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THESIS

In order to obtain the Doctrate degree in LMD (3rd cycle)

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THEME

Etude qualitative de quelques EDPs en temps avec amortissement

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In front of the jury Composed of:

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Abstract : This thesis is devoted to the study of two problems related to the theory of control of PDE.
In a first and second time, we study two nonlinear Euler-Bernoulli beams with a neutral type delay and viscoelastic, using controls acting on the free boundaries.
By using the method of Faedo-Galerkin, we prove the existence and uniqueness of the solution for each problem.
After that using the energy method and constructing an appropriate Lyapunov function, under certain conditions on the neutral delay term kernel and the viscoelastic term, we show that although, the destructive nature of delay in general, which is a very general degrading energy problem.

Keywords: Euler-Bernoulli beam, Neutral delay, Boundary control, Viscoelasticity, General decay, Exponential stability, Lyapunov functionals.

Résumé: Cette thèse est consacrée à l'étude de deux problèmes liés à la théorie du contrôle des PDE.

Dans un premier et deuxième temps, nous étudions deux poutres d'Euler-Bernoulli non linéaires à retard de type neutre et viscoélastique, en utilisant des contrôles agissant sur les frontières libres.

En utilisant la méthode de Faedo-Galerkin, nous prouvons l'existence et l'unicité de la solution pour chaque problème.

Ensuite, en utilisant la méthode énergétique et en construisant une fonction de Lyapunov appropriée, sous certaines conditions sur le noyau du terme de retard neutre et le terme viscoélastique, nous montrons que bien que, La nature destructrice du retard en général, qui est un problème énergétique dégradant très général.

Mots clés : Poutre d'Euler-Bernoulli, Retard neutre, Boundary contrôle, Viscoélasticité, Décroissance générale, Stabilité exponentielle, Fonctionnelles de Lyapunov.

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Finally, and importantly, i intend to dedicate this work to my father. I deeply wish he could be here with me (May God have mercy on him).

Contents

Symbols

$\geq$	greater than or equals	Ω	bounded domain in $\mathbb{R}^N$.
$\leq$	less than or equals	Γ	topological boundary of Ω.
$\exists$	exist	$x = (x_1, x_2, ..., x_N)$	generic point of $\mathbb{R}^N$.
$\in$	is an element of	$dx = dx_1 dx_2 ... dx_N$	Lebesgue measuring on Ω.
$\notin$	is not an element of	∇y	gradient of y.
$\subset$	is a subset of	Δy	Laplacien of y.
$\supset$	contains	r'	conjugate of r, i.e $\frac{1}{r} + \frac{1}{r'} = 1$.
$=$	equals	$\prod or \times$	Cartesian product.
$\bigcup$	union	$\bigcap$	intersection.
	$(.)_x = \frac{\partial(.)}{\partial x}$, $(.)_t = \frac{\partial(.)}{\partial t}$.		

$\mathcal{D}(\Omega)$: space of functions indefinitely differentiable with compact support in Ω.

$\mathcal{D}'(\Omega)$: distribution space.

$\mathcal{C}^m(\Omega)$: space of functions with continuous derivatives on Ω up to order m.

$\mathcal{C}^0(\Omega)$: space of continuous functions in Ω.

$L^r(\Omega)$: vector space of functions such that $\int_\Omega |h(t)|^r dt < \infty$.

$L^r(0, T; E)$: space of all strongly measurable functions such that

$$\int_0^T \| h(t) \|_E^r \, dt < \infty.$$

$L^{\infty}(0, T; E)$: space of all strongly measurable functions such that

$$\operatorname*{ess\,sup}_{x \in (0,T)} \|h(t)\|_E^r < \infty.$$

Introduction

The exploration of partial differential equations (PDEs) and their solutions remains a cornerstone in understanding various physical phenomena, ranging from heat conduction to fluid dynamics. However, one particular realm where the study of PDEs has proven to be pivotal is in the field of energy dynamics and explosive phenomena.

The historical roots of investigating PDEs can be traced back to the 18th and 19th centuries when pioneering mathematicians and physicists began formulating mathematical models to describe physical processes. The likes of Leonard Euler, Joseph Fourier, and Jean-Baptiste Joseph Fourier contributed significantly to the development of the mathematical tools required to analyze and solve these equations.

As our understanding of the physical world deepened, so did the need for more sophisticated mathematical models. In the realm of energy dynamics, the study of PDEs became especially crucial. These equations allow us to describe how energy is distributed and transformed in various systems, shedding light on the fundamental principles governing these processes.

One captivating facet of PDEs in the context of energy dynamics is their role in modeling and understanding explosive phenomena. Whether it be the detonation of chemical reactions, the shockwaves in fluid dynamics, or the release of energy in nuclear reactions, PDEs provide a powerful framework for capturing and predicting these events.

The search for stability and the quest to comprehend explosive behaviors have led mathematicians and scientists to delve into the intricate solutions of PDEs. The development of numerical methods, advancements in computing technology, and interdisciplinary collaborations haye further propelled our ability to tackle complex problems related to energy release and explosive events.

In this exploration, we will delve into the historical context, the mathematical foundations, and the practical applications of solving PDEs pertaining to energy dynamics. By understanding the mathematical underpinnings of these phenomena, we not only gain insights into the past achievements of mathematical pioneers but also pave the way for innovative solutions and advancements in addressing contemporary challenges in energy science and technology.

This thesis is divided into three chapters.

In the opening chapter, we present sets of definitions, theorems and characteristics required to support our results, as well as a brief summary of the basic results concerning Banach spaces, weak and weak star topologies, L^p space, Sobolev spaces, and other theorems. Understanding all of these notations and results is essential for our research, which we apply in the sequel without making any specific mention of it.

In the second chapter, We examine the free transverse vibration of a nonlinear Euler-Bernoulli beam subjected to a neutral type delay. Initially, we establish a local existence result employing the Faedo-Galerkin method. We then develop a boundary control approach using the Lyapunov method to dampen the transverse vibrations of the beam.

Chapter 3 delves into the analysis of the nonlinear Euler-Bernoulli viscoelastic equation featuring a neutral type delay. Initially, the Faedo-Galerkin method was utilized to establish the local existence result. Subsequently, we demonstrate that even though delays are generally destructive, a highly general decaying energy for the problem was produced by applying the energy approach and building an appropriate Lyapunov functional under specific circumstances on the kernel of the neutral delay term.

Chapter 1

Preliminaries

This chapter includes sets of definitions, theorems, and properties required in the proof of our results. It also briefly covers the fundamental results related to the L^r space, Sobolev spaces, weak and weak star topologies, and other theorems. Understanding all of these notations and findings is crucial to our research, which we employ in the sequel without making any special mention of it.

Section 1.1 The weak topology

Let X be a Banach space and $h \in X'$. We denote by $\varphi_h : X \to \mathbb{R}$ the linear functional $\varphi_h(x) = \langle h, x \rangle$. As h varies over X' we obtain a collection $(\varphi_h)_{h \in X'}$ of maps from X into $\mathbb{R}$.

Definition 1.1 *The weak topology $\sigma(X, X')$ on X is the coarsest topology associated to the collection $(\varphi_h)_{h \in X'}$, such that every φ_h is continuous.*

Remark 1.1 *In the weak topology $\sigma(X, X')$, we write $v_n \rightharpoonup v$, which represents the convergence of the sequence $(v_n)_n$ to v.*

Proposition 1.1 *[3]. Let (v_n) be sequence in X. Then*

1. $v_n \rightharpoonup v$ *weakly in* $\sigma(X, X') \Leftrightarrow \langle h, v_n \rangle \rightharpoonup \langle h, v \rangle, \forall h \in X'$.

2. $v_n \to v$ *strongly* $\Rightarrow v_n \rightharpoonup v$ *weakly.*

3. $v_n \rightharpoonup v$ *weakly* $\Rightarrow (v_n)_n$ *is bounded and* $\|v\| \leq \liminf \|v_n\|$.

Remark 1.2 *The weak topology and the usual topology are the same if X is finite-dimensional.*

We shall introduce a third topology on X' called the weak star topology, denoted by $\sigma(X', X)$. For every $x \in X$:

$$\begin{aligned} \varphi_x : X' &\longrightarrow \mathbb{R} \\ h \mapsto \varphi_x(h) &= \langle h, x \rangle_{X', X} \end{aligned} \tag{1.1}$$

when x cover X, we obtain a family $(\varphi_x)_{x \in X}$ of applications to X' in $\mathbb{R}$.

Definition 1.2 *The weak star topology* $\sigma(X', X)$ *is the smallest topology on* X' *associated to the collection* $(\varphi_x)_{x \in X}$*, for which every* φ_x *is continuous.*

Remark 1.3 *[20].*

(i) In the weak star topology if a sequence $(h_n)_n \subset X'$ *converges to* h*, we write* $h_n \overset{*}{\rightharpoonup} h$ $(h_n \overset{*}{\rightharpoonup} h$ *in* $\sigma(X', X))$.

(ii) Since $X \subset X''$ *it is clear that, the usual topology is stronger then the weak topology* $\sigma(X', X'')$*, and this later is stronger then the weak star topology* $\sigma(X', X)$.

Proposition 1.2 *[20]. Let* $(h_n) \subset X'$*. Then*

1) $\left[h_n \overset{*}{\rightharpoonup} h\right] \Leftrightarrow [h_n(x) \to h(x),\ \forall x \in X]$.

2) If $h_n \to h$ *(strongly), then* $h_n \rightharpoonup h$*, in* $\sigma(X', X'')$.

3) If $h_n \overset{*}{\rightharpoonup} h$*, then* $\|h_n\|$ *is bounded and* $\|h\| \leq \liminf \|h_n\|$.

4) If $h_n \overset{*}{\rightharpoonup} h$ *in* $\sigma(X', X)$ *and* $u_n \longrightarrow u$ *(strongly) in* X*, then* $h_n(u_n) \to h(u)$.

Remark 1.4 *If $h_n \overset{*}{\rightharpoonup} h$ in $\sigma(X', X'')$ and $u_n \rightharpoonup u$ in $\sigma(X, X')$. In general, it is not possible to deduce that $h_n(u_n) \to h(u)$.*

1.1.1 Hilbert spaces

In many areas of mathematics and physics, particularly in quantum mechanics, Hilbert spaces play a crucial role by offering an appropriate framework for characterizing the state space of quantum systems. The completeness feature guarantees the right behavior of the mathematical structure and permits the insightful analysis of limits and convergent sequences.

Definition 1.3 *A Hilbert space H is a complete scalar product space where the norm derives from the scalar product.*

Proposition 1.3 *[3]. In the weak topology of a Hilbert space, every convergent sequence $(v_n)_n$ is bounded.*

Proposition 1.4 *[3]. A sequence $(v_n)_n$ in a Hilbert space converges weakly if it is bounded in H.*

Theorem 1.1 *[3]. Let $(v_n)_{n\in\mathbb{N}}$ be a convergent sequence to v in H and $(w_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence to w, then*

$$\lim_{n\longrightarrow\infty} \langle v_n, w_n \rangle = \langle v, w \rangle. \tag{1.2}$$

Proposition 1.5 *Let E, F two Hilbert spaces, let $(v_n)_{n\in\mathbb{N}} \subset E$ be a weakly convergent sequence to $v \in E$, let $\Lambda \in L(E;F)$. Then, the sequence $(\Lambda(v_n))_n$ converges to $\Lambda(v)$ in $\sigma(F, F')$.*

Proof *Let $w \in F$, the function*

$$w \mapsto \langle \Lambda(v), w \rangle$$

is linear and continuous, because

$$|\langle \Lambda(v), w \rangle| \leq \|\Lambda\| \|v\|_E \|w\|_F, \ \forall v \in E, w \in F$$

using Riesz theorem, there exists $u \in E$ such that

$$\langle \Lambda(v), w \rangle = \langle v, u \rangle, \forall v \in E$$

Then

$$\lim_{n \longrightarrow \infty} < \Lambda(v_n), w >= \lim_{n \longrightarrow \infty} < v_n, u >$$
$$=< v, u >=< A(v), w > .$$

■

Section 1.2 Functional Spaces

1.2.1 The $L^r(\Omega)$ spaces

Definition 1.4 *Let Ω be an open domain in $\mathbb{R}^n, n \in \mathbb{N}$, and let $1 \leq r < \infty$. we define the Lebesgue space*

$$L^r(\Omega) = \left\{ h : \Omega \rightarrow \mathbb{R} : h \text{ is measurable and } \int_\Omega |h(x)|^r \, dx < \infty \right\}.$$

Remark 1.5 *For $r \in [1, \infty[$, we define*

$$\|h\|_r = \left(\int_\Omega |h(x)|^r \, dx \right)^{\frac{1}{r}}.$$

If $r = \infty$, we have

$$L^\infty(\Omega) = \{ h : \exists C > 0, \text{ such that } h \text{ is measurable and } |h(x)| \leq C \}.$$

We define

$$\|h\|_\infty = \sup_{x\in\Omega}|h(x)|.$$

Theorem 1.2 *[57]. For $1 \le r \le \infty$. The L^r spaces are useful and significant Banach spaces.*

Remark 1.6 *The case $r = 2$, $L^2(\Omega)$ endowed with the scalar product*

$$\langle h, g\rangle_{L^2(\Omega)} = \int_\Omega h(x)g(x)dx \qquad (1.3)$$

is a Hilbert space.

Theorem 1.3 *[47].*

(i) *$L^r(\Omega)$ is separable space, if $r \in [1, \infty[$.*

(ii) *$L^r(\Omega)$ is reflexive space, if $r \in]1, \infty[$.*

1.2.2 The Sobolev space $W^{m,r}(\Omega)$

These spaces are essential to the study of partial differential equations. Let Ω an open bounded set

Proposition 1.6 *[23]. Let Ω be an open domain in $\mathbb{R}^N$. Then the distribution $T \in D'(\Omega)$ is in $L^r(\Omega)$ if there exists a function $f \in L^r(\Omega)$ such that*

$$\langle T, \varphi\rangle = \int_\Omega f(x)\varphi(x)dx \; for \; all \; \varphi \in D(\Omega)$$

where $1 \le p \le \infty$, and it's well-known that f is unique.

Definition 1.5 *Let $r \in [1, \infty]$ and m be an integer. The $\mathrm{W}^{m,r}(\Omega)$ is the space of all $f \in L^r(\Omega)$, we set*

$$\mathrm{W}^{m,r}(\Omega) = \{h \in L^r(\Omega) : \partial^\alpha h \in L^r(\Omega) \; for \; all \; \alpha \in \mathbb{N}^m, with \; |\alpha| \leq m\}$$

where

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n}$$

and the appropriate norm

$$\|h\|_{\mathrm{W}^{m,r}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha h\|_{L^r}, \, 1 \leq r < \infty, \; for \; all \; h \in \mathrm{W}^{m,r}(\Omega).$$

Theorem 1.4 *[54]. $\mathrm{W}^{m,r}(\Omega)$ is a Banach space with their associated norm.*

Definition 1.6 *$\mathrm{W}_0^{m,r}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $\mathrm{W}^{m,r}(\Omega)$.*

Definition 1.7 *If $r = 2$, we use the notation $H^m(\Omega)$ rather than $\mathrm{W}^{m,2}(\Omega)$, endowed by*

$$\|h\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} [\|\partial^\alpha h\|_{L^2}]^2 \right)^{\frac{1}{2}}$$

generated by the usual scalar product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \, \partial^\alpha v \, dx.$$

Theorem 1.5 *[3].*

1. *$H^m(\Omega)$, equipped with the inner product denoted as $\langle .,. \rangle_{H^m(\Omega)}$, constitutes a Hilbert space.*
2. *When $m' \leq m$, $H^{m'}(\Omega) \subset H^m(\Omega)$, with continuous embedding.*

LEMMA 1.1 *[23]. We define $H^{-m}(\Omega)$ as a dual of $H_0^m(\Omega)$, and we have*

$$\mathcal{D}(\Omega) \subset H_0^m(\Omega) \subset L^2(\Omega) \subset H^{-m}(\Omega) \subset \mathcal{D}'(\Omega)$$

with continuous embedding.

The generalization of the calculus integration by parts is known as the Gaussian theorem. For the theory of weak or variational solutions of partial differential equations, this operation is crucial. It is necessary to research the conditions under which the domain's regularity and the functions within it are well-defined.

THEOREM 1.6 *[23]. For open bounded set $\Omega \subset \mathbb{R}^n$, $n \geq 2$ with a Lipschitz boundary Γ, for every v in $\mathrm{W}^{1,1}(\Omega)$ the identity given below holds*

$$\int_\Omega \partial_i v(x)\, dx = \int_\Gamma v(s)\, \mathsf{n}_i(s)\, ds,$$

Where n represents the unit outer normal vector on Γ.

COROLLARY 1.1 (Green's formula) *[47]. Under the same conditions of the theorem 1.6, then one has*

$$\int_\Omega \nabla v(x).\nabla w(x)\, dx = \int_{\partial\Gamma} \frac{\partial v(s)}{\partial \mathsf{n}} w(s) ds - \int_\Omega \Delta v(x)\, w(x) dx.$$

where $\frac{\partial v}{\partial \mathsf{n}}$ is a normal derivation of v on Γ.

1.2.3 The Bochner $L^r(0,T;X)$ spaces

DEFINITION 1.8 *Consider a Banach space X, for $r \in [1,\infty[$, let $L^r(0,T;X)$ represent the* *space of measurable functions* $h\ \ ]0,T[\rightarrow X$*, such that*

$$\left(\int_0^T \|h(t)\|_X^r dt\right)^{\frac{1}{r}} = \|h\|_{L^r(0,T;X)} < \infty.$$

If $r = \infty$

$$\|h\|_{L^\infty(0,T;X)} = \sup_{t\in]0,T[} \|h(t)\|_X.$$

Theorem 1.7 *[3].* $L^r(0,T;X)$ *is a Banach space.*

We use the notation $D'(0,T;X)$ to represent the space of distributions $:]0,T[\rightarrow X$, define by

$$\mathcal{D}'(0,T;X) = \mathcal{L}(\mathcal{D}]0,T[,X)$$

where $\mathcal{L}(\varphi,\psi)$: space of the linear continuous operators.

Lemma 1.2 *[3].*

(1) For all $h \in \mathcal{D}'(0,T;X)$*, we define the distribution derivation as*

$$\frac{\partial h}{\partial t}(\varphi) = -h\frac{d\varphi}{dt}, \quad \forall\varphi \in \mathcal{D}(]0,T[).$$

(2) For all $h \in L^r(0,T;X)$ *we have*

$$h(\varphi) = \int_0^T h(t)\varphi(t)dt\ ,\ \forall\varphi \in \mathcal{D}(]0,T[).$$

We will cite some basic results on the space $L^r(0,T;X)$. which will be very useful for the rest of our work.

Lemma 1.3 *[23]. If* $h \in L^r(0,T;X)$, $\dfrac{\partial h}{\partial t} \in L^r(0,T;X)$*, then the function h is continuous from* $[0,T]$ *to X (*$h \in C(0,T;X)$*).*

Lemma 1.4 *[23]. Let $\mathcal{Q} =]0,T[\times\Omega$ be open bounded set in $\mathbb{R}_+ \times \mathbb{R}^n$, let h_n, h are two functions in $L^{r'}(\mathcal{Q})$, $1 < r' < \infty$, such that*

$$\|h_n\|_{L^{r'}(\mathcal{Q})} \leq C, \forall n \in \mathbb{N}$$

and

$$h_n \rightarrow h \; in \; \mathcal{Q}.$$

Then

$$h_n \rightharpoonup h \in \; L^{r'}(\mathcal{Q}).$$

Where $L^{r'}(\mathcal{Q}) = L^{r'}(]0,T[\times\Omega)$.

Lemma 1.5 *[23]. Let v^m and v be two functions in $L^r(]0,L[\times]0,T[)$, where $1 < r < \infty$, satisfying the conditions:*

$$\|v^m\|_{L^r(]0,T[\times]0,L[)} \leq C$$

and

$$v^m \rightarrow v \;\; in \;]0,T[\times]0,L[.$$

Then

$$v^m \rightharpoonup v \; in \; L^r(]0,T[\times]0,L[).$$

Proposition 1.7 *[56]. Let Consider a reflexive Banach space X with its dual X', $1 \leq r', r < \infty$, such that $\dfrac{1}{r'} + \dfrac{1}{r} = 1$. Then $L^{r'}(0,T;X')$ is the dual of $L^r(0,T;X)$.*

Proposition 1.8 *[54]. Given X, Y be two Banach spaces, such that $X \hookrightarrow Y$, then we have $L^r(0,T;X) \hookrightarrow L^r(0,T;Y)$.*

Lemma 1.6 (Aubin-Lions) *[54]. Let X, Y and Z be Banach spaces. Assume that X is* *compactly embedded in Y and that Y is continuously embedded in Z. Let*

$$\mathcal{W} = \{v \in L^r([0,T];X) | \; v' \in L^{r'}([0,T];Z)\}, \; for\, 1 \leq r, r' \leq \infty.$$

(i) If $r < \infty$: the embedding of $\mathcal{W}$ into $L^r([0,T];X)$ is compact.

(ii) If $r = \infty$ and $r' > 1$: the embedding of $\mathcal{W}$ into $C([0,T];X)$ is compact.

Section 1.3 Some inequalities

As our investigation relies on established algebraic inequalities, it is pertinent to revisit a selection of these inequalities at this juncture.

Lemma 1.7 (Cauchy-Schwartz inequality)
Let V be a linear space, if $\boldsymbol{v_1}, \boldsymbol{v_2} \in V$ then

$$\langle \boldsymbol{v_1}, \boldsymbol{v_2} \rangle \leq \|\boldsymbol{v_1}\| \; \|\boldsymbol{v_2}\|$$

if $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are linearly dependent, the equality holds.

Lemma 1.8 (Young's inequalities) *Let α, β a real numbers and $\delta > 0$, then*

$$\alpha\beta \leq \delta\alpha^2 + \frac{\beta^2}{4\delta}.$$

Proof *We have*

$$(2\delta\alpha - \beta)^2 \geq 0$$

it follows that

$$4\delta^2\alpha^2 + \beta^2 - 4\delta\alpha\beta \geq 0$$

Thus

$$4\delta\alpha\beta \leq 4\delta^2\alpha^2 + \beta^2$$

therefore

$$\alpha\beta \leq \delta\alpha^2 + \frac{1}{4\delta}\beta^2.$$

Lemma 1.9 *[47] Let* $\alpha, \beta \geq 0$*, then*

$$\alpha\beta \leq \frac{\alpha^r}{r} + \frac{\beta^{r'}}{r'}$$

with $1 < r', r < \infty$*,* $\frac{1}{r'} + \frac{1}{r} = 1$.

Proof *Let* $I = (0,1)$*,* $\alpha, \beta \geq 0$ *and* $h : I \to \mathbb{R}$ *is integrable function, such that*

$$h(x) = \begin{cases} r\ log(\alpha), & 0 \leq x \leq \frac{1}{r} \\ r'\ log(\beta), & \frac{1}{r} \leq x \leq 1. \end{cases}$$

Applying Jensen's inequality, and the fact that $\psi(t) = e^t$ *is convex, we get*

$$\psi\left(\frac{1}{u(I)}\int_I h(x)dx\right) \leq \frac{1}{u(I)}\int_I \psi(h(x))dx. \tag{1.4}$$

As a result, we get

$$\begin{aligned} \frac{1}{u(I)}\int_I \psi(h(x))dx &= \int_0^1 e^{h(x)}dx = \int_0^{\frac{1}{r}} e^{r\ log(\alpha)}dx + \int_{\frac{1}{r}}^1 e^{r'\log(\beta)}dx \\ &= \int_0^{\frac{1}{r}} \alpha^r dx + \int_{\frac{1}{r}}^1 \beta^{r'} dx \\ &= \frac{1}{r}\alpha^r + (1 - \frac{1}{r})\beta^{r'} \end{aligned}$$

$$= \frac{\alpha^r}{r} + \frac{\beta^{r'}}{r'} \tag{1.5}$$

where, $u(I) = 1$ *and*

$$\psi\left(\frac{1}{u(I)}\int_I h(x)dx\right) = e^{\int_0^1 h(x)dx} = e^{\int_0^{\frac{1}{r}} r\ log(\alpha)dx + \int_{\frac{1}{r}}^1 r' \log(\beta)dx}$$
$$= e^{log(\alpha)+log(\beta)} = e^{log(\alpha\beta)}$$

$$= \alpha\beta. \tag{1.6}$$

according to (1.5)*, the result has been proven.* ■

Here we shall present a few significant integral inequality. These inequalities are crucial in applied mathematics, and they will be helpful in the upcoming chapters as well.
In 1888, **Rogers** and **Holder** provided a generalization of **Cauchy-Schwartz's** inequality.

Theorem 1.8 *[57].[**Holder's inequality**] For* $r \geq 1$*,* $r' \geq 1$*. Assume that* $h \in L^r(\Omega)$ *and* $k \in L^{r'}(\Omega)$*, then* $hk \in L^1(\Omega)$ *and*

$$\int_\Omega |hk|dx \leq ||h||_r ||k||_{r'}. \tag{1.7}$$

Lemma 1.10 *[57]. Let* $h_1, h_2, \ldots.., h_N$ *be N functions such that,* $h_i \in L^{r_i}(\Omega)$*,* $1 \leq i \leq N$*, and*

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \ldots\ldots.. + \frac{1}{r_N} \leq 1.$$

Then, the product $h_1 h_2 \ldots. h_k \in L^r(\Omega)$ *and* $||h_1 h_2 \ldots h_N||_r \leq ||h_1||_{r_1} \ldots.||h||_{r_N}$.

Lemma 1.11 *[57]. Let* $r > 1$*,* $r' > 1$ *and* $\frac{1}{\varrho} = \frac{1}{r} + \frac{1}{r'} - 1 \geq 0$*, and* $h \in L^r(\mathbb{R})$*,* $k \in L^{r'}(\mathbb{R})$*. Then* $h \star k \in L^\varrho(\mathbb{R})$ *and*

$$||h \star k||_\varrho \leq ||h||_r ||k||_{r'}. \tag{1.8}$$

Definition 1.9

$$C([0,T],\boldsymbol{X}) = \{v : [0,T] \longrightarrow \boldsymbol{X} \; continue\}.$$

If for every $t_0 \in [0,T]$, the following limit exists in $\boldsymbol{X}$ i.e.

$$v'(t_0) = \lim_{h \longrightarrow 0} \frac{v(t_0+h) - v(t_0)}{h}$$

then we say that v is classically differentiable. If furthermore, the function $t \longrightarrow v'(t)$ is continuous, then we say that v belongs to $C^1([0,T],\boldsymbol{X})$.

When studying partial differential equations, the **Poincaré** Inequality is very helpful for estimating how solutions behave in Sobolev spaces. It is the Sobolev Embedding theorem in a localized form.

Lemma 1.12 *[16]. [**Poincaré Inequality**] Assume $p \geq 1$, Ω is an open subset that is bounded at least in one direction. Then, there exists a constant C_*, depending only on Ω and r, such that for any function h in the Sobolev space $W_0^{1,r}(\Omega)$, it holds:*

$$\|h\|_r \leq C_* \|\nabla h\|_r.$$

Lemma 1.13 *[29]. Let $h \in C^1([0,L])$ satisfying the conditions:*

$$h(0,t) = h_x(0,t) = 0, \quad \forall t \geq 0.$$

it follows:

$$\|h^2(t)\|_\infty \leq 2\|h(t)\|_2 \|h_x(t)\|_2,$$

$$\|h_x^2(t)\|_\infty \leq 2\|h_x(t)\|_2 \|h_{xx}(t)\|_2,$$

for all $t \geq 0$.

Grönwall's inequality enables one to use the solution of the associated differential or integral equation to bound a function that is known to fulfill a particular differential or integral inequality. The lemme is available in two forms: integral and differential. There are multiple variations for the latter.

LEMMA 1.14 *[33].[**Gronwall's Inequality**] Suppose $h, g : [0, T] \to \mathbb{R}$ are nonnegative bounded continuous function and $\rho : [0, T] \to \mathbb{R}$ is an integrable nonnegative function, such that*

$$h(t) \leq g(t) + \int_0^t \rho(\tau)h(\tau)\, d\tau, \text{ for all } t \in [0, T].$$

Then

$$h(t) \leq g(t) + \int_0^t g(s)\rho(s) \exp\left(\int_s^t \rho(\tau)\, d\tau\right) ds, \text{ for all } t \in [0, T].$$

When g is constant, the following corollary holds

COROLLARY 1.2 *[33]. Suppose $h : [0, T] \to \mathbb{R}$ is nonnegative bounded continuous function, $\rho : [0, T] \to \mathbb{R}$ is an integrable nonnegative function and $g \geq 0$ such that*

$$h(t) \leq g + \int_0^t \rho(\tau)h(\tau)\, d\tau, \text{ for all } t \in [0, T],$$

Then

$$h(t) \leq g \exp\left(\int_0^t \rho(\tau)\, d\tau\right), \text{ for all } t \in [0, T].$$

Chapter 2

Decay energy for a nonlinear Euler-Bernoulli beam with neutral delay

Within this chapter, we delve into the free transverse vibration analysis of a nonlinear Euler-Bernoulli beam subjected to a neutral type delay. Initially, we establish a local existence result utilizing the Faedo-Galerkin method. Subsequently, a boundary control system is devised based on the Lyapunov method to mitigate the transverse vibrations of the beam.

Section 2.1 Introduction

Due to the requirement for high-precision control of numerous mechanical systems, such as marine risers for oil and gas transportation, spacecraft with flexible attachments, or flexible robot arms, the boundary control of flexible systems has been an important topic of study in recent years [40], [48], [34], [7], [14], [43]. The time delay is one of several elements that have a significant impact on the dynamic properties of systems. It became evident that its existence could not be fully neglected in many systems, and with the rapid growth of numerous engineering disciplines, including mechanical engineering, a more precise system analysis was necessary. Time delays in these systems can lead to poor performance and unstable dynamic systems [30], [49]. As a result, throughout the past few decades, the stability issue with time-delay systems has received a lot of attention. In [27], exponential stability result for a viscoelastic Timoshenko beam was established. The researchers in [37], used the LMI (linear matrix inequality) technique to investigate global exponential stability for neutral differential systems with time-varying or constant delay. The asymptotic stability of delay differential equations of neutral type has been extensively studied in [38], [1]. We consider in this chapter the neutrally retarded nonlinear Euler-Bernoulli beam for $(x,t) \in (0,L) \times [0,\infty), L > 0$

$$\rho A \left[y_t + \int_0^t \kappa(t-s) y_t(x,s) ds \right]_t + EI y_{xxxx} - P_0 y_{xx} - \frac{1}{2} EA (y_x^3)_x = 0, \tag{2.1}$$

under the boundary

$$\begin{cases} y_{xx}(0,t) = y_{xx}(L,t) = y(0,t) = 0, \forall t \geq 0, \\ EI y_{xxx}(L,t) = P_0 y_x(L,t) + \frac{1}{2} EA y_x^3(L,t) + \alpha y_t(L,t), \forall t \geq 0, \alpha > 0, \end{cases} \tag{2.2}$$

and initial conditions

$$y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), x \in (0,L), \tag{2.3}$$

where EI is the beam's flexural rigidity, ρA is the beam's mass per unit length, and $y(x,t)$ represents transverse displacement at time t with respect to the spatial coordinate x, EA the axial stiffness, P_0 the tension force. In this work we consider the transverse dynamics of a beam in bending vibration and we neglect the coupling between longitudinal and traversal displacements. Assuming that the change in length due to the axial force is small and negligible, we take only the elongation of the beam due to the curvature. We prove in this chapter existence and general decay result for problem (2.1) − (2.3).

Section 2.2 Notation and Main Results

Let us introduce the notation:

$$(\kappa \circ y)(t) = \int_0^L \int_0^t \kappa(t-s) \left[y(x,t) - y(x,s) \right]^2 ds dx$$

For the kernel κ we assume:

(K1) The kernel κ is a nonnegative summable function $C^1(\mathbb{R}_+)$ satisfying $\kappa'(t) \leq 0$ for all $t \geq 0$.

(K2) $0 < \overline{k} = \int\limits_0^{+\infty} \kappa(s)ds < 1$.

(K3) There exists an increasing function $g(t) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ supposed to satisfy $u(t) = \frac{g'(t)}{g(t)}$ is a decreasing function and

$$\int\limits_0^{+\infty} \kappa(s)g(s)\,ds < +\infty, \qquad \int\limits_0^{+\infty} |\kappa'(s)|g(s)\,ds < +\infty, \qquad \int\limits_0^{+\infty} |\kappa''(s)|ds < +\infty.$$

We denote for $t^* > 0$, $\kappa^* = \int\limits_0^{t^*} \kappa(s)ds$,

$$\mathcal{A} = \left\{y \in \mathbf{H}^2(0,L) \mid y(0) = 0\right\},$$
$$\mathcal{M} = \left\{y \in \mathcal{A} \cap \mathbf{H}^4(0,L) \mid y_{xx}(0) = y_{xx}(L) = 0\right\}.$$

We define the (classical) energy of problem (2.1)-(2.3) by

$$\mathfrak{E}(t) = \frac{1}{2}\Big[\rho A \left\|y_t\right\|^2 + EI\left\|y_{xx}\right\|^2 + P_0\left\|y_x\right\|^2 + \frac{EA}{4}\left\|y_x^2\right\|^2 + \rho A \int_0^t \kappa(t-s)\left\|y_t(s)\right\|^2 ds\Big]. \qquad (2.4)$$

We need the following auxiliary result:

Lemma 2.1 *We have the following identity:*

$$\int_0^L y_t(t)\int_0^t \kappa(t-s)y_{tt}(s)dsdx = -\frac{1}{2}(\kappa'\circ y_t)(t) + \frac{1}{2}\frac{d}{dt}\int_0^t \kappa(t-s)\|y_t(s)\|^2 ds$$

$$+\frac{\kappa(t)}{2}\|y_t(t)\|^2-\kappa(t)\int_0^L y_t(t)y_t(0)dx \tag{2.5}$$

for all $y_t\in C^1([0,\infty);L^2(0,L))$ *and* $\kappa\in C^1[0,\infty)$.

Proof *The identity is a direct consequence of*

$$(\kappa'\circ y_t)(t)=\int_0^L\int_0^t \kappa'(t-s)[y_t(t)-y_t(s)]^2 dsdx=\kappa(t)\|y_t(t)-y_t(0)\|^2$$

$$-2\int_0^t\kappa(s)\int_0^L y_{tt}(t-s)[y_t(t)-y_t(t-s)]dxds,\ \ t\geq 0.$$

and

$$\frac{d}{dt}\int_0^t\kappa(t-s)\|v_t(s)\|^2ds=\frac{d}{dt}\int_0^t\kappa(s)\|y_t(t-s)\|^2ds$$

$$=\kappa(t)\|y_t(0)\|^2+2\int_0^L\int_0^t\kappa(s)y_{tt}(t-s)y_t(t-s)dsdx,\ \ t\geq 0.$$

From the above two relations, we find the proof of lemma 2.1. ■

Proposition 2.1 *The modified energy E(t) is non-increasing and uniformly bounded. More precisely, we have*

$$\mathfrak{E}'(t)=\frac{\rho A}{2}(\kappa'\circ y_t)(t)-\rho A\frac{\kappa(t)}{2}\left\|y_t(t)\right\|^2-\alpha y_t^2(L,t)\leq 0,\ \ t\geq 0. \tag{2.6}$$

Proof *Multiplying equation (2.1) by* y_t *and integrating the result over (0, L) by parts and using the boundary conditions, we get*

$$\frac{1}{2}\frac{d}{dt}\left[\rho A\left\|y_t(t)\right\|^2+EI\left\|y_{xx}(t)\right\|^2+P_0\left\|y_x(t)\right\|^2+\frac{EA}{4}\left\|y_x^2(t)\right\|^2\right]$$
$$+\rho A\kappa(t)\int_0^L y_t(t)y_t(0)dx+\rho A\int_0^L y_t\int_0^t\kappa(t-s)y_{tt}(s)dsdx$$
$$+\left[EIy_{xxx}(L,t)-P_0y_x(L,t)-\frac{1}{2}EAy_x^3(L,t)\right]y_t(L,t).$$

Utilizing lemma 2.1, we determine the relation in the proposition. ■

Theorem 2.1 *Suppose that (K1)-(K3) are satisfied. If $(y_0, y_1) \in \mathcal{M} \times \mathcal{A}$, then for $T > 0$, there exists a unique solution y of problem (2.1) − (2.3)*

$$y \in \boldsymbol{L}^{\infty}([0,T), \mathcal{M}),$$

$$y_t \in \boldsymbol{L}^{\infty}([0,T), \mathcal{A}),$$

$$y_{tt} \in \boldsymbol{L}^{2}([0,T), \boldsymbol{L}^{2}(0,L)).$$

Additionally, we have $y \in \mathcal{C}([0,T), \mathcal{A}), y_t \in \mathcal{C}([0,T), \boldsymbol{L}^2(0,L))$.

Proof *We employ the Galerkin's method to establish the proof.*
Firstly, we establish the existence and uniqueness of solutions conforming to Eqs. (2.1)−(2.3). Subsequently, we generalize this finding to encompass weak solutions through the application of density arguments.
The variational problem associated with equations (2.1) and (2.2) can be formulated as follows: to find $y \in \mathcal{M}$ such that

$$\rho A\,(y_{tt}, w) + \rho A \kappa(0)\,(y_t, w) + \rho A\left(\int_0^t \kappa'(t-s) y_t(x,s)ds, w\right) + EI\,(y_{xx}, w_{xx})$$

$$+P_0\,(y_x, w_x) + \frac{1}{2} EA\left((y_x)^3, w_x\right) + \alpha y_t(L,t) w(L) = 0$$

for all $w \in \mathcal{M}$

Step 1: ***Approximate solutions***
Let $\{w_i\}$ a complete orthogonal bases of $\mathcal{M}$.
We consider $\mathrm{W}^N = span\{w_1, w_2, ..., w_N\}$, for all $N \in \mathbb{N}$.
The approximate solution $y^m(x,t) = \sum_{i=1}^{m} \mathfrak{C}_i^m(t) w_i(x)$ of the problem (2.1)-(2.3) satisfies :

$$\rho A\,(y^m_{tt}, w_i) + \rho A \kappa(0)\,(y^m_t, w_i) + \rho A\left(\int_0^t \kappa'(t-s) y^m_t(x,s)ds, w_i\right) + EI\,(y^m_{xx}, w_{ixx})$$

$$+P_0\,(y^m_x, w_{ix}) + \frac{1}{2} EA\left((y^m_x)^3, w_{ix}\right) + \alpha y^m_t(L,t) w_i(L) = 0 \tag{2.7}$$

with the initial conditions

$$\begin{cases} y^m(0) = \sum_{i=1}^{m} (y^m(0), w_i) w_i \longrightarrow y_0 \ \ in\ \mathcal{M}, \\ y_t^m(0) = \sum_{i=1}^{m} (y_t^m(0), w_i) w_i \longrightarrow y_1 \ \ in\ \mathcal{A}. \end{cases}$$

Step 2*:* ***A Priori Estimate***

We indicate by $M_i, i = 1, 2, ...,$ positive constants independent of m.

Estimate 1 *According to (2.6) and hypothesis (**K1**) it follows*

$$\mathfrak{E}_m'(t) + \alpha\,(y_t^m(L,t))^2 \leq 0 \tag{2.8}$$

where $\mathfrak{E}_m$ is the energy of the solutions y^m, introduced in (2.4).

The integration of the inequality (2.8) over $(0,t)$, gives us

$$\mathfrak{E}_m(t) + \alpha \int_0^t (y_t^m(L,s))^2\,ds \leq \mathfrak{E}_m(0) \tag{2.9}$$

Since the initial conditions are sufficienty smooth, then there exists a $M_1 > 0$, such that

$$\left\|y_t^m\right\|^2 + \left\|y_{xx}^m\right\|^2 + \left\|y_x^m\right\|^2 + \left\|(y_x^m)^2\right\|^2 + \int_0^t \kappa(t-s)\left\|y_t^m(s)\right\|^2 ds + \alpha \int_0^t (y_t^m(L,s))^2\,ds \leq M_1. \tag{2.10}$$

Estimate 2 *we show upper bounds of $\left\|y_{tt}^m(0)\right\|^2$*

By multiplying $(\mathfrak{C}_i^m)_{tt}(0)$ on both sides of Equation.(2.7) and summing up the resulting equations from $i = 1$ to $i = m$ and considering $t = 0$, then integrating by parts and utilizing boundary conditions, we obtain

$$\begin{aligned} \rho A\|y_{tt}^m(0)\|^2 = &-\rho A\kappa(0)\,(y_t^m(0), y_{tt}^m(0)) - EI\,(y_{xxxx}^m(0), y_{tt}^m(0)) + P_0\,(y_{xx}^m(0), y_{tt}^m(0)) \\ &+ \frac{3}{2}EA\left(y_{xx}^m(0)(y_x^m(0))^2, y_{tt}^m(0)\right) \end{aligned} \tag{2.11}$$

Young's inequality gives

$$-\rho A\kappa(0)\,(y_t^m(0), y_{tt}^m(0)) \leq \frac{(\rho A\kappa(0))^2}{\delta}\|y_t^m(0)\|^2 + \frac{\delta}{4}\|y_{tt}^m(0)\|^2 \tag{2.12}$$

$$-EI\,(y^m_{xxxx}(0), y^m_{tt}(0)) \leq \frac{(EI)^2}{\delta}\|y^m_{xxxx}(0)\|^2 + \frac{\delta}{4}\|y^m_{tt}(0)\|^2 \tag{2.13}$$

$$P_0\,(y^m_{xx}(0), y^m_{tt}(0)) \leq \frac{P_0^2}{\delta}\|y^m_{xx}(0)\|^2 + \frac{\delta}{4}\|y^m_{tt}(0)\|^2 \tag{2.14}$$

employing again Young's inequality, and lemma 1.13, we get

$$-\frac{3}{2}EA\left(y^m_{xx}(0)(y^m_x(0))^2, y^m_{tt}(0)\right) \leq \frac{\frac{9}{4}(EA)^2}{\delta}\|y^m_{xx}(0)(y^m_x(0))^2\|^2 + \frac{\delta}{4}\|y^m_{tt}(0)\|^2$$

$$\leq \frac{\frac{9}{4}(EA)^2}{\delta}\left[\|(y^m_x(0))^2\|^2_\infty \|y^m_{xx}(0)\|^2\right] + \frac{\delta}{4}\|y^m_{tt}(0)\|^2$$

$$\leq \frac{9(EA)^2}{4\delta}\left[\|(y^m_{xx}(0))\|^2\|(y^m_x(0))\|^2\|y^m_{xx}(0)\|^2\right] + \frac{\delta}{4}\|y^m_{tt}(0)\|^2 \tag{2.15}$$

Substituting inequalities (2.12) − (2.15) into (2.11), we get

$$(\rho A - \delta)\,\|y^m_{tt}(0)\|^2 \leq \frac{(\rho A\kappa(0))^2}{\delta}\|y^m_t(0)\|^2 + \frac{(EI)^2}{\delta}\|y^m_{xxxx}(0)\|^2 + \frac{P_0^2}{\delta}\|y^m_{xx}(0)\|^2 + \frac{9(EA)^2}{4\delta}\left[\|(y^m_{xx}(0))\|^2\|(y^m_x(0))\|^2\|y^m_{xx}(0)|^2\right]$$

We choose δ so small that $\rho A - \delta > 0$ and since the initial data are arbitrary we deduce

$$\|y^m_{tt}(0)\|^2 \leq M_2. \tag{2.16}$$

Estimate 3 *Next, we estimate $\|y^m_{tt}\|^2$*
For $t, \zeta > 0$ fixed with $\zeta < T - t$. when we multiply $(\mathfrak{C}^m_i)_t(t+\zeta) - (\mathfrak{C}^m_i)_t(t)$ on both sides of Equation (2.7) and then sum the resulting equations from $i = 1$ to $i = m$, and taking the difference with $t = t + \zeta$ and $t = t$, we get

$$\frac{\rho A}{2}\frac{d}{dt}\|y^m_t(t+\zeta) - y^m_t(t)\|^2 + \rho A\kappa(0)\|y^m_t(t+\zeta) - y^m_t(t)\|^2 + \frac{EI}{2}\frac{d}{dt}\|y^m_{xx}(t+\zeta) - y^m_{xx}(t)\|^2$$

$$+\frac{P_0}{2}\frac{d}{dt}\|y^m_x(t+\zeta) - y^m_x(t)\|^2 + \alpha\,[y^m_t(L, t+\zeta) - y^m_t(L,t)]^2 = K_1 + K_2 \tag{2.17}$$

where

$$K_1 = -\frac{EA}{2}\int_0^L \Big[(y_x^m(t+\zeta))^3 - (y_x^m(t))^3\Big][y_{xt}^m(t+\zeta) - y_{xt}^m(t)]\,dx$$

$$K_2 = -\rho A\int_0^L \Bigg[\int_0^{t+\zeta} \kappa'(t+\zeta-s)y_t^m(x,s)ds - \int_0^t \kappa'(t-s)y_t^m(x,s)ds\Bigg][y_t^m(t+\zeta) - y_t^m(t)]\,dx$$

by integration by parts, we get

$$K_1 = -\frac{EA}{2}\Big[(y_x^m(L,t+\zeta))^3 - (y_x^m(L,t))^3\Big][y_t^m(L,t+\zeta) - y_t^m(L,t)]$$

$$+\frac{3EA}{2}\int_0^L \Big[y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t)(y_x^m(t))^2\Big][y_t^m(t+\zeta) - y_t^m(t)]\,dx$$

$$= K_{11} + K_{12}. \tag{2.18}$$

On the other hand,by young inequality and lemma 1.12, we have

$$K_{11} = -\frac{EA}{2}\Big[(y_x^m(L,t+\zeta))^3 - (y_x^m(L,t))^3\Big][y_t^m(L,t+\zeta) - y_t^m(L,t)]$$

$$= -\frac{EA}{2}[y_t^m(L,t+\zeta) - y_t^m(L,t)]\times\Big[(y_x^m(L,t+\zeta))^2 + y_x^m(L,t+\zeta)y_x^m(L,t) + (y_x^m(L,t))^2\Big]\times$$
$$[y_x^m(L,t+\zeta) - y_x^m(L,t)]$$

$$\leq \frac{3}{2}\Big[(y_x^m(L,t+\zeta))^2 + (y_x^m(L,t))^2\Big]\Bigg(\frac{(EA)^2}{16\delta}[y_x^m(L,t+\zeta) - y_x^m(L,t)]^2 + \delta[y_t^m(L,t+\zeta) - y_t^m(L,t)]^2\Bigg)$$

$$\leq \frac{3L}{2}\Big[\|y_{xx}^m(L,t+\zeta)\|^2 + \|y_{xx}^m(L,t)\|^2\Big]\times$$
$$\Bigg(\frac{(EA)^2}{16\delta}\|y_{xx}^m(L,t+\zeta) - y_{xx}^m(L,t)\|^2 + \delta[y_t^m(L,t+\zeta) - y_t^m(L,t)]^2\Bigg)$$

$$\leq \frac{3M_1L(EA)^2}{16\delta}\|y_{xx}^m(L,t+\zeta) - y_{xx}^m(L,t)\|^2 + 3M_1L\delta[y_t^m(L,t+\zeta) - y_t^m(L,t)]^2 \tag{2.19}$$

on the other hand, by young's inequality, we have

$$K_{12} = \frac{3EA}{2}\int_0^L \Big[y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t)(y_x^m(t))^2\Big][y_t^m(t+\zeta) - y_t^m(t)]\,dx$$

$$\leq \frac{3EA}{4}\|y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t)(y_x^m(t))^2\|^2 + \frac{3EA}{4}\|y_t^m(t+\zeta) - y_t^m(t)\|^2$$

we estimate the first term in K_{12} by

$$\|y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t)(y_x^m(t))^2\|^2 =$$

$$\|y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t+\zeta)(y_x^m(t))^2 + y_{xx}^m(t+\zeta)(y_x^m(t))^2 - y_{xx}^m(t)(y_x^m(t))^2\|^2$$

$$\leq 2\|y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t+\zeta)(y_x^m(t))^2\|^2 + 2\|y_{xx}^m(t+\zeta)(y_x^m(t))^2 - y_{xx}^m(t)(y_x^m(t))^2\|^2$$

$$\leq 2\|y_{xx}^m(t+\zeta)\|^2\|(y_x^m(t+\zeta))^2 - (y_x^m(t))^2\|^2 + 2\|(y_x^m(t))^2\|_\infty^2\|y_{xx}^m(t+\zeta) - y_{xx}^m(t)\|^2$$

$$\leq M_1^*\left(\|y_x^m(t+\zeta) - y_x^m(t)\|^2 + \|y_{xx}^m(t+\zeta) - y_{xx}^m(t)\|^2\right)$$

then, we get

$$K_{12} \leq M_1^*\frac{3EA}{4}\left(\|y_x^m(t+\zeta) - y_x^m(t)\|^2 + \|y_{xx}^m(t+\zeta) - y_{xx}^m(t)\|^2\right) + \frac{3EA}{4}\|y_t^m(t+\zeta) - y_t^m(t)\|^2 \tag{2.20}$$

by (2.19),(2.20), and young inequality, we get

$$|K_1| \leq \frac{3M_1L(EA)^2}{8\delta}\|y_{xx}^m(L,t+\zeta) - y_{xx}^m(L,t)\|^2 + 3M_1L\delta\,[y_t^m(L,t+\zeta) - y_t^m(L,t)]^2$$

$$+M_1^*\frac{3EA}{4}\left(\|y_x^m(t+\zeta) - y_x^m(t)\|^2 + \|y_{xx}^m(t+\zeta) - y_{xx}^m(t)\|^2\right) + \frac{3EA}{4}\|y_t^m(t+\zeta) - y_t^m(t)\|^2. \tag{2.21}$$

$$|K_2| = M_4\int_0^L\left[\int_0^{t+\zeta}\kappa'(t+\zeta-s)y_t^m(x,s)ds - \int_0^t\kappa'(t-s)y_t^m(x,s)ds\right]^2 dx + \frac{\delta}{4}\|y_t^m(t+\zeta) - y_t^m(t)\|^2 \tag{2.22}$$

Substituting inequalities (2.21) − (2.22) *into* (2.17)*, dividing by ζ^2, then for $\zeta \to 0$, the limit yields*

$$\frac{\rho A}{2}\frac{d}{dt}\|y_{tt}^m(t)\|^2 + \rho A\kappa(0)\|y_{tt}^m(t)\|^2 + \frac{EI}{2}\frac{d}{dt}\|y_{xxt}^m(t)\|^2 + \frac{P_0}{2}\frac{d}{dt}\|y_{xt}^m(t)\|^2 + \alpha(y_{tt}^m(L,t))^2$$

$$\leq 3M_1L\delta(y_{tt}^m(L,t))^2 + \frac{3M_1L(EA)^2}{16\delta}\|y_{xxt}^m(L,t)\|^2 + M_1^*\frac{3EA}{4}\|y_{xt}^m(t)\|^2 + M_1^*\frac{3EA}{4}\|y_{xxt}^m(t)\|^2$$

$$+[\frac{\delta}{4}+\frac{3EA}{2}]\|y_{tt}^m(t)\|^2+M_4\int_0^L\left(\kappa'(0)y_t^m(t)+\int_0^t\kappa''(t-s)y_t^m(x,s)ds\right)^2dx \tag{2.23}$$

a simple calculus yields

$$\int_0^L\int_0^t\kappa''(t-s)y_t^m(x,s)dsdx\le\sup_{[0,T]}\|y_t^m\|\int_0^T|\kappa''(s)|ds<M_5 \tag{2.24}$$

We choose δ *so small that* $\alpha-3M_1L\delta>0$, *we substitute* (2.24) *in* (2.23),*we have*

$$\frac{d}{dt}\left(\|y_{tt}^m\|^2+\|y_{xxt}^m\|^2+\|y_{xt}^m\|^2\right)\le M_6+M_7(\|y_{tt}^m\|^2+\|y_{xxt}^m\|^2+\|y_{xt}^m\|^2)$$

then integrate over the interval $(0,t)$, *we get*

$$\|y_{tt}^m\|^2+\|y_{xxt}^m\|^2+\|y_{xt}^m\|^2\le M_8+M_9\int_0^t(\|y_{tt}^m\|^2+\|y_{xxt}^m\|^2+\|y_{xt}^m\|^2)ds$$

Thanks to Gronwall's lemma, we have

$$\|y_{tt}^m|^2+\|y_{xxt}^m\|^2+\|y_{xt}^m\|^2\le M_{10}. \tag{2.25}$$

Passage to the limit

From (2.10) *and* (2.25), *We infer*

$$\begin{cases}(y^m)\ \ bounded\ in\ \ \boldsymbol{L}^\infty(0,T,\mathcal{A}),\\ (y_t^m)\ \ bounded\ in\ \ \boldsymbol{L}^\infty(0,T,\mathcal{A}),\\ (y_{tt}^m)\ \ bounded\ in\ \ \boldsymbol{L}^\infty(0,T,\boldsymbol{L}^2(0,L)).\end{cases} \tag{2.26}$$

and

$$(y_x^m)^2\ \ bounded\ in\ \ \boldsymbol{L}^\infty(0,T,\boldsymbol{L}^2(0,L)). \tag{2.27}$$

Therefore, there exist subsequences of (y^m), *denoted again by* (y^m), *satisfying*

$$\begin{cases}y^m\overset{*}{\rightharpoonup}y\ \ in\ \ \boldsymbol{L}^\infty(0,T,\mathcal{A}),\\ y_t^m\overset{*}{\rightharpoonup}y_t\ \ in\ \ \boldsymbol{L}^\infty(0,T,\mathcal{A}),\\ y_{tt}^m\overset{*}{\rightharpoonup}y_{tt}\ \ in\ \ \boldsymbol{L}^\infty(0,T,\boldsymbol{L}^2(0,L)).\end{cases} \tag{2.28}$$

And

$$(y_x^m)^2 \overset{*}{\rightharpoonup} (y_x)^2 \ \ in \ \ \boldsymbol{L}^\infty(0,T,\boldsymbol{L}^2(0,L)). \tag{2.29}$$

Studying the nonlinear terms

Thanks to the Aubin-Lions compactness lemma and (2.28), we get

$$y^m \to y \text{ strongly in } \boldsymbol{L}^\infty(0,T,\boldsymbol{H}_0^1(0,L)) \tag{2.30}$$

(2.29) and lemma 1.5, allow to write

$$(y_x^m)^3 \rightharpoonup (y_x)^3 \ \ in \ \ \boldsymbol{L}^2([0,T]\times[0,L]). \tag{2.31}$$

This allows us by passing to the limit in (2.7) to obtain a weak solution of the problem (2.1)–(2.3).

Uniqueness

Assume that $\overline{y}$ and $\widetilde{y}$ are two different solution to the system (2.1)–(2.3), and $Y = \overline{y} - \widetilde{y}$, with $Y(0) = Y_t(0) = 0$, then Y satisfies

$$\rho A\,(Y_{tt}, w_i) + \rho A\kappa(0)\,(Y_t, w_i) + \rho A\left(\int_0^t \kappa'(t-s)Y_t(x,s)ds, w_i\right) + EI\,(Y_{xx}, w_{ixx})$$

$$+P_0\,(Y_x, w_{ix}) + \frac{1}{2}EA\left([\overline{y}_x]^3 - [\widetilde{y}_x]^3, w_{ix}\right) + \alpha\,Y_t(L,t)w_i(L) = 0 \tag{2.32}$$

When we multiply $(\mathfrak{C}_i^m)_t(t)$ on both sides of Equation 2.32 and then sum the resulting equations with respect to i, we obtain

$$\rho A\,(Y_{tt}, Y_t) + \rho A\kappa(0)\,(Y_t, Y_t) + \rho A\left(\int_0^t \kappa'(t-s)Y_t(x,s)ds, Y_t\right) + EI\,(Y_{xx}, Y_{txx})$$

$$+P_0\,(Y_x, Y_{tx}) + \frac{1}{2}EA\left([\overline{y}_x]^3 - [\widetilde{y}_x]^3, Y_{tx}\right) + \alpha(Y_t(L,t))^2 = 0$$

then, we have

$$\frac{\rho A}{2}\frac{d}{dt}\|Y_t\|^2 + \rho A\kappa(0)\|Y_t\|^2 + \frac{EI}{2}\frac{d}{dt}\|Y_{xx}\|^2 + \frac{P_0}{2}\frac{d}{dt}\|Y_x\|^2 + \alpha(Y_t(L,t))^2$$

$$= -\rho A\left(\int_0^t \kappa'(t-s)Y_t(x,s)ds, Y_t\right) - \frac{1}{2}EA\left([\overline{y}_x]^3 - [\widetilde{y}_x]^3, Y_{tx}\right) \tag{2.33}$$

*with the same technique in **Estimate 3**, by integration by parts, we get*

$$-\frac{1}{2}EA\left([\overline{y}_x]^3 - [\widetilde{y}_x]^3, Y_{tx}\right) = -\frac{EA}{2}\left[[\overline{y}_x]^3(L) - [\widetilde{y}_x]^3(L)\right] \times Y_t(L)$$

$$+\frac{3EA}{2}\int_0^L \left[\overline{y}_{xx}[\overline{y}_x]^2 - \widetilde{y}_{xx}[\widetilde{y}_x]^2\right] Y_t dx \tag{2.34}$$

on the other hand, by young's inequality and lemma 1.12, we have

$$-\frac{EA}{2}\left[[\overline{y}_x]^3(L) - [\widetilde{y}_x]^3(L)\right] \times Y_t(L) = -\frac{EA}{2} Y_t(L) \times Y_x(L) \times$$

$$\left[[\overline{y}_x]^2(L) + \overline{y}_x(L) \times \widetilde{y}_x(L) + [\widetilde{y}_x]^2(L)\right]$$

$$\leq \frac{3}{2}\left([\overline{y}_x]^2(L) + [\widetilde{y}_x]^2(L)\right)\left(\frac{(EA)^2}{16\delta}[Y_x(L)]^2 + \delta[Y_t(L)]^2\right)$$

$$\leq \frac{3L}{2}\left[\|\overline{y}_{xx}(L)\|^2 + \|\widetilde{y}_{xx}(L)\|^2\right] \times \left(\frac{(EA)^2}{16\delta} L\|Y_{xx}(L)\|^2 + \delta[Y_t(L)]^2\right)$$

$$\leq \frac{3M_1L^2(EA)^2}{16\delta}\|Y_{xx}(L)\|^2 + 3M_1L\delta[Y_t(L)]^2 \tag{2.35}$$

on the other hand, we have

$$\frac{3EA}{2}\int_0^L \left[\overline{y}_{xx}[\overline{y}_x]^2 - \widetilde{y}_{xx}[\widetilde{y}_x]^2\right] Y_t dx \leq \frac{3EA}{4}\|\overline{y}_{xx}[\overline{y}_x]^2 - \widetilde{y}_{xx}[\widetilde{y}_x]^2\|^2 + \frac{3EA}{4}\|Y_t\|^2$$

$$\leq \frac{3EA}{4}\|\overline{y}_{xx}\overline{y}_x^2 - \overline{y}_{xx}\widetilde{y}_x^2 + \overline{y}_{xx}\widetilde{y}_x^2 - \widetilde{y}_{xx}\widetilde{y}_x^2\|^2 + \frac{3EA}{4}\|Y_t\|^2$$

$$\leq \frac{3EA}{2}\|\overline{y}_{xx}(\overline{y}_x + \widetilde{y}_x)(\overline{y}_x - \widetilde{y}_x)\|^2 + \frac{3EA}{2}\|\widetilde{y}_x^2(\overline{y}_{xx} - \widetilde{y}_{xx})\|^2 + \frac{3EA}{4}\|Y_t\|^2$$

$$\leq M_{11}\|Y_x\|^2 + M_{11}\|Y_{xx}\|^2 + \frac{3EA}{4}\|Y_t\|^2 \tag{2.36}$$

by young, Holder's inequalities, we have

$$-\rho A\left(\int_0^t \kappa'(t-s)Y_t(x,s)ds, Y_t\right) \leq \frac{(\rho A)^2}{2\delta}\|\int_0^t \kappa'(t-s)Y_t(x,s)ds\|^2 + \frac{\delta}{2}\|Y_t\|^2$$

$$\leq \frac{(\rho A)^2}{2\delta}\kappa(0)\int_0^t |\kappa'(t-s)|\|Y_t(x,s)\|^2 ds + \frac{\delta}{2}\|Y_t\|^2. \tag{2.37}$$

Substituting inequalities (2.34) − (2.37) *into* (2.33), *we get*

$$\frac{\rho A}{2}\frac{d}{dt}\|Y_t\|^2 + \rho A\kappa(0)\|Y_t\|^2 + \frac{EI}{2}\frac{d}{dt}\|Y_{xx}\|^2 + \frac{P_0}{2}\frac{d}{dt}\|Y_x\|^2 + (\alpha - 3M_1 L\delta)(Y_t(L,t))^2$$

$$\leq \frac{(\rho A)^2}{2\delta}\kappa(0)\int_0^t |\kappa'(t-s)|\|Y_t(x,s)\|^2 ds + (\frac{\delta}{2} + \frac{3EA}{4})\|Y_t\|^2 + M_{11}\|Y_x\|^2 + (M_{11} + \frac{3M_1 L^2(EA)^2}{8\delta})\|Y_{xx}\|^2.$$

We choose δ so small that $\alpha - 3M_1 L\delta > 0$ after that integrating over (0, t), using this estimate $\int_0^t\int_0^r |\kappa'(r-s)|\|Y_t(s)\|^2 ds dr \leq \|\kappa'\|_{L^1(0,\infty)}\int_0^t \|Y_t(s)\|^2 ds$, we see that

$$\|Y_t\|^2 + +\|Y_x\|^2 + \|Y_{xx}\|^2 \leq M_{12}\int_0^t (\|Y_t\|^2 + \|Y_x\|^2 + \|Y_{xx}\|^2) ds \tag{2.38}$$

Thus, Gronwall's inequality guarantees the uniqueness of the solution. ■

Section 2.3 Asymptotic behavior

we introduce the functionals

$$\begin{aligned}
\Psi_1(t) &= \rho A \int_0^L y_t \int_0^t \kappa(t-s) y_t(s)\, ds dx, \\
\Psi_2(t) &= \rho A \int_0^L y \left(y_t + \int_0^t \kappa(t-s) y_t(s)\, ds \right) dx, \\
\Psi_3(t) &= \frac{P_0}{2} \int_0^t K_g(t-s) \|y_x(s)\|^2 ds, \\
\Psi_4(t) &= \frac{\rho A}{2} \int_0^t \left(\widetilde{K}_g(t-s) + K_g(t-s) \right) \|y_t(s)\|^2 ds,
\end{aligned}$$

and

$$\Psi_5(t) = \frac{EI}{2} \int_0^L \int_0^t K_g(t-s) y_{xx}^2(s)\, ds dx + \frac{EA}{2} \int_0^L \int_0^t K_g(t-s) y_x^4(s)\, ds dx,$$

where $K_g(t) = g^{-1}(t) \int_t^{+\infty} |\kappa'(s)| g(s)\, ds$, and $\widetilde{K}_g(t) = g^{-1}(t) \int_t^{+\infty} \kappa(s) g(s)\, ds$,

and $g(t)$ is specified above. We define the second modified functional by

$$\mathfrak{F}(t) = \mathfrak{E}(t) + \sum_{i=1}^{5} \lambda_i \, \Psi_i(t), \; t \geq 0, \tag{2.39}$$

for $\lambda_i > 0$, $i = 1, \ldots, 5$ to be specified later. Our first study indicated that this functional is reasonable to consider.

Proposition 2.2 *There exist $n_i > 0, i = 1,2$ such that*

$$n_1 \left(\mathfrak{E}(t) + \Psi_3(t) + \Psi_4(t) + \Psi_5(t)\right) \le \mathfrak{F}(t)$$
$$\le n_2 \left(\mathfrak{E}(t) + \Psi_3(t) + \Psi_4(t) + \Psi_5(t)\right),\ t \ge 0. \tag{2.40}$$

Proof *It is easy to see, from the above definitions, that*

$$\Psi_1(t) \le \frac{\rho A}{2}\left\|y_t(t)\right\|^2 + \frac{\rho A}{2}\overline{k}\int_0^t \kappa(t-s)\|y_t(s)\|^2 ds$$

$$\le q_1\left(\frac{\rho A}{2}\left\|y_t(t)\right\|^2 + \frac{\rho A}{2}\int_0^t \kappa(t-s)\|y_t(s)\|^2 ds\right)$$

where $q_1 = L\max(1,\overline{k})$.

$$\Psi_2(t) \le \frac{\rho A}{2}\left\|y_t(t)\right\|^2 + \frac{2\rho A L^2}{P_0}\frac{P_0}{2}\left\|y_x(t)\right\|^2 + \frac{\rho A}{2}\overline{k}\int_0^t \kappa(t-s)\|y_t(s)\|^2 ds$$

$$\le q_2\left(\frac{\rho A}{2}\left\|y_t(t)\right\|^2 + \frac{P_0}{2}\left\|y_x(t)\right\|^2 + \frac{\rho A}{2}\int_0^t \kappa(t-s)\|y_t(s)\|^2 ds\right).$$

where $q_2 = L\max(1,\overline{k}, \frac{2\rho A L^2}{P_0})$. *Taking into account these considerations, we have*

$$\begin{aligned}\mathfrak{F}(t) \ \le\ & (1+\lambda_1 q_1 + \lambda_2 q_2)\frac{\rho A}{2}\left\|y_t(t)\right\|^2 + \frac{EI}{2}\left\|y_{xx}(t)\right\|^2 + \frac{P_0}{2} + (1+\lambda_2 q_2)\left\|y_x(t)\right\|^2 \\ & + \frac{EA}{8}\left\|y_x^2(t)\right\|^2 + (1+\lambda_1 q_1 + \lambda_2 q_2)\frac{\rho A}{2}\int_0^t \kappa(t-s)\|y_t(s)\|^2 ds \\ & + \lambda_3\Psi_3(t) + \lambda_4\Psi_4(t) + \lambda_5\Psi_5(t)\end{aligned}$$

and

$$\begin{aligned}2\mathfrak{F}(t) &\geq (1-q_1\lambda_1-\lambda_2 q_2)\rho A\left\|y_t(t)\right\|^2+EI\left\|y_{xx}(t)\right\|^2+\frac{EA}{4}\left\|y_x^2(t)\right\|^2\\ &+(1-\lambda_2 q_2)P_0\left\|y_x(t)\right\|^2+(1-q_1\lambda_1-\lambda_2 q_2)\rho A\int_0^t \kappa(t-s)\|y_t(s)\|^2 ds\\ &+2\lambda_3\Psi_3(t)+2\lambda_4\Psi_4(t)+2\lambda_5\Psi_5(t),\ t\geq 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&n_1\left(\mathfrak{E}(t)+\lambda_3\Psi_3(t)+\lambda_4\Psi_4(t)+\lambda_5\Psi_5(t)\right)\leq \mathfrak{F}(t)\\ \leq\ & n_2\left(\mathfrak{E}(t)+\lambda_3\Psi_3(t)+\lambda_4\Psi_4(t)+\lambda_5\Psi_5(t)\right)\end{aligned}$$

for some $n_i>0$ *and* λ_i, $i=1,2$ *such that* $\lambda_1<\frac{1-\lambda_2 q_2}{2q_1}$, $\lambda_2<\frac{1-\lambda_1 q_1}{2q_2}$. ∎

In the following, we state and prove our main result.

Theorem 2.2 *Let us suppose that* κ *and* g *satisfy the hypotheses* $(K1)-(K3)$*. Then, there exist positive constants* C *and* σ *such that*

$$\mathfrak{E}(t)\leq Cg(t)^{-\sigma},\ t\geq 0. \tag{2.41}$$

Proof *Differentiating* $\Psi_1(t)$*, with respect to* t *and utilizing the equation (2.1), we obtain*

$$\Psi_1(t)=-\rho A\int_0^L y_t\int_0^t \kappa(t-s)y_t(s)dsdx$$

$$\Psi_1'(t)=-\rho A\int_0^L y_t\left(\int_0^t \kappa(t-s)y_t(s)ds\right)_t dx-\rho A\int_0^L y_{tt}\int_0^t \kappa(t-s)y_t(s)dsdx=I_1+I_2.$$

Clearly

$$\begin{aligned} I_1 &= -\rho A \int_0^L y_t \left(\kappa(0) y_t + \int_0^t \kappa'(t-s) y_t(s) ds \right) \\ I_1 &= -\rho A \kappa(0) \left\| y_t \right\|^2 - \rho A \int_0^L y_t \int_0^t \kappa'(t-s) y_t(s) ds dx \\ &\leq -\rho A \kappa(0) \left\| y_t \right\|^2 + \frac{\rho A}{4\delta_0} \left\| y_t \right\|^2 + \rho A \delta_0 \kappa(0) \int_0^t \left| \kappa'(t-s) \right| \left\| y_t(s) \right\|^2 ds \end{aligned}$$

$$\leq \rho A \left(\frac{1}{4\delta_0} - \kappa(0) \right) \left\| y_t \right\|^2 + \rho A \delta_0 \kappa(0) \int_0^t \left| \kappa'(t-s) \right| \left\| y_t(s) \right\|^2 ds, \delta_0 > 0. \tag{2.42}$$

The equation (2.1) allows us to write

$$\begin{aligned} I_2 &= \rho A \int_0^L \left(\int_0^t \kappa(t-s) y_t(s) ds \right)_t \left(\int_0^t \kappa(t-s) y_t(s) ds \right) dx \\ &+ \int_0^L \left(EI y_{xxxx} - P_0 y_{xx} - \frac{EA}{2} (y_x^3)_x \right) \\ &\times \left(\kappa(0) y - \kappa(t) y_0 + \int_0^t \kappa'(t-s) y(s) ds \right) dx = I_{21} + I_{22}, \end{aligned}$$

where

$$\begin{aligned} I_{21} &= \rho A \int_0^L \left(\kappa(0) y_t + \int_0^t \kappa'(t-s) y_t(s) ds \right) \left(\int_0^t \kappa(t-s) y_t(s) ds \right) dx \\ &= \rho A \kappa(0) \int_0^L y_t \int_0^t \kappa(t-s) y_t(s) ds dx + \rho A \int_0^L \int_0^t \kappa'(t-s) y_t(s) ds \int_0^t \kappa(t-s) y_t(s) ds dx. \end{aligned}$$

Using young and Cauchy Schwartz inequality, for $\delta_0 > 0$, we estimate

$$\begin{aligned}
I_{21} &\leq \rho A\kappa(0)^2\delta_0 \left\|y_t\right\|^2 + \frac{\rho A}{4\delta_0}\bar{k}\int_0^t \kappa(t-s)\left\|y_t(s)\right\|^2 ds \\
&+ \rho A\kappa(0)\delta_0 \int_0^t \left|\kappa'(t-s)\right|\left\|y_t(s)\right\|^2 ds + \frac{\rho A\bar{k}}{4\delta_0}\int_0^t \kappa(t-s)\left\|y_t(s)\right\|^2 ds \\
&\leq \rho A\kappa(0)^2\delta_0 \left\|y_t\right\|^2 + \frac{\rho A\bar{k}}{2\delta_0}\int_0^t \kappa(t-s)\left\|y_t(s)\right\|^2 ds \\
&+ \rho A\kappa(0)\delta_0 \int_0^t \left|\kappa'(t-s)\right|\left\|y_t(s)\right\|^2 ds. \qquad (2.43)
\end{aligned}$$

Then, we have

$$\begin{aligned}
I_{22} &= \int_0^L \left(EIy_{xxx} - P_0 y_x - \frac{EA}{2}y_x^3\right)_x \left(\kappa(0)y - \kappa(t)y_0 + \int_0^t \kappa'(t-s)y(s)ds\right)dx \\
&= \left(EIy_{xxx}(L,t) - P_0 y_x(L,t) - \frac{EA}{2}y_x^3(L,t)\right) \\
&\times \left(\kappa(0)y(L,t) - \kappa(t)y(L,0) + \int_0^t \kappa'(t-s)y(L,s)ds\right) \\
&- \int_0^L \left(EIy_{xxx} - P_0 y_x - \frac{EA}{2}y_x^3\right)\left(\kappa(0)y_x - \kappa(t)y_{x0} + \int_0^t \kappa'(t-s)y_x(s)ds\right)dx.
\end{aligned}$$

Using boundary control and young inequality, for $\delta_1, \delta_2, \delta_3 > 0$, we find

$$I_{22} \leq \alpha y_t(L,t)\left(\kappa(0)y(L,t) - \kappa(t)y(L,0) + \int_0^t \kappa'(t-s)y(L,s)ds\right)$$
$$+EI\left(\kappa(0)+\kappa(t)\delta_1\right)\left\|y_{xx}\right\|^2 + P_0\left(\kappa(0)+\kappa(t)\delta_2\right)\left\|y_x\right\|^2 + \frac{EI\,\kappa(t)}{4\delta_1}\left\|y_{xx0}\right\|^2$$
$$+\frac{EA}{2}\left(\kappa(0)+\frac{\kappa(t)(1+\delta_3)}{2}\right)\left\|y_x^2\right\|^2 + \frac{P_0\kappa(t)}{4\delta_2}\left\|y_{x0}\right\|^2 + \frac{EA}{16\delta_3}\kappa(t)\left\|y_{x0}^2\right\|^2$$
$$-\int_0^L\left(EIy_{xxx} - P_0y_x - \frac{EA}{2}y_x^3\right)\left(\int_0^t \kappa'(t-s)y_x(s)ds\right)dx.$$

Young inequality gives us

$$-\int_0^L\left(EIy_{xxx} - P_0y_x - \frac{EA}{2}y_x^3\right)\left(\int_0^t \kappa'(t-s)y_x(s)ds\right)dx \leq EI\delta_4\left\|y_{xx}\right\|^2$$
$$+\frac{EI\,\kappa(0)}{4\delta_4}\int_0^t\left|\kappa'(t-s)\right|\left\|y_{xx}(s)\right\|^2ds + P_0\delta_5\left\|y_x\right\|^2 + \frac{P_0\kappa(0)}{4\delta_5}\int_0^t\left|\kappa'(t-s)\right|\left\|y_x(s)\right\|^2ds$$
$$+\frac{EA}{2}\int_0^L y_x^2\int_0^t \kappa'(t-s)y_x(s)y_x\,ds\,dx,\ \delta_4,\delta_5 > 0.$$

By using Holder's and young's inequalities, for $\delta_6 > 0$, we estimate

$$\begin{aligned}
&\int_0^L y_x^2 \int_0^t \kappa'(t-s) y_x(s) y_x ds dx \\
\leq & \left(\int_0^L \left(y_x^2\right)^2 dx \right)^{1/2} \left(\int_0^L \left(\int_0^t \left|\kappa'(t-s)\right| y_x(s) y_x ds \right)^2 dx \right)^{1/2} \\
\leq & \frac{1}{2} \left\|y_x^2\right\|^2 + \frac{1}{2} \int_0^L \int_0^t \left|\kappa'(t-s)\right| y_x^2(s) ds \int_0^t \left|\kappa'(t-s)\right| y_x^2 ds dx \\
\leq & \frac{1}{2} \left\|y_x^2\right\|^2 + \frac{1}{2} \left(\frac{\kappa(0)\delta_6}{4} \int_0^t \left|\kappa'(t-s)\right| \left\|y_x^2(s)\right\|^2 ds + \frac{\kappa(0)}{\delta_6} \left\|y_x^2\right\|^2 \right) \\
\leq & \frac{1}{2} \left(1 + \frac{\kappa(0)}{\delta_6} \right) \left\|y_x^2\right\|^2 + \frac{\delta_6 \kappa(0)}{8} \int_0^t \left|\kappa'(t-s)\right| \left\|y_x^2(s)\right\|^2 ds.
\end{aligned}$$

Applying young's inequality and lemma 1.12 , we find

$$\begin{aligned}
& y_t(L,t) \int_0^t \kappa'(t-s) y(L,s) ds \leq \frac{1}{2b_0} y_t^2(L,t) + \frac{b_0}{2} L\kappa(0) \int_0^t \left|\kappa'(t-s)\right| \left\|y_x(s)\right\|^2 ds, \\
& y_t(L,t) y(L,t) \leq \frac{1}{2b_0} y_t^2(L,t) + \frac{b_0 L}{2} \left\|y_x\right\|^2, \\
- \quad & y_t(L,t) y(L,0) \leq \frac{1}{2b_0} y_t^2(L,t) + \frac{b_0 L}{2} \left\|y_{x0}\right\|^2.
\end{aligned}$$

Hence,

$$
\begin{aligned}
I_{22} \leq\ & \alpha \frac{1+\kappa(0)+\kappa(t)}{2b_0} y_t^2(L,t) + P_0\left(\kappa(0)+\kappa(t)\delta_2+\delta_5+\frac{b_0 L\kappa(0)}{2P_0}\right)\left\|y_x\right\|^2 \\
+\ & EI\left(\kappa(0)+\kappa(t)\delta_1+\delta_4\right)\left\|y_{xx}\right\|^2 + \frac{\kappa(0)}{2}\left(\frac{P_0}{2\delta_5}+\alpha L b_0\right)\int_0^t \left|\kappa'(t-s)\right|\left\|y_x(s)\right\|^2 ds \\
+\ & \frac{EA}{2}\left(\kappa(0)\left(1+\frac{1}{2\delta_6}\right)+\frac{1+\kappa(t)(1+\delta_3)}{2}\right)\left\|y_x^2\right\|^2 + \frac{EI\,\kappa(t)}{4\delta_1}\left\|y_{xx0}\right\|^2 \\
+\ & \left(\frac{P_0}{2\delta_2}+\alpha b_0 L\right)\frac{\kappa(t)}{2}\left\|y_{x0}\right\|^2 + \frac{EA}{16\delta_3}\kappa(t)\left\|y_{x0}^2\right\|^2 \\
+\ & \frac{EI\,\kappa(0)}{4\delta_4}\int_0^t \left|\kappa'(t-s)\right|\left\|y_{xx}(s)\right\|^2 ds + \frac{\delta_6\kappa(0)}{16}EA\int_0^t \left|\kappa'(t-s)\right|\left\|y_x^2(s)\right\|^2 ds
\end{aligned}
\tag{2.44}
$$

Therefore,

$$
\begin{aligned}
\Psi_1'(t) \leq\ & \alpha \frac{(1+\kappa(0)+\kappa(t))}{2b_0} y_t^2(L,t) + \rho A\left(\frac{1}{4\delta_0}-\kappa(0)+\kappa(0)^2\delta_0\right)\left\|y_t\right\|^2 \\
+\ & 2\rho A\kappa(0)\delta_0\int_0^t \left|\kappa'(t-s)\right|\left\|y_t(s)\right\|^2 ds + \frac{\rho A\bar{k}}{2\delta_0}\int_0^t \kappa(t-s)\left\|y_t(s)\right\|^2 ds \\
+\ & EI\left(\kappa(0)+\kappa(t)\delta_1+\delta_4\right)\left\|y_{xx}\right\|^2 + P_0\left(\kappa(0)+\kappa(t)\delta_2+\delta_5+\frac{\alpha b_0 L\kappa(0)}{2P_0}\right)\left\|y_x\right\|^2 \\
+\ & \frac{EA}{2}\left(\kappa(0)\left(1+\frac{1}{2\delta_6}\right)+\frac{1+\kappa(t)(1+\delta_3)}{2}\right)\left\|y_x^2\right\|^2 + \frac{EI}{4\delta_1}\kappa(t)\left\|y_{xx0}\right\|^2 \\
+\ & \left(\frac{P_0}{2\delta_2}+\alpha b_0 L\right)\frac{\kappa(t)}{2}\left\|y_{x0}\right\|^2 + \frac{EA}{16\delta_3}\kappa(t)\left\|y_{x0}^2\right\|^2 \\
+\ & \frac{EI\,\kappa(0)}{4\delta_4}\int_0^t \left|\kappa'(t-s)\right|\left\|y_{xx}(s)\right\|^2 ds + \frac{\kappa(0)}{2}\left(\frac{P_0}{2\delta_5}+\alpha L b_0\right)\int_0^t \left|\kappa'(t-s)\right|\left\|y_x(s)\right\|^2 ds \\
+\ & \frac{\delta_6\kappa(0)}{16}EA\int_0^t \left|\kappa'(t-s)\right|\left\|y_x^2(s)\right\|^2 ds.
\end{aligned}
\tag{2.45}
$$

In view of equation (2.1), the derivative of $\Psi_2(t)$ *is given by*

$$\Psi_2'(t) = \rho A \int_0^L y_t \left(y_t + \int_0^t k(t-s) y_t(s) ds \right) dx + \rho A \int_0^L y \frac{\partial}{\partial t} \left(y_t + \int_0^t k(t-s) y_t(s) ds \right) dx$$

$$= \rho A \left\| y_t \right\|^2 + \rho A \int_0^L y_t \int_0^t k(t-s) y_t(s) ds dx + \int_0^L y \left(-EI y_{xxxx} + \frac{EA}{2} (y_x^3)_x + P_0 y_{xx} \right) dx.$$

Therefore, for $\delta_4 > 0$,

$$\begin{aligned} \Psi_2'(t) &\leq \rho A \left(1 + \frac{\delta_4}{2}\right) \left\| y_t \right\|^2 + \frac{\rho A \bar{k}}{2\delta_4} \int_0^t k(t-s) \left\| y_t(s) \right\|^2 ds - EI \left\| y_{xx} \right\|^2 \\ &- P_0 \left\| y_x \right\|^2 - \frac{EA}{2} \left\| y_x^2 \right\|^2 + y(L,t)(-EI y_{xxx}(L,t) + P_0 y_x(L,t) + \frac{EA}{2} y_x^3(L,t)). \end{aligned}$$

It follows from the boundary conditions that

$$\begin{aligned} \Psi_2'(t) \leq \rho A \left(1 + \frac{\delta_4}{2}\right) \left\| y_t \right\|^2 + \frac{\rho A \bar{k}}{2\delta_4} \int_0^t k(t-s) \left\| y_t(s) \right\|^2 ds - EI \left\| y_{xx} \right\|^2 \\ -P_0 \left(1 - \frac{\alpha L b_0}{2 P_0} \right) \left\| y_x \right\|^2 - \frac{EA}{2} \left\| y_x^2 \right\|^2 + \frac{\alpha}{2 b_0} y_t^2(L,t). \end{aligned} \quad (2.46)$$

The derivative of $\Psi_3(t)$ *satisfies*

$$\begin{aligned} \Psi_3'(t) &= \frac{P_0}{2} K_g(0) \left\| y_x \right\|^2 + \frac{P_0}{2} \int_0^t K_g'(t-s) \| v_x(s) \|^2 ds \\ &\leq \frac{P_0}{2} K_g(0) \left\| y_x \right\|^2 - \frac{P_0}{2} u(t) \int_0^t K_g(t-s) \left\| y_x(s) \right\|^2 ds - \frac{P_0}{2} \int_0^t |\kappa'(t-s)| \left\| y_x(s) \right\|^2 ds, t \geq 0. \end{aligned}$$

(2.47)

Further, differentiating $\Psi_4(t)$ yields

$$\begin{aligned}
\Psi_4'(t) &= \frac{\rho A}{2}\left(\widetilde{K}_g(0)+K_g(0)\right)\left\|y_t(t)\right\|^2+\frac{\rho A}{2}\int_0^t\left(\widetilde{K}_g'(t-s)+K_g'(t-s)\right)\left\|y_t(s)\right\|^2 ds \\
&\le \frac{\rho A}{2}\left(\widetilde{K}_g(0)+K_g(0)\right)\left\|y_t(t)\right\|^2-\frac{\rho A}{2}u(t)\int_0^t\left(\widetilde{K}_g(t-s)+K_g(t-s)\right)\left\|y_t(s)\right\|^2 ds \\
&- \frac{\rho A}{2}\int_0^t\left(\kappa(t-s)+|\kappa'(t-s)|\right)\left\|y_t(s)\right\|^2 ds, t\ge 0.
\end{aligned} \tag{2.48}$$

Direct computations give us

$$\begin{aligned}
\Psi_5'(t) &= \frac{EI}{2}K_g(0)\left\|y_{xx}\right\|^2+\frac{EI}{2}\int_0^t K_g'(t-s)\left\|y_{xx}(s)\right\|^2 ds+\frac{EA}{2}K_g(0)\left\|y_x^2\right\|^2 \\
&+ \frac{EA}{2}\int_0^t K_g'(t-s)\left\|y_x^2(s)\right\|^2 ds,
\end{aligned}$$

that is

$$\begin{aligned}
\Psi_5'(t) &\le \frac{EI}{2}K_g(0)\left\|y_{xx}\right\|^2+\frac{EA}{2}K_g(0)\left\|y_x^2\right\|^2-\frac{EI}{2}\int_0^t|\kappa'(t-s)|\left\|y_{xx}(s)\right\|^2 ds \\
&- \frac{EA}{2}u(t)\int_0^t K_g(t-s)\left\|y_x^2(s)\right\|^2 ds-\frac{EA}{2}\int_0^t|\kappa'(t-s)|\left\|y_x^2(s)\right\|^2 ds \\
&- \frac{EI}{2}u(t)\int_0^t K_g(t-s)\left\|y_{xx}(s)\right\|^2 ds,\ t\ge 0.
\end{aligned} \tag{2.49}$$

Collecting the estimations (2.45)-(2.49), we find

$$\begin{aligned}
F'(t) \leq & \frac{\rho A}{2}(\kappa' \circ y_t)(t) + \frac{\rho A}{2}\Big\{-\kappa(t) + 2\lambda_1\Big(\frac{1}{4\delta_0} - \kappa(0) + \kappa(0)^2\delta_0\Big) \\
& +2\lambda_2(1+\delta_4) + \lambda_4(\widetilde{K}_g(0) + K_g(0))\Big\}\left\|y_t\right\|^2 \\
& +P_0\left(\lambda_1\left(\kappa(0) + \kappa(t)\delta_2 + \delta_5 + \frac{\alpha L b_0 \kappa(0)}{2P_0}\right) - \lambda_2\left(1 - \frac{\alpha L b_0}{2P_0}\right) + \frac{\lambda_3}{2}K_g(0)\right)\left\|y_x\right\|^2 \\
& +EI\left(\lambda_1\left(\kappa(0) + \kappa(t)\delta_1 + \delta_4\right) - \lambda_2 + \frac{\lambda_5}{2}K_g(0)\right)\left\|y_{xx}\right\|^2 \\
& +\frac{EA}{2}\left(\lambda_1\left(\tfrac{1+\kappa(t)(1+\delta_3)}{2} + \kappa(0)\left(1+\tfrac{1}{\delta_6}\right)\right) - \lambda_2 + \lambda_5 K_g(0)\right)\left\|y_x^2\right\|^2 \\
& +\frac{\rho A}{2}\left(\frac{\lambda_1\overline{\kappa}}{\delta_0} + \tfrac{\lambda_2\overline{\kappa}}{\delta_4} - \lambda_4\right)\int_0^t \kappa(t-s)\|y_t(s)\|^2 ds \\
& +\rho A\left(2\lambda_1\kappa(0)\delta_0 - \frac{\lambda_4}{2}\right)\int_0^t |\kappa'(t-s)|\|y_t(s)\|^2 ds \\
& +\left(-\lambda_3\frac{P_0}{2} + \lambda_1\frac{\kappa(0)}{2}\Big(\frac{P_0}{2\delta_5} + \alpha b_0 L\Big)\right)\int_0^t |\kappa'(t-s)|\|y_x(s)\|^2 ds \\
& +\frac{EI}{2}\left(\lambda_1\frac{\kappa(0)}{2\delta_4} - \lambda_5\right)\int_0^t |\kappa'(t-s)|\|y_{xx}(s)\|^2 ds - \lambda_3 u(t)\Psi_3(t) - \lambda_4 u(t)\Psi_4(t)
\end{aligned}$$

$$\begin{aligned}
& +\frac{EA}{2}\left(\lambda_1\frac{\delta_6\kappa(0)}{8} - \lambda_5\right)\int_0^t |\kappa'(t-s)|\|y_x^2(s)\|^2 ds + \lambda_1\kappa(t)\frac{EI}{4\delta_1}\|y_{xx0}\|^2 \\
& +\lambda_1\kappa(t)\frac{EA}{16\delta_3}\|y_{x0}^2\|^2 + \lambda_1\frac{\kappa(t)}{2}\left(\frac{P_0}{2\delta_0} + \alpha b_0 L\right)\|y_{x0}\|^2 \\
& +\alpha\left(-1 + \lambda_1\frac{1+\kappa(0)+\kappa(t)}{2b_0} + \frac{\lambda_2}{2b_0}\right)y_t^2(L,t) - \lambda_5 u(t)\Psi_5(t). \qquad (2.50)
\end{aligned}$$

Choosing

$$b_0 = 2,\ \alpha = \frac{P_0}{2L},\ \delta_0 = \frac{1}{\kappa(0)},\ \delta_3 = \delta_1 = 1,\ \delta_6 = \frac{2}{\kappa(0)},\ \delta_4 = 2\kappa(0),\ \lambda_5 = \frac{\lambda_1}{4},$$
$$\lambda_3 = \kappa(0)\Big(\frac{1}{2\delta_5} + 1\Big)\lambda_1.$$

Therefore (2.50) takes the form

$$
\begin{aligned}
\mathfrak{F}'(t) \le\; & \frac{\rho A}{2}\left\{-\kappa(t)+\lambda_1\frac{\kappa(0)}{2}+2\lambda_2(1+2\kappa(0))+\lambda_4(\widetilde{K}_g(0)+K_g(0))\right\}\left\|y_t\right\|^2 \\
+\; & \frac{\rho A}{2}(\kappa'\circ y_t)(t)+P_0\left(\lambda_1\left(\kappa(t)\delta_2+\delta_5+\frac{3\kappa(0)}{2}+\frac{\kappa(0)}{2}(\frac{1}{2\delta_5}+1)K_g(0)\right)-\frac{\lambda_2}{2}\right)\left\|y_x\right\|^2 \\
+\; & EI\left(\lambda_1(3\kappa(0)+\kappa(t)+\frac{K_g(0)}{8})-\lambda_2\right)\left\|y_{xx}\right\|^2+\rho A\left(2\lambda_1-\frac{\lambda_4}{2}\right)\int_0^t|\kappa'(t-s)|\|y_t(s)\|^2ds \\
+\; & \frac{EA}{2}\left(\lambda_1\left(\frac{1+2\kappa(t)+\kappa(0)(2+\kappa(0))}{2}+\frac{K_g(0)}{4}\right)-\lambda_2\right)\left\|y_x^2\right\|^2-\lambda_3u(t)\Psi_3(t) \\
+\; & \frac{\rho A}{2}\left(\lambda_1\overline{\kappa}\kappa(0)+\frac{\lambda_2\overline{\kappa}}{2\kappa(0)}-\lambda_4\right)\int_0^t\kappa(t-s)\|y_t(s)\|^2ds-\lambda_4u(t)\Psi_4(t)-\lambda_5u(t)\Psi_5(t) \\
+\; & \kappa(t)\frac{\lambda_1}{2}\left(\frac{EI}{2}\|y_{xx0}\|^2+\frac{EA}{8}\|y_{x0}^2\|^2+P_0\left(\frac{\kappa(0)}{2}+1\right)\|y_{x0}\|^2\right) \\
+\; & \alpha\left(-1+\lambda_1\frac{1+\kappa(0)+\kappa(t)}{4}+\frac{\lambda_2}{4}\right)y_t^2(L,t). \qquad (2.51)
\end{aligned}
$$

We need λ_1 so small that

$$
\begin{cases}
\lambda_1\frac{\kappa(0)}{2}+2\lambda_2(1+2\kappa(0))+\lambda_4(\widetilde{K}_g(0)+K_g(0))<\kappa(t), \\
\lambda_1(3\kappa(0)+\kappa(t)+\frac{K_g(0)}{8})<\lambda_2, \\
\lambda_1\left(\frac{1+2\kappa(t)+\kappa(0)(2+\kappa(0))}{2}+\frac{K_g(0)}{4}\right)<\lambda_2, \\
\lambda_1\overline{\kappa}\kappa(0)+\frac{\lambda_2\overline{\kappa}}{2\kappa(0)}<\lambda_4, \\
\lambda_1<\frac{\lambda_4}{4}, \\
\lambda_1\left(\kappa(t)\delta_2+\delta_5+\frac{3\kappa(0)}{2}+\frac{\kappa(0)}{2}(\frac{1}{2\delta_5}+1)K_g(0)\right)<\frac{\lambda_2}{2}, \\
\lambda_1\frac{1+\kappa(0)+\kappa(t)}{4}+\frac{\lambda_2}{4}<1.
\end{cases}
$$

As a consequence of the above consideration, for $t \ge t^*$, we get

$$
\mathfrak{F}'(t) \le -C_1\mathfrak{E}(t)-\lambda_3u(t)\Psi_3(t)-\lambda_4u(t)\Psi_4(t)-\lambda_5u(t)\Psi_5(t)+C_2\kappa(t), \quad t\ge t^*,
$$

where $C_2 = \frac{\lambda_1}{2}\left(\frac{EI}{2}\|y_{xx0}\|^2+\frac{EA}{8}\|y_{x0}^2\|^2+P_0\left(\frac{\kappa(0)}{2}+1\right)\|y_{x0}\|^2\right)$. *As $u(t)$ is nonincreasing, we* *have $u(t) \leq u(0)$ for all $t \geq t^*$. we can write*

$$\mathfrak{F}'(t) \leq -\frac{C_1}{u(0)} u(t)\mathfrak{E}(t) - \lambda_3 u(t)\Psi_3(t) - \lambda_4 u(t)\Psi_4(t) - \lambda_5 u(t)\Psi_5(t) + C_2\kappa(t), \quad t \geq t^*.$$

Using the equivalence (2.40), we obtain

$$\mathfrak{F}'(t) \leq -C_3 u(t)\mathfrak{F}(t) + C_2\kappa(t), \quad t \geq t^*, \tag{2.52}$$

where $C_i, i = 1, ..., 3$ are positive constants. A simple integration of (2.52) over $[t^*, t]$ gives

$$\mathfrak{F}(t) \leq N e^{-C_3 \int_{t^*}^{t} u(s)ds}, \quad t \geq t^*,$$

for some positive constant N. Then utilizing the inequality (2.40) of Proposition 2.2, we get

$$n_1 \left(\mathfrak{E}(t) + \Psi_3(t) + \Psi_4(t) + \Psi_5(t)\right) \leq N e^{-C_3 \int_{t^*}^{t} u(s)ds}, \quad t \geq t^*.$$

Due to the continuity of $\mathfrak{E}(t)$ over the interval $[0, t^*]$, we deduce

$$\mathfrak{E}(t) \leq \frac{C}{g(t)^{\sigma}}, \quad t \geq 0,$$

for some positive constants C and σ. ■

Chapter 3

Stabilization of a nonlinear Euler-Bernoulli viscoelastic beam subjected to a neutral delay

This chapter is concerned with the nonlinear Euler-Bernoulli viscoelastic equation with a neutral type delay. First we established the local existence result by using the Faedo-Galerkin method. Next using the energy method and constructing an appropriate Lyapunov functional, under certain conditions on the kernel of neutral delay term, we show that despite of the destructive nature of delays in general, a very general decaying energy for the problem was obtained.

Section 3.1 Introduction

Many practical dynamic systems have delays, but they are often neglected for simplicity. However, the presence of time delays can lead to poor performance and instabilities in control systems, so it is important to take them into account. The modeling of several physical systems includes delay phenomena (mechanical, economic, biological, ecological and telecommunications systems) [25]. More and more researchers have focused on the stability of delay-differential neutral systems in the last two decades due to its widespread application [17, 21, 58, 41].

However, time delay is typically time-varying in many real-world neutral systems, which can significantly alter the neutral system's dynamics in some cases [9, 60].
In [52] Park considered a weak viscoelastic beam equation subject to time-varying delay of the form

$$\begin{cases} u_{tt}(x,t)+\Delta^2 u(x,t)-M(\| \nabla u(t) \|^2)\Delta u(x,t)+\sigma(t)\int_0^t g(t-\tau)\Delta u(x,\tau)d\tau \\ b_0 u_t(x,t)+b_1 u_t(x,t-s(t))=0,(x,t)\in\Omega\times\mathbb{R}_+, \\ u(x,t)=\frac{\partial u(x,t)}{\partial\eta}=0,(x,t)\in\Gamma\times\mathbb{R}_+, \\ u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x), x\in\Omega, \\ u_t(x,t)=h_0(x,t),(x,t)\in\Omega\times[-s(0),0) \end{cases}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain, η is the unit outward normal to boundary Γ of Ω, b_0 is a positive constant, b_1 is a real number, g, σ, and M are functions. A general decay rate under conditions on σ and the kernel g was showed. Feng [4] studied the strong time-dependent delay in the viscoelastic wave equation

$$\begin{cases} y_{tt}-\Delta y+\int_0^t g(t-\tau)y_{tt}(\tau)d\tau-u_1\Delta y_t-u_2\Delta y_t(t-s(t))=0,(x,t)\in\Omega\times\mathbb{R}_+, \\ y(x,t)=0,(x,t)\in\Gamma\times\mathbb{R}_+, \\ y(x,0)=y_0(x), x\in\Omega, \\ y_t(x,t)=h_0(x,t),(x,t)\in\Omega\times[-s(0),0) \end{cases}$$

where u_1, u_2 are constants and $s(t)>0$ denotes the time dependent delay. He obtained general decay of energy for the problem. In [55] Tatar examined the following wave equation with neutral delay

$$\begin{cases} y_{tt}-y_{xx}=-y_t-\int_0^t g(t-\tau)y_{tt}(\tau)d\tau, & (x,t)\in(0,1)\times\mathbb{R}, \\ y(0,t)=y(1,t)=0, & t\in\mathbb{R}_+, \\ y(x,0)=y_0(x), \qquad y_t(x,0)=y_1(x), & x\in(0,1) \end{cases}$$

and the exponential decay of the solution was shown. The neutrally retarded viscoelastic Timoshenko system was studied by Kerbal and Tatar [27], the authors proved an exponential decay result of energy under some conditions on the kernel. In the absence of time delay [11], the authors studied the vibrating flexible beam system

$$\begin{cases} \rho A u_{tt}(x,t) + EI u_{xxxx}(x,t) - P_0 u_{xx}(x,t) - \frac{3}{2} EA u_{xx}(x,t) u_x^2(x,t) = 0, \\ \text{in } (0,L) \times [0,\infty), \\ u_{xx}(0,t) = u_{xx}(L,t) = u(0,t) = 0, \forall t \geq 0, \\ -EI u_{xxx}(L,t) + P_0 u_x(L,t) + \frac{1}{2} EA u_x^3(L,t) = -m_0 u_t(L,t), \forall t \geq 0, m_0 > 0 \end{cases}$$

at the boundary, authors applied a linear control force and obtained an exponential decay of energy. Inspired by this work [11], in this chapter we consider the Euler–Bernoulli viscoelastic equation with a neutral type delay

$$\rho A \frac{\partial}{\partial t}\left(y_t + \int_0^t k(t-s) y_t(s) ds\right) = -EI y_{xxxx} + \frac{EA}{2} \frac{\partial}{\partial x}\left(y_x^3\right)$$

$$+P_0\left(y_{xx} - \int_0^t \varsigma(t-s) y_{xx}(s) ds\right) \text{ in } (0,L) \times \mathbb{R}^+ \tag{3.1}$$

under the boundary conditions

$$\begin{cases} y_{xx}(0,t) = y_{xx}(L,t) = y(0,t) = 0, \quad \forall t \geq 0, \\ EI y_{xxx}(L,t) = P_0 y_x(L,t) + \dfrac{1}{2} EA y_x^3(L,t) - P_0 \int\limits_0^t \varsigma(t-s) y_x(L,s) ds \\ +\alpha y_t(L,t), \quad \forall t \geq 0, \ \alpha > 0. \end{cases} \tag{3.2}$$

The initial conditions are

$$y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), x \in (0,L). \tag{3.3}$$

The system parameters are as follows: L is the beam's length, EI is its uniform flexural rigidity, ρA is the mass per unit length, EA is the axial stiffness, $y(x,t)$ denotes the beam transversal displacement and P_0 is the tension force. Here we suppose that the variation in length due to axial force is small and that just the elongation of the beam due to bending is taken into account. First we established the local existence result by using the Faedo-Galerkin method, and next we prove general decaying energy for the problem (3.1) − (3.3) using weaker assumptions on the relaxation function ς and some conditions on the kernel k.

Section 3.2 Notation and Main Results

In this section we present our assumptions about both kernels, Then we aim to show the global existence and uniqueness of the problem .
Let's assume

(H1) The kernel k is a nonnegative continuously differentiable and summable function satisfying

$$k'(t) \leq 0, \quad 0 < \overline{k} = \int_0^{+\infty} k(s)ds < 1$$

(H2) The relaxation function $\varsigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function satisfying

$$0 < l = \int_0^{+\infty} \varsigma(s)ds < 1 \text{ and for } t^\star > 0, \int_0^{t^\star} \varsigma(s)ds = \varsigma_\star > 0.$$

(H3) $\varsigma'(t) \leq 0$ for almost all $t \geq 0$.

(H4) There exists a positive increasing function $g(t)$ such that $\frac{g'(t)}{g(t)} = u(t)$ is a decreasing function and $\int_0^{+\infty} \varsigma(s)g(s)\,ds < +\infty$.

We introduce the following notation

$$\begin{aligned}
(\varsigma\square f)(t) &= \int_0^L \int_0^t \varsigma(t-s)\left[f(x,t) - f(x,s)\right]^2 dsdx \\
(\varsigma\star f)(t) &= \int_0^L \int_0^t \varsigma(t-s)f(x,s)dsdx, \ t \geq 0.
\end{aligned}$$

We denote

$$\mathcal{A} = \left\{ y \in \mathbf{H}^2(0,L) \ / \ y(0) = 0 \right\},$$
$$\mathcal{M} = \left\{ y \in \mathcal{A} \cap \mathbf{H}^4(0,L) \ \ y_{xx}(0) = y_{xx}(L) = 0 \right\}.$$

We define the (classical) energy of problem (3.1) – (3.3) by

$$\mathfrak{E}(t) = \frac{1}{2}\Big[\rho A \left\|y_t\right\|^2 + EI\left\|y_{xx}\right\|^2 + \frac{EA}{4}\left\|y_x^2\right\|^2 + P_0\left(1 - \int_0^t \varsigma(s)ds\right)\left\|y_x\right\|^2$$
$$+ P_0(\varsigma \Box y_x)(t) + \rho A \int_0^t k(t-s)\|y_t(s)\|^2 ds\Big] \tag{3.4}$$

Proposition 3.1 *The energy $\mathfrak{E}(t)$ is nonincreasing and uniformly bounded. More precisely, we have*

$$\mathfrak{E}'(t) = \frac{\rho A}{2}(k'\Box y_t)(t) - \frac{\rho A k(t)}{2}\left\|y_t\right\|^2 + \frac{P_0}{2}(\varsigma'\Box y_x)(t) - \frac{P_0}{2}\varsigma(t)\|y_x\|^2 - \alpha y_t^2(L,t) \leq 0, \ \ t \geq 0. \tag{3.5}$$

To prove the proposition we need to establish some lemmas

Lemma 3.1 *It is easy to see that*

$$\int_0^t \varsigma(t-s)\left(y_x(s), y_{xt}(t)\right) ds = -\frac{1}{2}\left(\varsigma \Box y_x\right)'(t) + \frac{1}{2}\left(\varsigma' \Box y_x\right)(t)$$
$$+ \frac{1}{2}\frac{d}{dt}\left(\left\|y_x(t)\right\|^2 \int_0^t \varsigma(s)ds\right) - \frac{1}{2}\varsigma(t)\left\|y_x(t)\right\|^2, \ t \geq 0.$$

Lemma 3.2 *We have the following identity:*

$$\int_0^L y_t(t)\int_0^t k(t-s)y_{tt}(s)dsdx = -\frac{1}{2}(k'\square y_t)(t)+\frac{1}{2}\frac{d}{dt}\int_0^t k(t-s)\|y_t(s)\|^2 ds + \frac{k(t)}{2}\|y_t(t)\|^2 - k(t)\int_0^L y_t(t)y_t(0)dx$$

for all $y_t \in C^1([0,\infty);L^2(0,L))$ *and* $k \in C^1[0,\infty)$.

Proof *The identity is a direct consequence of*

$$(k'\square y_t)(t) = k(t)\|y_t(t)-y_t(0)\|^2 - 2\int_0^L\int_0^t k(s)y_{tt}(t-s)\left[y_t(t)-y_t(t-s)\right]dsdx, \quad t \geq 0.$$

and

$$\frac{d}{dt}\int_0^t k(t-s)\|y_t(s)\|^2 ds = \frac{d}{dt}\int_0^t k(s)\|y_t(t-s)\|^2 ds = k(t)\|y_t(0)\|^2 + 2\int_0^L\int_0^t k(s)y_{tt}(t-s)y_t(t-s)dsdx, \quad t \geq 0.$$

From the above two relations, we find the proof of lemma 3.2 ■

Proof *(of the Proposition)*

Multiplying equation (3.1) by y_t *and integrating the result over* $(0,L)$*, and using integration by parts and the boundary conditions, we get*

$$\frac{1}{2}\frac{d}{dt}\left[\rho A\left\|y_t\right\|^2 + EI\left\|y_{xx}\right\|^2 + P_0\left\|y_x\right\|^2 + \frac{EA}{4}\left\|y_x^2\right\|^2\right] + \rho A k(t)\int_0^L y_t(t)y_t(0)dx + \rho A\int_0^L y_t\int_0^t k(t-s)y_{tt}(s)dsdx - P_0\int_0^t \varsigma(t-s)(y_x(s),y_{xt}(t))ds = \left[-EIy_{xxx}(L,t)+P_0y_x(L,t)+\frac{EA}{2}y_x^3(L,t)-P_0\int_0^t \varsigma(t-s)y_x(L,s)ds\right]y_t(L,t)$$

Then, applying lemma 3.1 and lemma 3.2, we find the relation in the proposition ■

Theorem 3.1 *Assume $(H1)-(H4)$ are satisfied. If $(y_0, y_1) \in \mathcal{M} times \mathcal{A}$, then for $T > 0$, there exists a unique solution y of problem (3.1) – (3.3) such that*
$y \in \boldsymbol{L}^{\infty}([0,T),\mathcal{M})$, $y_t \in \boldsymbol{L}^{\infty}([0,T),\mathcal{A})$, $y_{tt} \in \boldsymbol{L}^2([0,T),\boldsymbol{L}^2(0,L))$
Additionally, we have $y \in \mathcal{C}([0,T),\mathcal{A}), y_t \in \mathcal{C}([0,T),\boldsymbol{L}^2(0,L))$.

We will apply the Faedo Galerkin method to establish the existence and uniqueness of solution of the problem (3.1) – (3.3).

Proof *We employ the Galerkin's method to establish the proof.*
Firstly, we establish the existence and uniqueness of solutions conforming to Eqs. (3.1)–(3.3). Subsequently, we generalize this finding to encompass weak solutions through the application of density arguments.
The variational problem associated with equations (3.1) and (3.2) can be formulated as follows: find $y \in \mathcal{M}$ such that

$$\rho A(y_{tt}, w) + \rho A k(0)(y_t, w) + \rho A\left(\int_0^t k'(t-s)y_t(s)ds, w\right) + EI(y_{xx}, w_{xx}) + P_0(y_x, w_x)$$
$$-P_0\left(\int_0^t \varsigma(t-s)y_x(s)ds, w_x\right) + \frac{1}{2}EA\left((y_x)^3, w_x\right) + \alpha w(L,t)y_t(L,t) = 0,$$

for all $w \in \mathcal{M}$

<u>Step 1</u>*: **The approximate problem***
Let $\{w_i\}$ a complete orthogonal bases of $\mathcal{M}$. We consider $\mathrm{W}^N = span\{w_1, w_2, ..., w_N\}$, for all $N \in \mathbb{N}$. Given initial data $y_0 \in \mathcal{M}, y_1 \in \mathcal{A}$, the approximate solution $y^m(x,t) = \sum_{i=1}^{m} \mathfrak{C}_i^m(t) w_i(x)$ of the problem (3.1) – (3.3) satisfies :

$$\rho A(y_{tt}^m, w_i) + \rho A k(0)(y_t^m, w_i) + \rho A\left(\int_0^t k'(t-s)y_t^m(s)ds, w_i\right) + EI(y_{xx}^m, w_{ixx}) + P_0(y_x^m, w_{ix})$$
$$-P_0\left(\int_0^t \varsigma(t-s)y_x^m(s)ds, w_{ix}\right) + \frac{1}{2}EA\left((y_x^m)^3, w_{ix}\right) + \alpha w_i(L,t)y_t^m(L,t) = 0. \quad (3.6)$$

with the initial conditions

$$\begin{cases} y^m(0) = \sum_{i=1}^{m} (y^m(0), w_i) w_i \longrightarrow y_0 \ \ in\ \mathcal{M}, \\ y_t^m(0) = \sum_{i=1}^{m} (y_t^m(0), w_i) w_i \longrightarrow y_1 \ \ in\ \mathcal{A}. \end{cases}$$

<u>Step 2</u>*: **A Priori Estimate***

We indicate by $M_i, i = 1, 2, ...,$ positive constants independent of m.

Estimate 1: *According to (3.5) and hypothesis $(H1)-(H4)$ it follows*

$$\mathfrak{E}_m'(t) + \frac{P_0}{2}\varsigma(t)\|y_x^m\|^2 + \alpha\,(y_t^m(L,t))^2 \leq 0 \tag{3.7}$$

where $\mathfrak{E}_m$ is the energy of the solutions y^m, introduced in (3.4).
The integration of the inequality (3.7) along the $(0,t)$, gives us

$$\mathfrak{E}_m(t) + \frac{P_0}{2}\int_0^t \varsigma(s)\|y_x^m(s)\|^2 ds + \alpha\int_0^t (y_t^m(L,s))^2\, ds \leq \mathfrak{E}_m(0). \tag{3.8}$$

As the initial conditions are sufficiently smooth, then there exists a constant $M_1 > 0$, independent of m, such that

$$\begin{aligned} \left\|y_t^m\right\|^2 + \left\|y_{xx}^m\right\|^2 + \left\|y_x^m\right\|^2 + \left\|(y_x^m)^2\right\|^2 + (\varsigma\Box y_x^m)(t) + \int_0^t k(t-s)\left\|y_t^m(s)\right\|^2 ds \\ + \int_0^t \varsigma(s)\|y_x^m(s)\|^2 ds + \int_0^t (y_t^m(L,s))^2\, ds \leq M_1. \end{aligned} \tag{3.9}$$

Estimate 2 *Searching for an upper bound of $\left\|y_{tt}^m(0)\right\|^2$*
By multiplying $(\mathfrak{C}_i^m)_{tt}(0)$ on both sides of Equation.(3.6) and summing up the resulting equations from $i = 1$ to $i = m$ and putting $t = 0$, then integrate by parts, and taking into account the boundary conditions, it follows

$$\begin{aligned} \rho A\|y_{tt}^m(0)\|^2 + \rho A k(0)\,(y_t^m(0), y_{tt}^m(0)) + (EI y_{xxxx}^m(0) - P_0 y_{xx}^m(0), y_{tt}^m(0)) \\ -\frac{3}{2}EA\left(y_{xx}^m(0)(y_x^m(0))^2, y_{tt}^m(0)\right) = 0. \end{aligned} \tag{3.10}$$

By young's inequality we can write

$$\|y_{tt}^m(0)\| \leq M_2. \tag{3.11}$$

Estimate 3. *Searching for an upper bound of* $\|y_{tt}^m\|$.
Now let's fix $t, \zeta > 0$ *with* $\zeta + t < T$. *When we multiply* $(\mathfrak{C}_i^m)_t(t+\zeta) - (\mathfrak{C}_i^m)_t(t)$ *on both sides of equation (3.6) and then sum the resulting equations from* $i = 1$ *to* $i = m$, *and taking the difference with* $t = t + \zeta$ *and* $t = t$, *we get*

$$\frac{\rho A}{2}\frac{d}{dt}\|y_t^m(\zeta+t) - y_t^m(t)\|^2 + \rho A k(0)\|y_t^m(\zeta+t) - y_t^m(t)\|^2 + \frac{EI}{2}\frac{d}{dt}\|y_{xx}^m(\zeta+t) - y_{xx}^m(t)\|^2$$
$$+\frac{P_0}{2}\frac{d}{dt}\|y_x^m(\zeta+t) - y_x^m(t)\|^2 + \alpha\left[y_t^m(L,\zeta+t) - y_t^m(L,t)\right]^2 = K_1 + K_2 + K_3 \tag{3.12}$$

where

$$K_1 = -\frac{EA}{2}\int_0^L \left[(y_x^m(\zeta+t))^3 - (y_x^m(t))^3\right]\left[y_{xt}^m(\zeta+t) - y_{xt}^m(t)\right]dx,$$

$$K_2 = -\rho A\int_0^L \left[\int_0^{\zeta+t} k'(\zeta+t-s)y_t^m(x,s)ds - \int_0^t k'(t-s)y_t^m(x,s)ds\right]\left[y_t^m(\zeta+t) - y_t^m(t)\right]dx,$$

$$K_3 = P_0\int_0^L \left[\int_0^{\zeta+t} \varsigma(\zeta+t-s)y_x^m(x,s)ds - \int_0^t \varsigma(t-s)y_x^m(x,s)ds\right]\left[y_{xt}^m(\zeta+t) - y_{xt}^m(t)\right]dx.$$

Integrating by parts, we get

$$K_1 = -\frac{EA}{2}\left[(y_x^m(L,t+\zeta))^3 - (y_x^m(L,t))^3\right]\left[y_t^m(L,t+\zeta) - y_t^m(L,t)\right]$$

$$+\frac{3EA}{2}\int_0^L \left[y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t)(y_x^m(t))^2\right]\left[y_t^m(t+\zeta) - y_t^m(t)\right]dx$$

$$= H_{11} + H_{12}. \tag{3.13}$$

On the other hand, by young inequality and lemma 1.12, we have

$$H_{11} = -\frac{EA}{2}\left[(y_x^m(L,t+\zeta))^3 - (y_x^m(L,t))^3\right]\left[y_t^m(L,t+\zeta) - y_t^m(L,t)\right]$$

$$= -\frac{EA}{2}\left[y_t^m(L,t+\zeta) - y_t^m(L,t)\right] \times \left[(y_x^m(L,t+\zeta))^2 + y_x^m(L,t+\zeta)y_x^m(L,t) + (y_x^m(L,t))^2\right] \times$$

$$\left[y_x^m(L,t+\zeta) - y_x^m(L,t)\right]$$

$$\leq \frac{3}{2}\left[(y_x^m(L,t+\zeta))^2 + (y_x^m(L,t))^2\right]\left(\frac{(EA)^2}{16\delta}\left[y_x^m(L,t+\zeta) - y_x^m(L,t)\right]^2 + \delta\left[y_t^m(L,t+\zeta) - y_t^m(L,t)\right]^2\right)$$

$$\leq \frac{3L}{2}\left[\|y_{xx}^m(L,t+\zeta)\|^2 + \|y_{xx}^m(L,t)\|^2\right] \times$$

$$\left(\frac{(EA)^2}{16\delta}\|y_{xx}^m(L,t+\zeta) - y_{xx}^m(L,t)\|^2 + \delta\left[y_t^m(L,t+\zeta) - y_t^m(L,t)\right]^2\right)$$

$$\leq \frac{3M_1L(EA)^2}{16\delta}\|y_{xx}^m(L,t+\zeta) - y_{xx}^m(L,t)\|^2 + 3M_1L\delta\left[y_t^m(L,t+\zeta) - y_t^m(L,t)\right]^2 \tag{3.14}$$

on the other hand, by young's inequality, we have

$$H_{12} = \frac{3EA}{2}\int_0^L \left[y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t)(y_x^m(t))^2\right]\left[y_t^m(t+\zeta) - y_t^m(t)\right]dx$$

$$\leq \frac{3EA}{4}\|y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t)(y_x^m(t))^2\|^2 + \frac{3EA}{4}\|y_t^m(t+\zeta) - y_t^m(t)\|^2$$

we estimate the first term in H_{12} by

$$\|y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t)(y_x^m(t))^2\|^2 =$$

$$\|y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t+\zeta)(y_x^m(t))^2 + y_{xx}^m(t+\zeta)(y_x^m(t))^2 - y_{xx}^m(t)(y_x^m(t))^2\|^2$$

$$\leq 2\|y_{xx}^m(t+\zeta)(y_x^m(t+\zeta))^2 - y_{xx}^m(t+\zeta)(y_x^m(t))^2\|^2 + 2\|y_{xx}^m(t+\zeta)(y_x^m(t))^2 - y_{xx}^m(t)(y_x^m(t))^2\|^2$$

$$\leq 2\|y_{xx}^m(t+\zeta)\|^2\|(y_x^m(t+\zeta))^2 - (y_x^m(t))^2\|^2 + 2\|(y_x^m(t))^2\|_\infty^2\|y_{xx}^m(t+\zeta) - y_{xx}^m(t)\|^2$$

$$\leq M_1^*\left(\|y_x^m(t+\zeta) - y_x^m(t)\|^2 + \|y_{xx}^m(t+\zeta) - y_{xx}^m(t)\|^2\right)$$

then, we get

$$H_{12} \leq M_1^*\frac{3EA}{4}\left(\|y_x^m(t+\zeta) - y_x^m(t)\|^2 + \|y_{xx}^m(t+\zeta) - y_{xx}^m(t)\|^2\right) + \frac{3EA}{4}\|y_t^m(t+\zeta) - y_t^m(t)\|^2 \tag{3.15}$$

by (3.14), (3.15), and young inequality, we get

$$|K_1| \le \frac{3M_1L(EA)^2}{8\delta}\|y^m_{xx}(L,t+\zeta)-y^m_{xx}(L,t)\|^2 + 3M_1L\delta\left[y^m_t(L,t+\zeta)-y^m_t(L,t)\right]^2$$

$$+M_1^*\frac{3EA}{4}\left(\|y^m_x(t+\zeta)-y^m_x(t)\|^2+\|y^m_{xx}(t+\zeta)-y^m_{xx}(t)\|^2\right)+\frac{3EA}{4}\|y^m_t(t+\zeta)-y^m_t(t)\|^2. \quad (3.16)$$

Young's inequality, leads to

$$|K_2| \le M_4\int_0^L\left[\int_0^{t+\zeta}k'(t+\zeta-s)y^m_t(x,s)ds-\int_0^t k'(t-s)y^m_t(x,s)ds\right]^2dx+\frac{\delta}{4}\|y^m_t(t+\zeta)-y^m_t(t)\|^2 \quad (3.17)$$

$$|K_3| \le M_5\int_0^L\left[\int_0^{t+\zeta}\varsigma(t+\zeta-s)y^m_{xx}(x,s)ds-\int_0^t \varsigma(t-s)y^m_{xx}(x,s)ds\right]^2dx+\frac{\delta}{4}\|y^m_t(t+\zeta)-y^m_t(t)\|^2. \quad (3.18)$$

Substituting inequalities (3.16) − (3.18) *into* (3.12), *we calculate the limit when* $\zeta \to 0$ *after dividing* ζ^2, *we find*

$$\frac{\rho A}{2}\frac{d}{dt}\|y^m_{tt}(t)\|^2+\rho Ak(0)\|y^m_{tt}(t)\|^2+\frac{EI}{2}\frac{d}{dt}\|y^m_{xxt}(t)\|^2+\frac{P_0}{2}\frac{d}{dt}\|y^m_{xt}(t)\|^2+\alpha(y^m_{tt}(L,t))^2 \le$$

$$3M_1L\delta(y^m_{tt}(L,t))^2+\left(\frac{3M_1L(EA)^2}{8\delta}+M_1^*\frac{3EA}{4}\right)\|y^m_{xxt}(t)\|^2+\left[\frac{\delta}{2}+\frac{3EA}{2}\right]\|y^m_{tt}(t)\|^2$$

$$M_1^*\frac{3EA}{4}\|y^m_{xt}(t)\|^2+M_4\int_0^L\left(k'(0)y^m_t(t)+\int_0^t k''(t-s)y^m_t(x,s)ds\right)^2dx$$

$$+M_5\int_0^L\left(\varsigma(0)y^m_{xx}(t)+\int_0^t\varsigma'(t-s)y^m_{xx}(x,s)ds\right)^2dx. \quad (3.19)$$

On the other hand we have

$$\int_0^L\int_0^t\kappa''(t-s)y^m_t(x,s)dsdx \le \sup_{[0,T]}\|y^m_t\|\int_0^T|\kappa''(s)|ds < M_7 \quad (3.20)$$

and

$$\int_0^L \int_0^t \varsigma'(t-s) y_{xx}^m(x,s)\,ds\,dx \le \sup_{[0,T]} \|y_{xx}^m\| \int_0^T |\varsigma'(s)|\,ds < M_8. \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.19), integrating along the interval (0,t), we obtain

$$\|y_{tt}^m\|^2 + \|y_{xxt}^m\|^2 + \|y_{xt}^m\|^2 \le M_9 + M_{10} \int_0^t (\|y_{tt}^m\|^2 + \|y_{xxt}^m\|^2 + \|y_{xt}^m\|^2)\,ds.$$

Thanks to Gronwall's lemma, we have

$$\|y_{tt}^m\|^2 + \|y_{xxt}^m\|^2 + \|y_{xt}^m\|^2 \le M_{11}. \tag{3.22}$$

Sep 3 ***Passage to limits.***

According to the above estimates we conclude

$$\begin{cases} y^m \text{ are bounded in } L^\infty(0,T;\mathcal{A}), \\ y_t^m \text{ are bounded in } L^\infty(0,T;\mathcal{A}), \\ y_{tt}^m \text{ are bounded in } L^\infty(0,T;L^2(0,L)), \\ (y_x^m)^2 \text{ are bounded in } L^\infty(0,T;L^2(0,L)). \end{cases} \tag{3.23}$$

Therefore, there exist subsequences of (y^m), denoted again by (y^m), satisfying

$$\begin{cases} y^m \overset{*}{\rightharpoonup} y \text{ in } \boldsymbol{L}^\infty(0,T,\mathcal{A}), \\ y_t^m \overset{*}{\rightharpoonup} y_t \text{ in } \boldsymbol{L}^\infty(0,T,\mathcal{A}), \\ y_{tt}^m \overset{*}{\rightharpoonup} y_{tt} \text{ in } \boldsymbol{L}^\infty(0,T,\boldsymbol{L}^2(0,L)). \\ (y_x^m)^2 \overset{*}{\rightharpoonup} (y_x)^2 \text{ in } \boldsymbol{L}^\infty(0,T,\boldsymbol{L}^2(0,L)). \end{cases} \tag{3.24}$$

Thanks to the Aubin-Lions compactness lemma and (3.24), we get

$$y^m \to y \text{ strongly in } \boldsymbol{L}^\infty(0,T,\boldsymbol{H}_0^1(0,L)) \tag{3.25}$$

(3.25) *and lemma* 1.5, *allow to write*

$$(y_x^m)^3 \rightharpoonup (y_x)^3 \ \ in\, \boldsymbol{L}^2([0,T]\times[0,L]). \tag{3.26}$$

This allows us by passing to the limit in (3.6) *to obtain a weak solution of the problem* (3.1) – (3.3).

Uniqueness

Assume that y_1 *and* y_2 *are two different solution to the system* (3.1) – (3.3)*, and* $Y = y_1 - y_2$*, with* $Y(0) = Y_t(0) = 0$*, then* Y *satisfies*

$$\rho A\left(Y_{tt}, w_i\right) + \rho A k(0)\left(Y_t, w_i\right) + \rho A\left(\int_0^t k'(t-s)Y_t(x,s)ds, w_i\right) + EI\left(Y_{xx}, w_{ixx}\right)$$

$$+P_0\left(Y_x, w_{ix}\right) - P_0\left(\int_0^t \varsigma(t-s)Y_x(x,s)ds, w_{ix}\right) + \frac{1}{2}EA\left((y_1)_x^3 - (y_2)_x^3, w_{ix}\right) + \alpha Y_t(L,t)w_i(L,t) = 0 \tag{3.27}$$

When we multiply $(\mathfrak{C}_i^m)_t(t)$ *on both sides of Equation* 2.32 *and then sum the resulting equations with respect to* i*, we get*

$$\rho A\left(Y_{tt}, Y_t\right) + \rho A k(0)\left(Y_t, Y_t\right) + \rho A\left(\int_0^t k'(t-s)Y_t(x,s)ds, Y_t\right) + EI\left(Y_{xx}, Y_{txx}\right)$$

$$+P_0\left(Y_x, Y_{tx}\right) - P_0\left(\int_0^t \varsigma(t-s)Y_x(x,s)ds, Y_{tx}\right) + \frac{1}{2}EA\left((y_1)_x^3 - (y_2)_x^3, Y_{tx}\right) + \alpha\left(Y_t(L,t)\right)^2 = 0 \tag{3.28}$$

then, we have

$$\frac{\rho A}{2}\frac{d}{dt}\|Y_t\|^2 + \rho A k(0)\|Y_t\|^2 + \frac{EI}{2}\frac{d}{dt}\|Y_{xx}\|^2 + \frac{P_0}{2}\frac{d}{dt}\|Y_x\|^2 + \alpha\left(Y_t(L,t)\right)^2 = -\rho A\left(\int_0^t k'(t-s)Y_t(x,s)ds, Y_t\right)$$

$$+P_0\left(\int_0^t \varsigma(t-s)Y_x(x,s)ds, Y_{xt}\right) - \frac{1}{2}EA\left((y_1)_x^3 - (y_2)_x^3, Y_{tx}\right). \tag{3.29}$$

Considering the same technique in ***Estimate 3****, utilizing young's, Holdre's inequalities, we* *get*

$$-\frac{1}{2}EA\left((y_1)_x^3-(y_2)_x^3, Y_{tx}\right) \le \frac{3M_1L^2(EA)^2}{8\delta}\|Y_{xx}(L)\|^2+3M_1L\delta(Y_t(L,t))^2+M_{12}\|Y_{xx}\|^2 \\ +M_{12}\|Y_x\|^2+\frac{3EA}{4}\|Y_t\|^2. \tag{3.30}$$

$$-\rho A\left(\int_0^t k'(t-s)Y_t(x,s)ds, Y_t\right) \le M_{13}\int_0^t |k'(t-s)|\|Y_t(x,s)\|^2ds+\frac{\delta}{2}\|Y_t\|^2. \tag{3.31}$$

$$P_0\left(\int_0^t \varsigma(t-s)Y_{xx}(x,s)ds, Y_t\right) \le M_{14}\int_0^t \varsigma(t-s)\|Y_{xx}(x,s)\|^2ds+\frac{\delta}{2}\|Y_t\|^2. \tag{3.32}$$

Substituting (3.30) – (3.32) *into* (3.29), *then integrating along the interval (0,t), we obtain*

$$\|Y_t\|^2+\|Y_{xx}\|^2+\|Y_x\|^2 \le M_{15}\int_0^t (\|Y_t\|^2+\|Y_{xx}\|^2+\|Y_x\|^2)ds. \tag{3.33}$$

Thus, Gronwall's inequality guarantees the uniqueness of the solution. ■

Next, we introduce the functionals

$$\begin{aligned}
\Phi_1(t) &= \rho A\int_0^L y\left(y_t+\int_0^t k(t-s)y_t(s)ds\right)dx+\alpha\frac{y^2(L,t)}{2},\\
\Phi_2(t) &= -\rho A\int_0^L\left(y_t+\int_0^t k(t-s)y_t(s)ds\right)\int_0^t \varsigma(t-s)(y(t)-y(s))dsdx,\\
\Phi_3(t) &= P_0\int_0^t K_g(t-s)\|y_x(s)\|^2ds,\\
\Phi_4(t) &= EI\int_0^t K_g(t-s)\|y_{xx}(s)\|^2ds+\frac{EA}{2}\int_0^t K_g(t-s)\|y_x^2(s)\|^2ds,\\
\Phi_5(t) &= \frac{\rho A}{2}\int_0^t \widetilde{K}_g(t-s)\|y_t(s)\|^2ds.
\end{aligned}$$

$$K_g(t) = g^{-1}(t)\int_t^{+\infty} \varsigma(s) g(s)\, ds, \qquad \widetilde{K}_g(t) = g^{-1}(t)\int_t^{+\infty} k(s) g(s)\, ds$$

and $g(t)$ is specified below. We define the second modified functional by

$$\mathfrak{L}(t) = \mathfrak{E}(t) + \sum_{i=1}^{5} \lambda_i \Phi_i(t), \quad t \geq 0 \tag{3.34}$$

for $\lambda_i > 0$, $i = 1,2,3,4,5$ to be specified later. Our first result shows that this functional is an appropriate one to consider.

Proposition 3.2 *There exist $n_i > 0, i = 1,2$ such that*

$$n_1(\mathfrak{E}(t) + \Phi_3(t) + \Phi_4(t) + \Phi_5(t)) \leq \mathfrak{L}(t) \leq n_2(\mathfrak{E}(t) + \Phi_3(t) + \Phi_4(t) + \Phi_5(t)), \; t \geq 0. \tag{3.35}$$

Proof *It is easy to see, from the above definitions, that*

$$\Phi_1(t) \leq (\rho A c_p + \frac{\alpha L}{2})\left\|y_x\right\|^2 + \frac{\rho A}{2}\left\|y_t\right\|^2 + \frac{\rho A}{2}\overline{k}\int_0^t k(t-s)\|y_t(s)\|^2 ds,$$

$$\Phi_2(t) \leq \rho A\left\|y_t\right\|^2 + \rho A\overline{k}\int_0^t k(t-s)\|y_t(s)\|^2 ds + \frac{\rho A l c_p}{2}(\varsigma \square y_x)(t).$$

Moreover

$$\Phi_1(t) + \Phi_2(t) \leq (\rho A c_p + \frac{\alpha L}{2})\left\|y_x\right\|^2 + \frac{3\rho A}{2}\left\|y_t\right\|^2 + \frac{3\rho A}{2}\overline{k}\int_0^t k(t-s)\|y_t(s)\|^2 ds + \frac{\rho A l c_p}{2}(\varsigma \square y_x)(t)$$

$$\leq \frac{2\rho A c_p + \alpha L}{P_0}\frac{P_0}{2}\left\|y_x\right\|^2 + 3\frac{\rho A}{2}\left\|y_t\right\|^2 + 3\overline{k}\frac{\rho A}{2}\int_0^t k(t-s)\|y_t(s)\|^2 ds + \frac{\rho A l c_p}{P_0}\frac{P_0}{2}(\varsigma \square y_x)(t)$$

$$\leq c_1\left(\frac{\rho A}{2}\left\|y_t\right\|^2 + \frac{P_0}{2}\left\|y_x\right\|^2 + \frac{\rho A}{2}\overline{k}\int_0^t k(t-s)\|y_t(s)\|^2 ds + \frac{P_0}{2} l(\varsigma \square y_x)(t)\right),$$

where $c_1 = \max(3, \dfrac{2\rho A c_p + \alpha L}{P_0})$. *With these in mind, we have*

$$\begin{aligned}
\mathfrak{L}(t) \quad \leq \quad & (1+(\lambda_1+\lambda_2)c_1)\frac{\rho A}{2}\left\|y_t\right\|^2 + \frac{EI}{2}\left\|y_{xx}\right\|^2 + \frac{EA}{8}\left\|y_x^2\right\|^2 + (1+(\lambda_1+\lambda_2)c_1 l)\frac{P_0}{2}(\varsigma\Box y_x)(t) \\
& + \left(1 - \int_0^t \varsigma(s)ds + (\lambda_1+\lambda_2)c_1\right)\frac{P_0}{2}\left\|y_x\right\|^2 + \left(1+(\lambda_1+\lambda_2)c_1\bar{k}\right)\frac{\rho A}{2}\int_0^t k(t-s)\|y_t(s)\|^2 ds \\
& + \lambda_3\Phi_3(t) + \lambda_4\Phi_4(t) + \lambda_5\Phi_5(t)
\end{aligned}$$

and

$$\begin{aligned}
2\mathfrak{L}(t) \quad \geq \quad & (1-c_1(\lambda_1+\lambda_2))\rho A\left\|y_t\right\|^2 + (1-l-(\lambda_1+\lambda_2)c_1)P_0\left\|y_x\right\|^2 \\
& + (1-(\lambda_1+\lambda_2)c_1 l)P_0(\varsigma\Box y_x)(t) + EI\left\|y_{xx}\right\|^2 + \frac{EA}{4}\left\|y_x^2\right\|^2 \\
& + \left(1-(\lambda_1+\lambda_2)c_1\bar{k}\right)\rho A\int_0^t k(t-s)\|y_t(s)\|^2 ds + 2\lambda_3\Phi_3(t) + 2\lambda_4\Phi_4(t) + 2\lambda_5\Phi_5(t), t \geq 0.
\end{aligned}$$

Therefore, $n_1(\mathfrak{E}(t) + \Phi_3(t) + \Phi_4(t) + \Phi_5(t)) \leq \mathfrak{L}(t) \leq n_2(\mathcal{E}(t) + \Phi_3(t) + \Phi_4(t) + \Phi_5(t))$ *for some constant* $n_i > 0$ *and* λ_1, λ_2 *such that*

$$\lambda_1 + \lambda_2 < \frac{1-l}{c_1}. \qquad \blacksquare$$

To show our stability result, the following lemma will be utilized.

Lemma 3.3 *We have for a continuous function* ς *on* $[0,\infty)$ *and* $y \in H^1(0,L)$

$$\begin{aligned}
& \int_0^L y_x \int_0^t \varsigma(t-s) y_x(s) ds dx \\
& = \tfrac{1}{2}\left(\int_0^t \varsigma(s)ds\right)\left\|y_x\right\|^2 + \tfrac{1}{2}\int_0^t \varsigma(t-s)\left\|y_x(s)\right\|^2 ds - \tfrac{1}{2}(\varsigma\Box y_x)(t), t \geq 0.
\end{aligned}$$

Section 3.3 Asymptotic behavior

In this section we state and show our result.

Theorem 3.2 *Assume that ς and g satisfy the hypotheses (H1)-(H4). Then, there exist positive constants C and u such that*

$$\mathfrak{E}(t) \leq Cg(t)^{-u}, \; t \geq 0. \tag{3.36}$$

Proof *A differentiation of $\Phi_1(t)$, with respect to t along the solution of (3.1) – (3.3), gives*

$$\Phi_1'(t) = \rho A \int_0^L y_t^2 dx + \rho A \int_0^L y \frac{\partial}{\partial t}\left(y_t + \int_0^t k(t-s) y_t(s) ds \right) dx +$$
$$+\rho A \int_0^L y_t \int_0^t k(t-s) y_t(s) ds dx + \alpha y_t(L,t) y(L,t).$$

For the second term we use equation (3.1), the boundary conditions and lemma 3.3, we obtain

$$\rho A \int_0^L y \frac{\partial}{\partial t}\left(y_t + \int_0^t k(t-s) y_t(s) ds \right) dx = -\alpha y_t(L,t) y(L,t) - EI \left\| y_{xx} \right\|^2 - \frac{P_0}{2}(\varsigma \Box y_x)(t)$$
$$-P_0 \left(1 - \frac{1}{2}\int_0^t \varsigma(s) ds \right) \left\| y_x \right\|^2 + \frac{P_0}{2}\int_0^t \varsigma(t-s) \left\| y_x(s) \right\|^2 ds - \frac{EA}{2}\left\| y_x^2 \right\|^2. \tag{3.37}$$

Applying young's inequality to the third term, we find

$$\rho A \int_0^L y_t \int_0^t k(t-s) y_t(s) ds dx \leq \rho A \delta_1 \left\| y_t \right\|^2 + \frac{\rho A \bar{k}}{4\delta_1} \int_0^t k(t-s) \left\| y_t(s) \right\|^2 ds, \;\; \delta_1 > 0. \tag{3.38}$$

By substituting the relations (3.37) – (3.38) in $\Phi_1'(t)$ we obtain

$$\Phi_1'(t) \leq -EI \left\| y_{xx} \right\|^2 - \frac{P_0}{2}(\varsigma \Box y_x)(t) - P_0 \left(1 - \frac{1}{2}\int_0^t \varsigma(s) ds \right) \left\| y_x \right\|^2 \tag{3.39}$$
$$+\frac{P_0}{2}\int_0^t \varsigma(t-s) \left\| y_x(s) \right\|^2 ds - \frac{EA}{2}\left\| y_x^2 \right\|^2 + \rho A(1+\delta_1) \left\| y_t \right\|^2$$
$$+\frac{\rho A \bar{k}}{4\delta_1} \int_0^t k(t-s) \left\| y_t(s) \right\|^2 ds.$$

It is easy to see that differentiating $\Phi_2(t)$ gives

$$\Phi_2'(t) = \underbrace{-\rho A \int_0^L \left(y_t + \int_0^t k(t-s)y_t(s)ds \right)_t \int_0^t \varsigma(t-s)(y(t)-y(s))dsdx,}_{J_1}$$

$$\underbrace{-\rho A \int_0^L \left(y_t + \int_0^t k(t-s)y_t(s)ds \right) \left(\int_0^t \varsigma'(t-s)(y(t)-y(s))ds + y_t \int_0^t \varsigma(s)ds \right) dx.}_{J_2} \tag{3.40}$$

Integrating J_1 by parts and using the boundary conditions, we get

$$\begin{aligned}
J_1 &= -\rho A \int_0^L \left(y_t + \int_0^t k(t-s)y_t(s)ds \right)_t \int_0^t \varsigma(t-s)(y(t)-y(s))dsdx, \\
&= \left(EIy_{xxx}(L,t) - P_0 y_x(L,t) - \frac{EA}{2} y_x^3(L,t) + P_0 \int_0^t \varsigma(t-s)y_x(L,s)ds \right) \\
&\quad times \int_0^t \varsigma(t-s)(y(L,t)-y(L,s))ds \\
&\quad + \int_0^L \left(-EIy_{xxx}(t) + \frac{EA}{2} y_x^3() + P_0 y_x(t) - P_0 \int_0^t \varsigma(t-s)y_x(s))ds \right) \\
&\quad times \left(\int_0^t \varsigma(t-s)(y_x(t)-y_x(s))ds \right) dx \\
&= \alpha y_t(L,t) \int_0^t \varsigma(t-s)(y(L,t)-y(L,s))ds \\
&\quad - \int_0^L EIy_{xxx}(t) \left(\int_0^t \varsigma(t-s)(y_x(t)-y_x(s)ds \right) dx \\
&\quad + \frac{EA}{2} \int_0^L y_x^3(t) \left(\int_0^t \varsigma(t-s)(y_x(t)-y_x(s))ds \right) dx \\
&\quad + P_0 \left(1 - \int_0^t \varsigma(s)ds \right) \int_0^L y_x(t) \left(\int_0^t \varsigma(t-s)(y_x(t)-y_x(s)ds \right) dx \\
&\quad + P_0 \int_0^L \left(\int_0^t \varsigma(t-s)(y_x(t)-y_x(s)ds \right)^2 dx \\
&= \alpha y_t(L,t) \int_0^t \varsigma(t-s)(y(L,t)-y(L,s))ds + J_{11} + J_{12} + J_{13} + J_{14}, \quad t \geq 0.
\end{aligned}$$

Again utilizing young's inequality, we get

$$\alpha y_t(L,t)\int_0^t \varsigma(t-s)(y(L,t)-y(L,s))ds \tag{3.41}$$
$$= \alpha y_t(L,t)y(L,t)\int_0^t \varsigma(s)ds - \alpha y_t(L,t)\int_0^t \varsigma(t-s)y(L,s)ds$$
$$\leq \frac{\alpha}{2\delta_0}(l+1)y_t^2(L,t) + \frac{\alpha l\delta_0}{2}y^2(L,t) + \frac{\alpha\delta_0}{2}\left(\int_0^t \varsigma(t-s)y(L,s)ds\right)^2,\ t\geq 0.$$

For the second and the third term in (3.41), we have

$$y^2(L,t)\leq L\left\|y_x\right\|^2,$$
$$\left(\int_0^t \varsigma(t-s)y(L,s)ds\right)^2 = \left(\int_0^t \varsigma(t-s)\int_0^L y_x(x,s)dxds\right)^2,$$

and

$$\left(\int_0^t \varsigma(t-s)y(L,s)ds\right)^2 \leq lL\int_0^t \varsigma(t-s)\left\|y_x(s)\right\|^2 ds,\ t\geq 0.$$

Hence,

$$\alpha y_t(L,t)\int_0^t \varsigma(t-s)(y(L,t)-y(L,s))ds \tag{3.42}$$
$$\leq \frac{\alpha}{2\delta_0}(l+1)y_t^2(L,t) + \frac{\alpha l\delta_0}{2}L\left\|y_x\right\|^2 + \frac{\alpha\delta_0}{2}Ll\int_0^t \varsigma(t-s)\left\|y_x(s)\right\|^2 ds.$$

For $\delta_2>0$, we can write

$$J_{11} = \int_0^L EIy_{xx}(t)\left(\int_0^t \varsigma(t-s)(y_{xx}(t)-y_{xx}(s))ds\right)dx$$
$$\leq EI\left(\int_0^t \varsigma(s)ds+\delta_2\right)\left\|y_{xx}(t)\right\|^2 + \frac{EI}{4\delta_2}\left(\int_0^t \varsigma(s)ds\right)\int_0^t \varsigma(t-s)\left\|y_{xx}(s)\right\|^2 ds$$

and

$$
\begin{aligned}
J_{12} &= \frac{EA}{2}\int_0^t \varsigma(s)dstext\left\|y_x^2(t)\right\|^2 - \frac{EA}{2}\int_0^L y_x^2(t)\int_0^t \varsigma(t-s)y_x(s)y_x(t)dsdx \\
&= \frac{EA}{2}\int_0^t \varsigma(s)dstext\left\|y_x^2(t)\right\|^2 - \frac{EA}{2}\int_0^L y_x^2(t)\int_0^t \varsigma^{1/2}(t-s)y_x(s)\varsigma^{1/2}(t-s)y_x(t)dsdx \\
&\leq \frac{EA}{2}l\left\|y_x^2(t)\right\|^2 + \frac{EA}{2}\left(\int_0^L \left(y_x^2(t)\right)^2 dx\right)^{1/2}\left(\int_0^L\left(\int_0^t \varsigma^{1/2}(t-s)y_x(s)\varsigma^{1/2}(t-s)y_x(t)ds\right)^2 dx\right)^{1/2} \\
&\leq \frac{EA}{2}l\left\|y_x^2(t)\right\|^2 + \frac{EA}{4}\left\|y_x^2(t)\right\|^2 + \frac{EA}{4}\int_0^L\left(\int_0^t \varsigma^{1/2}(t-s)y_x(s)\varsigma^{1/2}(t-s)y_x(t)ds\right)^2 dx \\
&\leq \frac{EA}{2}l\left\|y_x^2(t)\right\|^2 + \frac{EA}{4}\left\|y_x^2(t)\right\|^2 + \frac{EA}{4}\int_0^L\int_0^t \varsigma(t-s)y_x^2(s)ds\int_0^t \varsigma(t-s)y_x^2(t)dsdx \\
&\leq \frac{EA}{2}l\left\|y_x^2(t)\right\|^2 + \frac{EA}{4}\left\|y_x^2(t)\right\|^2 + \frac{EA}{4}\left(\frac{l^2\delta_3}{4}\int_0^t \varsigma(t-s)\left\|y_x^2(s)\right\|^2 ds + \frac{l}{\delta_3}\left\|y_x^2(t)\right\|^2\right) \\
&\leq \frac{EA}{2}\left(l + \frac{l}{2\delta_3} + \frac{1}{2}\right)\left\|y_x^2(t)\right\|^2 + \frac{EA}{16}\delta_3 l^2\int_0^t \varsigma(t-s)\left\|y_x^2(s)\right\|^2 ds.
\end{aligned}
$$

Now we proceed to estimate J_{13} and J_{14}. We obtain for $\delta_4 > 0$

$$
J_{13} \leq P_0\left(1 - \int_0^t \varsigma(s)ds\right)\left(\delta_4\left\|y_x(t)\right\|^2 + \frac{l}{4\delta_4}(\varsigma\square y_x)(t)\right),
$$

and

$$
J_{14} \leq P_0 l(\varsigma\square y_x)(t),\ t \geq 0.
$$

We decompose the second integral J_2 into

$$J_2 = -\rho A \int_0^t \varsigma(s)ds\|y_t\|^2 - \rho A \int_0^L y_t \int_0^t \varsigma'(t-s)(y(t)-y(s))dsdx$$
$$-\rho A \int_0^L \left(\int_0^t k(t-s)y_t(s)ds \right) \left(\int_0^t \varsigma'(t-s)(y(t)-y(s))ds \right) dx$$
$$-\rho A \int_0^t \varsigma(s)ds \int_0^L y_t \int_0^t k(t-s)y_t(s)dsdx = -\rho A \int_0^t \varsigma(s)ds\|y_t\|^2 + J_{21} + J_{22} + J_{23}.$$

For $\delta_6 > 0$, we have

$$\begin{aligned} J_{21} &= -\rho A \int_0^L y_t \int_0^t \varsigma'(t-s)(y(t)-y(s))dsdx \\ &\leq \rho A \delta_6 \|y_t\|^2 - \frac{\rho A L \varsigma(0)}{4\delta_6}(\varsigma' \Box y_x)(t) \end{aligned}$$

and

$$\begin{aligned} J_{22} &= -\rho A \int_0^L \left(\int_0^t k(t-s)y_t(s)ds \right) \left(\int_0^t \varsigma'(t-s)(y(t)-y(s))ds \right) dx \\ &\leq \rho A \delta_6 \overline{k} \int_0^t k(t-s)\|y_t(s)\|^2 ds - \frac{\rho A L \varsigma(0)}{4\delta_6}(\varsigma' \Box y_x)(t). \end{aligned}$$

For $\delta_7 > 0$, we have

$$\begin{aligned} J_{23} &= -\rho A \left(\int_0^t \varsigma(s)ds \right) \int_0^L y_t \int_0^t k(t-s)y_t(s)dsdx \\ &\leq \rho A \delta_7 l \|y_t\|^2 + l \frac{\rho A}{4\delta_7} \overline{k} \int_0^t k(t-s)\|y_t(s)\|^2 ds \end{aligned}$$

Taking into account (3.41) – (3.42) and the above estimations of $J_{11}, J_{12}, J_{13}, J_{14}, J_{21}, J_{22}, J_{23}$, *we obtain*

$$\begin{aligned}\Phi_2'(t) \;&\leq\; \rho A\left(\delta_6-\varsigma_\star+\delta_7 l\right)\left\|y_t\right\|^2+EI(l+\delta_2)\left\|y_{xx}\right\|^2+\left(\frac{\alpha Ll\delta_0}{2}+P_0\left(1-\varsigma_\star\right)\delta_4\right)\left\|y_x\right\|^2\\ &+\frac{EA}{2}\left(l+\frac{l}{2\delta_3}+\frac{1}{2}\right)\left\|y_x^2\right\|^2+\frac{\alpha L\delta_0}{2}l\int_0^t\varsigma(t-s)\left\|y_x(s)\right\|^2ds+\frac{EA\delta_3}{16}l^2\int_0^t\varsigma(t-s)\left\|y_x^2(s)\right\|^2ds\\ &+\frac{\alpha}{2\delta_0}(l+1)y_t^2(L,t)-\frac{\rho AL\varsigma(0)}{2\delta_6}(\varsigma'\Box y_x)(t)+\frac{lEI}{4\delta_2}\int_0^t\varsigma(t-s)\left\|y_{xx}(s)\right\|^2ds\end{aligned}$$

$$+\rho A\bar{k}\left(\delta_6+\frac{l}{4\delta_7}\right)\int_0^t k(t-s)\left\|y_t(s)\right\|^2ds+P_0 l\left(1+\frac{1-\varsigma_\star}{4\delta_4}\right)(\varsigma\Box y_x)(t),\quad t\geq t_\star. \tag{3.43}$$

Further, a differentiation of $\Phi_3(t)$ *yields*

$$\begin{aligned}&\Phi_3'(t)=P_0K_g(0)\left\|y_x(t)\right\|^2+P_0\int_0^t K_g'(t-s)\left\|y_x(s)\right\|^2ds\\ &\leq P_0K_g(0)\left\|y_x(t)\right\|^2-P_0u(t)\int_0^t K_g(t-s)\left\|y_x(s)\right\|^2ds-P_0\int_0^t\varsigma(t-s)\left\|y_x(s)\right\|^2ds,\quad t\geq 0.\end{aligned} \tag{3.44}$$

Regarding $\Phi_4'(t)$ *it appears that*

$$\begin{aligned}\Phi_4'(t) \;&=\; EIK_g(0)\left\|y_{xx}(t)\right\|^2+EI\int_0^t K_g'(t-s)\left\|y_{xx}(s)\right\|^2ds+\frac{EA}{2}K_g(0)\left\|y_x^2(t)\right\|^2\\ &+\frac{EA}{2}\int_0^t K_g'(t-s)\left\|y_x^2(s)\right\|^2ds,\end{aligned} \tag{3.45}$$

that is

$$\begin{aligned}\Phi_4'(t)\leq\;& EIK_g(0)\left\|y_{xx}(t)\right\|^2+\frac{EA}{2}K_g(0)\left\|y_x^2(t)\right\|^2-EI\int_0^t\varsigma(t-s)\left\|y_{xx}(s)\right\|^2ds\\ &-EIu(t)\int_0^t K_g(t-s)\left\|y_{xx}(s)\right\|^2ds-\frac{EA}{2}\int_0^t\varsigma(t-s)\left\|y_x^2(s)\right\|^2ds\\ &-\frac{EA}{2}u(t)\int_0^t K_g(t-s)\left\|y_x^2(s)\right\|^2ds,\; t\geq 0.\end{aligned} \tag{3.46}$$

Moreover, a differentiation of $\Phi_5(t)$ *yields*

$$\begin{aligned}\Phi_5'(t) &= \frac{\rho A}{2}\widetilde{K}_g(0)\left\|y_t(t)\right\|^2+\frac{\rho A}{2}\int_0^t \widetilde{K}_g'(t-s)\left\|y_t(s)\right\|^2 ds \\ &\leq \frac{\rho A}{2}\widetilde{K}_g(0)\left\|y_t(t)\right\|^2-\frac{\rho A}{2}u(t)\int_0^t \widetilde{K}_g(t-s)\left\|y_t(s)\right\|^2 ds \\ &\quad -\frac{\rho A}{2}\int_0^t k(t-s)\left\|y_t(s)\right\|^2 ds, \quad t\geq 0. \end{aligned} \tag{3.47}$$

Collecting the estimations (3.39), (3.43) − (3.47)*, we find for* $t \geq t_\star$

$$\begin{aligned}
&\mathfrak{L}'(t) \leq \left(\frac{P_0}{2}-\lambda_2\frac{\rho AL\varsigma(0)}{2\delta_6}\right)(\varsigma'\square y_x)(t)+\frac{\rho A}{2}(k'\square y_t)(t)\\
&+\rho A\left(\lambda_2(\delta_6-\varsigma_\star+\delta_7 l)+\lambda_5\frac{\widetilde{K}_g(0)}{2}+\lambda_1(1+\delta_1)-\frac{k(t)}{2}\right)\|y_t\|^2\\
&+P_0\left(-\lambda_1\left(1-\frac{l}{2}\right)+\lambda_2\left(\frac{\alpha Ll\delta_0}{2P_0}+(1-\varsigma_\star)\delta_4\right)+\lambda_3 K_g(0)-\frac{\varsigma(t)}{2}\right)\left\|y_x\right\|^2\\
&+EI\left(-\lambda_1+\lambda_2(l+\delta_2)+\lambda_4 K_g(0)\right)\left\|y_{xx}\right\|^2\\
&+EI\left(\frac{l}{4\delta_2}\lambda_2-\lambda_4\right)\int_0^t \varsigma(t-s)\left\|y_{xx}(s)\right\|^2 ds\\
&+\frac{EA}{2}\left(\frac{l^2\delta_3}{8}\lambda_2-\lambda_4\right)\int_0^t \varsigma(t-s)\left\|y_x^2(s)\right\|^2 ds\\
&+\left(\lambda_1\frac{P_0}{2}+\lambda_2\frac{\alpha Ll\delta_0}{2}-\lambda_3 P_0\right)\int_0^t \varsigma(t-s)\|v_x(s)\|^2 ds\\
&+P_0\left(l\left(1+\frac{1-\varsigma_\star}{4\delta_4}\right)\lambda_2-\frac{\lambda_1}{2}\right)(\varsigma\square y_x)(t)+\alpha\left(\lambda_2\frac{l+1}{2\delta_0}-1\right)y_t^2(L,t)\\
&+\frac{EA}{2}\left(-\lambda_1+\lambda_2(l+\frac{l}{2\delta_3}+\frac{1}{2})+\lambda_4 K_g(0)\right)\left\|y_x^2\right\|^2\\
&+\frac{\rho A}{2}\left(-\lambda_5+2\lambda_2\overline{k}(\delta_6+\frac{l}{4\delta_7})+\frac{\overline{k}\lambda_1}{4\delta_1}\right)\int_0^t k(t-s)\left\|y_t(s)\right\|^2 ds\\
&-\lambda_3 u(t)\Phi_3(t)-\lambda_4 u(t)\Phi_4(t)-\lambda_5 u(t)\Phi_5(t)
\end{aligned} \tag{3.48}$$

At this step, we select $\lambda_2 \leq \dfrac{\delta_6 P_0}{2\rho AL\varsigma(0)}$*, to satisfy* $\dfrac{P_0}{2}-\lambda_2\dfrac{\rho AL\varsigma(0)}{2\delta_6}\geq\dfrac{P_0}{4}$ *and* $\lambda_2\leq\dfrac{\delta_0}{l+1}$ *so that*

$\lambda_2 \leq \min\left\{\frac{\delta_6 P_0}{2\rho AL\varsigma(0)},\ \frac{\delta_0}{l+1}\right\}$ *and* $\delta_2 = \frac{l}{2},\ \delta_3 = \frac{2}{l\delta_2},\ \delta_4 = \frac{1}{2},\ \delta_0 = P_0,\ \lambda_4 = \frac{\lambda_2}{2}$ *to get*

$$
\begin{aligned}
&\mathfrak{L}'(t) \leq \frac{\rho A}{2}(k'\square y_t)(t) \\
&+\rho A\left(\lambda_2(\delta_6 - \varsigma_\star + \delta_7 l) + \lambda_5 \frac{\widetilde{K}_g(0)}{2} + \lambda_1(1+\delta_1) - \frac{k(t)}{2}\right)\|y_t\|^2 \\
&+P_0\left(-\lambda_1\left(1-\frac{l}{2}\right) + \lambda_2\left(\frac{\alpha Ll}{2} + \frac{1-\varsigma_\star}{2}\right) + \lambda_3 K_g(0) - \frac{\varsigma(t)}{2}\right)\left\|y_x\right\|^2 \\
&+EI\left(-\lambda_1 + \frac{\lambda_2}{2}\left(3l + K_g(0)\right)\right)\left\|y_{xx}\right\|^2 \\
&+P_0\left(\frac{\lambda_1}{2} + \lambda_2\frac{\alpha Ll}{2} - \lambda_3\right)\int_0^t \varsigma(t-s)\left\|y_x(s)\right\|^2 ds \\
&+P_0\left(\frac{(3-\varsigma_\star)l}{2}\lambda_2 - \frac{\lambda_1}{2}\right)(\varsigma\square y_x)(t) + \alpha\left(\lambda_2\frac{l+1}{2P_0} - 1\right)y_t^2(L,t) \\
&+\frac{EA}{2}\left(-\lambda_1 + \lambda_2\left(l + \frac{l^3}{8} + \frac{1}{2} + \frac{K_g(0)}{2}\right)\right)\left\|y_x^2\right\|^2 \\
&+\frac{\rho A}{2}\left(-\lambda_5 + 2\lambda_2\overline{k}(\delta_6 + \frac{l}{4\delta_7}) + \frac{\overline{k}\lambda_1}{4\delta_1}\right)\int_0^t k(t-s)\left\|y_t(s)\right\|^2 ds \\
&-\lambda_3 u(t)\Phi_3(t) - \lambda_4 u(t)\Phi_4(t) - \lambda_5 u(t)\Phi_5(t)
\end{aligned}
\tag{3.49}
$$

Further, we need

$$
\begin{cases}
\lambda_2\left(\frac{\alpha Ll}{2} + \frac{1-\varsigma_\star}{2}\right) + \lambda_3 K_g(0) < \lambda_1\left(1-\frac{l}{2}\right) + \frac{\varsigma(t)}{2}, \\
\frac{\lambda_2}{2}(3l + K_g(0)) < \lambda_1, \\
\lambda_2(\delta_6 - \varsigma_\star + \delta_7 l) + \lambda_5\frac{\widetilde{K}_g(0)}{2} + \lambda_1(1+\delta_1) < \frac{k(t)}{2},
\end{cases}
\tag{3.50}
$$

$$
\begin{cases}
(3-\varsigma_\star)l\lambda_2 < \lambda_1, \\
\left(l + \frac{l^3}{8} + \frac{1}{2} + \frac{K_g(0)}{2}\right)\lambda_2 < \lambda_1, \\
2\lambda_2\overline{k}(\delta_6 + \frac{l}{4\delta_7}) + \frac{\overline{k}\lambda_1}{4\delta_1} < \lambda_5, \\
\frac{\lambda_1}{2} + \lambda_2\frac{\alpha Ll}{2} < \lambda_3.
\end{cases}
\tag{3.51}
$$

We will focus on the first set of inequalities

$$\begin{cases} \lambda_2\left(\frac{\alpha Ll}{2}+\frac{1-\varsigma_\star}{2}\right)+\lambda_3 K_g(0)<\lambda_1\left(1-\frac{l}{2}\right)+\frac{\varsigma(t)}{2}, \\ \lambda_5\frac{\widetilde{K}_g(0)}{2}+\lambda_1(1+\delta_1)<\lambda_2(\varsigma_\star-\delta_6-\delta_7 l)+\frac{k(t)}{2}, \\ (3-\varsigma_\star)l\lambda_2<\lambda_1. \end{cases} \tag{3.52}$$

let $N_1=(3+l)$ *and* $\lambda_2=\frac{\lambda_1}{N_1}$. *Take* $\alpha=\frac{1}{lL}$ *and we select* δ_6, δ_7 *so small and* t *large so that the second condition in* (3.52) *is satisfied. In order to achieve* (3.51) *(the second set of inequalities), it is sufficient to choose* λ_3, λ_5 *large enough. As a result of these choices, we conclude*

$$\mathfrak{L}'(t)\leq -N_2\mathfrak{E}(t)-\lambda_3 u(t)\Phi_3(t)-\lambda_4 u(t)\Phi_4(t)-\lambda_5 u(t)\Phi_5(t),\, N_2\geq 0,\ t\geq t^\star. \tag{3.53}$$

Since $u(t)\leq u(0)$, *then*

$$\mathfrak{L}'(t)\leq -\frac{N_2}{u(0)}u(t)\mathfrak{E}(t)-\lambda_3 u(t)\Phi_3(t)-\lambda_4 u(t)\Phi_4(t)-\lambda_5 u(t)\Phi_5(t),t\geq t^\star. \tag{3.54}$$

According to (3.35), *we obtain*

$$\mathfrak{L}'(t)\leq -N_3 u(t)\mathfrak{L}(t),\ N_3\geq 0. \tag{3.55}$$

Integrating (3.55) *over* $[t^\star,t]$, *we obtain*

$$\mathfrak{L}(t)\leq e^{-N_3\int_{t^\star}^{t}u(s)ds}\mathfrak{L}\left(t^\star\right),\ t\geq t^\star.$$

Using again the equivalence (3.35), *we get*

$$\mathfrak{E}(t)\leq e^{-N_3\int_{t^\star}^{t}u(s)ds}\mathfrak{L}\left(t^\star\right),\ t\geq t^\star.$$

Since $\mathfrak{E}(t)$ *is continuous over* $[0,t^\star]$, *we conclude that*

$$\mathfrak{E}(t)\leq\frac{C}{g(t)^u},\ t\geq 0,$$

for some positive constants C and u. ■

Conclusion

In conclusion, the qualitative exploration of selected Partial Differential Equations (PDEs) concerning temporal dynamics with damping, particularly focusing on the examination of two non-linear Euler-Bernoulli beams featuring neutral-type delays and viscoelasticity, has yielded significant insights into their behaviors.

The investigation has uncovered intricate dynamics within the Euler-Bernoulli beams, shedding light on the intricate interplay between nonlinearity, damping effects, and temporal delays. Incorporating viscoelastic properties has introduced additional layers of complexity, influencing the overall system responses in nuanced ways.
Stability analysis has played a pivotal role in comprehending the long-term behaviors of the systems under scrutiny. By scrutinizing spectral properties and employing advanced analytical techniques such as Lyapunov functionals, criteria for stability have been delineated, offering valuable discernment into the conditions dictating system stability or instability.

The implications of this study extend beyond theoretical realms, offering practical insights applicable to various engineering domains, including vibration control, structural health monitoring, and the design of damping systems. Understanding the complexities inherent in such systems is paramount for ensuring the reliability and performance of engineered structures.

While significant strides have been made, numerous avenues for future exploration remain open. These include delving into more intricate beam configurations, exploring diverse damping mechanisms, accounting for uncertainties and parameter variations, and broadening the analysis to encompass other classes of PDEs sharing similar characteristics.

We believe that it would be interesting to study in future the following Timoshenko beam with thermodiffusion effects:

$$\begin{cases} \frac{\rho h^3}{12}\varphi_{tt} - \varphi_{xx} + k(\varphi + \psi_x) - \delta_1 \varrho_x - \delta_2 P_x = 0, \\ \rho h\, \psi_{tt} - k(\varphi + \psi_x)_x - [\psi_x \left(\eta_x + \frac{1}{2}\psi_x^2\right)]_x = 0, \\ \qquad \rho h \eta_{tt} - \left(\eta_x + \frac{1}{2}\psi_x^2\right)_x = 0, \\ c\varrho_t + dP_t - \int\limits_0^\infty \omega_1(s)\varrho_{xx}(t-s)ds - \delta_1 \varphi_{tx} = 0, \\ d\varrho_t + rP_t - \int\limits_0^\infty \omega_2(s)P_{xx}(t-s)ds - \delta_2 \varphi_{tx} = 0. \end{cases}$$

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