K_5r_1t_1^2\Big)+y_2^2\Big(-K_1K_2r_1t_2t_3+2K_1K_3r_1t_1t_2-2K_1K_3r_1t_1t_3+K_1K_3r_2t_1t_2 \\
& +K_1K_3r_2t_1t_3+K_1K_4r_1t_2t_3-2K_1K_5r_1t_1t_2+2K_1K_5r_1t_1t_3+K_1K_5r_2t_1t_2+K_1K_5r_2t_1t_3 \\
& +K_2K_4r_2t_1t_2+K_2K_4r_2t_1t_3-K_2K_5r_1t_1^2+K_3K_4r_1t_1^2\Big)+y_2(-2K_1K_3r_1t_2t_3+2K_1K_5r_1t_2 \\
& -K_2K_5r_1t_1t_2+K_2K_5r_1t_1t_3+K_2K_5r_2t_1t_2+K_2K_5r_2t_1t_3+K_3K_4r_1t_1t_2-K_3K_4r_1t_1t_3 \\
& +K_3K_4r_2t_1t_2+K_3K_4r_2t_1t_3)+K_2K_5r_1t_2t_3-K_3K_4r_1t_2t_3+K_3K_5r_2t_1t_2+K_3K_5r_2t_1t_3.
\end{aligned}
$$

We denote by $(C_{\tilde{f}_2})$ and $(C_{\tilde{g}_2})$ the graphics of $\tilde{f}_2$ and $\tilde{g}_2$, respectively.

According with the sign of $\left(\tilde{f}_2\right)'(y_2)$ which depends on m_1 and with the sign of $\delta_1 = (s_1s_4 - s_2s_3)(s_1s_6+s_3s_5)$, we obtain that all the possible topologically distinct graphics $(C_{\tilde{f}_2})$ of the function $\tilde{f}_2(y_2)$ are given in what follows.

For $s_5 \neq -s_2$ the function $\tilde{f}_2(y_2)$ can have two vertical asymptotes and all the distinct topologically equivalent graphics of the function $\tilde{f}_2(y_2)$ are given in Figure 2.26 as follows.

If $\delta_1 > 0$ the graphics $(C_{\tilde{f}_2})$ are given in (a), or (b), or (c), or (d), or (e), or (f) of Figure 2.26.

If $\delta_1 < 0$ the function $\tilde{f}_2(y_2)$ has one graphic shown in Figure 2.26(g).

If $\delta_1 = 0$ the function $\tilde{f}_2(y_2)$ has one graphic shown in Figure 2.26(h).

For $s_5 = -s_2$ the graphic $(C_{\tilde{f}_2})$ has only one vertical asymptote, then the graphics $(C_{\tilde{f}_2})$ vary according to the sign of δ_1.

If $\delta_1 \leq 0$, the function $\tilde{f}_2(y_2)$ has one graphic given in Figure 2.26(h).

If $\delta_1 > 0$, the graphics are given in (i), or (j), or (k) of Figure 2.26.

According with the derivative $\left(\tilde{g}_2\right)'(y_2)$ and the parameters r_1, r_2, and due to the fact that the sign of the discriminants Δ_1 and Δ_2 of the equations $K_1y_2^2+K_2y_2+K_3=0$, and $K_1y_2^2+K_4y_2+K_5=0$, respectively, are negative, where $\Delta_1=\Delta_2=-L^2(b\gamma_1\delta_2-b\gamma_2\delta_1-\gamma_1)^2$, and knowing the different kind of the roots x_i with $i\in\{1,\dots,4\}$ of the polynomial $P_2(y_2)$, we get all the possible topologically distinct graphics $(C_{\tilde{g}_2})$ of the function $\tilde{g}_2(y_2)$ which are given in Figures 2.14, 2.20, 2.21, 2.22, 2.23 and 2.24 if $\Delta_1 = 0$ as we proved in the first case of *statements (I) and (II). For* $\Delta_1 < 0$ *the topologically distinct possible graphics of* $\tilde{g}_2(y_2)$ *are given in what follows.*

If either r_2 *is an even integer or* $r_2 = (2p)/(2q+1)$ *with* $p, q \in \mathbb{Z}$*, and* $P_2(y_2)$ *has four simple real roots, then the graphics of* $\tilde{g}_2(y_2)$ *are given in (a), or (b), or (c), or (d), or (e), or (f) of Figure 2.27.*

If $P_2(y_2)$ *has two complex and two simple real roots, then the graphics of* $\tilde{g}_2(y_2)$ *are given in (g), or (h), or (i), or (j) of Figure 2.27.*

If $P_2(y_2)$ *has four complex roots, then the graphics of* $\tilde{g}_2(y_2)$ *are shown in (k), or (l) of Figure 2.27.*

If $P_2(y_2)$ *has one double and two complex roots, then the graphics of* $\tilde{g}_2(y_2)$ *are given in (a), or (b), or (c) of Figure 2.28.*

If $P_2(y_2)$ *has two double real roots, then the graphics of* $\tilde{g}_2(y_2)$ *are shown in (d), or (e), or (f), or (g) of Figure 2.28.*

If $P_2(y_2)$ *has one triple and one simple real root, or one double and two simple real roots, then the graphic of* $\tilde{g}_2(y_2)$ *is given in (h) of Figure 2.28.*

If r_2 *is an odd integer or* $r_2 = (2p+1)/(2q+1)$ *with* $p, q \in \mathbb{Z}$ *we have the same graphics as the case when* r_2 *is an even integer where* $x_0 = -(t_2/t_1)$ *represents an inflexion point of the function* $\tilde{g}_2(y_2)$*.*

if r_2 *is irrational or* $r_2 = p/(2q)$ *with* $p, q \in \mathbb{Z}$ *the sign of the derivative* $\left(\tilde{g}_2\right)'(y_2)$ *depends only on the sign of* $P_2(y_2)$*, then the possible graphics of the function* $\tilde{g}_2(y_2)$ *are the same as the ones of the case where* r_2 *is an odd integer on its definition domain.*

For $r_2 < 0$ *and by a similar way we find the same graphics as in the case* $r_2 > 0$*.*

The graphics of $\tilde{f}_2(y_2)$ *are given in Figure 2.15 and the graphics of* $\tilde{g}_2(y_2)$ *are given in Figures 2.14, 2.20, 2.21, 2.22, 2.23, 2.24, 2.27 or 2.28.*

For the function $\tilde{g}_2(y_2)$ *we remark that its graphics can have at most five local extremes in (a), or (b), or (c), or (d), or (e) or (f) of Figure 2.27, we know also that the function* $\tilde{f}_2(y_2)$ *is positive and its graphics can have at most two extremes in (a), or (b), or (c), or (d), or (f), or (i), or (j) or (k) of Figure 2.15. Therefore we guarantee that the maximum number of intersection points between the graphics of* $\tilde{f}_2(y_2)$ *and* $\tilde{g}_2(y_2)$ *can be precisely between (a), or (b), or (c), or (d), or (e) or (f) of Figure 2.27 and (a), or (b), or (c), or (d), or (f), or (i), or (j) or (k) of Figure 2.15. Due to the fact that there are no intersection points at infinity because of the common horizontal asymptote* $\tilde{f}_2(y_2) = \tilde{g}_2(y_2) = 1$*. Then the maximum number of solutions of system (2.17) is at most seven see for example Figure 2.8, this provides at most* *three limit cycles of the discontinuous piecewise differential system* (1.4)–(2.2).

Now we construct an example with exactly seven intersections points between the two functions $\tilde{f}_2(y_2)$ *and* $\tilde{g}_2(y_2)$ *by taking* $\{K_1, K_2, K_3, K_4, K_5, t_1, t_2, t_3, r_1, r_2, s_1, s_2, s_3, s_4, s_5, s_6, m_1\} \to \{0.576282, -3.2, 4.1, 0.1, 0.02, -5, 5.4, 93, 1, 2, 1.2, 4, 1, 100, 5, 2, -3.8.\}$, *see Figure* 2.8.

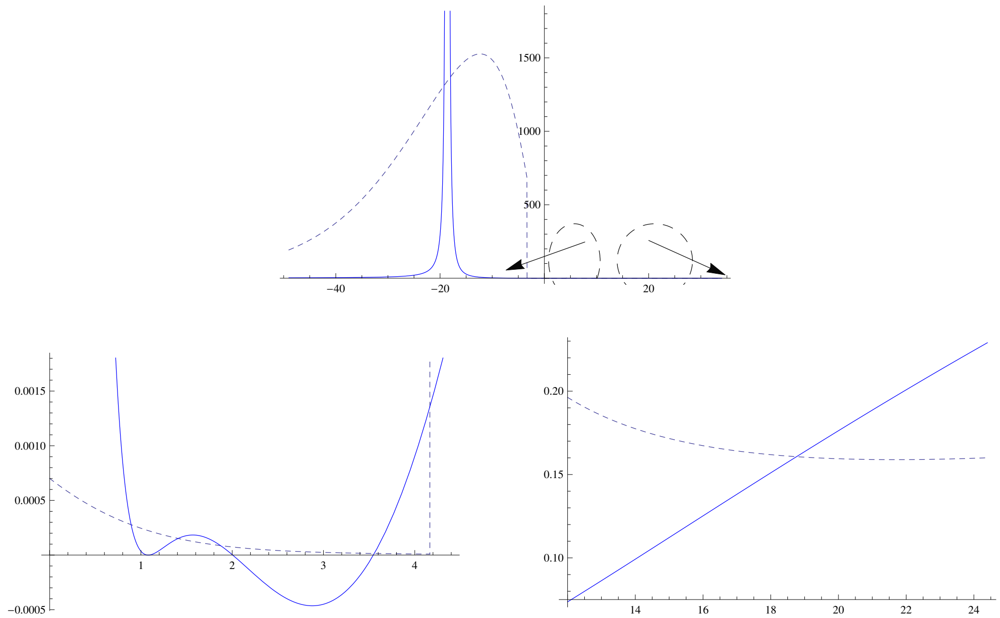


Figure 2.8: The seven intersection poitns between the functions $\tilde{f}_2(y_2)$ drawn in continuous line and $\tilde{g}_2(y_2)$ drawn in dashed line.

Case 3. If $AbC\Delta \neq 0$ *and* $a = 0$ *with* $\Delta = 4b(A+b)+C^2 > 0$, *then* $k = 3$ *and* $j = 2$ *in system* (2.17), *the first integral of the quadratic center* (2.2) *is* $H_3^{(2)}(x,y)$ *given in* (2.9). *Then the solutions* y_2 *satisfying* $F(y_2) = 0$ *are equivalent to the ones of the equation* $\tilde{f}_3(y_2) = \tilde{g}_3(y_2)$ *with* $\tilde{f}_3(y_2) = f_1(y_2)$ *and* $\tilde{g}_3(y_2) = \tilde{f}_1(y_2)$, *where*

$$
\begin{aligned}
m_1 &= \frac{1}{2}\delta_1\left(C+\sqrt{4b(A+b)+C^2}\right) - b\delta_2 + 1, \quad m_2 = b\gamma_2 - \frac{1}{2}\gamma_1\left(C+\sqrt{4b(A+b)+C^2}\right),\\
m_3 &= \frac{1}{2}\left(C+\sqrt{4b(A+b)+C^2}\right)\left(\frac{8\alpha\gamma_1\sigma_1}{4\beta^2+\omega^2}+\delta_1\right) - \frac{8\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2} - b\delta_2 + 1,\\
n_1 &= \frac{1}{2}\delta_1\left(C-\sqrt{4b(A+b)+C^2}\right) - b\delta_2 + 1, \quad n_2 = b\gamma_2 - \frac{1}{2}\gamma_1\left(C-\sqrt{4b(A+b)+C^2}\right),\\
n_3 &= \frac{1}{2}\left(C-\sqrt{4b(A+b)+C^2}\right)\left(\frac{8\alpha\gamma_1\sigma_1}{4\beta^2+\omega^2}+\delta_1\right) - \frac{8\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2} - b\delta_2 + 1,\\
r_1 &= \frac{1}{2b}\left(1+\frac{C}{\sqrt{4b(A+b)+C^2}}\right), \quad r_2 = \frac{1}{2b}\left(1-\frac{C}{\sqrt{4b(A+b)+C^2}}\right),\\
L_1 &= A\left(\frac{8\alpha\gamma_2\sigma_1}{4\beta^2+\omega^2}+\delta_2\right)+1, \quad L_2 = A\gamma_2, \quad L_3 = A\delta_2 + 1, \quad r = \frac{1}{A}.
\end{aligned}
$$

The graphics of the function $\tilde{g}_3(y_2)$ are shown in Figures 2.14, 2.20, 2.21, 2.22, 2.23 or 2.24. For the function $\tilde{f}_3(y_2)$ all its graphics are given in Figure 2.14.

It is obvious that the graphics of $\tilde{f}_3(y_2)$ can have at most four local extrem in Figure 2.20 and due to the fact that the function $\tilde{g}_3(y_2)$ is positive and its graphics can have at most one extremes in (a) or (b) of Figure 2.14, we guarantee that the maximum number of intersection points between the graphics of $\tilde{f}_3(y_2)$ and $\tilde{g}_3(y_2)$ take place between Figure 2.20 and (a) or (b) of Figure 2.14. In this case we know that the two functions have the common horizontal asymptote $\tilde{f}_3(y_2) = \tilde{g}_3(y_2) = 1$. Hence the maximum number of intersection points between these two functions is at most seven see for example Figure 2.9. Due to the symmetry of solutions of system (2.17) we conclude that the maximum number of limit cycles is at most three.

In Figure 2.9 we build an example that shows exactly seven points of intersection between the two functions $\tilde{f}_3(y_2)$ and $\tilde{g}_3(y_2)$ by choosing $\{m_1, m_2, m_3, r_1, r_2, n_1, n_2, n_3, L_1, L_2, L_3, r\} \to \{7, -1, 2, 2, 6, -0.53, -0.17, -1.71, 4, -3.25, -0.15, 2\}$.

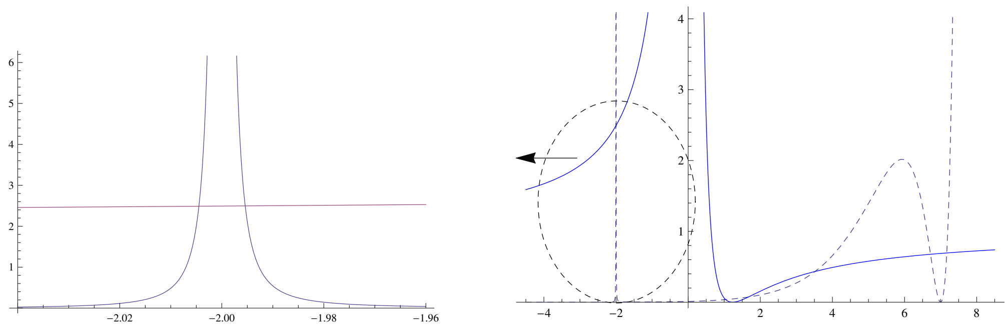


Figure 2.9: The seven intersection poitns between the functions $\tilde{f}_3(y_2)$ drawn in continuous line and $\tilde{g}_3(y_2)$ drawn in dashed line.

Case 4. If $b = C = 0$ then $k = 4$ and $j = 2$ in system (2.17), where (2.10) is the first integral of the quadratic center (2.2). Then the solutions of $F(y_2) = 0$ are the solutions of the equation $\tilde{f}_4(y_2) = \tilde{g}_4(y_2)$, where
$\tilde{f}_4(y_2) = \left(\dfrac{m_0\ y_2^2 + m_1\ y_2 + m_2}{m_0\ y_2^2 + n_1\ y_2 + n_2}\right)^r$ *and* $\tilde{g}_4(y_2) = e^{k_1 + k_2 y_2 + 2A\left(\tanh^{-1}\left(\frac{S_1 y_2 + S_2}{S_1 y_2 + S_4}\right) - \tanh^{-1}\left(\frac{S_5 - S_1 y_2}{S_6 - S_1 y_2}\right)\right)}$. *Where*

$$
\begin{aligned}
m_0 = &\ \ a^2\gamma_1^2 - a^2\gamma_2^2 + aA\gamma_1\gamma_2, \quad m_1 = 2a^2\gamma_1\delta_1 - 2a^2\gamma_2\delta_2 + aA\gamma_1\delta_2 + aA\gamma_2\delta_1 + 2a\gamma_1 + A\gamma_2, \\
m_2 = &\ \ a^2\delta_1^2 - a^2\delta_2^2 + aA\delta_1\delta_2 + 2a\delta_1 + A\delta_2 + 1, \\
n_1 = &\ -a(\gamma_1(2a\delta_1 + A\delta_2 + 2) + \gamma_2(A\delta_1 - 2a\delta_2)) - \frac{16a\alpha\sigma_1(a(\gamma_1^2 - \gamma_2^2) + A\gamma_1\gamma_2)}{4\beta^2 + \omega^2} - A\gamma_2,
\end{aligned}
$$

$$
\begin{aligned}
n_2 = {} & a^2\delta_1^2 - a^2\delta_2^2 + \frac{1}{\left(4\beta^2+\omega^2\right)^2}\left(64a\alpha^2\sigma_1^2(a(\gamma_1-\gamma_2)(\gamma_1+\gamma_2)+A\gamma_1\gamma_2)\right) + \frac{1}{4\beta^2+\omega^2}(8\alpha \\
& \sigma_1((a\delta_1+1)(2a\gamma_1+A\gamma_2)+a\delta_2(A\gamma_1-2a\gamma_2))) + aA\delta_1\delta_2 + 2a\delta_1 + A\delta_2 + 1, \\
r = {} & -\sqrt{4a^2+A^2}, \;\; k_1 = \frac{(16a\beta^2\gamma_1+4a\gamma_1\omega^2)\sqrt{4a^2+A^2}}{4\beta^2+\omega^2}, \;\; k_2 = -\frac{16a\alpha\gamma_1\sigma_1\sqrt{4a^2+A^2}}{4\beta^2+\omega^2}, \\
S_1 = {} & aA\gamma_1 - 2a^2\gamma_2, \;\; S_2 = -2a^2\delta_2 + aA\delta_1 + A, \;\; S_3 = a\gamma_1\sqrt{4a^2+A^2}, \\
S_4 = {} & a\delta_1\sqrt{4a^2+A^2} + \sqrt{4a^2+A^2}, \;\; S_5 = -\frac{16\alpha a^2\gamma_2\sigma_1}{4\beta^2+\omega^2} - 2a^2\delta_2 + \frac{8\alpha aA\gamma_1\sigma_1}{4\beta^2+\omega^2} + aA\delta_1 + A, \\
S_6 = {} & \frac{8a\alpha\gamma_1\sqrt{4a^2+A^2}}{4\beta^2+\omega^2} + a\delta_1\sqrt{4a^2+A^2} + \sqrt{4a^2+A^2}.
\end{aligned}
$$

We see that the discriminants of the numerator and the denominator of $\tilde{f}_4(y_2)$ are equal, and equal to $\Delta = \left(4a^2+A^2\right)(a\gamma_1\delta_2 - a\gamma_2\delta_2 - \gamma_2)^2$. Since $\Delta \geq 0$ we obtain that all the possible topologically different graphics $(C_{\tilde{f}_4})$ of the function $\tilde{f}_4(y_2)$ are given in Figures 2.14, 2.20, 2.21, 2.22, 2.23 or 2.24 as we proved in the first cases of statements (I) and (II).
Now we study the function $\tilde{g}_4(y_2)$ where its derivative is $\left(\tilde{g}_4\right)'(y_2) = \tilde{g}_4(y_2)\dfrac{P_1(y_2)}{P_2(y_2)}$, with

$$
\begin{aligned}
P_1(y_2) = {} & -2AS_1^3S_4y_2^2 - 2AS_1^3S_6y_2^2 + 2AS_1^2S_2S_3y_2^2 - 4AS_1^2S_2S_6y_2 + 2AS_1^2S_3S_5y_2^2 + 4AS_1^2S_4 \\
& S_5y_2 - 2AS_1S_2^2S_6 + 2AS_1S_3^2S_4y_2^2 + 2AS_1S_3^2S_6y_2^2 + 2AS_1S_4^2S_6 - 2AS_1S_4S_5^2 + 2AS_1 \\
& S_4S_6^2 + 2AS_2^2S_3S_5 - 2AS_2S_3^3y_2^2AS_2S_3^2S_6y_2 + 2AS_2S_3S_5^2 - 2AS_2S_3S_6^2 - 2AS_3^3S_5y_2^2 \\
& -2AS_3S_4^2S_5 + k_2S_1^4y_2^4 + 2k_2S_1^3S_2y_2^3 - 2k_2S_1^3S_5y_2^3 + k_2S_1^2S_2^2y_2^2 - 4k_2S_1^2S_2S_5y_2^2 - 2 \\
& k_2S_1^2S_3^2y_2^4 - 2k_2S_1^2S_3S_4y_2^3 + 2k_2S_1^2S_3S_6y_2^3 - k_2S_1^2S_4^2y_2^2 + k_2S_1^2S_5^2y_2^2 - 2k_2S_1S_2S_3^2y_2^3 \\
& -k_2S_1^2S_6^2y_2^2 - 2k_2S_1S_2^2S_5y_2 + 4k_2S_1S_2S_3S_6y_2^2 + 2k_2S_1S_2S_5^2y_2 - 2k_2S_1S_2S_6^2y_2 + 2 \\
& k_2S_1S_3^2S_5y_2^3 + 4k_2S_1S_3S_4S_5y_2^2 + 2k_2S_1S_4^2S_5y_2 - k_2S_2^2S_3^2y_2^2 + 2k_2S_2^2S_3S_6y_2 + k_2S_2^2 \\
& -k_2S_2^2S_6^2 + k_2S_3^4y_2^4 + 2k_2S_3^3S_4y_2^3 - 2k_2S_3^3S_6y_2^3 + k_2S_3^2S_4^2y_2^2 - 4k_2S_3^2S_4S_6y_2^2 - k_2S_3^2y_2^2 \\
& +k_2S_3^2S_6^2y_2^2 + k_2S_4^2S_6^2 - 2k_2S_3S_4^2S_6y_2 - 2k_2S_3S_4S_5^2y_2 + 2k_2S_3S_4S_6^2y_2 - 4AS_3^2S_4S_5y_2 \\
& -k_2S_4^2S_5^2, \\
P_2(y_2) = {} & (y_2(S_1-S_3)+S_2-S_4)(y_2(S_1+S_3)+S_2+S_4)(y_2(S_3-S_1)+S_5-S_6)(-y_2(S_1+S_3) \\
& +S_5+S_6).
\end{aligned}
$$

According with the sign of $\left(\tilde{g}_4\right)'(y_2)$ and the kind of roots of the quartic polynomial $P_1(y_2)$ and with their position with respect to the two vertical asymptotes $y_{21} = \dfrac{S_5+S_6}{S_1+S_3}$ and $y_{22} =$

$\frac{S_4 - S_2}{S_1 - S_3}$, we give all the possible topologically different graphics of the function $\tilde{g}_4(y_2)$ in what follows.

If $P_1(y_2)$ has four simple real roots the graphics of $\tilde{g}_4(y_2)$ are given in (a), or (b), or (c), or (d), or (e), or (f), or (g), or (h), or (i) of Figure 2.30.

If $P_1(y_2)$ has two simple real roots and two complex roots the graphics of $\tilde{g}_4(y_2)$ are given in (j), or (k), or (l) of Figure 2.30, or in (a) of Figure 2.29.

If $P_1(y_2)$ has four complex roots the unique graphic of $\tilde{g}_4(y_2)$ is shown in (b) of Figure 2.29.

If $P_1(y_2)$ has one triple and one simple real root the unique graphic of $\tilde{g}_4(y_2)$ is shown in (c) of Figure 2.29.

If $P_1(y_2)$ has one double real root and two complex roots the unique graphic of $\tilde{g}_4(y_2)$ is shown in (d) of Figure 2.29.

If $P_1(y_2)$ has one double and two simple real roots the graphics of $\tilde{g}_4(y_2)$ are shown in (e), or (f), or (g) of Figure 2.29.

If $P_1(y_2)$ has one real root of order four or two double real roots the unique graphic of $\tilde{g}_4(y_2)$ is shown in (h) of Figure 2.29.

Since the graphics of $\tilde{f}_4(y_2)$ can have at most four local extremes in Figure 2.20 and by knowing that the function $\tilde{g}_4(y_2)$ is positive and its graphics can have at most four local extremes in (a), or (b), or (c), or (d), or (e), or (f), or (g), or (h) of Figure 2.30. Therefore we conclude that the maximum number of intersection points between the graphics of $\tilde{f}_4(y_2)$ and $\tilde{g}_4(y_2)$ can be precisely between Figure 2.20 and (a), or (b), or (c), or (d), or (e), or (f), or (g), or (h) of Figure 2.30. It results that the maximum number of intersection points between the functions $\tilde{f}_4(y_2)$ and $\tilde{g}_4(y_2)$ is at most seven see for example Figure 2.10. Due to symmetry of solutions (y_1, y_2) of (2.17), we know that the maximum number of limit cycles of the discontinous piecewise differential system (1.4)–(2.2) is at most three.

In figure 2.10 we build an example that shows exactly seven intersection points between the functions $\tilde{f}_4(y_2)$ and $\tilde{g}_4(y_2)$ by considering $\{m_0, m_1, m_2, n_1, n_2, r, k_1, k_2, A, S_1, S_2, S_3, S_4, S_5, S_6\} \to \{1, -1.23283, 0.265293, -1.65, 6.63, 4, 5, -4.5, 4, 4, -0.5, 0.5, 1, 0.5, 9.4\}$.

Case 5. If $A = a = 0$, $C \neq 0$ and $b \neq 0$, then $k = 5$ and $j = 2$ in system (2.17), where (2.2) has the first integral $H_5^{(2)}(x, y)$ given in (2.11). Studying the solutions of $F(y_2) = 0$ is equivalent to study the solutions of the equation $\tilde{f}_5(y_2) = \tilde{g}_5(y_2)$ where $\tilde{f}_5(y_2) = f_2(y_2)$ and $\tilde{g}_5(y_2) = \tilde{f}_1(y_2)$,

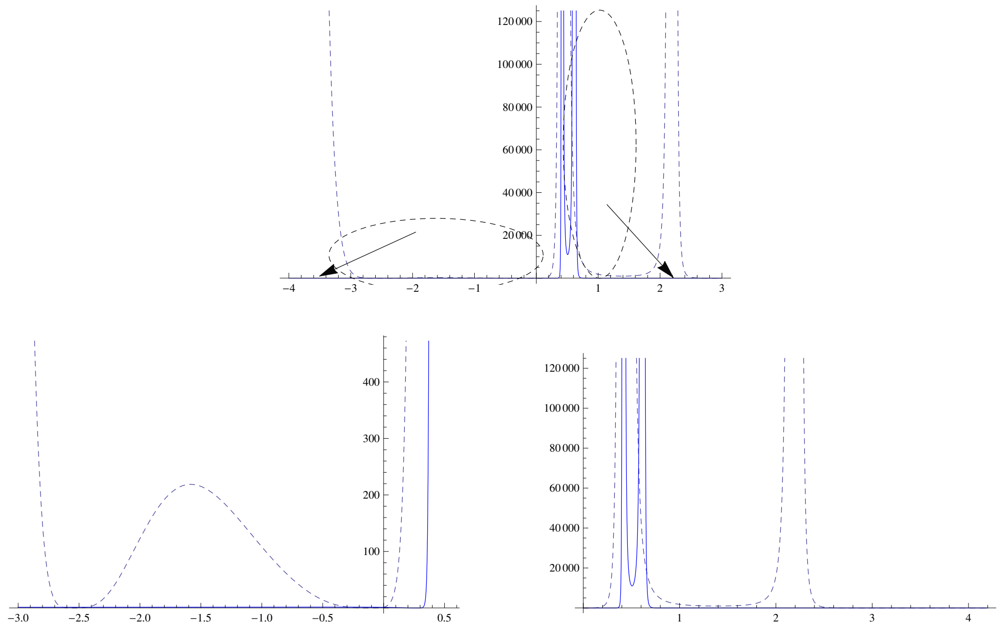


Figure 2.10: The seven intersection poitns between the functions $\tilde{f}_4(y_2)$ drawn in continuous line and $\tilde{g}_4(y_2)$ drawn in dashed line.

with

$$
\begin{aligned}
m_1 =& \ \frac{1}{2}\delta_1\left(C+\sqrt{4b^2+C^2}\right)-b\delta_2+1, \ \ m_2=\frac{1}{2}\gamma_1\left(C+\sqrt{4b^2+C^2}\right)-b\gamma_2, \\
m_3 =& \ \frac{4\alpha\gamma_1\sigma_1}{4\beta^2+\omega^2}\left(C+\sqrt{4b^2+C^2}\right)+\frac{1}{2}\delta_1\left(C+\sqrt{4b^2+C^2}\right)-\frac{8\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2}-b\delta_2+1, \\
n_1 =& \ \frac{1}{2}\delta_1\left(C-\sqrt{4b^2+C^2}\right)-b\delta_2+1, \ \ n_2=\frac{1}{2}\gamma_1\left(C-\sqrt{4b^2+C^2}\right)-b\gamma_2, \\
n_3 =& \ \frac{4\alpha\gamma_1\sigma_1}{4\beta^2+\omega^2}\left(C-\sqrt{4b^2+C^2}\right)+\frac{1}{2}\delta_1\left(C-\sqrt{4b^2+C^2}\right)-\frac{8\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2}-b\delta_2+1, \\
r_1 =& \ \frac{1}{2b}\left(1+\frac{C}{\sqrt{4b^2+C^2}}\right), \ \ r_2=\frac{1}{2b}\left(1-\frac{C}{\sqrt{4b^2+C^2}}\right), \ \ l_0=-2\gamma_2, \ \ l_1=\frac{8\alpha\gamma_2\sigma_1}{4\beta^2+\omega^2}.
\end{aligned}
$$

We know that the graphics of $\tilde{f}_5(y_2)$ are shown in Figure 2.15, and the graphics of $\tilde{g}_5(y_2)$ are shown in Figures 2.14, 2.20, 2.21, 2.22, 2.23 and *2.24.*

As in the previous case we ensure that the graphics $(C_{\tilde{g}_5})$ shown in Figure 2.20 are the ones that guarentee the maximum number of intersection points between the graphics of the functions $g_5(y_2)$ and $f_5(y_2)$ which has the horizontal asymptote $\tilde{f}_5(y_2)=0$. Then we guarantee that the maximum number of intersection points between the graphics (C_{f_5}) and (C_{g_5}) takes place between Figure 2.15 and Figure 2.20. Thus the maximum number of the intersection points of these graphics is at most seven, which provides at most three limit cycles of the discontinuous piecewise differential system (1.4)–(2.2).

In what follows we build an example provides the seven intersection points between the tow functions $\tilde{f}_5(y_2)$ and $\tilde{g}_5(y_2)$ when we consider $\{m_1, m_2, m_3, n_1, n_2, n_3, r_1, r_2, l_0, l_1\} \to \{5, -1.3, 3, -0.5, -0.1703, -1.1, -2, -4, 2.19, -0.18\}$, *these points are shown in Figure 2.11.*

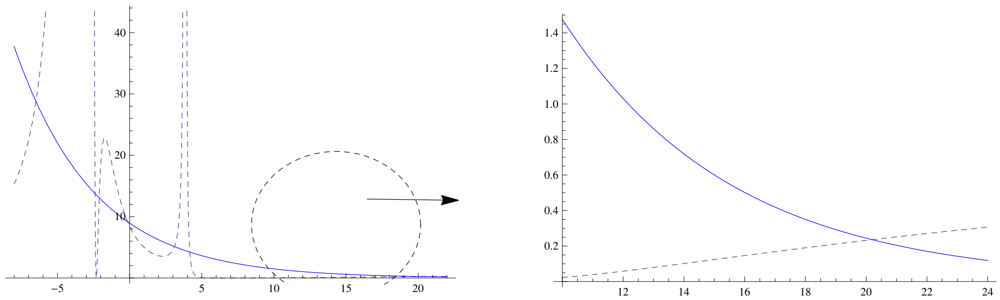


Figure 2.11: The seven intersection poitns between graphics of the functions $\tilde{f}_5(y_2)$ drawn in continuous line and $\tilde{g}_5(y_2)$ drawn in dashed line.

Case 6. If $b = a = 0$, $C \neq 0$ and $A \neq 0$, then $k = 6$ and $j = 2$ in system (2.17), the first integral of (2.2) is $H_6^{(2)}(x, y)$ given in (2.12). The equation $F(y_2) = 0$ is equivalent to the equation $\tilde{f}_6(y_2) = \tilde{g}_6(y_2)$ where $\tilde{f}_6(y_2) = f_2(y_2)$, $\tilde{g}_6(y_2) = \tilde{f}_1(y_2)$,

and

$$l_0 = -\frac{16\alpha AC\sigma_1}{4\beta^2 + \omega^2}(A\gamma_1 + \gamma_2 C),\ \ l_1 = 4AC(A\gamma_1 + \gamma_2 C),\ \ r_1 = 2C^2,\ \ r_2 = 2A^2,\ \ m_1 = A\delta_2 + 1,$$

$$m_2 = A\gamma_2,\ \ m_3 = \frac{8\alpha A\gamma_2\sigma_1}{4\beta^2 + \omega^2} + A\delta_2 + 1,\ \ n_1 = C\delta_1 + 1,\ \ n_2 = \gamma_1 C,\ \ n_3 = \frac{8\alpha\gamma_1 C\sigma_1}{4\beta^2 + \omega^2} + C\delta_1 + 1.$$

The graphics of $\tilde{f}_6(y_2)$ are given in Figure 2.15 and the graphics of $\tilde{g}_6(y_2)$ are given in Figures 2.14, 2.20, 2.21, 2.22, 2.23 or 2.24. Then the maximum number of solutions of system (2.17) is at most seven which provides at most three limit cycles of the discontinuous piecewise differential system (1.4)–(2.2).

Since the maximum number of limit cycles of all these six cases is at most three, we will build only one example with three limit cycles of the discontinuous piecewise differential system (1.4)–(2.2) of type $b + d = 0$ with $A = a = 0$, $C \neq 0$ and $b \neq 0$.

Three limit cycles of the discontinuous piecewise differential system (1.4)–(2.2) of type $b + d = 0$ with $A = a = 0$, $C \neq 0$ and $b \neq 0$.

In the half-plane Σ^+ we consider the quadratic center

$$\begin{aligned}\dot{x} \simeq\ & -0.0273949x^2 + 3.3482 + x(0.0133241y + 3.85736) + (-0.001506y \\ & -0.845815)y,\end{aligned} \tag{2.22}$$

$$\dot{y} \simeq \quad -0.273949x^2 + x(0.133241y + 18.5736) + (-0.0150606y - 3.7726)y + 25.7332,$$

with its corresponding first integral

$$H_6^{(2)}(x,y) \simeq \quad -0.0000384239e^{\frac{1}{200}(y-10x)}(x - 0.30772y - 34.9503)(x - 0.17866y + 27.1508)^2.$$

In the half-plane Σ^- we consider the linear differential center

$$\dot{x} = \frac{1}{5}x - \frac{29}{120}y + \frac{19}{20}, \quad \dot{y} = \frac{6}{5}x - \frac{1}{5}y + \frac{11}{5}, \tag{2.23}$$

with the first integral $\quad H(x,y) = 4\left(\frac{6}{5}x - \frac{1}{5}y\right)^2 + \frac{48}{5}\left(\frac{11}{5}x - \frac{19}{20}y\right) + y^2.$

In this case system (2.17) *has the three solutions* $(y_1, y_2) \simeq \{(0.592968, 7.2691), (1.31716, 6.54491), (2.34295, 5.51911)\}$ *which provide the three limit cycles for the discontinuous piecewise differential system* (2.22)–(2.23)*, see Figure* 2.2(*b*).

Case 7. If $a = 0 \neq C$ and $\Delta = 4b(A+b) + C^2 = 0$, then $k = 7$ and $j = 2$ in system (2.17)*, where* (2.13) *is the first integral of the quadratic center* (2.2)*. Then the solutions of $F(y_2) = 0$ are the ones of the equation $\tilde{f}_7(y_2) = \tilde{g}_7(y_2)$ where $\tilde{f}_7(y_2) = \tilde{f}_1(y_2)$, $\tilde{g}_7(y_2) = \tilde{g}_1(y_2)$, and*

$$\begin{aligned}
m_1 = &\quad \frac{8\alpha\sigma_1}{4\beta^2 + \omega^2}(\gamma_1 C - 2b\gamma_2) - 2b\delta_2 + C\delta_2 + 2, \;\; m_2 = 2b\gamma_2 - \gamma_1 C, \;\; m_3 = -2b\delta_2 + C\delta_2 + 2, \\
r_1 = &\quad 1, \;\; r_2 = -\frac{4b^2}{4b^2 + C^2}, \;\; n_1 = 1 - \frac{\left(4b^2 + C^2\right)}{4b\left(4\beta^2 + \omega^2\right)}\left(\delta_2\left(4\beta^2 + \omega^2\right) + 8\alpha\gamma_2\sigma_1\right), \\
n_2 = &\quad \frac{\gamma_2}{4b}\left(4b^2 + C^2\right), \;\; n_3 = 1 - \frac{\delta_2}{4b}\left(4b^2 + C^2\right), \\
k_0 = &\quad -\frac{16\alpha C\sigma_1}{4\beta^2 + \omega^2}(b\gamma_1\delta_2 - b\gamma_2\delta_2 - \gamma_1), \;\; k_1 = 4C(b\gamma_1\delta_2 - b\gamma_2\delta_2 - \gamma_1).
\end{aligned}$$

Since $r_1 = 1$, the graphics of the function $\tilde{f}_7(y_2)$ are given in (*h*), (*i*), (*j*), (*k*) *or* (*l*) *of Figure* 2.21 *and in Figures* 2.14, 2.22, 2.23 *or* 2.24*. All the graphics of the function $\tilde{g}_7(y_2)$ are shown in Figure* 2.25.

Therefore the graphics of the two functions $\tilde{f}_7(y_2)$ and $\tilde{g}_7(y_2)$ intersect at most in five points see for example Figure 2.12*. Consequently, the maximum number of limit cycles of the discontinuous piecewise differential system* (1.4)–(2.2) *is at most two.*

Now we construct an example with exactly five intersection points between $\tilde{f}_7(y_2)$ and $\tilde{g}_7(y_2)$ by taking $\{m_1, m_2, m_3, r_1, r_2, n_1, n_2, n_3, k_0, k_1\} \to \{-0.5, 7.2, -2, 1, -2, 0.12, -9.5, -0.3, 1.2, 0\}$. These points are shown in Figure 2.12.

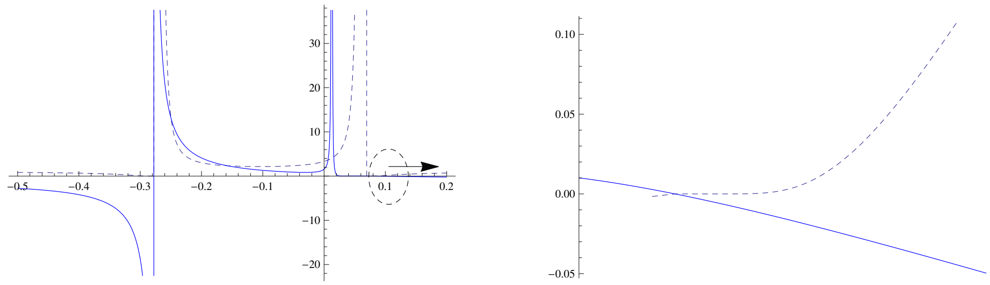


Figure 2.12: The five intersection poitns between the functions $\tilde{f}_7(y_2)$ drawn in continuous line and $\tilde{g}_7(y_2)$ drawn in dashed line.

Case 8. If $A = b = 0$ and $a = 0 \neq C$, then $k = 8$ and $j = 2$ in system (2.17), the first integral of (2.2) is $H_8^{(2)}(x, y)$ given in (2.14), and the solutions of $F(y_2) = 0$ are the same as the solutions of the equation $\tilde{f}_8(y_2) = \tilde{g}_8(y_2)$ where

$$\tilde{f}_8(y_2) = M^2 f_2(y_2), \;\; \tilde{g}_8(y_2) = f_1(y_2), \;\; \textit{with} \;\; M = \frac{1}{4\beta^2 + \omega^2},$$

and

$$l_0 = \;\; -M^2(16\alpha C\sigma_1)\Big(\big(4\beta^2 + \omega^2\big)(\gamma_1 + \gamma_2 C\delta_2) + 4\alpha\gamma_2^2 C\sigma_1\Big), \;\; L_3 = C\left(\frac{8\alpha\gamma_1\sigma_1}{4\beta^2 + \omega^2} + \delta_1\right) + 1,$$

$$l_1 = \;\; M(4C\Big(\big(4\beta^2 + \omega^2\big)(\gamma_1 + \gamma_2 C\delta_2) + 4\alpha\gamma_2^2 C\sigma_1\Big)), \;\; L_1 = C\delta_1 + 1, \;\; L_2 = \gamma_1 C, \;\; r = 2.$$

The graphics of $\tilde{f}_8(y_2)$ are given in Figure 2.15. Since $\tilde{g}_8(y_2)$ is a sub-case of $f_1(y_2)$ with the particular parameters given previously, then its graphics are shown in Figures 2.14(a) and 2.14(b). Clearly that the maximum number of intersection points of their corresponding graphics is at most three see for example Figure 2.13. Then the upper bound of the number of limit cycles in this case is at most one.

In what follows we consider $\{l_0, l_1, L_1, L_2, L_3, r, M\} \to \{-0.2, -1.75, 1, -5.6, -0.4, 2, 1.53\}$ for building an example with exactly three intersection points between the two functions $\tilde{f}_8(y_2)$ and $\tilde{g}_8(y_2)$, see Figure 2.13. Now we will prove that the result of case 8 is reached by giving an example of system (1.4)–(2.2) of type $b + d = 0$ with $A = b = 0$ and $a = 0 \neq C$, with exactly one limit cycle.

One limit cycle of the PWS (1.4)–(2.2) of type $b + d = 0$ with $A = b = 0$ and $a = 0 \neq C$.

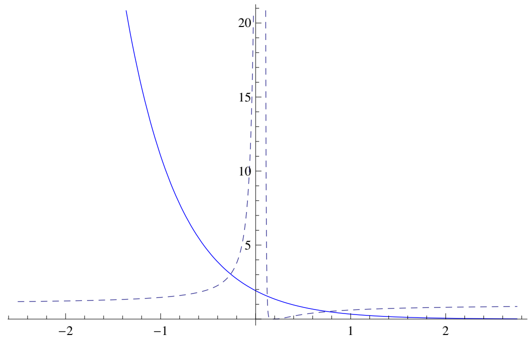


Figure 2.13: The three intersection poitns between the functions $\tilde{f}_8(y_2)$ drawn in continuous line and $\tilde{g}_8(y_2)$ drawn in dashed line.

In the half-plane Σ^+ *we consider the quadratic center*

$$
\begin{aligned}
\dot{x} &= \frac{1}{770}\Big(-24x^2 + x(1537-2490y) - 20(5y(90y-151)+241)\Big),\\
\dot{y} &= \frac{3320}{3850}\Big(xy + x(32x-1921) + 12000y^2 - 7300y + 10\Big),
\end{aligned}
\tag{2.24}
$$

this system has the first integral $H_8^{(2)}(x,y) = e^{-(1/100)(8x+30y+1)^2+((1/5)x+20y-10}(x+100y-60)^2$.

In the half-plane Σ^- *we consider the linear differential center*

$$
\dot{x} \simeq (1/2)x - 1.57968y + 2, \quad \dot{y} = 2x - (1/2)y + 1/2, \tag{2.25}
$$

with the first integral $H(x,y) = x((1/2)-(1/2)y) + x^2 + (0.789842y-2)y$.

In this case system (2.17) *has the unique solution* $(y_1, y_2) \simeq (0.894524, 1.63763)$ *which provides the unique limit cycle for the discontinuous piecewise differential system* (2.24)–(2.25), *see Figure* 2.2(*b*). *This example completes the proof of statement* (II).

Proof of statement (III) ***of Theorem 2.2****. In this statement* $F(y_2)=0$ *is a cubic equation in the variable* y_2, *where*

$$
\begin{aligned}
F(y_2) = \; & \frac{1}{(4\beta^2+\omega^2)^3}\Big(2\Big(y_2\Big(4\beta^2+\omega^2\Big) - 4\alpha\sigma_1\Big)\Big(64\alpha^2\sigma_1^2\Big(a\Big(\gamma_1^3 - 3\gamma_1\gamma_2^2\Big) + 3b\gamma_1^2\gamma_2 + \gamma_2^3 d\Big)\\
& -4\alpha\sigma_1\Big(4\beta^2+\omega^2\Big)\Big(\gamma_1^2(-6a\sigma_1 - 6b\delta_2 + 6b\gamma_2 y_2 - 3) - 6\gamma_1\gamma_2(-2a\delta_2 + a\gamma_2 y_2 + 2b\delta_1)\\
& +\gamma_2^2(6a\delta_1 - 6d\delta_2 + 2\gamma_2 d y_2 - 3) + 2a\gamma_1^3 y_2\Big) + \Big(4\beta^2+\omega^2\Big)^2\Big(a(3\gamma_1(\delta_1-\delta_2)(\delta_1+\delta_2)\\
& -6\gamma_2\delta_1\delta_2 + y_2^2\Big(\gamma_1^3 - 3\gamma_1\gamma_2^2\Big)\Big) + 3b(\delta_1(2\gamma_1\delta_2 + \gamma_2\delta_1) + \gamma_1^2\gamma_2 y_2^2) + 3\gamma_1\delta_1 + 3\gamma_2\delta_2\\
& +3\gamma_2 d\delta_2^2 + \gamma_2^3 d y_2^2\Big)\Big)\Big).
\end{aligned}
$$

Therefore this equation has at most three real solutions. Eventually the planar discontinuous piecewise differential system (1.4)–(2.2) has at most one limit cycle.

To confirm that, we present in what follows a discontinuous piecewise differential systems with exactly one limit cycle.

One limit cycle of PWS (1.4)–(2.2) of type $C+2a=A-2b=0$**.**

In the half-plane Σ^- *we consider the quadratic center*

$$\begin{aligned}\dot{x} = &\ \frac{1}{3400}\Big(-262(174x+19)y+20(x(841x+1963)-3198)+24185y^2\Big),\\ \dot{y} = &\ \frac{1}{1700}\Big(6500x^2+20x(665-841y)+y(11397y-19630)+9000\Big),\end{aligned} \tag{2.26}$$

its first integral is

$$\begin{aligned}H_1^{(3)}(x,y) = &\ -393(174x+19)y^2+60(x(841x+1963)-3198)y+(x(x(130x+399)+540)\\ &-2869)+24185y^3.\end{aligned}$$

In the half-plane Σ^+ *we consider the linear differential center*

$$\dot{x} \simeq x+0.618871y-1, \quad \dot{y} \simeq -2.01981x-y+1.3, \tag{2.27}$$

with the first integral $H(x,y) \simeq 16.3185x^2+x(16.1584y-21.006)+5(y-3.23169)y$.

In this case system (2.17) has the unique solution $(y_1,y_2) \simeq (0.567633, 2.66406)$ *which provides the unique limit cycle for the discontinuous piecewise differential system (2.26)–(2.27), see Figure 2.3(a). This example completes the proof of statement* (III).

Proof of statement (IV) ***of Theorem 2.2****. In this statement the solutions of* $F(y_2)=0$ *are equivalent to the solutions of an equation of degree nine and due to the big expression of this equation we omit it. This equation has at most nine real solutions which provide at most four limit cycles for the discontinuous piecewise differential system (1.4)–(2.2).*

In what follows we give a discontinuous piecewise differential system of the class (1.4)–(2.2) of type (IV) *with four limit cycles.*

Four limit cycles of PWS (1.4)–(2.2) of type (IV)

In the half-plane Σ^+ *we consider the linear differential center*

$$\dot{x} = -8x-(25601/40)y+50, \quad \dot{y} = (1/10)x+8y+20, \tag{2.28}$$

with the first integral $H(x,y) = 4(x+80y)^2 + 800(2x-5y) + y^2$.

In the half-plane Σ^- *we consider the quadratic center*

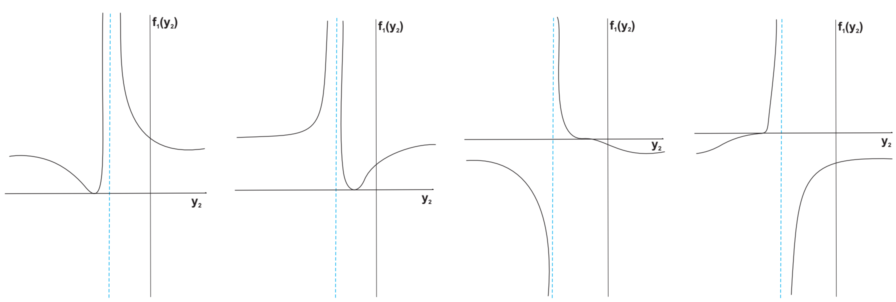
$$\begin{aligned} \dot{x} \simeq &\quad x(525.153 - 2.477y) - 0.0005176x^2 + y(5365y - 850504) + 66467, \\ \dot{y} \simeq &\quad x(0.0686512 - 0.0004798y) + y(104.576 - 0.024y) - 1.924 * 10^{-8}x^2, \end{aligned} \tag{2.29}$$

with its first integral

$$\begin{aligned} H_1^{(4)}(x,y) \simeq &\quad (0.990099(x^3 + x^2(3.62455 * 10^6 - 33461.5y) + x(y(3.73225 * 10^8 y - 10^{10} \\ &*8.09237) + 4.38777 * 10^{12}) + y((4.51686 * 10^{14} - 1.38763 * 10^{12}y)y - 10^{16} \\ &*4.90232) + 1.77406 * 10^{18})^2)/((x^2 + x(2.41637 * 10^6 - 22307.7y) + y(10^8 \\ &*1.24408y - 2.69973 * 10^{10}) + 1.46548 * 10^{12})^3). \end{aligned}$$

∎

In this case system (2.17) *has the four solutions* $(y_1, y_2) \simeq \{(0.00817805, 0.148066), (0.0177713, 0.138473), (0.0292114, 0.127033), (0.0443241, 0.11192)\}$ *which provide the four limit cycles for the discontinuous piecewise differential system* (2.28)–(2.29), *see Figure* 2.3(b). *This example completes the proof of statement* (IV).



(a) (b) (c) (d)

Figure 2.14: The graphics of the function $f_1(y_2)$. The dashed straight line is the vertical asymptote straight line.

(a) (b)

Figure 2.15: The graphic of the function $f_2(y_2)$.

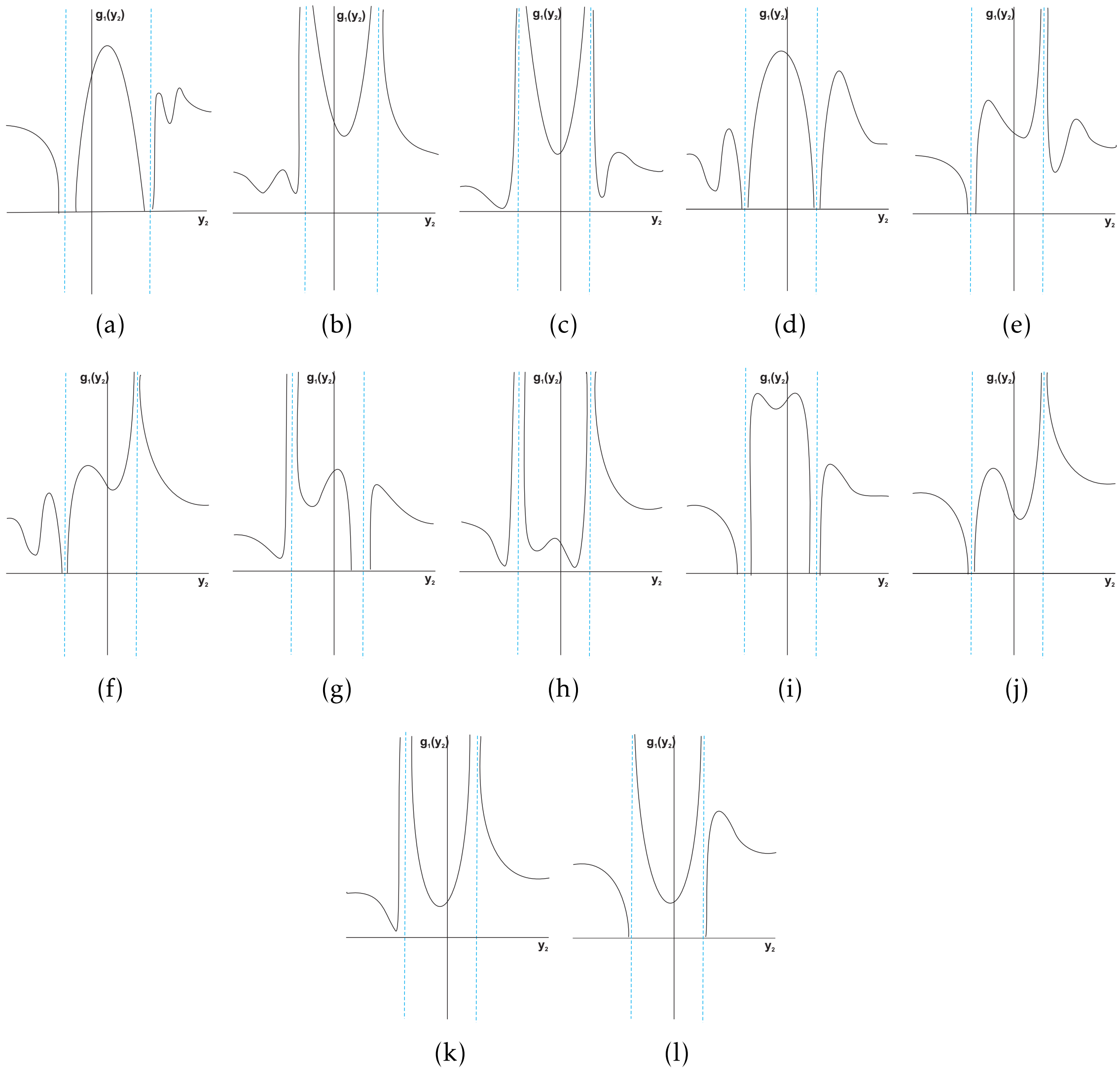


Figure 2.16: The graphics of the function $g_1(y_2)$.

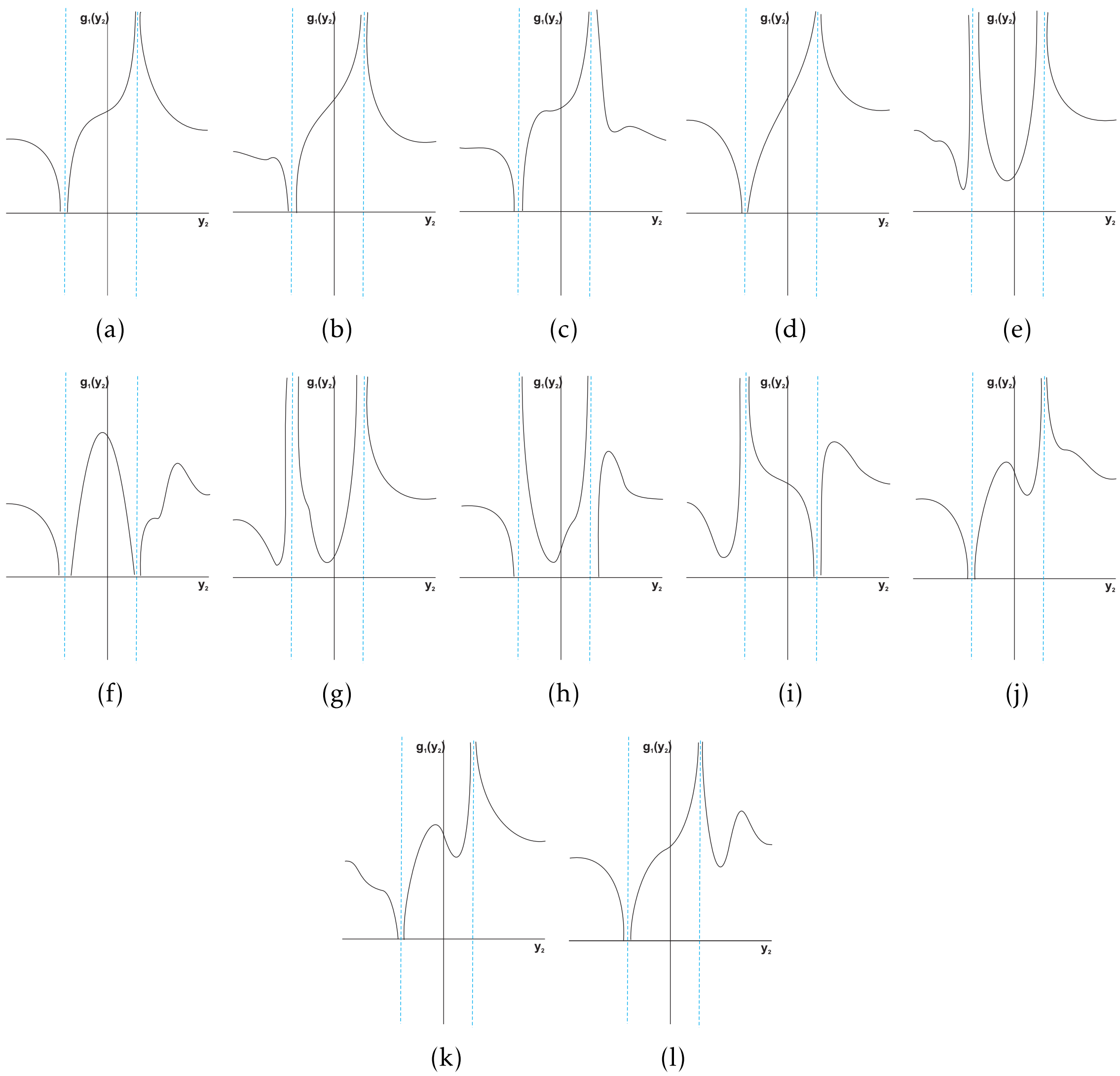


Figure 2.17: The graphics of the function $g_1(y_2)$.

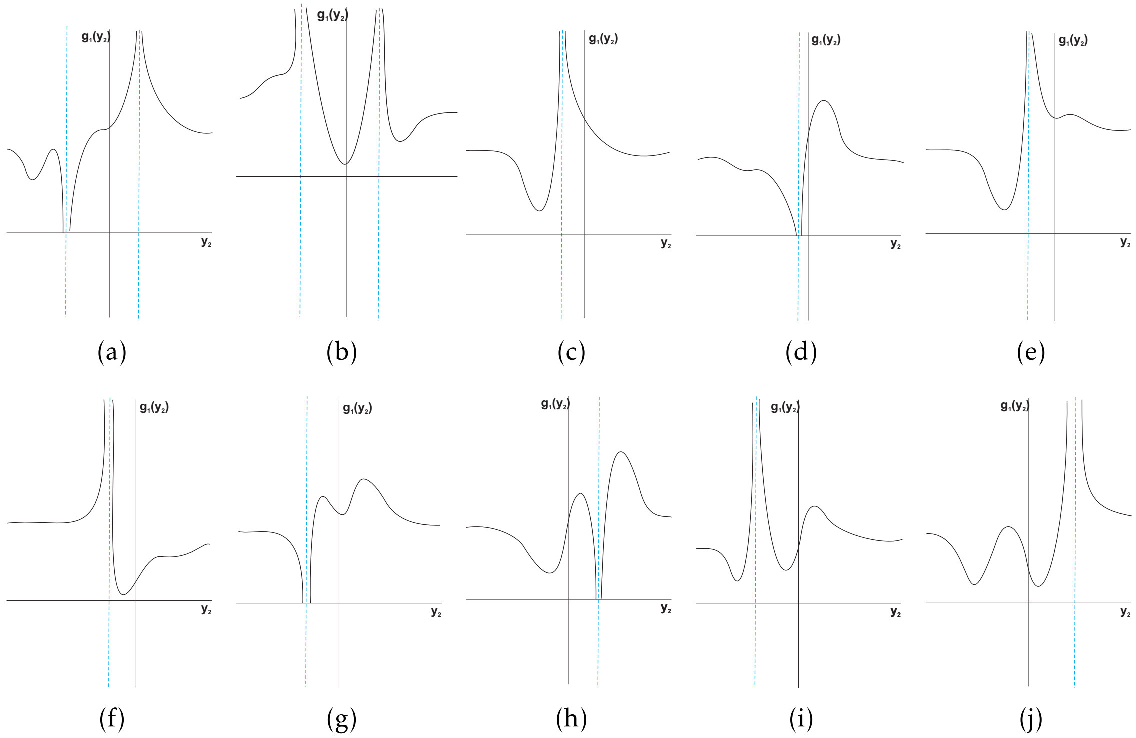


Figure 2.18: The graphics of the function $g_1(y_2)$.

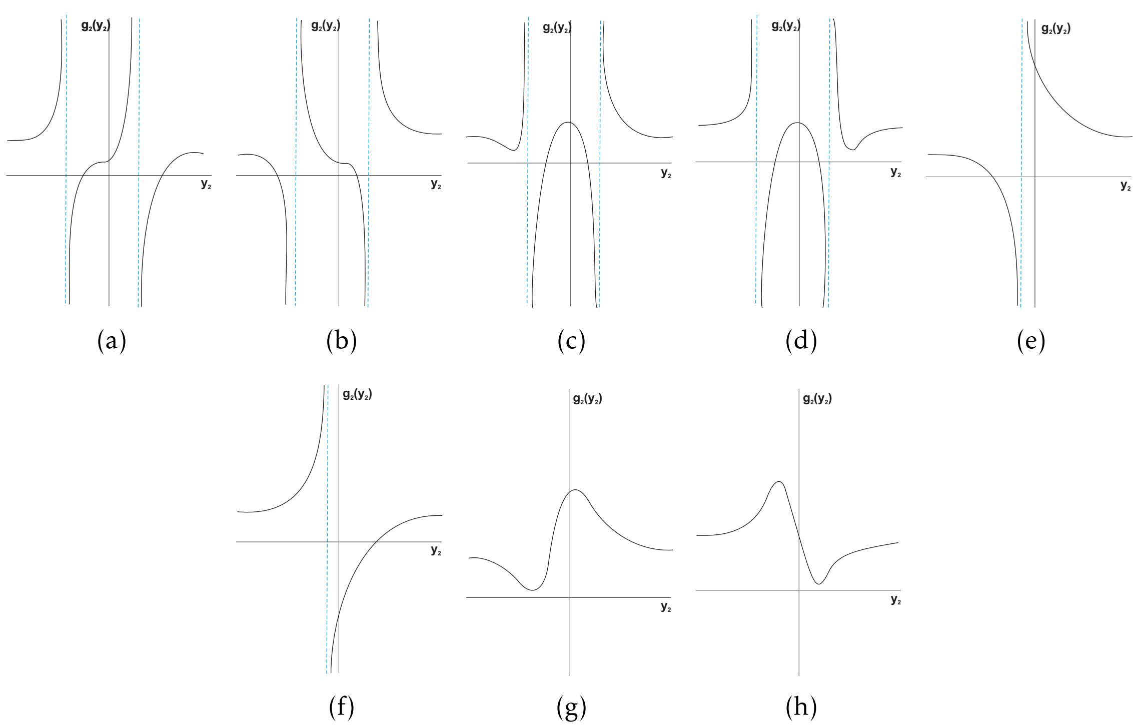


Figure 2.19: The graphics of the function $g_2(y_2)$.

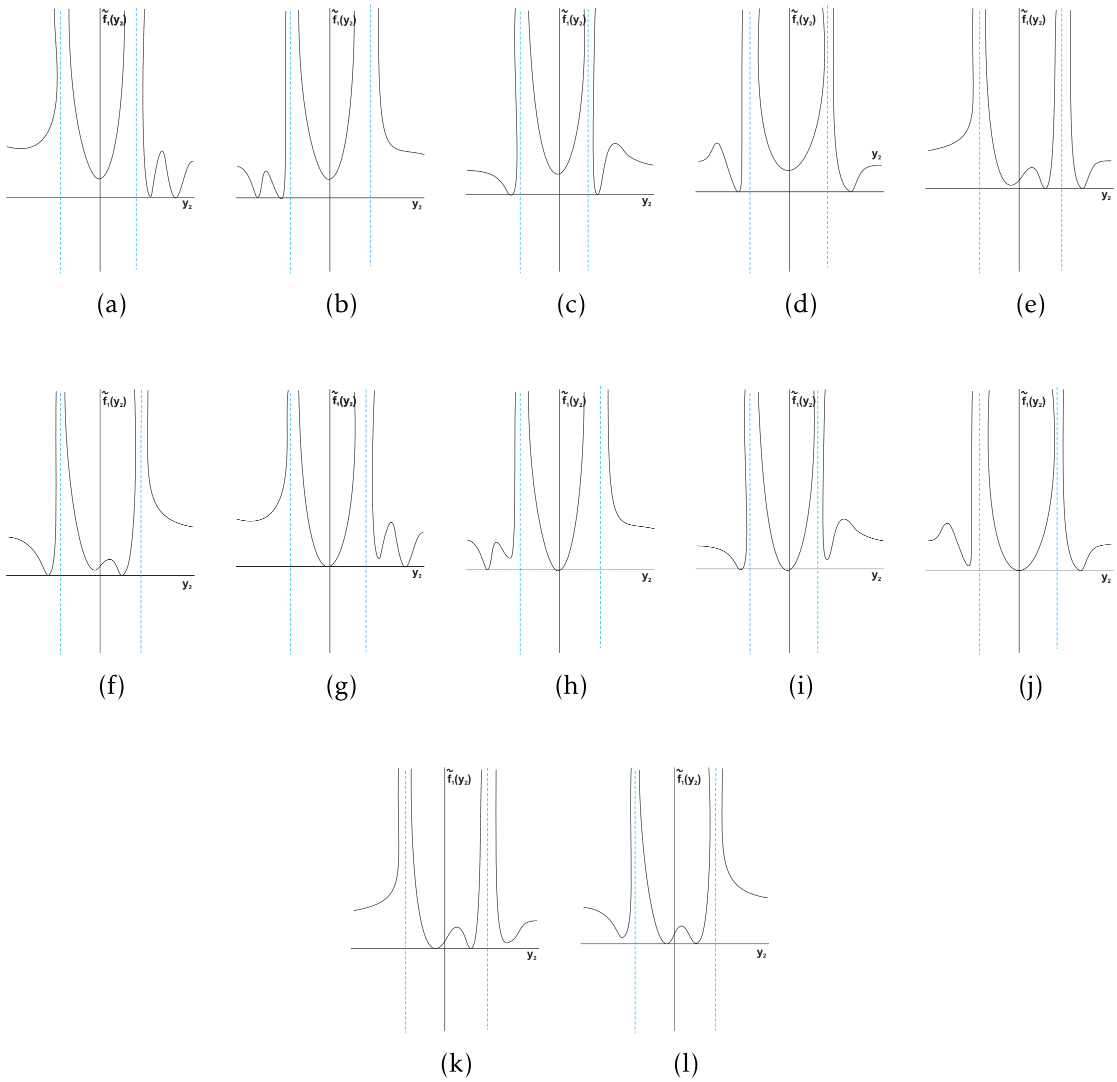


Figure 2.20: The graphics of the function $\tilde{f}_1(y_2)$.

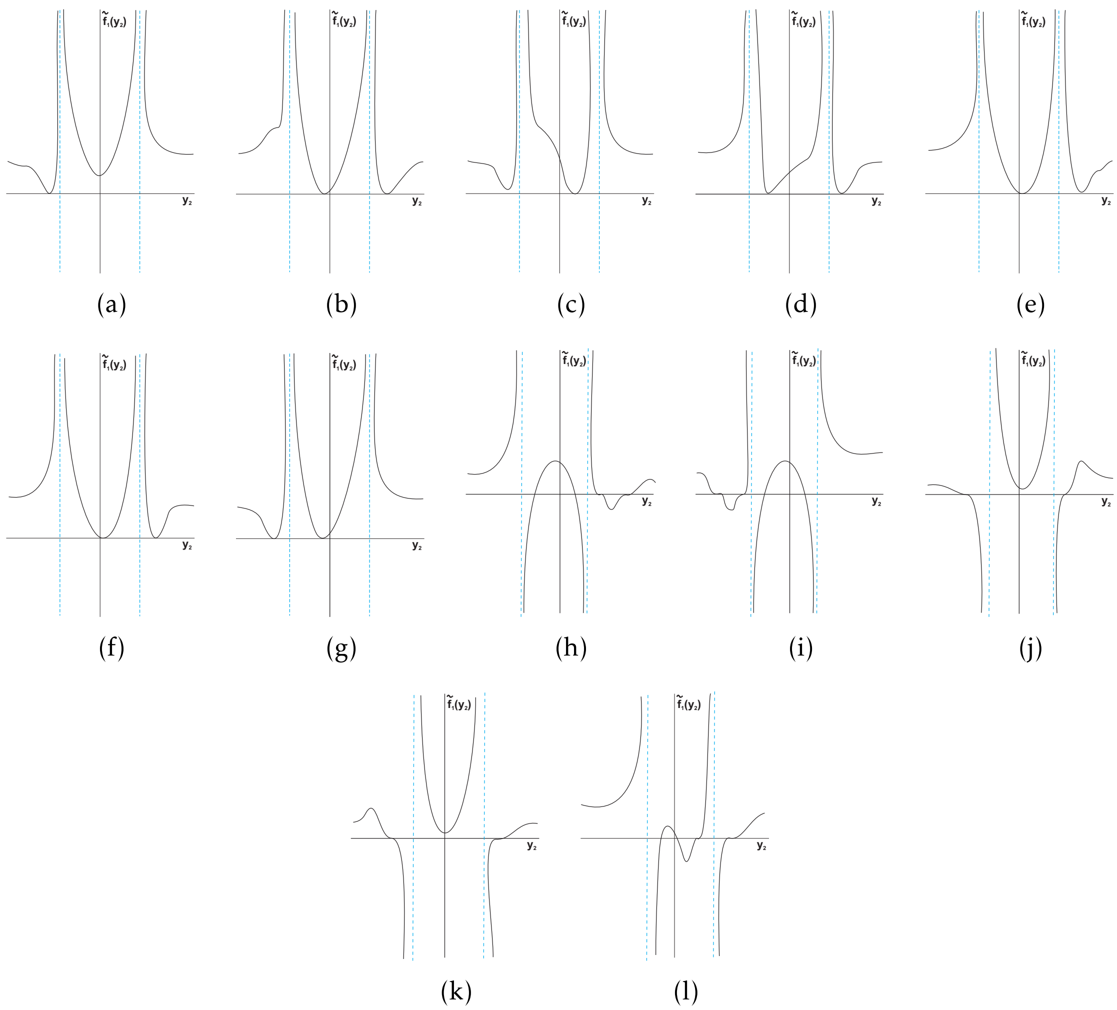


Figure 2.21: The graphics of the function $\tilde{f}_1(y_2)$.

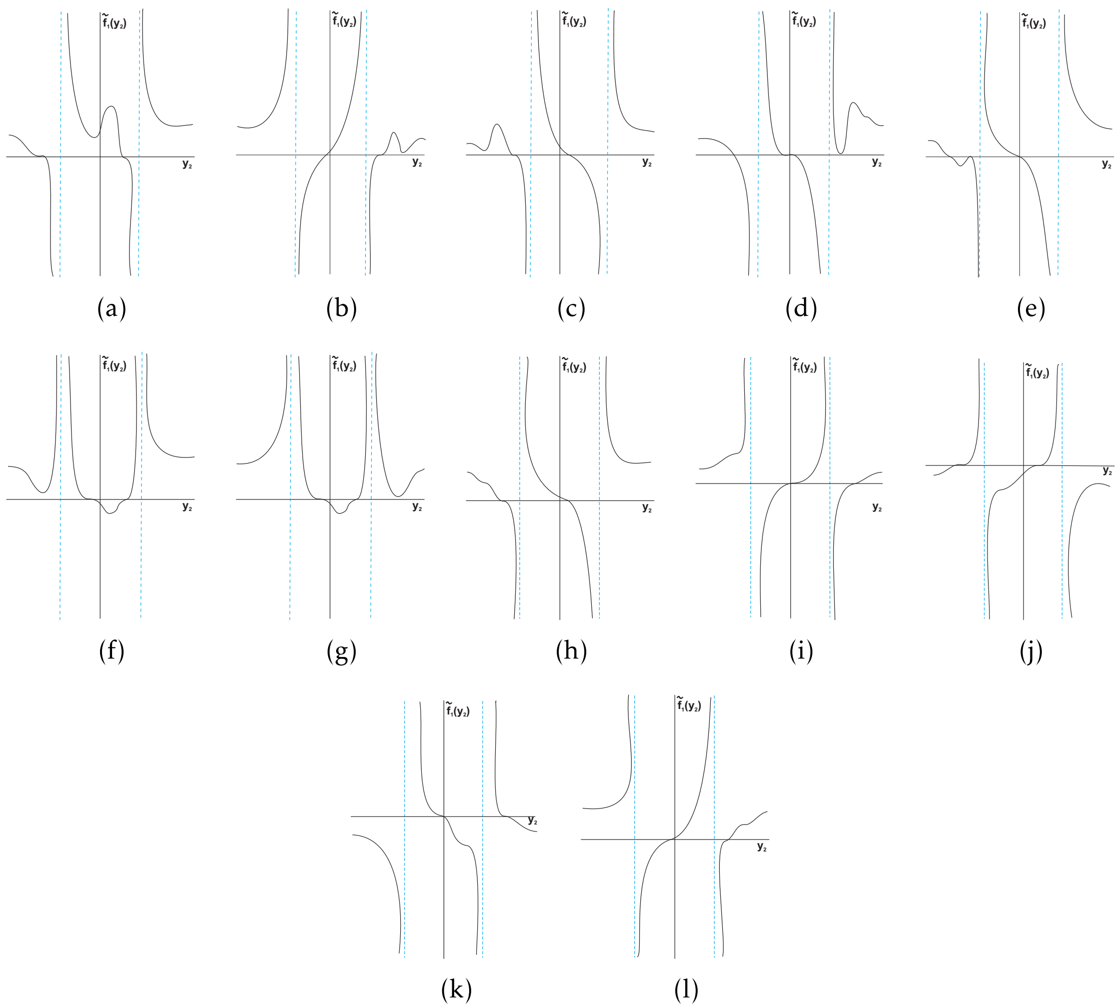


Figure 2.22: The graphics of the function $\tilde{f}_1(y_2)$.

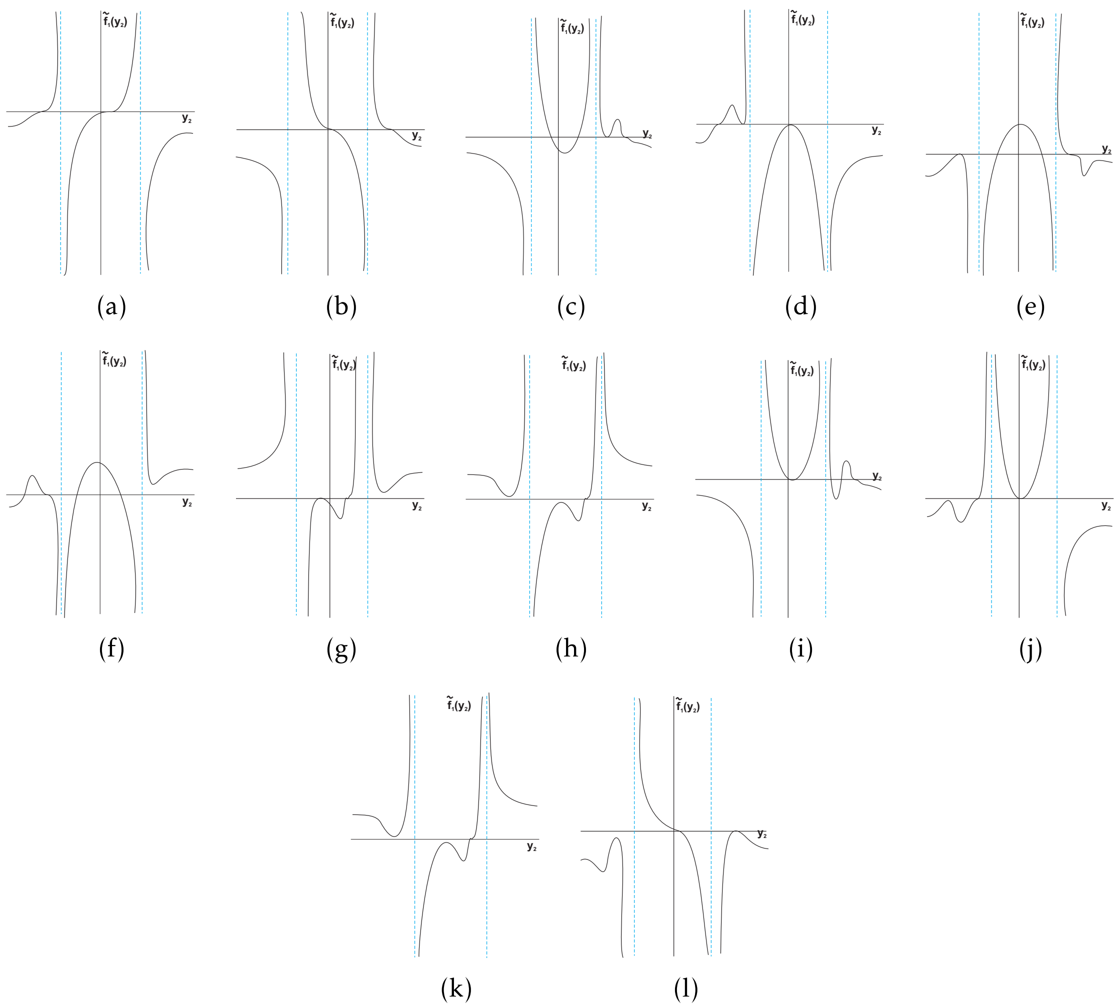


Figure 2.23: The graphics of the function $\tilde{f}_1(y_2)$.

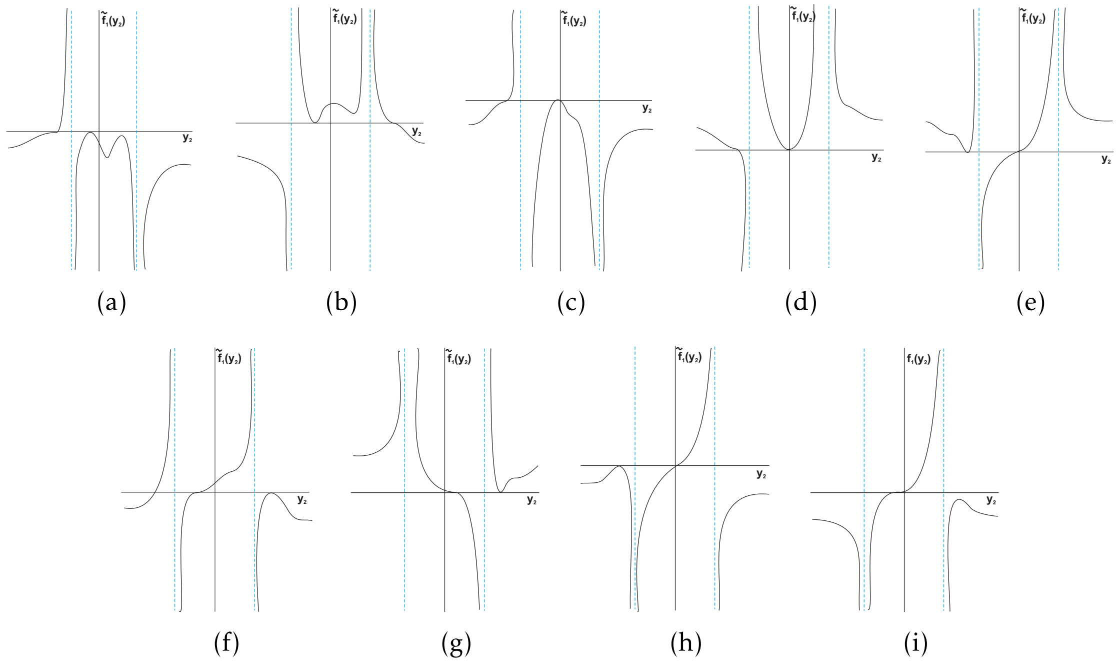


Figure 2.24: The graphics of the function $\tilde{f}_1(y_2)$.

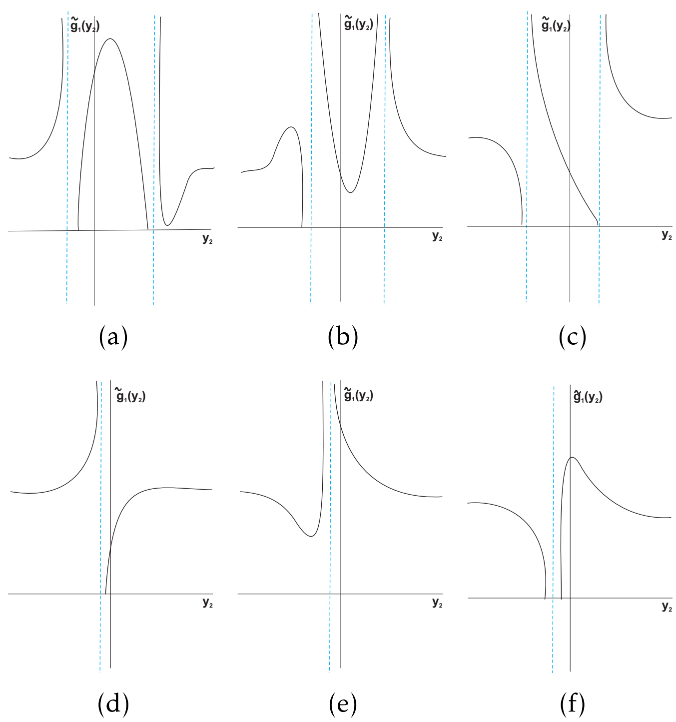


Figure 2.25: The graphics of the function $\tilde{g}_1(y_2)$.

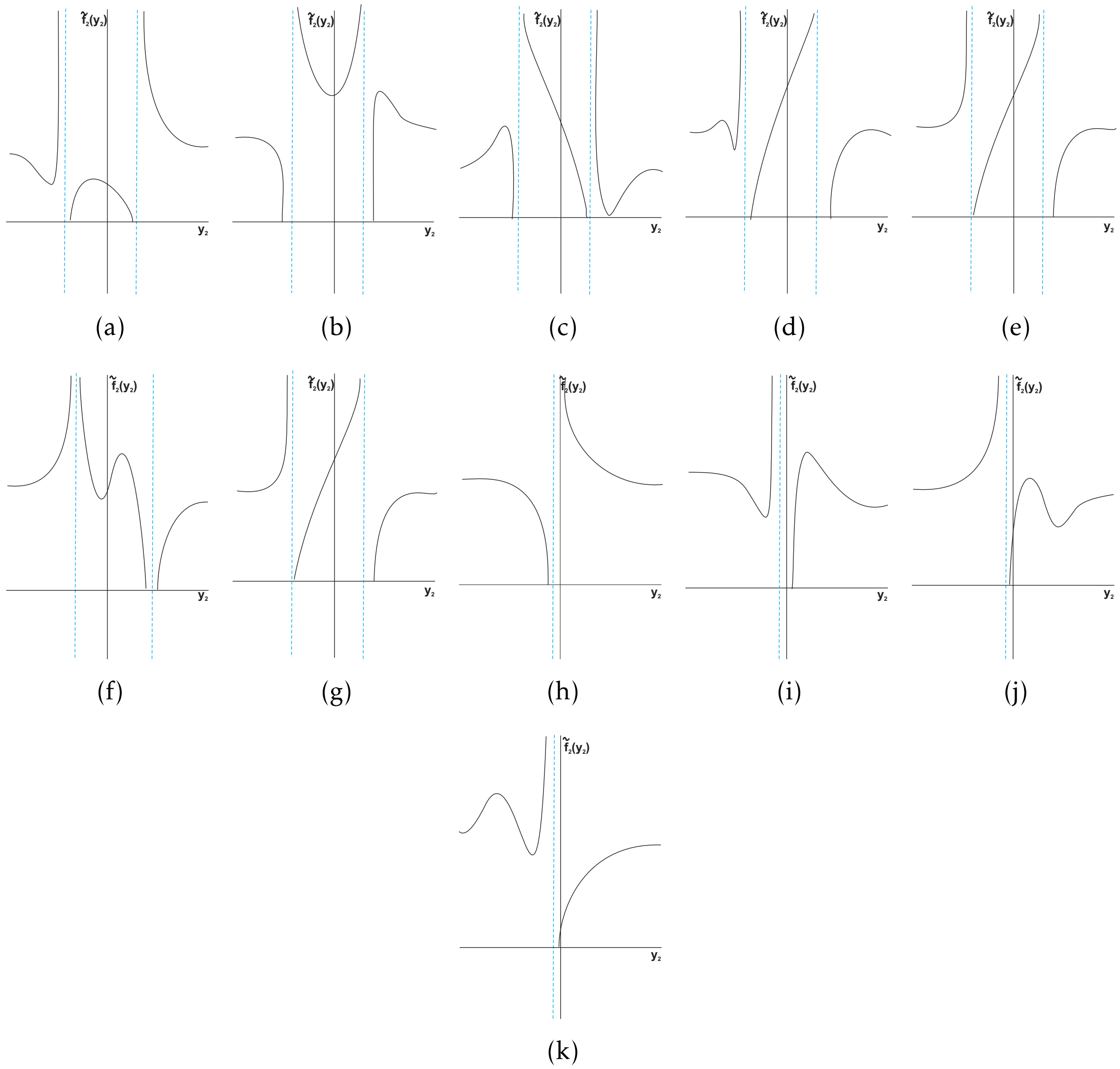


Figure 2.26: The graphics of the function $\tilde{f}_2(y_2)$.

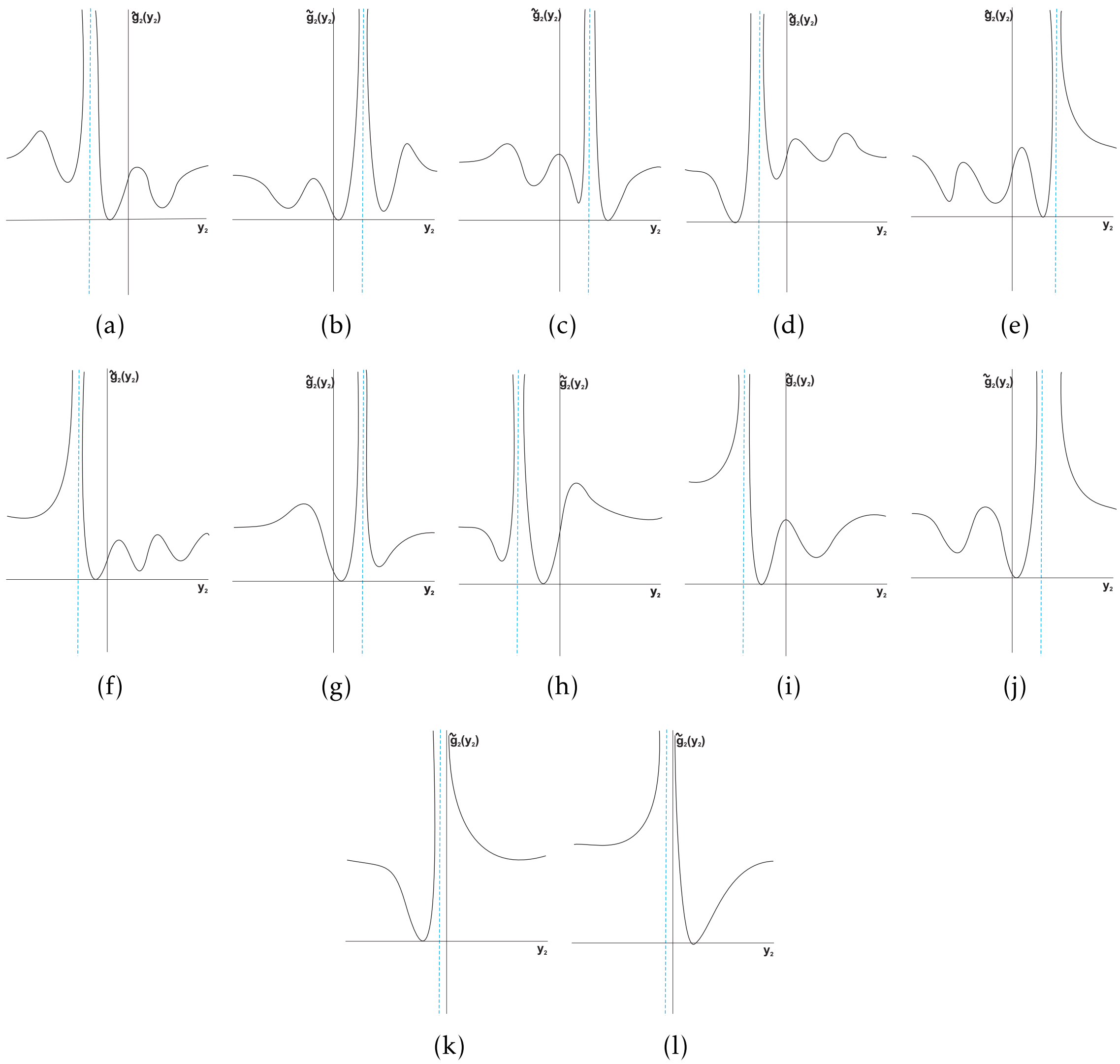


Figure 2.27: The graphics of the function $\tilde{g}_2(y_2)$.

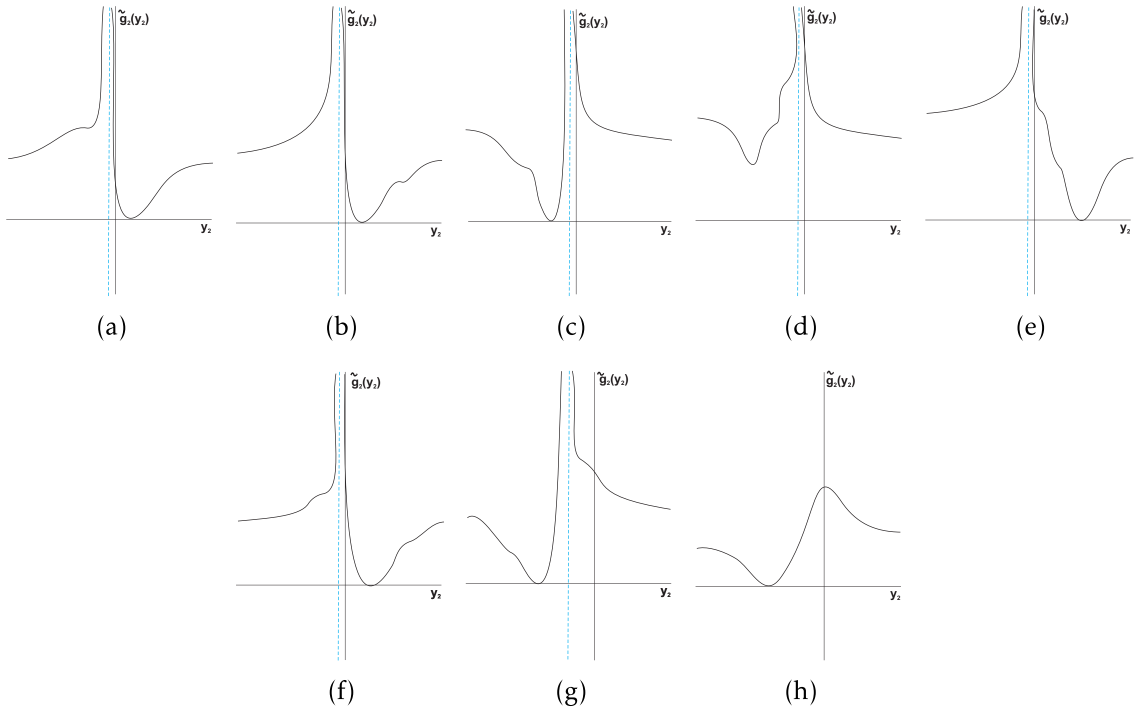


Figure 2.28: The graphics of the function $\tilde{g}_2(y_2)$.

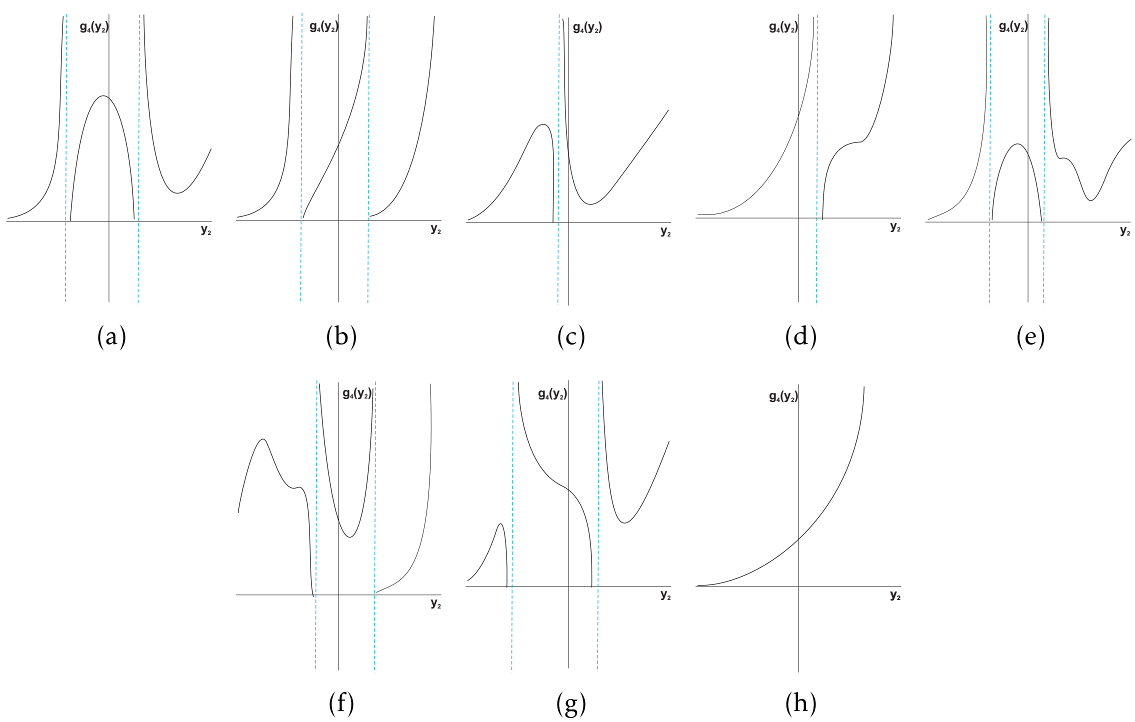


Figure 2.29: The graphics of the function $\tilde{g}_4(y_2)$.

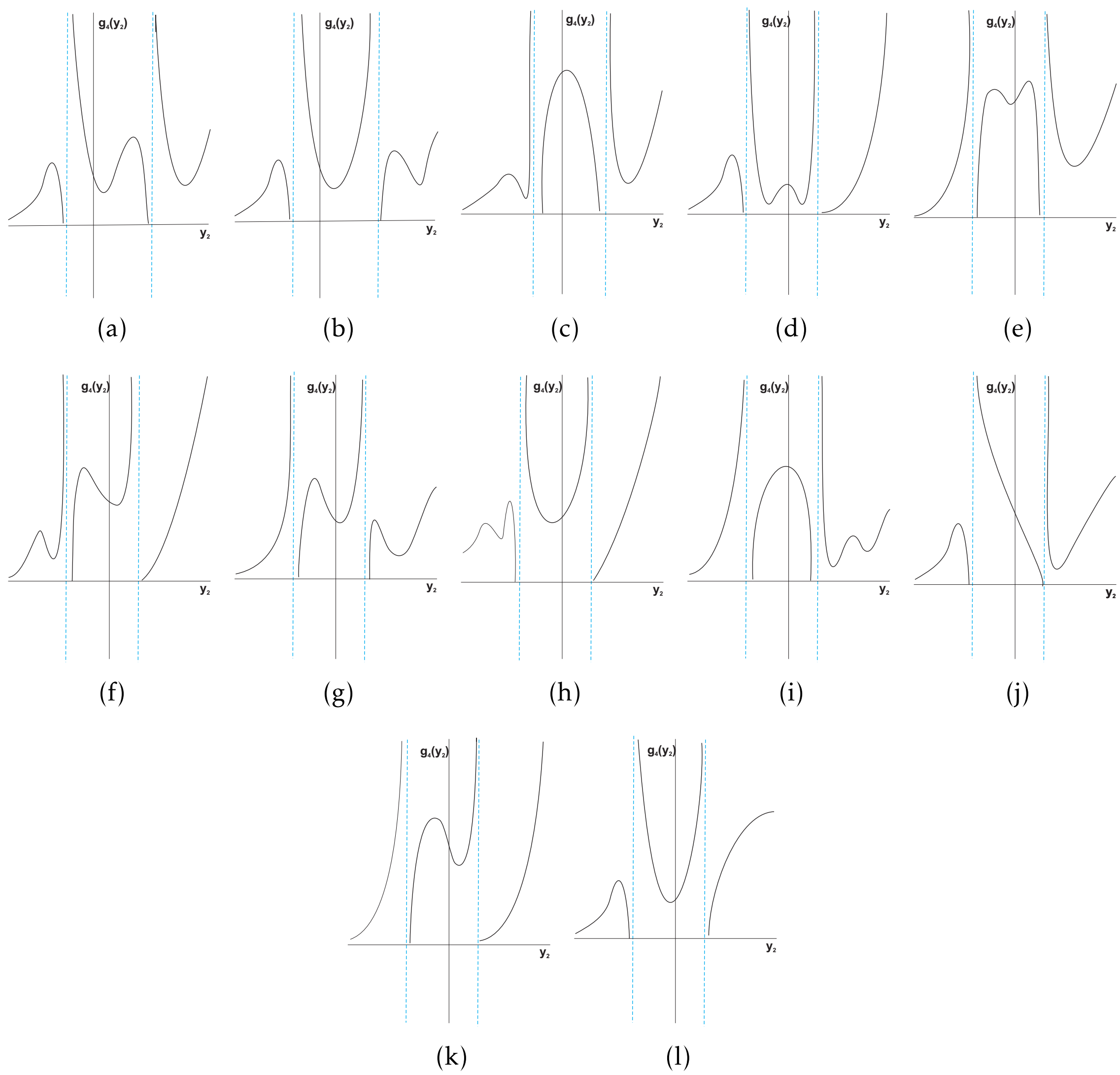


Figure 2.30: The graphics of the function $\tilde{g}_4(y_2)$.

Chapter 3

LC of PWS Separated by a Straight Line and Formed by Cubic Reversible Isochronous Centers

The obdjectif of this chapter is to solve the extension of the second part of the sixteenth Hilbert problem for all classes of discontinuous piecewise differential systems with cubic reversible isochronous centers having rational first integrals, where the separation curve is the straight line $\Sigma = \{(x, y) : x = 0\}$.

Firstly, we found the maximum number of limit cycles that the three classes of discontinuous piecewise differential systems formed by an arbitrary linear center and one of the three cubic reversible isochronous centers can exhibit.

Secondly, we provided the upper bound for the maximum number of limit cycles of discontinuous piecewise differential systems formed by two cubic reversible isochronous centers.

Section 3.1 The cubic reversible isochronous centers

The normal forms of the three cubic reversible isochronous centers with a rational first integral are given in the following theorem.

Theorem 3.1 *After an affine change of variables and a rescaling of the independent variable the three cubic reversible isochronous centers with rational first integrals can be ex-*

pressed as one of the following three differential systems.

$$(\mathcal{C}_1)\quad \dot{x} = y(-1+2ax+2bx^2),\quad \dot{y} = x + a(y^2-x^2) + 2bxy^2,$$

$$(\mathcal{C}_2)\quad \dot{x} = -y(1-x)(1-2x),\quad \dot{y} = x - 2x^2 + y^2 + 2x^3,$$

$$(\mathcal{C}_3)\quad \dot{x} = y\Big(-1+\frac{8}{3}x-\frac{32}{9}y^2\Big),\quad \dot{y} = x - \frac{4}{3}y^2.$$

For a proof of Theorem 3.1 see [17].

Cubic reversible isochronuous centers after an affine change of variables

Let $(\tilde{\mathcal{C}}_i)$ with $i = 1,2,3$, be the three cubic reversible isochronous centers having a rational first integral after a general affine change of variables $\{x \to a_1x + b_1y + c_1, y \to \alpha_1x + \beta_1y + \gamma_1\}$, with $a_1\beta_1 - \alpha_1b_1 \neq 0$. In what follows the expressions of the cubic reversible isochronous centers $(\tilde{\mathcal{C}}_1)$, $(\tilde{\mathcal{C}}_2)$ and $(\tilde{\mathcal{C}}_3)$ as well as of their first integrals are given in this section.

The isochronous center $(\tilde{\mathcal{C}}_1)$ is

$$\begin{aligned}
\dot{x} =& \frac{-1}{\alpha_1b_1-a_1\beta_1}(\beta_1(\gamma_1+\alpha_1x+\beta_1y)(2(a_1x+b_1y+c_1)(a+b(a_1x+b_1y+c_1))-1)-b_1(a\\
&\Big((\gamma_1+\alpha_1x+\beta_1y)^2-(a_1x+b_1y+c_1)^2\Big)+2b(a_1x+b_1y+c_1)(\gamma_1+\alpha_1x+\beta_1y)^2+a_1x\\
&+b_1y+c_1)),\\
\dot{y} =& \frac{-1}{\alpha_1b_1-a_1\beta_1}(a_1\Big(a\Big((\gamma_1+\alpha_1x+\beta_1y)^2-(a_1x+b_1y+c_1)^2\Big)+2b(a_1x+b_1y+c_1)(\alpha_1x\\
&+\beta_1y+\gamma 1)^2+a_1x+b_1y+c_1\Big)-\alpha_1(\gamma_1+\alpha_1x+\beta_1y)(2(a_1x+b_1y+c_1)(a+b(a_1x+b_1y\\
&+c_1))-1)),
\end{aligned}$$

with the first integral

$$\tilde{H}_1(x,y) = \frac{1-2(a_1x+b_1y+c_1)\Big(a+b(a_1x+b_1y+c_1)\Big)}{(a_1x+b_1y+c_1)^2+(\gamma_1+\alpha_1x+\beta_1y)^2}. \tag{3.1}$$

The isochronous center $(\tilde{\mathcal{C}}_2)$ is written as

$$\begin{aligned}
\dot{x} =& \frac{-1}{\alpha_1b_1-a_1\beta_1}\Big(\beta_1(-(a_1x+b_1y+c_1-1))(2a_1x+2b_1y+2c_1-1)(\alpha_1x+\beta_1y+\gamma_1)-b_1\Big(2\\
&(a_1x+b_1y+c_1)^3-2(a_1x+b_1y+c_1)^2+a_1x+b_1y+c_1+(\gamma_1+\alpha_1x+\beta_1y)^2\Big)\Big),\\
\dot{y} =& \frac{-1}{\alpha_1b_1-a_1\beta_1}\Big(\alpha_1(a_1x+b_1y+c_1-1)(2a_1x+2b_1y+2c_1-1)(\gamma_1+\alpha_1x+\beta_1y)+a_1\Big(2(a_1x\\
&+b_1y+c_1)^3-2(a_1x+b_1y+c_1)^2+a_1x+b_1y+c_1+(\gamma_1+\alpha_1x+\beta_1y)^2\Big)\Big),
\end{aligned}$$

with the first integral

$$\tilde{H}_2(x,y) = \frac{1}{(2(a_1x+b_1y+c_1)-1)^2}(a_1x+b_1y+c_1-1)^2\Big((a_1x+b_1y+c_1)^2+(\alpha_1 x +\beta_1 y+\gamma_1)^2\Big). \tag{3.2}$$

The isochronous center $(\tilde{C}_3)$ is

$$\begin{aligned}\dot{x} = & \frac{-1}{\alpha_1 b_1 - a_1\beta_1}\Big(\beta_1(\gamma_1+\alpha_1x+\beta_1y)\Big(\frac{8}{3}(a_1x+b_1y+c_1)-\frac{32}{9}(\gamma_1+\alpha_1x+\beta_1y)^2-1\Big)-b_1 \\ & \Big(a_1x+b_1y+c_1-\frac{4}{3}+(\gamma_1+\alpha_1x+\beta_1y)^2\Big)\Big),\\ \dot{y} = & \frac{-1}{\alpha_1 b_1 - a_1\beta_1}\Big(a_1\Big(a_1x+b_1y+c_1-\frac{4}{3}(\gamma_1+\alpha_1x+\beta_1y)^2\Big)-\alpha_1(\gamma_1+\alpha_1x+\beta_1y)\Big(\frac{8}{3}(a_1x \\ & +b_1y+c_1)-\frac{32}{9}(\gamma_1+\alpha_1x+\beta_1y)^2-1\Big)\Big),\end{aligned}$$

with the first integral

$$\tilde{H}_3(x,y) = \Big(3(a_1x+b_1y+c_1)-4(\gamma_1+\alpha_1x+\beta_1y)^2\Big)^2+9(\gamma_1+\alpha_1x+\beta_1y)^2. \tag{3.3}$$

Section 3.2 LC of PWS with cubic reversible isochronous centers

The purpose of this subsection is to provide the upper bounds of the maximum number of crossing limit cycles of the three classes of discontinuous piecewise differential systems, are formed by an arbitrary linear center and one of the three cubic reversible isochronous center having rational first integral, and separated by the straight line Σ.

3.2.1 Statement of the first main result

We study the crossing limit cycles of the discontinuous piecewise differential formed by one of the three cubic reversible isochronous centers with rational first integrals and a linear differential center, separated by the straight line Σ.

Our first result is the following.

Theorem 3.2 *The maximum number of crossing limit cycles of the class of planar discontinuous piecewise differential systems separated by the straight line Σ and formed by the linear center* (1.6) *and*

(I) the cubic reversible isochronous center $(\tilde{C}_1)$ is one, there are systems with exactly one limit cycle, see Figure 3.1(a);

(II) the cubic reversible isochronous center $(\tilde{C}_2)$ is two, there are systems with exactly two limit cycles, see Figure 3.1(c);

(III) the cubic reversible isochronous center $(\tilde{C}_3)$ is one, there are systems with exactly one limit cycle, see Figure 3.1(b).

3.2.2 Proof of theorem 3.2

Now we will prove Theorem 3.2 for the class of planar discontinuous piecewise differential systems separated by the straight line Σ and formed by a linear center and one of the three cubic reversible isochronous centers $(\tilde{C}_i)$ with $i = 1, 2, 3$.

Proof.

In the first half-plane $\Sigma^+ = \{(x, y) : x \geq 0\}$ we consider the cubic reversible isochronous center $(\tilde{C}_i)$ with its first integral $\tilde{H}_i(x, y)$ given in either (3.1), or (3.2), or (3.3), and in the second half-plane $\Sigma^- = \{(x, y) : x \leq 0\}$ we consider the planar linear differential center (1.6) with its first integral $H(x, y)$ given in (1.7).

To prove that the discontinuous piecewise differential systems (1.6)–$(\tilde{C}_i)$ has at most one crossing limit cycle that crosses the line of discontinuity Σ at two different points $(0, y)$ and $(0, Y)$, with $y \neq Y$. These two points must satisfy the following system of equations

$$\begin{aligned} e_1 &= H(0, y) - H(0, Y) = (y - Y)(2B - M(y + Y)) = 0, \\ e_2 &= \tilde{H}_i(0, y) - \tilde{H}_i(0, Y) = (y - Y)F_i(y, Y) = 0. \end{aligned} \tag{3.4}$$

Where $M = A^2 + \omega^2$.

From $e_1 = 0$ we obtain $Y = -y + \dfrac{2B}{M} = f(y)$ for $i = 1, 2, 3$. Substituting the expression of Y in $F_i(y, Y) = 0$, we get an equation $K_i(y) = F_i(y, f(y)) = 0$ in the variable y, which varies according to the expressions of the first integrals $\tilde{H}_i(x, y)$.

***Proof of statement** (I) **of Theorem 3.2**. For $i = 1$ we consider the class of discontinuous piecewise differential systems formed by the linear differential center (1.6) with the first integral $H(x, y)$ and the cubic reversible isochronous center $(\tilde{C}_1)$ with its corresponding first integral $\tilde{H}_1(x, y)$. We find that $K_1(y) = 0$ is the quadratic equation in the variable y, where*

$$K_1(y) = b_1^2\Big(M(2ac_1-2b\gamma_1(\gamma_1+2\beta_1 y)-1)+4b\beta_1\gamma_1 y^2\Big)+2b_1\Big(ac_1^2+a\Big(\beta_1^2My-\gamma_1^2-\beta_1^2y^2\Big)$$
$$+c_1\left(2b(-\gamma_1^2+\beta_1^2My-\beta_1^2y^2)-1\right)\Big)+\beta_1\left(2ac_1+2bc_1^2-1\right)(2\gamma_1+\beta_1 M)+2ab_1^3y(M$$
$$-y).$$

This equation can have at most two real solutions y_1 and y_2 for the variable y. Thus system (3.4) has at most two real solutions. Since $(y_1, f(y_1)) = (f(y_2), y_2)$, then the class of the discontinuous piecewise differential systems (1.6)–($\tilde{\mathcal{C}}_1$) has at most one limit cycle.

In order to complete the proof of this statement we give an example with exactly one limit cycle. In the half-plane Σ^- we consider the linear center

$$\dot{x} = -1-\frac{5}{2}x-\frac{41}{4}y, \qquad \dot{y} = \frac{1}{2}+x+\frac{5}{2}y, \tag{3.5}$$

with the first integral $H(x,y)=\left(x+\frac{5}{2}y\right)^2+2\left(\frac{1}{2}x+y\right)+4y^2.$

In the other half-plane Σ^+ we consider the cubic reversible isochronous center

$$\begin{aligned}\dot{x} \simeq\ & -\frac{21}{500}x^3+x^2(0.632258y+1.02182)+x(-2.05837y^2-4.49659y-2.69497)\\ & -8.34146y^2-15.1823y-2.29501,\\ \dot{y} \simeq\ & x^2(-\frac{21}{500}y-0.066818)+x(0.632258y^2+2.31636+1.93406)-2.05837y^3\\ & -10.8976y^2-\frac{3461}{200}y-8.48195,\end{aligned} \tag{3.6}$$

with the first integral $\tilde{H}_1(x,y) \simeq \dfrac{1-2(0.21x-y-1)(-0.21x+y+1.5)}{(0.21x-y-1)^2+(0.1x-1.02919y-2)^2}.$

Eventually, system (3.4) for $i=1$ has the unique real solution $(y,Y)\simeq(-0.4248, 0.22967)$ which provides the unique limit cycle of the discontinuous piecewise differential system (3.5)–(3.6) shown in Figure 3.1(a). Thus statement (I) of Theorem 3.2 is proved.

***Proof of statement** (II) **of Theorem 3.2**. For $i=2$ we consider the class of discontinuous piecewise differential system formed by the linear differential center (1.6) with its first integral (1.7) and the cubic reversible isochronous center ($\tilde{\mathcal{C}}_2$) with its first integral $\tilde{H}_2(x,y)$. We obtain that $K_2(y)=0$ is the quartic equation in the variable y, where*

$$K_2(y) = 4b_1^6My^2(M-y)^2+4b_1^5(2c_1-1)y\left(M^3-2My^2+y^3\right)+b_1^4\Big((1-2c_1)^2M(M^2+6My$$
$$-6y^2)+4\beta_1^2My^2(-y+M)^2+8\beta_1\gamma_1y^2(M-y)^2\Big)+2b_1^3\Big(M^2(2c_1(4c_1^2-6c_1-2\beta_1^2y^2$$
$$+3)+4\beta_1(2c_1-1)\gamma_1y-1)+My\Big(2\left(\gamma_1^2+c_1\left(4c_1^2-6c_1+3\right)-4\beta_1c_1\gamma_1y\right)+4\beta_1y(\gamma_1$$
$$+\beta_1y)-1\Big)-y^2\left(2\gamma_1^2+2c_1\left(4c_1^2-6c_1+3\right)+2\beta_1^2y^2-1\right)+2\beta_1^2(2c_1-1)M^3y\Big)+b_1^2\Big(M$$

$$\Big(20c_1^4-40c_1^3+30c_1^2+\beta_1^2(2c_1-1)((2c_1-1)M^2+2(2c_1-3)My+2(3-2c_1)y^2)+1 -10c_1\Big)+2\beta_1\gamma_1((1-2c_1)^2M^2+(4c_1(2c_1-3)+3)My+(4(3-2c_1)c_1-3)y^2)+(4c_1 -3)\gamma_1^2M\Big)+2b_1(c_1-1)(2c_1-1)\Big(2c_1^3-2c_1^2+\gamma_1^2+c_1(2\beta_1M(2\gamma_1+\beta_1M)+1)-\beta_1^2 (M^2+My-y^2)-2\beta_1\gamma_1M\Big)+\beta_1(2c_1^2-3c_1+1)^2(2\gamma_1+\beta_1M).$$

The equation $K_2(y)=0$ can have at most four real solutions, and due to the symmetry stated in the proof of statement (I) we know that system (3.4) has at most two different real solutions. As a result the class of discontinuous piecewise differential systems (1.6)–($\tilde{\mathcal{C}}_2$) has at most two limit cycles.

Now we prove that this result is reached by giving an example with exactly two limit cycles. So in Σ^- we consider the linear center

$$\dot{x}=-\frac{1}{10}-\frac{5}{2}x-\frac{29}{4}y,\qquad \dot{y}=-\frac{1}{10}+x+\frac{5}{2}y, \tag{3.7}$$

its first integral is $H(x,y)=\Big(x+\frac{5}{2}y\Big)^2+2\Big(\frac{1}{10}y-\frac{1}{10}x\Big)+y^2.$

In Σ^+ we consider the cubic reversible isochronous center

$$\begin{aligned}\dot{x}\simeq\;& 0.2x^3+x^2(-16y-1.65628)+x(114.847y+4.14776)-205.093y-2.82886,\\ \dot{y}\simeq\;& 0.2525x^3+x^2(-0.2y-2.39869)+x(1.23559y+7.63844)-2.34297y+8y^2\\ &-8.12409,\end{aligned} \tag{3.8}$$

which has the first integral

$$\tilde{H}_2(x,y)\simeq\quad \frac{(x-3.83898)^2}{(2(x-2.83898)-1)^2}\Big(\Big(-\frac{1}{10}x+8y+0.110345\Big)^2+(x-2.83898)^2\Big).$$

The discontinuous piecewise differential system (3.7)–(3.8) has two limit cycles because system (3.4) when $i=2$ has the two real solutions $(y,Y)\simeq\{(-0.27676,0.24918),(-0.38544,$ $0.357854)\}$, these two limit cycles are drawn in Figure 3.1(c). Hence this statement is proven.

***Proof of statement** (III) **of Theorem 3.2**. For $i=3$ we consider the class of discontinuous piecewise differential systems composed by the linear differential center (1.6) with the first integral $H(x,y)$ and the cubic reversible isochronous center ($\tilde{\mathcal{C}}_3$) with its first integral $\tilde{H}_3(x,y)$. We realize that $K_3(y)=0$ is the quadratic equation in the variable y, where*

$$\begin{aligned}K_3(y)=\;& 9b_1^2M+6b_1\Big(-4\gamma_1^2+3c_1-4\beta_1^2\Big(M^2-My+y^2\Big)-8\beta_1\gamma_1M\Big)+\beta_1(2\gamma_1+\beta_1M)\\ &\Big(-24c_1+16(2\gamma_1^2+\beta_1^2\Big(M^2-2My+2y^2\Big)+2\beta_1\gamma_1M)+9\Big).\end{aligned}$$

Therefore system (3.4) has at most one distinct real solution. Consequently the discontinuous piecewise differential systems (1.6)–($\tilde{\mathcal{C}}_3$) has at most one limit cycle.

Now we will build an example with exactly one limit cycle to prove that this maximum is reached.

In Σ^- we consider the linear center

$$\dot{x} = -1 - \frac{5}{2}x - \frac{11}{4}y, \qquad \dot{y} = 1 + x + \frac{5}{2}y, \tag{3.9}$$

which has the first integral $H(x,y) = \left(x + \frac{5}{2}y\right)^2 + 2(x+y) + 4y^2.$

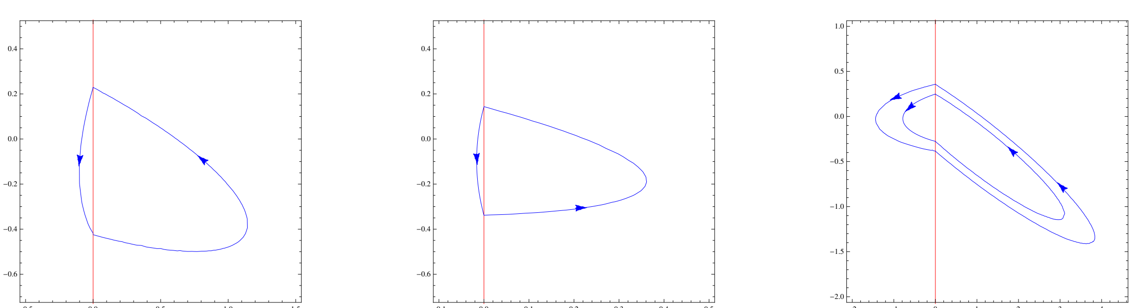
In Σ^+ we consider the cubic reversible isochronous center

$$\begin{aligned} \dot{x} \simeq\ & x^2(0.0257383 - 0.00119261y) - 0.00002265x^3 + x(-0.0209315y^2 - 0.7837 \\ & +0.43544y) - 0.122457y^3 - 0.28597y^2 - 20.1494y - 1.9583, \\ \dot{y} \simeq\ & 1.290533111466 * 10^{-6}x^3 + x^2(0.00006795y - 0.0022262) + x(0.918295 \\ & +0.00119261y^2 - 0.0514766y) + 0.00697717y^3 - 0.21772y^2 - 35.9736 \\ & +0.783717y, \end{aligned} \tag{3.10}$$

with the first integral

$$\tilde{H}_3(x,y) \simeq \left(3\left(\frac{1}{10}x - y - 1\right) - 4\left(\frac{1}{100}x + 0.175511y - 2\right)^2\right)^2 + 9\left(\frac{1}{100}x + 0.175511y - 2\right)^2.$$

The pair $(y, Y) \simeq (-0.339012, 0.14389)$ is the unique real solution of system (3.4) for $i = 3$, therefore the discontinuous piecewise differential system (3.9)–(3.10) has the unique limit cycle shown in Figure 3.1(b). With this example we complete the proof of Theorem 3.2.



(a) (b) (c) ∎

Figure 3.1: The unique limit cycle of the discontinuous piecewise differential system, (a) for (3.5)–(3.6), (b) for (3.9)–(3.10), and (c) the two limit cycles of the discontinuous piecewise differential system (3.7)–(3.8).

Section 3.3 LC of PWS formed by two cubic centers

In this section we are interesting in studying the crossing LC of the piecewise differential systems separated by Σ and formed by $(\tilde{C}_i)$ in each region.

3.3.1 The maximum number of LC of PWS formed by two cubic centers

Our second result is as follows.

Theorem 3.3 *The maximum number of crossing limit cycles of the class of planar discontinuous piecewise cubic reversible isochronous centers separated by the straight line Σ and formed by*

(I) *the cubic reversible isochronous center $(\tilde{C}_1)$ in each half plane is at most two, this maximum is reached, see Figure 3.2(a);*

(II) *the cubic reversible isochronous centers $(\tilde{C}_1)$ and $(\tilde{C}_2)$ is at most three, this maximum is reached, see Figure 3.2(b);*

(III) *the cubic reversible isochronous centers $(\tilde{C}_1)$ and $(\tilde{C}_3)$ is at most three, this maximum is reached, see Figure 3.2(c);*

(IV) *the cubic reversible isochronous center $(\tilde{C}_2)$ in each half plane is at most eight, this maximum is reached, see Figure 3.3(a);*

(V) *the cubic reversible isochronous centers $(\tilde{C}_2)$ and $(\tilde{C}_3)$ is at most seven, this maximum is reached, see Figure 3.3(b);*

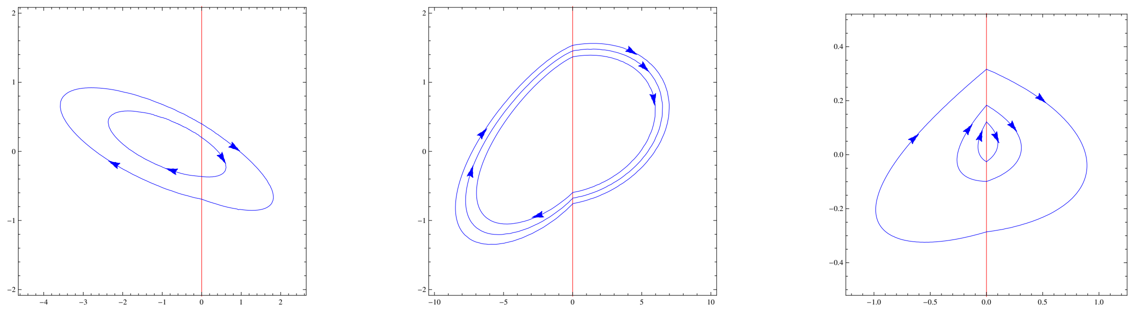
(VI) *the cubic reversible isochronous center $(\tilde{C}_3)$ in each half plane is at most three, this maximum is reached, see Figure 3.4(a).*

3.3.2 Proof of theorem 3.3

This subsection is devoted to give the proof of Theorem 3.3.

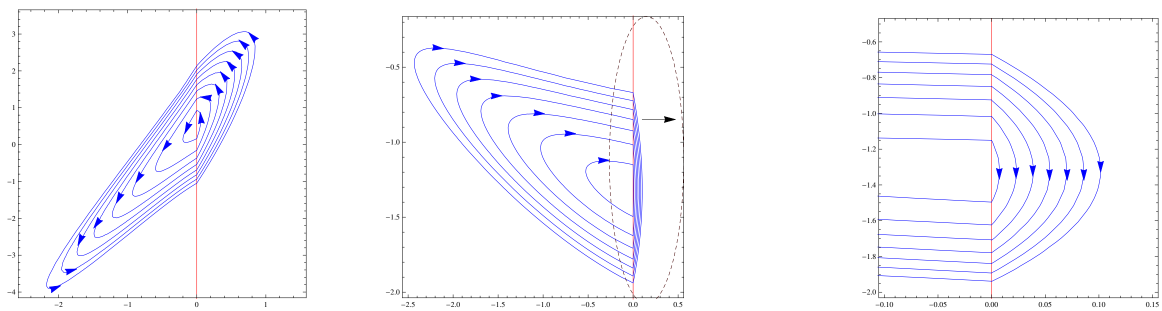
Proof.

For the proof of this theorem we start with the class of discontinuous piecewise differential



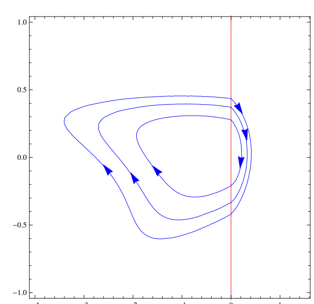
(*a*) (*b*) (*c*)

Figure 3.2: (*a*) The two limit cycles of the discontinuous piecewise differential system (3.13)–(3.14), the three limit cycles of the discontinuous piecewise differential system (*b*) for (3.15)–(3.16), and (*c*) for (3.17)–(3.18).



(*a*) (*b*)

Figure 3.3: (*a*) The eight limit cycles of the discontinuous piecewise differential system (3.19)–(3.20), and (*b*) the seven limit cycles of the discontinuous piecewise differential system (3.21)–(3.22).



(*a*)

Figure 3.4: (*a*) The three limit cycles of the discontinuous piecewise differential system (3.23)–(3.24).

system created by systems $(\tilde{\mathcal{C}}_i)$– $(\tilde{\tilde{\mathcal{C}}}_i)$ *with* $i = 1,2,3$. *In* Σ^+ *we consider the cubic reversible isochronous center* $(\tilde{\mathcal{C}}_i)$ *with its corresponding first integral* $\tilde{H}_i(x,y)$ *given either in* (3.1), *or* (3.2), *or* (3.3), *and in* Σ^- *we consider the second differential cubic reversible isochronous center* $(\tilde{\tilde{\mathcal{C}}}_i)$ *with its first integral* $\tilde{\tilde{H}}_i(x,y)$ *but with changing the parameters* $(a,b,a_1,b_1,c_1,\alpha_1,\beta_1,\gamma_1)$ *by* $(\alpha,\beta,a_2,b_2,c_2,\alpha_2,\beta_2,\gamma_2)$. *The following system of equations must be satisfied at the points*

$(0,y)$ and $(0,Y)$ with $y<Y$, to prove that the discontinuous piecewise differential system created by the systems $(\tilde{C}_i)$– $(\tilde{\tilde{C}}_i)$ for $i=1,2,3$ has a limit cycle intersecting the line of discontinuity Σ in these two different points

$$e_1=\tilde{H}_i(0,y)-\tilde{H}_i(0,Y)=0,\;\; e_2=\tilde{\tilde{H}}_i(0,y)-\tilde{\tilde{H}}_i(0,Y)=0. \tag{3.11}$$

Later on we shall give the proof of Theorem 3.3 for the class of the discontinuous piecewise differential system formed by systems $(\tilde{C}_i)$– $(\tilde{C}_j)$, for $i\neq j$ and $i,j=1,2,3$. In the first half-plane we consider the cubic reversible isochronous center $(\tilde{C}_i)$ with its corresponding first integral $\tilde{H}_i(x,y)$. In the second one we consider the differential cubic reversible isochronous center $(\tilde{C}_j)$ with its corresponding first integral $\tilde{H}_j(x,y)$, but with the parameters $(\alpha,\beta,a_2,b_2,c_2,\alpha_2,\beta_2,\gamma_2)$ instead of the parameters $(a,b,a_1,b_1,c_1,\alpha_1,\beta_1,\gamma_1)$. Thus if we assume the existence of a limit cycle of the discontinuous piecewise differential system $(\tilde{C}_i)$–$(\tilde{C}_j)$, it must intersects the discontinuity line Σ in the two points $(0,y)$ and $(0,Y)$ with $y\neq Y$, then these points are the solutions of the following system of equations

$$e_1=\tilde{H}_i(0,y)-\tilde{H}_i(0,Y)=0,\;\; e_2=\tilde{H}_j(0,y)-\tilde{H}_j(0,Y)=0. \tag{3.12}$$

Proof of statement** (I) **of Theorem 3.3. *We consider the discontinuous piecewise differential system $(\tilde{C}_1)$– $(\tilde{\tilde{C}}_1)$, we know that system (3.11) for $i=1$ is equivalent to*

$$\begin{aligned} e_1 = {} & 2ab_1^3yY+2ab_1^2c_1y+2ab_1^2c_1Y+2ab_1c_1^2-2ab_1\gamma_1^2+2a\beta_1^2b_1yY+4a\beta_1c_1\gamma_1+2a\beta_1^2c_1Y \\ & -\beta_1^2Y-2bb_1^2\gamma_1^2y-4b\beta_1b_1^2\gamma_1yY-2bb_1^2\gamma_1^2Y-4bb_1c_1\gamma_1^2+4b\beta_1^2b_1c_1yY+4b\beta_1c_1^2\gamma \\ & +2b\beta_1^2c_1^2y+2b\beta_1^2c_1^2Y-b_1^2y-b_1^2Y-2b_1c_1-2\beta_1\gamma_1-\beta_1^2y+2a\beta_1^2c_1y=0, \end{aligned}$$

$$\begin{aligned} e_2 = {} & 2\alpha b_2^3yY+2\alpha b_2^2c_2y+2\alpha b_2^2c_2Y-2\beta b_2^2\gamma_2^2y-4\beta\beta_2b_2^2\gamma_2yY-2\beta b_2^2\gamma_2^2Y-b_2^2Y-2\alpha b_2\gamma_2^2 \\ & -b_2^2y+2\alpha b_2c_2^2-4\beta b_2c_2\gamma_2^2+4\beta\beta_2^2b_2c_2yY-2b_2c_2-2\beta_2\gamma_2+2\alpha\beta_2^2b_2yY+4\beta\beta_2c_2^2\gamma_2 \\ & +2\beta\beta_2^2c_2^2y+2\beta\beta_2^2c_2^2Y+4\alpha\beta_2c_2\gamma_2+2\alpha\beta_2^2c_2y+2\alpha\beta_2^2c_2Y-\beta_2^2y-\beta_2^2Y=0. \end{aligned}$$

Now by doing the change of variable $yY\to z$ in e_1 and e_2, and solving $e_1=0$ with respect to the variable z and by substituting the value of z in $e_2=0$ we get an equation $P(y,Y)=0$ in the variables y and Y. Due to the big expression of $P(y,Y)$ we omit it.

From $P(y,Y)=0$ we obtain $Y=h(y)$, and by substituting it in $e_1=0$ we get the quadratic equation $E(y)=0$ in the variable y, where

$$\begin{aligned} E(y) = {} & \frac{1}{M}(2b_1y^2(a(b_1^2+\beta_1^2)+2b\beta_1(c_1\beta_1-b_1\gamma_1))(((2b\alpha\gamma_1^2+\alpha)b_2^3-2(\beta\beta_2\gamma_2+b\gamma_1(2\beta\beta_2\gamma_1 \\ & +\beta_1(2c_2\alpha-2\beta\gamma_2^2-1)))b_2^2+(\alpha+2c_2\beta)\beta_2^2(2b\gamma_1^2+1)b_2-2b((2\beta c_2^2+2\alpha c_2-1)\beta_1\beta_2^2) \end{aligned}$$

$b_1^2+2bc_1\beta_1^2((-2\beta\gamma_2^2+2c_2\alpha-1)b_2^2+((2\beta c_2^2+2\alpha c_2-1)\beta_2^2)b_1-b_2c(2bc_1^2-1)\beta_1^2(\alpha b_2^2$
$-2\beta\beta_2\gamma_2b_2+(\alpha+2c_2\beta)\beta_2^2)+a(b_1^2+\beta_1^2)(b_1((-2\beta\gamma_2^2+2c_2\alpha-1)b_2^2+(2\beta c_2^2+2\alpha c_2$
$-1)\beta_2^2)-2b_2c_1(\alpha b_2^2-2\beta\beta_2\gamma_2b_2+(\alpha+2c_2\beta)\beta_2^2)))y^2+4b_1(a(b_1^2+\beta_1^2)+2b\beta_1(c_1\beta_1$
$-b_1\gamma_1))(2b\beta_1\gamma_1((1-2\beta c_2^2-2\alpha c_2)\beta_2\gamma_2+b_2(-\alpha c_2^2+2\beta\gamma_2^2c_2+c_2+\alpha\gamma_2^2))b_1^2+c_1((\alpha$
$+2b\alpha\gamma_1^2)b_2^3-2\beta\beta_2(2b\gamma_1^2+1)\gamma_2b_2^2+((\alpha+2c_2\beta)\beta_2^2+2b(-\alpha\gamma_2^2\beta_1^2-2c_2\beta\gamma_2^2\beta_1^2-c_2\beta_1^2$
$+c_2^2\alpha\beta_1^2+\alpha\beta_2^2\gamma_1^2+2c_2\beta\beta_2^2\gamma_1^2))b_2+2b(2\beta c_2^2+2\alpha c_2-1)\beta_1^2\beta_2\gamma_2)b_1-b_2(2bc_1^2-1)\beta_1$
$\gamma_1(\alpha b_2^2-2\beta\beta_2\gamma_2b_2+(\alpha+2c_2\beta)\beta_2^2)+a(((2\beta c_2^2+2\alpha c_2-1)\beta_2\gamma_2+b_2(\alpha c_2^2-2\beta\gamma_2^2c_2$
$-c_2-\alpha\gamma_2^2)b_1^3-(\alpha(c_1^2-\gamma_1^2)b_2^3+2\beta\beta_2(\gamma_1^2-c_1^2)\gamma_2b_2^2+(-c_2^2\alpha\beta_1^2+\alpha(c_1^2\beta_2^2-\gamma_1^2\beta_2^2+\beta_1^2$
$\gamma_2^2)+c_2((2\beta\gamma_2^2+1)\beta_1^2+2\beta\beta_2^2(c_1^2-\gamma_1^2)))b_2+((-2\beta c_2^2-2\alpha c_2+1)\beta_1^2\beta_2\gamma_2)b_1-2b_2c_1$
$\beta_1\gamma_1((\alpha b_2^2-2\beta\beta_2\gamma_2b_2+(\alpha+2c_2\beta)\beta_2^2)))y+2b_1((a((b_1^2+\beta_1^2)+2b\beta_1(c_1\beta_1-b_1\gamma_1))$
$((-2b\gamma_1^2+2ac_1-1)((2\beta c_2^2+2\alpha c_2-1)\beta_2\gamma_2+b_2(\alpha c_2^2-((2\beta\gamma_2^2+1)c_2-\alpha\gamma_2^2))b_1^2+(1$
$-2bc_1^2-2ac_1)\beta_1((\gamma_1(-2\beta\gamma_2^2+2c_2\alpha-1)b_2^2+\beta_1(-\alpha c_2^2+2\beta\gamma_2^2c_2+c_2+\alpha\gamma_2^2)b_2+(2\beta$
$c_2^2+2\alpha c_2-1)\beta_2(\beta_2\gamma_1-\beta_1\gamma_2))-b_1((ac_1^2-((2b\gamma_1^2+1)c_1-a\gamma_1^2t)((2c_2\alpha-2\beta\gamma_2^2-1)$
$b_2^2+(2\beta c_2^2+2\alpha c_2-1)\beta_2^2))),$

with

$$M = a(b_1^2+\beta_1^2)(b_1(b_2^2(-2\beta\gamma_2^2+2\alpha c_2-1)+\beta_2^2(2c_2(\alpha+\beta c_2)-1))-2b_2c_1(\alpha b_2^2-2\beta\beta_2b_2\gamma_2 +\beta_2^2(\alpha+2\beta c_2)))+b_1^2(b_2^3(\alpha+2\alpha b\gamma_1^2)-2b_2^2(\beta\beta_2\gamma_2+b\gamma_1(2\beta\beta_2\gamma_1\gamma_2+\beta_1(-2\beta\gamma_2^2-1 +2\alpha c_2)))+\beta_2^2b_2(2b\gamma_1^2+1)(\alpha+2\beta c_2)-2b\beta_1\beta_2^2\gamma_1(2c_2(\alpha+\beta c_2)-1))+2b\beta_1^2b_1c_1(b_2^2 (-2\beta\gamma_2^2+2\alpha c_2-1)+\beta_2^2(2c_2(\alpha+\beta c_2)-1))-\beta_1^2b_2(2bc_1^2-1)(\alpha b_2^2-2\beta\beta_2b_2\gamma_2+\beta_2^2 (\alpha+2\beta c_2)).$$

Using Descartes's Theorem, we know that $E(y)=0$ can have at most two positive real solutions y_1 and y_2. Therefore system (3.11) when $i=1$ has at most two distinct real solutions (y_1,Y_1) and (y_2,Y_2). Consequently the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_1)$– $(\tilde{\tilde{\mathcal{C}}}_1)$ can have at most two limit cycles.

In order to complete the proof of this statement we build an example with exactly two limit cycles. In the half-plane Σ^- we consider the cubic reversible isochronous differential center $(\tilde{\mathcal{C}}_1)$

$$\begin{aligned} \dot{x} &\simeq (0.0967917-0.292708y)x^2+0.0390278x^3+(-0.975694y^2-0.644167y \\ &\quad +0.955083)x-1.29306y^2+\frac{779}{200}y+0.222722, \\ \dot{y} &\simeq (0.0390278y+0.0161167)x^2+(-0.292708y^2+0.236694y-0.406908)x \\ &\quad -0.975694y^3+0.122083y^2-0.718972y-0.233283, \end{aligned} \tag{3.13}$$

which has the first integral

$$\tilde{H}_1(x,y) \simeq -\frac{0.750534\left(x^2+5xy+1.51246x+6.25y^2+3.78114y-4.61833\right)}{x^2+4.03846xy+1.11538x+9.85577y^2+0.865385y+0.394231}.$$

In the half-plane Σ^+ *we consider the cubic reversible isochronous center* $(\tilde{\tilde{\mathcal{C}}}_1)$

$$\begin{aligned} \dot{x} &= \frac{1}{100}x^3+\left(-\frac{3}{40}y-\frac{31}{1000}\right)x^2+x\left(-\frac{1}{4}y^2-\frac{22}{25}y+\frac{433}{500}\right)-\frac{49}{40}y^2+\frac{369}{100}y \\ &\quad +\frac{211}{1000}, \\ \dot{y} &\simeq \left(\frac{1}{100}y+\frac{101}{2500}\right)x^2+\left(-\frac{3}{40}y^2+\frac{121}{500}y-\frac{1887}{5000}\right)x-\frac{1}{4}y^3+\frac{1}{25}y^2-0.666y \\ &\quad -0.2264, \end{aligned} \tag{3.14}$$

its first integral is $\tilde{\tilde{H}}_1(x,y) = -\dfrac{5\left(4x^2+20xy+20x+25y^2+50y-71\right)}{104x^2+420xy+116x+1025y^2+90y+41}.$

For $i=1$ *the two real solutions* $(0.2,-0.3625)$ *and* $(0.395217,-0.690869)$ *of system* (3.11) *provide the two limit cycles of discontinuous piecewise differential system* (3.13)–(3.14) *shown in Figure 3.2(a). This example completes the proof of statement* (I).

Proof of statement (II) ***of Theorem 3.3****. Now we consider the class of discontinuous piecewise differential system created by the cubic reversible isochronous center* $(\tilde{\mathcal{C}}_1)$ *with the first integral* $\tilde{H}_1(x,y)$ *and the cubic reversible isochronous differential center* $(\tilde{\mathcal{C}}_2)$ *with its first integral* $\tilde{H}_2(x,y)$. *Then system* (3.12) *when* $i=1$ *and* $j=2$ *is given by*

$$\begin{aligned} e_1 = {} & 2ab_1^3yY+2ab_1^2c_1y+2ab_1^2c_1Y+2ab_1c_1^2-2ab_1\gamma_1^2+2a\beta_1^2b_1yY+4a\beta_1c_1\gamma_1+2a\beta_1^2c_1y \\ & +2a\beta_1^2c_1Y-2bb_1^2\gamma_1^2y-4b\beta_1b_1^2\gamma_1yY-2bb_1^2\gamma_1^2Y-4bb_1c_1\gamma_1^2+4b\beta_1^2b_1c_1yY-\beta_1^2Y \\ & +2b\beta_1^2c_1^2y+2b\beta_1^2c_1^2Y-b_1^2y-b_1^2Y-2b_1c_1-2\beta_1\gamma_1-\beta_1^2y+4b\beta_1c_1^2\gamma_1=0, \\ e_2 = {} & 4b_2^6y^2Y^2(y+Y)+4b_2^5(-1+2c_2)yY\left(y^2+3yY+Y^2\right)+b_2^4\Big((y+Y)\Big((1-2c_2)^2(y^2+8yY \\ & +Y^2)+4\beta_2^2y^2Y^2\Big)+8\beta_2\gamma_2y^2Y^2\Big)+2b_2^3\Big(y^2\Big(8c_2^3-12c_2^2+c_2(8\beta_2Y(\gamma_2+\beta_2Y)+6)-2\beta_2 \\ & (2\gamma_2+3\beta_2Y)Y-1\Big)+2\beta_2^2(2c_2-1)y^3Y+yY(2\gamma_2^2+2\beta_2^2(2c_2-1)Y^2+4\beta_2\gamma_2(2c_2-1)Y \\ & +3(-1+2c_2)^3)+(-1+2c_2)^3Y^2\Big)+b_2^2\Big((y+Y)\Big(20c_2^4-40c_2^3+30c_2^2+\beta_2^2(2c_2-1)(y^2(2c_2 \\ & -1)+8(-1+c_2)yY+(2c_2-1)Y^2)-10c_2+1\Big)+2\beta_2\gamma_2((1-2c_2)^2y^2+yY(4c_2(4c_2-5) \\ & +5)+(1-2c_2)^2Y^2)+(4c_2-3)\gamma_2^2(y+Y)\Big)+2b_2(c_2-1)(2c_2-1)\Big(2c_2^3-2c_2^2+\gamma_2^2+c_2(\beta_2 \\ & (2y+2Y)(2\gamma_2+\beta_2(y+Y))+1)-\beta_2^2\left(y^2+3yY+Y^2\right)(-2\beta_2\gamma_2+y+Y)\Big)+\beta_2(-3c_2+1 \\ & +2c_2)^2\Big(2\gamma_2+\beta_2(y+Y)\Big)=0. \end{aligned}$$

From $e_1 = 0$ we obtain $Y = f(y)$, substituting it in $e_2 = 0$ we find an equation of the variable y of degree six. Therefore system (3.12) when $i = 1$ and $j = 2$ has at most six real solutions namely (y_i, Y_i) with $i \in \{1, \dots, 6\}$. Since $(y_i, Y_i) = (y_j, Y_j)$ with $i \in \{1,3,5\}$ and $j \in \{2,4,6\}$, then these solutions provide at most three limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_1)$– $(\tilde{\mathcal{C}}_2)$.

Now we prove that this result is reached by giving an example with exactly three limit cycles. In Σ^- we consider the cubic reversible isochronous center of type $(\tilde{\mathcal{C}}_2)$

$$\begin{aligned}\dot{x} \simeq\ & -0.514085x^3 + (3.75423y + 2.2093)x^2 + x(-0.735423y^2 - 25.5419y \\ & +3.44704) + 0.0365141y^3 + 2.17709y^2 + 43.1293y - 16.5836, \\ \dot{y} \simeq\ & -0.140845x^3 + (0.542254y + 0.792958)x^2 + x(-0.104225y^2 - 1.42958 \\ & -2.65859y) + 0.00514085y^3 - 3.29207y^2 + 8.88296y - 1.43575,\end{aligned} \tag{3.15}$$

with the first integral

$$\tilde{H}_2(x,y) = \frac{1}{14400(5x-8)^2}(15x-14)^2\Big(12325x^2 - 12100xy + 5580x + 3025y^2 - 3300y + 2056\Big).$$

In Σ^+ we consider the cubic reversible isochronous center of type $(\tilde{\mathcal{C}}_1)$

$$\begin{aligned}\dot{x} \simeq\ & -0.00141019x^2 + (0.000171292y - 0.217157)x + 0.0496747y^2 + 6.04893y \\ & -2.29839, \\ \dot{y} \simeq\ & -2.42082291791.10^{-6}x^2 + x(-0.00280815y - 0.170976) + 0.00362918y^2 \\ & +0.214343y - 0.0817112,\end{aligned} \tag{3.16}$$

its corresponding first integral is

$$\begin{aligned}\tilde{H}_1(x,y) \simeq\ & \frac{1}{(-10x + 13.2y - 5)^2 + (0.1x + 58y + 7084.68)^2}\Bigg(1 - 0.000281426\Bigg(\frac{1}{10}x + 58y \\ & + \frac{177117}{25}\Bigg)\Bigg).\end{aligned}$$

For the discontinuous piecewise differential system (3.15)–(3.16), system (3.12) when $i = 1$ and $j = 2$ has the three real solutions $(y, Y), \simeq \{(-0.598492, 1.36883), (-0.682118, 1.45531), (-0.759451, 1.53549)\}$. These solutions provide the three limit cycles of system (3.15)–(3.16) drawn in Figure 3.2(b). Then this statement holds.

***Proof of statement** (III) **of Theorem 3.3**. We consider the class of discontinuous piece-*

wise differential system composed by the cubic reversible isochronous center ($\tilde{\mathcal{C}}_1$) *with the first integral* $\tilde{H}_1(x,y)$ *and the cubic reversible isochronous differential center* ($\tilde{\mathcal{C}}_3$) *with its first integral* $\tilde{H}_3(x,y)$, *so system* (3.12) *when* $i=1$ *and* $j=3$ *is equivalent to*

$$
\begin{aligned}
e_1 = \;& 2ab_1^3yY + 2ab_1^2c_1y + 2ab_1^2c_1Y + 2ab_1c_1^2 - 2ab_1\gamma_1^2 + 2a\beta_1^2b_1yY + 4a\beta_1c_1\gamma_1 + 2a\beta_1^2c_1y \\
& + 2a\beta_1^2c_1Y - 2bb_1^2\gamma_1^2y - 4b\beta_1b_1^2\gamma_1yY - 2bb_1^2\gamma_1^2Y - 4bb_1c_1\gamma_1^2 - b_1^2y - 2b_1c_1 - 2\beta_1\gamma_1 \\
& - b1^2Y + 4b\beta_1^2b_1c_1yY + 4b\beta_1c_1^2\gamma_1 + 2b\beta_1^2c_1^2y + 2b\beta_1^2c_1^2Y - \beta_1^2y - \beta_1^2Y = 0, \\
e_2 = \;& 9b_2^2(y+Y) + 6b_2(3c_2 - 4(\beta_2^2y^2 + \beta_2y(2\gamma_2+\beta_2Y) + (\gamma_2+\beta_2Y)^2)) + \beta_2(2\gamma_2 + \beta_2(y+Y)) \\
& \Big(-24c_2 + 16\Big(2\gamma_2^2 + \beta_2^2\Big(y^2+Y^2\Big) + 2\beta_2\gamma_2(y+Y)\Big) + 9\Big) = 0.
\end{aligned}
$$

From $e_1 = 0$ *we obtain* $Y = f(y)$, *substituting it in* $e_2 = 0$ *we find an equation of the variable* y *of degree six. Thus system* (3.12) *when* $i=1$ *and* $j=3$ *has at most six real solutions, due to the symmetry of the solutions of this system, we conclude that the discontinuous piecewise differential system* ($\tilde{\mathcal{C}}_1$)– ($\tilde{\mathcal{C}}_3$) *has at most three limit cycles.*

In what follows we give a class of discontinuous piecewise differential system whith three limit cycles. So in Σ^- *we consider the cubic reversible isochronous center*

$$
\begin{aligned}
\dot{x} \simeq \;& -0.013446x^3 + x^2(0.305282y + 0.45729) + x(-2.31041y^2 - 4.90347y \\
& -2.46581) + 5.82847y^3 + 10.9181y^2 + 9.12441y - 0.481199, \\
\dot{y} \simeq \;& -0.0017767x^3 + x^2(0.04034y + 0.078041) + x(-0.305282y^2 - 0.84173 \\
& -0.914579y) + 0.770136y^3 + 2.45173y^2 + 2.46581y + 0.233753,
\end{aligned}
\tag{3.17}
$$

with the first integral

$$
\begin{aligned}
\tilde{H}_3(x,y) \simeq \;& \left(3\left(\frac{1}{2}x - 1.7828y\right) - 4\left(\frac{1}{10}x - 0.75681y + 0.410816\right)^2\right)^2 + 9\left(\frac{1}{10}x - 0.75681\right. \\
& \left. y + 0.410816\right)^2.
\end{aligned}
$$

In Σ^+ *we consider the cubic reversible isochronous center*

$$
\begin{aligned}
\dot{x} \simeq \;& \left(\frac{7}{200} - \frac{9}{40}y\right)x^2 + \frac{1}{100}x^3 + \left(-\frac{5}{8}y^2 - \frac{27}{20}y + 0.405909\right)x - \frac{15}{8}y^2 + 5.07727 \\
& y - 0.247727, \\
\dot{y} \simeq \;& \left(\frac{1}{100}y + \frac{9}{500}\right)x^2 + x\left(-\frac{9}{40}y^2 + \frac{3}{20}y - 0.173364\right) - \frac{5}{8}y^3 - \frac{17}{40}y^2 - 0.10591 \\
& y - 0.113909,
\end{aligned}
\tag{3.18}
$$

its corresponding first integral is $\tilde{H}_1(x,y) = -\dfrac{5\left(4x^2+20xy+20x+25y^2+50y-71\right)}{104x^2+300xy+124x+3125y^2-350y+61}$.

The pairs $(y,Y) \simeq \{(0.316546,-0.28577),(0.183095,-0.0988845),(0.12139,-0.026152)\}$ *are the distinct three real solution of system* (3.12) *when* $i=1$ *and* $j=3$. *Then the discontinuous piecewise differential system* (3.17)–(3.18) *has three limit cycles shown in Figure* 3.2*(c). This example completes the proof of statement* (III) *of Theorem* 3.3.

Proof of statement (IV) ***of Theorem 3.3***. *We consider the discontinuous piecewise differential system* $(\tilde{\mathcal{C}}_2)$– $(\tilde{\tilde{\mathcal{C}}}_2)$. *Then system* (3.11) *when* $i=2$ *is written as*

$$e_1 = \quad (y-Y)E_Y = 0,\ \ e_2 = (y-Y)E_y = 0.$$

We denote by E_y *and* E_Y *the polynomials of variables* y *and* Y *where*

$$\begin{aligned}
E_Y = \ & \Big(4b_1^6y^2Y^2(y+Y)+4b_1^5(2c_1-1)yY\left(y^2+3yY+Y^2\right)+b_1^4\Big((y+Y)\Big((1-2c_1)^2(y^2+8yY\\
&+Y^2)+4\beta_1^2y^2Y^2\Big)+8\beta_1\gamma_1y^2Y^2\Big)+2b_1^3\Big(y^2\Big(8c_1^3-12c_1^2+c_1(8\beta_1Y(\gamma_1+\beta_1Y)+6)-1\\
&-2\beta_1Y(2\gamma_1+3\beta_1Y)\Big)+2\beta_1^2(2c_1-1)y^3Y+yY(2\gamma_1^2+2\beta_1^2(2c_1-1)Y^2+4\beta_1\gamma_1Y(2c_1\\
&-1)+3(2c_1-1)^3)+(2c_1-1)^3Y^2\Big)+b_1^2\Big((y+Y)\Big(20c_1^4-40c_1^3+30c_1^2+\beta_1^2(2c_1-1)\Big((2c_1\\
&-1)y^2+8(c_1-1)yY+(2c_1-1)Y^2\Big)-10c_1+1\Big)+2\beta_1\gamma_1((1-2c_1)^2y^2+(4c_1(4c_1-5)\\
&+5)yY+(1-2c_1)^2Y^2)+(4c_1-3)\gamma_1^2(y+Y)\Big)+2b_1(c_1-1)(2c_1-1)\Big(2c_1^3-2c_1^2+\gamma_1^2+c_1\\
&(2\beta_1(y+Y)(2\gamma_1+\beta_1(y+Y))+1)-\beta_1^2(y^2+3yY+Y^2)-2\beta_1\gamma_1(y+Y)\Big)+\beta_1(2c_1^2-3c_1\\
&+1)^2(2\gamma_1+\beta_1(y+Y))\Big),\\
E_y = \ & \Big(4b_2^6y^2Y^2(y+Y)+4b_2^5(2c_2-1)yY\left(y^2+3yY+Y^2\right)+b_2^4\Big((y+Y)\Big((1-2c_2)^2(y^2+8yY\\
&+Y^2)+4\beta_2^2y^2Y^2\Big)+8\beta_2\gamma_2y^2Y^2\Big)+2b_2^3\Big(y^2\Big(8c_2^3-12c_2^2+c_2(8\beta_2Y(\gamma_2+\beta_2Y)+6)-2\beta_2\\
&(2\gamma_2+3\beta_2Y)Y-1\Big)+2\beta_2^2(2c_2-1)y^3Y+yY(2\gamma_2^2+2\beta_2^2(2c_2-1)Y^2+4\beta_2\gamma_2(2c_2-1)Y\\
&+3(2c_2-1)^3)+(2c_2-1)^3Y^2\Big)+b_2^2\Big((y+Y)\Big(20c_2^4-40c_2^3+30c_2^2+\beta_2^2(2c_2-1)((2c_2-1)y^2\\
&+8(c_2-1)yY+(2c_2-1)Y^2)-10c_2+1\Big)+2\beta_2\gamma_2((1-2c_2)^2y^2+(4c_2(4c_2-5)+5)yY+\\
&(1-2c_2)^2Y^2)+(4c_2-3)\gamma_2^2(y+Y)\Big)+2b_2(c_2-1)(2c_2-1)\Big(2c_2^3-2c_2^2+\gamma_2^2+c_2(2\beta_2(y\\
&+Y)(2\gamma_2+\beta_2(y+Y))+1)-\beta_2^2\left(y^2+3yY+Y^2\right)(y-2\beta_2\gamma_2+Y)\Big)+\beta_2(2c_2^2-3c_2+1)^2\\
&\Big(2\gamma_2+\beta_2(y+Y)\Big)\Big).
\end{aligned}$$

The number of the common zeros (y, Y) of E_y and E_Y show the existence and the number of limit cycles of the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_2)$– $(\tilde{\tilde{\mathcal{C}}}_2)$. To find this number, we calculate the two Resultants. $R_y = [E_y, E_Y, y]$ and $R_Y = [E_y, E_Y, Y]$ of E_y and E_Y with respect to y and Y, respectively. Knowing that E_y and E_Y are symmetry with respect to y and Y, it results that the resultant R_y and R_Y have the same expression. So we only need to calculate one of them, and in this case we consider R_y which is a polynomial of degree sixteen in the variable Y, and because of the big expression of R_y we omit it. Consequently the maximum number of solutions of system (3.11) when $i = 2$ is at most sixteen. Due to the symmetry of these solutions it results that the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_2)$– $(\tilde{\tilde{\mathcal{C}}}_2)$ can have at most eight limit cycles.

In what follows we construct a class of discontinuous piecewise differential system which has exactly eight limit cycles.
In the first half-plane Σ^- we consider the cubic reversible isochronous center $(\tilde{\mathcal{C}}_2)$

$$\begin{aligned}
\dot{x} \simeq\ & -1.33333\Big(\frac{3}{4}x-\frac{7}{10}\Big)\Big(\frac{24}{10}-\frac{1}{2}x\Big)\Big(\frac{11}{2}x-\frac{11}{4}y+\frac{3}{2}\Big),\\
\dot{y} \simeq\ & 0.484848\Big(\frac{11}{2}\Big(\frac{24}{10}-\frac{3}{2}x\Big)\Big(\frac{7}{10}-\frac{3}{4}x\Big)\Big(\frac{11}{2}x-\frac{11}{4}y+\frac{3}{2}\Big)-\frac{3}{4}\Big(2\Big(\frac{17}{10}-\frac{3}{4}x\Big)^3\\
& -2\Big(\frac{17}{10}-\frac{3}{4}x\Big)^2+\Big(\frac{11}{2}x-\frac{11}{4}y+\frac{3}{2}\Big)^2-\frac{3}{4}x+\frac{17}{10}\Big)\Big),
\end{aligned} \tag{3.19}$$

with the first integral

$$\tilde{H}_2(x,y) = \frac{1}{14400(5x-8)^2}(15x-14)^2(12325x^2-12100xy+5580x+3025y^2-3300y$$
$$+2056).$$

In Σ^+ we consider the cubic reversible isochronous center $(\tilde{\tilde{\mathcal{C}}}_2)$

$$\begin{aligned}
\dot{x} \simeq\ & x^2(4.19651-0.367605y)+\frac{9}{10}x^3+x((-1.08819*10^{-15}y-1.63216)y\\
& +5.10083)+y((-8.05325*10^{-31}y-1.87169*10^{-15})y-1.7198)\\
& +0.938074,\\
\dot{y} \simeq\ & x^2\Big(10.8999-\frac{9}{10}y\Big)+2.88353x^3+x((-2.66421*10^{-15}y-2.196)y\\
& +15.8829)+y((-1.97166*10^{-30}y-0.367605)y-3.8095)+4.1256,
\end{aligned} \tag{3.20}$$

its corresponding first integral is

$$\tilde{\tilde{H}}_2(x,y) \simeq \frac{1}{(2(-0.5x-7.40057*10^{-16}y-0.36)-1)^2}\Bigg(\Bigg(\Bigg(-\frac{1}{2}x-7.40057*10^{-16}y -\frac{36}{100}\Bigg)^2+(\frac{9}{10}x-0.367605y+0.200512)^2\Bigg)\Bigg(-\frac{1}{2}x-7.40057*10^{-16}y -\frac{36}{100}\Bigg)^2\Bigg).$$

The eight pairs $(y,Y) \simeq \{(0.162904, 0.928005), (0.162904, 0.928005), (-0.36574, 1.45665), (-0.537248, 1.62816), (-0.685078, 1.77599), (-0.816962, 1.90787), (-0.93716, 2.02807), (-1.04832, 2.13923)\}$ *are solutions of system* (3.11) *when* $i = 2$*. Consequently the discontinuous piecewise differential system* (3.19)–(3.20) *has eight limit cycles, see Figure* 3.3(*a*)*. Thus, the proof of statement* (IV) *holds.*

Proof of statement (V) ***of Theorem 3.3****. We consider the two classes of discontinuous piecewise differential system formed by the cubic reversible isochronous center* $(\tilde{\mathcal{C}}_2)$ *and* $(\tilde{\mathcal{C}}_3)$*, with the first integral* $\tilde{H}_2(x,y)$ *and* $\tilde{H}_3(x,y)$*, respectively, so system* (3.12) *for* $i = 2$ *and* $j = 3$ *becomes*

$$\begin{aligned} e_1 = \;& 4b_2^6y^2Y^2(y+Y)+4b_2^5(2c_2-1)yY\left(y^2+3yY+Y^2\right)+b_2^4\Big((y+Y)\Big((-2c_2+1)^2(y^2+8yY \\ &+Y^2)+4\beta_2^2y^2Y^2\Big)+8\beta_2\gamma_2y^2Y^2\Big)+2b_2^3\Big(y^2\Big(8c_2^3-12c_2^2+c_2(8\beta_2Y(\gamma_2+\beta_2Y)+6)-2\beta_2Y \\ &(2\gamma_2+3\beta_2Y)-1\Big)+2\beta_2^2(2c_2-1)y^3Y+yY(2\gamma_2^2+2\beta_2^2(2c_2-1)Y^2+4\beta_2\gamma_2(-1+2c_2)Y \\ &+3(2c_2-1)^3)+(2c_2-1)^3Y^2\Big)+b_2^2\Big((y+Y)\Big(20c_2^4-40c_2^3+30c_2^2+\beta_2^2(2c_2-1)((2c_2-1)y^2 \\ &+8(-1+c_2)yY+(-1+2c_2)Y^2)-10c_2+1\Big)+2\beta_2\gamma_2((1-2c_2)^2y^2+(4c_2(4c_2-5)+5)yY \\ &+(1-2c_2)^2Y^2)+(4c_2-3)\gamma_2^2(y+Y)\Big)+2b_2(c_2-1)(2c_2-1)\Big(2c_2^3-2c_2^2+\gamma_2^2+c_2(2\beta_2(y \\ &+Y)(2\gamma_2+\beta_2(y+Y))+1)-\beta_2^2(y^2+3yY+Y^2)(y-2\beta_2\gamma_2+Y)\Big)+\beta_2\left(-3c_2+2c_2^2+1\right)^2 \\ &\left(2\gamma_2+\beta_2(y+Y)\right)=0, \\ e_2 = \;& 9b_1^2(y+Y)+6b_1\left(3c_1-4\left(\beta_1^2y^2+\beta_1y(2\gamma_1+\beta_1Y)+(\gamma_1+\beta_1Y)^2\right)\right)+(32\gamma_1^2+16\beta_1^2y^2 \\ &-24c_1+32\beta_1\gamma_1y+16\beta_1^2Y^2+32\beta_1\gamma_1Y+9)(2\gamma_1\beta_1+\beta_1^2y+\beta_1^2Y)=0. \end{aligned}$$

Using Bézout Theorem 1.1 we know that the maximum number of solutions of system (3.12) *when* $i = 2$ *and* $j = 3$ *is at most fifteen. As the solutions of this system are symmetric, therefore we conclude that the maximum number of solutions of system* (3.12) *is at most seven. Hence the maximum number of limit cycles of the discontinuous piecewise differential system* $(\tilde{\mathcal{C}}_2)$– $(\tilde{\mathcal{C}}_3)$ *is at most seven.*

Now we give a class of discontinuous piecewise differential system that has exactly seven *limit cycles.*

In Σ^+ we consider the cubic reversible isochronous center

$$\begin{aligned} \dot{x} \simeq\ & 0.0000870489x^3 + x^2(-0.00792718y - 0.079887) + x(0.235678y^2 + \\ & 5.98489y - 39.2858) - 2.29883y^3 - 103.885y^2 + 862.313y + 1319.27, \\ \dot{y} \simeq\ & 2.963419321 * 10^{-6}x^3 + x^2(-0.000241147y - 0.00641714) - 2352.32 \\ & -2352.32 + x(0.00652981y^2 + 0.351331y + 4.42621) - 0.0588451y^3 \\ & -4.88435y^2 - 100.652y, \end{aligned} \tag{3.21}$$

its first integral is

$$\begin{aligned} \tilde{H}_2(x,y) \simeq\ & \frac{1}{(26y - x + 831.968)^2}(6.5 * 10^{-6}x^4 + x^3(-0.000719868y - 0.0160353) + x^2 \\ & ((0.0317101y + 0.864893)y + 44.7714) + x(y((-0.646011y - 15.2721) \\ & y - 1692.73) - 49965) + y(y(y(5.04228y + 94.2432) + 14776.6) + 1.05795 \\ & *10^6) + 1.7008 * 10^7). \end{aligned}$$

In Σ^- we consider the cubic reversible isochronous center

$$\begin{aligned} \dot{x} =\ & \frac{1}{9590625}(531250x^3 + 1875x^2(578y - 1535) + 75x(9826y^2 - 110160y \\ & -65075) + 167042y^3 - 4287315y^2 + 21974850y + 37022000), \\ \dot{y} =\ & \frac{1}{383625}(-31250x^3 - 3750x^2(17y + 5) - 75x(578y^2 - 3070y - 2825) - \\ & 9826y^3 + 165240y^2 + 195225y - 41500), \end{aligned} \tag{3.22}$$

which has the first integral

$$\tilde{H}_3(x,y) = 9\left(\frac{1}{4}x + \frac{17}{100}y + \frac{1}{5}\right)^2 + \left(3\left(\frac{1}{10}x + \frac{3}{4}y + 1\right) - 4\left(\frac{1}{4}x + \frac{17}{100}y + \frac{1}{5}\right)^2\right)^2.$$

System (3.12) *when $i = 2$ and $j = 3$ has the seven solutions* $(y, Y) \simeq \{(-1.89267, -0.723701),$ $(-1.9409, -0.669484), (-1.83955, -0.78291), (-1.77978, -0.84871), (-1.71019, -0.92431),$ $(-1.62382, -1.0167), (-1.49637, -1.15015)\}$. *Therefore the discontinuous piecewise differential system* (3.21)–(3.22) *has seven limit cycles shown in Figure* 3.3*(b). Thus, the proof of this statement holds.*

Proof of statement (VI) ***of Theorem 3.3*** *For the discontinuous piecewise differential* *system* $(\tilde{\mathcal{C}}_3)$– $(\tilde{\tilde{\mathcal{C}}}_3)$ *we obtain that the system* (3.11) *when* $i = 3$ *is given by*

$$e_1 = (y - Y)E_Y = 0, \quad e_2 = \quad (y - Y)E_y = 0.$$

Where

$$\begin{aligned} E_Y = \; & 9b_1^2(y+Y) + 6b_1\Big(3c_1 - 4\Big(\beta_1^2 y^2 + \beta_1 y(2\gamma_1 + \beta_1 Y) + (\gamma_1 + \beta_1 Y)^2\Big)\Big) + \beta_1(2\gamma_1 + \beta_1 y \\ & + \beta_1 Y)(32\gamma_1^2 - 24c_1 + 16\beta_1^2 y^2 + 32\beta_1\gamma_1 y + 16\beta_1^2 Y^2 + 32\beta_1\gamma_1 Y + 9), \\ E_y = \; & 9b_2^2(y+Y) + 6b_2\Big(3c_2 - 4\Big(\beta_2^2 y^2 + \beta_2 y(2\gamma_2 + \beta_2 Y) + (\gamma_2 + \beta_2 Y)^2\Big)\Big) + \beta_2(2\gamma_2 + \beta_2 y \\ & + \beta_2 Y)(32\gamma_2^2 - 24c_2 + 16\beta_2^2 y^2 + 32\beta_2\gamma_2 y + 16\beta_2^2 Y^2 + 32\beta_2\gamma_2 Y + 9). \end{aligned}$$

As in statement (IV) *and by computing the resultants, Resultant* $R_y = [E_y, E_Y, y]$ *and* $R_Y[E_y, E_Y, Y]$. *Due to their symmetry, we obtain that the resultant* R_y *is a polynomial of degree six. Consequently the maximum number of solutions of system* (3.11) *is at most six. Since their solutions are symmetric we know that the discontinuous piecewise differential system* $(\tilde{\mathcal{C}}_3)$– $(\tilde{\tilde{\mathcal{C}}}_3)$ *has at most three limit cycles.*

To prove that our result is reached we give an example of discontinuous piecewise differential system with exactly three limit cycles.

In the first half-plane Σ^- *we consider the cubic reversible isochronous center*

$$\begin{aligned} \dot{x} \simeq \; & -0.00343827x^3 + x^2(0.0908048y - 0.258721) + x(-0.799387y^2 + 2.20767 \\ & y + 0.10374) + 2.34576y^3 + 0.615762y^2 + 0.895054y - 0.0485279, \\ \dot{y} \simeq \; & -0.00039056x^3 + x^2(0.010315y - 0.044535) - 0.10374y + x(-0.090805y^2 \\ & + 0.517442y - 1.13956) + 0.266462y^3 - 1.10383y^2 - 0.00943529, \end{aligned} \tag{3.23}$$

with the first integral

$$\begin{aligned} \tilde{H}_3(x,y) \simeq \; & \left(3(-x - 0.300305y) - 4\left(0.880335y - 0.0508927 - \frac{1}{10}x\right)^2\right)^2 + 9\Big(-\frac{1}{10}x \\ & + 0.880335y - 0.0508927\Big)^2. \end{aligned}$$

In Σ^+ *we consider the cubic reversible isochronous center* $(\tilde{\tilde{\mathcal{C}}}_3)$

$$\begin{aligned} \dot{x} \simeq \; & -0.01185x^3 + 4.5222y + x^2(0.35556y - 0.23556) + x(-3.5556y^2 + 2.0444y \\ & -0.218889) + 11.8519y^3 + 3.11111y^2 - 0.245185, \end{aligned}$$

$$\begin{aligned}\dot{y} \simeq \; & -0.0011852x^3 + x^2(0.035556y - 0.0368889) - 0.137852 + x(-0.355556y^2 \\ & +0.471111y - 0.298556) + 1.18519y^3 - 1.02222y^2 + 0.218889y,\end{aligned} \tag{3.24}$$

its corresponding first integral is

$$\tilde{\tilde{H}}_3(x,y) = \; 9\Big(-\frac{1}{10}x + y - \frac{1}{10}\Big)^2 + \Big(3\Big(-\frac{1}{4}x - \frac{1}{2}y - \frac{1}{10}\Big) - 4\Big(-\frac{1}{10}x + y - \frac{1}{10}\Big)^2\Big)^2.$$

The discontinuous piecewise differential system (3.23)–(3.24) *has three limit cycles because system* (3.11) *when* $i = 3$ *has the three real solutions* $(y, Y) \simeq \{(-0.420112, 0.436464), (-0.20979, 0.278363), (-0.33396, 0.37219)(-0.33396, 0.37219)\}$. *Figure 3.4(a) shows these three limit cycles. Thus, the proof of Theorem 3.3 is done.* ■

Chapter 4

Four LC of PWS with Nilpotent Saddles Separated by a Straight Line

One of the most difficult tasks in the qualitative theory of planar differential systems is determining the maximum number of limit cycles that a class of planar differential systems can have. Therefore, tis chapter is devoted to solve the extinction of Hilbert problem for all classes of discontinuous piecewise differential systems with Hamiltonian nilpotent saddles, separated by the straight line $\Sigma = \{(x,y) : x = 0\}$.

Section 4.1 The cubic Hamiltonian nilpotent saddles

Montserrat Corbera and Claudia Valls characterized the global phase portraits in the Poincaré disk for all planar Hamiltonian vector fields with linear plus cubic homogeneous terms having nilpotent saddles at the origin, they proved that there are six classes of system Hamiltonian nilpotent saddles, which are presented in the following theorem.

Theorem 4.1 *A Hamiltonian planar polynomial vector field with linear plus cubic homogeneous terms has a nilpotent saddle at the origin if and only if, after a linear change of variables and a rescaling of its independent variable it can be written as one of the following six classes:*

$(\mathcal{C}_1)$ $\dot{x} = ax + by, \quad \dot{y} = -\frac{a^2}{b}x - ay + x^3$, *with* $b > 0$.

$(\mathcal{C}_2)$ $\dot{x} = ax + by - x^3, \quad \dot{y} = -\dfrac{a^2}{b}x - ay + 3x^2y$, *with* $a < 0$.

$(\mathcal{C}_3)$ $\dot{x} = ax + by - 3x^2y + y^3, \quad \dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + 3xy^2$, *with either* $a = b = 0$ *and* $c > 0$, *or* $c = 0 \quad ab \neq 0$ *and* $a^2/b - 6b < 0$.

$(\mathcal{C}_4)$ $\dot{x} = ax + by - 3x^2y - y^3, \quad \dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + 3xy^2$, *with either* $a = b = 0$ *and* $c < 0$, *or* $c = 0 \quad ab \neq 0$ *and* $b > 0$.

$(\mathcal{C}_5)$ $\dot{x} = ax + by - 3\mu x^2y + y^3, \quad \dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + x^3 + 3\mu xy^2$, *with either* $a = b = 0$ *and* $c > 0$, *or* $c = 0$ $b \neq 0$ *and* $(a^4 - b^4 - 6a^2b^2\mu)/b < 0$.

$(\mathcal{C}_6)$ $\dot{x} = ax + by - 3\mu x^2y - y^3, \quad \dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + x^3 + 3\mu xy^2$, *with either* $a = b = 0$ *and* $c < 0$, *or* $c = 0$ $b \neq 0$, *and* $(a^4 - b^4 - 6a^2b^2\mu)/b > 0$, *where* a, b, c, $\mu \in \mathbb{R}$.

Now we give the new expressions of the six classes of the Hamiltonian nilpotent saddles $(\mathcal{C}_i)$, with $i = \{1, \ldots, 6\}$ after doing a general affine change of variables.

LEMMA 4.1 *By doing a linear change of variable* $\{x \to \delta_1 + \alpha_1 X + \gamma_1 Y, y \to \delta_2 + \alpha_2 X + \gamma_2 Y\}$, *with* $\alpha_2\gamma_1 - \alpha_1\gamma_2 \neq 0$, *we obtain the new expressions* $(\tilde{\mathcal{C}_1})$, $(\tilde{\mathcal{C}_2})$, $(\tilde{\mathcal{C}_3})$, $(\tilde{\mathcal{C}_4})$, $(\tilde{\mathcal{C}_5})$ *and* $(\tilde{\mathcal{C}_6})$ *of the Hamiltonian nilpotent saddles mentioned in Theorem 4.1.*

Thus, the differential system $(\tilde{\mathcal{C}_1})$ *is*

$$
\begin{aligned}
\dot{X} = \quad & \frac{1}{b(\alpha_2\gamma_1 - \alpha_1\gamma_2)}\Big(- a^2\gamma_1(\delta_1 + \alpha_1 X + \gamma_1 Y) + ab(-\gamma_1\delta_2 - \gamma_2\delta_1 - \alpha_1\gamma_2 X \\
& -\alpha_2\gamma_1 X - 2\gamma_1\gamma_2 Y) + b\Big(- b\gamma_2\delta_2 + X\Big(3\alpha_1\gamma_1(\delta_1 + \gamma_1 Y)^2 - \alpha_2 b\gamma_2\Big) - b\gamma_2^2 Y \\
& +\gamma_1\delta_1^3 + \alpha_1^3\gamma_1 X^3 + 3\alpha_1^2\gamma_1 X^2(\delta_1 + \gamma_1 Y) + \gamma_1^4 Y^3 + 3\gamma_1^3\delta_1 Y^2 + 3\gamma_1^2\delta_1^2 Y\Big)\Big), \\
\dot{Y} = \quad & \frac{1}{b(\alpha_2\gamma_1 - \alpha_1\gamma_2)}\Big(a^2\alpha_1(\delta_1 + \alpha_1 X + \gamma_1 Y) + ab(\alpha_1\delta_2 + 2\alpha_1\alpha_2 X + \alpha_1\gamma_2 Y \\
& +\alpha_2\delta_1 + \alpha_2\gamma_1 Y) + b\Big(\alpha_2 b\delta_2 + X\Big(\alpha_2^2 b - 3\alpha_1^2(\delta_1 + \gamma_1 Y)^2\Big) + \alpha_2 b\gamma_2 Y - \alpha_1^4 X^3 \\
& -\alpha_1\delta_1^3 - 3\alpha_1^3 X^2(\delta_1 + \gamma_1 Y) - \alpha_1\gamma_1^3 Y^3 - 3\alpha_1\gamma_1^2\delta_1 Y^2 - 3\alpha_1\gamma_1\delta_1^2 Y\Big)\Big),
\end{aligned}
\tag{4.1}
$$

with the first integral

$$
\begin{aligned}
H_1(x,y) = \frac{1}{4}\Big(&\frac{2a^2}{b}(\delta_1 + \alpha_1 x + \gamma_1 y)^2 + 4a(\delta_1 + \alpha_1 x + \gamma_1 y)(\delta_2 + \alpha_2 x + \gamma_2 y) + 2b \\
&(\delta_2 + \alpha_2 x + \gamma_2 y)^2 - (\delta_1 + \alpha_1 x + \gamma_1 y)^4\Big).
\end{aligned}
\tag{4.2}
$$

The differential system $(\tilde{\mathcal{C}}_2)$ is written as

$$\begin{aligned}
\dot{X} = & \frac{1}{b(\alpha_2\gamma_1-\alpha_1\gamma_2)}\Big(-a^2\gamma_1(\delta_1+\alpha_1X+\gamma_1Y)-ab(\gamma_1\delta_2+\gamma_2\delta_1+\alpha_1\gamma_2X \\
& +\alpha_2\gamma_1X+2\gamma_1\gamma_2Y)+b\Big(X\Big(3\delta_1(2\alpha_1\gamma_1\delta_2+\alpha_1\gamma_2\delta_1+\alpha_2\gamma_1\delta_1)-\alpha_2b\gamma_2 \\
& +3\gamma_1^2Y^2(3\alpha_1\ \gamma_2+\alpha_2\gamma_1)+6\gamma_1Y(\alpha_1\gamma_1\delta_2+2\alpha_1\gamma_2\delta_1+\alpha_2\gamma_1\delta_1)\Big)+\gamma_2 \\
& \Big(-b\gamma_2Y+\delta_1^3+4\gamma_1^3Y^3+9\gamma_1^2\delta_1Y^2+6\gamma_1\delta_1^2Y\Big)+\delta_2\Big(3\gamma_1(\delta_1+\gamma_1Y)^2 \\
& -b\gamma_2\Big)+\alpha_1^2X^3(\alpha_1\gamma_2+3\alpha_2\gamma_1)+3\alpha_1X^2(\alpha_1\gamma_1\delta_2+\alpha_1\gamma_2\delta_1+2\alpha_2\gamma_1\delta_1 \\
& +2\gamma_1Y(\alpha_1\gamma_2+\alpha_2\gamma_1))\Big)\Big), \\
\dot{Y} = & \frac{1}{b(\alpha_2\gamma_1-\alpha_1\gamma_2)}\Big(a^2\alpha_1(\delta_1+\alpha_1X+\gamma_1Y)+ab(\alpha_1\delta_2+2\alpha_1\alpha_2X+\alpha_1\gamma_2Y \\
& +\alpha_2\delta_1+\alpha_2\gamma_1Y)+b\Big(\alpha_2b\delta_2-3\alpha_1\delta_1^2\delta_2-\alpha_2\delta_1^3+X\Big(\alpha_2^2b-6\alpha_1(\delta_1+\gamma_1Y) \\
& (\alpha_1\delta_2+\alpha_2\delta_1+\alpha_1\gamma_2Y+\alpha_2\gamma_1Y)\Big)+Y(\alpha_2b\gamma_2-3\delta_1(2\alpha_1\gamma_1\delta_2+\alpha_1\gamma_2\delta_1 \\
& +\alpha_2\gamma_1\delta_1))-4\alpha_1^3\alpha_2X^3-3\alpha_1^2X^2(\alpha_1\delta_2+3\alpha_2\delta_1+\alpha_1\gamma_2Y+3\alpha_2\gamma_1Y)-\gamma_1^2 \\
& (3\alpha_1\gamma_2+\alpha_2\gamma_1)Y^3-3\gamma_1Y^2(\alpha_1\gamma_1\delta_2+2\alpha_1\gamma_2\delta_1+\alpha_2\gamma_1\delta_1)\Big)\Big),
\end{aligned} \tag{4.3}$$

its corresponding first integral is

$$\begin{aligned}
H_2(x,y) = \frac{1}{2}\Big(\frac{a^2}{b}(\delta_1+\alpha_1x+\gamma_1y)^2+2a(\delta_1+\alpha_1x+\gamma_1y)(\delta_2+\alpha_2x+\gamma_2y)+b \\
(\delta_2+\alpha_2x+\gamma_2y)^2-2(\delta_1+\alpha_1x+\gamma_1y)^3(\delta_2+\alpha_2x+\gamma_2y)\Big).
\end{aligned} \tag{4.4}$$

The differential system $(\tilde{\mathcal{C}}_3)$ is given by

$$\begin{aligned}
\dot{X} = & -\frac{1}{(b+c)(\alpha_2\gamma_1-\alpha_1\gamma_2)}\Big(a^2\gamma_1(\delta_1+\alpha_1X+\gamma_1Y)+a(b+c)(\gamma_2\delta_1+\alpha_1\gamma_2X \\
& +\gamma_1\delta_2+\alpha_2\gamma_1X+2\gamma_1\gamma_2Y)+(b+c)\Big((\delta_2+\alpha_2X+\gamma_2Y)\Big(\gamma_2(b-3\delta_1^2+Y^2 \\
& (\gamma_2^2-6\gamma_1^2)-9\gamma_1\delta_1Y)+\gamma_2\delta_2^2+X^2(-3\alpha_1^2\gamma_2-3\alpha_1\alpha_2\gamma_1+\alpha_2^2\gamma_2)-X\Big(Y \\
& (9\alpha_1\gamma_1\gamma_2+3\alpha_2\gamma_1^2-2\alpha_2\gamma_2^2)+3\alpha_1\gamma_1\delta_2+6\alpha_1\gamma_2\delta_1+3\alpha_2\gamma_1\delta_1-2\alpha_2\gamma_2\delta_2\Big) \\
& +\delta_2(-3\gamma_1\delta_1-3\gamma_1^2Y+2\gamma_2^2Y)\Big)-c\gamma_1(\delta_1+\alpha_1X+\gamma_1Y)\Big)\Big), \\
\dot{Y} = & \frac{1}{(b+c)(\alpha_2\gamma_1-\alpha_1\gamma_2)}\Big(a^2\alpha_1(\delta_1+\alpha_1X+\gamma_1Y)+a(b+c)(\alpha_2\delta_1+2\alpha_1 \\
& \alpha_2X+\alpha_1\gamma_2Y+\alpha_1\delta_2+\alpha_2\gamma_1Y)+(b+c)\Big((\delta_2+\alpha_2X+\gamma_2Y)\Big(-3\alpha_1\delta_1\delta_2
\end{aligned}$$

$$
\begin{aligned}
&-3\alpha_2\delta_1^2+\alpha_2\delta_2^2+\alpha_2 b+X^2(\alpha_2^3-6\alpha_1^2\alpha_2)-X(3\alpha_1^2\delta_2+9\alpha_1\alpha_2\delta_1-2\alpha_2^2\delta_2\\
&+3\alpha_1^2\gamma_2 Y+9\alpha_1\alpha_2\gamma_1 Y-2\alpha_2^2\gamma_2 Y)-3\alpha_1\gamma_1\gamma_2 Y^2-3\alpha_2\gamma_1^2 Y^2+\alpha_2\gamma_2^2 Y^2\\
&-3\alpha_1\gamma_1\delta_2 Y-3\alpha_1\gamma_2\delta_1 Y-6\alpha_2\gamma_1\delta_1 Y+2\alpha_2\gamma_2\delta_2 Y\Big)-\alpha_1 c(\delta_1+\alpha_1 X\\
&+\gamma_1 Y)\Big)\Big),
\end{aligned}
\tag{4.5}
$$

which has the first integral

$$
\begin{aligned}
H_3(x,y)= &\ \frac{1}{4}\Big(-2(c-\frac{a^2}{b+c})(\delta_1+\alpha_1 x+\gamma_1 y)^2+4a(\delta_1+\alpha_1 x+\gamma_1 y)(\delta_2+\alpha_2 x\\
&+\gamma_2 y)+2b(\delta_2+\alpha_2 x+\gamma_2 y)^2-6(\delta_1+\alpha_1 x+\gamma_1 y)^2(\delta_2+\alpha_2 x+\gamma_2 y)^2\\
&+(\delta_2+\alpha_2 x+\gamma_2 y)^4\Big).
\end{aligned}
\tag{4.6}
$$

The differential system $(\tilde{C_4})$ *is*

$$
\begin{aligned}
\dot{X}= &\ -\frac{1}{(b+c)(\alpha_2\gamma_1-\alpha_1\gamma_2)}\Big(a^2\gamma_1(\delta_1+\alpha_1 X+\gamma_1 Y)+a(b+c)(\gamma_1\delta_2+\gamma_2\delta_1\\
&+\alpha_1\gamma_2 X+\alpha_2\gamma_1 X+2\gamma_1\gamma_2 Y)+(b+c)\Big(-(\delta_2+\alpha_2 X+\gamma_2 Y)\Big(3\gamma_1\delta_1\delta_2-b\gamma_2\\
&+3\gamma_2\delta_1^2+\gamma_2\delta_2^2+X^2(3\alpha_1^2\gamma_2+3\alpha_1\alpha_2\gamma_1+\alpha_2^2\gamma_2)+X\Big(3\alpha_1\gamma_1\delta_2+6\alpha_1\gamma_2\delta_1\\
&+3\alpha_2\gamma_1\delta_1+2\alpha_2\gamma_2\delta_2+Y(9\alpha_1\gamma_1\gamma_2+3\alpha_2\gamma_1^2+2\alpha_2\gamma_2^2)\Big)+Y^2(6\gamma_1^2\gamma_2+\gamma_2^3)\\
&+Y(3\gamma_1^2\delta_2+9\gamma_1\gamma_2\delta_1+2\gamma_2^2\delta_2)\Big)-c\gamma_1(\delta_1+\alpha_1 X+\gamma_1 Y)\Big)\Big),\\
\dot{Y}= &\ \frac{1}{(b+c)(\alpha_2\gamma_1-\alpha_1\gamma_2)}\Big(a^2\alpha_1(\delta_1+\alpha_1 X+\gamma_1 Y)+a(b+c)(\alpha_2\delta_1+2\alpha_1\alpha_2 X\\
&+\alpha_1\delta_2+\alpha_1\gamma_2 Y+\alpha_2\gamma_1 Y)+(b+c)\Big(\alpha_1(-c)(\delta_1+\alpha_1 X+\gamma_1 Y)-(\delta_2+\alpha_2 X\\
&+\gamma_2 Y)\Big(3\alpha_1\delta_1\delta_2+3\alpha_2\delta_1^2+\alpha_2\delta_2^2-\alpha_2 b+X^2(6\alpha_1^2\alpha_2+\alpha_2^3)+X(9\alpha_1\alpha_2\gamma_1 Y\\
&+3\alpha_1^2\delta_2+9\alpha_1\alpha_2\delta_1+2\alpha_2^2\delta_2+3\alpha_1^2\gamma_2 Y+2\alpha_2^2\gamma_2 Y)+3\alpha_2\gamma_1^2 Y^2+\alpha_2\gamma_2^2 Y^2\\
&+3\alpha_1\gamma_1\delta_2 Y+3\alpha_1\gamma_2\delta_1 Y+6\alpha_2\gamma_1\delta_1 Y+2\alpha_2\gamma_2\delta_2 Y+3\alpha_1\gamma_1\gamma_2 Y^2\Big)\Big)\Big),
\end{aligned}
\tag{4.7}
$$

with the first integral

$$
\begin{aligned}
H_4(x,y)= &\ \frac{1}{4}\Big(-2\Big(c-\frac{a^2}{b+c}\Big)(\delta_1+\alpha_1 x+\gamma_1 y)^2+4a(\delta_1+\alpha_1 x+\gamma_1 y)(\delta_2+\alpha_2 x\\
&+\gamma_2 y)+2b(\delta_2+\alpha_2 x+\gamma_2 y)^2-6(\delta_1+\alpha_1 x+\gamma_1 y)^2(\delta_2+\alpha_2 x+\gamma_2 y)^2\\
&-(\delta_2+\alpha_2 x+\gamma_2 y)^4\Big).
\end{aligned}
\tag{4.8}
$$

The differential system $(\tilde{\mathcal{C}}_5)$ is given by

$$\begin{aligned}
\dot{X} = & -\frac{1}{(b+c)(\alpha_2\gamma_1-\alpha_1\gamma_2)}\Big(a^2\gamma_1(\delta_1+\alpha_1X+\gamma_1Y)+a(b+c)(\gamma_1\delta_2+\gamma_2\delta_1+\alpha_1\\
& \gamma_2X+\alpha_2\gamma_1X+2\gamma_1\gamma_2Y)+(b+c)\Big(b\gamma_2\delta_2+\alpha_2b\gamma_2X+b\gamma_2^2Y-c\gamma_1(\delta_1+\alpha_1\\
& X+\gamma_1Y)-\gamma_1\delta_1^3+\gamma_2\delta_2^3-\alpha_1^3\gamma_1X^3+\alpha_2^3\gamma_2X^3-3\alpha_1^2\gamma_1\delta_1X^2+3\alpha_2^2\gamma_2\delta_2X^2\\
& -3\alpha_1^2\gamma_1^2X^2Y+3\alpha_2^2\gamma_2^2X^2Y-3\alpha_1\gamma_1\delta_1^2X+3\alpha_2\gamma_2\delta_2^2X-3\alpha_1\gamma_1^3XY^2+3\alpha_2\\
& \gamma_2^3XY^2-3\mu(\delta_1+\alpha_1X+\gamma_1Y)(\delta_2+\alpha_2X+\gamma_2Y)(\gamma_2\delta_1+2\gamma_1\gamma_2Y+\alpha_1\gamma_2X\\
& +\alpha_2\gamma_1X+\gamma_1\delta_2)-6\alpha_1\gamma_1^2\delta_1XY+6\alpha_2\gamma_2^2\delta_2XY+\gamma_2^4Y^3-3\gamma_1^3\delta_1Y^2-\gamma_1^4Y^3\\
& +3\gamma_2^3\delta_2Y^2-3\gamma_1^2\delta_1^2Y+3\gamma_2^2\delta_2^2Y\Big)\Big),\\
\dot{Y} = & \frac{1}{(b+c)(\alpha_2\gamma_1-\alpha_1\gamma_2)}\Big(a^2\alpha_1(\delta_1+\alpha_1X+\gamma_1Y)+a(b+c)(\alpha_1\delta_2+\alpha_2\delta_1+2\alpha_1\\
& \alpha_2X+\alpha_1\gamma_2Y+\alpha_2\gamma_1Y)+(b+c)\Big(-\alpha_1\delta_1^3+\alpha_2\delta_2^3+\alpha_2b\delta_2+\alpha_2^2bX+\alpha_2b\gamma_2\\
& Y-\alpha_1c(\delta_1+\alpha_1X+\gamma_1Y)-\alpha_1^4X^3+\alpha_2^4X^3-3\alpha_1^3\delta_1X^2+3\alpha_2^3\delta_2X^2-3\alpha_1^3\gamma_1\\
& X^2Y+3\alpha_2^3\gamma_2X^2Y-3\alpha_1^2\delta_1^2X+3\alpha_2^2\delta_2^2X-3\alpha_1^2\gamma_1^2XY^2+3\alpha_2^2\gamma_2^2XY^2-6\alpha_1^2\\
& \gamma_1\delta_1XY-3\mu(\delta_1+\alpha_1X+\gamma_1Y)(\delta_2+\alpha_2X+\gamma_2Y)(2\alpha_1\alpha_2X+\alpha_1\gamma_2Y+\alpha_2\\
& \gamma_1Y+\alpha_2\delta_1+\alpha_1\delta_2)+6\alpha_2^2\gamma_2\delta_2XY-\alpha_1\gamma_1^3Y^3-3\alpha_1\gamma_1^2\delta_1Y^2+3\alpha_2\gamma_2^2\delta_2Y^2\\
& -3\alpha_1\gamma_1\delta_1^2Y+\alpha_2\gamma_2^3Y^3+3\alpha_2\gamma_2\delta_2^2Y\Big)\Big),
\end{aligned} \tag{4.9}$$

which has the first integral

$$\begin{aligned}
H_5(x,y) = & \frac{1}{4}\Big(\Big(-2c+\frac{2a^2}{b+c}\Big)(\delta_1+\alpha_1x+\gamma_1y)^2+4a(\delta_1+\alpha_1x+\gamma_1y)(\delta_2+\alpha_2x\\
& +\gamma_2y)+2b(\delta_2+\alpha_2x+\gamma_2y)^2+6\mu(-\delta_1-\alpha_1x-\gamma_1y)^2(\alpha_2x+\gamma_2y\\
& +\delta_2)^2-(\delta_1+\alpha_1x+\gamma_1y)^4+(\delta_2+\alpha_2x+\gamma_2y)^4\Big).
\end{aligned} \tag{4.10}$$

The differential system $(\tilde{\mathcal{C}}_6)$ is

$$\begin{aligned}
\dot{X} = & -\frac{1}{(b+c)(\alpha_2\gamma_1-\alpha_1\gamma_2)}\Big(a^2\gamma_1(\delta_1+\alpha_1X+\gamma_1Y)+a(b+c)(\gamma_1\delta_2+\gamma_2\delta_1\\
& +\alpha_1\gamma_2X+\alpha_2\gamma_1X+2\gamma_1\gamma_2Y)+(b+c)\Big(b\gamma_2\delta_2+\alpha_2b\gamma_2X+b\gamma_2^2Y-c\gamma_1\\
& (\delta_1+\alpha_1X+\gamma_1Y)-\gamma_1\delta_1^3-\gamma_2\delta_2^3-\alpha_1^3\gamma_1X^3-\alpha_2^3\gamma_2X^3-3\alpha_1^2\gamma_1\delta_1X^2\\
& -3\alpha_2^2\gamma_2\delta_2X^2-3\alpha_1^2\gamma_1^2X^2Y-3\alpha_2^2\gamma_2^2X^2Y-3\alpha_1\gamma_1\delta_1^2X-3\alpha_2\gamma_2\delta_2^2X\\
& -3\alpha_1\gamma_1^3XY^2-3\alpha_2\gamma_2^3XY^2-3\mu(\delta_1+\alpha_1X+\gamma_1Y)(\delta_2+\alpha_2X+\gamma_2Y)(\gamma_1
\end{aligned} \tag{4.11}$$

$$\delta_2 + \gamma_2\delta_1 + \alpha_1\gamma_2 X + \alpha_2\gamma_1 X + 2\gamma_1\gamma_2 Y) - 6\alpha_1\gamma_1^2\delta_1 XY - 6\alpha_2\gamma_2^2\delta_2 XY$$
$$-\gamma_1^4 Y^3 - \gamma_2^4 Y^3 - 3\gamma_1^3\delta_1 Y^2 - 3\gamma_2^3\delta_2 Y^2 - 3\gamma_1^2\delta_1^2 Y - 3\gamma_2^2\delta_2^2 Y\Big)\Big),$$
$$\dot{Y} = \frac{1}{(b+c)(\alpha_2\gamma_1 - \alpha_1\gamma_2)}\Big(a^2\alpha_1(\delta_1 + \alpha_1 X + \gamma_1 Y) + a(b+c)(2\alpha_1\alpha_2 X + \alpha_2\delta_1$$
$$+\alpha_1\delta_2 + \alpha_1\gamma_2 Y + \alpha_2\gamma_1 Y) + (b+c)\Big(- \alpha_1\delta_1^3 + \alpha_2 b\gamma_2\alpha_2^2 bX + \alpha_2 b\delta_2 Y$$
$$-\alpha_2\delta_2^3 - \alpha_1 c(\delta_1 + \alpha_1 X + \gamma_1 Y) - \alpha_1^4 X^3 - \alpha_2^4 X^3 - 3\alpha_1^3\delta_1 X^2 - 3\alpha_2^3\delta_2 X^2$$
$$-3\alpha_1^3\gamma_1 X^2 Y - 3\alpha_2^3\gamma_2 X^2 Y - 3\alpha_1^2\delta_1^2 X - 3\alpha_2^2\delta_2^2 X - 3\alpha_1^2\gamma_1^2 XY^2 - 3\alpha_2^2\gamma_2^2$$
$$-6\alpha_1^2\gamma_1\delta_1 XY - 3\mu(\delta_1 + \alpha_1 X + \gamma_1 Y)(\delta_2 + \alpha_2 X + \gamma_2 Y)(\alpha_1\delta_2 + \alpha_2\delta_1$$
$$+2\alpha_1\alpha_2 X + \alpha_1\gamma_2 Y + \alpha_2\gamma_1 Y) - 6\alpha_2^2\gamma_2\delta_2 XY - \alpha_1\gamma_1^3 Y^3 - 3\alpha_1\gamma_1^2\delta_1 Y^2$$
$$-\alpha_2\gamma_2^3 Y^3 - 3\alpha_2\gamma_2^2\delta_2 Y^2 - 3\alpha_1\gamma_1\delta_1^2 Y - 3\alpha_2\gamma_2\delta_2^2 Y\Big)\Big),$$

its first integral is

$$H_6(x,y) = \frac{1}{4}\Big(- 2\Big(c - \frac{a^2}{b+c}\Big)(\delta_1 + \alpha_1 x + \gamma_1 y)^2 + 4a(\delta_1 + \alpha_1 x + \gamma_1 y)(\delta_2 + \alpha_2 x$$
$$+\gamma_2 y) + 2b(\delta_2 + \alpha_2 x + \gamma_2 y)^2 + 6\mu(-\delta_1 - \alpha_1 x - \gamma_1 y)^2(\alpha_2 x + \gamma_2 y \tag{4.12}$$
$$+\delta_2)^2 - (\delta_1 + \alpha_1 x + \gamma_1 y)^4 - (\delta_2 + \alpha_2 x + \gamma_2 y)^4\Big).$$

Section 4.2 LC of PWS with cubic Hamiltonian saddles

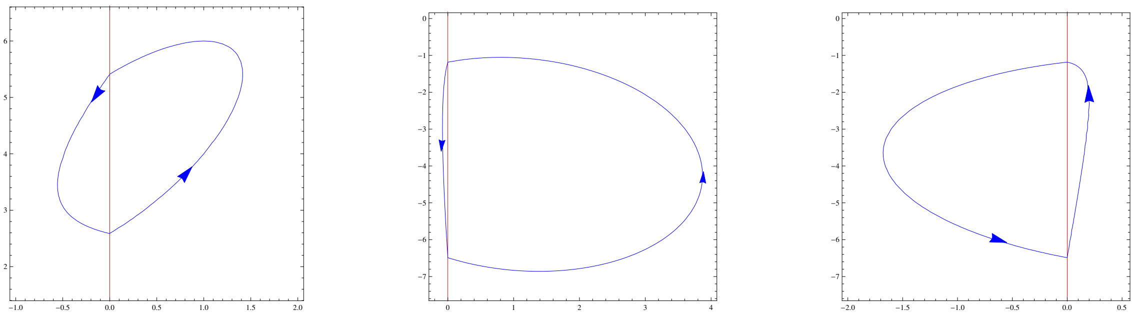
Our first goal in this chapter is to provide the maximum number of limit cycles for each of the six classes of discontinuous piecewise differential systems created by linear center and a cubic Hamiltonian nilpotent saddles having the straight line Σ as a switching curve.

4.2.1 Statement of the first main result

Our first main result is presented in the following theorem.

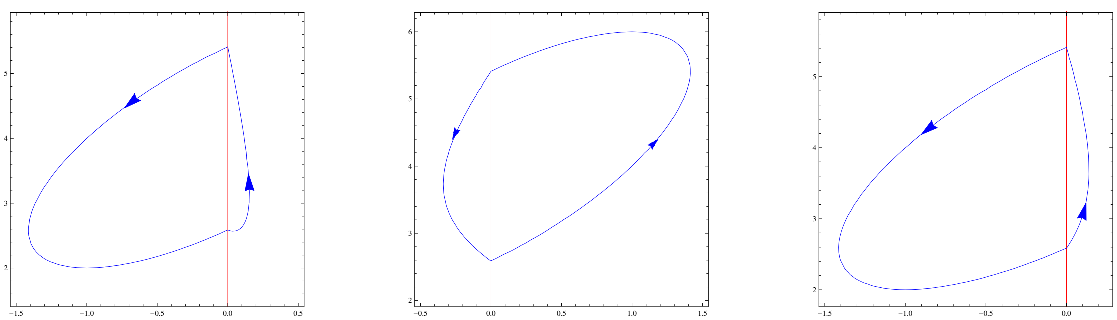
Theorem 4.2 *The maximum number of crossing limit cycles of the discontinuous piecewise differential systems separated by the straight line Σ, and formed by a linear differential center* (1.6) *in one half-plane, and by one of the six classes of the Hamiltonian nilpotent saddles $(\tilde{C}_k)$ for $k \in \{1,2,3,4,5,6\}$ is at most one. There are examples with ex-*

actly one limit cycle for all these discontinuous piecewise systems, see Figures 4.1 and 4.2.



(*a*) (*b*) (*c*)

Figure 4.1: The unique limit cycle of the discontinuous piecewise differential system, (*a*) for (4.14)–(4.15), (*b*) for (4.16)–(4.17), and (*c*) for (4.18)–(4.19).



(*a*) (*b*) (*c*)

Figure 4.2: The unique limit cycle of the discontinuous piecewise differential system, (*a*) for (4.20)–(4.21), (*b*) for (4.22)–(4.23), and (*c*) for (4.24)–(4.25).

4.2.2 Proof of theorem 4.2

Now we are going to give the proof of theorem 4.2 for the class of discontinuous piecewise differential systems separated by the straight line Σ, and formed by the linear center and one of the six classes of the Hamiltonian nilpotent saddles.

Proof.

In the first half-plane $\Sigma^+ = \{(x,y) : x \geq 0\}$ we consider the Hamiltonian nilpotent saddle $(\tilde{C}_k)$ for $k \in \{1,2,3,4,5,6\}$ with its first integral $H_k(x,y)$, and in the second half-plane $\Sigma^- = \{(x,y) : x \leq 0\}$ we consider the planar linear differential center (1.6) with its first integral $H(x,y)$ given by (1.7).

To show that the discontinuous piecewise differential system (1.6)–$(\tilde{C}_k)$ for $k \in \{1,2,3,4,5,6\}$ has at most a crossing limit cycle intersecting the line of discontinuity $x = 0$ in two different *points $(0,y_1)$ and $(0,y_2)$, with $y_1 \neq y_2$. We can immediately realize that these two points must satisfy the system of equations*

$$\begin{aligned} e_1 &= H(0,y_1) - H(0,y_2) = (y_1 - y_2)\Big((A^2+\omega^2)(y_1+y_2) - 2B\Big) = 0, \\ e_2 &= H_k(0,y_1) - H_k(0,y_2) = (y_1 - y_2)h_k(y_1,y_2) = 0. \end{aligned} \tag{4.13}$$

From $e_1 = 0$ we get $y_1 = f(y_2)$ for $k \in \{1,2,3,4,5,6\}$, and by substituting the expression of y_1 in $h_k(y_1,y_2) = 0$, we obtain a quadratic equation $p_k(y_2) = 0$ in the variable y_2 for $k \in \{1,2,3,4,5,6\}$. In particular for $k = 1$,

$$\begin{aligned} p_1(y_2) = \; & \frac{1}{b\,(A^2+\omega^2)^3}\Big(a^2\gamma_1\left(A^2+\omega^2\right)^2\left(\delta_1\left(A^2+\omega^2\right)+B\gamma_1\right)+ab\left(A^2+\omega^2\right)^2\Big(\left(A^2+\omega^2\right) \\ & (\gamma_1\delta_2+\gamma_2\delta_1)+2B\gamma_1\gamma_2\Big)+b\Big(-B\left(A^2+\omega^2\right)^2\left(-b\gamma_2^2+3\gamma_1^2\delta_1^2+\gamma_1^4y_2^2-2\gamma_1^3\delta_1y_2\right) \\ & -\left(A^2+\omega^2\right)^3(\gamma_1\delta_1^3-b\gamma_2\delta_2+\gamma_1^3\delta_1y_2^2)+2B^2\gamma_1^3\left(A^2+\omega^2\right)(\gamma_1y_2-2\delta_1)-2B^3\gamma_1^4\Big)\Big). \end{aligned}$$

This equation has at most two real solutions for the variable y_2. Therefore, system (4.13) has at most two real solutions. We can easily prove that these solutions verify the relation $(y_1, f(y_1)) = (f(y_2), y_2)$, which means that both solutions are symmetric and provide one limit cycle for the discontinuous piecewise differential system (1.6)–($\tilde{\mathcal{C}}_1$).

By a similar way we conclude that for $k \in \{2,3,4,5,6\}$, system (4.13) has at most two real solutions, Consequently the discontinuous piecewise differential system (1.6)–($\tilde{\mathcal{C}}_k$) for $k \in \{2,3,4,5,6\}$ has at most one limit cycle.

One limit cycle for the discontinuous piecewise differential system (1.6) –($\tilde{\mathcal{C}}_1$). *In the half-plane Σ^+ we consider the Hamiltonian nilpotent saddle*

$$\begin{aligned} \dot{x} \simeq \; & -0.519884\Big(\frac{1}{2}\Big(-0.934804x^3+2.80441(0.934804y-1)x^2+x(-2.80441 \\ & (0.9348y-1)^2-1.8875)+0.763627y^3-2.8223-2.45066y^2+1.48152y\Big) \\ & -0.934804(-x-1+0.934804y)+\frac{1}{2}(-0.827009x-2.82311y-0.827009)\Big), \\ \dot{y} \simeq \; & -0.519884\Big(\frac{1}{2}\Big(x(3.125-3(0.934804y-1)^2)-x^3+3(0.934804y-1)x^2 \\ & +0.816885y^3-2.62157y^2+4.69191y+2.125\Big)-\frac{1}{2}(5x+5-0.82701y)+x \\ & -0.934804y+1\Big), \end{aligned} \tag{4.14}$$

with the first integral

$$H_1(x,y) \simeq \frac{1}{4}\Big(-(-x+0.934804y-1)^4 + 4(-x+0.934804y-1)^2 - 4\Big(-\frac{5}{2}x - \frac{151}{100}y - \frac{5}{2}\Big)$$
$$(-x+0.934804y-1) + \Big(-\frac{5}{2}x - \frac{151}{100}y - \frac{5}{2}\Big)^2\Big).$$

In the half-plane Σ^- we consider the linear differential center

$$\dot{x} = 2 + \frac{1}{2}x - \frac{1}{2}y, \qquad \dot{y} = 2 + x - \frac{1}{2}y, \tag{4.15}$$

its first integral is $H(x,y) = 2(2x-2y) + \Big(x - \frac{1}{2}y\Big)^2 + \frac{1}{4}y^2$.

The unique real solution of system (4.13), when $k = 1$ is $(y_1, y_2) \simeq (2.58579, 5.41421)$. So, the discontinuous piecewise differential system (4.14)–(4.15) *has one limit cycle shown in Figure 4.1(a).*

One limit cycle for the discontinuous piecewise differential system (1.6)–($\tilde{\mathcal{C}}_2$).

In the half-plane Σ^+ we consider the linear differential center

$$\dot{x} = -\frac{1}{10}x - \frac{73}{8}y - \frac{7}{2}, \qquad \dot{y} = x + \frac{1}{10}y - \frac{7}{10}, \tag{4.16}$$

its corresponding first integral is given by $H(x,y) = \Big(x + \frac{1}{10}y\Big)^2 + 2\Big(\frac{7}{2}y - \frac{10}{7}x\Big) + \frac{361}{400}y^2$.

In the half-plane Σ^- we consider the Hamiltonian nilpotent saddle

$$\begin{aligned}
\dot{x} \simeq\ & -1.1432\Big(\frac{1}{2}(1.55789 - 1.27055x - 0.28925y) + 0.09578(1 - x - 0.09578y) \\
& -1/2(-0.79166x^3 + 1/2(-0.755 + 0.287336(1 - 0.09578y)^2) - 3x^2(-1.079 \\
& +0.243384y) + x(-5.9865 + 1.62543y - 0.118079y^2) + \frac{151}{100}(1 + 0.18033y \\
& +0.082562y^2 - 0.00351452y^3))\Big), \\
\dot{y} \simeq\ & -1.14322\Big(-\frac{1}{2}\Big(-10x^3 - 3x^2(-0.79166y - 7) + x(-3.125 - 6(1.27055y + 2) \\
& (1 - 0.09578y)) + 0.039359y^3 - 0.81271y^2 + 5.9865y + 0.375\Big) - \frac{1}{2}(5x - 2 \\
& -1.27055y) + x + 0.0957787y - 1\Big),
\end{aligned} \tag{4.17}$$

with the first integral

$$\begin{aligned}
H_2(x,y) \simeq\ & -\frac{5}{2}x^4 + (0.79166y + 7)x^3 + (0.365076y^2 - 3.23699y - 6.0625)x^2 + (0.039359y^3 \\
& -0.812713y^2 + 4.52439y + 0.375)x + 0.00132673y^4 + 0.473279y - 0.041995y^3 \\
& -0.276187y^2 - 1.0625.
\end{aligned}$$

The discontinuous piecewise differential system (4.16)–(4.17) *has one limit cycle, because system (4.13) has the unique real solution* $(y_1, y_2) \simeq (-6.48905, -1.18218)$, *when* $k = 2$. *We illustrate this limit cycle in Figure 4.1(b).*

One limit cycle for the discontinuous piecewise differential system (1.6)–($\tilde{\mathcal{C}}_3$)**.** *In the half-plane* Σ^+ *we consider the Hamiltonian nilpotent saddle*

$$\begin{aligned}\dot{x} \simeq\ & 0.364018\Big(\Big(-\frac{5}{2}x + \frac{151}{100}y - \frac{1}{2}\Big)\Big((16.2889y - 9.86591)x - 8.61886x^2\\ & -1.22439y^2 - 4.8122y + 4.89477\Big) - 0.247424(-x + 1 - 0.494848y)\Big),\\ \dot{y} \simeq\ & -0.364018\Big(\Big(-\frac{5}{2}x + \frac{151}{100}y - \frac{1}{2}\Big)\Big(x(\frac{109}{4} - 25.4791y) + \frac{5}{8}x^2 + 6.10535y^2\\ & -5.375 - 1.62455y\Big) - \frac{1}{2}(-x - 0.494848y + 1)\Big),\end{aligned} \tag{4.18}$$

this system has the first integral

$$\begin{aligned}H_3(x, y) \simeq\ & \frac{1}{4}\Big(\Big(\frac{5}{2}x - \frac{151}{100}y + \frac{1}{2}\Big)^4 - \frac{75}{2}\Big(x - 0.604y + \frac{1}{5}\Big)^2(x + 0.494848y - 1)^2 - (x - 1\\ & +0.494848y)^2\Big).\end{aligned}$$

In the half-plane Σ^- *we consider the linear differential center*

$$\dot{x} = -\frac{1}{10}x - \frac{73}{80}y - \frac{7}{2}, \qquad \dot{y} = x + \frac{1}{10}y - \frac{7}{10}, \tag{4.19}$$

its corresponding first integral is $H(x, y) = \Big(x + \frac{1}{10}y\Big)^2 + 2\Big(\frac{7}{2}y - \frac{7}{10}x\Big) + \frac{361}{400}y^2.$

The unique real solution of system (4.13), *for* $k = 3$ *is* $(y_1, y_2) \simeq (-6.48905, -1.18218)$, *which provides the unique limit cycle of the discontinuous piecewise differential system* (4.18)–(4.19), *see Figure 4.1(c).*

One limit cycle for the discontinuous piecewise differential system (1.6)–($\tilde{\mathcal{C}}_4$)**.** *In the half-plane* Σ^+ *we consider the linear differential center*

$$\dot{x} = 2 + \frac{1}{2}x - \frac{1}{2}y, \qquad \dot{y} = x - 0.5y + 2, \tag{4.20}$$

with its first integral $H(x, y) = \Big(x - \frac{1}{2}y\Big)^2 + 2(2x - 2y) + \frac{1}{4}y^2.$
In the half-plane Σ^- *we consider the Hamiltonian nilpotent saddle*

$$\dot{x} \simeq\ -0.914733\Big(\frac{1}{2}\Big(-\frac{3}{2}x - 0.157142y + \frac{1}{2}\Big)(x(-11.4023 + 8.6346) - 3.86892x^2$$

$$-2.15367y^2+5.5804y-2.69714)-\frac{1}{4}(-2.34357x-0.474567y+0.91214)$$
$$-0.3775\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)\Big),$$
$$\dot{y}\simeq -0.914733\Big(-\frac{1}{2}(-\frac{3}{2}x-0.157142y+\frac{1}{2})\Big(x(-11.0175y+9.375)-5.625x^2 \quad (4.21)$$
$$-4.875-10.6534y^2+15.1939y-4.875\Big)+\frac{1}{8}\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)+\frac{1}{4}\Big(-\frac{3}{2}x$$
$$-2.34357y+\frac{7}{4}\Big)\Big),$$

which has the first integral

$$H_4(x,y)\simeq (5.625-5.80339y)x^3-2.10938x^4+x^2(-8.85571y^2+14.8864y-6.3125)+x$$
$$(-1.67409y^3+7.7143y^2-9.15731y+3.0625)-0.0846078y^4+0.651249y^3$$
$$-1.15563y^2+1.04964y-0.328125.)$$

The discontinuous piecewise differential system (4.20)–(4.21) has exactly one limit cycle which is shown in Figure 4.2(a), because system (4.13), when $k=4$ has the unique real solution $(y_1,y_2)\simeq(2.58579, 5.41421)$.

One limit cycle for the discontinuous piecewise differential system (1.6)–($\tilde{C}_5$).

In the half-plane Σ^+ we consider the linear differential center

$$\dot{x}=2+\frac{1}{2}x-\frac{1}{2}y, \qquad \dot{y}=x-\frac{1}{2}y+2, \quad (4.22)$$

with its first integral $H(x,y)=\Big(x-\frac{1}{2}y\Big)^2+2(2x-2y)+\frac{1}{4}y^2.$

In the half-plane Σ^- we consider the Hamiltonian nilpotent saddle

$$\dot{x}\simeq -0.99224\Big(-8.29721x^3-40.9634x^2y+7.35346x^2-66.3665xy^2$$
$$+21.6087xy+0.755\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)+\frac{3}{2}\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)\Big(-\frac{3}{2}x+0.5$$
$$-2.51436y\Big)(-3.52218x-7.59336y+3.26936)-0.563652x-34.7688y^3$$
$$+13.5148y^2+2.0988y-1.19571\Big), \quad (4.23)$$
$$\dot{y}\simeq 0.99224\Big(-5x^3-24.8916x^2y+4.6875x^2-40.9634xy^2+14.7069$$
$$xy+\frac{1}{4}\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)+\frac{3}{2}\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)\Big(-\frac{3}{2}x-2.51436y+\frac{1}{2}\Big)\Big(-\frac{3}{2}x$$

$$-3.52218y+\frac{7}{4}\Big)-0.9375x-22.1222y^3+10.8043y^2-0.563652y$$
$$-0.3125\Big),$$

this system has the first integral

$$H_5(x,y)\simeq\ \frac{1}{4}\Big(-\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)^4-3\Big(-\frac{151}{100}x-2.51436y+\frac{1}{2}\Big)^2(\frac{1}{2}x+\frac{151}{100}y-1)^2-\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)^2+\Big(-\frac{3}{2}x-2.51436y+\frac{1}{2}\Big)^4\Big).$$

The pair $(y_1,y_2)\simeq(2.58579,5.41421)$ *is the unique real solution of system (4.13), when* $k=5$. *So, the discontinuous piecewise differential system* (4.22)–(4.23) *has one limit cycle shown in Figure 4.2(b).*

One limit cycle for the discontinuous piecewise differential system (1.6)–($\tilde{\mathcal{C}}_6$).

In the half-plane Σ^+ *we consider the Hamiltonian nilpotent saddle*

$$\begin{aligned}\dot{x}\simeq\ & 0.302806\Big(-6.81399x^3+30.7698x^2y+5.87024x^2-35.033xy^2-26.212xy\\ &-\frac{3}{2}(1.22756x-6.26615y+1.31989)\Big(\frac{3}{2}x-2.07489y-\frac{1}{2}\Big)\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)\\ &+\frac{151}{200}\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)-0.069246x+23.7332y^3+3.07019y^2+10.0692y\\ &-1.25064\Big),\\ \dot{y}\simeq\ & -0.302806\Big(\frac{41}{8}x^3-20.442x^2y-5.4375x^2+30.7698xy^2+11.7405xy-\frac{3}{2}\\ &\Big(\frac{3}{2}x-2.07489y-\frac{1}{2}\Big)\Big(\frac{3}{2}x+1.22756y-1.75\Big)\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)+\frac{1}{4}\Big(\frac{1}{2}x-1\\ &+\frac{151}{100}y\Big)+2.4375x-11.6776y^3-13.1067y^2-0.0692464y-0.6875\Big),\end{aligned}\tag{4.24}$$

its first integral is

$$H_6(x,y)=\ \frac{1}{4}\Big(-\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)^4+3\Big(\frac{3}{2}x-2.07489y-\frac{1}{2}\Big)^2\Big(\frac{1}{2}x+\frac{151}{100}y-1\Big)^2-\Big(-1+\frac{1}{2}x+\frac{151}{100}y\Big)^2\Big(\frac{3}{2}x-2.07489y-\frac{1}{2}\Big)^4\Big).$$

In the half-plane Σ^- *we consider the linear differential center*

$$\dot{x}=2+\frac{1}{2}x-\frac{1}{2}y,\qquad \dot{y}=x-\frac{1}{2}y+2,\tag{4.25}$$

with the first integral $H(x,y)=\Big(x-\frac{1}{2}y\Big)^2+2(2x-2y)+\frac{1}{4}y^2.$

The pair $(y_1,y_2)\simeq(2.58579,5.41421)$ *is the unique real solution of system (4.13), when* $k=6$. *This proves that the discontinuous piecewise differential system* (4.24)–(4.25) *has the* *unique limit cycle shown in Figure 4.2(c). These examples complete the proof of Theorem 4.2.* ■

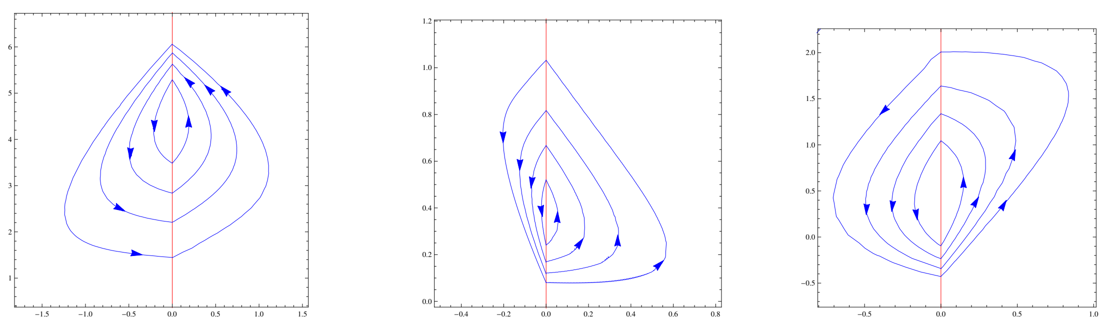
Section 4.3 LC of PWS formed by two cubic Hamiltonian saddles

This subsection aims to give the upper bound of the maximum number of crossing limit cycles of piecewise differential Hamiltonian systems that are separated by the straight line Σ and formed by two cubic Hamiltonian saddles.

4.3.1 Statement of the second main result

Our second result consists in solving the sixteenth extended Hilbert problem for discontinuous piecewise differential systems formed by nilpotent Hamiltonian saddles of cubic linear homogeneous polynomials in each piece separated by the straight line Σ. The results are presented in the following theorem.

Theorem 4.3 *The maximum number of crossing limit cycles of the discontinuous piecewise differential systems separated by the straight line Σ, and formed by two Hamiltonian nilpotent saddles $(\tilde{C}_i)$ and $(\tilde{C}_j)$ for $i, j \in \{1, 2, 3, 4, 5, 6\}$ is at most four. For all these classes, there are systems exhibiting exactly four limit cycles, see Figures 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9 and 4.10.*



(a) (b) (c)

Figure 4.3: The four limit cycles of the discontinuous piecewise differential system, (a) for (4.28)–(4.29), (b) for (4.30)–(4.31), and (c) for (4.32)–(4.33).

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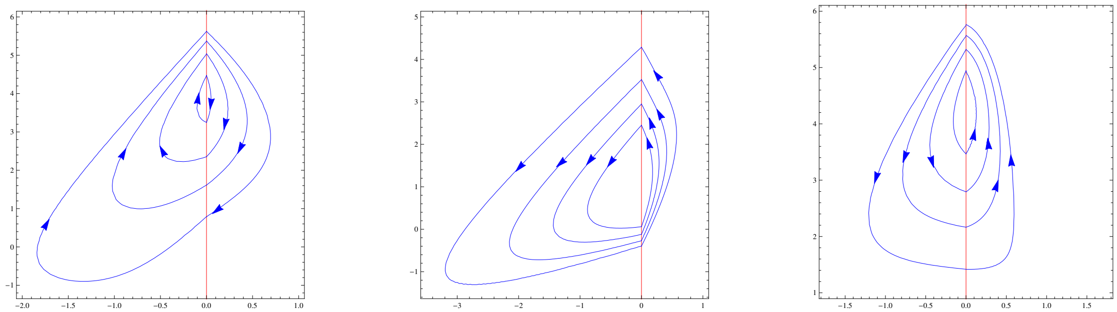
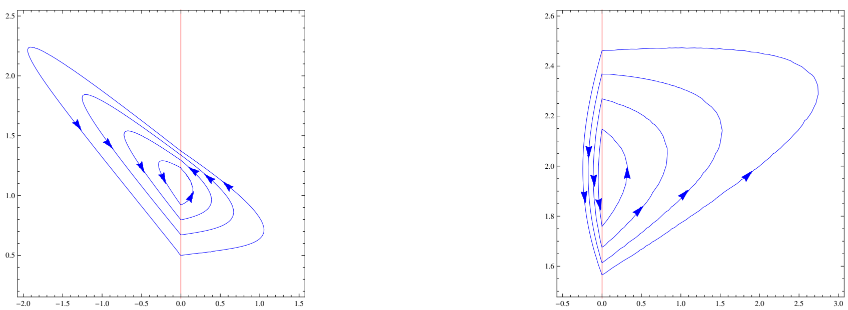


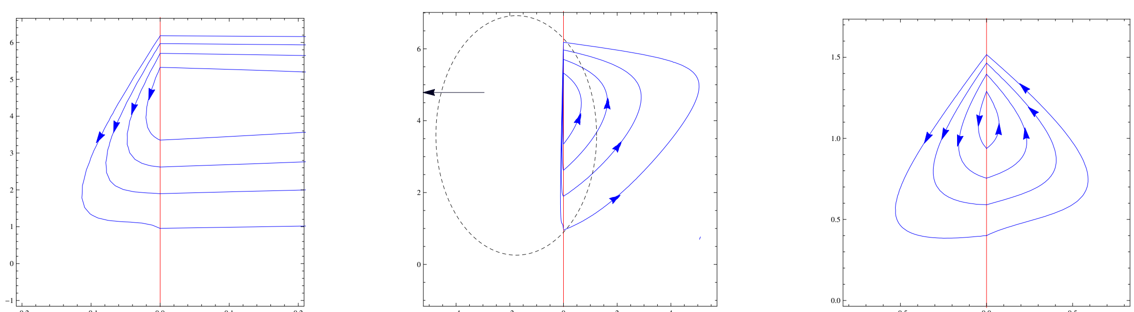
(a) (b) (c)

Figure 4.4: The four limit cycles of the discontinuous piecewise differential system, (a) for (4.34)–(4.35), (b) for (4.36)–(4.37), and (c) for (4.38)–(4.39).



(a) (b)

Figure 4.5: The four limit cycles of the discontinuous piecewise differential system, (a) for (4.40)–(4.41), and (b) for (4.42)–(4.43).

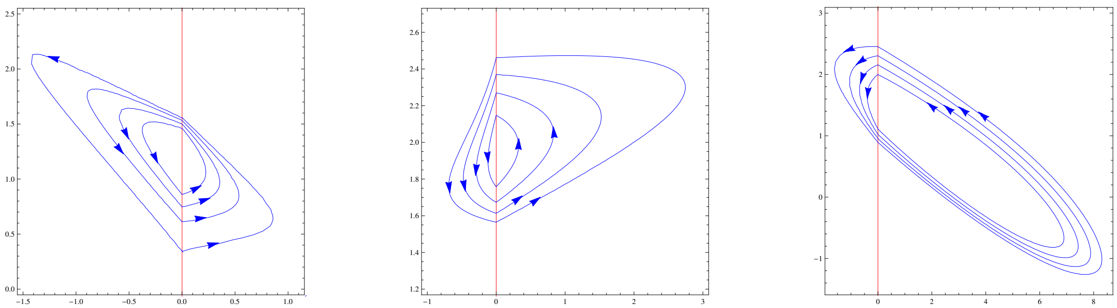


(a) (b)

Figure 4.6: The four limit cycles of the discontinuous piecewise differential system, (a) for (4.44)–(4.45), and (b) for (4.46)–(4.47).

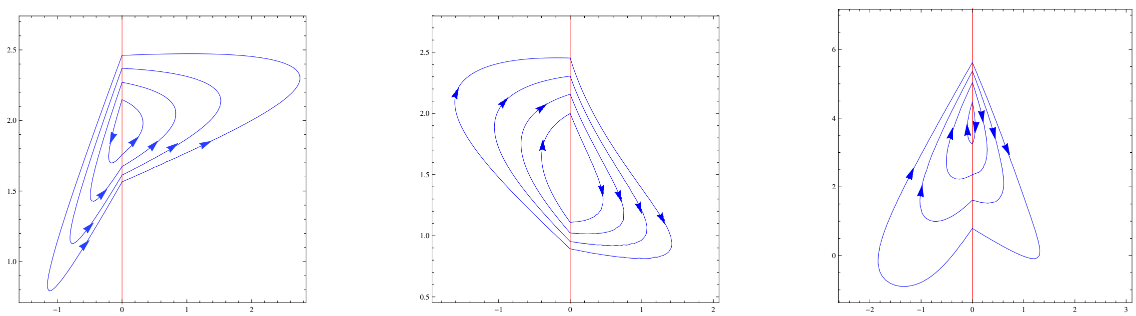
4.3.2 Proof of theorem 4.3

In this subsection we give the proof of theorem 4.3 for the discontinuous piecewise differential systems formed by either $(\tilde{\mathcal{C}}_i)$–$(\tilde{\tilde{\mathcal{C}}}_i)$, or $(\tilde{\mathcal{C}}_i)$–$(\tilde{\mathcal{C}}_j)$, with $i \neq j$ and $i, j \in \{1,2,3,4,5,6\}$. The system $(\tilde{\tilde{\mathcal{C}}}_i)$ and its corresponding first integral $\tilde{H}_i(x,y)$, are obtained by changing the parameters $(a, b, c, \mu, \alpha_1, \delta_1, \gamma_1, \alpha_2, \delta_2, \gamma_2)$ of system $(\tilde{\mathcal{C}}_i)$ and of the inte-



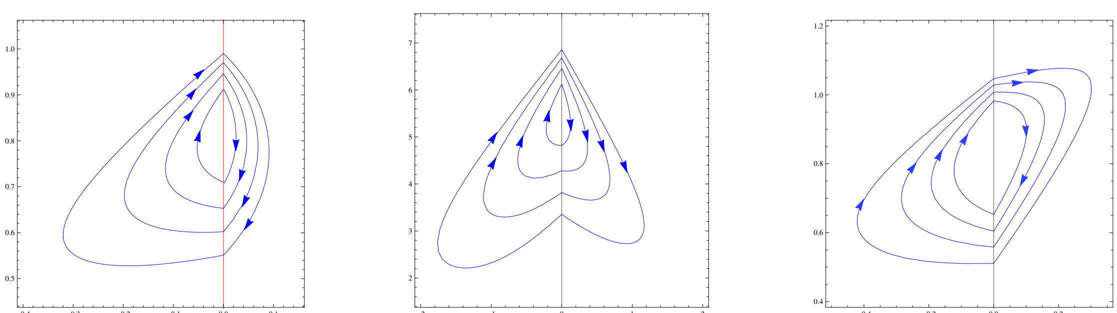
(*a*) (*b*) (*c*)

Figure 4.7: The four limit cycles of the discontinuous piecewise differential system, (*a*) for (4.48)–(4.49), (*b*) for (4.50)–(4.51), and (*c*) for (4.52)–(4.53).



(*a*) (*b*) (*c*)

Figure 4.8: The four limit cycles of the discontinuous piecewise differential system, (*a*) for (4.54)–(4.55), (*b*) for (4.56)–(4.57), and (*c*) for (4.58)–(4.59).



(*a*) (*b*) (*c*)

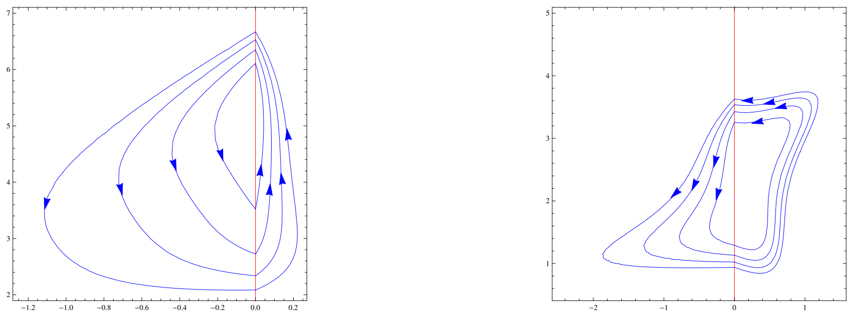
Figure 4.9: The four limit cycles of the discontinuous piecewise differential system, (*a*) for (4.60)–(4.61), (*b*) for (4.62)–(4.63), and (*c*) for (4.64)–(4.65).

gral $H_i(x,y)$ by the parameters $(a_1, b_1, c_1, \mu_1, \tilde{\alpha_1}, \tilde{\delta_1}, \tilde{\gamma_1}, \tilde{\alpha_2}, \tilde{\delta_2}, \tilde{\gamma_2})$.

Proof.

Firstly, we prove the Theorem for the discontinuous piecewise differential system formed by systems $(\tilde{C}_i)$–$(\tilde{\tilde{C}}_i)$ *with their corresponding first integrals* $H_i(x,y)$ *and* $\tilde{H}_i(x,y)$*, respectively.*

In one half-plane we consider the Hamiltonian nilpotent saddle $(\tilde{C}_i)$ *with its corresponding first integral* $H_i(x,y)$ *given by* (4.2)*, or* (4.4)*, or* (4.6)*, or* (4.8)*, or* (4.10) *or* (4.12)*, and in the*



(a) (b)

Figure 4.10: The four limit cycle of the discontinuous piecewise differential system, (a) for (4.66)–(4.67), and (b) for (4.68)–(4.69).

other half-plane we consider the second differential Hamiltonian nilpotent saddle $(\tilde{\tilde{\mathcal{C}_i}})$ *with its first integral* $\tilde{H}_i(x,y)$.

In order to prove that the discontinuous piecewise differential systems $(\tilde{\mathcal{C}_i})$–$(\tilde{\tilde{\mathcal{C}_i}})$ *for* $i \in \{1,2,3,4,5,6\}$ *have a limit cycle intersecting the line of discontinuity* $x = 0$ *in two different points* $(0,y_1)$ *and* $(0,y_2)$ *with* $y_1 < y_2$, *these points should satisfy the system of equations*

$$e_1 = H_i(0,y_1) - H_i(0,y_2) = 0, \quad e_2 = \tilde{H}_i(0,y_1) - \tilde{H}_i(0,y_2) = 0, \tag{4.26}$$

where e_1 *and* e_2 *for* $i = 1$ *are cubic equations such that*

$$\begin{aligned} e_1 = \; & \frac{1}{4b}\Big(4a^2\gamma_1\delta_1 + 2a^2\gamma_1^2 y_1 + 2a^2\gamma_1^2 y_2 + 4ab\gamma_1\delta_2 + 4ab\gamma_2\delta_1 + 4ab\gamma_1\gamma_2 y_1 + 4ab\gamma_1\gamma_2 y_2 \\ & +4b^2\gamma_2\delta_2 + 2b^2\gamma_2^2 y_1 + 2b^2\gamma_2^2 y_2 - 4b\gamma_1\delta_1^3 - b\gamma_1^4 y_1^3 - 4b\gamma_1^3\delta_1 y_1^2 - b\gamma_1^4 y_1^2 y_2 - b\gamma_1^4 y_2^3 \\ & -6b\gamma_1^2\delta_1^2 y_1 - b\gamma_1^4 y_1 y_2^2 - 4b\gamma_1^3\delta_1 y_1 y_2 - 4b\gamma_1^3\delta_1 y_2^2 - 6b\gamma_1^2\delta_1^2 y_2\Big) = 0, \\ e_2 = \; & -\frac{1}{4b_1}\Big(-4a_1^2\tilde{\gamma_1}\tilde{\delta_1} - 2a_1^2\tilde{\gamma_1}^2 y_1 - 2a_1^2\tilde{\gamma_1}^2 y_2 - 4a_1b_1\tilde{\gamma_1}\tilde{\delta_2} - 4a_1b_1\tilde{\gamma_2}\tilde{\delta_1} - 4a_1b_1\tilde{\gamma_1}\tilde{\gamma_2}y_1 \\ & -4a_1b_1\tilde{\gamma_1}\tilde{\gamma_2}y_2 - 4b_1^2\tilde{\gamma_2}\tilde{\delta_2} - 2b_1^2\tilde{\gamma_2}^2 y_1 - 2b_1^2\tilde{\gamma_2}^2 y_2 + 4b_1\tilde{\gamma_1}\tilde{\delta_1}^3 + 6b_1\tilde{\gamma_1}^2\tilde{\delta_1}^2 y_2 + b_1\tilde{\gamma_1}^4 y_1^3 \\ & +4b_1\tilde{\gamma_1}^3\tilde{\delta_1}y_1^2 + b_1\tilde{\gamma_1}^4 y_1^2 y_2 + 6b_1\tilde{\gamma_1}^2\tilde{\delta_1}^2 y_1 + b_1\tilde{\gamma_1}^4 y_1 y_2^2 + 4b_1\tilde{\gamma_1}^3\tilde{\delta_1}y_1y_2 + 4b_1\tilde{\gamma_1}^3\tilde{\delta_1}y_2^2 \\ & +b_1\tilde{\gamma_1}^4 y_2^3\Big) = 0. \end{aligned}$$

Using Bézout's Theorem 1.1, we obtain that the maximum number of solutions of system (4.26) is at most nine. Since these solutions are symmetric, we know that the maximum number of the solutions satisfying $y_1 < y_2$ *is at most four. Consequently, the discontinuous piecewise differential system* $(\tilde{\mathcal{C}_1})$–$(\tilde{\tilde{\mathcal{C}_1}})$ *can have at most four limit cycles.*

For the remaining cases $i \in \{2,3,4,5,6\}$, *we know that* e_1 *and* e_2 *are also cubic equations. So by a similar way we prove that the discontinuous piecewise differential system* $(\tilde{\mathcal{C}_i})$–$(\tilde{\tilde{\mathcal{C}_i}})$ *for*

$i \in \{2,3,4,5,6\}$ can have at most four limit cycles.

Secondly, we prove the Theorems for the discontinuous piecewise differential system formed by systems $(\tilde{\mathcal{C}_i})$–$(\tilde{\mathcal{C}_j})$ with their corresponding first integrals $H_i(x,y)$ and $H_j(x,y)$, for $i \neq j$, respectively.

In one half-plane we consider the differential Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}_i})$ with its first integral $H_i(x,y)$. In the other half-plane we consider the differential Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}_j})$ with its corresponding first integral $H_j(x,y)$, we recall that we performed a different change of variables to obtain the systems $(\tilde{\mathcal{C}_i})$ and $(\tilde{\mathcal{C}_j})$ and their corresponding first integrals. We suppose the existence of limit cycle of the discontinuous piecewise differential system $(\tilde{\mathcal{C}_i})$–$(\tilde{\mathcal{C}_j})$, which intersects the discontinuity line $x=0$ in two distinct points $(0,y_1)$ and $(0,y_2)$ with $y_1 \neq y_2$. Then these points satisfy the system of equations

$$e_1 = H_i(0,y_1) - H_i(0,y_2) = 0, \quad e_2 = H_j(0,y_1) - H_j(0,y_2) = 0. \tag{4.27}$$

For $i=1$ and $j=2$ we obtained that e_1 and e_2 are cubic equations, where

$$\begin{aligned}
e_1 = \ & -\frac{1}{4b}\Big(b\gamma_1^4y_2^3 - 2a^2\gamma_1^2y_1 - 2a^2\gamma_1^2y_2 - 4ab\gamma_1\delta_2 - 4ab\gamma_2\delta_1 - 4ab\gamma_1\gamma_2y_1 - 4ab\gamma_1\gamma_2y_2 \\
& -4b^2\gamma_2\delta_2 - 2b^2\gamma_2^2y_1 - 2b^2\gamma_2^2y_2 + 4b\gamma_1\delta_1^3 + b\gamma_1^4y_1^3 + 4b\gamma_1^3\delta_1y_1^2 + b\gamma_1^4y_1^2y_2 - 4a^2\gamma_1\delta_1 \\
& +6b\gamma_1^2\delta_1^2y_1 + b\gamma_1^4y_1y_2^2 + 4b\gamma_1^3\delta_1y_1y_2 + 4b\gamma_1^3\delta_1y_2^2 + 6b\gamma_1^2\delta_1^2y_2\Big) = 0, \\
e_2 = \ & -\frac{1}{2b}\Big(-2a^2\gamma_1\delta_1 - a^2\gamma_1^2y_1 - a^2\gamma_1^2y_2 - 2ab\gamma_1\delta_2 - 2ab\gamma_2\delta_1 - 2ab\gamma_1\gamma_2y_1 - 2ab\gamma_1\gamma_2y_2 \\
& +6b\gamma_1\gamma_2\delta_1^2y_2 - b^2\gamma_2^2y_1 - b^2\gamma_2^2y_2 + 6b\gamma_1\delta_1^2\delta_2 + 2b\gamma_2\delta_1^3 + 2b\gamma_1^3\gamma_2y_1^3 + 6b\gamma_1^2\gamma_2\delta_1y_1^2 \\
& +2b\gamma_1^3\delta_2y_1^2 + 2b\gamma_1^3\gamma_2y_1^2y_2 + 6b\gamma_1^2\delta_1\delta_2y_1 + 6b\gamma_1\gamma_2\delta_1^2y_1 + 2b\gamma_1^3\gamma_2y_1y_2^2 + 2b\gamma_1^3\delta_2y_1y_2 \\
& +6b\gamma_1^2\gamma_2\delta_1y_1y_2 + 2b\gamma_1^3\gamma_2y_2^3 + 2b\gamma_1^3\delta_2y_2^2 + 6b\gamma_1^2\gamma_2\delta_1y_2^2 + 6b\gamma_1^2\delta_1\delta_2y_2 - 2b^2\gamma_2\delta_2\Big) = 0.
\end{aligned}$$

Using again Bézout's Theorem 1.1, and due to the symmetry of the solutions of system (4.27), we know that the maximum number of limit cycles of the discontinuous piecewise differential system $(\tilde{\mathcal{C}_1})$–$(\tilde{\mathcal{C}_2})$ is at most four.

Likewise for $i,j \in \{1,2,3,4,5,6\}$ and $i \neq j$, the discontinuous piecewise differential system $(\tilde{\mathcal{C}_i})$–$(\tilde{\mathcal{C}_j})$ has at most four limit cycles.

Now we have to complete the proof of Theorem 4.3 by building discontinuous piecewise differential systems for all the classes $(\tilde{\mathcal{C}_i})$–$(\tilde{\mathcal{C}_j})$, with $i,j \in \{1,2,3,4,5,6\}$, exhibiting exactly four limit cycles.

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}_1})$–$(\tilde{\tilde{\mathcal{C}_1}})$.

In the half-plane Σ^- we consider the Hamiltonian nilpotent saddle $(\tilde{\tilde{\mathcal{C}_1}})$

$$
\begin{aligned}
\dot{x} = \ & \frac{1}{75200}\Big(43740(x+1)y^2+(60961-48600x(x+2))y+50(x+1)(360x(x+2) \\
& +691)-13122y^3\Big), \\
\dot{y} = \ & \frac{1}{7520}\Big(4860(x+1)y^2-5(1080x(x+2)+1411)y+250(x+1)(8x(x+2)+7) \\
& -1458y^3\Big),
\end{aligned}
\tag{4.28}
$$

its corresponding first integral is

$$
\begin{aligned}
H_1(x,y) \ = \ & \frac{1}{4}\Big(-\Big(-x+\frac{9}{10}y-1\Big)^4+4\Big(-x+\frac{9}{10}y-1\Big)^2-4\Big(-\frac{5}{2}x-\frac{151}{100}y-\frac{5}{2}\Big)\Big(-x \\
& +\frac{9}{10}y-1\Big)+\Big(-\frac{5}{2}x-\frac{151}{100}y-\frac{5}{2}\Big)^2\Big).
\end{aligned}
$$

In the half-plane Σ^+ we consider the Hamiltonian differential nilpotent saddle $(\tilde{\mathcal{C}}_1)$

$$
\begin{aligned}
\dot{x} \simeq \ & -0.020184\Big(-\frac{69}{10}\Big(-\frac{23}{10}y+2.5556\Big)x^2+x\Big(\frac{69}{10}\Big(2.5556-\frac{23}{10}y\Big)^2-51.8443\Big) \\
& +\frac{23}{10}x^3+15.8951(54.1443x+238.484y-220.608)+581.102\Big(-x-\frac{23}{10}y \\
& +2.55556\Big)+27.9841y^3-93.2803y^2-2584.19y+1947.86\Big), \\
\dot{y} \simeq \ & -0.020184\Big(3x^2(2.55556-\frac{23}{10}y)+\Big(1-3\Big(2.55556-\frac{23}{10}y\Big)^2\Big)x-x^3-15.8951 \\
& (-40.8673+2x+54.1443y)-252.653\Big(-x-\frac{23}{10}y+2.55556\Big)-12.167y^3 \\
& +40.5567y^2+6.78134y-21.6217\Big),
\end{aligned}
\tag{4.29}
$$

which has the first integral

$$
\begin{aligned}
\tilde{H}_1(x,y) \ \simeq \ & \frac{1}{4}\Big(-63.5803(-x-51.8443y+38.3117)\Big(-x-\frac{23}{10}y+2.55556\Big)+2(-x \\
& -51.8443y+38.3117)^2-\Big(-x-\frac{23}{10}y+2.55556\Big)^4+505.306\Big(-x-\frac{23}{10}y \\
& +2.55556\Big)^2\Big).
\end{aligned}
$$

The discontinuous piecewise differential system (4.28)–(4.29) has four limit cycles, because system (4.26) in this case has exactly four real solutions $(y_1,y_2) \simeq \{(1.44782, 6.06606), (2.2075, 5.87454), (2.837, 5.63494), (3.48469, 5.29375)\}$, *see Figure 4.3(a).*

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_1)$–$(\tilde{\mathcal{C}}_2)$.

In Σ^- we consider the Hamiltonian nilpotent saddle which takes the form of system $(\tilde{\mathcal{C}}_1)$

$$
\begin{aligned}
\dot{x} \simeq\ & -0.0101267\Big(625x^3 + 375x^2\Big(5y - \frac{35}{8}\Big) + \Big(75\Big(5y - \frac{35}{8}\Big)^2 + 18.7499\Big)x \\
& -16.8134\Big(5(x+y) - \frac{35}{8}\Big) + 625y^3 + 1.83376(88.7493x + 187.499y \\
& -183.003..) - 1640.63..y^2 + 1083.99y - 40.0591\Big), \\
\dot{y} \simeq\ & -0.010126\Big(-625x^3 - 375x^2\Big(5y - \frac{35}{8}\Big) + \Big(1 - 75\Big(5y - \frac{35}{8}\Big)^2\Big)x + 16.8134 \\
& \Big(5x + 5y - \frac{35}{8}\Big) - 1.83376(-10x + 88.7493y - 96.597) + 1640.63y^2 - 625y^3 \\
& -\frac{14543}{10}y + 438.896\Big),
\end{aligned}
\tag{4.30}
$$

with the first integral

$$
\begin{aligned}
H_1(x,y) \simeq\ & \frac{1}{4}\Big(-\Big(5x + 5y - \frac{35}{8}\Big)^4 + 6.7253\Big(5x + 5y - \frac{35}{8}\Big)^2 - 7.33504(-x + 18.7499y \\
& -20.1944)\Big(5x + 5y - \frac{35}{8}\Big) + 2(-x + 18.7499y - 20.1944)^2\Big).
\end{aligned}
$$

In the half-plane Σ^+ we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}}_2)$

$$
\begin{aligned}
\dot{x} \simeq\ & -0.0647249\Big(-\frac{3}{2}\Big(6\Big(\frac{216}{5} - \frac{291}{5}y\Big)x^2 - \frac{182}{5}x^3 + x\Big(-\frac{8019}{10}y^2 + 1333.8y \\
& -532.65\Big) - 5(108y^3 - 243y^2 + 154.5y - 27) + \frac{5}{2}\Big(9(3y-3)^2 - \frac{15}{2}\Big)\Big) - \frac{363}{4} \\
& (2x + 3y - 3) - \frac{33}{4}\Big(-\frac{97}{10}x - 30y + 22.5\Big)\Big), \\
\dot{y} \simeq\ & -0.0647249\Big(-\frac{3}{2}(-12x^2\Big(\frac{41}{10} - \frac{91}{10}y\Big) + \Big(-12(3y-3)\Big(\frac{47}{10} - \frac{97}{10}y\Big) - \frac{3}{200}\Big)x \\
& -\frac{16}{5}x^3 + \frac{2673}{10}y^3 - \frac{6669}{10}y^2 + \frac{10653}{20}y - 132.675\Big) + \frac{33}{4}\Big(\frac{2}{5}x - \frac{97}{10}y + \frac{47}{10}\Big) \\
& +\frac{121}{2}(2x + 3y - 3)\Big),
\end{aligned}
\tag{4.31}
$$

which has the first integral

$$
\begin{aligned}
H_2(x,y) \simeq\ & \frac{1}{2}\Big(-2\Big(\frac{1}{10}x - 5y + \frac{5}{2}\Big)(2x + 3y - 3)^3 - 20.1667(2x + 3y - 3)^2 - 11(\frac{1}{10}x - 5y \\
& +\frac{5}{2})(2x + 3y - 3) - \frac{3}{2}\Big(0.1x - 5y + \frac{5}{2}\Big)^2\Big).
\end{aligned}
$$

The discontinuous piecewise differential system (4.30)–(4.31) has exactly four limit cycles which are shown in Figure 4.3(b), because system (4.27) for $i = 1$ and $j = 2$ has the four solutions $(y_1, y_2) \simeq \{(0.0795, 1.03166), (0.11693, 0.81620), (0.16528, 0.66715), (0.24045, 0.52019)\}$.

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}_1})$–$(\tilde{\mathcal{C}_3})$.

In the half-plane Σ^- we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}_1})$

$$
\begin{aligned}
\dot{x} \simeq\ & x(16.4752-6(4.39497-2y)^2)-0.0690837\Big(-6x^2(4.39497-2y)-2x^3\\
&+258.974(x-2y+4.39497)-11.3792(18.4752x+137.491-65.9008y)\\
&+16y^3-105.479y^2-39.6431y+366.341\Big),\\
\dot{y} \simeq\ & -0.069084\Big(-3x^2(-2y+4.3949)+x(1-3(-2y+4.39497)^2)-x^3+11.3792\\
&(18.4752y-2x-36.9363)+129.487(x-2y+4.39497)+8y^3-52.7396y^2\\
&+99.4193y-52.3507\Big),
\end{aligned}
\tag{4.32}
$$

with the first integral

$$
\begin{aligned}
H_1(x,y) \ \simeq\ & \frac{1}{4}\Big(-(-2y+x+4.39497)^4+258.974(-2y+x+4.39497)^2+45.517(-x\\
&+16.4752y-32.5414)(x-2y+4.39497)+2(16.4752y-x-32.5414)^2\Big).
\end{aligned}
$$

In the half-plane Σ^+ we consider the Hamiltonian differential nilpotent saddle $(\tilde{\mathcal{C}_3})$

$$
\begin{aligned}
\dot{x} =\ & \frac{1}{10996000}\Big(12(3810997x+20797042)y^2+(9x(810333x-33733324)\\
&-455004604)y-37856096y^3-250(x(x(6506x+48597)-1943212)\\
&-567980)\Big),\\
\dot{y} =\ & \frac{1}{10996000}\Big(62500(x(x(23x-372)+223)+7734)+500(9759x^2+48597x\\
&-971606)y+9(16866662-810333x)y^2-15243988y^3\Big),
\end{aligned}
\tag{4.33}
$$

its first integral is

$$
H_3(x,y) \ =\ \frac{1}{4}\Big(\frac{1}{4}x+y-\frac{7}{2}\Big)^4-\frac{3}{2}\Big(\frac{1}{2}x-\frac{749}{1000}y+\frac{5}{4}\Big)^2\Big(\frac{1}{4}x+y-\frac{7}{2}\Big)^2-\frac{1}{2}\Big(\frac{1}{2}x-\frac{749}{1000}y+\frac{5}{4}\Big)^2.
$$

The solutions of system (4.27) for $i=1$ and $j=3$, are $(y_1,y_2)\simeq\{(-0.43595,2.00604),$ $(-0.34729,1.63833),(-0.24004,1.33811),(-0.09903,1.04468)\}$, *and these four solutions provide the four limit cycles for the discontinuous piecewise system (4.32)–(4.33) shown in Figure 4.3(c).*

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}_1})$–$(\tilde{\mathcal{C}_4})$.

In the half-plane Σ^+ *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}}_1)$

$$
\begin{aligned}
\dot{x} \simeq\ & 0.08643\Big(x^3 - 3x^2(-y - 0.224122) + x\Big(3(-y - 0.224122)^2 - 27.1399\Big)\\
&+95.9099(-x - y - 0.224122) + y^3 - 9.79336(-15.57x - 27.1399y\\
&-9.13537) + 0.672365y^2 - 183.993y - 82.6848\Big),\\
\dot{y} \simeq\ & 0.0864307\Big(- x^3 - y^3 - 3x^2(y + 0.224122) + x\Big(4 - 3(-y - 0.224122)^2\Big)\\
&-95.9099(-x - y - 0.224122) + 9.79336\Big(- 4x - \frac{1557}{100}y - 6.54229\Big)\\
&-0.672365y^2 + 26.9892y + 12.1768\Big),
\end{aligned}
\tag{4.34}
$$

this system has the first integral

$$
\begin{aligned}
H_1(x,y) \ \simeq\ & \frac{1}{4}\Big(- (-x - y - 0.224122)^4 + \frac{9591}{50}(-x - y - 0.224122)^2 + 39.1734(2x +\\
&13.57y + 6.09405)(-x - y - 0.224122) + 2\Big(2x + \frac{1357}{100}y + 6.09405\Big)^2\Big).
\end{aligned}
$$

In the half-plane Σ^- *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}}_4)$

$$
\begin{aligned}
\dot{x} \simeq\ & 2.9301\Big(\frac{1}{2}\Big(\frac{2}{5}x - 0.15714y + \frac{1}{2}\Big)(0.763x^2 + x(1.6881y + 8.4428) - 2.15367y^2\\
&-8.3007y - 1.76381) - \frac{1}{4}(0.525429x - 0.474567y - 0.10928) - 0.3775\Big(\frac{1}{2}x\\
&+\frac{151}{100}y + \frac{11}{2}\Big)\Big),\\
\dot{y} \simeq\ & 2.9301\Big(- \frac{1}{2}\Big(\frac{2}{5}x - 0.15714y + \frac{1}{2}\Big)(0.664x^2 + x(2.54986y + 10.435) + 40.325\\
&+2.3901y^2 + 19.7052y) + \frac{1}{4}\Big(\frac{2}{5}x + \frac{49}{20} + 0.525429y\Big) + \frac{1}{8}\Big(\frac{11}{2} + \frac{151}{100}y + \frac{1}{2}x\Big)\Big),
\end{aligned}
\tag{4.35}
$$

with the first integral

$$
\begin{aligned}
H_4(x,y) \ \simeq\ & -\frac{1}{4}\Big(\frac{2}{5}x - 0.157142y + \frac{1}{2}\Big)^4 - \frac{3}{2}\Big(\frac{1}{2}x + \frac{151}{100}y + \frac{11}{2}\Big)^2\Big(\frac{2}{5}x - 0.157142y + \frac{1}{2}\Big)^2\\
&+\frac{1}{4}\Big(\frac{2}{5}x - 0.157142y + \frac{1}{2}\Big)^2 + \frac{1}{2}\Big(\frac{1}{2}x + \frac{151}{100}y + \frac{11}{2}\Big)\Big(\frac{2}{5}x - 0.157142y + \frac{1}{2}\Big)\\
&+\frac{1}{4}\Big(\frac{1}{2}x + \frac{151}{100}y + \frac{11}{2}\Big)^2.
\end{aligned}
$$

The discontinuous piecewise system (4.34)–(4.35) *has four limit cycles, because system* (4.27) *for* $i = 1$ *and* $j = 4$, *has the four real solutions* $(y_1, y_2) \simeq \{(0.79064, 5.62516), (1.61348, 5.37496), (2.35323, 5.04467), (3.24928, 4.47718)\}$, *see Figure 4.4(a).*

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_1)$–$(\tilde{\mathcal{C}}_5)$.

In the half-plane Σ^+ we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}}_1)$

$$\begin{aligned}\dot{x} = &\ \frac{1}{3200}\Big(11240x^3-24x^2(1541y-4702)+8x\Big(4935y^2-32772y+45034\Big)\\ &-13800y^3+144960y^2-462716y+335197\Big),\\ \dot{y} = &\ \frac{1}{16000}\Big(49032x^3-72x^3(7025y-18468)(46230y^2-282120y+279031)\\ &-65800y^3+655440y^2-1801360y+373051\Big),\end{aligned} \tag{4.36}$$

its corresponding first integral is given by

$$\begin{aligned}H_1(x,y) \ = \ &\frac{1}{4}\Big(\Big(\frac{7}{10}x-\frac{1}{2}y+\frac{83}{20}\Big)^4+\frac{3}{10}(x-y+4)^2\Big(\frac{7}{10}x-\frac{1}{2}y+\frac{83}{20}\Big)^2-(x-y+4)^4\\ &-4(x-y+4)^2\Big).\end{aligned}$$

In the half-plane Σ^- we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}}_5)$

$$\begin{aligned}\dot{x} \simeq &\ -5\Big(-0.8285x^3+2.6325x^2y-8.94975x^2-2.7375xy^2+19.6425xy-2(x\\ &-y+4)-\frac{3}{20}(x-y+4)\Big(\frac{7}{10}x-\frac{1}{2}y+\frac{83}{20}\Big)\Big(-\frac{6}{5}x+y-\frac{123}{20}\Big)-29.9164x\\ &+0.9375y^3-10.4438y^2+35.0831y-28.2633\Big),\\ \dot{y} \simeq &\ 5\Big(0.7599x^3-2.4855x^2y+7.72965x^2+2.6325xy^2-17.8995xy-\frac{3}{20}\Big(\frac{7}{5}x\\ &-\frac{6}{5}y+\frac{139}{20}\Big)(x-y+4)\Big(\frac{7}{10}x-\frac{1}{2}y+\frac{83}{20}\Big)+2(x-y+4)+13.9686+22.6829x\\ &-0.9125y^3+9.82125y^2-29.9164y\Big),\end{aligned} \tag{4.37}$$

which has the first integral

$$\begin{aligned}H_5(x,y) \ = \ &\frac{1}{4}\Big(\Big(\frac{7}{10}x-\frac{1}{2}y+\frac{83}{20}\Big)^4+\frac{3}{10}(x-y+4)^2\Big(\frac{7}{10}x-\frac{1}{2}y+\frac{83}{20}\Big)^2-(x-y+4)^4-4(x-y\\ &+4)^2\Big).\end{aligned}$$

System (4.27) *for $i=1$ and $j=5$, has the four real solutions* $(y_1,y_2)\simeq\{(-0.39789,4.28545),$ $(-0.27168,3.52194),(-0.123785,2.94818),(0.0586396,2.44949)$, *which provide the four limit cycles for the discontinuous piecewise system* (4.36)–(4.37) *shown in Figure 4.4(b).*

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_1)$–$(\tilde{\mathcal{C}}_6)$.

In the half-plane Σ^- we consider the Hamiltonian nilpotent saddle ($\tilde{\mathcal{C}}_1$)

$$
\begin{aligned}
\dot{x} \simeq\ & -0.519884\Big(\frac{1}{2}(-0.934804x^3 + 2.80441x^2(0.934804y-1) + x(-2.80441 \\
& (0.934804y-1)^2 - 1.8875) + 0.76363y^3 - 2.8223 - 2.4507y^2 + 1.4815y) \\
& -0.934804(-x+0.934804y-1) + \frac{1}{2}(-0.827009x - 2.82311y - 0.827009)\Big), \\
\dot{y} \simeq\ & -0.519884\Big(1 + \frac{1}{2}\Big(x(3.125 - 3(0.934804y-1)^2) - x^3 + 3x^2(-1+0.934804y) \\
& +0.816885y^3 - 2.62157y^2 + 4.69191y + \frac{17}{18}\Big) - \frac{1}{2}(-0.827009y + 5x + 5) + x \\
& -0.934804y\Big),
\end{aligned}
\tag{4.38}
$$

its corresponding first integral is

$$
\begin{aligned}
H_1(x,y) \ \simeq\ & \frac{1}{4}\Big(-(-x+0.934804y-1)^4 + 4(-x+0.934804y-1)^2 - 4\Big(-\frac{5x}{2} - \frac{151y}{100} \\
& -\frac{5}{2}\Big)(-x+0.934804y-1) + \Big(-\frac{5x}{2} - \frac{151y}{100} - \frac{5}{2}\Big)^2\Big).
\end{aligned}
$$

In the half-plane Σ^+ we consider the Hamiltonian nilpotent saddle ($\tilde{\mathcal{C}}_6$)

$$
\begin{aligned}
\dot{x} \simeq\ & -0.2393\Big(-0.1789x^3 + 26.2423x^2y - 9.69614x^2 - 7.03195xy^2 + 42.2531xy \\
& -17.7297(x+2y) - 0.934364(x+2y)(-x+2.17886y-1.48337)(0.17886x \\
& +8.71544y - 2.96673) - 14.3829x - 7.11171 + 38.5381y^3 - 46.0317y^2 \\
& +31.3384y\Big), \\
\dot{y} \simeq\ & 0.2393\Big(2x^3 - 0.53658x^2y + 4.4501x^2 + 26.2423xy^2 - 19.3923xy - 8.86486 \\
& (x+2y) - 0.93436(0.17886y - 2x - 1.48337)(x+2y)(2.1789y - x - 1.4834) \\
& +6.6011x - 2.3439y^3 + 21.1265y^2 + 3.2639 - 14.3829y\Big),
\end{aligned}
\tag{4.39}
$$

with its corresponding first integral

$$
\begin{aligned}
H_6(x,y) \ \simeq \frac{1}{4}\ & \Big(-(x+2y)^4 + 1.86873(-x+2.17886y-1.48337)^2(x+2y)^2 + 17.7297 \\
& (x+2y)^2 - (-x+2.17886y-1.48337)^4\Big).
\end{aligned}
$$

The discontinuous piecewise differential system (4.38)–(4.39) has four limit cycles, due to the fact that system (4.27) when $i = 1$ and $j = 6$, has exactly the four real solutions $(y_1, y_2) \simeq \{(1.41808, 5.76618), (2.16491, 5.57153), (2.79194, 5.32269), (3.46607, 4.94562)\}$, the limit cycles are shown in Figure 4.4(c).

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}_2})$–$(\tilde{\tilde{\mathcal{C}_2}})$.

In the half-plane Σ^- we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}_2})$

$$\begin{aligned}
\dot{x} \simeq\ & -0.325492\Big(-\frac{3}{2}x^2(5.45467-11.2704y)+0.933642x^3-x\Big(-\frac{325}{4}y^2\\
&+84.7104y-20.8767\Big)-1.65574\Big(\frac{3}{2}x+1.65574y-1\Big)+3.40342x\\
&+6.47569(-1+18.1567y^3-24.6733y^2+3.45876y)+21.4441y\\
&-2.27774(4.96722(1.65574y-1)^2-6.47569)-10.247\Big),\\
\dot{y} \simeq\ & -0.325492\Big(\frac{3}{4}(1.43887-3.73457y)x^2+\Big(3(1.65574y-1)(1.23887\\
&-3.40342y)+\frac{1}{100}\Big)x-\frac{1}{20}x^3+\frac{1}{2}\Big(\frac{1}{2}x+1.65574y-1\Big)-\frac{1}{10}x+4.52771\\
&-27.0833y^3+42.3552y^2-24.2802y\Big),
\end{aligned}\tag{4.40}$$

its corresponding first integral is

$$\begin{aligned}
H_2(x,y)\ \simeq\ & \frac{1}{2}\Big(-2\Big(-\frac{1}{10}x-6.47569y+2.27774\Big)\Big(\frac{1}{2}x+1.65574y-1\Big)^3-\Big(-1+\frac{1}{2}x+\\
& 1.65574y-1)^2-2\Big(-\frac{1}{10}x-6.47569y+2.27774\Big)(-1+\frac{1}{2}x+1.65574y)\\
& -(-0.1x-6.47569y+2.27774)^2\Big).
\end{aligned}$$

In the half-plane Σ^+ we consider the Hamiltonian nilpotent saddle $(\tilde{\tilde{\mathcal{C}_2}})$

$$\begin{aligned}
\dot{x} \simeq\ & -1.8018\Big(\frac{3}{4}\Big(25.4464x^3+\frac{24}{5}x^2(24.564y-14.166)+x(181.235y^2\\
&-202.129y+41.2235)+\frac{19}{10}(48.668y^3-30.9465y^2+4.4055y\\
&-0.274625)-\frac{5}{2}\Big(\frac{69}{10}\Big(\frac{23}{10}y-\frac{13}{20}\Big)^2-1.425\Big)\Big)-2.16407\Big(-\frac{13}{20}+\frac{23}{10}y\\
&+\frac{8}{5}x\Big)+0.7275\Big(\frac{267}{50}x+\frac{437}{50}y-6.985\Big)\Big),\\
\dot{y} \simeq\ & -1.8018\Big(\frac{3}{4}\Big(x\Big(\frac{3}{4}-\frac{48}{5}\Big(\frac{23}{10}y-\frac{13}{20}\Big)\Big(\frac{267}{50}y-\frac{93}{20}\Big)\Big)-16.384x^3-\frac{192}{25}x^2\\
&\Big(\frac{497}{50}y-\frac{119}{20}\Big)-60.4118y^3+101.064y^2-41.2235y+3.46963\Big)\\
&+1.50544\Big(\frac{8}{5}x+\frac{23}{10}y-\frac{13}{20}\Big)-0.7275\Big(\frac{16}{5}x+\frac{267}{50}y-\frac{93}{20}\Big)\Big),
\end{aligned}\tag{4.41}$$

with its first integral

$$\tilde{H_2}(x,y)\ \simeq\ \frac{1}{2}\Big(-2\Big(x+1.9y-\frac{5}{2}\Big)\Big(\frac{8}{5}x+\frac{23}{10}y-\frac{13}{20}\Big)^3+\frac{9409}{7500}\Big(\frac{8}{5}x+\frac{23}{10}y-\frac{13}{20}\Big)^2-\frac{97}{50}$$

$$\left(x+1.9y-\frac{5}{2}\right)\left(\frac{8}{5}x+\frac{23}{10}y-\frac{13}{20}\right)+\frac{3}{4}\left(x+1.9y-\frac{5}{2}\right)^2\Big).$$

The solutions of system (4.26) for $i=2$, are $(y_1,y_2)\simeq\{(0.4992,1.37208),(0.67124,1.33776),(0.7963,1.29447),(0.92016,1.2316)\}$, which implies that the discontinuous piecewise differential system (4.40)–(4.41) has exactly four limit cycles shown in Figure 4.5(a).

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_2)$–$(\tilde{\mathcal{C}}_3)$. *In the half-plane Σ^- we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}}_2)$*

$$\begin{aligned}
\dot{x}\simeq\;& -0.0005804\Big(-269.125\Big(15.206x^3+3(47.5593y-113.808)x^2+x\\
&(210.284y^2-710.853y+135.232)-630.558y^3+3769.11y^2-6406.29y\\
&+4.97794\Big(16.206(5.40201y-14.3511)^2-269.125\Big)+2955.68\Big)-5.40201\\
&(x+5.40201y-14.3511)-269.125(4.40201x-10.804y+41.242)\Big),\\
\dot{y}\simeq\;& -0.0005804\Big(-269.125\Big(-4x^3-3x^2(15.206y-38.0754)+x(-6(4.40201y\\
&-9.3732)(5.40201y-14.3511)-269.125)-1459.7-70.0946y^3-135.232\\
&y+355.426y^2\Big)+269.125(2x+4.40201y-9.3732)+5.40201y-14.3511\\
&+x\Big),
\end{aligned}\tag{4.42}$$

with the first integral

$$\begin{aligned}
H_2(x,y)\;\simeq\;& \frac{1}{2}\Big(-2(-y+x+4.97794)(-14.3511+x+5.40201y)^3-0.00371574(x\\
&+5.40201y-14.3511)^2-2(x-y+4.97794)(x+5.40201y-14.3511)\\
&-269.125(x-y+4.97794)^2\Big).
\end{aligned}$$

In the half-plane Σ^+ we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}}_3)$

$$\begin{aligned}
\dot{x}=\;& -\frac{10}{7}\bigg(\left(\frac{1}{4}x+y-\frac{89}{20}\right)\left(-\frac{97}{400}x^2+x\left(\frac{629}{100}-\frac{11}{10}y\right)+23y^2-121y+\frac{7393}{50}\right)\\
&-2\left(\frac{1}{5}x-2y+\frac{17}{4}\right)\bigg),\\
\dot{y}=\;& \frac{10}{7}\bigg(\left(\frac{71}{1600}x^2+x\left(\frac{7739}{4000}-\frac{181}{200}y\right)+\frac{31}{20}y^2-\frac{527}{200}y-\frac{2201}{800}\right)\left(\frac{1}{4}x+y-\frac{89}{20}\right)\\
&+\frac{1}{5}\left(\frac{1}{5}x-2y+\frac{17}{4}\right)\bigg),
\end{aligned}\tag{4.43}$$

which has the first integral

$$H_3(x,y)=\frac{1}{4}\left(\frac{1}{4}x+y-\frac{89}{20}\right)^4-\frac{3}{2}\left(\frac{1}{5}x-2y+\frac{17}{4}\right)^2\left(\frac{1}{4}x+y-\frac{89}{20}\right)^2-\frac{1}{2}\left(\frac{1}{5}x-2y+\frac{17}{4}\right)^2.$$

Due to the fact that system (4.27) for $i=2$ and $j=3$, has the four real solutions $(y_1, y_2) \simeq \{(1.56431, 2.46132), (1.6127, 2.37006), (1.6728, 2.27025), (1.75781, 2.14818)\}$, we conclude that the discontinuous piecewise differential system (4.42)–(4.43) has four limit cycles, see Figure 4.5(b).

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}_2})$–$(\tilde{\mathcal{C}_4})$**.** *In the half-plane Σ^- we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}_2})$*

$$\begin{aligned}
\dot{x} \simeq\ & 6.5621*10^6\Big(0.00001458\Big(-0.000086547x^3-0.3(-0.00008566y \\
&-0.0000863273)x^2+x(-1.78261y^2*10^{-6}+4.88504y*10^{-7}+10^{-6} \\
&*7.63022)+\frac{1}{10}\Big(3.60655y^3*10^{-7}-1.80966y^2*10^{-6}-10^{-6}+10^{-6} \\
&*1.23197y\Big)+0.395527\Big(0.0134524(0.00448412y-0.01)^2-10^{-6} \\
&*1.4585\Big)\Big)\Big), \\
\dot{y} \simeq\ & 6.5621*10^6\Big(0.000014585(0.0000125427+0.0004x^3+x(1.4585*10^{-7} \\
&+\frac{3}{5}(-0.0405527-0.00955159y)(-0.01+0.00448412y))-x^2\Big(\frac{3}{100} \\
&(-0.0425527-0.00865476y)\Big)-7.63022*10^{-6}y-2.44252*10^{-7}y^2 \\
&+5.94204*10^{-7}y^3)\Big),
\end{aligned} \tag{4.44}$$

which has the first integral

$$\begin{aligned}
H_2(x,y) \simeq\ & 0.5\Big(-2(0.1x+0.1y+0.395527)(-0.1x+0.00448412y-0.01)^3 \\
&+0.000014585(0.1x(0.1x+0.1y+0.395527)^2\Big).
\end{aligned}$$

In the half-plane Σ^+ we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}_4})$

$$\begin{aligned}
\dot{x} =\ & \frac{1}{185850000}\Big(10135503x^3+189x^2(375289y-1802450)-72\Big(5103308y^2 \\
&-17232425y-19565375\Big)x+16\Big(19449472y^3-39895800y^2-78087500 \\
&-190896375y\Big)\Big), \\
\dot{y} =\ & \frac{1}{991200000}\Big(-11340189x^3-5292x^2(30644y-113575)-144x(2627023 \\
&y^2-25234300y+13326500)+64(10206616y^3-51697275y^2+44331250 \\
&-117392250y)\Big),
\end{aligned} \tag{4.45}$$

its corresponding first integral is

$$H_4(x,y) = -\frac{1}{4}\Big(-\frac{3}{200}x-\frac{4}{25}y+\frac{1}{2}\Big)^4-\frac{3}{2}\Big(\frac{3x}{2}-\frac{17y}{10}-3\Big)^2\Big(-\frac{3}{200}x-\frac{4}{25}y+\frac{1}{2}\Big)^2+\frac{7}{40}\Big(-\frac{3}{200}x-\frac{4}{25}y+\frac{1}{2}\Big)^2+\frac{1}{2}\Big(\frac{3}{2}x-\frac{17}{10}y-3\Big)\Big(-\frac{3}{200}x-\frac{4}{25}y+\frac{1}{2}\Big)+\frac{5}{14}\Big(-3-\frac{17}{10}y+\frac{3}{2}x\Big)^2.$$

System (4.27) *for* $i=2$ *and* $j=4$, *has the four real solutions* $(y_1,y_2)\simeq\{(0.9575,6.18244),(1.89796,5.97084),(2.62004,5.70563),(3.3489,5.32627)\}$, *which provide the four limit cycles for the discontinuous piecewise system* (4.44)–(4.45) *shown in Figure 4.6(a).*

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_2)$–$(\tilde{\mathcal{C}}_5)$**.** *In the half-plane* Σ^+ *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}}_2)$

$$\begin{aligned}\dot{x}\simeq\ & -0.204161\Big(\frac{1}{2}\Big(9.79621x^3+3x^2(30.7882y+3.41307)+x(-49.8671\\&+217.718y^2-60.1842y)x+9.79621(-1+15.5221y^3+4.53051y\\&-22.2247y^2)+8.40587(4.71431(-1+1.57144y)^2\Big)-4.89811\\&+1.57144(x+1.57144y-1)+0.5(9.79621x+30.7882y+3.41307)\Big),\\ \dot{y}\simeq\ & -0.204161\Big(x+\frac{1}{2}\Big(x(-6(1.57144y-1)(9.79621y+8.40587))-3x^2\\&(9.79621y+8.40587)-72.5726y^3+30.0921y^2+49.8671y-25.2176\Big)\\&+1.57144y-\frac{1}{2}(9.79621y+8.40587)-1\Big),\end{aligned}\tag{4.46}$$

its first integral is

$$\begin{aligned}H_2(x,y)\simeq\ & \frac{1}{2}\Big(-2(9.79621y+8.40587)(x+1.57144y-1)^3+2(-1+x+1.57144y)^2\\&-2(9.79621y+8.40587)(x+1.57144y-1)+\frac{1}{2}(9.79621y+8.40587)^2\Big).\end{aligned}$$

In the half-plane Σ^- *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}}_5)$

$$\begin{aligned}\dot{x}=\ & \frac{1}{296000}\Big(2512000x^3-4800x^2(3880y+49)+20(2065641y^2-273084y\\&-112804)x-24325359y^3+19174620y^2+8688868y+765520\Big),\\ \dot{y}=\ & \frac{1}{14800}\Big(48000x^3-2400x^2(157y+8)-688547+160x(5820y^2+147y\\&-143)y^3+136542y^2+112804y+13736\Big),\end{aligned}\tag{4.47}$$

with the first integral

$$H_5(x,y) = \frac{1}{4}\Big(\Big(-x+\frac{77}{20}y+1\Big)^4-6\Big(x-2y+\frac{3}{10}\Big)^2\Big(-x+\frac{77}{20}y+1\Big)^2-\Big(x-2y+\frac{3}{10}\Big)^4-\frac{2}{5}\Big(x -2y+\frac{3}{10}\Big)^2\Big).$$

System (4.27) *for* $i=2$ *and* $j=5$, *has the four real solutions* $(y_1,y_2)\simeq\{(0.59048,1.46437),(0.4015,1.5171),(0.75426,1.39625),(0.93725,1.28925)\}$, *which provide the four limit cycles for the discontinuous piecewise system* (4.46)–(4.47) *shown in Figure* 4.6*(b).*

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_2)$–$(\tilde{\mathcal{C}}_6)$**.**
In the half-plane Σ^- *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}}_2)$

$$\begin{aligned}\dot{x} \simeq\ & -3.0441\Big(\frac{3}{20}\Big(33.8979x^3+\frac{57}{10}\Big(\frac{579}{25}y-\frac{7977}{500}\Big)x^2+x\Big(\frac{4131}{25}y^2-207.216y \\ & +47.0097\Big)+\frac{21}{10}\Big(32y^3-\frac{108}{5}y^2+\frac{801}{200}y-\frac{27}{125}\Big)-3\Big(6\Big(2y-\frac{3}{5}\Big)^2-\frac{63}{200}\Big)\Big) \\ & +\frac{3}{40}\Big(\frac{579}{100}x+\frac{42y}{5}-\frac{363}{50}\Big)+\frac{1}{2}\Big(-\frac{19}{10}x-2y+\frac{3}{5}\Big)\Big), \\ \dot{y} \simeq\ & -3.0441\Big(\frac{3}{20}\Big(-24.6924x^3-\frac{1083}{100}x^2\Big(\frac{9.39}{100}y-\frac{183}{25}\Big)+x\Big(\frac{243}{2000}+\frac{57}{5}\Big(2y-\frac{3}{5}\Big) \\ & \Big(\frac{579}{100}y-\frac{156}{25})\Big)-\frac{1377}{25}y^3+103.608y^2-47.0097y+5.9454\Big)+\frac{19}{40}\Big(\frac{19}{10}x-\frac{3}{5} \\ & +2y\Big)-\frac{3}{40}\Big(\frac{171}{50}x+\frac{579}{100}y-\frac{156}{25}\Big)\Big)\Big),\end{aligned} \tag{4.48}$$

its corresponding first integral is given by

$$H_2(x,y) = \frac{1}{2}\Big(-2\Big(\frac{9}{10}x+\frac{21}{10}y-3\Big)\Big(1.9x+2y-\frac{3}{5}\Big)^3+\frac{5}{3}\Big(1.9x+2y-\frac{3}{5}\Big)^2-\Big(\frac{9}{10}x+\frac{21y}{10}-3\Big) \Big(1.9x+2y-\frac{3}{5}\Big)+\frac{3}{20}\Big(\frac{9}{10}x+\frac{21y}{10}-3\Big)^2\Big).$$

In the half-plane Σ^+ *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}}_6)$

$$\begin{aligned}\dot{x} \simeq\ & -0.0817147\Big(4.23771x^3+251.579x^2y-74.5707x^2+1485.03xy^2 \\ & -1228.58xy-1.3192(4.23771x-65.9016y+12.0698)(x-4y)(8.23771y \\ & +x-3.01746)+50.3792(-4y+x)+225.014x+4860.95y^3-5060.35y^2 \\ & +1853.6y-226.323\Big),\end{aligned} \tag{4.49}$$

$$\dot{y} = 0.0817147\Big(-9.05237x^2+12.7131x^2y+251.579xy^2-149.141xy+2x^3$$

$$-1.3192(-4y+x)(2x+4.23771y-3.01746)(x+8.23771y-3.01746)$$
$$-12.5948(-4y+x)+27.3151x+495.009y^3-614.292y^2+225.014y$$
$$-27.474\Big),$$

its corresponding first integral is

$$H_6(x,y) \simeq \frac{1}{4}\Big(-(x-4y)^4+2.6384(-x-8.23771y+3.01746)^2(x-4y)^2+25.1896 (x-4y)^2-(-x-8.23771y+3.01746)^4\Big).$$

The discontinuous piecewise system (4.48)–(4.49) *has four limit cycles, because system* (4.27) *for $i=2$ and $j=6$, has the four real solutions* $(y_1,y_2)\simeq\{(0.34545,1.56086),(0.61254,1.53336),(0.74912,1.50166),(0.86017,1.46364)\}$, *these limit cycles are shown in Figure* 4.7*(a).*

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_3)$**–**$(\tilde{\tilde{\mathcal{C}}}_3)$**.**

In the half-plane Σ^- we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}}_3)$

$$\begin{aligned}\dot{x} &= \frac{1}{5600}\Big(485x^3+9x^2(460y-2357)-4x(9300y^2-38130y+17149)-8\\ &\quad(23000y^3-223350y^2+690310y-666477)\Big),\\ \dot{y} &= \frac{1}{112000}\Big(1775x^3+x^2(45795-29100y)-8x(10350y^2-106065y\\ &\quad+185149)+10(24800y^3-152520y^2+137192y+209489)\Big),\end{aligned} \tag{4.50}$$

its first integral is

$$H_3(x,y) = \frac{1}{4}\left(\frac{1}{4}x+y-\frac{89}{20}\right)^4-\frac{3}{2}\left(\frac{1}{5}x-2y+\frac{17}{4}\right)^2\left(\frac{1}{4}x+y-\frac{89}{20}\right)^2-\frac{1}{2}\left(\frac{x}{5}-2y+\frac{17}{4}\right)^2.$$

In the half-plane Σ^+ we consider the Hamiltonian nilpotent saddle $(\tilde{\tilde{\mathcal{C}}}_3)$

$$\begin{aligned}\dot{x} &\simeq 2.87846\Big(-\frac{2}{5}\Big(\frac{1}{10}x+6y-3.4929\Big)\Big((18.0611-19.537y)x+0.03944x^2\\ &\quad+43.5706y^2-397.744y+322.273\Big)+601.947\Big(\frac{1}{10}x-2.6852y+3.9564\Big)\\ &\quad+5.98895(0.33148x-32.2224y+33.1174)\Big),\\ \dot{y} &\simeq 2.87846\Big(-\frac{2}{5}\Big(\frac{1}{10}x+6y-3.4929\Big)\Big(-x(0.321145-0.181668y)-\frac{1}{200}x^2\\ &\quad+6.27027y^2-7.7525y+0.62992\Big)-5.98895\Big(0.33148y+\frac{1}{50}x+0.04635)\\ &\quad+22.4172\Big(\frac{1}{10}x-2.6852y+3.95638\Big)\Big),\end{aligned} \tag{4.51}$$

and its first integral is

$$\widetilde{H_3}(x,y) \simeq \frac{1}{4}\Big(\Big(\frac{1}{10}x+6y-3.4929\Big)^4-6\Big(\frac{1}{10}x-2.6852y+3.95638)^2\Big(\frac{1}{10}x+6y-3.4929\Big)^2-\frac{4}{5}\Big(\frac{1}{10}x+6y-3.4929\Big)^2+59.8895\Big(\frac{1}{10}x-2.6852y+3.9564\Big)\Big(\frac{1}{10}x+6y-3.4929\Big)-1120.86\Big(\frac{1}{10}x-2.6852y+3.95638\Big)^2\Big).$$

For $i=3$, system (4.26) has the four real solutions $(y_1,y_2)\simeq\{(1.56431,2.46132),(1.6127,2.37006),(1.6728,2.27025),(1.75781,2.14818)\}$, which provide the four limit cycles for the discontinuous piecewise system (4.50)–(4.51) shown in Figure 4.7(b).

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}_3})$–$(\tilde{\mathcal{C}_4})$. *In Σ^- we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}_3})$*

$$\begin{aligned}\dot{x} = &\frac{16}{7}\Big(\Big(-\frac{1}{4}x+\frac{3}{4}y-\frac{7}{2}\Big)\Big(-\frac{3}{32}x^2+x\Big(\frac{39}{32}y-\frac{75}{16}\Big)+\frac{261y^2}{64}-\frac{321y}{16}+\frac{333}{16}\Big)+\frac{1}{2}\Big(\frac{1}{4}x+y-2\Big)\Big),\\ \dot{y} = &\frac{-16}{7}\Big(\Big(-\frac{1}{4}x+\frac{3y}{4}-\frac{7}{2}\Big)\Big(-\frac{5}{64}x^2+x\Big(\frac{29}{32}-\frac{33}{64}y\Big)-\frac{3}{64}y^2-\frac{33}{16}y+\frac{85}{16}\Big)+\frac{1}{8}\Big(\frac{1}{4}x+y-2\Big)\Big),\end{aligned} \tag{4.52}$$

which has the first integral

$$H_3(x,y) = \frac{1}{4}\Big(-\frac{x}{4}+\frac{3y}{4}-\frac{7}{2}\Big)^4-\frac{3}{2}\Big(-\frac{x}{4}-y+2\Big)^2\Big(-\frac{x}{4}+\frac{3y}{4}-\frac{7}{2}\Big)^2-\frac{1}{4}\Big(-\frac{x}{4}-y+2\Big)^2.$$

In the half-planeΣ^+ we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}_4})$

$$\begin{aligned}\dot{x} \simeq\ &0.005186\Big(85.1089(-x-4y+14.1679)\Big(x(300.205-103.454y)-21.2026x^2-136.18y^2+823.105y-928.242\Big)\Big),\\ \dot{y} \simeq\ &0.005186\Big(-85.1089(14.1679-x-4y)\Big(x(105.712-35.6079y)-45.833y^2-7x^2+273.867y-325.348\Big)\Big),\end{aligned} \tag{4.53}$$

its corresponding first integral is

$$H_4(x,y) \simeq -\frac{1}{4}(-x-4y+14.1679)^4-\frac{3}{2}(x+1.73421y-3.8747)^2(-x-4y+14.1679)^2+42.5544(-x-4y+14.1679)^2.$$

The four solutions of system (4.27) for $i = 3$ and $j = 4$, are $(y_1, y_2) \simeq \{(0.89241, 2.45296),$ $(0.95203, 2.30495), (1.02213, 2.15712), (1.10946, 2)\}$, these solutions provide the four limit cycles for the discontinuous piecewise system (4.52)–(4.53) shown in Figure 4.7(c).

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}_3})$–$(\tilde{\mathcal{C}_5})$**.** *In Σ^+ we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}_3})$*

$$\begin{aligned} \dot{x} = {} & \frac{-10}{7}\Big(\Big(\frac{1}{4}x + y - \frac{89}{20}\Big)\Big(-\frac{97}{400}x^2 + x\Big(\frac{629}{100} - \frac{11}{10}y\Big) + 23y^2 - 121y + \frac{7393}{50}\Big) \\ & -2\Big(\frac{1}{5}x - 2y + \frac{17}{4}\Big)\Big), \\ \dot{y} = {} & \frac{10}{7}\Big(\Big(\frac{71}{1600}x^2 + x\Big(\frac{7739}{4000} - \frac{181}{200}y\Big) + \frac{31}{20}y^2 - \frac{527}{200}y - \frac{2201}{800}\Big)\Big(\frac{1}{4}x + y - \frac{89}{20}\Big) \\ & +\frac{1}{5}\Big(\frac{1}{5}x - 2y + \frac{17}{4}\Big)\Big), \end{aligned} \tag{4.54}$$

which has the first integral

$$H_3(x,y) = \frac{1}{4}\Big(\frac{x}{4} + y - \frac{89}{20}\Big)^4 - \frac{3}{2}\Big(\frac{x}{5} - 2y + \frac{17}{4}\Big)^2\Big(\frac{x}{4} + y - \frac{89}{20}\Big)^2 - \frac{1}{2}\Big(\frac{x}{5} - 2y + \frac{17}{4}\Big)^2.$$

In Σ^- we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}_5})$

$$\begin{aligned} \dot{x} \simeq {} & 20\Big(-0.189875x^3 + 0.3375x^2y - 1.58576x^2 - \frac{3}{20}xy^2 + 1.7895xy - 0.7576 \\ & \Big(\frac{23}{20}x - y + 3.48716\Big) - 0.290453\Big(\frac{11}{10}x + 3.37453 - y\Big)\Big(\frac{23}{20}x - y + 3.48716) \\ & \Big(-\frac{9}{24}x + 2y - 6.86169\Big) - 4.3744x - 0.337891y^2 + 2.3185y - 3.97757\Big), \\ \dot{y} \simeq {} & -20\Big(0.284906x^3 - 0.569625x^2y + 2.43611x^2 + 0.3375xy^2 - 3.17153xy \\ & -0.29045\Big(\frac{253}{100}x - \frac{9}{4}y + 7.71658\Big)\Big(\frac{11}{10}x + 3.37453 - y\Big)\Big(\frac{23}{20}x - y + 3.48716\Big) \\ & +0.871283\Big(\frac{23}{20}x - y + 3.48716\Big) + 6.90948x - \frac{1}{20}y^3 + 0.89475y^2 - 4.3744y \\ & +6.49557\Big) \end{aligned} \tag{4.55}$$

it has the first integral

$$\begin{aligned} H_5(x,y) \simeq {} & \frac{1}{4}\Big(\Big(\frac{11}{10}x - y + 3.3745\Big)^4 + 0.580905\Big(\frac{23}{20}x - y + 3.48716\Big)^2\Big(-y + 3.3745 \\ & +\frac{11}{10}x\Big)^2 - \Big(\frac{23}{20}x - y + 3.48716\Big)^4 - 1.51527\Big(\frac{23}{20}x - y + 3.48716\Big)^2\Big). \end{aligned}$$

The discontinuous piecewise system (4.54)–(4.55) has four limit cycles shown in Figure

4.8(a), *because system* (4.27) *for* $i = 3$ *and* $j = 5$, *has the four real solutions* $(y_1, y_2) \simeq \{(1.56431, 2.46132), (1.6127, 2.37006), (1.6728, 2.27025), (1.75781, 2.14818)\}$.

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}_3})$–$(\tilde{\mathcal{C}_6})$.
In Σ^- *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}_3})$

$$
\begin{aligned}
\dot{x} \simeq\ & 2.28571\bigg(\Big(-\frac{1}{4}x+\frac{3}{4}y-\frac{7}{2}\Big)\Big(-0.09375x^2+x(1.21875y-4.6875)+20.8125 \\
& +4.07813y^2-20.0625y\Big)-\frac{1}{2}\Big(-\frac{1}{4}x-y+2\Big)\bigg), \\
\dot{y} \simeq\ & -2.28571\bigg(\Big(-\frac{1}{4}x+\frac{3}{4}y-\frac{7}{2}\Big)\Big(x(0.90625-0.515625y)+5.3125-2.0625y \\
& -0.078125x^2-0.046875y^2\Big)-\frac{1}{8}\Big(-\frac{1}{4}x-y+2\Big)\bigg),
\end{aligned}
\tag{4.56}
$$

it has the first integral

$$
H_3(x,y) = \frac{1}{4}\Big(-\frac{x}{4}+\frac{3}{4}y-\frac{7}{2}\Big)^4-\frac{3}{2}\Big(-\frac{x}{4}-y+2\Big)^2\Big(-\frac{1}{4}x+\frac{3}{4}y-\frac{7}{2}\Big)^2-\frac{1}{4}\Big(-\frac{1}{4}x-y+2\Big)^2.
$$

In Σ^+ *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}_6})$

$$
\begin{aligned}
\dot{x} \simeq\ & 2132.8\Big(10.0005x^3+150.014x^2y-2.83183x^2+750.106xy^2-28.3209xy \\
& -0.999999(-10.0005x-50.0047y+0.943854)(-x-5.00047y+0.188771) \\
& (x+5y)-0.34858(x+5y)+1250.23y^3+0.5346x-70.8089y^2+2.67308y \\
& -0.0336368\Big), \\
\dot{y} \simeq\ & -2132.8\Big(2x^3+30.0014x^2y-0.566312x^2+150.014xy^2-5.66365xy \\
& -0.999999(-2x-10.0005y+0.188771)(-x-5.00047y+0.188771)(5y \\
& x)-0.0697159(x+5y)+0.106903x+250.035y^3-14.1605y^2+0.534566y \\
& -0.00672673\Big),
\end{aligned}
\tag{4.57}
$$

its corresponding first integral is

$$
\begin{aligned}
H_6(x,y) \simeq\ & \frac{1}{4}\Big(-(x+5y)^4+2(-x-5.00047y+0.188771)^2(x+5y)^2+0.139432 \\
& (x+5y)^2-(-x-5.00047y+0.188771)^4\Big).
\end{aligned}
$$

The discontinuous piecewise differential system (4.56)–(4.57) *has four limit cycles, due to the fact that system* (4.27) *for* $i = 3$ *and* $j = 6$, *has the four real solutions* $(y_1, y_2) \simeq$

$\{(0.89241, 2.45296), (0.95203, 2.30495), (1.02213, 2.15712), (1.10946, 2)\}$, *see Figure 4.8(b).*

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_4)$–$(\tilde{\tilde{\mathcal{C}}}_4)$**.**

In Σ^- *we consider the Hamiltonian nilpotent saddle of type* $(\tilde{\mathcal{C}}_4)$

$$\begin{aligned}
\dot{x} \simeq\ & -0.001187\Big(450.18\Big(-\Big(2x+\frac{39}{20}y-\frac{1}{4}\Big)\Big(13.8833x^2+x(15.9014y+135.352)\\
&+7.43256y^2+5.95029y-103.297\Big)-3.8880548*10^{-14}(-x-0.03888y\\
&-11.5082)\Big)-4.41286(-x-0.03888y-11.5082)-4796.01(-2.02776x\\
&-0.151634y-22.4313)\Big),\\
\dot{y} \simeq\ & 0.001187\Big(450.18\Big(\Big(\frac{1}{4}-2x-\frac{39}{20}y\Big)\Big(20x^2+x(22.1498y+204.398)-114.233\\
&+7.84152y^2+70.7132y-114.233\Big)-10^{-12}(-x-0.0388805y-11.5082)\Big)\\
&-113.498(-x-0.03888y-11.5082)+4796.01(4x+2.02776y+22.7664)\Big),
\end{aligned} \tag{4.58}$$

its first integral is

$$\begin{aligned}
H_4(x,y)\ \simeq\ & \frac{1}{4}\Big(-\Big(2x+\frac{39}{20}y-\frac{1}{4}\Big)^4-6(-x-0.0388805y-11.5082)^2\Big(2x+\frac{39}{20}y-0.25\Big)^2\\
&+900.36\Big(2x+\frac{39}{20}y-\frac{1}{4}\Big)^2-42.6141(-x-0.0388805y-11.5082)\Big(2x\\
&+\frac{39}{20}y-\frac{1}{4}\Big)+0.504233(-x-0.0388805y-11.5082)^2\Big).
\end{aligned}$$

In Σ^+ *we consider the Hamiltonian nilpotent saddle of type* $(\tilde{\tilde{\mathcal{C}}}_4)$

$$\begin{aligned}
\dot{x} \simeq\ & 2.9301\Big(\frac{1}{2}\Big(\frac{2}{5}x-0.157142y+\frac{1}{2}\Big)\Big(0.763001x^2+x(1.6881y+8.44281)\\
&-2.15367y^2-8.3007y-1.76381\Big)-\frac{1}{4}(0.525429x-0.474567y-0.109278)\\
&-0.3775\Big(\frac{1}{2}x+\frac{151}{100}y+\frac{11}{2}\Big)\Big),\\
\dot{y} \simeq\ & 2.9301\Big(-\frac{1}{2}\Big(\frac{2}{5}x-0.15714y+\frac{1}{2}\Big)\Big(0.664x^2+x(2.54986y+10.435)+40.325\\
&+2.39007y^2+19.7052y+40.325\Big)+\frac{1}{4}\Big(\frac{2}{5}x+0.525429y+\frac{49}{20}\Big)+\frac{1}{8}\Big(\frac{1}{2}x+\frac{11}{2}\\
&+\frac{151}{100}y\Big)\Big),
\end{aligned} \tag{4.59}$$

which has the first integral

$$\widetilde{H}_4(x,y)\ \simeq\ -\frac{1}{4}\Big(\frac{2}{5}x-0.157142y+\frac{1}{2}\Big)^4-\frac{3}{2}\Big(\frac{1}{2}x+\frac{151}{100}y+\frac{11}{2}\Big)^2\Big(0.4x-0.157142y+\frac{1}{2}\Big)^2$$

$$+\frac{1}{4}\Big(\frac{2}{5}x-0.157142y+\frac{1}{2}\Big)^2+\frac{1}{2}\Big(\frac{1}{2}x+\frac{151}{100}y+\frac{11}{2}\Big)\Big(\frac{2}{5}x-0.157142y+\frac{1}{2}\Big)+\frac{1}{4}$$
$$(\frac{1}{2}x+\frac{151}{100}y+\frac{11}{2}\Big)^2.$$

System (4.26) *for* $i=4$, *has the following four real solutions* $(y_1,y_2)\simeq\{(0.79064,5.62516),$ $(1.61348,5.37496),(2.35323,5.04467),(3.24928,4.47718)\}$, *which provide the four limit cycles for the discontinuous piecewise system* (4.58)–(4.59) *shown in Figure 4.8(c).*

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}_4})$–$(\tilde{\mathcal{C}_5})$**.** *In* Σ^+ *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}_4})$

$$\begin{aligned}\dot{x}\simeq\;& 0.017928\Big(9.95432(-x-4y-0.00302329)\Big(-11.1894x^2+x(18.0134y\\ &-45.3307)-125.712y^2+136.219y-27.0844\Big)\Big),\\ \dot{y}\simeq\;& 0.017928\Big(-9.95432(-x-4y-0.00302329)\Big(-7x^2+x(-21.2711\\ &-5.56816y)-4.47155y^2-5.62777y-6.80111\Big)\Big),\end{aligned}\tag{4.60}$$

which has the first integral

$$\begin{aligned}H_4(x,y)\simeq\;& -\frac{1}{4}(-x-4y-0.00302329)^4-\frac{3}{2}(x-1.60354y+2.36178)^2(-x-4y\\ &-0.00302329)^2+4.97716(-x-4y-0.00302329)^2.\end{aligned}$$

In Σ^- *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}_5})$

$$\begin{aligned}\dot{x}=\;& \frac{1}{1958000}\Big(5063409x^3-30x^2(428262y-552439)-1500x(133164y^2\\ &-21372y-8387)+200\Big(1479978y^3-1602558y^2+317646y+241\Big)\Big),\\ \dot{y}=\;& \frac{1}{293700000}\Big(-528871721x^3-150x^2(15190227y+8097032)+1500x\\ &(1284786y^2-3314634y-614891)+25000(399492y^3-96174y^2\\ &-75483y-11584)\Big),\end{aligned}\tag{4.61}$$

with its corresponding first integral

$$\begin{aligned}H_5(x,y)\;=\;& \frac{1}{4}\Big(-\Big(\frac{81}{100}x-\frac{3}{2}y+1\Big)^4-30\Big(-\frac{1}{2}x-\frac{27}{10}y-\frac{1}{10}\Big)^2\Big(\frac{81}{100}x-\frac{3}{2}y+1\Big)^2-3\Big(\frac{81}{100}x-\\ &\frac{3y}{2}+1\Big)^2+\Big(-\frac{1}{2}x-\frac{27}{10}y-\frac{1}{10}\Big)^4\Big).\end{aligned}$$

The discontinuous piecewise system (4.60)–(4.61) *has four limit cycles, due to the fact that* *system* (4.27) *when* $i = 4$ *and* $j = 5$, *has the four real solutions* $(y_1, y_2) \simeq \{(0.70964, 0.912115), (0.65295, 0.945775), (0.602076, 0.970419), (0.551174, 0.990484)\}$, *see Figure* 4.9*(a).*

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}_4})$–$(\tilde{\mathcal{C}_6})$**.** *In* Σ^- *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}_4})$

$$
\begin{aligned}
\dot{x} = \quad & \frac{1}{38650000}\Big(-840(116965x+298018)y^2+100(3x(100927x+382900)\\
& +142306)y+125(x(x(120295x+1159926)+3330420)+2620392)\\
& +42884408y^3\Big),\\
\dot{y} = \quad & \frac{1}{3092000}\Big(-24(100927x+191450)y^2-30(x(120295x+773284)\\
& +1110140)y-25(x(5x(6005x+63546)+1047948)+1052440)\\
& 2620016y^3\Big),
\end{aligned}
\tag{4.62}
$$

which has the first integral

$$
\begin{aligned}
H_4(x,y) = \quad & -\frac{1}{4}\left(\frac{x}{4}-\frac{7y}{50}+\frac{7}{10}\right)^4-\frac{3}{2}\left(\frac{7x}{10}+\frac{27y}{10}+3\right)^2\left(\frac{x}{4}-\frac{7y}{50}+\frac{7}{10}\right)^2+\frac{1}{2}\left(\frac{x}{4}-\frac{7y}{50}+\frac{7}{10}\right)^2\\
& +\frac{1}{2}\left(\frac{7x}{10}+\frac{27y}{10}+3\right)\left(\frac{x}{4}-\frac{7y}{50}+\frac{7}{10}\right)+\frac{1}{8}\left(\frac{7x}{10}+\frac{27y}{10}+3\right)^2.
\end{aligned}
$$

In Σ^+ *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}_6})$

$$
\begin{aligned}
\dot{x} \simeq \quad & 0.2033\Big(1.08085x^3+38.0494x^2y+27.0421x^2+59.7945xy^2-103.796xy\\
& -1.03369(1.08085x-11.5149y-14.0907)(-1.91915y+x-4.69689)(x\\
& +3y)-9.96074(x+3y)-127.014x+94.5655y^3+99.5997y^2+243.758y\\
& +198.856\Big),\\
\dot{y} \simeq \quad & -0.2033\Big(-14.0907x^2+2x^3+3.24255x^2y+38.0494xy^2+54.0842xy\\
& -1.03369(x-1.91915y-4.69689)(2x+1.08085y-4.69689)(x+3y)\\
& -3.32025(x+3y)+66.1823x+19.9315y^3-51.8978y^2-127.014y\\
& -103.617\Big),
\end{aligned}
\tag{4.63}
$$

its corresponding first integral is

$$
\begin{aligned}
H_6(x,y) \quad \simeq \quad & \frac{1}{4}\Big(-(x+3y)^4+2.06739(x-1.91915y-4.69689)^2(x+3y)^2+6.64049\\
& (x+3y)^2-(x-1.91915y-4.69689)^4\Big).
\end{aligned}
$$

System (4.27) for $i=4$ and $j=6$, has the four real solutions $(y_1,y_2)\simeq\{(4.81888, 6.12412),$ $(4.28035, 6.45928), (3.81918, 6.68363), (3.35366, 6.86035)\}$, which provide the four limit cycles for the discontinuous piecewise system (4.62)–(4.63) shown in Figure 4.9(b).

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_5)$–$(\tilde{\tilde{\mathcal{C}}}_5)$.

In Σ^- we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}}_5)$

$$\begin{aligned}\dot{x} \simeq\ & 63515.6\Big(0.00331513\Big(-0.00628079x^3+0.0327017x^2y-0.01361x^2\\ &-0.0425664xy^2+0.0277152xy+0.000683986x+0.0164168y^3\\ &-0.0138017y^2-0.000330382y+0.000281957\Big)\Big),\\ \dot{y} \simeq\ & 63515.6\Big(0.00331513\Big(0.0188424x^2y-0.0129654x^2-0.0327017xy^2\\ &+0.0272201xy+0.00113769x+0.0141888y^3-0.0138576y^2\\ &-0.000683986y+0.000340256\Big)\Big),\end{aligned} \tag{4.64}$$

its first integral is

$$\begin{aligned}H_5(x,y) \ \simeq\ & \frac{1}{4}\Big(\Big(\frac{23}{20}x-0.99587y+0.142561\Big)^4+0.00663026\Big(\frac{23}{20}x-0.99587y\\ &+0.142561\Big)^2-(-\frac{23}{20}x+y-0.145403\Big)^4\Big).\end{aligned}$$

In Σ^+ we consider the Hamiltonian nilpotent saddle $(\tilde{\tilde{\mathcal{C}}}_5)$

$$\begin{aligned}\dot{x} =\ & \frac{1}{26500}\Big(15(1271x-147257)y^2-225(x(2405x+5584)+235)y-125(x(x\\ &(367x+42)-1254)-361)+2627384y^3\Big),\\ \dot{y} =\ & \frac{1}{5300}\Big(45(2405x+2792)y^2+75(x(367x+28)-418)y-125(x(9x(9x+38)\\ &+404)+142)-1271y^3\Big),\end{aligned} \tag{4.65}$$

which has the first integral

$$\begin{aligned}\widetilde{H}_5\,(x,y) \ =\ & \frac{1}{4}\Big(-\Big(\frac{x}{2}-\frac{3y}{2}+1\Big)^4-24\Big(-x-\frac{23y}{10}-\frac{1}{2}\Big)^2\Big(\frac{x}{2}-\frac{3y}{2}+1\Big)^2-4\Big(\frac{x}{2}-\frac{3y}{2}+1\Big)^2\\ &+\Big(x+\frac{23}{10}y+\frac{1}{2}\Big)^4\Big).\end{aligned}$$

The discontinuous piecewise differential system (4.64)–(4.65) has four limit cycles, because system (4.26) has the four real solutions for $i=5$, given by $(y_1,y_2)\simeq\{(0.65276, 0.98274),$

$(0.60463, 1.00802), (0.55822, 1.02903), (0.51112, 1.04723)\}$, *which provide the four limit cycles for the discontinuous piecewise system* (4.64)–(4.65) *shown in Figure 4.9(c).*

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}_5})$–$(\tilde{\mathcal{C}_6})$**.** *In* Σ^+ *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}_5})$

$$\begin{aligned}\dot{x} \simeq\;& 7.13171\Big(0.00023x^3 + 0.0255614x^2y + 0.00394378x^2 + 0.405655xy^2 \\ & -1.93117xy - 1.35472(-0.104619x - 8.71622y - 9.55726)\Big(\frac{1}{100}x \\ & -2.44838y - 5.36925\Big)\Big(\frac{1}{20}x + \frac{89}{50}y\Big) - 375.296\Big(\frac{1}{20}x + \frac{89}{50}y\Big) - 2.11751x \\ & +45.9734y^3 + 236.412y^2 + 518.447y + 378.982\Big), \\ \dot{y} \simeq\;& -7.13171\Big(6.26*10^{-6}x^3 + 0.00066015x^2y - 0.0000162x^2 + 0.02556xy^2 \\ & +0.00788756xy - 1.35472\Big(\frac{1}{100}x - 2.44838y - 5.36925\Big)\Big(-0.104619y \\ & \frac{1}{1000}x - 0.268462\Big)\Big(\frac{1}{20}x + \frac{89}{50}y\Big) - 10.542\Big(\frac{1}{20}x + \frac{89}{50}y\Big) + 0.00864864x \\ & +0.13522y^3 - 1.54789 - 0.96559y^2 - 2.11751y\Big),\end{aligned} \tag{4.66}$$

with the first integral

$$\begin{aligned}H_5(x,y) \;\simeq\;& \frac{1}{4}\Big(-\Big(\frac{1}{20}x + \frac{89}{50}y)^4 + 2.70944\Big(\frac{1}{100}x - 2.44838y - 5.36925\Big)^2\Big(\frac{1}{20}x \\ & +\frac{89}{50}y\Big)^2 + 421.681(0.05x + 1.78y)^2 - (0.01x - 2.44838y - 5.36925)^4\Big).\end{aligned}$$

In Σ^- *we consider the Hamiltonian nilpotent saddle* $(\tilde{\mathcal{C}_6})$

$$\begin{aligned}\dot{x} \simeq\;& -\frac{5}{3}\Big(-\frac{81}{80}x^3 + \frac{1197}{400}x^2y - \frac{4863}{400}x^2 - \frac{6003}{2000}xy^2 + 23.937xy - 2(x - y + 4) \\ & -\frac{3}{10}(x - y + 4)\Big(-\frac{2}{5}x - \frac{1}{5}y - \frac{17}{10}\Big)\Big(\frac{1}{2}x + \frac{1}{10}y + \frac{21}{10}\Big) - 48.6615x \\ & +0.9999y^3 - 12.0063y^2 + 47.8677y - 64.9261\Big), \\ \dot{y} \simeq\;& \frac{5}{3}\Big(\frac{15}{16}x^3 - \frac{243}{80}x^2y + \frac{897}{80}x^2 + 2.9925xy^2 - 24.315xy + 2(x - y + 4) - \frac{3}{10} \\ & (x - y + 4)\Big(x - \frac{2}{5}y + \frac{41}{10}\Big)\Big(\frac{1}{2}x + \frac{1}{10}y + \frac{21}{10}\Big) + \frac{17877}{400}x - \frac{2001}{2000}y^3 + 59.3695 \\ & +11.9685y^2 - 48.6615y\Big),\end{aligned} \tag{4.67}$$

and its first integral is

$$H_6(x,y) \simeq \frac{1}{4}\Big(\frac{3}{5}\Big(\frac{1}{2}x + \frac{1}{10}y + \frac{21}{10}\Big)^2(x - y + 4)^2 - (x - y + 4)^4 - 4(x - y + 4)^2 + \Big(\frac{1}{2}x + \frac{1}{10}y + \frac{21}{10}\Big)^4\Big)$$

The discontinuous piecewise differential system (4.66)–(4.67) has exactly four limit cycles, due to the fact that system (4.27) for $i = 5$ and $j = 6$, has the four real solutions $(y_1, y_2) \simeq \{(2.08477, 6.67049), (2.33749, 6.52949), (2.72322, 6.35381), (3.51937, 6.11069)\}$, see Figure 4.10(a).

Four limit cycles for the discontinuous piecewise differential system $(\tilde{\mathcal{C}}_6)$–$(\tilde{\tilde{\mathcal{C}}}_6)$**.** *In Σ^- we consider the Hamiltonian nilpotent saddle $(\tilde{\mathcal{C}}_6)$*

$$\begin{aligned}
\dot{x} \simeq\ & -430.527\Big(-1.43808x^3+49.6338x^2y-2.2836x^2-571.02xy^2+52.5653\\
& xy-0.999968(5.75232x-66.1784y+3.04235)\Big(-\frac{23}{4}y+\frac{1}{2}x\Big)(5.75465y\\
& -\frac{1}{2}x-0.529104)+3.03025\Big(\frac{1}{2}x-\frac{23}{4}y\Big)-2.41653x+2189.79y^3-302.495\\
& y^2+27.8125y-0.852396\Big),\\
\dot{y} \simeq\ & 430.527\Big(-4.31424x^2y+0.198414x^2+49.6338xy^2-4.56721xy+\frac{1}{8}x^3\\
& -0.999968\Big(\frac{1}{2}x-\frac{23}{4}y\Big)\Big(-\frac{1}{2}x+5.75232y-0.264552\Big)\Big(-\frac{1}{2}x+5.75465y\\
& -0.5291\Big)-0.2635\Big(\frac{1}{2}x-\frac{23}{4}y\Big)+0.20996x+0.07406-190.34y^3+26.2827\\
& y^2-2.41653y\Big),
\end{aligned} \tag{4.68}$$

its corresponding first integral is

$$\begin{aligned}
H_6(x,y) \simeq\ & \frac{1}{4}\Big(-\Big(-\frac{1}{2}x+5.75465y-0.529104\Big)^4+1.99994\Big(\frac{1}{2}x-\frac{23}{4}y\Big)^2\Big(-\frac{1}{2}x\\
& +5.75465y-0.529104\Big)^2-\Big(\frac{1}{2}x-\frac{23}{4}y\Big)^4+\frac{527}{500}\Big(\frac{1}{2}x-\frac{23}{4}y\Big)^2\Big).
\end{aligned}$$

In Σ^+ we consider the Hamiltonian nilpotent saddle $(\tilde{\tilde{\mathcal{C}}}_6)$

$$\begin{aligned}
\dot{x} =\ & -\frac{2}{3}\Big(-\frac{1}{2}x^3+\frac{15}{4}x^2y-9x^2-\frac{21}{8}xy^2+\frac{27}{2}xy-\frac{1}{2}\Big(-x+y-\frac{5}{2}\Big)+9\Big(-x-\frac{1}{2}y\\
& +1\Big)\Big(-x+y-\frac{5}{2}\Big)\Big(-\frac{1}{2}x-y+\frac{9}{4}\Big)-\frac{69}{4}x+\frac{17}{16}y^3-\frac{63}{8}y^2+\frac{39}{2}y-\frac{129}{8}\Big),\\
\dot{y} =\ & \frac{2}{3}\Big(2x^3-\frac{3}{2}x^2y+\frac{9}{2}x^2+\frac{15}{4}xy^2-18xy+9\Big(-x-\frac{1}{2}y+1\Big)(2x-\frac{1}{2}y+\frac{3}{2}\Big)\Big(-x\\
& +y-\frac{5}{2}\Big)+\frac{1}{2}\Big(-x+y-\frac{5}{2}\Big)+\frac{87}{4}x-\frac{7}{8}y^3+\frac{27}{4}y^2-\frac{64}{4}y+\frac{117}{8}\Big),
\end{aligned} \tag{4.69}$$

which has the first integral

$$\widetilde{H}_6\,(x,y)=\frac{1}{4}\Big(-\Big(1-x-\frac{1}{2}y\Big)^4-18\Big(y-x-\frac{5}{2}\Big)^2\Big(1-x-\frac{1}{2}y\Big)^2-\Big(y-x-\frac{5}{2}\Big)^4+\Big(y-x-\frac{5}{2}\Big)^2\Big).$$

System (4.26) *for* $i = 6$, *has the four real solutions* $(y_1, y_2) \simeq \{(1.29085, 3.25778), (1.13235, 3.42658), (1.02312, 3.54042), (0.93752, 3.62878)\}$, *which provide the four limit cycles for the discontinuous piecewise system* (4.68)–(4.69) *shown in Figure 4.10(b).*

These examples complete the proof of Theorem 4.3. ∎

Conclusion

In this work, based on the first integral, we have solved the second part of the sixteenth Hilbert problem for five families of discontinuous piecewise differential systems having the straight line $x = 0$ as a switching curve. The first one is created by an arbitrary linear and quadratic center, and the second family is composed of a linear center and cubic isochronous center having a rational first integral. The third family is formed by two cubic isochronous centers having a rational first integral in each half-plane. We also solved the extinction of the sixteenth Hilbert problem for the family of the discontinuous piecewise differential system formed by a linear center and cubic Hamiltonian nilpotent saddle. The last one that we gave its maximum number of limit cycles is the one formed by two cubic Hamiltonian nilpotent saddles in each half-plane.

Even though we have solved the extinction of the sixteenth Hilbert problem for some families of discontinuous piecewise differential systems separated by a straight line. It's still difficult to solve this problem for the nonlinear discontinuous piecewise differential systems, especially if we consider a quadratic differential center in each half-plane.

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ملخص:

تهتم هذه الأطروحة بحل مسألة مهمة وصعبة في النظرية النوعية للأنظمة التفاضلية، و التي تسمى مسألة هيلبرت السادسة عشر. قمنا تحديدا باستعمال التكاملات الأولية لايجاد الحد الأقصى لعدد الدورات الحدية لبعض الأنظمة التفاضلية غير الخطية والمتقطعة والمفصولة بخط مستقيم.

الكلمات المفتاحية:

أنظمة تفاضلية متعددة الحدود، النظام المركزي الخطي، الأنظمة التفاضلية المكعبة المتزامنة ذات مركز والتي تقبل تكامل أولي كسري، الأنظمة التفاضلية الهاملتونية المكعبة التي تقبل نقطة توازن عديمة القوى ومن نوع سرج، دورات الحدود المعزولة.

Abstract:

Our thesis is devoted to solving a significant and challenging issue in the qualitative theory of differential systems called the sixteenth Hilbert problem. More precisely, we use the first integrals to determine the maximum number of limit cycles of some families of discontinuous piecewise nonlinear differential systems separated by a straight line.

Keywords:

Piecewise differential system, linear center, cubic reversible isochronous centers having rational first integrals, Hamiltonian system with linear plus cubic homogeneous terms with a nilpotent saddle, limit cycle.

Résumé:

Notre thèse porte sur la résolution d'un problème crucial et difficile dans la théorie qualitative des systèmes différentiels, connu sous le nom de seizième problème de Hilbert.

En d'autres termes, nous utilisons les intégrales premières pour déterminer le nombre maximal de cycles limites de certaines familles de systèmes différentiels non linéaires par morceaux séparés par une ligne droite.

Mots clés:

Systèmes différentiels par morceaux, centre linéaire, centre cubiques isochrone réversible admet une intégrale première rationnelle, système Hamiltonien avec un terme linéaire plus des termes cubiques homogènes admet un point d'equilibre de type selle nilpotent, cycle limite.