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Department of Civil Engineering

Handout

Strength of Materials

Theory and Solved Problems

A teaching tool designed for second-year engineering students and preparatory classes in civil engineering and mechanical engineering (LMD).

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PREFACE

This handout is intended for students of the second year of engineering, in civil engineering, and even for LMD (civil engineering, mechanical engineering) students. It is a fundamental concept in science and engineering that studies the behavior of deformable materials. The resistance of materials (RDM) is a fundamental module that goes into the training of future civil engineering engineers; without the knowledge and application of these assumptions, it is impossible to design safe structures.

As a future engineer, the student must predict how the structure reacts to lateral forces (wind), inertial forces (earthquake), temperature changes (expansion)... etc. The primary purpose of this handout is to clarify for students the simple phenomena, deformations, and the main parameters that govern the mechanical behavior of solids in the reversible (elastic) perturbation zone., to design and analyze the structures in the various stages of development, under the assumptions of small perturbations (HPP) and the notion of elastic limit, which constitutes the acceptance end of the material to have reversible deformations.

This Strength of Materials handout, taught by the author to second-year civil engineering students, is the result of 10 years of experience in technical high schools and universities. The course is simplified to achieve the following objectives:

- Give students the principles of strength of materials in simple and understandable forms.
- Understand the main demands and their manifestations in the structures.
- Introduce students to the pre-sizing of elements (conditions of resistance, rigidity, and the phenomenon of stress concentration).

This handout is divided into eight chapters (07) which deal with the main solicitations and their influences on the structures.

Finally, I hope this work will help students broaden their knowledge and serve as a support for preparations for classes and tutorials.

Dr. Hocine Ziani

What you plant today you will harvest tomorrow

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Semestre	Intitulé de la matière		Coefficient	Crédits	Code
	Résistance des matériaux I		3	6	
VHH	Cours	Travaux dirigés	Travaux pratiques		
67h30	1h30	3h00	-		

prendre les notions de base de la résistance des matériaux, les buts et les hypothèses de la RDM, la notion des efforts internes, caractéristiques géométriques des sections, la loi de comportement des matériaux, notion de contraintes admissibles et le dimensionnement des pièces sous sollicitations simples.

Contenu de la matière

Chapitre 1: Introduction et généralités (2 Semaines)

Buts et hypothèses de la résistance des matériaux, Différents types de chargements, Liaisons (appuis, encastremets, rotules), Principe Général d'équilibre - Équations d'équilibres, Méthode des sections – Notion des efforts internes : Effort normal N, Effort tranchant T, Moment fléchissant M, Définitions, conventions de signes et unités.

Chapitre 2: Caractéristiques géométriques des sections droites (2 Semaines)

Centre de gravité, Moments statiques, Moments d'inertie d'une section droite, Transformation des moments d'inertie. Axes principaux centraux, moments d'inertie principaux.

Chapitre 3: Traction simple et compression simple (3 Semaines)

Définitions, Efforts normaux de traction et de compression, Contrainte normale, Déformation élastique, Loi de Hooke, Module de Young, Diagramme contrainte-déformation, Condition de résistance et notion de contrainte admissible.

Chapitre 4: Flexion simple (4 Semaines)

Définitions et hypothèses, Effort tranchants, Moments fléchissant, Relation différentielle entre la charge, Effort tranchant et Moment fléchissant. Diagramme des efforts tranchants et moments fléchissant, Contraintes en flexion simple, Notion de l'axe neutre et dimensionnement. Déformée d'une poutre soumise à la flexion simple (notion de la flèche), Calcul de la contrainte tangentielle.

Chapitre 5: Cisaillement (2 Semaines)

Définitions, Cisaillement simple, Cisaillement pur, Contrainte de cisaillement, Déformation élastique en cisaillement, Condition de résistance au cisaillement.

Chapitre 6: Torsion (2 Semaines)

Définitions, Contrainte tangentielle ou de glissement, Déformation élastique en torsion, Condition de résistance à la torsion.

Mode d'évaluation: Contrôle Continu : 40%; Examen: 60%.

Références bibliographiques: 1. F. Beer, Mécanique à l'usage des ingénieurs – statique, Mc Graw-Hill, 1981. 2. G. Pissarenko et al, Aide-mémoire de résistance des matériaux.

3. I. Miropolioubov et coll, "Problèmes de résistance des matériaux", Editions de Moscou.

4. L. Aleinik & J. Durler, "Résistance des matériaux", Ed. Spes, Dunod. 5. M. Kerguignas & G.

Caignaert, "Résistance des matériaux", Ed. Dunod Université. 6. P. Stepine, Résistance des

matériaux, Editions MIR ; Moscou, 1986. 7. S. Timoshenko, Résistance des matériaux, Dunod, 1986.

8. William et Nash, Résistance des matériaux, cours et problème, série Schaum, 1983.

Introduction and Generalities

Introduction

Strength of materials (SOM) is a branch of deformable solid mechanics that deals with their static behavior (such as drive shafts, buildings, engineering structures, dams, etc.) under all types of external loading in order to

- Size or check the parts, structures or elements under the condition of safely resisting the stresses caused by external loads applied and at the best cost (optimization of shapes, dimensions, materials...).
- Evaluate the deformations in order to verify the deformability limit of the elements.

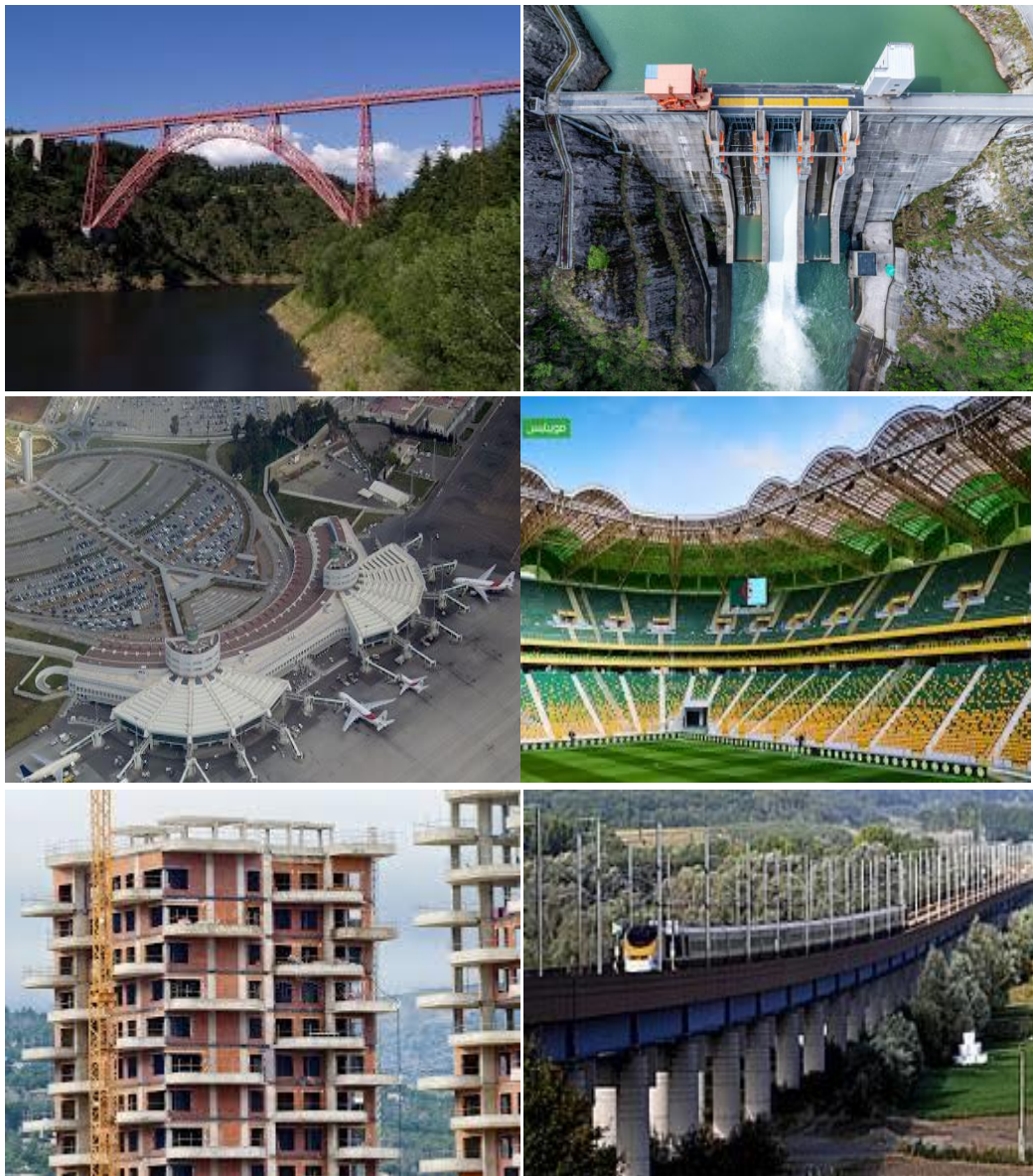


Fig1.1 Some images of the structures, in concrete, timber frame and mixed materials

The purpose of strength of materials is to evaluate the internal stability of a solid under multiple stresses, that is, to determine the stresses and strains within, and even the displacements of, the neutral axis of the structures generated (machines in mechanical engineering, buildings in civil engineering, etc.). This course material is intended for second-year engineering students (under the Bachelor's, Master's, and Doctorate) in Civil Engineering. It comprises six chapters, the content of which conforms to the curriculum for the second year of Civil Engineering as adopted by the National Pedagogical Committee for the Science and Technology sector.

1.1 General information

The strength of materials evaluates the following criteria:

- **Resistance:** the ability of a material or structure (bridges, buildings, machinery) to withstand external forces (forces or moments) without ruining or excessive permanent deformations
- **Rigidity (Modulus of elasticity):** is the ability of a work (material or structure) to withstand elastic deformations under external loads (no excessive deformation);
- **Stability:** is the ability of a structure or material to remain in balance or retain its original shape under loads, without major changes (irreversible deformations).

1.2 Assumptions of strength of materials

For the laws of strength of materials to be applied, the material must meet certain conditions (assumptions). Otherwise, finite element calculations, based on the theory of elasticity and plasticity, can be performed beyond these limits.

- **Homogeneous:** the mechanical properties are the same at all points on the material's surface (counter-example: concrete).
- **Isotropic:** the mechanical properties are independent of the direction considered (counter-example: wood, fiberglass).
- **Continuity:** The particles are so tightly packed that they can be considered a continuous whole.
- **Shapes:** The solids studied can be approximated as beams (length $\gg \gg$ width and height).
- **Small Perturbation Hypothesis (SPH):** The displacements and deformations of the beam are infinitesimally small compared to the dimensions of the cross-section (Fig. 1.2).

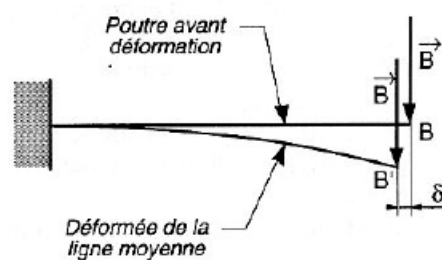


Fig 1.2 Small perturbation principle

• **Navier and Bernoulli hypothesis:** the straight, flat, and perpendicular sections remain flat and perpendicular after deformation. If we know the deformation of the neutral fiber, we can therefore deduce the displacement of any point of the beam. In the following, we will therefore only represent the neutral fiber to represent a beam (Fig 1.3).

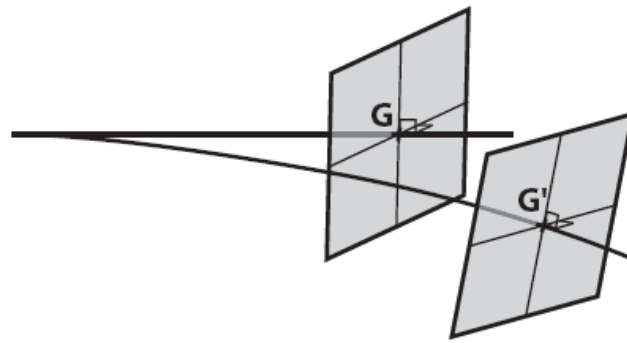


Fig 1.3 Navier- Bernoulli principle

- *Barré-Saint Venant hypothesis*: the stresses and deformations in a section far from the points of application of the external loads applied to the beam depend only on the resultants of the external forces and not on the precise local distribution of these (Fig 1.4).

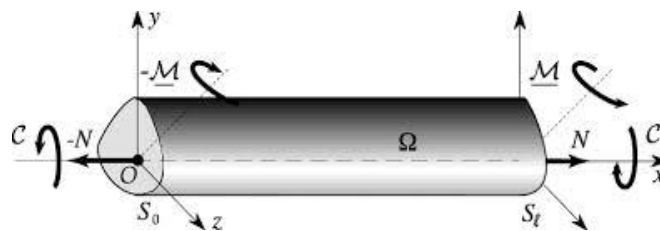


Fig 1.4 Barré-Saint Venant principle

Note: the materials that meet these assumptions are mainly metals, alloys and plastics.

1.3 Classification of solids in Strength of Materials

The classification of solids is primarily linked to their geometry, that is, the ratio between their three dimensions (length, width, and thickness). Three (03) main families are distinguished:

1.3.1 One-dimensional solids: in which one dimension (length L) is clearly dominant in relation to other dimensions of the cross section (width b and height h), (rails, posts, beam and bar).

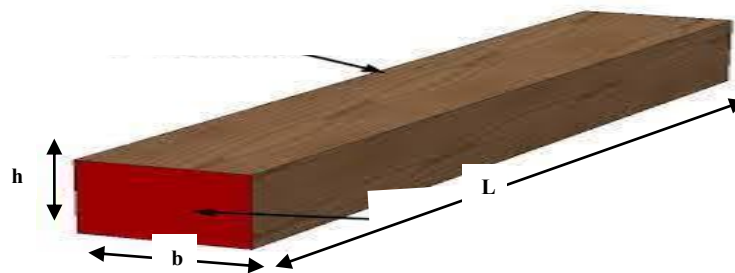


Fig 1.5 Beam element

Beams: structural elements elongated horizontally or inclined along their longest dimension L , designed to support vertical loads and transmit them to columns or load-bearing walls.

Their cross-section (Σ) has a center of gravity G that varies gradually, and the neutral axis describes a curve (deformation) after loading (Fig. 1.5).

- The beam has a large length L compared to its transverse dimensions (b , h). The beam possesses a plane of symmetry.

- Formed by the superposition of identical fibers that create lines on the cross-sections called fibers and the median line called the neutral fiber (neutral axis) (Fig. 1.6).
- The cross-sections of the beams studied have a plane of symmetry and are loaded in this plane.

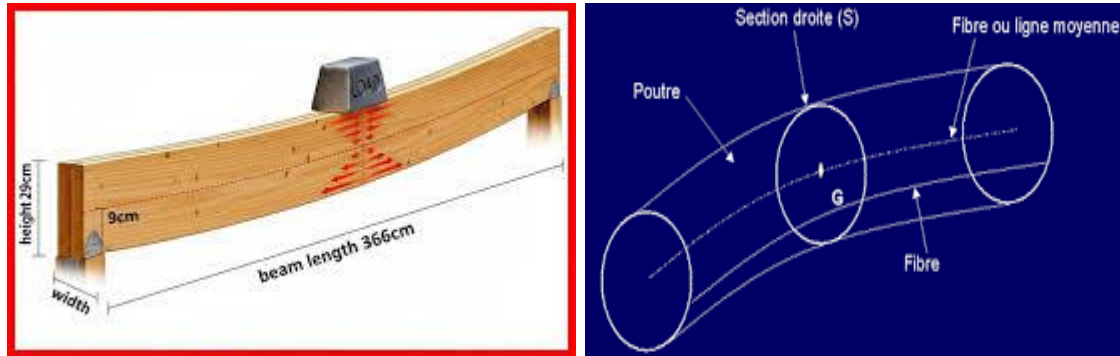


Fig 1.6 Deformation beam (notion of fiber)

1.3.2 Two-dimensional solids (plates and shells): are mainly modelled as surface elements, where the thickness is small compared to other dimensions (plates, shells). The average plane will be the plane (O, x, y) , the transverse displacement will be according to z . It is assumed that the hypothesis of small displacements is verified.

Plate: Planar structures subjected to loads perpendicular to their plane (bending) (Fig. 1.7a).

Shells: Thin structures with a curved surface, transmitting loads mainly by membrane forces (traction/compression in the plane) (Fig. 1.7b).

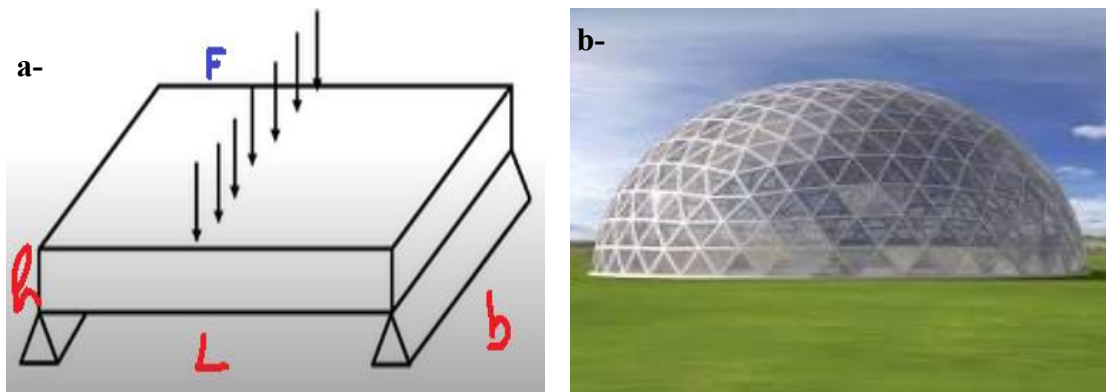


Fig 1.7 Plate element

1.3.3 Massifs (or three-dimensional elements): volumes whose three dimensions are of the same order of magnitude. 3D models (the space (x, y, z)), commonly used in numerical simulation, calculation of structures (finite elements). Basic geometric shapes include the tetrahedron, hexahedron and pentahedron (prism) (Fig 1.8).

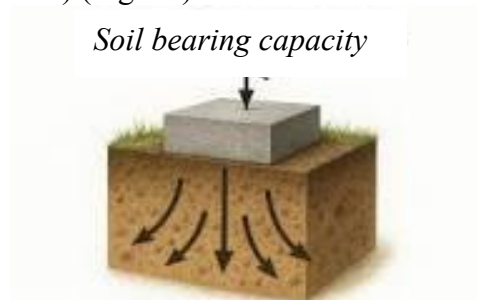


Fig 1.8 Load-bearing soil mass

1.4 Different Types of Loads and Stresses

1.4.1 Types of Loads (External Loads): This is the action of a force on a supporting surface (beam). Loads can be classified in different ways. There are two types of mechanical actions (External):

- Surface contact forces (distributed in two directions): pressure, temperature effect, deformation of a surface under load (N/m^2 ...). Examples: roof loads, floor loads, wind loads, or snow loads (Fig. 1.9).

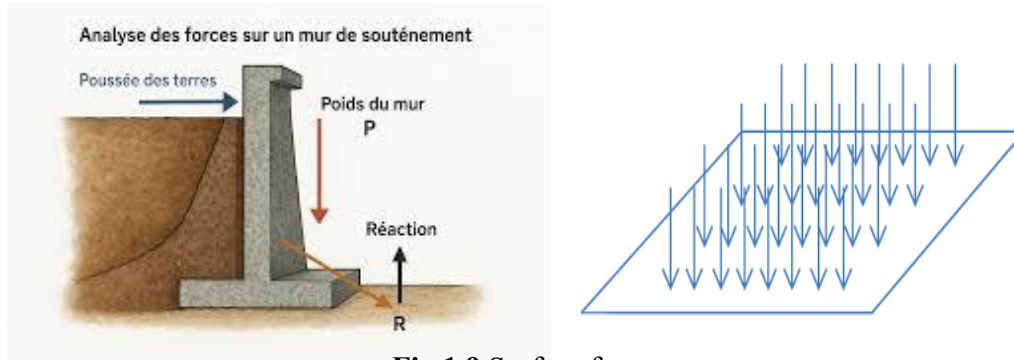


Fig 1.9 Surface forces

◆ *Forces volumique ou massique* (ρ) répartie sur trois directions : champ de pesanteur, force électro- magnétique, charge électrique,... (N/m^3 , KN/m^3) (Fig 1.10).

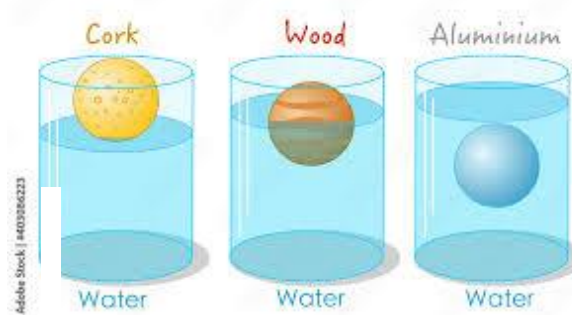


Fig 1.10 Surface force

These forces can be concentrated, distributed (N/ml) or a moment (N.m) which tends to rotate a surface (tightening a bolt) (Fig 1.10).

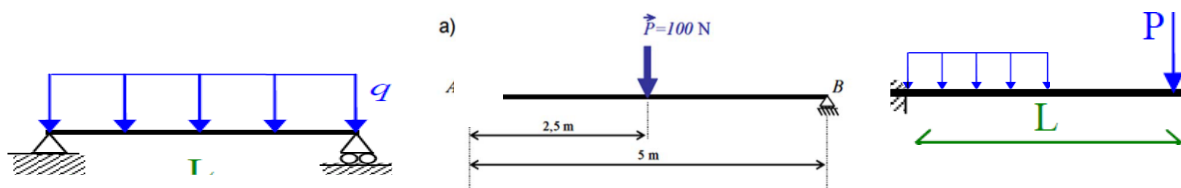
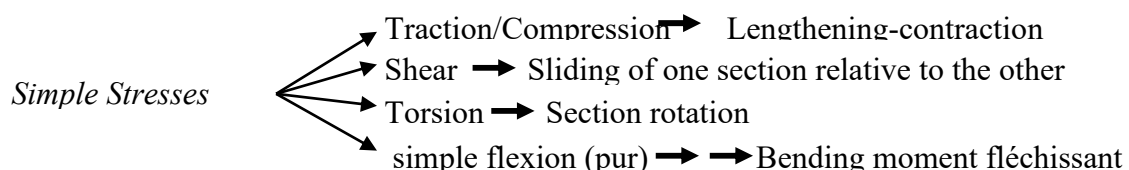


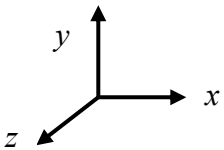
Fig. 1.11 Load Types

1.4.2 Stresses (Internal Forces): Under the action of external loads, internal forces appear in the loaded section (normal force, shear force, bending moment, torsional moment). They are generally classified into two categories: simple stresses (a single internal force) and combined stresses (a combination of several forces).



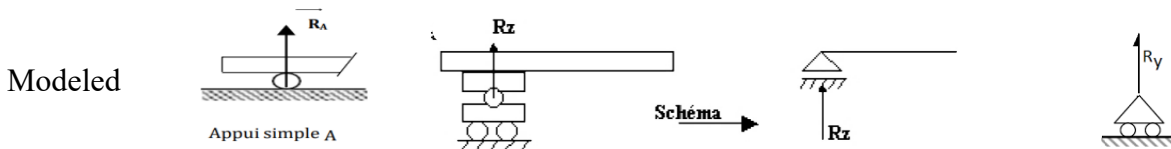
1.5 General Equilibrium Principle-Equilibrium Equations

A solid (S) under the action of an external force system $\{\vec{F}_{Ext}\}$ is in equilibrium if and only

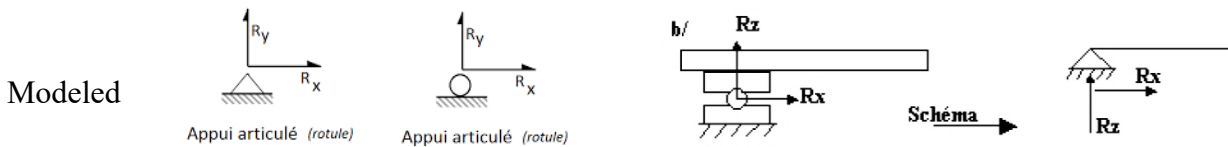
$$\{\vec{F}_{Ext}\} = \vec{0} \Rightarrow \begin{cases} \sum \vec{F}_{/x} = \vec{0}; & \sum \vec{M}_{/x} = \vec{0} \\ \sum \vec{F}_{/y} = \vec{0}; & \sum \vec{M}_{/y} = \vec{0} \\ \sum \vec{F}_{/z} = \vec{0}; & \sum \vec{M}_{/z} = \vec{0} \end{cases} \quad (1.1)$$


1.5.1 Types of connections: There are three (3) types of supports (connections):

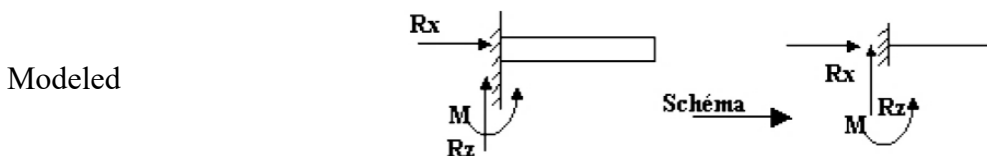
- **Simple support (sliding or roller) without friction:** A support that restricts only one degree of freedom.



- **Double support (joint, ball joint):** blocks two degrees of freedom.



- **Embedding:** blocks three degrees of freedom.



Summary of linkage modeling

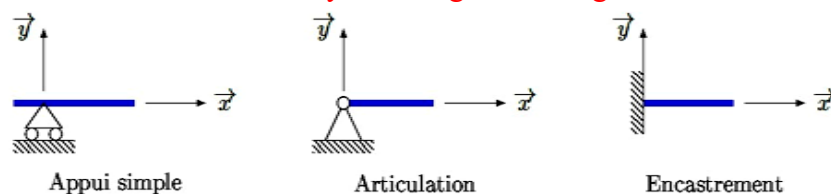


Fig. 1.12 Types of links and their modelling

1.5.2 Local model/ Global model: the representation and study of mechanical actions are done according to two (02) models, the local model and the global model.

- **The local model:** makes it possible to study the action and its effect at any point in the area where it is exerted. Example a force distributed over a surface or in a volume, the effect of hydrostatic pressure on the walls of the dam at a given depth. (Fig. 1.13a)

- **The global model:** replaces the infinity of local forces with a single resulting force and a moment, says said Torsor of Mechanical Action. This model eliminates the local effect of the action but makes its use practical for the study of equilibrium or dynamics (Fig. 1.13b).

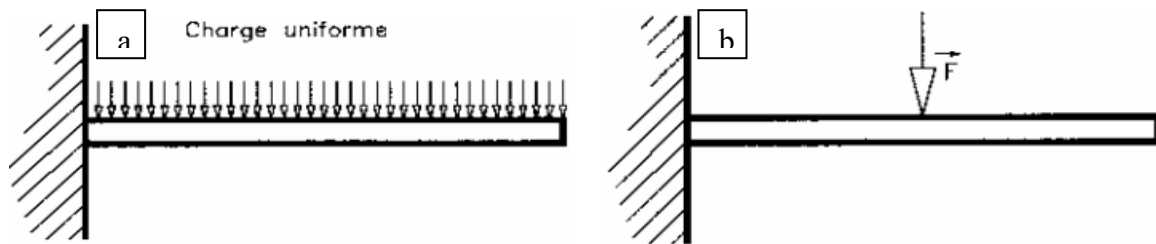


Fig. 1.13 local and global model

Table 1 Liaisons and their unknowns

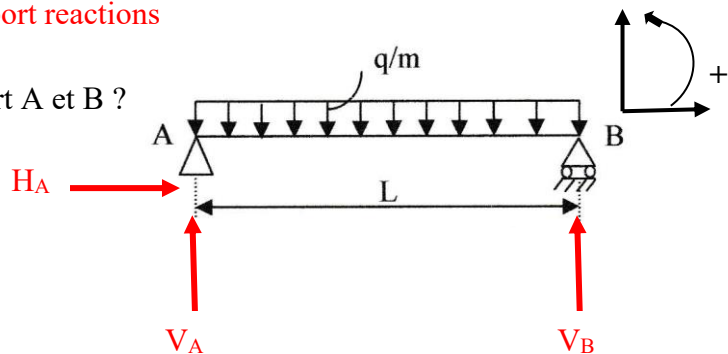
Link type	Modeled (schéma)	unknowns
Simple support (sliding or roller) (1 DDL)		1 unknown V_A ?
Double support (joint, ball joint) (2DDL)		2 unknowns H_A et V_A ?
Embedding (3DDL)		3 unknowns H_A , V_A et M_A ?

1.5.3 Examples of calculation of support reactions

1- Consider the beam bellow.

Determine the reactions at the support A et B ?

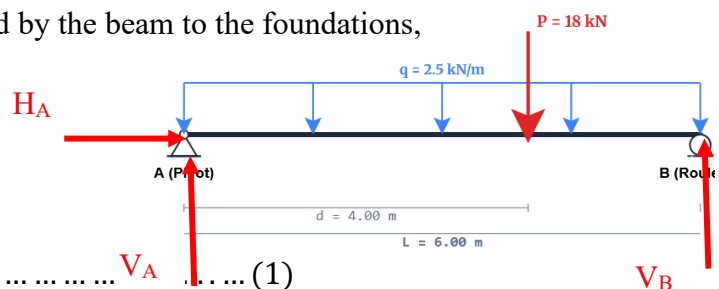
$$\begin{cases} \sum \vec{F}_{/x} = \vec{0} \Rightarrow H_A = 0 \\ \sum \vec{F}_{/y} = \vec{0} \Rightarrow V_A + V_B - q \cdot l = 0 \\ \sum \vec{M}/_A = \vec{0} \Rightarrow -V_B \cdot l + \frac{q \cdot l^2}{2} = 0 \\ \Rightarrow V_A = V_B = \frac{q \cdot l}{2} \end{cases}$$



2- Déterminer the efforts transmittés by the beam to the foundations,

$$\begin{cases} \sum \vec{F}_{/x} = \vec{0} \\ \sum \vec{F}_{/y} = \vec{0} \\ \sum \vec{M}/_A = \vec{0} \end{cases}$$

$$\begin{cases} \sum \vec{F}_{/x} = \vec{0} \Rightarrow H_A = 0 \dots \dots \dots V_A \dots \dots (1) \\ \sum \vec{F}_{/y} = \vec{0} \Rightarrow V_A + V_B - q \cdot L - P = 0 = 25.6 + 18 = 168 \text{ KN} \dots \dots (2) \\ \sum \vec{M}/_A = \vec{0} \Rightarrow -V_B \cdot L + \frac{q \cdot L^2}{2} + P \cdot d = 0 \dots \dots (3) \end{cases}$$



From (3) we will have: $V_B = \frac{q \cdot L}{2} + \frac{P \cdot d}{L}$

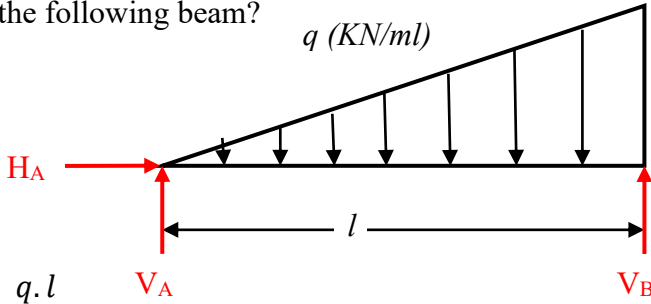
Remplacant V_B with its value in (2), we will have:

$$V_A + V_B = 168 \Rightarrow V_A = 168 - V_B$$

$$\Rightarrow V_A = 168 - \left(\frac{q \cdot L}{2} + \frac{P \cdot d}{L} \right)$$

So
$$\begin{cases} V_B = \frac{25.6}{2} + \frac{18.4}{6} = 87 \text{ KN} \\ V_A = q \cdot L + P - \frac{q \cdot L}{2} - \frac{P \cdot d}{L} = 25.6 + 18 - \left(\frac{25.6}{2} + \frac{18.4}{6} \right) = 81 \text{ KN} \end{cases}$$

3- Determine the support reactions of the following beam?

$$\begin{cases} \sum \vec{F}_x = \vec{0} \Rightarrow H_A = 0 \\ \sum \vec{F}_y = \vec{0} \\ \sum \vec{M}_A = \vec{0} \end{cases}$$


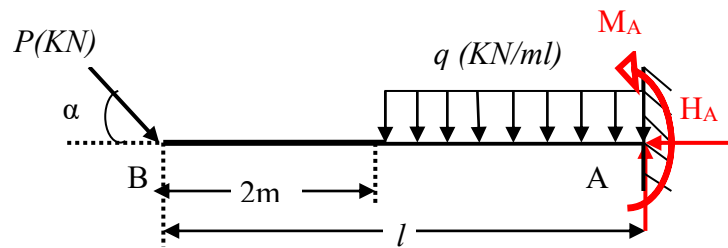
$$\sum \vec{F}_y = \vec{0} \Rightarrow V_A + V_B = q \cdot \frac{l}{2}$$

$$\sum \vec{M}_B = \vec{0} \Rightarrow V_A \cdot l - \frac{q \cdot l}{2} \left(\frac{l}{3} \right) = 0 \Rightarrow V_A = \frac{q \cdot l}{6}$$

Or, $V_A + V_B = q \cdot \frac{l}{2} \Rightarrow V_B = \frac{q \cdot l}{3}$

4- Determine the support reactions of the following beam??

$$\begin{cases} \sum \vec{F}_x = \vec{0} \dots \dots \dots (1) \\ \sum \vec{F}_y = \vec{0} \dots \dots \dots (2) \\ \sum \vec{M}_A = \vec{0} \dots \dots \dots (3) \end{cases}$$



$$(1) \dots \sum \vec{F}_x = \vec{0} \Rightarrow H_A + P \cdot \cos \alpha = 0$$

$$\Rightarrow H_A = -P \cdot \cos \alpha$$

$$(2) \dots \sum \vec{F}_y = \vec{0} \Rightarrow V_A - P \cdot \sin \alpha - q(2) = 0$$

$$\Rightarrow V_A = P \cdot \sin \alpha + 2q$$

$$(3) \dots \sum \vec{M}_A = \vec{0} \Rightarrow -P \cdot \sin \alpha \cdot l - \frac{q \cdot (l-2)^2}{2} - M_A = 0$$

$$\Rightarrow M_A = -P \cdot \sin \alpha \cdot l - \frac{q \cdot (l-2)^2}{2}$$

Si, $\alpha = 30^\circ$, $P = 15 \text{ KN}$, $q = 10 \text{ KN/ml}$, $l = 3.5 \text{ m}$

$$H_A = P \cdot \cos \alpha = 15 \cdot \cos 30 = 13 \text{ KN}$$

$$V_A = 15 \cdot \sin 30 + 10(3.5 - 2) = 22.5 \text{ KN}$$

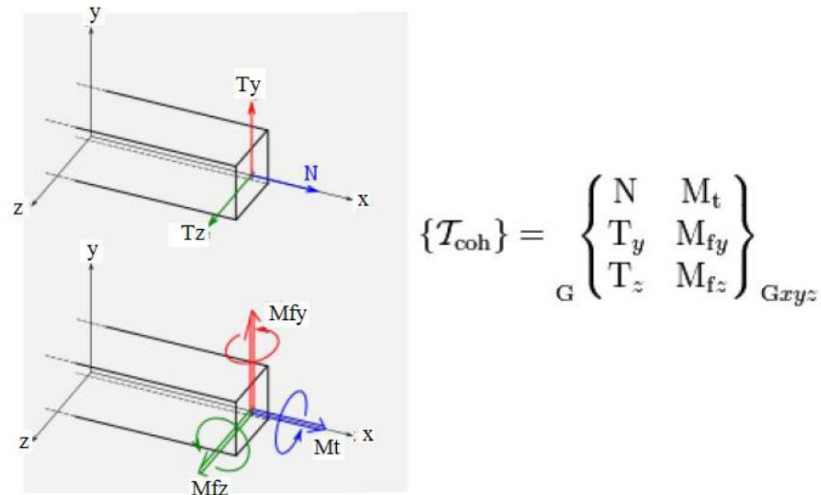
$$M_A = -15 \cdot 3.5 \cdot \sin 30 - \frac{10 \cdot (3.5 - 2)^2}{2}$$

$$M_A = -26.25 - 11.25 = -37.5 \text{ KN.m}$$

1.5.4 Method of Sections: The method of sections in Strength of Materials involves making imaginary (fictitious) cross-sections where the external force(s) are applied in order to determine the internal forces (the internal forces): normal force N_x , shear force T_y , and bending moment M_{fz} at any point of a structure (beam). It consists of dividing the beam into two parts after each external force, isolating one of the parts, and replacing the other part with internal (unknown) forces. Then, static

equilibrium is applied ($\sum \vec{F} = \vec{0}$ and $\sum \vec{M} = \vec{0}$). Si $\vec{R}$ is the resultant of the internal forces, it is more convenient to decompose M and R along the ox, oy, and oz axes.

If the external forces do not lie in the same plane (spatial problem-reference frame oxyz) and in the general case, the cross section can be the site of six internal forces: the normal force N, the shear force Tx, the shear force Ty and three moments Mx, My and Mz, the first two being bending moments and the third Mz acting in the plane of the section called torsional moment because it is produced by the torsion of the bar.



$$\vec{R} = \begin{cases} \vec{N} \rightarrow \text{Normal effort relative to the } x - \text{axis} \\ \vec{T}_y \rightarrow \text{Shear effort relative to the } y - \text{axis} \\ \vec{T}_z \rightarrow \text{Shear effort relative to the } z - \text{axis} \\ \vec{M}_{t/x} \rightarrow \text{Moment of torsion relative to the } x - \text{axis} \\ \vec{M}_{f/y,z} \rightarrow \text{Bending moments along the axes } y \text{ and } z \end{cases}$$

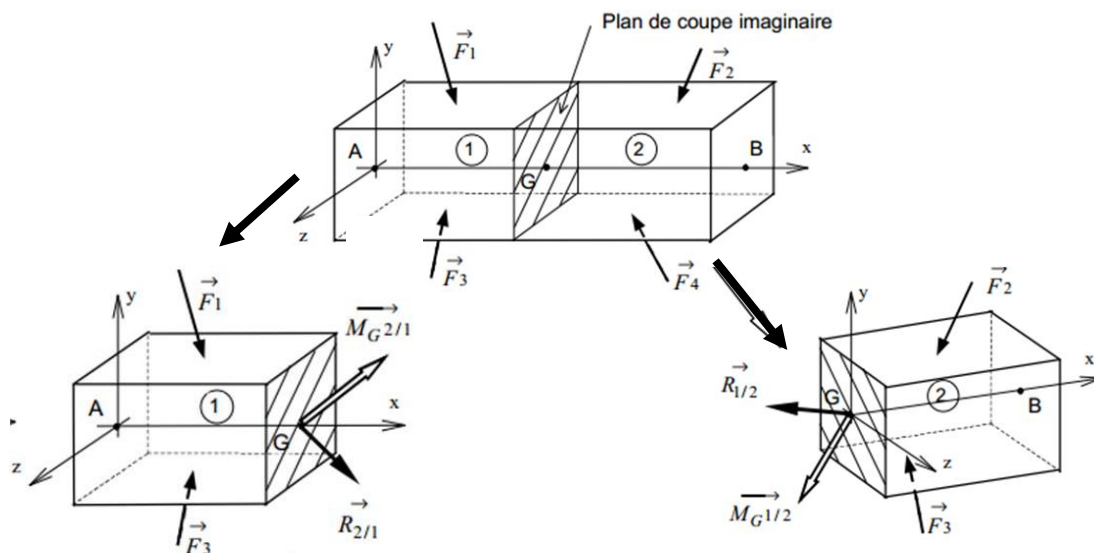


Fig. 1.14 Fictitious section on a beam and representation of internal forces

Normal force (Nx): is the sum of the horizontal projections of all forces resulting from the decomposition of the resultant F. The normal force causes a longitudinal deformation of the element. Nx is considered positive if it is a traction and negative otherwise.

Application

Find the normal forces in A and B in the beam below

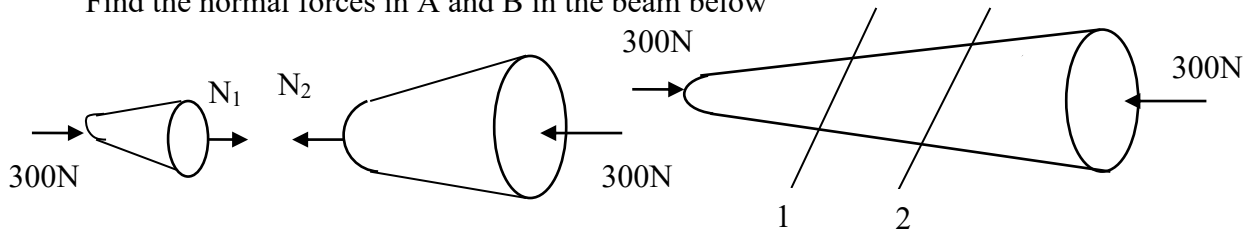


Fig1.15 Determination of N_x on a beam of variable shape.

Solution

First, let's isolate the left section for the cut in 1 and replace the ignored part with N_1 , assuming it is in tension (by convention):

$$\sum \vec{F}_{/x} = \vec{0} \Rightarrow 300 + N_1 = 0$$

$$\Rightarrow N_1 = -300N \rightarrow \text{Compression effort}$$

let's isolate the left section for the cut in 2 and replace the ignored part with N_2 , assuming it is in tension:

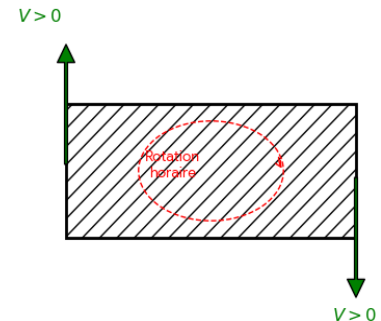
$$\sum \vec{F}_{/x} = \vec{0} \Rightarrow -300 - N_2 = 0$$

$$\Rightarrow N_2 = -300N \rightarrow \text{Compression effort}$$

The change in shape does not affect the nature of the internal force N , which always remains a compressive force. The normal force N depends only on the applied external loads.

- **Shear effort (T)**

The transverse forces T_z and T_y (V_z and V_y) are the sums of the projections of all internal forces within the section onto its principal central axes. These forces tend to cause one part of the section to slide vertically relative to the other, resulting in a shear effect. The shear force T is considered positive when it tends to rotate a beam element clockwise.

**Application1**

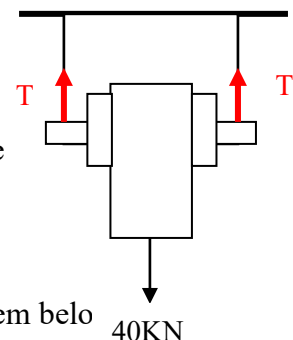
Find the shear force in the pin of the following system.

Solution

$$\sum \vec{F}_{/y} = \vec{0} \Rightarrow 40 + 2T = 0$$

$$\Rightarrow T = 20KN$$

The pin, which supports the rod and its load, is subjected to a shear force of 20 kN..

**Application 2**

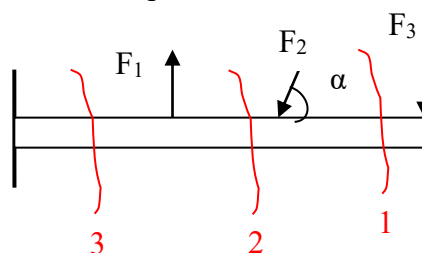
Find the shear force and the normal force at points 1, 2 and 3 of the system below

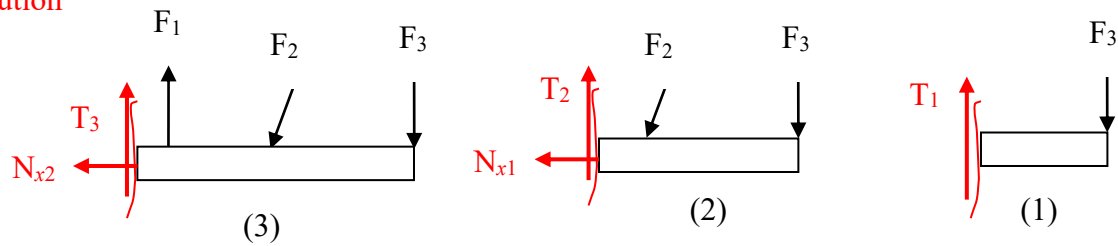
$$F_1 = 100N$$

$$F_2 = 260N$$

$$F_3 = 500N$$

$$\alpha = 30^\circ$$



Solution

$$(1) \dots \sum \vec{F}_{/y} = \vec{0} \Rightarrow -F_3 + T_1 = 0$$

$$\Rightarrow T_1 = F_3 = 100\text{N}$$

$N_x = 0 \rightarrow$ No component in the x direction

$$(2) \dots \sum \vec{F}_{/y} = \vec{0} \Rightarrow -F_3 - F_2 \sin \alpha + T_2 = 0$$

$$\Rightarrow T_2 = F_3 + F_2 \sin \alpha = 100 + 260 \cdot \sin 30^\circ$$

$$\Rightarrow T_2 = 230\text{KN}$$

$$\sum \vec{F}_{/x} = \vec{0} \Rightarrow +F_2 \cos \alpha + N_{x1} = 0$$

$$\Rightarrow N_{x1} = -F_2 \cos \alpha = -260 \cdot \cos 30^\circ = -199,18\text{KN}$$

$$\Rightarrow N_{x1} = -199,18\text{KN} \rightarrow \text{compressive force.}$$

$$(3) \dots \sum \vec{F}_{/y} = \vec{0} \Rightarrow -F_3 - F_2 \sin \alpha + F_1 + T_3 = 0$$

$$\Rightarrow T_3 = F_3 + F_2 \sin \alpha - F_1 = 100 + 260 \cdot \sin 30^\circ - 500$$

$$\Rightarrow T_3 = -270\text{KN}$$

$$\sum \vec{F}_{/x} = \vec{0} \Rightarrow N_{x2} = N_{x1} = -199,18\text{KN}$$

• **Bending moment (M_f)**

The bending moment M_f reflects the tendency of a force to "bend" the beam. The moment is considered positive ($M_f > 0$), if the beam curves upward (a smile shape).

Negative Moment ($M_f < 0$), the beam curves downward ("sad" shape).

The internal forces on the beam that can be modeled by a torsor called *cohesion torsor* $\{T_{coh} 1/2\}_{(G,x,y,z)}$ and whose reduction elements at point G are:

-the resultant $\vec{R}$ and - the moment $\vec{M}$

The nature of a stress is determined by the content of the cohesion wrench (table 2).

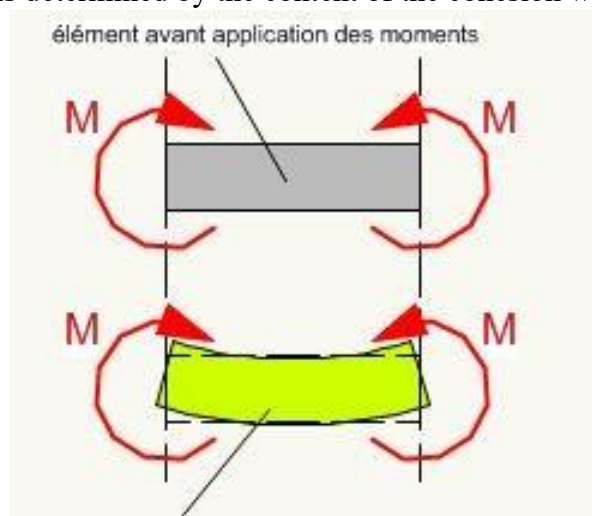
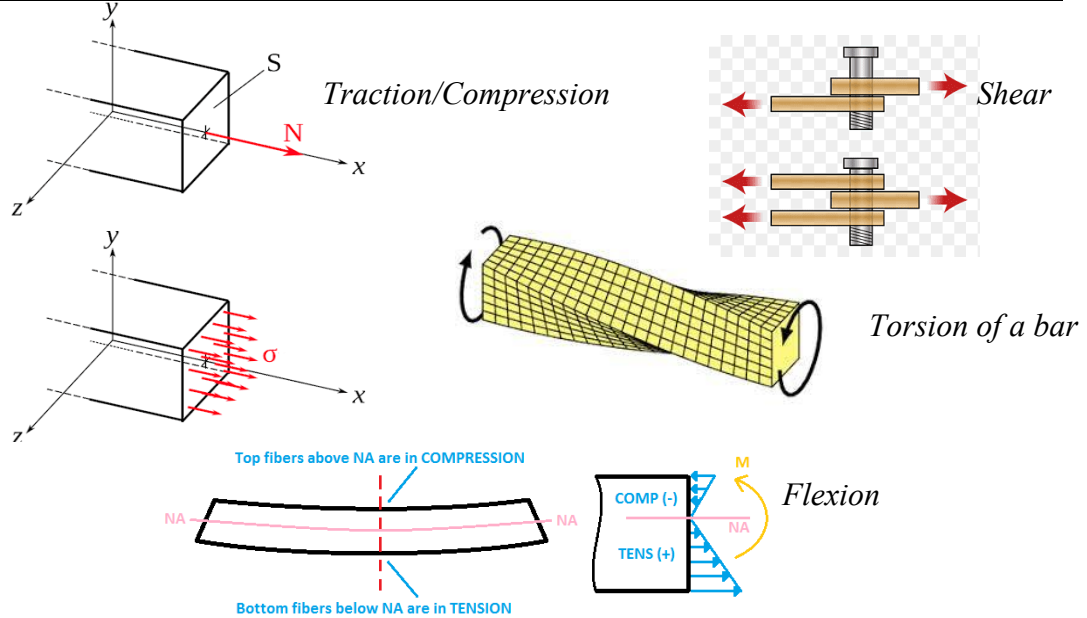


Fig. 1.16 Conventional sign of M_f

Table 2 illustration of different requests

<i>Simple traction</i>	<i>Simple compression</i>	<i>Simple shear</i>
$\mathcal{T}_{coh\ G} \begin{Bmatrix} N_x > 0 \\ 0 \\ 0 \end{Bmatrix} \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$	$\mathcal{T}_{coh\ G} \begin{Bmatrix} N_x < 0 \\ 0 \\ 0 \end{Bmatrix} \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$	$\mathcal{T}_{coh\ G} \begin{Bmatrix} 0 \\ T_y \\ 0 \end{Bmatrix} \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$
<i>Simple flexion</i>	<i>Compound flexion</i>	<i>Pure torsion</i>
$\mathcal{T}_{coh\ G} \begin{Bmatrix} 0 \\ T_y \\ 0 \end{Bmatrix} \begin{vmatrix} 0 \\ 0 \\ M_{fz} \end{vmatrix}$	$\mathcal{T}_{coh\ G} \begin{Bmatrix} N_x \\ T_y \\ 0 \end{Bmatrix} \begin{vmatrix} 0 \\ 0 \\ M_{fz} \end{vmatrix}$	$\mathcal{T}_{coh\ G} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \begin{vmatrix} T_z \\ 0 \\ 0 \end{vmatrix}$

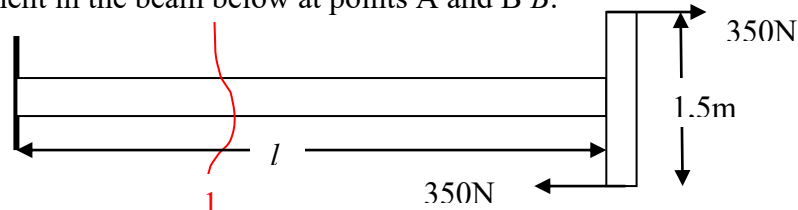
**Fig. 1.17** Nature de Sollicitations sur une section

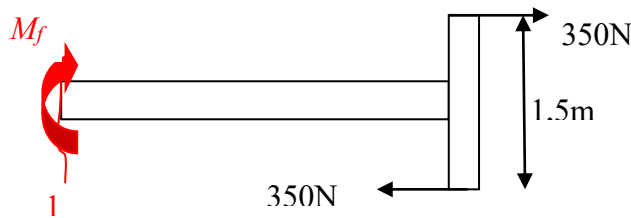
The cohesion torsor $\mathcal{T}_{coh\ G}$, sera will be simplified as follows in the case of planar solicitations.

$$\mathcal{T}_{coh\ G} \begin{Bmatrix} N(x) \\ T(x) \\ M_f(x) \end{Bmatrix} \quad (1.2)$$

Fig. 1.18 *Simplification* of the cohesion torsor in the planar state**Application 1**

Find the bending moment in the beam below at points A and B.



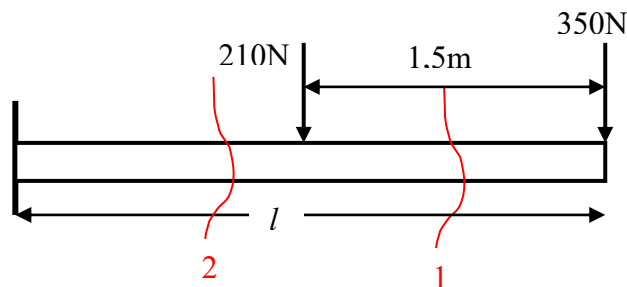
Solution

$$\sum \vec{M}_{/1} = \vec{0} \Rightarrow M_f + 350 \times 0,75 + 350 \times 0,75 = 0$$

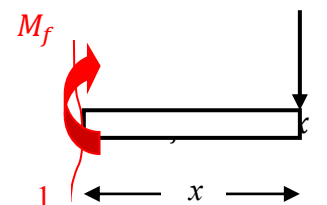
$$\Rightarrow M_f = -350 \times 1,5 = -525 \text{ N.m}$$

So the direction of M_f est is the opposite of clockwise..

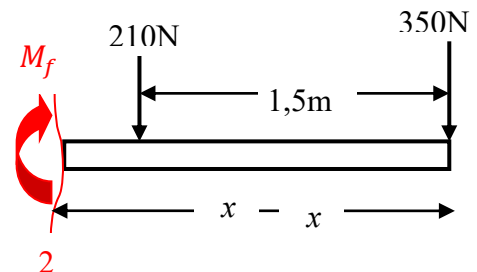
$$M_f = 525 \text{ N.m}$$

Application 2**Solution**

$$(1) \dots \dots \dots \sum \vec{M}_{/1} = \vec{0} \Rightarrow M_f + 350 \cdot x = 0$$



$$(2) \dots \dots \dots \sum \vec{M}_{/2} = \vec{0} \Rightarrow M_f + 350 \cdot x - 210(x - 1.5) = 0$$



We note that the bending moment will depend on the position of the fictitious cut (x) in the beam.

1.6 Constraints

In civil engineering, stress (often denoted σ or τ) represents the intensity of internal forces within a material, in addition to the effects caused by the application of external loads. From a physical perspective, it is the distribution of a force over a surface. It is the same concept as pressure, but manifests itself within a solid (concrete, steel, wood). However, in mechanics, stress is defined mathematically by the formula:

$$\text{Constraint} = \frac{\text{Force (N)}}{\text{Surface (mm}^2\text{)}}$$

The standard unit is the Pascal (Pa), but the Megapascal (MPa) is most commonly used in construction.

$$1 \text{ MPa} = 1 \text{ N/mm}^2$$

Or it is the limit of the resultant of the internal forces Δp , by an infinitely small section Δs (tends towards zero):

$$\lim_{\Delta s \rightarrow 0} \frac{\Delta p}{\Delta s} = \sigma$$

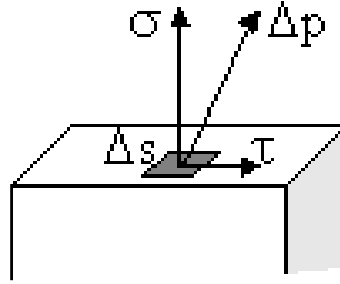


Fig 1.19 Decomposition of the resultant stress

La Decomposition of the resultant stress allows the appearance of two types of stresses: normal stress, often tensile or compressive, designated by σ , and tangential or shear stress, denoted by τ .

1.6.1 Normal Constraint (σ): Normal stress σ (sigma) is defined as the uniform distribution of the normal force in a section perpendicular to the mean line of the beam.

$$\sigma = \frac{N}{A} \left[\frac{N}{m^2} \right]$$

Where,

σ : Normal stress applied to the surface (MPa).

N : Normal force applied to the section (N). $\Rightarrow \begin{cases} N > 0 \Rightarrow \sigma > 0 \rightarrow \text{traction} \\ N < 0 \Rightarrow \sigma < 0 \rightarrow \text{compression} \end{cases}$

A : Section of the surface considered (m^2).

1.6.2 Tangential constraint (τ): The tangential (shear) stress τ (tau), represents the intensity of internal forces acting parallel to the surface of a material. Tangential stresses tend to cause the layers of materials to slide, whereas normal stress causes shrinkages or stretches depending on the nature of external force applied.

NB: The resistance of a material or structure is the ability to support the applied load(s) without breaking. It is therefore expressed as stress in MPa.

The unit of measurement for pressure in the International System (SI) is Pascal (Pa).

Pascal is the same as newton per square meter (N/m^2).

$$\frac{1N}{m^2} = 1Pa$$

$$1KPa = 1 \times 10^3 Pa$$

$$1MPa = 1 \times 10^6 Pa$$

$$1GPa = 1 \times 10^9 Pa$$

The expression for stress, pure or simple shear, is:

$$\tau = \frac{T}{A} \left[\frac{N}{m^2} \right] \text{ Ou } [MPa]$$

T : Shear force (N).

A : Area of the sheared section (mm^2).

Under the influence of τ , the material undergoes an angular deformation, denoted γ (gamma). In the elastic range, it follows Hooke's law for shear:

$$\tau = G \gamma$$

G : is the transverse (Coulomb) modulus of elasticity; it represents the material's stiffness against sliding.

1.7 Poutre cantilever et poutre gerber

- *Cantilever beam*: also known as a cantilever beam, is a horizontal structural element fixed at one end by a rigid support (fixed end), leaving the other end free. These beams have spans ranging from 1 to 5 m in residential buildings, and exceeding 100 m in civil engineering projects, depending heavily on the material and the load (Fig. 1.20).



Fig 1.20 Quebec Bridge (Canada) (Cantilever Bridge)

- *Gerber beam*: is a continuous beam on several supports (naturally hyperstatic) into which joints (ball joints) are inserted at strategic points to make it isostatic. It is generally designed to span lengths of 15 to 40 meters in buildings, and can reach more than 150 meters for bridges (Fig 1.21).

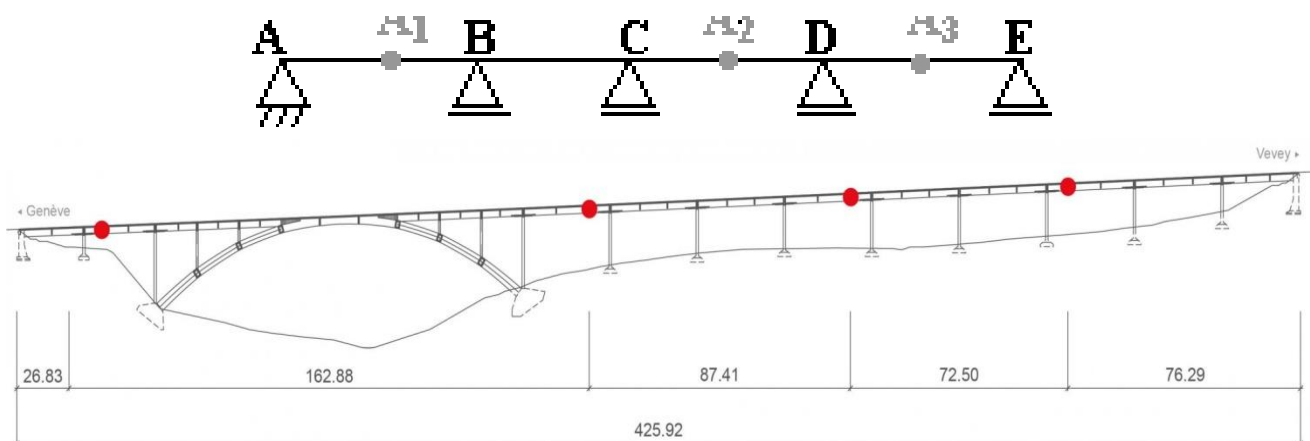


Fig 1.22 Reinforcement of the bridge over the Flon river in Lausanne (Switzerland)

● Articulation Gerber

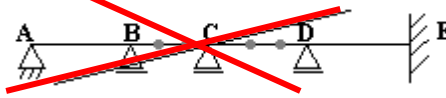
- *Conditions for placing joints*: Four (04) conditions must be verified,

1. It is forbidden to have two joints in the first and/or last span, except in the case of fixed joints.

2. It is forbidden to have more than two joints in the intermediate sections.



3. It is forbidden to have two joints in two intermediate and successive sections.



4. It is forbidden to have a joint in the first or last bay if the adjacent bay has two joints.

SERIES OF EXERCICES

Exercise 1

Study the equilibrium of this beam and calculate the reactions at the supports?

Solution

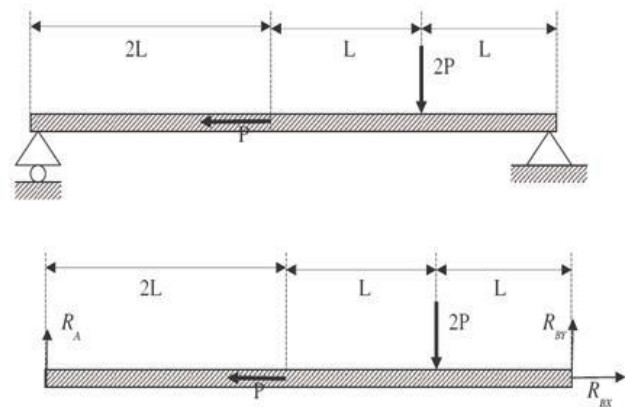
Equilibrium condition (equation 1.1)

$$\begin{cases} \sum \vec{F}_X = \vec{0} & (1) \\ \sum \vec{F}_Y = \vec{0} & (2) \\ \sum \vec{M}_{A,B,\dots} = \vec{0} & (3) \end{cases}$$

Equation (1) give : $R_A - 2P + R_{BY} = 0 \dots \dots \dots (2')$

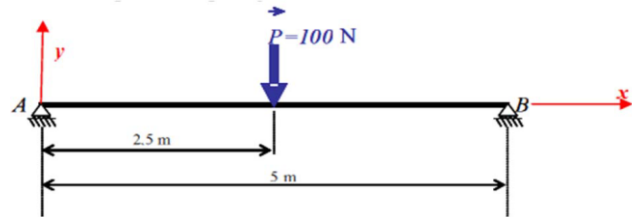
Equation (3) give : $\sum \vec{M}_{f/z} = \vec{0} \Rightarrow -R_A(4L) + 2P(L) = R_A = \frac{P}{2}$
 $\Rightarrow R_A = \frac{P}{2}$

for (2') we have ; $R_{BY} = 3\frac{P}{2}$



Exercise 2

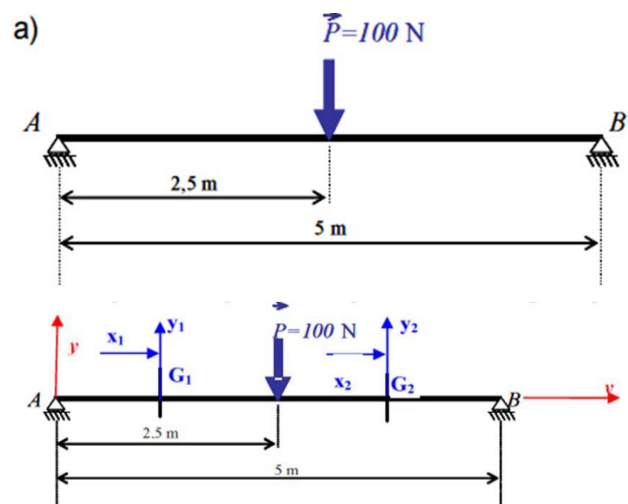
1. Identify the connections present on the beam.
2. Establish the general coordinate system.
3. Determine the forces transmitted by the connections.
4. Determine the number of sections required for the structural analysis of the beam.
5. Establish the cuts and the coordinate systems associated with each cut.
6. Determine the internal force wrench in each section.
7. Draw the stress diagrams.



Solution

- 1- Point connections.
- 2- 2
- 3 and 4- The action P is centered on the beam, therefore the reactions at A and B are equal to P/2, i.e., 50 N, and directed along the y-axis.
- 5- Two sections are necessary for the structural analysis of the beam: AC and CB.

◆ Section AC - Algebraic Method:



◆ $T_{y1} = -\text{left reaction } T_{y1} = -Y_A = -50 \text{ N}$; $M_{fz1} = 50x$:

- **Vector Method:**

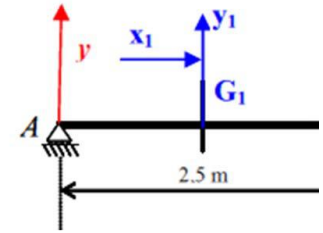
$$T_{coh G} = \begin{pmatrix} N & M_t \\ T_y & M_{fy} \\ T_x & M_{fz} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -Y_A & 0 \\ 0 & Y_A \cdot x \end{pmatrix}$$

◆ **Section CB - Algebraic Method:**

$T_{y2} = \text{right reaction}$

$T_{y1} = Y_B = 50$

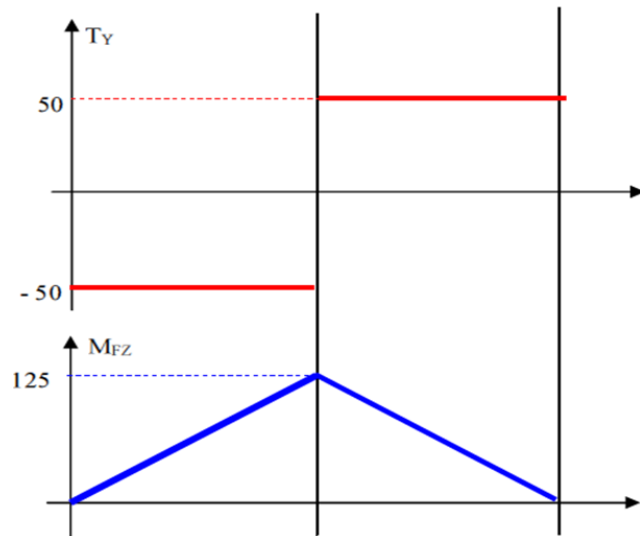
$M_{fz1} = 50(5-x)$



- **Vector Method:** $T_{coh G} = \begin{pmatrix} 0 & 0 \\ Y_B & 0 \\ 0 & 50(5-x) \end{pmatrix}$

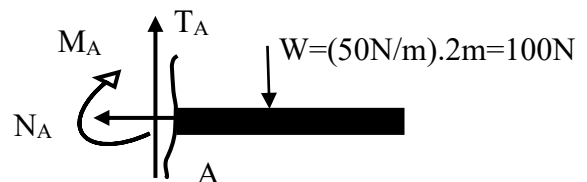
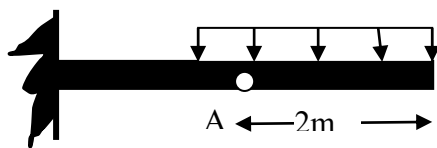
6- Stress diagrams

X (m)	T_y (N)	M_{fz} (N.m)
0	-50	0
2.5	-50	125
5	50	0



Exercise 3

Calculate the forces and stresses in the beam in the figure below at point A. The beam is square with sides of 10 cm.



Solution

Cut in A :

$$A = 0,1 \times 0,1 = 0,01 \text{ m}^2$$

$$W = 50 \times 2 = 100 \text{ N à 1 m du bord} : \begin{cases} \sum \vec{F}_{/X} = \vec{0} \Rightarrow -N_A = 0 \\ \sum \vec{F}_{/Y} = \vec{0} \Rightarrow T_A - 100 \Rightarrow T_A = 100 \text{ N} \\ \sum \vec{M}_{/A} = \vec{0} \Rightarrow -M_A - (100 \times 1) = 0 \Rightarrow M_A = -100 \text{ N.m} \end{cases}$$

Donc : $\sigma_A = \frac{N_A}{A} = 0$ et $\tau_A = \frac{T_A}{A} = -\frac{100}{0,01} \text{ N/m}^2 = 1 \text{ KPa}$

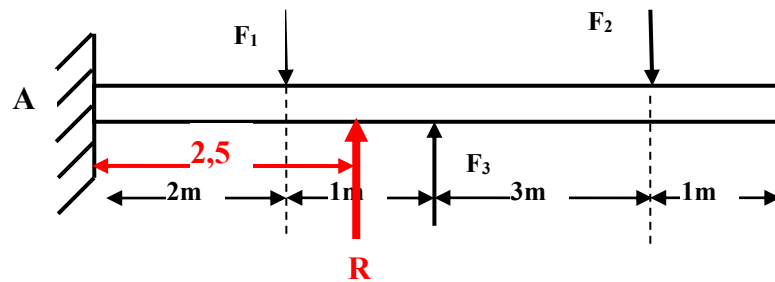
Exercise 4

Determine the characteristics of the resultant $\vec{R}$, (magnitude, direction, sense, and point of application) of the forces $\vec{F}_1$, $\vec{F}_2$ et $\vec{F}_3$ acting on the beam (figure below).

$$F_1 = 200\text{N}$$

$$F_2 = 100\text{N}$$

$$F_3 = 500\text{N}$$

**Solution**

1-Module

The resultant of forces : $\vec{R} = \sum_{i=1}^3 \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \dots \dots \dots (1)$

$$\vec{R} = \vec{R}_x + \vec{R}_y \begin{cases} \vec{R}_x = 0 \text{ (vertical forces)} \\ \vec{R}_y = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = -200 - 100 + 500 = 200\text{N} \end{cases}$$

The module is : $\vec{R} = \vec{R}_y = 200\text{N}$

- Direction: vertical,
- Meaning: according to the greatest force (in the sense of $y > 0$),
- Point of application: $M_{R/A} = \vec{R} \cdot d = M(\vec{F}_1) + M(\vec{F}_2) + M(\vec{F}_3)$

$$200 \cdot d = -2 \cdot 200 - 6 \cdot 100 + 3 \cdot 500$$

$$\Rightarrow d = \frac{-400 - 600 + 1500}{200} = 2,5\text{m to point A}$$

Exercise 5

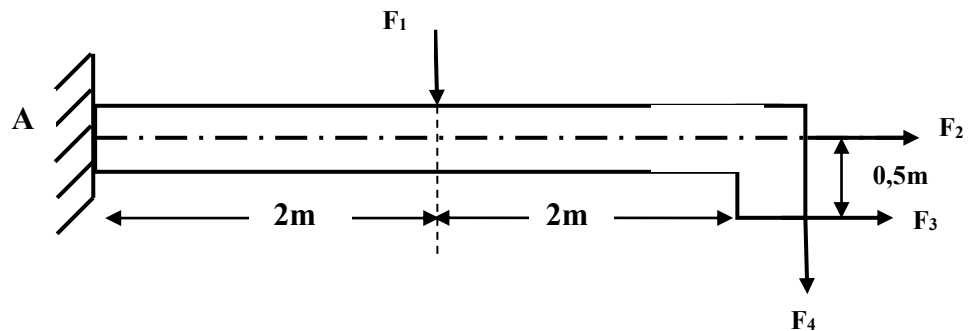
Find the resultant and the resulting moment at point O of the forces $\vec{F}_1$ exerted on the beam

$$F_1 = 50\text{N}$$

$$F_2 = 60\text{N}$$

$$F_3 = 20\text{N}$$

$$F_4 = 40\text{N}$$

**Solution**

The resultante of forces $\vec{R}$ is:

$$: \vec{R} = \sum_{i=1}^4 \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 \dots \dots \dots (1)$$

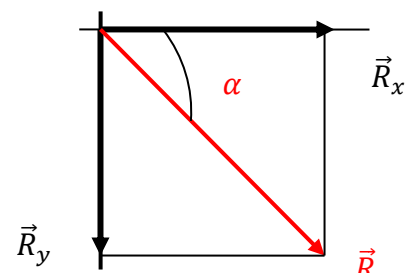
$$\vec{R} = \vec{R}_x + \vec{R}_y \begin{cases} \vec{R}_x = \vec{F}_2 + \vec{F}_3 = 60 + 20 = 80\text{N} \\ \vec{R}_y = \vec{F}_1 + \vec{F}_4 = -50 - 40 = -90\text{N} \end{cases}$$

• Module: $\vec{R} = \sqrt{(\vec{R}_x)^2 + (\vec{R}_y)^2} = \sqrt{(80)^2 + (90)^2} = 120,42\text{N}$

Direction:

$$\text{tg } \alpha = \frac{R_y}{R_x} = -\frac{90}{80} = -1,125$$

$$\Rightarrow \alpha = 48,37^\circ$$



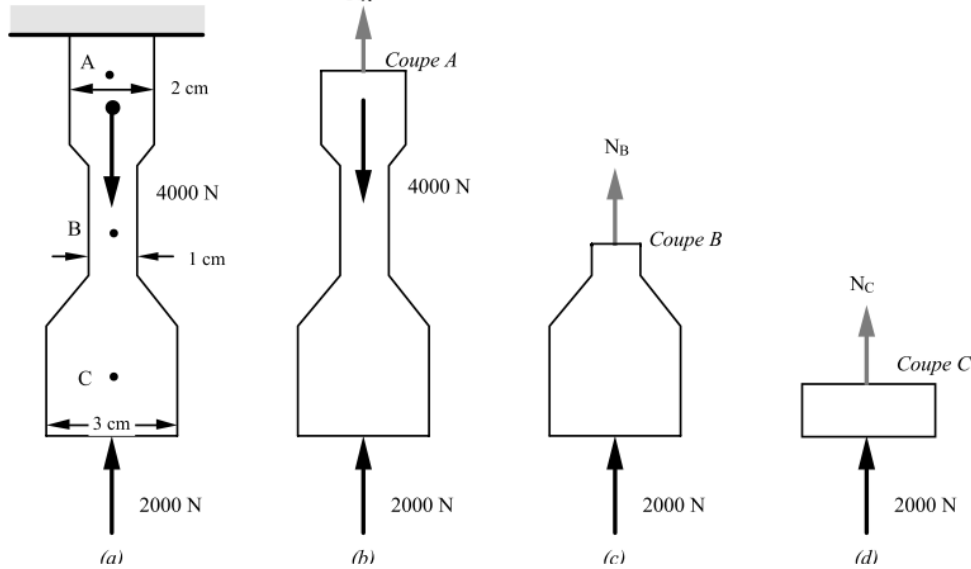
- Resulting moment is:

$$M/A = M(\vec{F}_1) + M(\vec{F}_2) + M(\vec{F}_3) + M(\vec{F}_4)$$

$$= -50.2 - 40.4 + 2.05 = -250 \text{ N.m}$$

Exercise 6

Calculate the normal stress at points A, B, and C of the rod below.



Solution

$$\sum \vec{F}_{/y} = \vec{0} \Rightarrow -4000 + 2000 + N_A = 0$$

$$\Rightarrow N_A = 4000 - 2000 = 2000 \text{ N}$$

- Cut A

$$\sigma_1 = \frac{N_A}{A_1}$$

$$A_1 = \frac{\pi d_1^2}{4} = \frac{\pi \cdot 2^2}{4} = 3.14 \text{ cm}^2 = 3.14 \cdot 10^{-4} \text{ m}^2$$

$$\Rightarrow \sigma = \frac{2000}{3.14 \cdot 10^{-4}} = 336,22 \cdot \frac{10^4 \text{ N}}{\text{m}^2} = 3,37 \text{ MPa} \rightarrow \text{traction}$$

- Cut B

$$\sum \vec{F}_{/y} = \vec{0} \Rightarrow N_B + 2000 = 0$$

$$\Rightarrow N_B = -2000 \text{ N}$$

$$A_2 = \frac{\pi d_2^2}{4} = \frac{\pi \cdot (0.01)^2}{4} = 7,85 \cdot 10^{-5} \text{ m}^2$$

$$\sigma_2 = \frac{-2000}{7,85 \cdot 10^{-5}} = -254,77 \cdot \frac{10^5 \text{ N}}{\text{m}^2} = -2,55 \cdot 10^7 \text{ Pa} = -25,5 \text{ MPa}$$

$$\sigma_2 = -25,5 \text{ MPa} \rightarrow \text{compression}$$

- Cut A

$$\sum \vec{F}_{/y} = \vec{0} \Rightarrow N_C + 2000 = 0$$

$$\Rightarrow N_C = -2000 \text{ N}$$

$$A_3 = \frac{\pi d_3^2}{4} = \frac{\pi \cdot (0.03)^2}{4} = 7,1 \cdot 10^{-4} \text{ m}^2$$

$$\sigma_3 = \frac{-2000}{7,1 \cdot 10^{-4}} = -281,69 \cdot \frac{10^4 \text{ N}}{\text{m}^2} = -28,2 \cdot 10^7 \text{ Pa} = -2,82 \text{ MPa}$$

$$\sigma_3 = -2,82 \text{ MPa} \rightarrow \text{compression}$$

Chapter 2

Geometric Characteristics of Straight Sections

2.1 Introduction

For solicitations, simple traction or compression, only the data of the area of the cross-section is necessary to perform calculations. For all other stresses, the shape and dimensions of the cross-section of the beam play a preponderant role in the behavior under the various torsional or bending stresses.

2.2 Geometric characteristics

2.2.1 Static moment of a straight section

The static moment S_x of a cross-section with respect to any x -axis (Fig. 2.1), is the geometric characteristic mathematically defined by the integral:

$$S_x = \int \int_A y \cdot dA = A \cdot y_G \quad \text{et} \quad S_y = \int \int_A x \cdot dA = A \cdot x_G$$

Where A is the total section and $y_G(x_G)$ is the distance between the center of gravity G of the figure and the x -axis (y).

The unit of static moment (SI) is the length at power three (m^3).

We can also use submultiples (cm^3 , mm^3).

$$1000 \text{ mm}^2 = 10 \times 10^2 \text{ mm}^2 = 10 \text{ cm}^2$$

$$1000 \text{ mm}^2 = 1000 \times 10^{-6} \text{ m}^2$$

Generally, these static moments are linked to reference axes: X , Y , ...

If surface A can be decomposed into elementary surfaces (Fig. 2.2), the static moment is the algebraic sum of these surfaces multiplied by their distance from the reference axis.

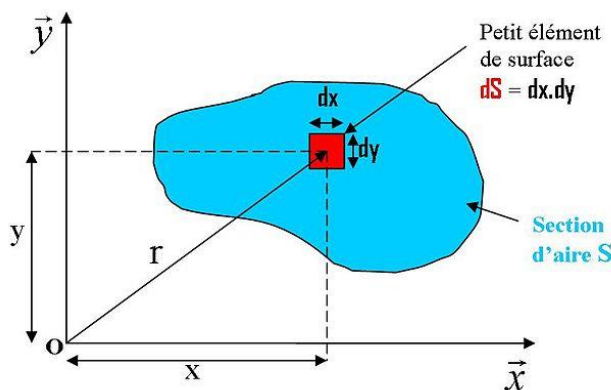


Fig. 2.1 S_x of an elementary surface

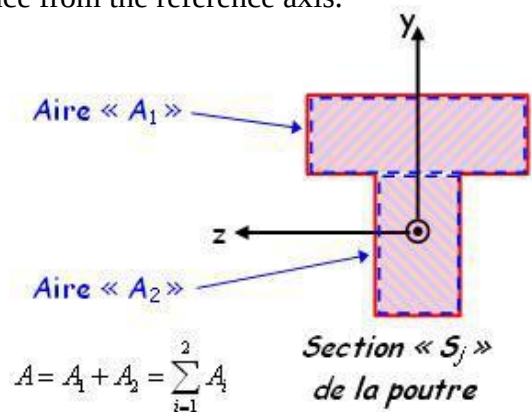


Fig. 2.2 S_x of a decomposable surface

$$A = \sum_{i=1}^n A_i = A_1 + A_2 + A_3 + \dots$$

$$S_x = A_1 \cdot y_1 + A_2 \cdot y_2 + A_3 \cdot y_3 + \dots = \sum_{i=1}^n A_i \cdot y_i$$

$$S_y = A_1 \cdot x_1 + A_2 \cdot x_2 + A_3 \cdot x_3 + \dots = \sum_{i=1}^n A_i \cdot x_i$$

$S_x \rightarrow$ Static moment with respect to x

$S_y \rightarrow$ static moment with respect to y

Note:

- 1-The static moment of a section S with respect to any axis passing through its centered is zero.
- 2-If another axis (Δ) parallel to and a distance d from the x-axis exists such that: $y_c = y + d$, the static moment with respect to the axis (Δ) is: $S_\Delta = S_x + A \cdot d$ Or $S_\Delta = S_x + A \cdot d$
- 3-If the x-axis passes through the censored of A, the static moment with respect to Δ is given by: $S_\Delta = A \cdot d$.

2.2.2 Center of Gravity

The distance y_G from the center of gravity of a plane section A to a line x is defined by the following relation:

$$y_G = \frac{S_x}{A}$$

Similarly, with respect to y

$$x_G = \frac{S_y}{A}$$

This relation also allows calculating the static moment of a section knowing the position of its center of gravity.

NB: These two formulas show that if the x and y axes pass through the center of gravity of any body, its static moment with respect to these axes is zero and the x and y axes are called central axes.

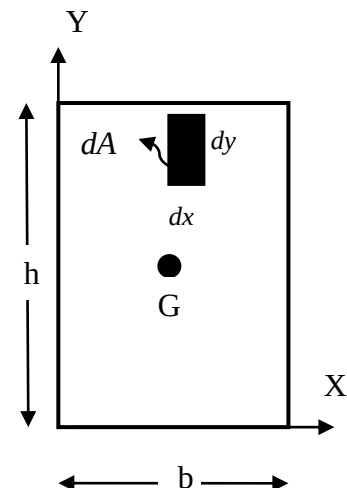
Example

Determine the center of gravity of the rectangular section of dimension b.h?

Solution

$$\begin{aligned} S_x &= \int \int_A y \cdot dA = \int \int_A y \cdot dx \cdot dy \\ &= \int_0^b dx \int_0^h y dy \\ &= [x]_0^b \int_0^h y dy = b \int_0^h y dy \\ &= b \left[\frac{y^2}{2} \right]_0^h = \frac{bh^2}{2} \\ S_x &= \frac{bh^2}{2} \end{aligned}$$

$$\begin{aligned} \text{Or, } S_x &= y_G \cdot A \Rightarrow y_G = \frac{S_x}{A} = \frac{bh^2}{2} \cdot \frac{1}{bh} \\ &\Rightarrow y_G = \frac{h}{2} \end{aligned}$$



Since the section remains constant, we can proceed as follows:

$$\begin{cases} x_G = \frac{S_y}{A} = \frac{bh \cdot \frac{b}{2}}{bh} = \frac{b}{2} \\ y_G = \frac{S_x}{A} = \frac{bh \cdot \frac{h}{2}}{bh} = \frac{h}{2} \end{cases}$$

In the case where the surface in question is decomposable into several parts of which the dimensions and their centre of gravity are easily known, the following general formulas can be used:

$$\begin{cases} x_G = \frac{\sum_i^n A_i \cdot x_i}{\sum_i^n A_i} = \frac{A_1 x_1 + A_2 x_2 + \dots}{A_1 + A_2 + \dots} = \frac{\text{sum of static moments/OY}}{\text{sum of surfaces}} \\ y_G = \frac{\sum_i^n A_i \cdot y_i}{\sum_i^n A_i} = \frac{A_1 y_1 + A_2 y_2 + \dots}{A_1 + A_2 + \dots} = \frac{\text{sum of static moments/OX}}{\text{sum of surfaces}} \end{cases}$$

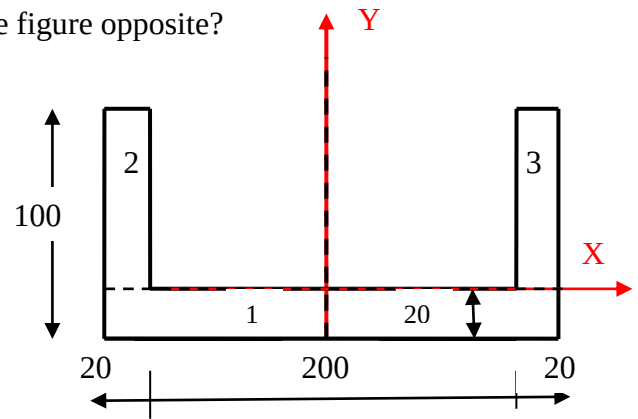
Example

Calculate the position of the center of gravity of the figure opposite?

Solution

The center of gravity G , is located on the axis of Y symmetry.

So let's take as a system of axes, the axis OX passing through the base of the figure and as OY is the axis of symmetry.



Let's break it down into simple surfaces.

A_1 the 'large' rectangle (240 x 20) with $y_{G1} = 10\text{mm}$

$A_2 = A_3$ the 'small' rectangle (80 x 20) with $y_{G2,3} = 60\text{mm}$

$$\begin{cases} x_G = \frac{A_1x_1 + A_2x_2 + A_3x_3}{A_1 + A_2 + A_3} = \frac{(240 \cdot 20)0 - (80 \cdot 20)110 + (80 \cdot 20)110}{(240 \cdot 20) + (80 \cdot 20) + (80 \cdot 20)} = 0 \\ y_G = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A_1 + A_2 + A_3} = \frac{(240 \cdot 20)10 + (80 \cdot 20)60 + (80 \cdot 20)60}{(240 \cdot 20) + (80 \cdot 20) + (80 \cdot 20)} = 30\text{mm} \end{cases}$$

We can deduce:

$$\begin{cases} S_y = 0 \\ S_x = 240000\text{mm}^3 \end{cases}$$

Here are some centers of gravity of some flat surfaces

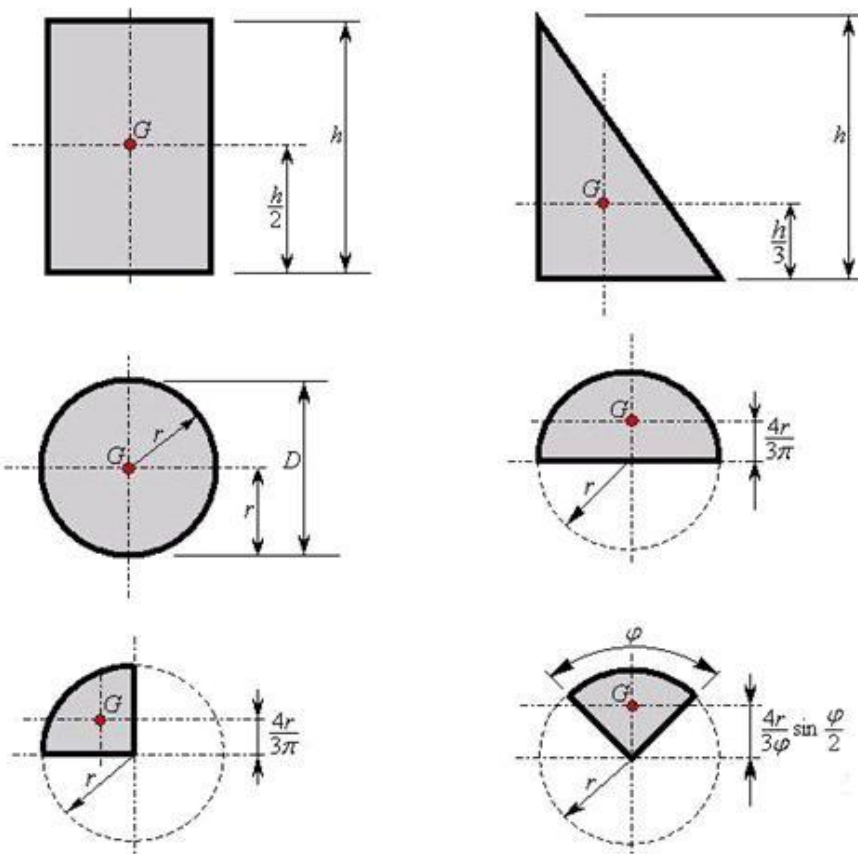


Fig.2.3 Center of gravity of some flat surfaces

Steps to determine the center of gravity of a surface

If the section has an axis of symmetry, the centre of gravity G is located on this axis.

Otherwise, we proceed to:

- Choose a reference system (O, x, y);
- Calculate the static moment S of the section with respect to this axis;
- Calculate the total area of the section;
- Use the static moments:

$$S_Y = X_G \cdot A, \quad S_X = Y_G \cdot A$$

2.3 Moment of inertia (quadratic moment)

The quadratic or inertia moments (axial inertia moment) of a surface with respect to the horizontal axis (o; x) and the vertical axis (o; y) are:

$$I_x = \int_A y^2 dA \quad I_y = \int_A x^2 dA$$

I_x , I_y Always positive, because y^2 , x^2 are positive.

Use to calculate the flexural strength of a beam (bending moment). The unit of moment of inertia is the unit of power length 4: in cm^4 or mm^4).

2.3.1 Moment of inertia/Product of inertia with respect to xy (xy or yx)

The moment of inertia of a body is the sum of the products of elementary masses of this body by the product of the square of their distance from a reference element R which can be a point, a line or a plane.

$$I_{xy} = \int_A xy dA = \int \int_A xy dx dy$$

-Where x and y are the distances between the area dA , the axes x and y.

-Measures symmetry. If the shape is symmetrical with respect to one of the axes x or y, I_{xy} is zero.

- I_{xy} can be positive, negative or null.

In summary, I_x describes rigidity with respect to bending, while I_y measures the asymmetry of the material distribution with respect to the two axes.

Note: the moment of inertia is often confused with the quadratic moment, widely used in material resistance (bending calculations), but it's not the same thing: the quadratic moment only takes into account the geometry, but not the mass.

2.3.2 Polar moment of inertia

According to the figure: $\rho^2 = x^2 + y^2$

By substituting this value of ρ^2 into the formula:

$$\begin{aligned} I_\rho &= \int_A \rho^2 dA = \int_A (x^2 + y^2) dA \\ &= \underbrace{\int_A x^2 dA}_{I_y} + \underbrace{\int_A y^2 dA}_{I_x} \\ I_\rho &= I_y + I_x \end{aligned}$$

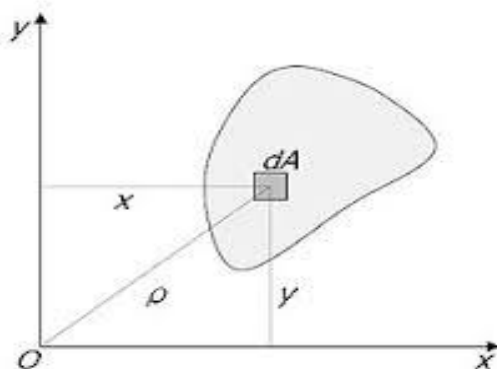


Fig.2.4 Polar moment of inertia

2.3.3 Huygens' theorem (parallel axes)

The moment of inertia of a flat surface A with respect to any axis (Δ) of its plane is equal to the sum (Fig.2.5):

- The moment of inertia of this surface relative to the $x'x'$ axis, parallel to the axis (Δ) and passing through its center of gravity G.
- Of the product of this surface by the square of the distance d of the two axes.

$$I_{\Delta} = I_{xx'/G} + d^2 \cdot dA$$

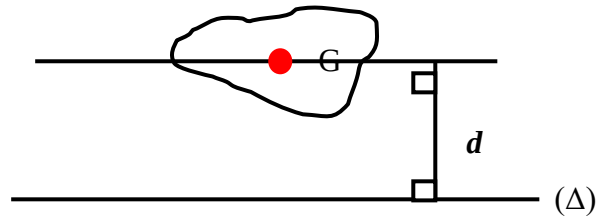


Fig.2.5 Representation of the Huygens theorem

2.3.4 Transformation of moments of inertia

Consider a section A; its moments of inertia in the xoy system: I_x , I_y , I_{xy} are known. We propose to calculate the moments of inertia of section A in the $x'oy'$ system by performing translations of the ox and oy axes according to figure.4.6.

$$I_{x'} = I_x + b^2 \cdot A$$

$$I_{y'} = I_y + a^2 \cdot A$$

$$I_{x'y'} = I_{xy} + ab \cdot A$$

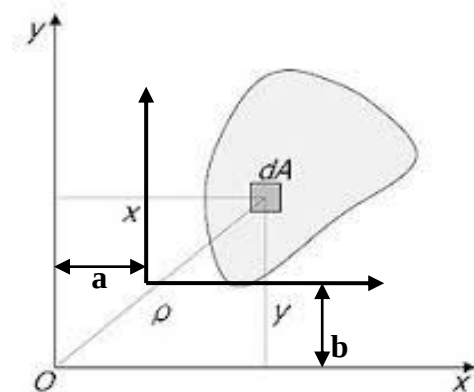


Fig. 2.6 Transformation of the moment of inertia

Note: The second moment of inertia (I_{xy}) and the polar moment of inertia (I_{xy}) are always positive.

- I_{xy} can be positive, negative, or zero (depending on the position of the axes). At each point of a plane section, there are two orthogonal axes to which the product of inertia is zero ($I_{xy} = 0$), the axes thus defined are called *principal axes*.

-Axes are principal when at least one of them is an axis of symmetry of the section.

Example: Case where one of the two axes of a section is an axis of symmetry

Consider the following section of symmetry axis Gy . Compare the inertia products of the 2 symmetrical dS sections 1 and 2 with respect to Gy .

$$dI_{xy}(1) = (-x)(-y) dS = x \cdot y \cdot dS$$

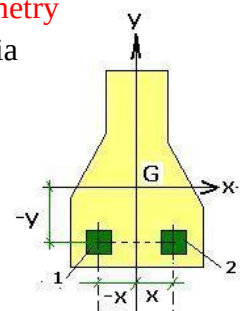
And

$$dI_{xy}(2) = (x)(-y) \cdot dS = -x \cdot y \cdot dS$$

And so

$$I_{xy}(1) + dI_{xy}(2) = 0$$

So the product of inertia of the whole section S is zero.



2.3.5 Moment of inertia of some simple sections

a-Rectangle with base b and height h (homogeneous material).

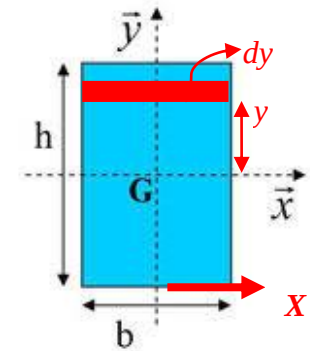
Let's calculate the moment of inertia of the rectangular section with respect to the x-axis passing through the center of gravity.

$$\begin{aligned} S_{y_G} &= \int \int_A x dA = \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} x dy \right] dx = \int_{-\frac{b}{2}}^{\frac{b}{2}} [xy]_{y=-\frac{h}{2}}^{y=\frac{h}{2}} dx \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} x h dx = \left[h \frac{x^2}{2} \right]_{-\frac{b}{2}}^{\frac{b}{2}} = \frac{h}{2} \left[\frac{b^2}{4} - \frac{b^2}{4} \right] = 0 \\ I_{y_G} &= \int \int_A x^2 dA = \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} x^2 dy \right] dx = \int_{-\frac{b}{2}}^{\frac{b}{2}} [x^2 y]_{y=-\frac{h}{2}}^{y=\frac{h}{2}} dx \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} x^2 h dx = \left[h \frac{x^3}{3} \right]_{-\frac{b}{2}}^{\frac{b}{2}} = \frac{h}{3} \left[\frac{b^3}{8} - \frac{b^3}{8} \right] = \frac{bh^3}{12} \end{aligned}$$

Similarly, $I_{x_G} = \frac{b}{3} \left[\frac{h^3}{8} - \frac{h^3}{8} \right] = \frac{hb^3}{12}$

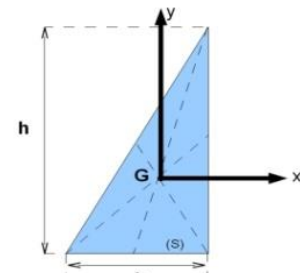
Calculation of the moment of inertia with respect to the X-axis

$$I_X = \int \int_A y^2 dA = \int_0^h \left[\int_0^b y^2 b dy \right] = b \frac{y^3}{3} \Big|_0^h = \frac{bh^3}{3}$$



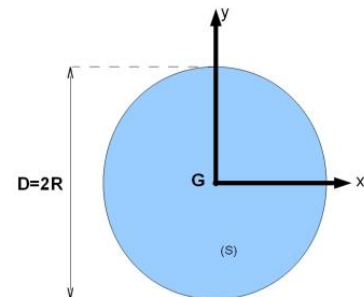
b- Right triangle with base b and height h (homogeneous material).

$$I_{Gx} = \frac{bh^3}{36} \quad I_{Gy} = \frac{hb^3}{36}$$



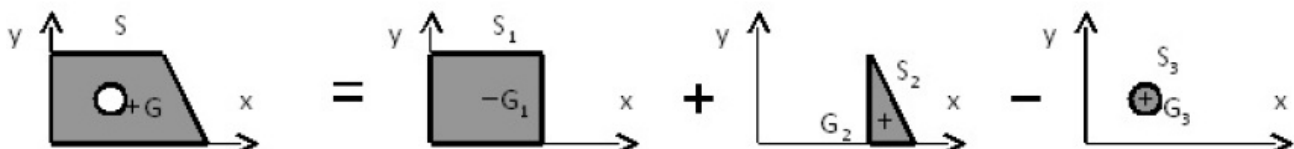
c-Disk with radius R and center G (homogeneous material).

$$I_{Gy} = I_{Gx} = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}$$



d- Perforated trapezoid (example of composite figures)

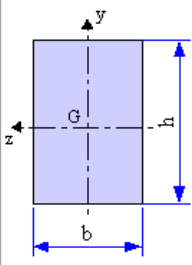
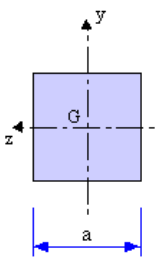
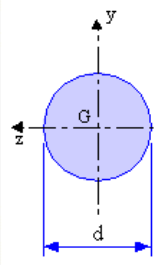
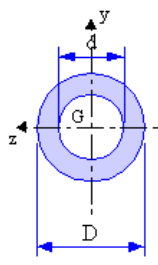
Decomposition of a solid but perforated trapezoid into the sum of several figures: a solid rectangle, a solid triangle and the subtraction of a solid circle.



We have the following properties:

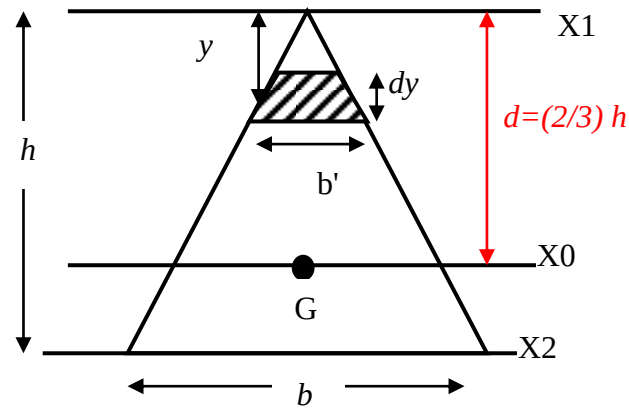
$$\begin{aligned} I_{Gx}(A_1 + A_2 - A_3) &= I_{Gx}(A_1) + I_{Gx}(A_2) - I_{Gx}(A_3) \\ I_{Gy}(A_1 + A_2 - A_3) &= I_{Gy}(A_1) + I_{Gy}(A_2) - I_{Gy}(A_3) \end{aligned}$$

The following table lists the second moments of area of sections commonly used in strength of materials.

				
IG_z (mm ⁴)	$\frac{bh^3}{12}$	$\frac{a^4}{12}$	$\frac{\pi d^4}{64}$	$\frac{\pi (D^4 - d^4)}{64}$
I_0 (mm ⁴)	$\frac{bh^3 + hb^3}{12}$	$\frac{a^4}{6}$	$\frac{\pi d^4}{32}$	$\frac{\pi (D^4 - d^4)}{32}$

Application

Determine the moment of inertia in relation to axis x_1 , x_0 and x_2 , passing respectively through the vertex, the centre of gravity and the base of the triangle (figure below).



Solution

Let's adopt as dA the area of a hatched trapeze (infinitely small). We can admit that its area is equal to that of a rectangle; hence

$$dA = b' \cdot dy \text{ or } b', \text{ is the length of the rectangle}$$

The similarity of the triangles allows us to derive b' ; hence:

$$\frac{b'}{b} = \frac{y}{h} \Rightarrow b' = \frac{y \cdot b}{h}$$

So:

$$I_{X1} = \int_0^h y^2 \cdot b' \cdot dy = \int_0^h y^2 \cdot \frac{y \cdot b}{h} \cdot dy = \int_0^h y^3 \cdot \frac{b}{h} \cdot dy$$

$$\Rightarrow I_{X1} = \frac{b}{h} \int_0^h y^3 \cdot dy = \frac{bh^3}{4}$$

Determine the moment of inertia relative to the central axis passing through the center of gravity:

$$I_{X1} = I_{X0} + A \cdot d^2 \text{ (Theorem of parallel axes)}$$

$$\Rightarrow I_{X0} = I_{X1} - A \cdot d^2 = \frac{bh^3}{4} - \frac{bh}{2} \cdot \left(\frac{2}{3} \cdot h\right)^2 = \frac{bh^3}{36}$$

Let's calculate the moment of inertia about the axis passing through the base:

$$I_{X2} = I_{X0} + A \cdot d^2 = \frac{bh^3}{36} + \frac{bh}{2} \cdot \left(\frac{1}{3} \cdot h\right)^2 = \frac{bh^3}{12}$$

2.4 Radius of gyration

In strength of materials, the radius of gyration i is a quantity, dimensionally equivalent to a length, whose square is equal to the ratio of the second moment of area I to the area A of a section:

It quantifies the ratio between the elastic properties of a section with respect to bending (EI) and compression (EA), and thus naturally appears in calculations of beam elongation with respect to buckling:

$$I_x = \int_A y^2 \cdot dA = i_x^2 \cdot A$$

$$\Rightarrow i_x = \sqrt{\frac{I_x}{A}} \quad \text{or} \quad i_y = \sqrt{\frac{I_y}{A}} \quad (\text{cm, mm})$$

i_x or i_y being the radius of inertia or radius of gyration relative to the x or y axis,

2.5 Resistance module

The ratio between the moment of inertia in relation to a given axis and the distance from that axis to the point furthest from the cross section:

$$W_x = \frac{I_x}{y_{max}}$$

The unit of resisting moment is a unit of length power 3 (cm^3 ; mm^3)

- Resistance modules of common sections

1-Rectangular section (base b , height h)

$$W = \frac{bh^2}{6}$$

2-Full circular section (diameter d):

$$W = \frac{\pi d^3}{32}$$

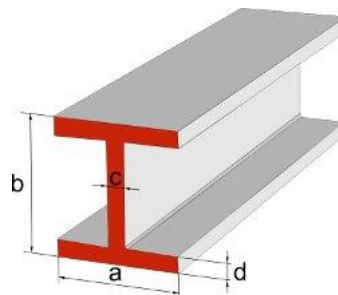
3-Hollow circular section (outer diameter D , inner diameter d):

$$W = \frac{\pi(D^4 - d^4)}{32D}$$

4- Square section (side a):

$$W = \frac{a^3}{6}$$

5- I section (Profiles type IPE or IPN): For these complex shapes, we usually use the values given by the constructor tables.



2.6 Variation of the moments of inertia in relation to the rotation of the axes

Consider a section A ; its moments of inertia in the XOY system I_x , I_y , I_{xy} are known. We propose to calculate the moments of inertia of section A in the X_1OY_1 system, which makes an angle α with the XOY system. When α changes, I_{x1} , I_{y1} , I_{x1y1} also change. Let's determine the angle α for which the values of I_{x1} , I_{y1} are maximum (Fig.2.7).

According to figure 2.7, we have:

$$\begin{cases} x_1 = x \cos \alpha + y \sin \alpha \\ y_1 = -x \sin \alpha + y \cos \alpha \end{cases}$$

Let's evaluate the moment of inertia with respect to the X_1 and Y_1 axes.

$$\begin{aligned} \bullet I_{X_1} &= \int_A y_1^2 \cdot dA = \int_A (-x \sin \alpha + y \cos \alpha)^2 dA \\ I_{X_1} &= I_X \cdot \cos^2 \alpha + I_Y \cdot \sin^2 \alpha - 2I_{XY} \cos \alpha \cdot \sin \alpha \\ I_{X_1} &= \left(\frac{I_X + I_Y}{2} \right) + \left(\frac{I_X - I_Y}{2} \right) \cos 2\alpha - I_{XY} \cdot \sin 2\alpha \dots \dots \dots (1) \end{aligned}$$

And the same for I_{Y_1}

$$\begin{aligned} \bullet I_{Y_1} &= \int_A x_1^2 \cdot dA = \int_A (x \cos \alpha + y \sin \alpha)^2 dA \\ I_{Y_1} &= I_X \cdot \sin^2 \alpha + I_Y \cdot \cos^2 \alpha + 2I_{XY} \cos \alpha \cdot \sin \alpha \\ I_{Y_1} &= \left(\frac{I_X + I_Y}{2} \right) + \left(\frac{I_X - I_Y}{2} \right) \cos 2\alpha + I_{XY} \cdot \sin 2\alpha \dots \dots \dots (2) \end{aligned}$$

$$\begin{aligned} (1) + (2): I_{X_1} + I_{Y_1} &= I_X (\cos^2 \alpha + \sin^2 \alpha) + I_Y (\sin^2 \alpha + \cos^2 \alpha) \\ &= I_X + I_Y = I_\rho \end{aligned}$$

$$(1) - (2): I_{X_1} - I_{Y_1} = (I_X - I_Y) \cos 2\alpha - 2I_{XY} \sin 2\alpha$$

$$\begin{aligned} \bullet I_{X_1 Y_1} &= \int_A x_1 y_1 dA = \int_A (x \cos \alpha + y \sin \alpha)(-x \sin \alpha + y \cos \alpha) dA \\ &= I_X \cdot \cos \alpha \cdot \sin \alpha - I_Y \cdot \sin \alpha \cdot \cos \alpha + I_{XY} (\cos^2 \alpha - \sin^2 \alpha) \\ &= I_X \frac{\sin 2\alpha}{2} - I_Y \frac{\sin 2\alpha}{2} + I_{XY} \cdot \cos 2\alpha \\ I_{X_1 Y_1} &= \left(\frac{I_X - I_Y}{2} \right) \sin 2\alpha + I_{XY} \cdot \cos 2\alpha \dots \dots \dots (3) \end{aligned}$$

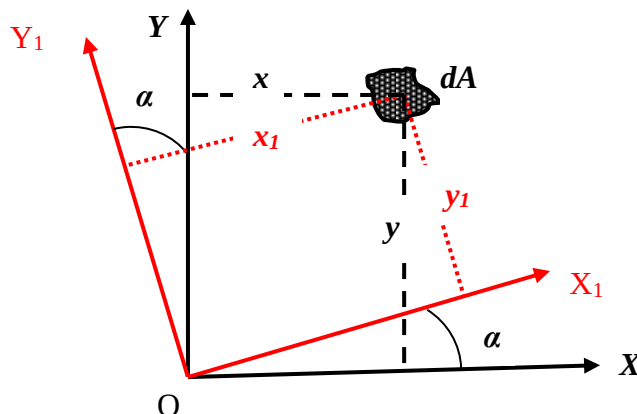


Fig.2.7 Moment of inertia for a section and rotation axes.

2.7 Main axes of inertia and moments of inertia

When, the angle α changes, I_{x_1} , I_{y_1} and I_{xy} also change. Let's determine the angle α for which the values of I_{x_1} , I_{y_1} are maximum.

So the primary derivatives of I_{x_1} and I_{y_1} compared to α are zero.

$$\begin{aligned} \frac{dI_{X_1}}{d\alpha} &= 0 \\ \Rightarrow \frac{dI_{X_1}}{d\alpha} &= -2I_X \cos \alpha \cdot \sin \alpha + 2I_Y \sin \alpha \cdot \cos \alpha - 2I_{XY} \cos 2\alpha = 0 \\ &\Rightarrow \tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{2I_{XY}}{I_Y - I_X} \\ &\Rightarrow 2\alpha = \tan^{-1} \left(\frac{2I_{XY}}{I_Y - I_X} \right) \end{aligned}$$

$$\Rightarrow \alpha = \frac{1}{2} \left[\tan^{-1} \left(\frac{2I_{XY}}{I_Y - I_X} \right) \right]$$

This formula determines the position of two axes whose moment of inertia is maximum for one and minimum for the other. These axes are called principal. The moments of inertia relative to the principal axes are called principal.

Using trigonometric formulas and replacing α with its value, we will have:

$$I_{(max,min)} = \frac{I_X + I_Y}{2} \pm \frac{1}{2} \sqrt{(I_X - I_Y)^2 + 4I_{XY}^2}$$

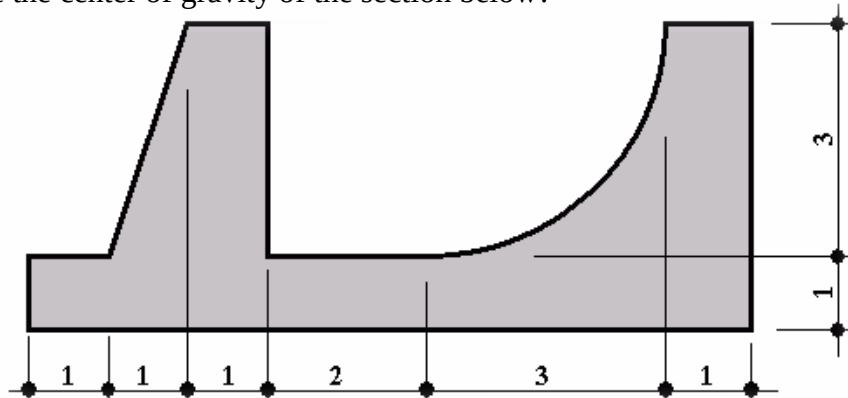
The principal axes are those axes with the following properties:

- 1- With respect to these axes, the product of inertia is zero ($I_{XY}=0$).
- 2- With respect to these axes, the moments of inertia have limiting values (maximum for one and minimum for the other).
- 3- The principal axes that pass through the centre of gravity of the section are called central axes.

SERIES OF EXERCISES

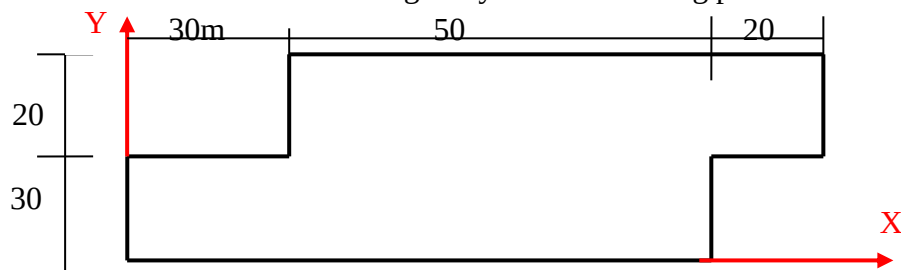
Exercise 1

Determine the center of gravity of the section below?



Exercise 2

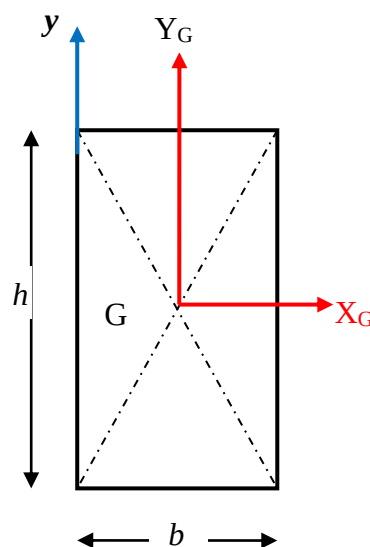
Calculate the coordinates of the center of gravity of the following plane section.



Exercise 3

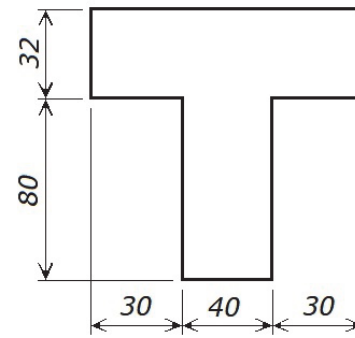
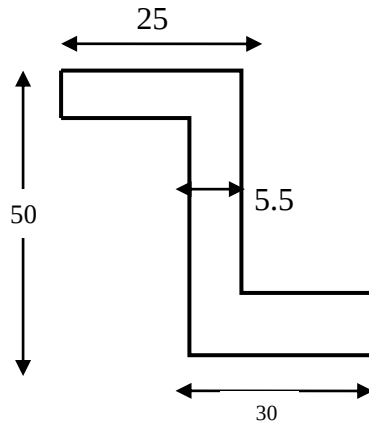
Let be a rectangular section of width b and height h . We ask to calculate the static moment and the moment of inertia of this section in relation to the following axes:

- A vertical (y) axis passing through the left side of the section.
- A vertical axis (y_G) passing through the centre of gravity of the section.



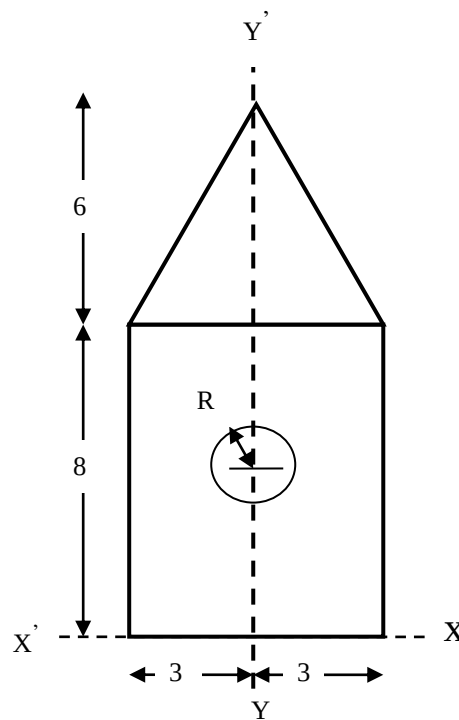
Exercise 4

Determine the position of the center of gravity of the surfaces opposite (dimensions in mm).

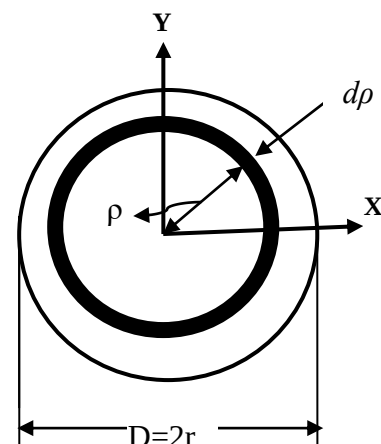
**Exercise 5**

For the plane section shown in the figure below, given that $I_{X'X} = 2690.44 \text{ cm}^4$ and $I_{Y'Y} = 158.44 \text{ cm}^4$, determine:

- 1-The radius "R" of the circular hollow.
- 2-The position "d" of the center of gravity for the circular hollow relative to the X'X axis.

**Exercise 6**

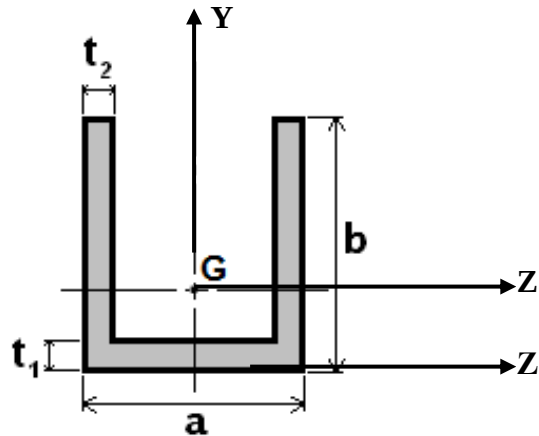
Determine the polar moment of inertia I_p in relation to the centre of gravity of the circle (C). then I_x and I_y .

**Exercise 7**

Determine the moment of inertia I_z of the following section:

$$t_1 = t_2 = 5 \text{ mm}$$

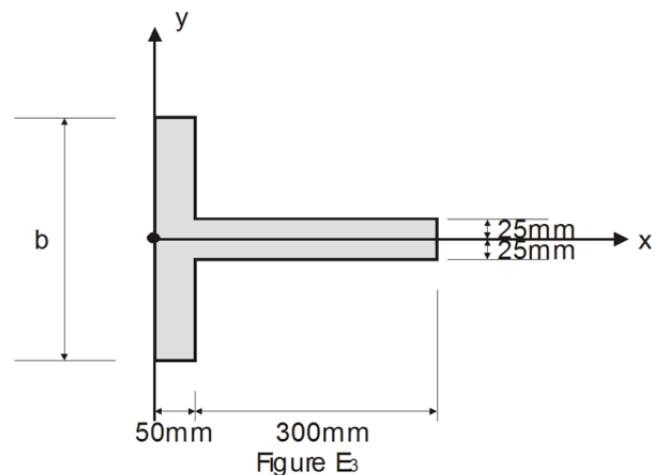
$$a = b = 25 \text{ mm}$$



Exercise 8

Calculate for the surface area shown:

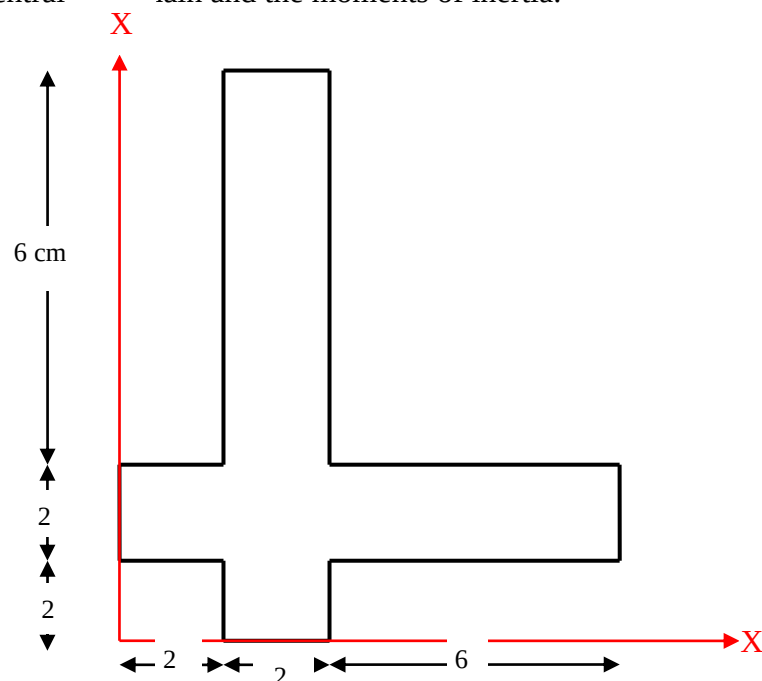
- 1- I_{xx} and I_{yy} if $b = 25 \text{ cm}$.
- 2- b for $I_{xx} = I_{yy}$.



Exercise 9

Calculate, for the surface below (the dimensions are in cm):

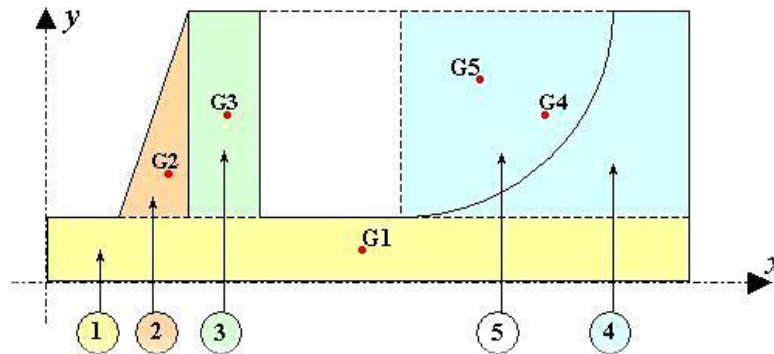
- 1- The position of the center of gravity in the mark x, y.
- 2- The moments of inertia and the product of inertia relative to the central axes parallel to x, y.
- 3- The position of the central axes main and the moments of inertia.



Solutions

Exercise 1

- 1- Choose a referential axis (ox , oy).
- 2- Break down the figure into simple surfaces whose centers of gravity are known.
- 3- Draw a table grouping: the area, the partial distances to the axis (ox , oy) and the static moments of each area.



POSITION DU CENTRE DE GRAVITE PAR RAPPORT A L'AXE DES X

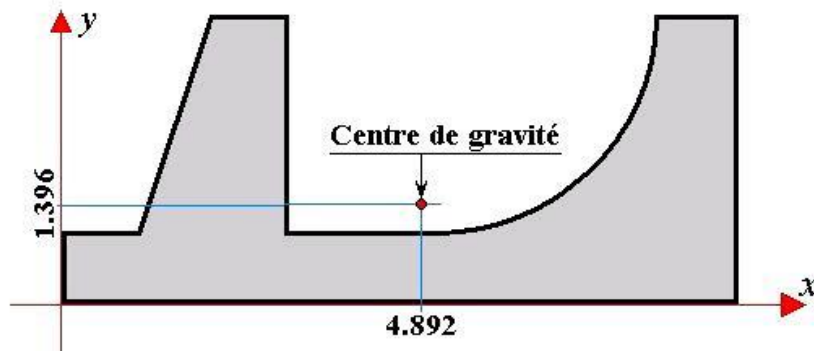
	Aire	Centre de gravité	Moment statique
Rectangle 1	9	0.5	$9 \times 0.5 = 4.5$
Triangle 2	1.5	2	$1.5 \times 2 = 3$
Rectangle 3	3	2.5	$3 \times 2.5 = 7.5$
Rectangle 4	12	2.5	$12 \times 2.5 = 30$
Quart de cercle 5	-7.068	2.727	$-7.068 \times 2.727 = -19.274$
TOTAUX	18.432		25.726

Centre de gravité par rapport à l'axe des x : $y = \frac{25.726}{18.432} = 1.396$

POSITION DU CENTRE DE GRAVITE PAR RAPPORT A L'AXE DES Y

	Aire	Centre de gravité	Moment statique
Rectangle 1	9	4.5	$9 \times 4.5 = 40.5$
Triangle 2	1.5	1.667	$1.5 \times 1.667 = 2.5$
Rectangle 3	3	2.5	$3 \times 2.5 = 7.5$
Rectangle 4	12	7	$12 \times 7 = 84$
Quart de cercle 5	-7.068	6.273	$-7.068 \times 6.273 = -44.338$
TOTAUX	18.432		90.162

Centre de gravité par rapport à l'axe des y : $x = \frac{90.162}{18.432} = 4.892$



Note: The area of section S is equal to $S_1 + S_2 + S_3 + S_4 - S_5$. Therefore, the static moment of element 5 must be subtracted.

Exercise 2

$$S_x = 2,5(5 \times 10) - 4(2 \times 3) - 1,5(3 \times 2) = 125 - 24 - 9 = 92 \text{ cm}^3$$

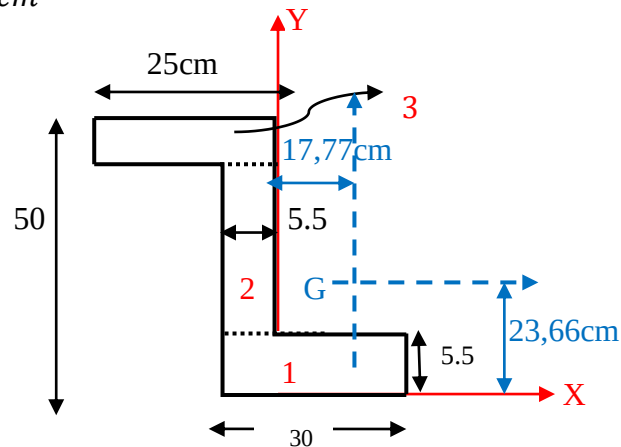
$$S_y = 5(5 \times 10) - 1,5(2 \times 3) - 9(3 \times 2) = 250 - 9 - 54 = 187 \text{ cm}^3$$

$$X_G = S_y / A = 187 / 38 = 4,9 \text{ cm}$$

$$Y_G = S_x / A = 92 / 38 = 2,4 \text{ cm}$$

Exercise 4

A-



Position of CG with respect to X axes.

	Area (cm ²)	CG	S _x
Rectangle 1	5,5.30= 165	2,25	371,25
Rectangle 2	39.5,5=214,5	25	5362,5
Rectangle 3	25.5,5=137.5	47,25	6496,875
	517		12230,625

$$Y_G = \frac{12230,625}{517} = 23,66 \text{ cm}$$

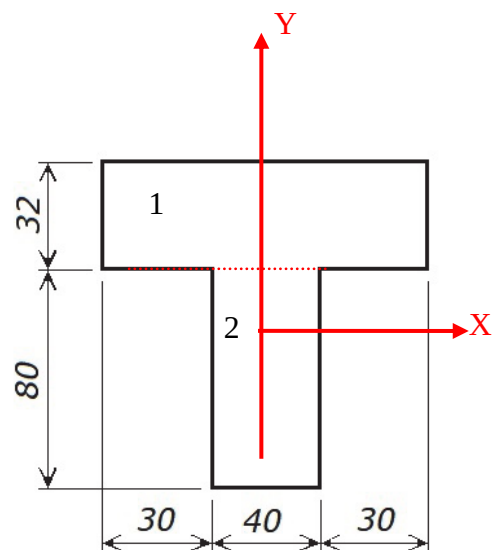
Position of CG with respect to Y axes.

	Area (cm ²)	CG	S _y
Rectangle 1	5,5.30= 165	9,5	1567,5
Rectangle 2	39.5,5=214,5	2,75	5898,75
Rectangle 3	25.5,5=137.5	12,5	1718,75
	517		9185

$$X_G = \frac{9185}{517} = 17,77 \text{ cm}$$

B-

The section is symmetrical with respect to the Y axes.
The choice of axis is arbitrary.



	Area (cm ²)	CG	S _x
Rectangle 1	100.32= 3200	96	307200
Rectangle 2	40.80=3200	40	128000
	6400		435200

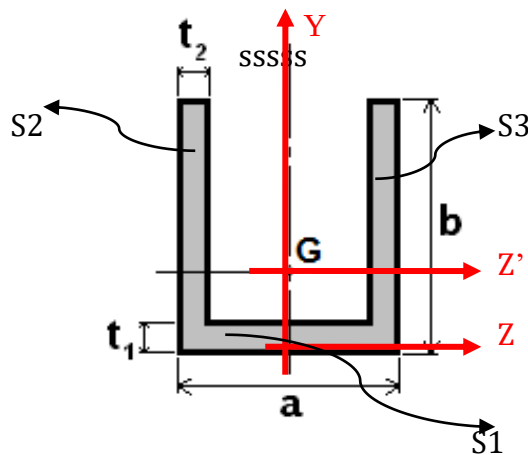
$$Y_G = \frac{435200}{6400} = 68mm$$

	Area (cm ²)	CG	S _y
Rectangle 1	100.32= 3200	0	0,00
Rectangle 2	40.80=3200	0	0,00
	6400		0,00

$$X_G = 0$$

The y axis is an axis of symmetry

Exercise 7



$$S_1 = t_1 \cdot a \Rightarrow S_1 = 5.25 = 125 \text{ mm}^2$$

$$Y_1 = \frac{t_1}{2} = \frac{5}{2} = 2,5mm$$

$$I_{S1} = \frac{a \cdot t_1^3}{12} = \frac{25 \cdot 5^3}{12} = 260,42mm^4$$

$$S_2 = S_3 = t_2 \cdot b \Rightarrow S_2 = S_3 = 20.5 = 100 \text{ mm}^2$$

$$Y_2 = Y_3 = \frac{t_2}{2} + 5 = \frac{25}{2} + 5 = 15mm$$

$$I_{S2} = \frac{t_2 \cdot b^3}{12} = \frac{5 \cdot 20^3}{12} = 3333,33mm^4$$

$$Y_G = \frac{\sum S_i \cdot X_i}{\sum S_i} = \frac{125 \cdot 2,5 + 100 \cdot 15 + 100 \cdot 15}{125 + 100 + 100} = 10,2mm$$

$$I_Z = \sum (I_{Gi} + (Y_i - Y_{Gi})^2 \cdot S_i) = I_{G1} + (Y_1 - Y_{G1})^2 \cdot S_1 + 2 \cdot (I_{G2} + (Y_2 - Y_{G2})^2 \cdot S_2)$$

$$= 260,42 + (10,2 - 2,5)^2 \cdot 125 + 2 \cdot (3333,33 + (15 - 10,2)^2 \cdot 100)$$

$$I_Z = 18946,33 \text{ mm}^4$$

Exercise 8

1- b=25cm

Let's decompose the surface S into two S₁, and S₂

$$I_{XX}^{S1} = \frac{30 \times 5^3}{12} = 312,5cm^4$$

$$I_{XX}^{S2} = \frac{5 \times 25^3}{12} = 6510,42cm^4$$

$$\Rightarrow I_{XX}^S = I_{XX}^{S_1} + I_{XX}^{S_2} = 312,5 + 6510,42$$

$$\Rightarrow I_{XX}^S = 6822,92 \text{ cm}^4$$

Similarly, after applying Huygens' theorem:

$$I_{YY}^{S_1} = \frac{5 \times 30^3}{12} + 20^2 \times 30 \times 5 = 71250 \text{ cm}^4$$

$$I_{YY}^{S_2} = \frac{25 \times 5^3}{12} + 2,5^2 \times 5 \times 25 = 1041,67 \text{ cm}^4$$

$$\Rightarrow I_{YY}^S = I_{YY}^{S_1} + I_{YY}^{S_2} = 71250 + 1041,67$$

$$\Rightarrow I_{YY}^S = 72291,67 \text{ cm}^4$$

b

$$2- \text{ b? } I_{YY}^S = I_{YY}^S$$

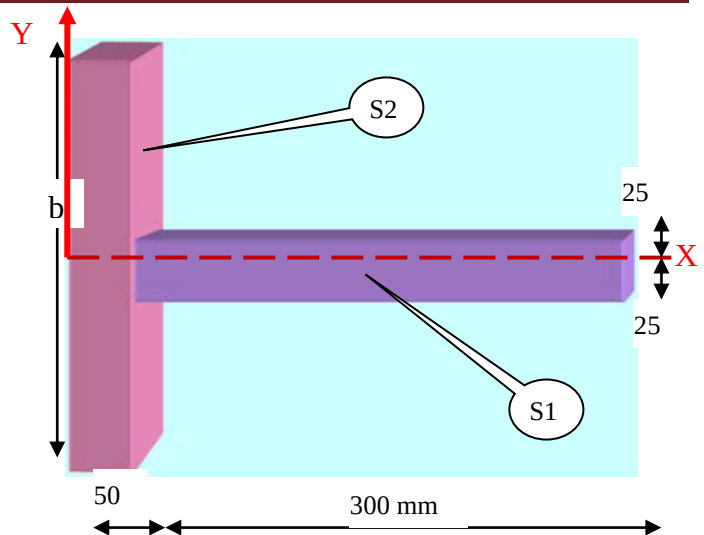
$$\frac{30 \times 5^3}{12} + \frac{5 \times b^3}{12} = \frac{5 \times 30^3}{12} + 20^2 \times 30 \times 5 + \frac{b \times 5^3}{12} + 2,5^2 \times 5 \times b$$

$$312,5 + \frac{5 \times b^3}{12} - 71250 - \frac{b \times 5^3}{12} - 31,25 \times b = 0$$

$$5b^3 - 500b - 851250 = 0$$

A cubic equation with two imaginary solutions and one real solution:

$$\Rightarrow b = 56 \text{ cm}$$



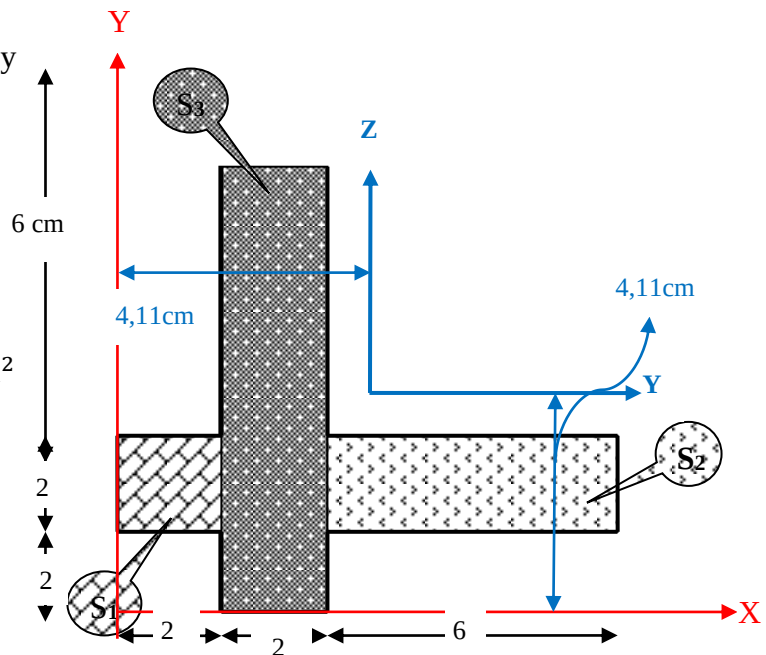
Exercise 9

1- The position of the center of gravity in the x, y coordinate system.

$$G_{1/XY} = \begin{pmatrix} 1 \text{ cm} \\ 3 \text{ cm} \end{pmatrix} \rightarrow A_1 = 2 \times 2 = 4 \text{ cm}^2$$

$$G_{2/XY} = \begin{pmatrix} 7 \text{ cm} \\ 3 \text{ cm} \end{pmatrix} \rightarrow A_2 = 6 \times 2 = 12 \text{ cm}^2$$

$$G_{3/XY} = \begin{pmatrix} 3 \text{ cm} \\ 5 \text{ cm} \end{pmatrix} \rightarrow A_3 = 10 \times 2 = 20 \text{ cm}^2$$



$$X_G = \frac{x_{G1} \times A_1 + x_{G2} \times A_2 + x_{G3} \times A_3}{A_1 + A_2 + A_3} = \frac{1 \times 4 + 7 \times 12 + 3 \times 20}{4 + 12 + 20} = \frac{148}{36}$$

$$\Rightarrow X_G = 4,11 \text{ cm}$$

$$Y_G = \frac{y_{G1} \times A_1 + y_{G2} \times A_2 + y_{G3} \times A_3}{A_1 + A_2 + A_3} = \frac{3 \times 4 + 3 \times 12 + 5 \times 20}{4 + 12 + 20} = \frac{148}{36}$$

$$\Rightarrow Y_G = 4,11 \text{ cm}$$

S_i	S_1	S_2	S_3
$A_i(\text{cm}^2)$	4	12	20
$G_{i/YZ}(\text{cm}^2)$	-3,11 -1,11	2,89 -1,11	-1,11 0,89
$I_{YiYi}(\text{cm}^4)$	$\frac{2^4}{12} = 1,33$	$\frac{6 \times 2^3}{12} = 4,0$	$\frac{2 \times 10^3}{12} = 166,67$
$I_{YZ}(\text{cm}^4)$	0	0	0

2- The moments of inertia and the product of inertia with respect to the central axes parallel to X, Y. $\Rightarrow$

$$\bullet I_{YY}^{S_i} = I_{YY}^{S_1} + I_{YY}^{S_2} + I_{YY}^{S_3}$$

$$\text{Or, } I_{YY}^{S_i} = I_{YiYi}^{S_i} + S_i \cdot Y_{Gi}^2$$

$$\Rightarrow I_{YY}^{S_i} = 1,33 + 4 \times (1,11)^2 + 4 + 12 \times (-1,11)^2 + 166,67 \times 20 \times (0,89)^2$$

$$\Rightarrow I_{YY}^{S_i} = 207,75 \text{ cm}^4$$

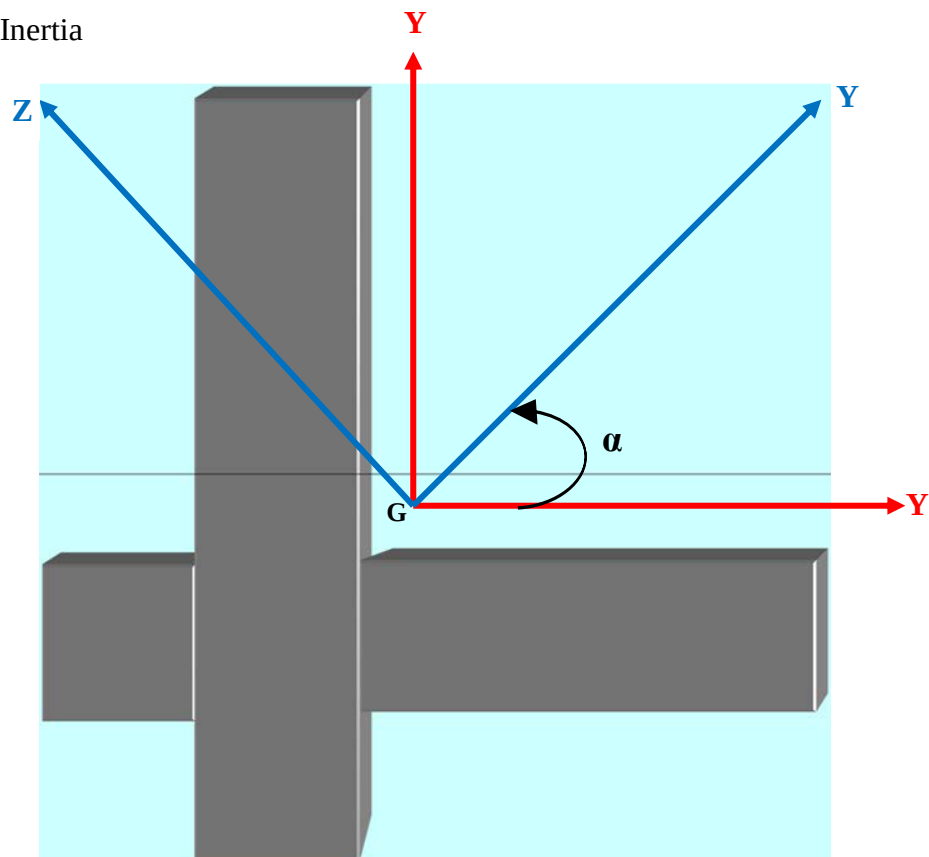
$$\bullet I_{YZ}^{S_i} = I_{YZ}^{S_1} + I_{YZ}^{S_2} + I_{YZ}^{S_3}$$

$$\text{Or, } I_{YZ}^{S_i} = I_{YiZi}^{S_i} + S_i \cdot Y_i \cdot Z_i$$

$$\Rightarrow I_{YZ}^{S_i} = 0 + (-3,11) \times (-1,11) \times 4 + (2,89) \times (-1,11) \times 12 + (-1,11) \times (0,89) \times 20$$

$$\Rightarrow I_{YZ}^{S_i} = -44,44 \text{ cm}^4$$

3- Main elements of Inertia



$\varphi = 45^\circ \rightarrow$ Because Y is the axis of symmetry of the surface S.

$$I_{max,min} = \frac{I_Y + I_Z}{2} \pm \frac{1}{2} \sqrt{\frac{I_Y - I_Z}{2}^2 + 4 \times I_{YZ}^2}$$

$$I_{max} = \frac{I_Y + I_Z}{2} + \frac{1}{2} \sqrt{0 + 4 \times I_{YZ}^2}$$

$$I_{max} = \frac{207,56 + 207,56}{2} + \frac{1}{2} \sqrt{4 \times (-44,44)^2}$$

$$\Rightarrow I_{max} = 207,56 + 44,44 = 252,00 \text{ cm}^4$$

$$I_{min} = \frac{I_Y + I_Z}{2} - \frac{1}{2} \sqrt{\frac{I_Y - I_Z}{2}^2 + 4 \times I_{YZ}^2}$$

$$\Rightarrow I_{min} = 207,56 - 44,44 = 163,12 \text{ cm}^4$$

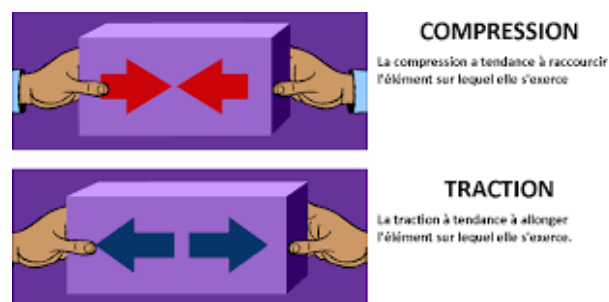
Simple Traction and Compression

3.1 Generalities

Strength of Materials is the foundation upon which all modern engineering rests. It is the discipline that allows for the optimization of materials in civil, mechanical, and aeronautical engineering, while ensuring that the structure does not break or deform irreversibly under the effect of applied loads, i.e. its durability.

A section (S) is subjected to *traction* if it is stressed by a normal force N applied in G (center of gravity) and directed outwards from (S), inducing an *elongation*. *Compression* is the application of inward-balanced forces at different points on a body to reduce its initial size in one or more directions. However, if these forces are directed outward, the body is said to be in traction.

These forces (pull) cause an increase in size (length). The compressive strength of materials and structures is an important engineering consideration. In the following, it is assumed that the applied forces act perpendicularly to the axis passing through the centre of gravity of the section studied. Compressive or tensile forces are uniaxial forces.



3.2 Simple traction and compression

Definition: Simple tension or compression are two simple and fundamental stresses in Strength of Materials (SM), resulting in any part (beam, bar, etc.) subjected to two equal and opposite external forces, applied along its longitudinal axis. If a fictitious cut is applied to the beam that divides it into two parts, the action of the ignored (right) section is replaced by the normal force N .

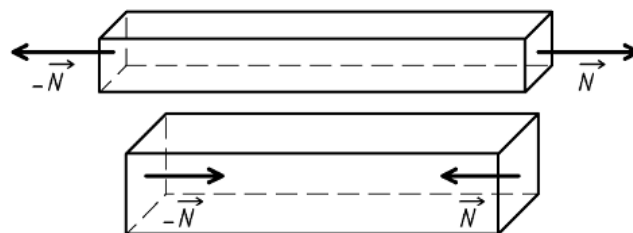


Fig.3.1 Beams subjected to tension/compression

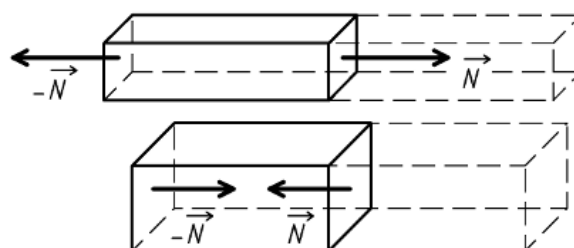


Fig.3.2 Fictitious section on beam

Note: 1- Simple tension (or compression) satisfies three conditions.
$$\begin{cases} N_x \neq 0 \\ T_y = 0 \\ M = 0 \end{cases}$$

2- In compression, the risk of buckling must be checked if the beam is too slender.

3.3 Normal tensile and compressive forces

In the coordinate system (G, x, y, z) linked to the right-section (S), the elements of the cohesive force torsor are expressed by:

◆ *Traction*
$$\{\mathcal{T}_{\text{coh}}\}_G = \begin{pmatrix} \vec{R} \\ \vec{0} \end{pmatrix} \quad \text{tel que dans } \mathcal{R} : \quad \{\mathcal{T}_{\text{coh}}\}_G = \begin{pmatrix} N & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}_{(\vec{i}, \vec{j}, \vec{z})} \quad \text{avec } N > 0.$$

◆ *Compression*
$$\{\mathcal{T}_{\text{coh}}\}_G = \begin{pmatrix} \vec{R} \\ \vec{0} \end{pmatrix} \quad \text{tel que dans } \mathcal{R} : \quad \{\mathcal{T}_{\text{coh}}\}_G = \begin{pmatrix} N & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}_{(\vec{i}, \vec{j}, \vec{z})} \quad \text{Avec } N < 0$$

Each infinitesimal element of the surface ΔS supports a tensile force, parallel to the line passing through the beam's center of gravity G. The uniform stress distribution in the cross-section is:

$$\sigma = \frac{N}{S}$$

With,

σ: the normal stress uniformly distributed over (S) in N/mm² or MPa.

N: the normal force at section S, expressed in N, KN,

S: the cross-section perpendicular to the axis passing through G, expressed in mm².

If $\begin{cases} N > 0 \Rightarrow \sigma \text{ is a tensile stress } (\sigma > 0) \\ N < 0 \Rightarrow \sigma \text{ is a compressive stress } (\sigma < 0) \end{cases}$

For the bar to withstand stresses, the deformations must remain within the elastic zone. This zone is limited by a value called the practical tensile strength R_{pe} or R_{pc} in the case of compression.

◆ *The strength condition of a part is:*
$$\begin{cases} \sigma = \frac{N}{S} \leq R_{pe}, \text{ for traction} \\ \sigma = \frac{N}{S} \leq R_{pc}, \text{ for compression} \end{cases}$$

$$R_{pe} = \frac{\sigma_e}{FS} \quad (\text{Traction case})$$

$$R_{pc} = \frac{\sigma_e}{FS} \quad (\text{Compression case})$$

R_{pe}, R_{pc}: practical resistance to extension and compression respectively

σ_e: limit elastic stress in MPa.

FS: safety factor

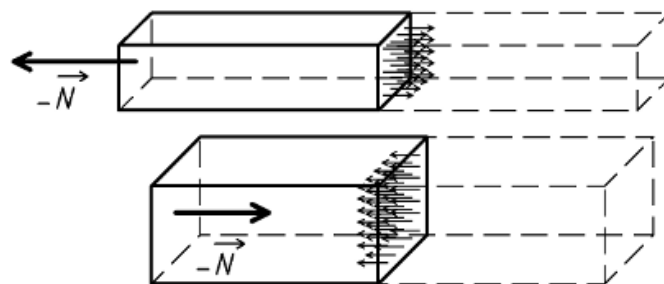


Fig.3.3 Appearance of normal stress N in the fictitious section

Tab 1 Elastic tensile strength of common materials

Matter	Nuance	R _e (MPa)
Aluminum alloy	1000 series to 7000 series	90 to 470
Tempered low-alloy steel	30Cr Ni MO 16 (30 CND 8)	700 to 1450
common non-alloy construction steel	S235 to S355	235 to 355
Titanium alloy	TA6V (Ti-6Al-4V)	900
Fiberglass	-	2500 to 3200
glued laminated timber	GL24 to GL32	24 to 32
Fiber/matrix composites	Glass or carbon	1000 to 1800
Polypropylene	-	24 to 32

In practice, the maximum stress which will be applied to the part without risk of rupture or excessive deformation is fixed, this maximum stress is called: admissible stress noted σ_{adm} ($\bar{\sigma}$).

In general: $\frac{1}{2}\sigma_e < \bar{\sigma} < \frac{3}{2}\sigma_e$ ou $\frac{1}{5}\sigma_u < \bar{\sigma} < \frac{1}{3}\sigma_u \rightarrow$ for steels

σ_u : the ultimate stress (given).

Therefore, always check the strength condition for tensile or compression or both at the same time

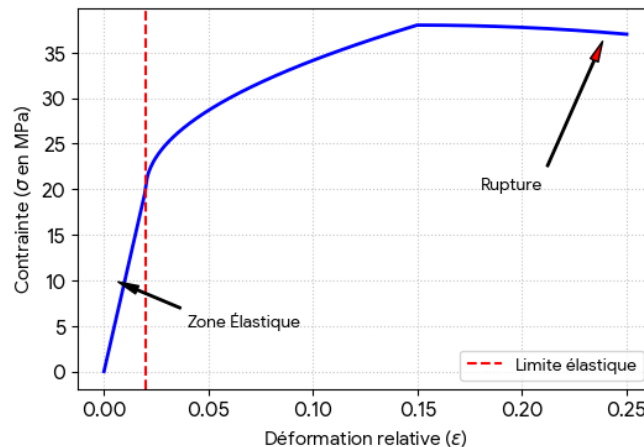
$$\sigma_{max} \leq \bar{\sigma} = \frac{\sigma_e}{FS}$$

For functional reasons (support alignment issues, specifications, etc.), the elongation Δl is limited so that it is less than a limit value Δl_{lim} .

$$\Delta l \leq \Delta l_{lim}.$$

The elongation Δl is determined as follows:

According to Hooke's Law:
$$\begin{cases} \sigma = \frac{N}{S} \\ \sigma = E \cdot \varepsilon = E \cdot \frac{\Delta l}{l} = \frac{N}{S} \Rightarrow \Delta l = \frac{N \cdot l}{ES} \end{cases}$$

**Fig. 3.4** Stress-strain diagram (σ - ε)

$$1 \text{ N/m}^2 = 1 \text{ Pa} \Rightarrow 1 \text{ kN/m}^2 = 10^3 \text{ Pa}$$

$$1 \text{ kPa} = 10^3 \text{ Pa}$$

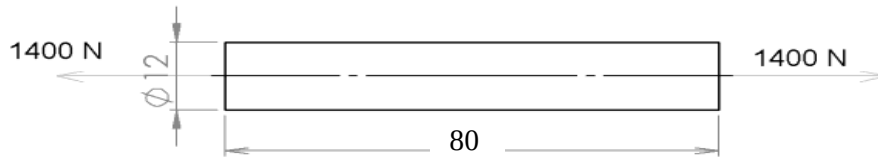
$$1 \text{ MPa} = 10^6 \text{ Pa}$$

$$1 \text{ GPa} = 10^9 \text{ Pa}$$

Application

Calculate the stress due to a tensile force of 1400 N on a steel shaft with diameter 12 mm and length 80 mm. Knowing that: $\sigma_e = 500$ MPa and FS=5

Then check the safety condition for the shaft.



Solution

-Determination of tensile stress:

$$\sigma = \frac{N}{S} = \frac{1400}{\pi \cdot r^2} = \frac{1400}{\pi \cdot 6^2} = 12.38 \left[\frac{N}{mm^2} \right] = 13.38 \text{ MPa}$$

- Determination of the resistance condition of a part in traction

$$\sigma = \frac{N}{S} \leq R_{pe}$$

$$R_{pe} = \frac{\sigma_e}{FS} = \frac{500}{5} = 100 \text{ MPa}$$

$$\sigma = 13.38 \text{ MPa} < R_{pe} = 100 \text{ MPa}$$

⇒ The shaft resists to the applied tensile stress.

3.4 Elastic deformations

3.4.1 Tensile test: A tensile test is a physics experiment that provides information on the elastic behavior of a material, such as its yield strength and ultimate strength under uniaxial loading. This test involves placing a bar of the material between the jaws of a tensile testing machine, which pulls on the bar until it breaks. The elongation Δl and the applied force F are recorded, which are then converted into strain and stress.

The tensile test allows us to plot the stress-strain curve (σ - ϵ) in order to obtain the elasticity parameters:

- The Young's modulus, E , or longitudinal elasticity modulus, expressed in (MPa) or (GPa);
- The elastic limit, R_e , σ_e , which is used to characterize an area where deformations are reversible (conventionally $R_{e0.2}$),
- Tensile strength, R_m , σ_m , the limiting value at which the material fails (fracture);
- Elongation at break, $A\%$, characterizes the final ability of the material to elongate under load before breaking;

Example: $E_{\text{Steel}} = 21000 \text{ daN/mm}^2$, $E_{\text{Beton}} = 2000 \text{ daN/mm}^2$, $E_{\text{Aluminium}} = 7000 \text{ daN/mm}^2$

- Tensile strength, R_m , σ_m , the material failure limit value (fracture);
- The elongation at rupture $A\%$, characterizes the final ability of the material to stretch under load before it breaks;

$$A\% = \epsilon = \frac{l-l_0}{l_0} = \frac{\Delta l}{l_0} \text{ (without unit)}$$

- The Poisson coefficient ν , coefficient of proportionality between the transverse deformations (section decrease) and longitudinal deformations (relative elongation) of the part in the elastic field.

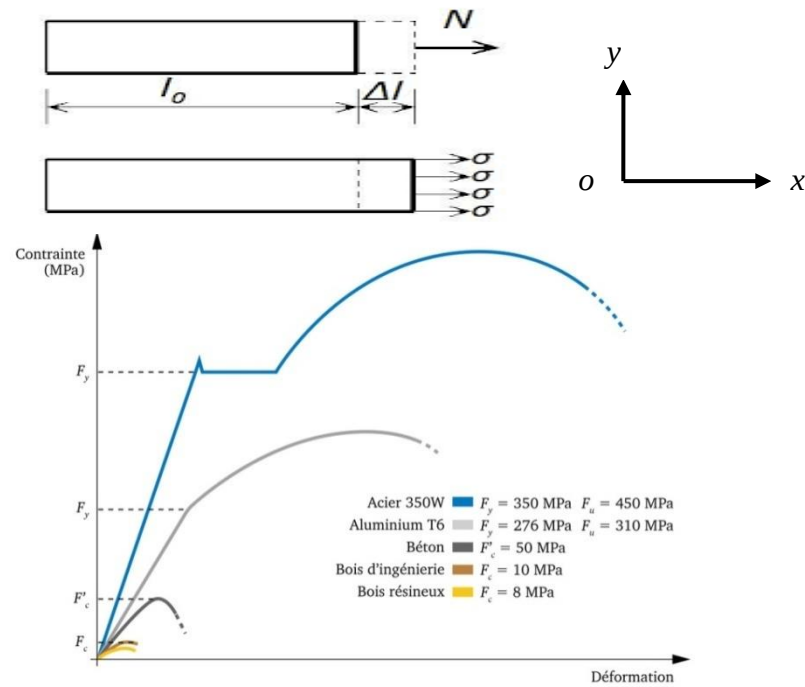


Fig.3.4 Strain diagram of a ductile material

3.4.2 Elastic behavior law (Hooke's law): We place on the straight line which passes through the origin (OA) of the stress-strain curve, equation:

$$y = ax + b, \quad b = 0 \text{ (Line passes through the origin o)}$$

$$y = ax \Leftrightarrow \sigma = E \cdot \frac{\Delta l}{l_0} = E \cdot \varepsilon_x$$

Hence the general form of Hooke's law: $\sigma = E \cdot \varepsilon$

ε_x : is the longitudinal strain (along the longest length), in our case, it is the x coordinate system.

ε_y : is the transverse strain (along the width), in our case, it is the y coordinate system, which is often neglected.

$$\varepsilon_y = -\nu \cdot \varepsilon_x \Rightarrow \nu = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_{transverse}}{\varepsilon_{longitudinal}}$$

Note: ν Poisson's ratio (for construction materials with elastic behavior varies ν from 0.30 to 0.25).

3.4.3 Stress Concentration Phenomenon

When the cross-section of the part varies abruptly, the hypothesis about the strength of the materials are no longer verified, and subsequently the normal stress distribution σ is no longer uniform. The stress value increases in the vicinity of these stress concentration points.

To account for this increase, the nominal constraint σ_{nom} is multiplied by the parameter k_t called the stress concentration coefficient. k_t is a multiplier factor that allows to calculate the actual maximum stress σ_{max} in the vicinity of a geometric discontinuity (hole, shoulder, groove). The fundamental relationship is:

$$\sigma_{max} = k_t \cdot \sigma_{nom}$$

◆ Piece with a circular hole

1-Infinite plate in tension: $k_t=3$

2-Round bar with cross hole: $2,5 \leq k_t \leq 6,5$

♦ **Connecting shoulders and fillets:** The values depend on the ratio between the large diameter (D), the small diameter (d) and the connecting radius r

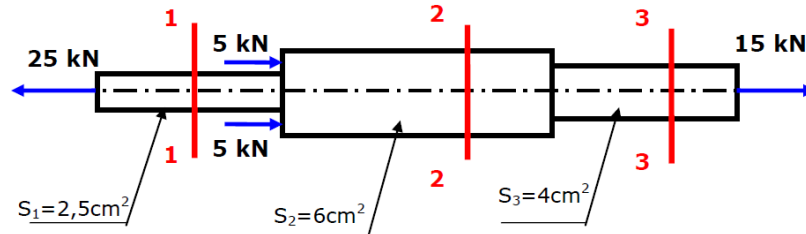
1-Traction/Compression: $1,5 \leq k_t \leq 3,0$. Plus, the radius r is small (sharp angle), plus k_t is high.

2- Flexion: $1,2 \leq k_t \leq 2,5$

3-Torsion: generally weaker, $1,1 \leq k_t \leq 2,0$

Application

Consider the bar below. Calculate the stresses at sections 1-1, 2-2, and 3-3.

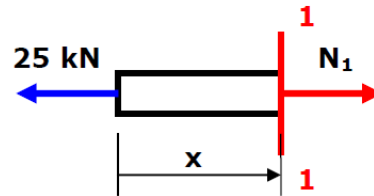


Solution

Section 1-1

$$\sum F_x = 0 \Rightarrow N_1 = 25 \text{ kN}$$

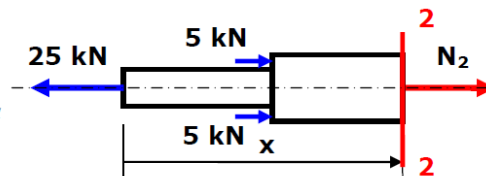
$$\sigma_{1-1} = \frac{N_1}{S_1} = \frac{25}{2,5} = 10 \text{ kN/cm}^2 = 100 \text{ MPa}$$



Section 2-2

$$\sum F_x = 0 \Rightarrow N_2 = 15 \text{ kN}$$

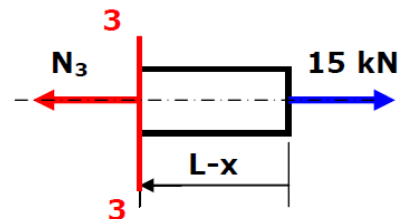
$$\sigma_{2-2} = \frac{N_2}{S_2} = \frac{15}{6} = 2,5 \text{ kN/cm}^2 = 25 \text{ MPa}$$



Section 3-3

$$\sum F_x = 0 \Rightarrow N_3 = 15 \text{ kN}$$

$$\sigma_{3-3} = \frac{N_3}{S_3} = \frac{15}{4} = 3,75 \text{ kN/cm}^2 = 37,5 \text{ MPa}$$



3.4.4 Normal force diagram (NFD)

• The normal force diagram, expresses the value of the normal force in all sections perpendicular to the centerline of the section under study.

• The normal force in a section is the resultant of the axial loads acting on the section.

• The DEN is obtained using the section method by making fictitious cuts after each concentrated force and at the beginning and end, as well as at the minimum and maximum (if applicable), of each distributed load.

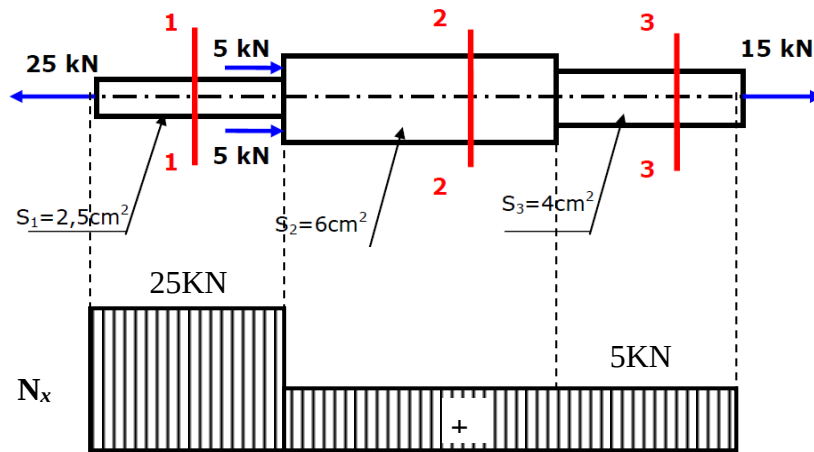


Fig.3.5 Diagram of normal force N_x

Application

A 10 mm diameter steel bar receives a tensile force of 12,560 N. What will be the elongation of the bar by 5 meters if $E = 210,000 \text{ N/mm}^2$? What will then be the stress in this bar?

Solution

- Bar section: $A = \frac{\pi D^2}{4} = \frac{\pi \cdot 10^2}{4} = 78.54 \text{ mm}^2$

- he elongation (Δl) of the bar, we know that: $\sigma = E \cdot \frac{\Delta l}{l} = \frac{N}{A}$

$$\Rightarrow E \cdot \frac{\Delta l}{l_0} = \frac{N}{A}$$

$$\Rightarrow \Delta l = \frac{N \cdot l_0}{EA} = \frac{12560 \cdot 5000}{210000 \cdot 78,57} = 3,8 \text{ mm}$$

-Normal stress is: $\sigma = \frac{N}{A} = \frac{12560}{78,54} \approx 160 \text{ N/mm}^2$

Or else

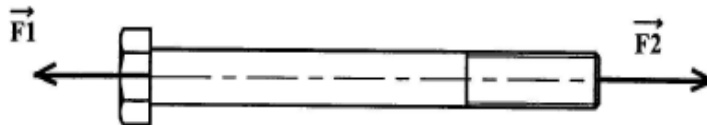
$$\sigma = E \cdot \varepsilon = E \cdot \frac{\Delta l}{l_0} = 210000 \cdot \frac{3,8}{5000} = 159,6 \text{ N/mm}^2 \approx 160 \text{ N/mm}^2$$

SERIES OF EXERCISES

Exercise 1

Consider the screw below, 150 mm long and 16 mm in diameter, in equilibrium under the action of two forces F_1 and F_2 , each with an intensity of 1000 daN. The screw is made of steel and its longitudinal modulus of elasticity is 200 GPa.

- 1- What type of stress is the screw subjected to?
- 2- Calculate the stress value.
- 3- If a safety factor of 4 is used, calculate the yield strength of the steel.
- 4- Determine the elongation of the screw.



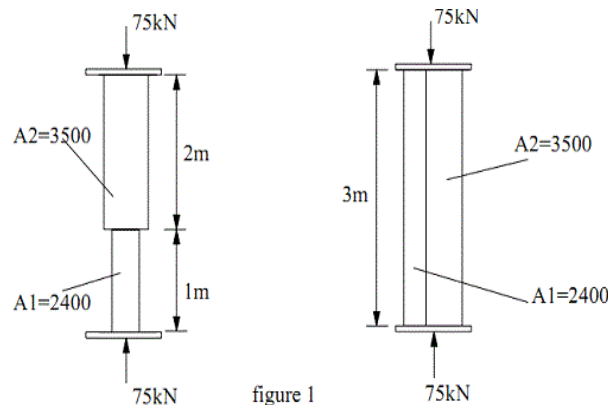
Exercise 2

Given two columns of equal height L , one consisting of two parallel bars, and the other of two coaxially welded bar sections, each bar supports a compressive force P (Figure below).

- Given $E_1 = 200 \text{ kN/mm}^2$, $E_2 = 120 \text{ kN/mm}^2$, $A_1 = 2400 \text{ mm}^2$, $A_2 = 3500 \text{ mm}^2$, $L_1 = 1 \text{ m}$, $L_2 = 2 \text{ m}$, and $P = 75 \text{ kN}$.

- 1- Calculate the axial forces and deformations in each bar (and section).

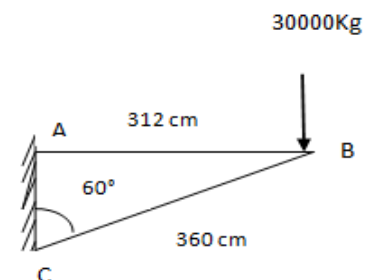
$$K_i = E_i A_i / L_i \quad \text{of each section.}$$



Exercise 3

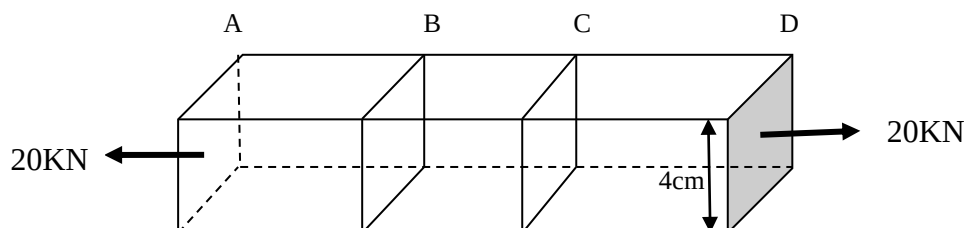
The two steel bars AB and BC are pinned together at each end and support a load of 30000 kg (Fig). The metal is tempered cast steel with yield strength of 4200 kg/cm². The permissible safety factors are 2 for tension members and 3.5 for compression members.

- Calculate the cross-sections of these bars and their deformations.



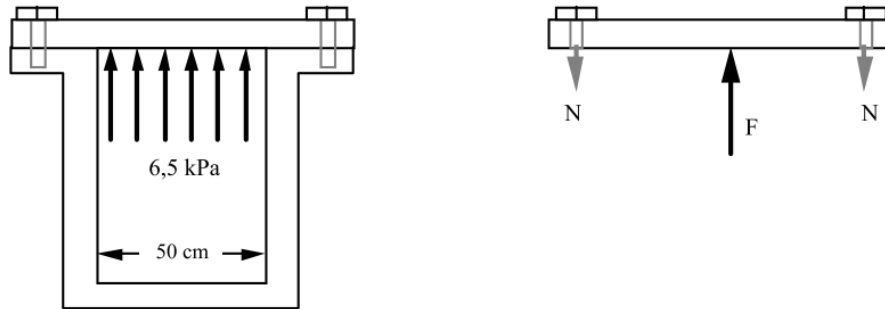
Exercise 4

Find the normal stresses at B and C of the square beam below.



Exercise 5

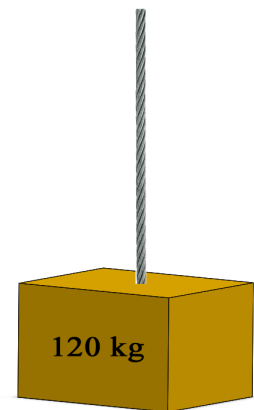
The lid of a 50 cm diameter tank is secured with 10 bolts (Figure below). Each 1.5 cm² in cross-section. What is the stress in the bolts if the pressure in the tank is 6.5 kPa?

**Exercise 6**

A 3-meter-long steel cable supports a weight of 120 kg. The normal stress $\sigma = 20$ MPa and the modulus of elasticity

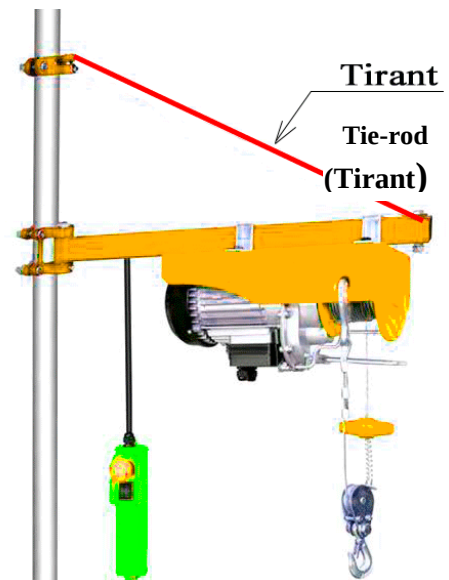
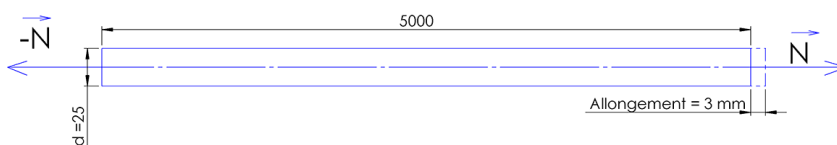
$E = 1.55 \times 10^5$ MPa are given.

- Determine the minimum diameter of this cable that will support this load.
- Determine its elongation.

**Exercise 7**

A 25 mm diameter, 5 meter long tie rod for an electric hoist is subjected to a tensile force that stretches it by 3 mm. Modulus of elasticity $E = 2.1 \times 10^5$ MPa.

- Determine the tensile force experienced by the rod.

**Exercise 8**

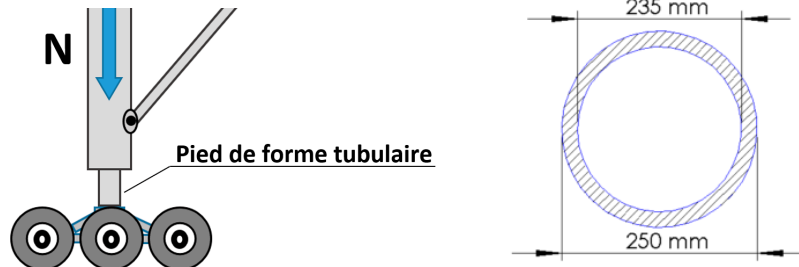
A chain is composed of a series of steel links, all identical. We have the elastic limit of steel = 350 MPa, the safety coefficient $ES = 5$ and the link diameter $d = 16$ mm. Determine the maximum extension force that the chain can withstand under satisfactory safety conditions.



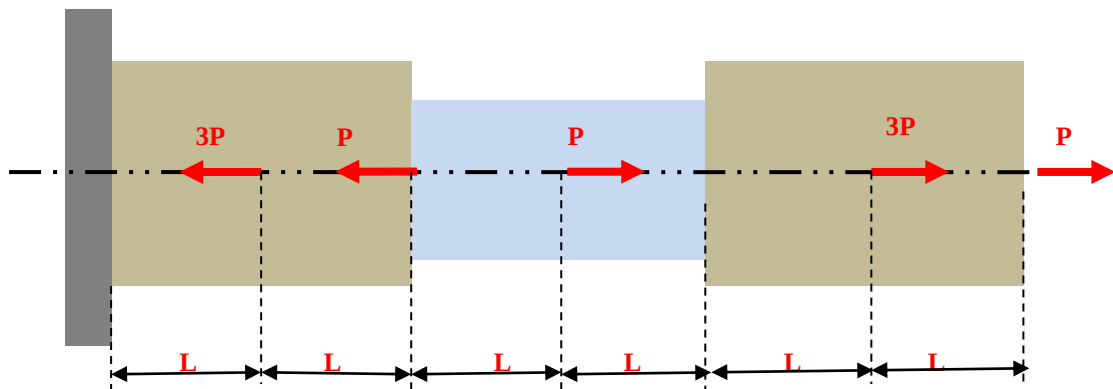
Exercise 9

An aircraft landing gear is subjected to a compressive force. On a tubular foot ($D = 250$ mm, $d = 235$ mm and modulus of elasticity $E = 0.85 \times 10^5$ MPa), the relative shortenings are: $\varepsilon_1 = 73.10^{-5}\%$ in unloaded position and: $\varepsilon_2 = 145.10^{-5}\%$ in load.

-Determine the additional load.

**Exercise 10**

Consider a steel bar of diameter $2d$ with a copper bar of diameter d inserted between them. It is subjected to an axial loading system. Plot the diagrams of normal forces, stresses, and elongations. $E_{\text{steel}} = 2.1 \times 10^5$ MPa, $E_{\text{copper}} = 1.2 \times 10^5$ MPa, $L = 0.5$ m, $d = 20$ mm, $P = 10$ kN.



CORRECTION ELEMENTS

Exercise 1

1- Traction –extension-Elongation.

2- Calculation of the value of the constraint:

$$S = \pi \cdot r^2 = 8^2 \cdot \pi = 201,06 \text{ mm}^2$$

$$\sigma = \frac{F}{S} = \frac{10000}{201,06} \left(\frac{\text{N}}{\text{mm}^2} \right) = 49,74 \text{ MPa}$$

3- If we adopt a safety coefficient of 4, calculate the elastic resistance of the steel.

$$\sigma \leq \frac{R_e}{FS} \Rightarrow R_e > \sigma \cdot FS = 49,74 \times 4 = 198,96 \text{ MPa}$$

4- Determine the elongation of the screw.

$$\frac{\Delta l}{l} = \frac{\sigma}{E} \Rightarrow \Delta l = \frac{\sigma \cdot l}{E} = \frac{49,74 \times 150}{200 \times 10^3} = 0,0373 \text{ mm}$$

Exercise 2

1- Calculation of axial forces and deformations in each bar.

a) **Bars in series:** The normal effort is the same in both sections.

$$\sum F_y = 0 \Rightarrow N + 75 = 0 \Rightarrow N = -75 \text{ KN}$$

The deformations in each section are:

$$\frac{\Delta l_1}{l_1} = \frac{\sigma}{E_1} \Rightarrow \Delta l_1 = \frac{\sigma \cdot l_1}{E_1} = \frac{N \cdot l_1}{E_1 \cdot A_1}$$

$$\Rightarrow \Delta l_1 = \frac{75 \cdot 1000}{200 \cdot 2400} = 0,156 \text{ mm}$$

$$\Delta l_2 = \frac{\sigma \cdot l_2}{E_2} = \frac{N \cdot l_2}{E_2 \cdot A_2} = \frac{75 \cdot 2000}{120 \cdot 3500} = 0,357 \text{ mm}$$

b) **Parallel bars**

$$\sum F_y = 0 \Rightarrow N_1 + N_2 + 75 = 0 \dots \dots \dots (1)$$

both bars undergo the same deformation

$$\Delta l_1 = \Delta l_2 \Rightarrow \frac{N_1 \cdot l_1}{E_1 \cdot A_1} = \frac{N_2 \cdot l_2}{E_2 \cdot A_2} \dots \dots \dots (2)$$

From (1) and (2) we obtain:

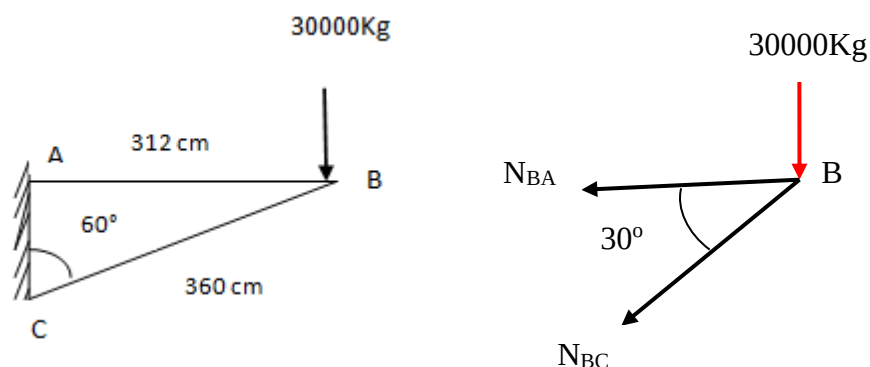
$$N_2 = \frac{-75 \cdot l_1 \cdot E_2 \cdot A_2}{l_2 \cdot A_1 \cdot E_1 + l_1 \cdot A_2 \cdot E_2} \Rightarrow \frac{315 \cdot 10^8}{96 \cdot 10^6 + 42 \cdot 10^6} = 22,83 \text{ KN}$$

$$\Rightarrow N_1 = 275 - 22,83 = 52,17 \text{ KN}$$

Hence

$$\Delta l_1 = \Delta l_2 \Rightarrow \frac{N_1 \cdot l_1}{E_1 \cdot A_1} = \frac{52,17 \times 1000}{200 \times 2400} = \frac{52,17}{480} = 0,11 \text{ mm}$$

Exercise 3



Assuming that the unknown forces are of tension, the free body diagram of the B-joint is shown in figure above.

$$\begin{cases} \Sigma F/y = -30000 - N_{BC} \sin 30^\circ = 0 \Rightarrow N_{BC} = -60000 \text{ Kg} \\ \Sigma F/x = -N_{BA} - N_{BC} \cos 30^\circ = 0 \Rightarrow N_{BA} = 52000 \text{ Kg} \end{cases}$$

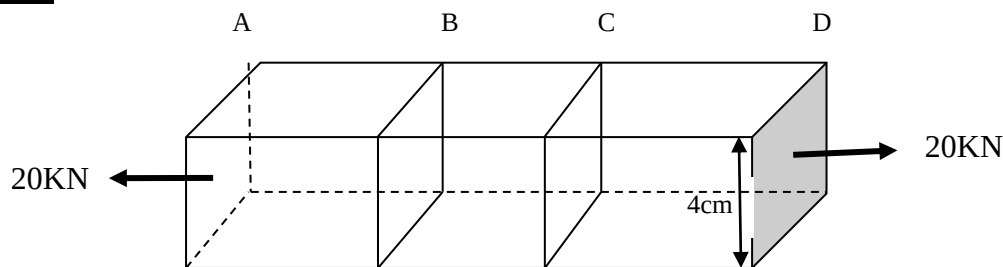
The practical constraints are:

$$\begin{cases} 4200 / 2 = 2100 \text{ Kg/cm}^2 \rightarrow \text{traction} \\ 4200 / 3.5 = 1200 \text{ Kg/cm}^2 \rightarrow \text{Compression} \end{cases}$$

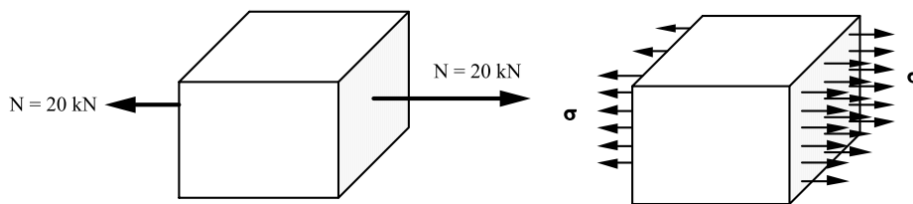
The required sections are calculated by dividing the axial force in each bar by the constraint admissible practice.

$$\sigma = \frac{N}{A} \Rightarrow A = \frac{N}{\sigma} \Rightarrow \begin{cases} A_1 = \frac{5200}{2100} = 24,7 \text{ cm}^2 \\ A_2 = \frac{6000}{1200} = 50 \text{ cm}^2 \end{cases}$$

Exercise 4



By making cuts at B and C, we can easily find that the force N is 20 kN in tension. Let's consider section CD

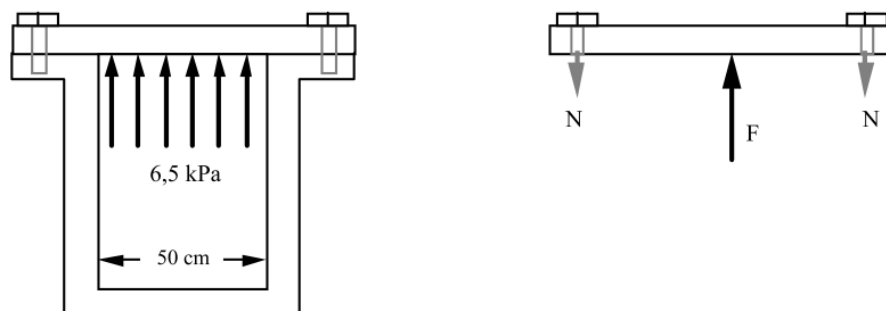


The area: $A = 4 \text{ cm} \times 4 \text{ cm} = 16 \text{ cm}^2 = 0.0016 \text{ m}^2$

$$\sigma = \frac{N}{A} = \frac{20}{0.0016} = 12500 \text{ KN/m}^2 = 12,5 \text{ MPa}$$

Exercise 5

First let's calculate the force exerted by the pressure in the tank on the lid.



$$\sigma = \frac{F}{A_1}$$

Where A_1 : Area on which the thrust is exerted

$$A_1 = \pi (0,25)^2 = 0,196 \text{ m}^2$$

$$\text{So: } F = 6,5 \cdot 10^3 \text{ (N/m}^2\text{)} \times 0,196 \text{ (m}^2\text{)} = 1274 \text{ N}$$

If we calculate the force supported by each bolt, we have:

$$\sum F_y = 1274 - 10N = 0$$

$$\Rightarrow N = 127,4N$$

Hence:

Each bolt has a section of 1.5 cm^2 , so:

$$\sigma = \frac{127,4}{1,5 \cdot 10^{-4}} = 849,33 \times 10^3 \text{ Pa} = 849,33 \text{ kPa}$$

Exercise 6

$$\sigma = \frac{N}{S}$$

σ : normal stress = 20 MPa;

N: normal tensile effect: $120 \text{ kg} \times 9.81 = 1177,2 \text{ N}$

S: area of the cross-section of the piece (mm^2)

Either:

Now, we will determine its elongation:

$$\varepsilon = \frac{\Delta l}{l_0}$$

Either:

$$\Delta l = \varepsilon \cdot l_0$$

Δl : Elongation (mm);

l_0 : Length before deformation = 3000 mm;

ε : relative elongation

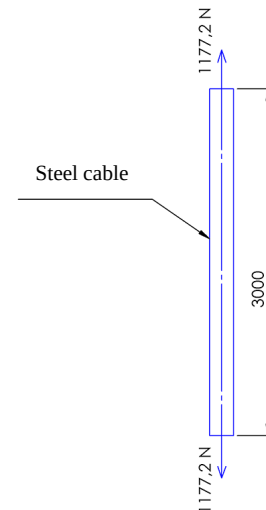
Hooke's law: $\sigma = E \cdot \varepsilon \Rightarrow \varepsilon = \frac{\sigma}{E}$

$$E = 1,55 \cdot 10^5 \text{ MPa}$$

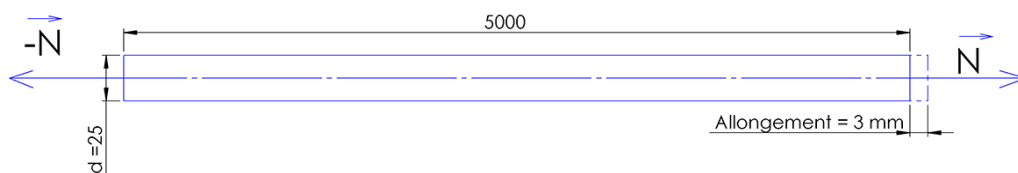
$$\sigma = 20 \text{ MPa}$$

$$\varepsilon = \frac{20}{1,55 \cdot 10^5} = 0,000129\%$$

Thus : $\Delta l = \varepsilon \cdot l_0 = 0.000129 \times 3000 = 0,387 \text{ mm}$.



Exercise 7



$$\sigma = \frac{N}{S} \Rightarrow N = \sigma \cdot S$$

N: tensile force

σ : normal constraint?

S: area of the right section of the piece?

1-Determination of normal stress:

Hooke's Law

E: modulus of elasticity = $2.1 \times 10^5 \text{ MPa}$;

ε : (epsilon) relative elongation?

$$\varepsilon = \frac{\Delta l}{l_0}$$

Δl : elongation (mm) = 3 mm;

l_0 : length before deformation = 5000 mm

So, we have the relative elongation:

$$\varepsilon = \frac{3}{5000} = 6.10^{-4}\%$$

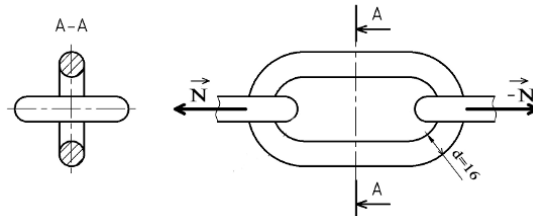
$$\sigma = E \cdot \varepsilon = 2,1 \cdot 10^5 \cdot 6.10^{-4} = 126 \text{ MPa}$$

2- determination n of the section of the tie-rod

$$S = \pi \cdot r^2 = \pi \cdot 12,5^2 = 490,87 \text{ mm}^2$$

$$\sigma = \frac{N}{S} \Rightarrow N = \sigma \cdot S = 126 \cdot 490,87 = 61849,62 \text{ N} \approx 62 \text{ KN}$$

Exercise 8



The resistance

condition of a tensile part:

$$\sigma \leq R_{pe}$$

1- Determination of practical tensile strength. $R_{pe} = \frac{R_e}{FS} = \frac{350}{5} = 70 \text{ MPa}$

2- Determination of the maximum extension force $F_{\max}$:

$$\begin{aligned} \sigma &= \frac{F_{\max}}{S} \Rightarrow F_{\max} = \sigma \cdot S \\ &= \sigma \cdot 2 \cdot \pi \cdot r^2 = 70 \cdot 2 \cdot \pi \cdot 8^2 = 28148,67 \text{ N} \\ F_{\max} &\approx 28,15 \text{ KN} \end{aligned}$$

Exercise 9

- Section search:

$$\begin{aligned} S &= S_D - S_d \\ &= \pi \frac{D^2}{4} - \pi \frac{d^2}{4} = \pi \cdot \frac{250^2}{4} - \pi \cdot \frac{235^2}{4} = 5713,77 \text{ mm}^2 \end{aligned}$$

-Determination of normal stress:

$\sigma = E \cdot \varepsilon$: Hooke's law;

$\sigma_1 = E \cdot \varepsilon_1$: Normal stress in the unloaded position;

$\sigma_2 = E \cdot \varepsilon_2$: Normal stress under load;

$$\sigma_1 = 0,85 \cdot 10^5 \cdot 73 \cdot 10^{-5} = 62,05 \text{ MPa}$$

$$\sigma_2 = 0,85 \cdot 10^5 \cdot 145 \cdot 10^{-5} = 123,25 \text{ MPa}$$

- The additional load N_{add} is:

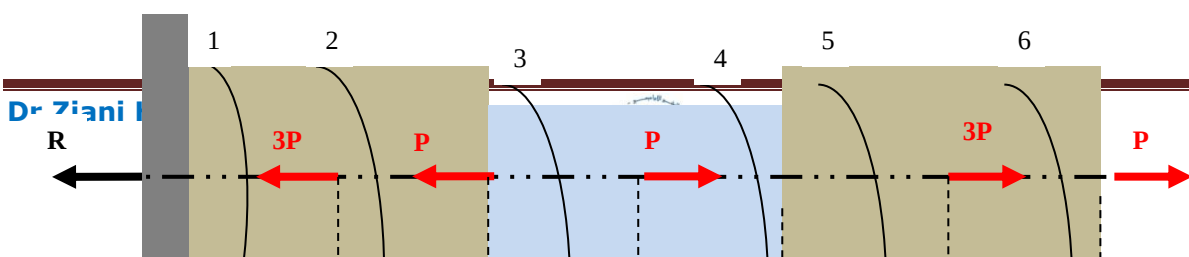
$$\sigma = \frac{N}{S} \Rightarrow N = \sigma \cdot S \Rightarrow \begin{cases} N_1 = \sigma_1 \cdot S \\ N_2 = \sigma_2 \cdot S \end{cases} \rightarrow N_{\text{add}} = N_2 - N_1$$

$$N_1 = 62,05 \times 5713,77 = 354539,43 \text{ N} = 354,54 \text{ KN}$$

$$N_2 = 123,25 \times 5713,77 = 70422,15 \text{ N} = 704,22 \text{ KN}$$

$$N_{\text{add}} = 704,22 - 354,54 = 349,68 \text{ KN}$$

Exercise 10



$$\Sigma F_x = 0 \Leftrightarrow R + 3P + P - P - 3P - P = 0$$

$$\Rightarrow R = P = 10 \text{ kN}$$

Section 2-2 $0 \leq X_2 \leq L$

Normal effort:

$$N_2 - R - 3P = 0$$

$$\Rightarrow N_2 = 3P + R = 4P$$

$$N_2 = 4P$$

Normal stress:

$$\sigma_2 = \frac{N_2}{S_2} = \frac{4P}{S_2} = \frac{4 \times 10}{\frac{\pi(2d)^2}{4}} = \frac{40}{\pi(20)^2 \times 10^{-6}} = 31,83 \text{ MPa} = 31831 \text{ KN/m}^2$$

$$\Delta l_1 = \frac{\sigma_1 \cdot X_2}{E_s}$$

$$\sigma_1 = \frac{P}{S_1} = \frac{10}{\pi \times (20)^2 \times 10^{-6}} = 7,96 \text{ MPa} = 7960 \text{ KN/m}^2$$

Elongation:

$$\Rightarrow \Delta l_1 = \frac{\sigma_1 \times l}{E_s} = \frac{7960 \times 0,5 \times 10^3}{2,1 \times 10^8} = 0,0189 \text{ mm}$$

$$\Delta l_2 = \Delta l_1 + \frac{\sigma_2 \cdot X_2}{E_s}$$

$$\text{For } X_2 = 0 \dots\dots\dots \Delta l_2 = \Delta l_1 = 0,0189 \text{ mm}$$

$$\text{For } X_2 = L = 0,5 \text{ m} \dots\dots\dots \Delta l_2 = \Delta l_1 + \frac{\sigma_2 \cdot X_1}{E_s} = 0,0189 + \frac{31831 \times 0,5 \times 10^3}{2,1 \times 10^8} = 0,0758 \text{ mm}$$

$$\Rightarrow \Delta l_2 = 0,0189 + 0,0758 = 0,094 \text{ mm}$$

Section 3-3 $0 \leq X_3 \leq L$

Normal effort:

$$R + 3P + P - N_3 = 0$$

$$N_3 = P + 3P + P = 5P$$

$$N_3 = 5P$$

Normal Stress:

$$\sigma_3 = \frac{N_3}{S_3} = \frac{5P}{S_3} = \frac{5 \times 10}{\frac{\pi(d)^2}{4}} = \frac{50}{\pi(10)^2 \times 10^{-6}} = 159,154 \text{ MPa} = 159154,94 \text{ KN/m}^2$$

Elongation:

$$\Delta l_3 = \Delta l_2 + \frac{\sigma_3 \cdot X_3}{E_c} = 0,094 + \frac{159154,94 \times 0,5 \times 10^3}{1,2 \times 10^8} = 0,757 \text{ mm}$$

Section 4-4 $0 \leq X_4 \leq L$

Normal effort

$$N_4 + P - P - 3P - R = 0$$

$$N_4 = 4P$$

Normal stress

$$\sigma_4 = \frac{N_4}{S_4} = \frac{4P}{S_3} = \frac{4 \times 10}{\frac{\pi(d)^2}{4}} = \frac{40}{\pi(10)^2 \times 10^{-6}} = 127,32 \text{ MPa} = 127320 \text{ KN/m}^2$$

$$\Delta l_4 = \Delta l_3 + \frac{\sigma_4 \cdot X_4}{E_c} = 0,757 + \frac{127320 \times 0,5 \times 10^3}{1,2 \times 10^8}$$

$$\Delta l_4 = 0,757 + 0,5303 = 1,2875 \text{ mm}$$

Section 5-5 $0 \leq X_5 \leq L$

Normal effort

$$N_5 + P - P - 3P - R = 0$$

$$N_5 = 4P$$

Normal stress

$$\sigma_5 = \frac{N_5}{S_5} = \frac{4P}{S_5} = \frac{4 \times 10}{\frac{\pi(2d)^2}{4}} = \frac{40}{\pi(20)^2 \times 10^{-6}} = 31,83 \text{ MPa} = 31831 \text{ KN/m}^2$$

Elongation

$$\Delta l_5 = \Delta l_4 + \frac{\sigma_5 \cdot l}{E_s} = 1,2875 + \frac{31831 \times 0,5 \times 10^3}{2,1 \times 10^8}$$

$$\Delta l_5 = 1,2875 + 0,0758 = 1,3633 \text{ mm}$$

Section 6-6 $0 \leq X_6 \leq L$

Normal effort

$$N_6 + 3P + P - P - 3P - P = 0$$

$$N_6 = P$$

Normal stress

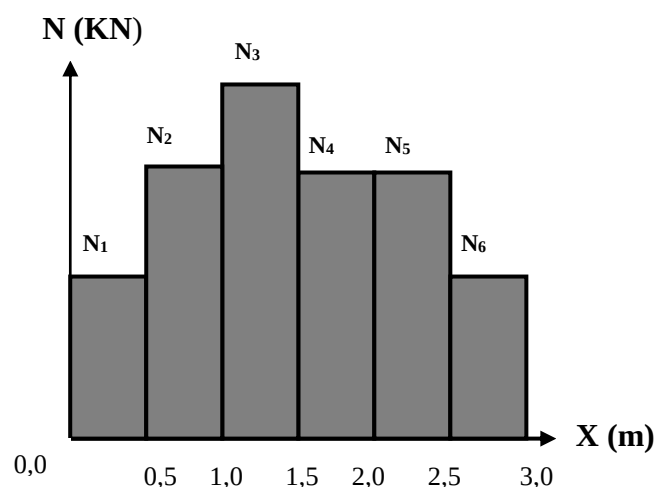
$$\sigma_6 = \frac{N_6}{S_6} = \frac{4P}{S_5} = \frac{10}{\frac{\pi(2d)^2}{4}} = \frac{10}{\pi(20)^2 \times 10^{-6}} = 7,9577 \text{ MPa} = 7957,75 \text{ KN/m}^2$$

$$\Delta l_6 = \Delta l_5 + \frac{\sigma_6 \cdot l}{E_s} = 1,3633 + \frac{7,9577 \times 0,5 \times 10^3}{2,1 \times 10^5}$$

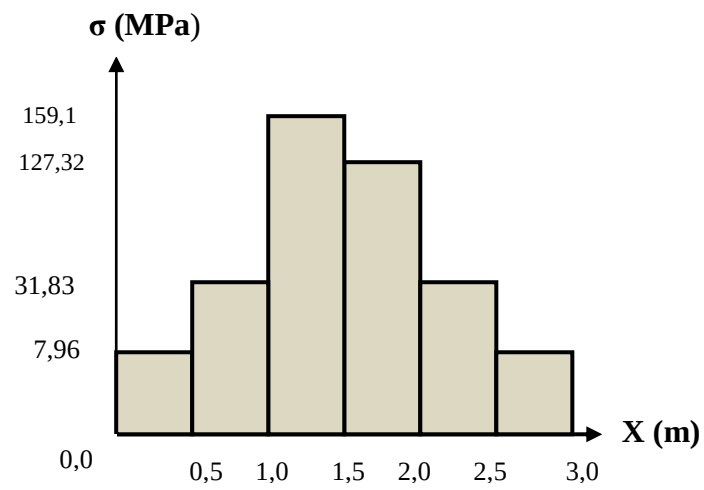
$$\Delta l_6 = 1,3633 + 0,0189 = 1,3822 \text{ mm}$$

Diagrams

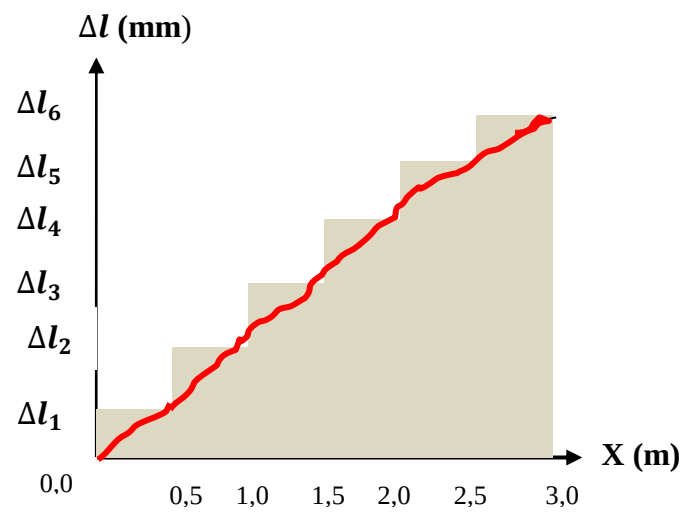
a- Normal effort diagram



b-Stresses diagram



c- Diagram of elongations



Simple Flexion

4.1 Introduction

Bending is the deformation of an object under the action of forces applied perpendicularly to its neutral axis (forces acting at a distance, constraint forces). These forces, whether located in the plane of symmetry or distributed symmetrically with respect to it, manifest as curvature. In the case of a beam, this curvature tends to bring its two ends closer together. The bending test of a beam is a mechanical test used to assess its flexural strength.

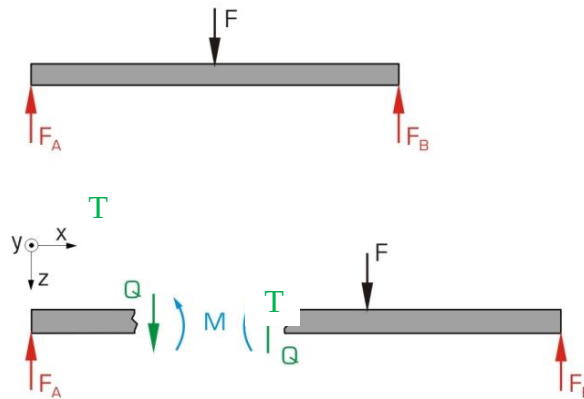


Fig. 4.1 Modeling of external forces on a beam in simple bending

4.2 Pure and simple bending

We call bending the effect produced in the straight sections of a beam, by a bending moment whose external forces acting to the left of a fictitious right section SS' and reduced to its center of gravity are equivalent to M, N, and T.

4.2.1 Definitions

A beam is subjected to bending when the forces applied to it tend to modify its curvature.

Pure bending is a state of load such that, in any straight section of a part, there exists only one bending moment M_f . This bending moment must be constant. Fig 4.2 represents a bar subjected to pure bending. The two moments C act in a plane of longitudinal symmetry of the beam. M_f being constant, the bar will deform in the same way along its length. It takes the form of an arc of a circle:

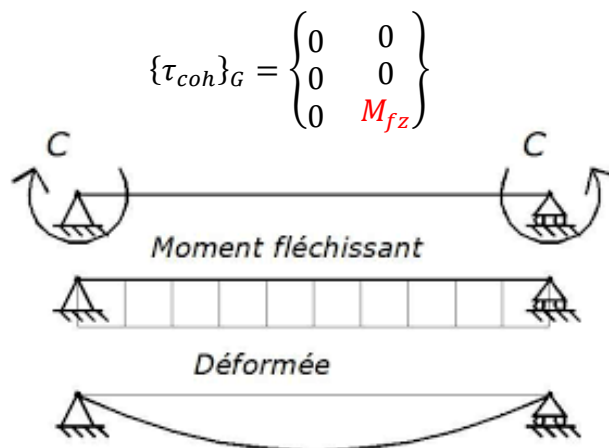


Fig 4.2 Beam at pure flexion

Simple bending (or transverse bending) is a state of load, such that in any section of a part there exists a bending moment M_f (M_{fy} or M_{fz}) and an associated shear force T (T_y or T_z) (Fig 4.3). The torsor associated with the cohesion efforts can be reduced to G , barycenter of the right section S .

$$\{\tau_{coh}\}_G = \begin{Bmatrix} 0 & 0 \\ T_y & 0 \\ 0 & M_{fz} \end{Bmatrix}$$

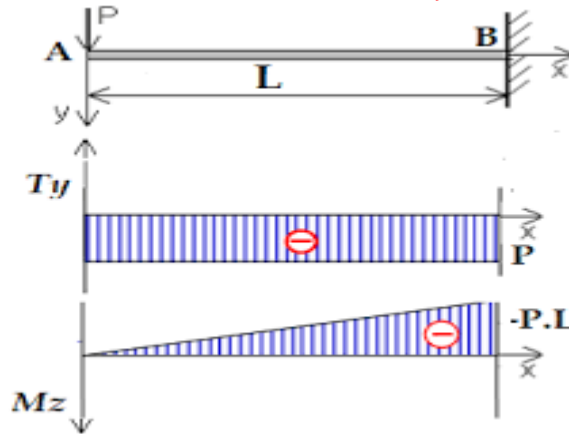


Fig 4.3 Beam in simple bending

4.2.2 Longitudinal sliding

This deformation can be demonstrated by placing two nearby supports (example: 1 m apart) three (3) boards of 1.20 m in length and the same thickness (2.2 cm). It is observed that the end sections remain in the same vertical plane; however, if we take the 3 boards of the same thickness (2.2cm) but with a span of 3m subject to overload, there is a relative longitudinal slip of the boards (Fig. 4.4).

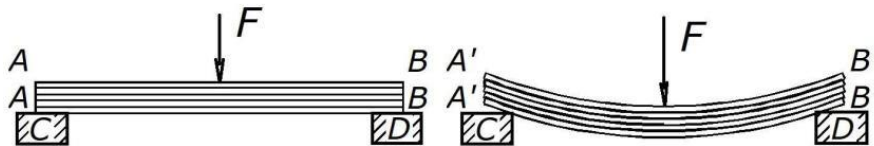


Fig. 4.4 Longitudinal sliding in a beam in flexion

The failure of a beam occurs through longitudinal or transverse shearing due to loads applied to it after it has exceeded its breaking strength. In this course, we will limit ourselves to the study of the bending of simply supported straight beams, i.e., those for which the equilibrium equations are sufficient to determine the support reactions, and also to beams whose plane of symmetry is vertical (G_{xy}) and whose bending moments act in this plane, perpendicular to the major axis of the beam (Fig. 4.5).

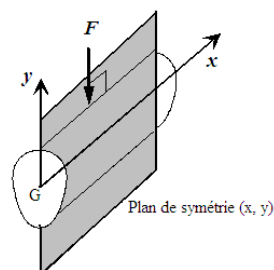


Fig 4.5 Beam having a plane of symmetry

Note

1. The study of bending begins with pure bending, then the more general case of simple bending is considered.

2. However, the deformation of the plane of the cross-sections does not significantly affect the magnitude of the normal stresses.

3. A bar subjected primarily to bending is called a beam.

4.2.3 Calculation assumptions

- Deformations are elastic and obey the Assumptions of Small Deformations (ASD).
- Any fiber contained in a plane of symmetry remains in that plane during deformation.
- Navier-Bernoulli hypothesis (1705): the straight sections of the beam remain flat and perpendicular to its axis after deformation.
- Any fiber contained in a plane parallel to the plane of symmetry remains in this plane during deformation. That is why bending is called planar.

4.2.4 Reduction of forces

Let the following bar be in equilibrium under the effect of the forces including the support reactions applied; let SS' be a straight section with abscissa x, and G its center of gravity (Fig 4.6).

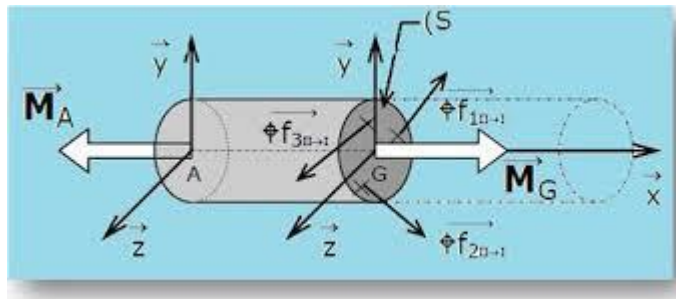


Fig 4.6 reduction of forces relative to G

Since the part is in equilibrium, the sum of the projections of all external forces onto the Ox and Oy axes is zero, as is the sum of their moments with respect to G. The plane containing the cross-section SS' delimits the two zones such that the external forces are distributed into two groups:

- 1- The forces located to the left of SS have a resultant force R_g .
- 2- The forces located to the right of SS have a resultant force R_d .

However, the part is in equilibrium:

$$MR_g + MR_d = 0 \Rightarrow MR_g = -MR_d \dots \dots \dots (I)$$

Therefore: The sum of the moments of the external forces acting to the left of a section is equal to the sum, with the sign reversed, of the moments of the external forces acting to the right of that section. If we reduce to G the forces acting to the right of the section SS', we will likewise obtain the three (03) quantities: M' , N' , and T' , and by virtue of equation (I), we can write:

$$\Rightarrow \begin{cases} N + N' = 0 \\ T + T' = 0 \\ M + M' = 0 \end{cases} \begin{cases} N = -N' \\ T = -T' \\ M = -M' \end{cases}$$

4.3 Bending moment and shear force diagrams

4.3.1 Bending moment and shear force

The bending moment M_f is the result of the application of a force that tends to bend a structural element (beam). It is calculated by multiplying the force by its distance perpendicular to the point under consideration, whose graphical representation makes it possible to locate the critical areas where the risk of rupture is highest.

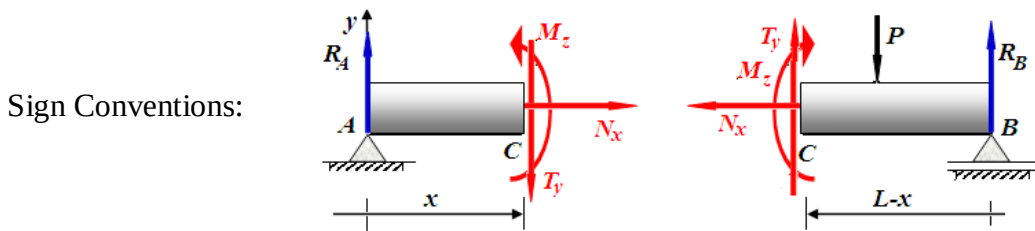


Fig. 4.7 Sign convention for $M_f(x)$ and $T(x)$

$$\begin{array}{l|l}
 \begin{array}{l}
 \blacksquare \sum F_x = 0 \Rightarrow N(x) = 0 \\
 \blacksquare \sum F_y = 0 \Rightarrow T(x) = \frac{pb}{L} \\
 \blacksquare \sum M_f(x) / C = 0 \Rightarrow M_f(x) = \frac{pb}{L} \cdot x
 \end{array}
 &
 \begin{array}{l}
 \blacksquare \sum F_x = 0 \Rightarrow N(x) = 0 \\
 \blacksquare \sum F_y = 0 \Rightarrow T(x) = P - \frac{Pa}{L} \Rightarrow T(x) = \frac{Pa}{L} \\
 \blacksquare \sum M_f(x) / C = 0 \Rightarrow M_f(x) = \left(\frac{Pa}{L}\right) \cdot (L-x) - p(L-x-b) \\
 \blacksquare \sum M_f(x) / C = 0 \Rightarrow M_f(x) = \left(\frac{Pa}{L}\right) \cdot (L-x) - p(L-x-b) \\
 \Rightarrow M_f(x) = \frac{Pa}{L} \cdot x
 \end{array}
 \end{array}$$

→ The bending moment M_f : is positive $\Rightarrow$ if it tends to pull the lower longitudinal fibers of the beam.

→ The associated shear force T : is positive $\Rightarrow$ if it tends to make the small element rotate clockwise.

The load $p(x)$ is positive: if it acts downwards.

4.3.2 Physical meaning

The bending moment (M_f) and shear force (T) are crucial in the strength of materials for dimensioning beams: they determine the bending (cracking/rupture) and the shearing, respectively.

The M_{fmax} indicates the breakpoint, while T represents the shear force.

→ The shear force T represents the integral transmissions of loads to supports.

→ The bending moment M_f depends on the position of the loads and the spacing between the supports. This is the only effort that depends on the length of the beam. The shear force allows us to determine the transverse reinforcement, while the bending moment gives the longitudinal reinforcement in a reinforced concrete beam.

4.3.3 M, T, N diagrams

The values of M , T , N depend on the abscissa of the point of the part under consideration; the diagrams for these elements are their graphical representations. The relationships between M , T and p make it possible to specify the nature and form of the diagrams of $M(x)$ and $T(x)$ represented in the 2D system.

To construct the diagrams of bending moments and shear forces, make dummy cuts after each loading and determine the expression of $M_f(x)$ and $T(x)$ by balancing the section between one end of the beam and the cut-off. The graphical representation of $M_f(x)$ and $T(x)$ is the translation of these equations into diagrams while ensuring the continuity of the beam and respecting the sign convention.

▪ Shear effort in a section : $\sum F_{yi} = 0$

→ Sum of forces (including reactions) located to the right (or left) of the section considered.

▪ Bending moment in a section: $\sum M_{fz} = 0$

→Sum of the moments of all forces (including reactions) located to the right (or left) of the section considered.

◆ $M_f(x)$ Diagrams

When $T(x) = 0 \Rightarrow M_f'(x) = 0 \rightarrow$ therefore the tangent to $M_f(x)$ is horizontal and passes through an extremum. $T(x)$ represents the direction parameter, with a change of sign, of the tangent to the diagram $M_f(x)$ at the point with the considered abscissa. In particular, note:

1/ for $x=0$, $T(0) = R_A \rightarrow$ left support reaction.

2/ for $x=l$, $T(l) = -R_B \rightarrow$ right-hand support reaction, changed in direction since it changed sign.

◆ $T(x)$ Diagrams

1- If $p=0$ (no charge distributed between two concentrated charges), $T'(x)=0$, therefore $T(x)=$ constant; the graph of $T(x)$ is a straight line parallel to the x -axis, and the graph of $M_f(x)$ will be a straight line of the form: $y = ax + b$, with $a > 0$ if $T(x) < 0$ and vice versa;

2- If $p =$ constant, a uniformly distributed charge, $T(x)$ is a straight line of the form: $px + \alpha$, and $M_f(x)$ a quadratic parabola of the form: $-\frac{px^2}{2} - \alpha x + \beta$;

3- If $p = ax + b$, trapezoidal variable overload, we will have:

$$T(x) = \frac{ax^2}{2} + bx + c \text{ (Second-degree parabola)}$$

and

$$M_f(x) = -\frac{ax^3}{6} - \frac{bx^2}{2} - cx + d \text{ (Second-degree parabola)}$$

4.3.4 Key points for drawing diagrams

Concentrated force: Produces a jump, level change in the diagram of $T(x)$ and an angular point (slope change) in the diagram of $M_f(x)$.

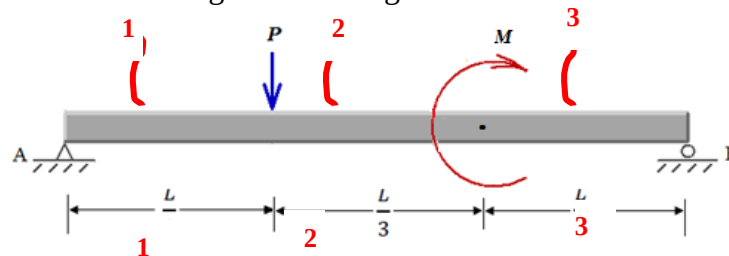
Distributed load: Produces a linear variation of $T(x)$ and a parabolic variation of the bending moment $M_f(x)$.

Application 1

Let the beam (below) of length L simply be supported at both ends A and B, is subjected to a concentrated load P at $x = L/3$ from point A and a moment $M_0 = PL/4$ at $x = L/3$ from point B.

1-Determine the support reactions and the equations of $T(x)$ and $M(x)$.

2- Draw the shear force and bending moment diagrams.



Solution

1-Determination of support reactions at A and B

$$\begin{cases} \sum M_B = 0 \Rightarrow R_A \cdot L - \frac{2PL}{3} + \frac{PL}{4} = 0 \Rightarrow R_A = \frac{5P}{12} \\ \sum M_A = 0 \Rightarrow R_B \cdot L - \frac{PL}{4} - \frac{PL}{3} = 0 \Rightarrow R_B = \frac{7P}{12} \end{cases}$$

-Determination of internal efforts $T(x)$ and $M(x)$

Section 1-1

$$0 \leq x \leq \frac{L}{3}$$

$$\sum F_{yi} = 0 \Rightarrow R_A - T = 0$$

$$\Rightarrow T = R_A = \frac{5P}{12}$$

$$\sum M_{/c} = 0 \Rightarrow R_A \cdot x - M_f(x) = 0$$

$$\Rightarrow M_f(x) = R_A \cdot x = \frac{5P}{12} \cdot x \Rightarrow \begin{cases} M_f(0) = 0 \\ M_f\left(\frac{L}{3}\right) = \frac{5PL}{36} \end{cases}$$

Section 2-2

$$\frac{L}{3} \leq x \leq \frac{2L}{3}$$

$$\sum F_{yi} = 0 \Rightarrow R_A - P - T = 0$$

$$\Rightarrow T = R_A - P = \frac{5P}{12} - P = -\frac{7P}{12}$$

$$\sum M_{/c} = 0 \Rightarrow R_A \cdot x - P \left(x - \frac{L}{3}\right) - M_f(x) = 0$$

$$\Rightarrow M_f(x) = \frac{5P}{12} \cdot x - P \left(x - \frac{L}{3}\right) \Rightarrow \begin{cases} M_f\left(\frac{L}{3}\right) = \frac{5PL}{36} \\ M_f\left(\frac{2L}{3}\right) = -\frac{2PL}{36} \end{cases}$$

We observe that $M_f x$ changes sign over the interval $x \in \left[\frac{L}{3}, \frac{2L}{3}\right]$

$$\Rightarrow M_f(x) = \frac{5P}{12} \cdot x - P \left(x - \frac{L}{3}\right) = 0 \Rightarrow x_0 = \frac{4L}{7}$$

Section 3-3

$$0 \leq x \leq \frac{L}{3}$$

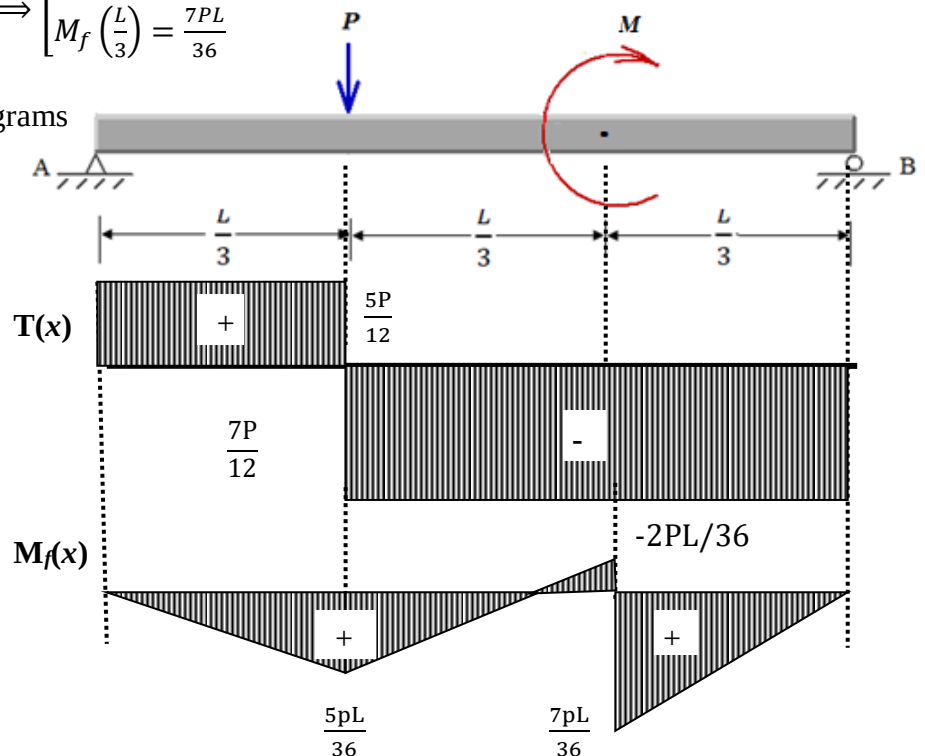
$$\sum F_{yi} = 0 \Rightarrow R_B + T = 0$$

$$\Rightarrow T = -R_B = -\frac{7P}{12}$$

$$\sum M_{/c} = 0 \Rightarrow -R_B \cdot x + M_f(x) = 0$$

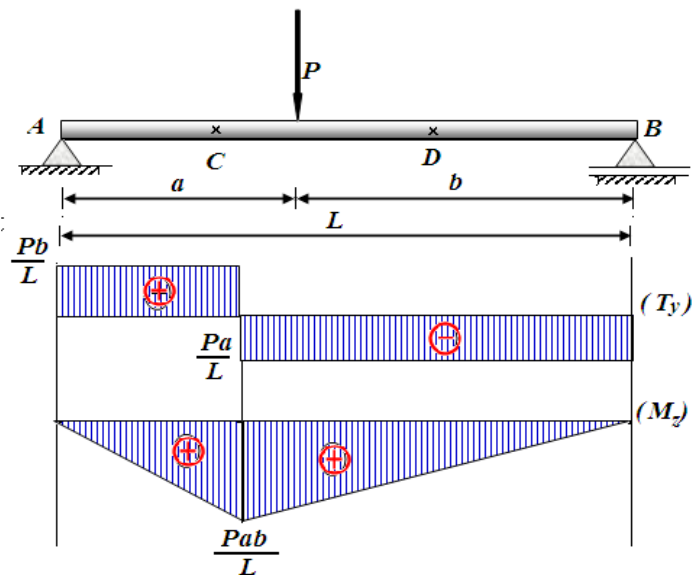
$$\Rightarrow M_f(x) = \frac{7P}{12} \cdot x \Rightarrow \begin{cases} M_f(0) = 0 \\ M_f\left(\frac{L}{3}\right) = \frac{7PL}{36} \end{cases}$$

2- T and M_f Diagrams



Application 2

Determine the equations of internal forces for the beam below and verify the correspondence of these diagrams with those drawn under the beam.



Solution

1-Determination of support reactions at A and B

$$\begin{cases} \sum F_{/x} = 0 \Rightarrow R_A + R_B = P \\ \sum M_{/A} = 0 \Rightarrow R_B \cdot L - P \cdot a = 0 \Rightarrow R_B = \frac{P \cdot a}{L} \end{cases}$$

$$\Rightarrow R_A = P - R_B = P - \frac{P \cdot a}{L} = \frac{P(L - a)}{L}$$

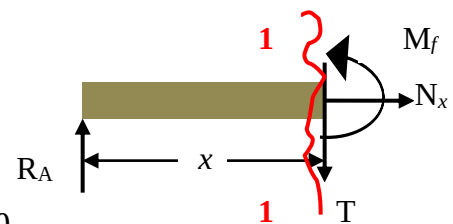
$$\Rightarrow R_A = \frac{Pb}{L}$$

- $T(x)$ and $M_f(x)$ equations

Section 1-1

$$\begin{cases} 0 \leq x \leq a \\ \sum F_{/x} = 0 \Rightarrow N_x = 0 \\ \sum F_{/y} = 0 \Rightarrow R_A - T_y = 0 \Rightarrow T_y = R_A = \frac{P \cdot b}{L} \\ \sum M_{1-1} = 0 \Rightarrow R_A \cdot x - M_f(x) = 0 \end{cases}$$

$$\Rightarrow M_f(x) = \frac{Pb}{L}x \Rightarrow \begin{cases} M_f(0) = 0 \\ M_{f1}(a) = \frac{P \cdot b}{L} \cdot a \end{cases}$$

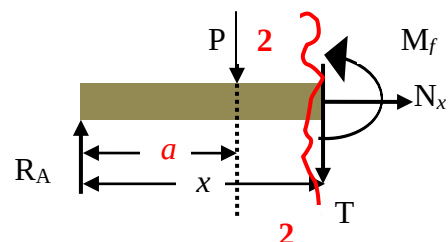


So : $T_y = R_A = \frac{P \cdot b}{L} = C^{ste}$ sur $0 \leq x \leq a \rightarrow T_y$ is a straight line of equation : $T_y = \frac{P \cdot b}{L}$

$M_f(x) = \frac{Pb}{L}x$, is a linear function of variable x

Section 2-2

$$\begin{cases} a \leq x \leq L \\ \sum F_{/x} = 0 \Rightarrow N_x = 0 \\ \sum F_{/y} = 0 \Rightarrow R_A - P - T_y = 0 \Rightarrow T_y = R_A - P \\ \sum M_{2-2} = 0 \Rightarrow R_A \cdot x - P(x - a) - M_{f2}(x) = 0 \end{cases}$$



$$\begin{aligned}
 T_y &= R_A - P = \frac{P.b}{L} - \frac{P.L}{L} = \frac{P}{L}(b - L) = -\frac{P.a}{L} = C^{\text{ste}} \text{ on : } a \leq x \leq L \\
 \Rightarrow M_f(x) &= R_A \cdot x - P(x - a) \\
 \Rightarrow M_f(x) &= \frac{P.b}{L} \cdot x - P \cdot x + P \cdot a = x \left(\frac{P.b}{L} - P \right) + P \cdot a \\
 \Rightarrow M_f(x) &= x \left(\frac{P.b}{L} - P \right) + P \cdot a \Rightarrow \begin{cases} M_f(a) = \frac{P.b}{L} \cdot a \\ M_f(L) = 0 \end{cases}
 \end{aligned}$$

What justified the good correspondence with the diagrams of M_f and T_y

4.4 Relationship between bending moment and shear force

Consider a beam element between two infinitely close sections (Σ) and (Σ') separated by a distance dx . As shown in Fig. 4.8, where: The ignored right-hand portion of the element is replaced by T and M_f . The ignored left-hand portion of the element is replaced by T' and M_f' .

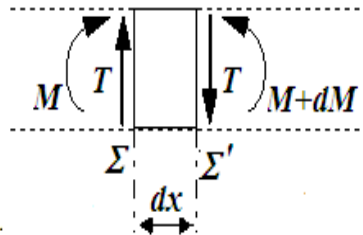


Fig.4.8 Unloaded isolated beam section

There is no force applied to the beam between sections (Σ) and (Σ').

So, their shear forces are equal ($T' = T$). On the other hand, the bending moments M and M' ($M' = M + dM$) are different.

The equilibrium of the element is written: $\uparrow \sum F_y = 0 \Rightarrow T + T - dT = 0$

$$\Rightarrow T = \frac{dM}{dx} = M'_f(x)$$

The shear force T is the derivative with respect to the abscissa x of the bending moment.

4.5 Relationship between shear force and distributed loading

Let a beam element support a load uniformly distributed between two sections (Σ) and (Σ'), infinitely close, as shown in the following figure.

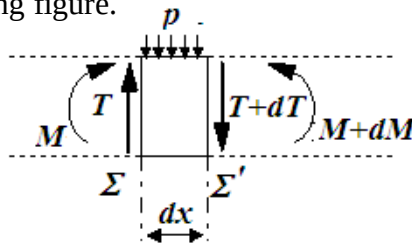


Fig.4.9 Uniformly loaded insulated beam section (p)

- The equilibrium of forces gives: $\sum F_y = 0 \Rightarrow T - p \, dx - T - dT = 0$
 $\Rightarrow T - p \, dx = 0$
 $\Rightarrow dT + p \, dx = 0$

$$\Rightarrow p(x) = -\frac{dT}{dx}$$

- The equilibrium of moments gives:

$$\sum M_y = 0 \Rightarrow M + T \, dx - p \, dx \cdot \frac{dx}{2} - M - dM = 0$$

negligible

We find the same result, i.e.:

$$\Rightarrow T = \frac{dM}{dx} = M'(x)$$

Note: When a concentrated charge is exerted between (Σ) and (Σ'), (Fig. 6.8), the equilibrium is written:

$$T' = T - F$$

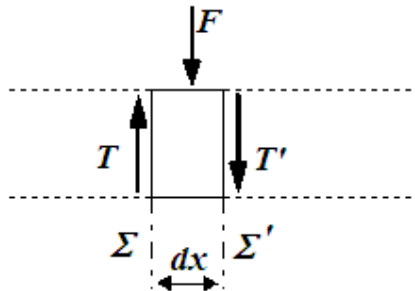


Fig.4.10 Isolated beam section under concentrated load F

Application 1

Consider a beam AB of length 3m, uniformly loaded.

- 1- Determine the support reactions at A and B.
- 2- Determine the equations of T and M_f , then draw their diagrams.

Solution

1-Détermination of support reactions.

- Calculation of F_q and x

$$F_q = q \times L = 3 \times 30 = 90 \text{ daN}$$

$$x = \frac{L}{2} = \frac{3}{2} = 1.5 \text{ m}$$

Using equilibrium equation, we have:

- $\sum F_y = 0 \Rightarrow R_A - F_q + R_B = 0 \Rightarrow R_A + R_B = F_q$
- $\sum M_A = 0 \Rightarrow F_q \times x - R_B \times l = 0$
 $\Rightarrow 1.5 F_q - 3 R_B = 0$
 $\Rightarrow R_B = 45 \text{ daN}$ et $R_A = 45 \text{ daN}$

$T(x)$ et $M_f(x)$ diagrams

$$0 \leq x \leq 3$$

$$\sum F_y = 0 \Rightarrow R_A - q \cdot x - T = 0$$

$$\Rightarrow T(x) = -q \cdot x + R_A$$

$$\Rightarrow T(x) = 45 - 30x$$

$$\Rightarrow T(x) = \begin{cases} T(0) = +45 \text{ daN} \\ T(3) = -45 \text{ daN} \end{cases}$$

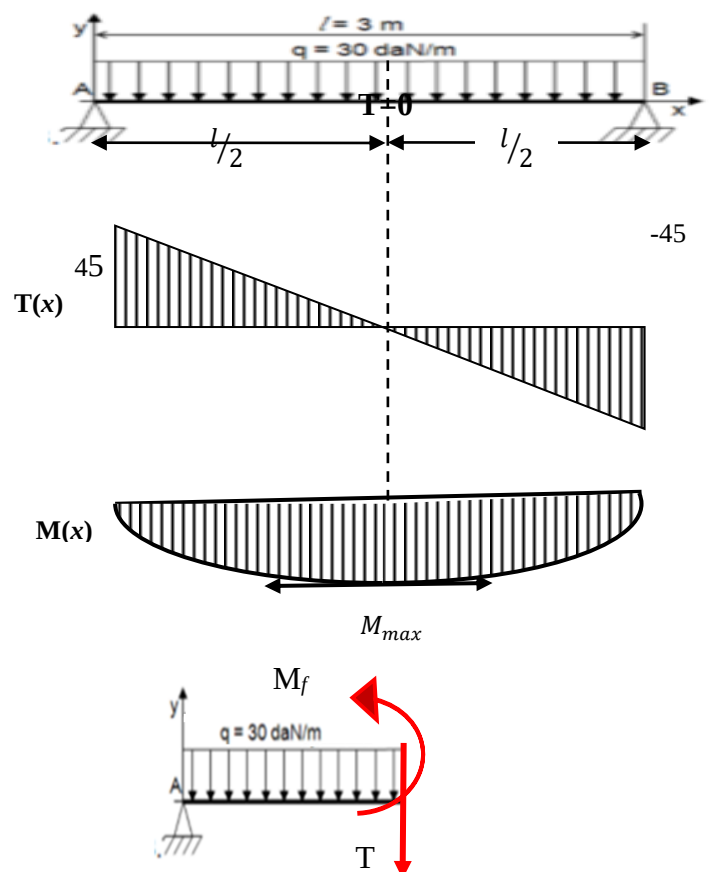
$$\sum M_C = 0 \Rightarrow -M_f(x) - q \cdot x \cdot \frac{x}{2} + R_A \cdot x = 0$$

$$\Rightarrow M_f(x) = 45 \cdot x - 30 \cdot \frac{x^2}{2}$$

$$\Rightarrow M_f(x) = \begin{cases} M_f(0) = 0 \\ M_f(3) = 0 \end{cases}$$

$$\text{Or: } T(x) = \frac{dM}{dx} = \left[45 \cdot x - 30 \cdot \frac{x^2}{2} \right]' = 45 - 30x$$

$$T(x) = 0 \Rightarrow 45 - 30x = 0$$



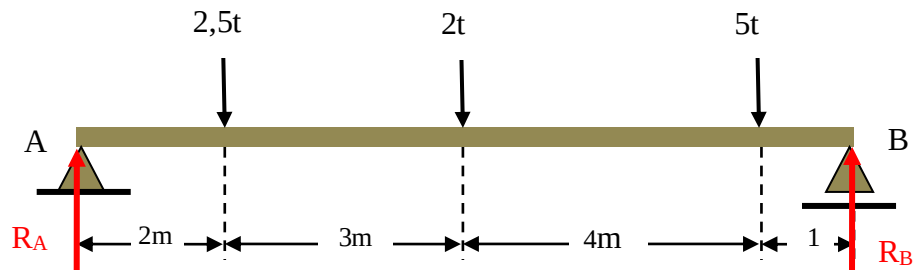
$$\Rightarrow x = 45/30 = 1.5\text{m}$$

$$\Rightarrow M(1.5) = 45 \cdot (1.5) - 30 \cdot \frac{(1.5)^2}{2} = 33,75 \text{ daN.m}$$

$$q = -\frac{dT}{dx} = -[45 - 30x]' = 30 \text{ daN/m}$$

Application 2

One beam on two supports receives several concentrated loads.
Determine the reactions and draw the diagrams of M and T.



Solution

1- Determination of the support reactions

$$\bullet \sum F_y = 0 \Rightarrow R_A - 2,5 - 2 - 5 + R_B = 0 \Rightarrow R_A + R_B = 9,5t$$

$$\bullet \sum M_A = 0 \Rightarrow 10 \cdot R_B - 5 \cdot 9 - 2,5 - 2,5 \cdot 2 = 0$$

$$\Rightarrow 10 \cdot R_B = 60$$

$$\Rightarrow R_B = \frac{60}{10} = 6t$$

$$\Rightarrow R_A = 9,5 - 6 = 3,5t$$

2- Internal forces in the beam

Siction1-1

$$0 < x < 2$$

$$\sum F_y = 0 \Rightarrow R_A - T = 0 \Rightarrow T = R_A = 3,5t$$

$$\sum M_{/1-1} = 0 \Rightarrow 3,5x - M_f(x) = 0$$

$$\Rightarrow M_f(x) = 3,5x \Rightarrow \begin{cases} M_f(0) = 0 \\ M_f(2) = 7t.m \end{cases}$$

• Siction2-2

$$\sum M_{/2-2} \Rightarrow 3,5x - 2,5(x-2) - M_f(x) = 0$$

$$\Rightarrow M_f(x) = 3,5x - 2,5(x-2)$$

$$\Rightarrow M_f(x) = x + 5 \Rightarrow \begin{cases} M_f(2) = 7t.m \\ M_f(5) = 10t.m \end{cases}$$

$$T = \frac{dM_f(x)}{dx} = (x+5)' = 1t$$

• Siction 3-3

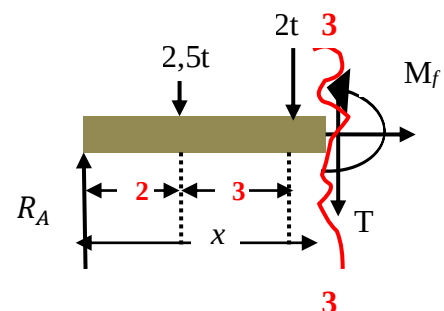
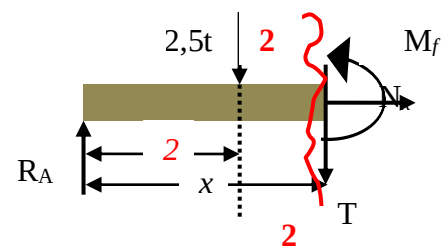
$$5 < x < 9$$

$$\sum M_{/3-3} \Rightarrow 3,5x - 2,5(x-2) - 2(x-5) - M_f(x) = 0$$

$$\Rightarrow M_f(x) = 3,5x - 2,5(x-2) - 2(x-5)$$

$$M_f(x) = \begin{cases} M_f(5) = 10t.m \\ M_f(9) = 6t.m \end{cases}$$

$$T = \frac{dM_f(x)}{dx} = (3,5x - 2,5(x-2) - 2(x-5))' = -1t$$



• Siction 4-4

$0 < x < 1$ (change of origin (from right to left))

$$\sum M/_{3-3} \Rightarrow 6x - M_f(x) = 0$$

$$\Rightarrow M_f(x) = 6x \Rightarrow \begin{cases} M_f(0) = 0 \\ M_f(5) = 6t.m \end{cases}$$

$$T = -\frac{dM_f(x)}{dx} = (6x)' = -6t$$

Or ;

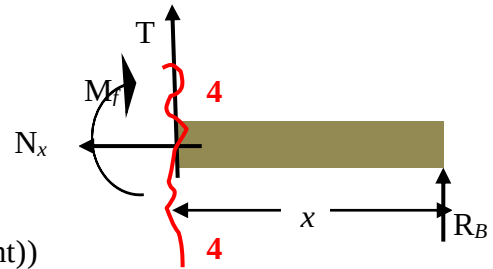
$9 < x < 10$ (Continuing along the same path (from left to right))

$$\sum M/_{4-4} \Rightarrow 3,5x - 2,5(x-2) - 2(x-5) - 5(x-9) - M_f(x) = 0$$

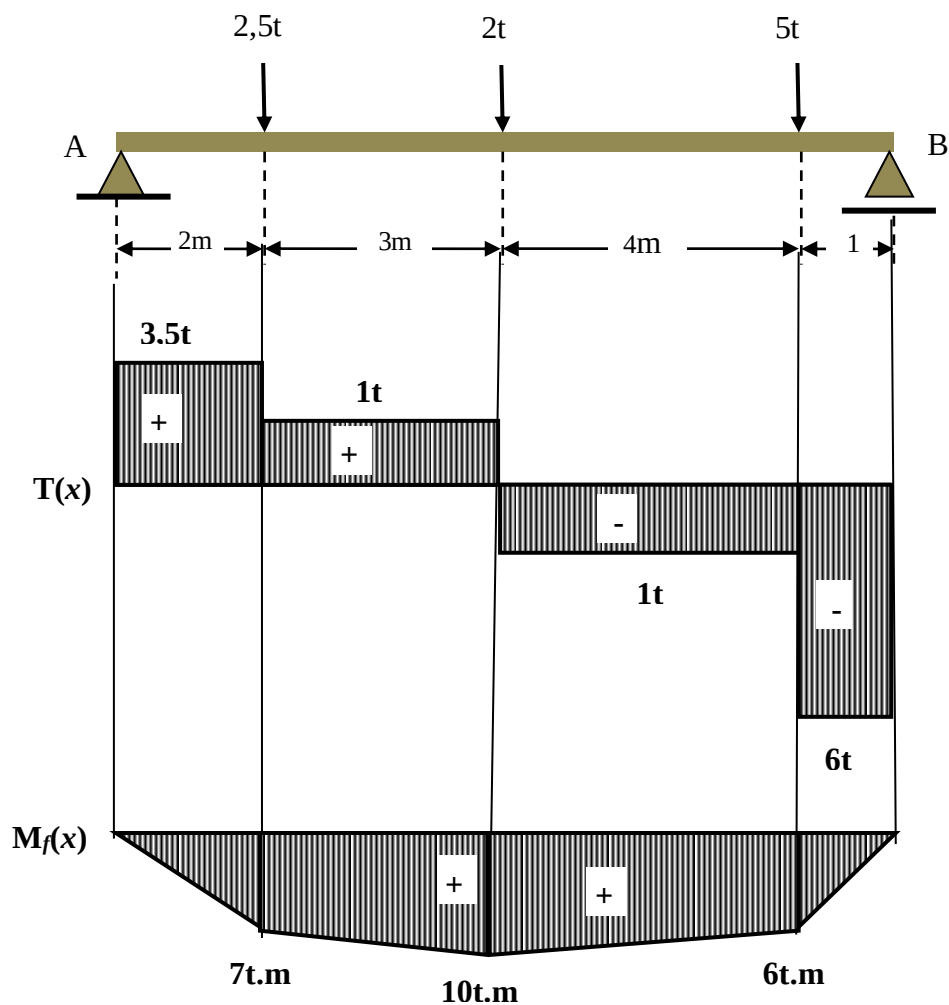
$$M_f(x) = 3,5x - 2,5x + 5 - 2x + 10 - 5x + 45$$

$$\Rightarrow M_f(x) = -6x + 60 = \begin{cases} M_f(9) = 6t.m \\ M_f(5) = 0 \end{cases}$$

$$\Rightarrow T = \frac{dM_f(x)}{dx} = (-6x + 60)' = -6t$$



2- $M_f(x)$ and $T(x)$ Diagrams



Application 3

A beam fixed at one end receives a load uniformly distributed along its entire length. Determine M_f and T .

Solution

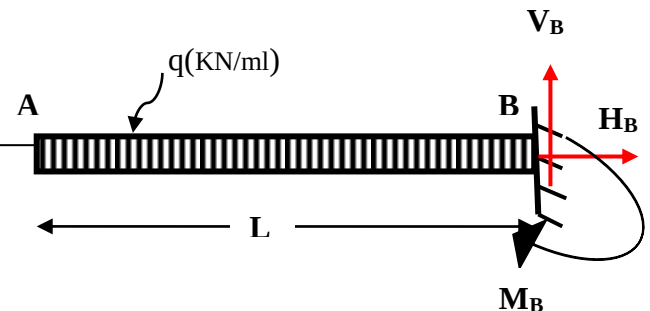
1- Support reactions, we have:

$$\sum F_x = 0 \Rightarrow H_B = 0$$

$$\sum F_y = 0 \Rightarrow V_B - q \cdot L = 0 \Rightarrow V_B = q \cdot L$$

$$\sum M_B = 0 \Rightarrow \frac{q \cdot L^2}{2} - M_B = 0$$

$$\Rightarrow M_B = \frac{q \cdot L^2}{2}$$



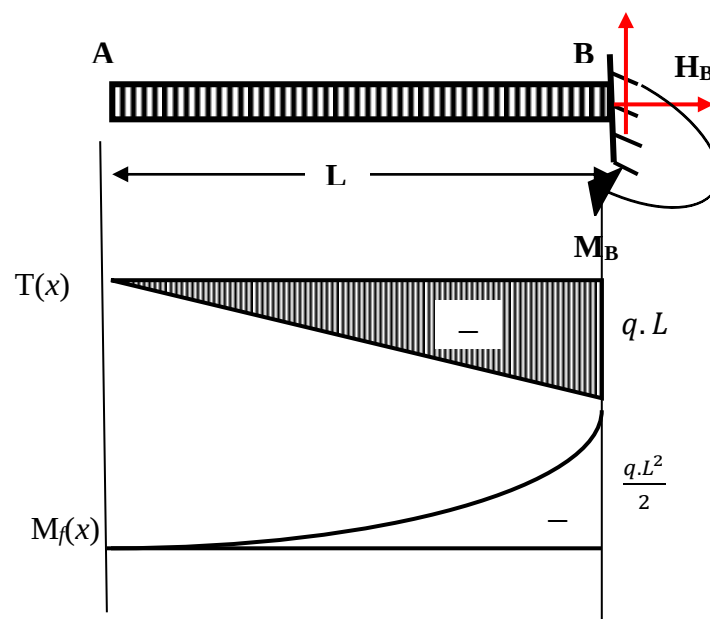
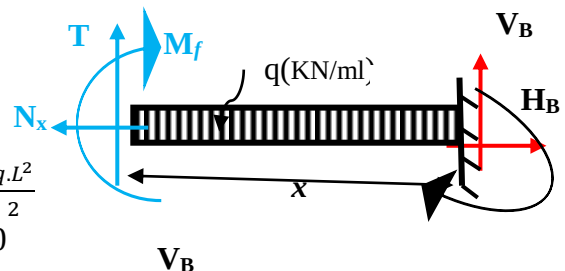
2- Determination M_f et T diagrams

$$0 < x < L$$

$$\sum F_x = 0 \Rightarrow H_B - N_x = 0 \Rightarrow H_B = N_x = 0$$

$$\sum M_{1-1} = 0 \Rightarrow M_f(x) + \frac{q \cdot x^2}{2} + M_B - q \cdot L \cdot x = 0$$

$$\Rightarrow M_f(x) = -\frac{q \cdot x^2}{2} - \frac{q \cdot L^2}{2} + q \cdot L \cdot x = \begin{cases} M_f(0) = -\frac{q \cdot L^2}{2} \\ M_f(L) = 0 \end{cases}$$



4.6 Deflection of a beam subjected to simple bending (concept of deflection)

Under the effect of stresses, a beam deforms in the same direction as those stresses; this is called the deflection at abscissa x , which is the vertical displacement of the center of gravity of the section corresponding to that abscissa. The new locus of the centers of gravity of all the beam's sections is called the deflection curve (Fig 4.11).

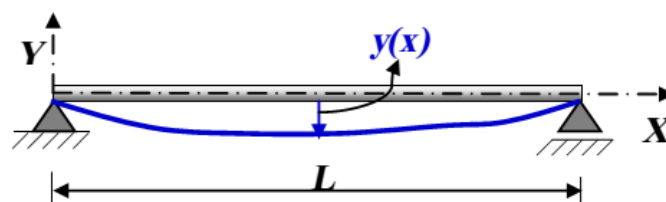


Fig. 4.11 Beam deflection under its own weight

The following relationship allows us to calculate the deformation $y''(x)$ based on the ratio $M_f(x)$ on the inertia EI_z of the beam in question.

$$y''(x) = \frac{M_f(x)}{EI_z}$$

Where:

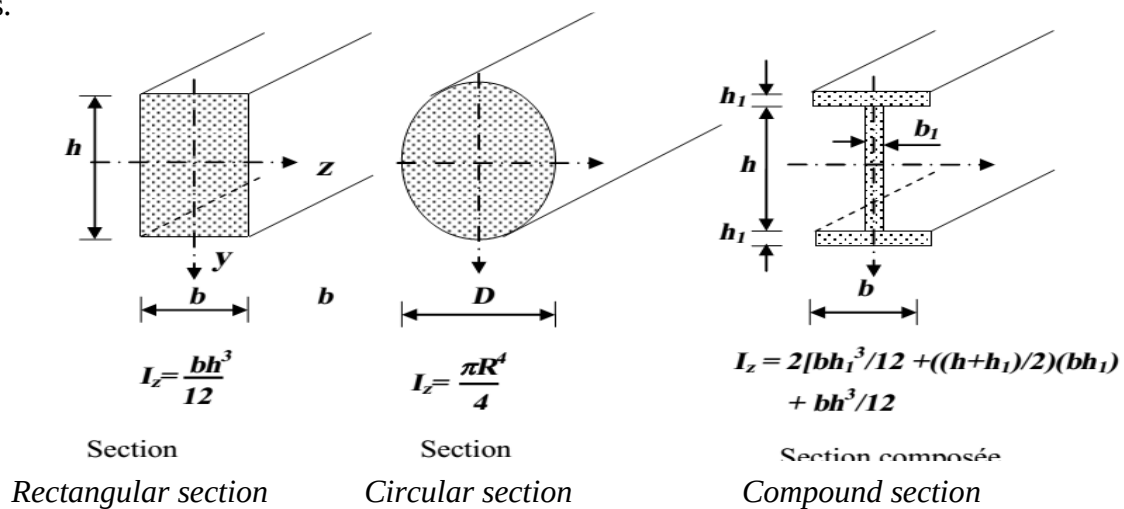
$y''(x)$: is the second derivative of the arrow with respect to x

$M(x)$: the bending moment at the section with abscissa x .

E : the longitudinal elasticity modulus (Young's modulus).

I_z : the moment of inertia of the section with respect to the z -axis passing through the center of gravity of the section and perpendicular to the mean plane of the beam.

The Fig. 4.12 shows the expressions for the central moment of inertia for three (03) common sections.

**Fig.4.12** Expressions for the moment of inertia/ z of common cross-sections

To determine the arrow y (or v), we must integrate this equation twice, and obtain an equation with two constants obtained by the boundary conditions. These are generally written as:

- For a support (single or double): $y = 0$
- For an embedding: $y = 0$ and $y' = 0$ (Bresse formulas).

The slope of the tangent to the elastic line is the first derivative of y (or 1st integration of y''):

$$y'_x = -f'_x(x)$$

The bending angle θ , represents the rotational displacement by bending: $\theta = tg\theta$

A second integration gives the curve equation, deformation of the beam axis:

$$y_x = -f_x(x)$$

The arrow formula is :

$$f_x = \iint \left(\frac{M_f(x)}{E I_z} dx \right) + D$$

4.7 Calculation of constraints

4.7.1 Normal constraint (σ)

Consider a beam resting on two supports and subjected to a concentrated vertical load. After deformation, the beam's upper fibers are compressed, while the lower fibers are stretched (tension).

Between these two parts, there is a fiber that is neither in tension nor compression; this is the neutral fiber (elongation $\Delta l = 0$). The normal stress in the beam, acting as a bending stress (simple or pure bending), is given by:

- beam of symmetrical section: $y_T = y_C$

$$\sigma_{ft} = \frac{M_{fz}}{I} \cdot y_T = \frac{M_{fz}}{I} \cdot y_C = \sigma_f$$

$$\sigma_f = \frac{M_{fz}}{I_z} \cdot y$$

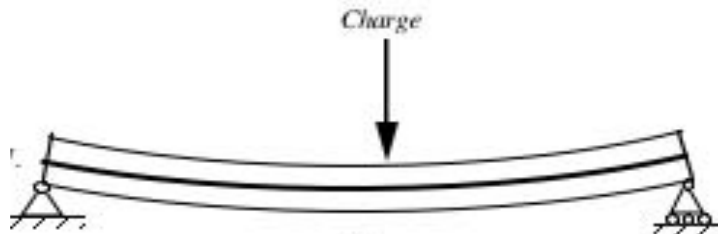


Fig. 4.13 Beam deflected under a concentrated load P

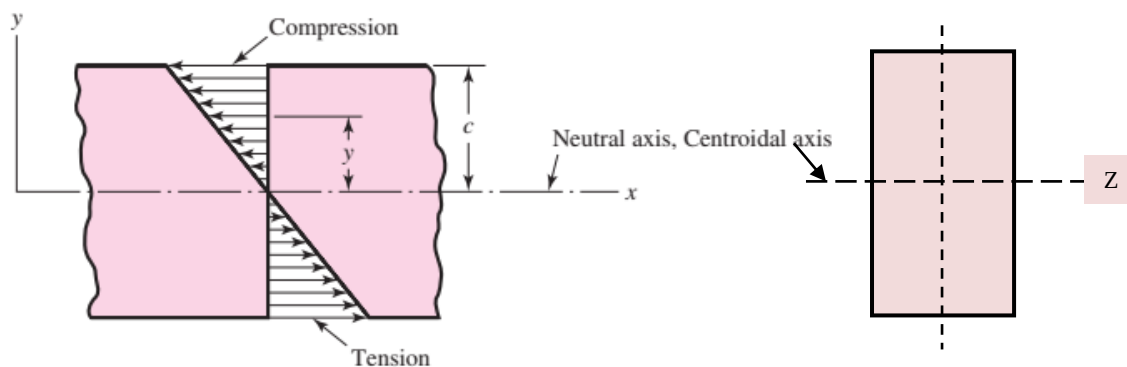


Fig.4.14 Normal stress state, symmetrical section

- Non-symmetrical beam section: $y_t > y_c$

$$\sigma_{ft} = \frac{M_{fz}}{I_z} \cdot y_t \quad \text{and} \quad \sigma_{fc} = \frac{M_{fz}}{I_z} \cdot y_c$$

The maximum compressive stress σ_{fc} is lower than the maximum tensile stress σ_{ft} ($\sigma_{ft} > \sigma_{fc}$).

So the constraint is maximum ($\sigma_{\max}$) on the furthest fiber.

To check the resistance condition of a beam in bending, it is necessary to know the maximum stress at the place where it undergoes the maximum bending moment.

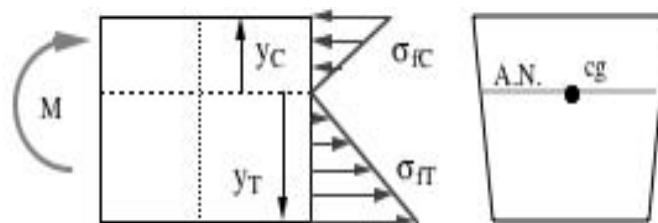


Fig.4.15 Normal stress state, non-symmetrical section

With:

M_{fz} : bending moment in the right section considered.

I_z : moment of inertia of the section relative to the z-axis.

y : distance from the point in question to the neutral axis.

$$\sigma_{max} \leq \bar{\sigma}$$

$$\sigma_{max} = \frac{M_{fz}}{I_z} \leq \bar{\sigma}$$

Table 2 Summarizes the geometric characteristics of common

Profile Shape	I (mm ⁴)	ν (mm)	Distance module W (mm ³)
Rectangle (b,h)	$\frac{b \cdot h^3}{12}$	$\frac{h}{2}$	$\frac{bh^2}{6}$
square (Cote a)	$\frac{a^4}{12}$	$\frac{a}{2}$	$\frac{a^3}{6}$
Dis / Round ($\varnothing$ d)	$\frac{\pi d^4}{64}$	$\frac{d}{2}$	$\frac{\pi d^3}{32}$
Circulaire tube (D, d)	$\frac{\pi \cdot (D^4 - d^4)}{64}$	$\frac{D}{2}$	$\frac{\pi \cdot (D^4 - d^4)}{32 \cdot D}$
Hollow rectangle (ext B,H et int b,h)	$\frac{B \cdot H^3 - b \cdot h^3}{12}$	$\frac{H}{2}$	$\frac{B \cdot H^3 - b \cdot h^3}{6 \cdot H}$

Application

Calculate the maximum normal stress in a rectangular beam having a base of 2 cm and a height of 4 cm and being subjected to a maximum bending moment of 2000 N.m

Solution

Here, since the neutral axis is symmetrical, we have, by referring to the tables in the previous chapter, the value of the section modulus of a rectangular section:

$$S_z = \frac{bd^2}{6}$$

$$\sigma_{fmax} = \frac{M_{fz}}{I_z} \cdot y_{max}$$

$$I_z = \frac{bh^3}{12} \text{ et } y_{max} = \frac{h}{2}$$

$$\sigma_{fmax} = \frac{M_{fz}}{I_z} \cdot y_{max}$$

$$W = \frac{I_z}{y_{max}} \Rightarrow \sigma_{fmax} = \frac{M_{fz}}{W}$$

$$W = \frac{I_z}{y_{max}} = \frac{\frac{bh^3}{12}}{\frac{h}{2}} = \frac{bh^2}{6} = \frac{2 \cdot 4^2}{6} = 5.33 \text{ cm}^3$$

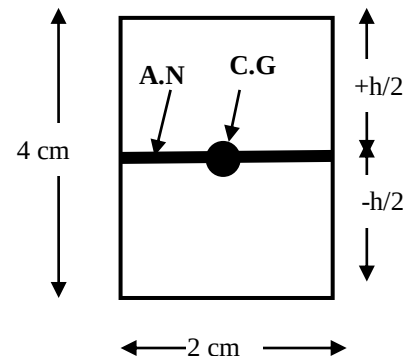
We ask:

$$\sigma_{fmax} = \frac{M_{fz}}{W} = \frac{2000 \cdot 10^2}{5.33} = 37,52 \frac{\text{N}}{\text{cm}^2} = 375 \text{ MPa}$$

$$W = \frac{I_z}{y_{max}} = \frac{I_z}{\nu}$$

W: is called the modulus of inertia, always given in the table of design characteristics for metal

profiles. However, the expression for the deformation is given by: $\varepsilon = \frac{\sigma_f}{E} = \frac{\frac{M_{fz}}{I_z} \cdot y}{E} = \frac{M_{fz}}{EI_z} \cdot y$



$$\varepsilon = \frac{M_{fz}}{EI_z} \cdot y$$

4.7.2 Tangential stresses (τ)

Any beam stressed in simple bending, in addition to the variable bending moment that develops a normal stress σ , there is also the shear force T , which causes tangential stresses τ_{xy} .

It is expressed by the Jouravsky equation:

$$\tau_{xy} = \frac{T_y \cdot S_z}{I \cdot b} \dots \dots \dots (II)$$

Where:

τ_{xy} : shear tangential stress in daN/mm² or (MPa);

T_y : shear force in N;

S_z : static moment of the area located either below or above the layer considered in mm³.

I : moment of inertia of the entire section considered in relation to the neutral fiber in mm⁴.

b : width of the section in the considered layer in mm.

Note

1/ Considering formula (II), we observe that T_y and (I) are constant for a given section, therefore τ varies with S_z/b .

2/ The shear stress τ is maximum (τ_{max}) for a surface at height y that maximizes the value of S_z/b .

3/ If the width b is constant over the entire height of the section, this ratio is maximum when S_z is maximum, i.e., for $y = 0$ (at the neutral axis). For the outer fibers, $S_z = 0$, the τ values are zero.

For, the rectangular beam ($b \times h$), of length l supporting a concentrated load P .

$$\tau_{xy} = \frac{T_y \cdot S_z}{I \cdot b} = \frac{P \cdot b \cdot h^2/8}{b^2 \cdot h^3/12} = \frac{3}{2} \cdot \frac{P}{b \cdot h}$$

$$\tau_{xy} = \frac{3P}{2b \cdot h}$$

Or, we have:

$$\sigma = \frac{M_{max}}{I/v} = \frac{6Pl}{b \cdot h^2}$$

So,
$$\frac{\tau}{\sigma} = \frac{h}{4 \cdot l} \Rightarrow \tau = \left(\frac{h}{4 \cdot l} \right) \cdot \sigma$$

That is, τ is proportional to σ ; this proportionality ratio $\frac{h}{4 \cdot l}$ increases as l decreases and vice versa.

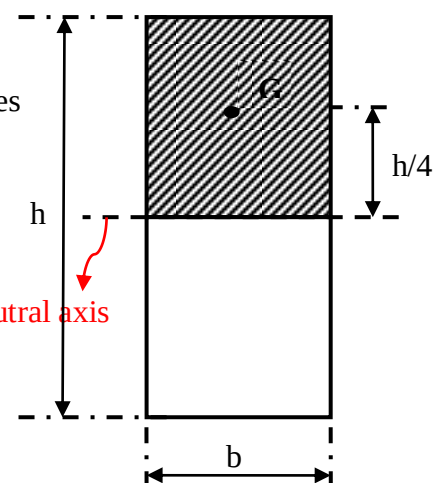
4.7.3 Stability condition

If τ_{max} is the largest shear stress that develops in the beam, and if $\bar{\tau}$ (τ_{adm}) is the allowable stress, it is necessary to verify that:

$$\tau_{max} \leq \tau_{adm}$$

Application 1

Calculate the maximum normal stress in a rectangular beam with a base of 2 cm and a height of 4 cm, subjected to a maximum bending moment of 2000 N.m. If $T_{max} = 10 \cdot 2$ N, calculate τ_{max} ?



Solution

The maximum stress is found on the neutral fiber:

$$S_z = A \cdot Y_G = \left(b \cdot \frac{h}{2}\right) \cdot \frac{h}{4} = \frac{b \cdot h^2}{8}$$

$$I_z = \frac{b \cdot h^3}{12}$$

$$\sigma_{max} = \frac{M_{max}}{I/v} = \frac{M_{max}}{\frac{b \cdot h^3}{12} \cdot \frac{h}{4}} = \frac{6 \cdot M_{max}}{b \cdot h^2}$$

$$\sigma_{max} = \frac{6 \cdot 2000}{0,02 \cdot (0,04)^2} = 375 \text{ N/m}^2$$

$$\tau_{max} = \frac{T_y \cdot S_z}{I \cdot b} = \frac{\frac{b \cdot h^2}{8} \cdot T_y}{\frac{b \cdot h^3}{12} \cdot b} = \frac{3}{2} \cdot \frac{T_{max}}{b \cdot h}$$

$$\tau_{max} = \frac{3}{2} \cdot \frac{T_{max}}{b \cdot h} = \frac{3}{2} \cdot \frac{10^{-2}}{0,02 \cdot 0,04} = 18,75 \text{ N/m}^2$$

4.8 Beam sizing

To size a beam, two criteria can be used:

- 1- a criterion under normal stress (strength condition)
- 2- a criterion on the maximum deflection (rigidity condition)

The criterion on rigidity reflects the fact that the maximum deflection $v(P)$ at a point P must remain below a given value depending on the conditions of use: $\text{Max}(v(P)) \leq [v]$

For ordinary beams (of length L), the permissible deflection value is in the order of:

$$[v] = [f] = \frac{L}{100} \div \frac{L}{1000}$$

4.8.1 Sizing to the strength condition

The following steps should be followed:

- 1- Plot of the M_f diagram (M_z or M_y) along the beam,
- 2- Determination of the hazardous section from the diagram of M_f ,
- 3- Calculation of the maximum stress σ_{max} , that is, the stress at the hazardous point along the beam cross-section,
- 4- Satisfaction of the resistance condition, which is written according to the admissible stress method as follows:

$$\sigma_{max} \leq \bar{\sigma}$$

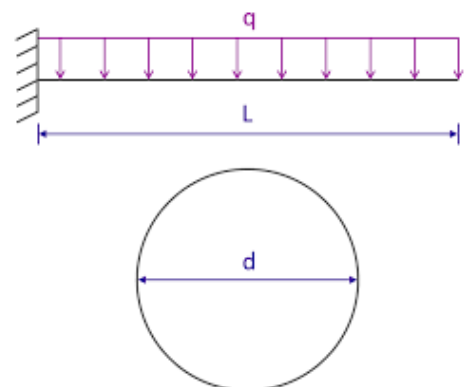
Application 2

A steel beam of length $L = 3\text{m}$, recessed at one end A and free at the other B, with a circular section, receives an evenly distributed load of 2000 kg/ml .

Determine the minimum section and the arrow.

We give:

$$\sigma = 12 \text{ kg/mm}^2, E = 21000 \text{ kg/mm}^2$$

**Solution**

- Support reactions

$$\sum F_x = 0 \Rightarrow H_A = 0$$

$$\sum F_y = 0 \Rightarrow V_A - q \cdot L = 0 \Rightarrow V_A = q \cdot L (\downarrow)$$

$$\sum M/A = 0 \Rightarrow \frac{q.L^2}{2} - M_A = 0$$

$$\Rightarrow M_A = \frac{q.L^2}{2} \text{ (against clockwise) } q.L$$

$$\sum M/1-1 = 0 \Rightarrow V_A \cdot x - M_A - \frac{q.x^2}{2} - M_f(x) = 0$$

$$\Rightarrow M_f(x) = q.L.x - \frac{q.x^2}{2} - \frac{q.L^2}{2}$$

$$\Rightarrow M_f(x) = q.L.x - \frac{q.L^2}{2} - \frac{q.x^2}{2} = -\frac{q}{2}(L-x)^2$$

$$\Rightarrow \begin{cases} M_f(0) = -\frac{q.L^2}{2} \\ M_f(L) = 0 \end{cases}$$

$$T(x) = -q(L-x) \Rightarrow \begin{cases} T(0) = -qL \\ T(L) = 0 \end{cases}$$

1- Minimum section: $\frac{l}{v} = \frac{M}{\sigma}$

$$0,1.D^3 \geq \frac{20.300^2}{2.100} \Rightarrow D^3 \geq 7,5.10^3$$

$$\Rightarrow D \geq 19,57mm$$

• Shear verification

$$\tau = \frac{4}{3} \cdot \frac{T}{\pi.R^2} \left[1 - \left(\frac{y}{R} \right)^2 \right] \text{ soit:}$$

$$\tau_{max} = \frac{4}{3} \cdot \frac{T}{\pi.R^2} = \frac{4}{3} \tau_m$$

$$\tau_{max} \leq \frac{2.T}{\pi.R^2} = \frac{2.6000}{\pi.100} = 38,20kg/cm^2 < \bar{\tau} = 1000kg/cm^2$$

2- Calculation of the Arrow

$$y'' = \frac{q}{2.EI} (L-x)^2 = \frac{q}{2.EI} (L^2 - 2.L.x + x^2)$$

$$\Rightarrow y' = \frac{q}{2.EI} (L^2x - Lx^2 + \frac{x^3}{3} + c_1) \quad x=0, y'=0 \Rightarrow c_1 = 0$$

$$\Rightarrow y = \frac{q}{2.EI} (L^2 \frac{x^2}{2} - L \frac{x^3}{3} + \frac{x^4}{12} + c_1.x + c_2) \quad x=0, y=0 \Rightarrow c_2 = 0$$

• Hence the deformed fiber equation is: $y = \frac{q.x^2}{24.EI} (x^2 - 4Lx + 6L^2)$

The direction of variation of y is given by the derivative y'.

$$y' = \frac{q.x}{6.EI} (3L^2 - 3Lx + x^2) = 0$$

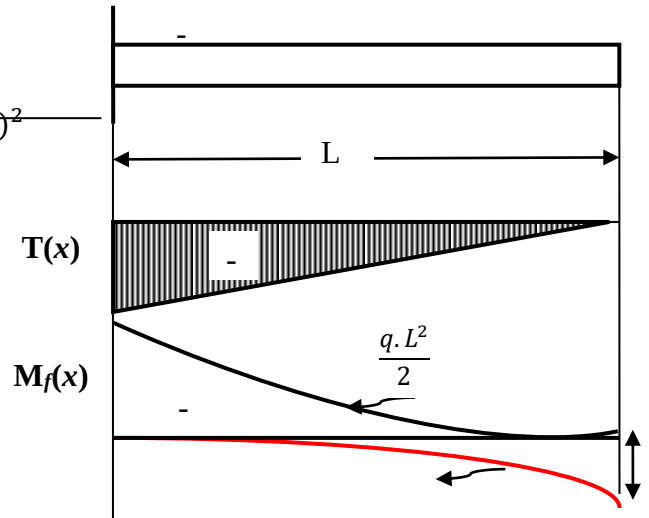
$$\Delta < 0 \Rightarrow y' > 0 \text{ over the interval } 0-L$$

The max of y will be for : $x=L$;

$$y = \frac{q.L^2}{24.EI} (L^2 - 4L^2 + 6L^2) = \frac{3.q.L^4}{24.EI}$$

$$\Rightarrow y = f_{max} = \frac{q.L^4}{8.EI}$$

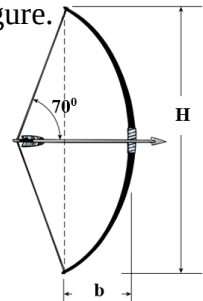
$$\Rightarrow f_{max} = \frac{q.L^4}{8.EI} = \frac{20 \times 300^4 \times 64}{8 \times 2100000 \times \pi \times 20^4} = 12,28cm$$



SERIES OF EXERCISES

Exercise 1

An archer applies a 150 N pull on the fully tensioned bowstring shown in the figure. Determine the bending moment in the middle of the arc



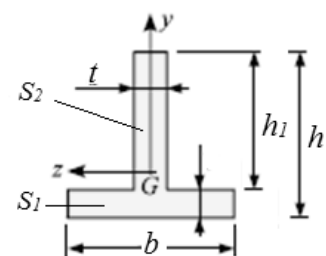
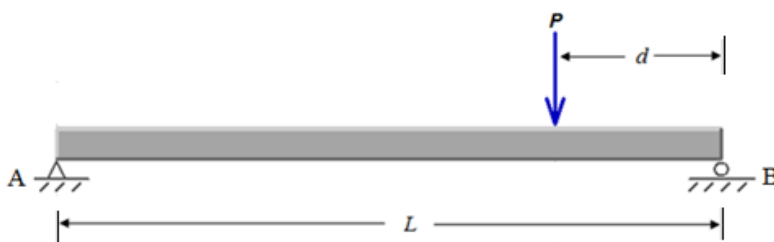
Exercise 2

Consider a beam fixed at point A, as shown in the figure. It is subjected to a concentrated load $P = 3 \text{ kN}$ at 0.8 m from the fixed end, and a uniformly distributed load $q = 1 \text{ kN/m}$ over a distance of 1.6 m from point B. Draw the shear force and bending moment diagrams.

Exercise 3

Determine the maximum tensile stress σ_{max}^t and the maximum compressive stress σ_{max}^c due to the load P acting on a beam with a T-shaped cross-section, simply supported at points A and B.

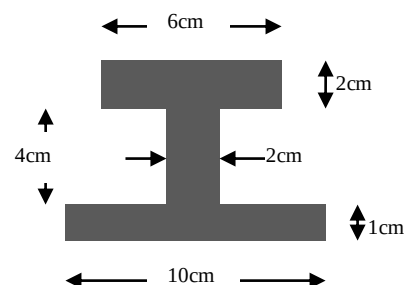
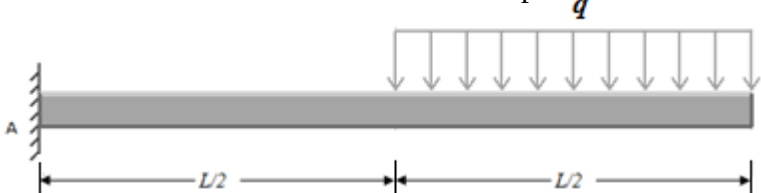
Given: $P=5.4\text{kN}$, $L=3\text{m}$, $d=1.2\text{m}$, $b=75 \text{ mm}$, $t=25 \text{ mm}$, $h=100 \text{ mm}$ and $h_1=75 \text{ mm}$.



Exercise 4

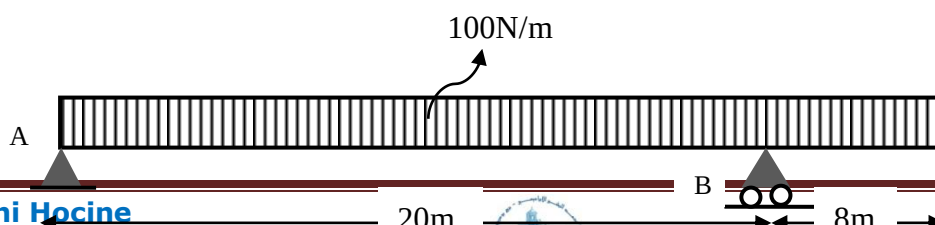
Consider a cantilever beam of length L subjected to a uniformly distributed load q along section BC. Determine the general expression for the deflection.

Determine the maximum deflection at point B?



Exercise 5

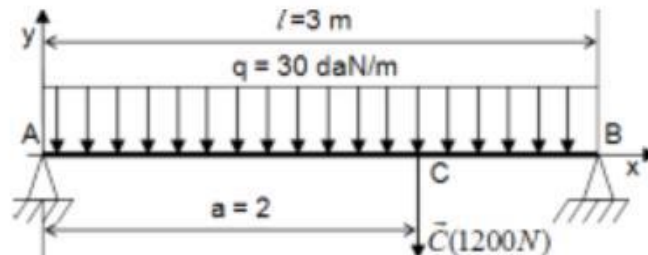
Find the maximum normal bending stress σ_{fzmax} in the beam whose section is given below.



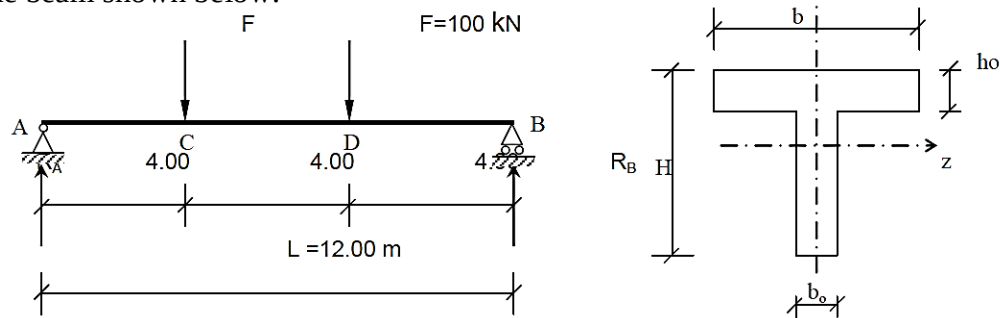
Exercise 6

Consider a beam AB subjected to three external forces: R_A (support force at A), R_B (support force at B), and the distributed load.

- 1- Determine the forces at A and B.
- 2- Determine T and M and draw their diagrams along AB.
- 3- Calculate the maximum normal stress.
- 4- Represent the normal stresses if the diameter of AB is 40 on a diagram.

**Exercise 7**

Consider the beam shown below.



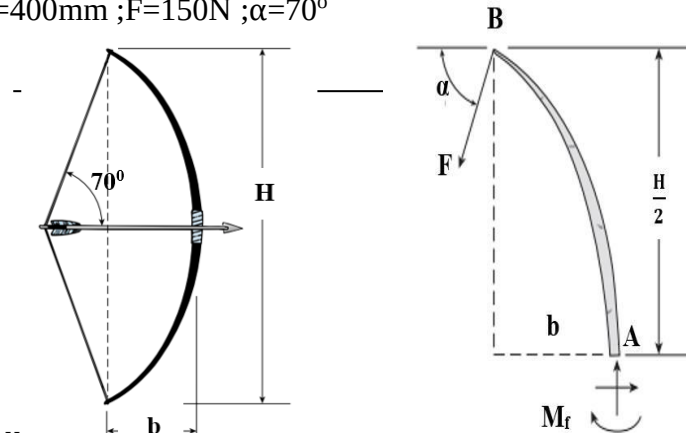
1. Determine the shear force and bending moment diagrams along the beam;
2. Draw the normal stress diagram σ at the most stressed section;
3. Sketch the shear stress diagram at section C and calculate $\tau_{\max}$.

Given: $H = 80$ cm, $b = 100$ cm, $h_o = 20$ cm, and $b_o = 30$ cm

CORRECTION ELEMENTS

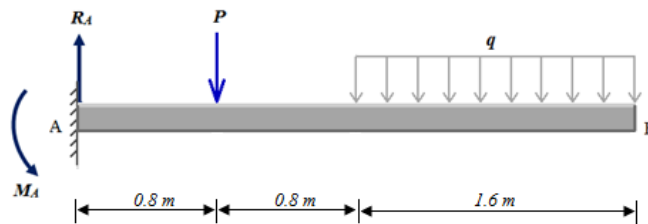
Exercise 1

Given: $H=1450\text{mm}$; $B=400\text{mm}$; $F=150\text{N}$; $\alpha=70^\circ$



$$\begin{aligned}\sum M/A = 0 &\Rightarrow (F \cdot \cos \alpha) \cdot \frac{H}{2} + (F \cdot \sin \alpha) \cdot b - M_f = 0 \\ &\Rightarrow M_f = F \cdot \cos \alpha \cdot \frac{H}{2} + (F \cdot \sin \alpha) \cdot b = 150 \left[\cos 70^\circ \cdot \frac{1450}{2} + \sin 70^\circ \cdot 400 \right] = 93,58 \text{ N} \cdot \text{mm} \\ M_f &= 93,58 \text{ N} \cdot \text{mm}\end{aligned}$$

Exercise 2



$$\begin{aligned}\sum F/y = 0 &\Rightarrow R_A - P - q \cdot 1,6 = 0 \\ &\Rightarrow R_A = P + q \cdot 1,6 = 3 + 1,1 \cdot 1,6 = 4,6 \text{ KN} \\ \sum M/A = 0 &\Rightarrow M_A - P \cdot 0,8 - q \cdot 1,6 \cdot 2,4 = 0 \\ &\Rightarrow M_A = P \cdot 0,8 + q \cdot 1,2,4 \\ &\Rightarrow M_A = 3 \cdot 0,8 + 1,1 \cdot 1,2,4 = 6,24 \text{ KN} \cdot \text{m}\end{aligned}$$

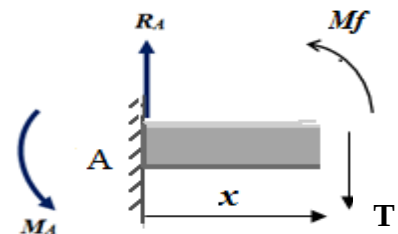
• Section 1-1

$$0 \leq x \leq 0,8 \text{ m}$$

$$M_f(x) = R_A \cdot x - M_A$$

$$M_f(x) = 4,6 \cdot x - 6,25 = \begin{cases} M_f(0) = -6,225 \text{ KN} \cdot \text{m} \\ M_f(0,8) = -2,56 \text{ KN} \cdot \text{m} \end{cases}$$

$$\Rightarrow T(x) = [M_f(x)]' = 4,6 \text{ KN}$$

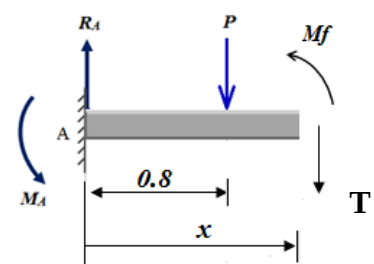


• Section 2-2

$$0,8 \leq x \leq 1,6 \text{ m}$$

$$T(x) = R_A - P = 4,6 - 3 = 1,6 \text{ KN}$$

$$M_f(x) = R_A \cdot x - P(x - 0,8) - M_A$$



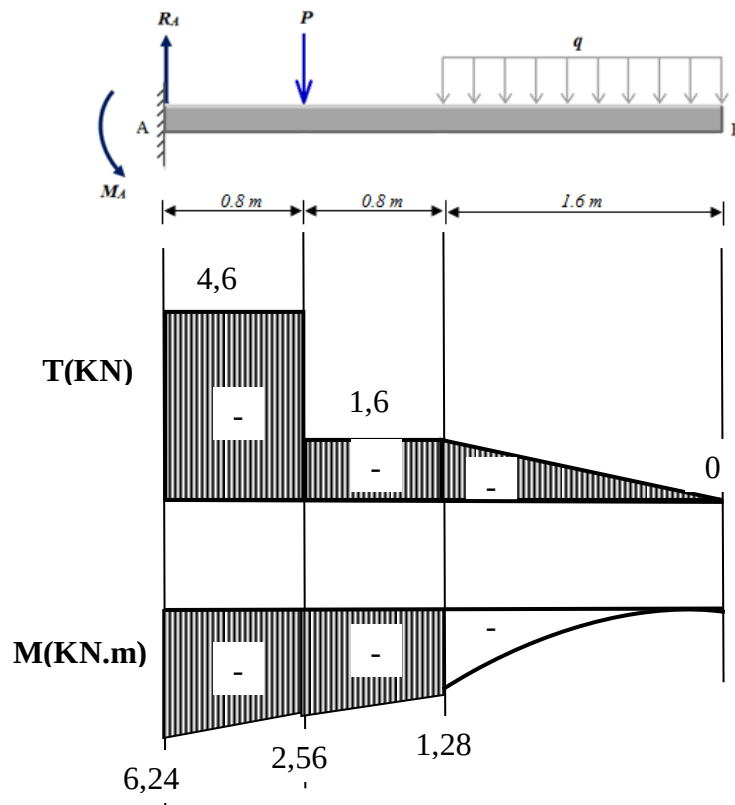
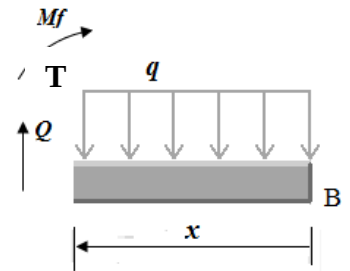
$$M_f(x) = 4,6x - 3(x - 0,8) - 6,24 = \begin{cases} M_f(0,8) = -2,6 \text{ KN.m} \\ M_f(1,6) = -1,28 \text{ KN.m} \end{cases}$$

• **Section 3-3**

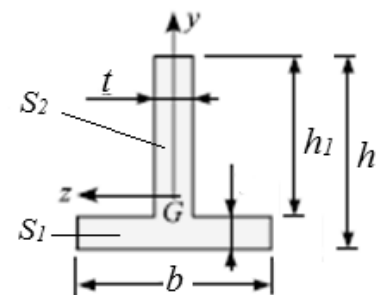
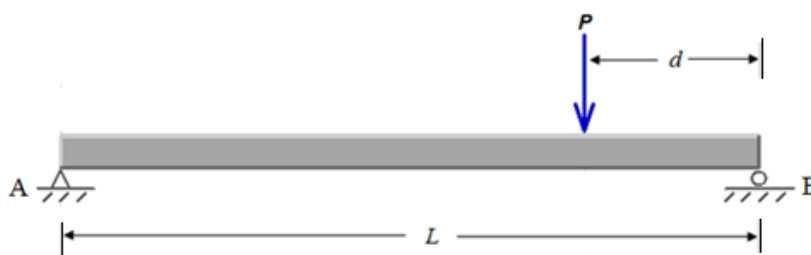
$$0 \leq x \leq 1,6 \text{ m}$$

$$T(x) = q \cdot x = 1 \cdot x = \begin{cases} T(0) = 0 \\ T(1,6) = 1,6 \text{ KN} \end{cases}$$

$$M_f(x) = -\frac{x^2}{2} \begin{cases} M_f(0) = 0 \\ M_f(0,8) = -1,28 \text{ KN.m} \end{cases}$$



Exercise 3



$$\sum M/B = 0 \Rightarrow R_A \cdot L - P \cdot d = 0$$

$$\Rightarrow R_A = \frac{P \cdot d}{L} = \frac{5,4 \cdot 1,2}{3} = 2,16 \text{ KN}$$

$$\sum M/A = 0 \Rightarrow R_A \cdot L - P \cdot (L - d) = 0$$

$$\Rightarrow R_B = \frac{P \cdot (L - d)}{L} = \frac{5,4 \cdot 1,28}{3} = 3,24 \text{ KN}$$

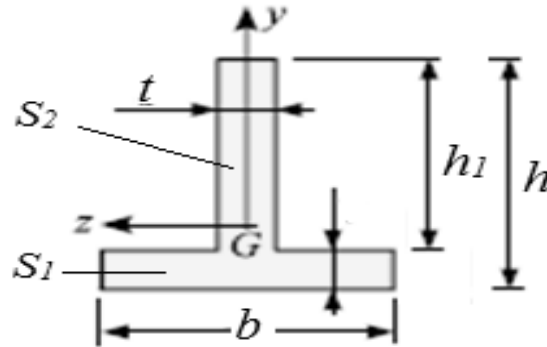
• $0 \leq x \leq L - d$

$$M_f(x) = R_A \cdot x = 2,16x = \begin{cases} M_f(0) = 0 \\ M_f(L-d) = 2,16 \cdot (L-d) = 2,6 \cdot 1,8 = 3,89 \text{ KN} \cdot \text{m} \end{cases}$$

$$\text{Donc: } M_{f_{\max}} = 3,89 \text{ KN} \cdot \text{m} = 3,89 \cdot 10^6 \text{ N} \cdot \text{mm}$$

- Position of the center of gravity of the section.

$$I_z = \sum_{i=1}^2 [I_{Gi} + (y_{Gi} - y_G)^2 \cdot S_i] \\ = [I_{G1} + (y_{G1} - y_G)^2 \cdot S_1] + [I_{G2} + (y_{G2} - y_G)^2 \cdot S_2]$$



$$I_z = [97656,25 + (25)^2 \cdot 25 \cdot 75] + [878906,25 + (25)^2 \cdot 25 \cdot 75] = 3320312,5 \text{ mm}^4$$

$$I_{G1} = \frac{b \cdot h^3}{12} = \frac{75 \cdot 25^3}{12} = 97656,25 \text{ mm}^4$$

$$I_{G2} = \frac{h \cdot b^3}{12} = \frac{25 \cdot 75^3}{12} = 878906,25 \text{ mm}^4$$

$$y_{G1} = \frac{b}{2} = \frac{75}{2} = 37,5 \text{ mm}$$

$$y_{G1} = \frac{b}{2} = \frac{75}{2} = 37,5 \text{ mm}$$

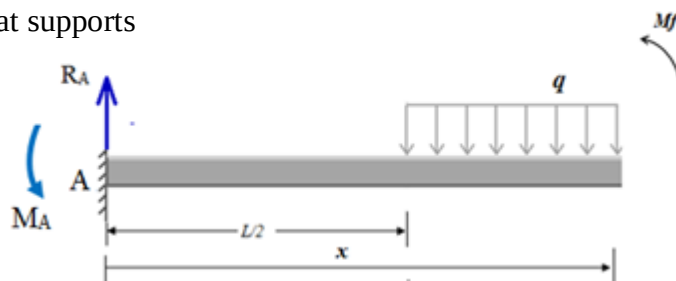
$$y_{G2} = \frac{h_1}{2} + t = \frac{75}{2} + 25 = 62,5 \text{ mm}$$

$$\sigma_{\max}^t = \frac{M_{\max}}{\frac{I_z}{v_1}} = \frac{3,89 \cdot 10^6}{3320312,5} \cdot 37,5 = 43,6 \text{ N/mm}^2$$

$$\sigma_{\max}^c = -\frac{M_{\max}}{\frac{I_z}{v_2}} = -\frac{3,89 \cdot 10^6}{3320312,5} \cdot 62,5 = -73,22 \text{ N/mm}^2$$

Exercise 4

Calculation of reactions at supports



$$\bullet \quad \Sigma F = 0 \rightarrow R_A = \frac{qL}{2}$$

$$\bullet \quad \Sigma M/A = 0 \rightarrow M_A = \frac{3qL^2}{8}$$

$$\bullet \quad M_f = R_A \cdot x - q \frac{(x-L/2)^2}{2} - M_A$$

$$EIy'' = EI \frac{d^2y}{dx^2} = -M_f = -R_A \cdot x + q \frac{(x-L/2)^2}{2} + R_A \cdot x - q \frac{(x-L/2)^2}{2} - M_A$$

$$EIy' = EI \frac{dy}{dx} = -R_A \cdot \frac{x^2}{2} + \frac{q(x - L/2)^3}{6} + R_A \cdot x - \frac{q(x - L/2)^2}{2} - M_A \frac{x^2}{2} + C_1$$

$$EIy = -R_A \cdot \frac{x^3}{6} + P \frac{(x - a)^4}{24} + M_A \frac{x^2}{2} + C_1 x + C_1$$

- Boundary conditions:

$$\text{For : } x_A = 0, y'_A = 0 \rightarrow C_1 = 0$$

$$\text{For : } x_A = 0, y_A = 0 \rightarrow C_2 = 0$$

$$\text{The expression of the arrow is: } y = \frac{1}{EI} \left[2 \left(x - \frac{L}{2} \right)^4 - 4Lx^3 - 9L^2x^2 \right]$$

$$\text{At B } (x_B = L) \text{ the arrow is maximum: } y_B = \frac{41qL^4}{384EI}$$

Exercise 5

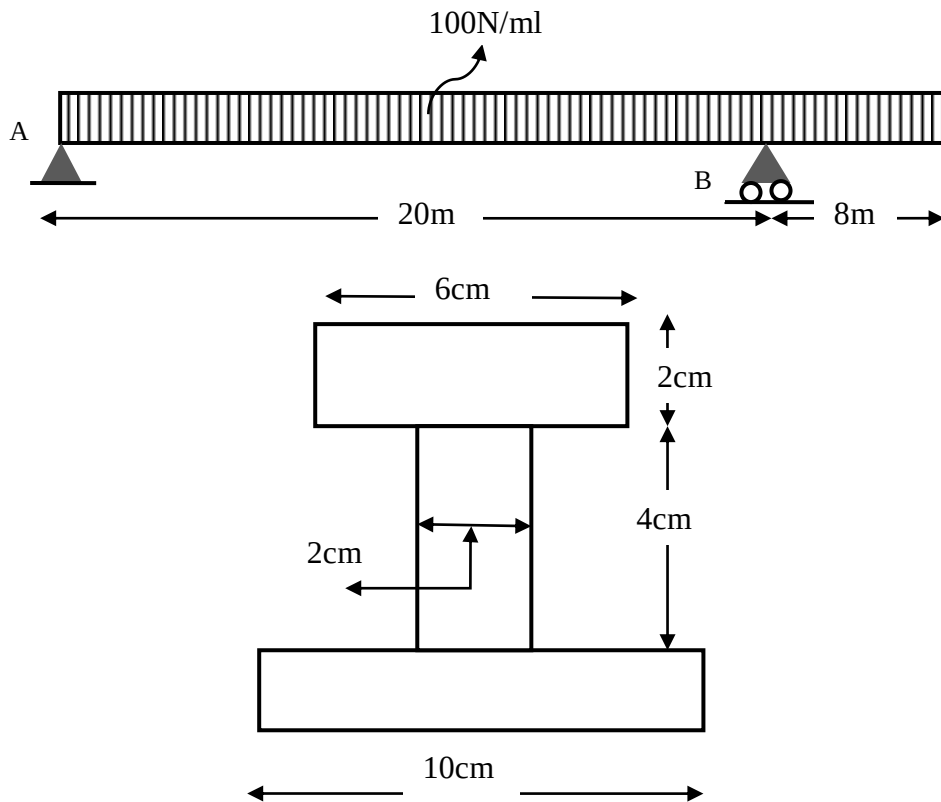


Fig 4.16 Section of the beam to study

- 1- Let's calculate the support reactions.

$$\Sigma F/x = H_A = 0$$

$$\Sigma M/A = -(2800 \times 14) + (R_B \times 20) = 0$$

$$\Rightarrow R_B = 1960 \text{ N}$$

$$\Sigma Fy = R_A - 2800 + 1960 = 0$$

$$\Rightarrow R_A = 840 \text{ N}$$

Calculation of M and T

- **Section 1-1**

$$0 < x \leq 20 \text{ m}$$

$$T(x) = 840 - 100 \cdot x = \begin{cases} T(0) = 840 \text{ N} \\ T(20) = -1160 \text{ N} \end{cases}$$

$$M_f(x) = 840x - 50x^2 = \begin{cases} M_f(0) = 0 \\ M_f(20) = -1160 \text{ N.m} \end{cases}$$

Or $T(x) = [M_f(x)]' = (840x - 50x^2)'$
 $T(x) = 840 - 100x$ (The same for both methods).

• **Section 2-2**

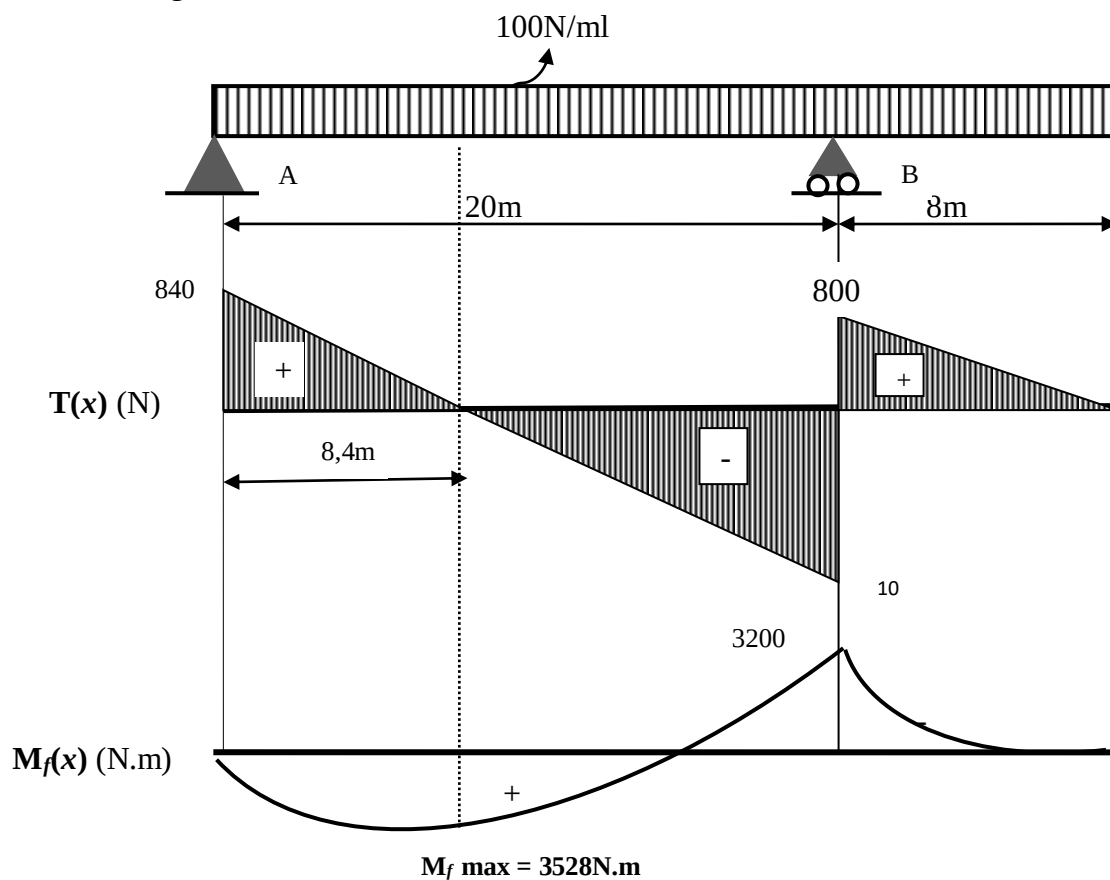
$$0 < x \leq 8 \text{ m}$$

$$T(x) = 100 \cdot x = \begin{cases} T(0) = 0 \text{ N} \\ T(20) = 800 \text{ N} \end{cases}$$

$$M_f(x) = -50x^2 = \begin{cases} M_f(0) = 0 \\ M_f(20) = -3200 \text{ N.m} \end{cases}$$

$$T(x) = -[M_f(x)]' = (-50x^2)' = 100x \text{ (Change of direction).}$$

2- M and T diagrams



3- Determining the neutral axis of the section:

$$A_1 = 2 \times 6 = 12 \text{ cm}^2$$

$$A_2 = 2 \times 4 = 8 \text{ cm}^2$$

$$A_3 = 1 \times 10 = 10 \text{ cm}^2$$

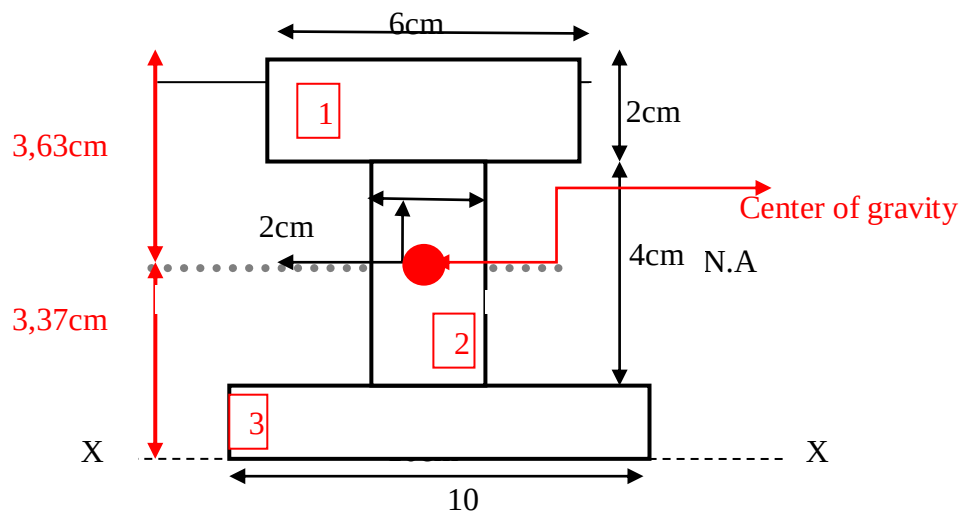
The total area is:

$$A_{\text{total}} = A_1 + A_2 + A_3$$

$$A_{\text{total}} = 12 + 8 + 10 = 30 \text{ cm}^2$$

$$\Sigma Q_{XX} = (12 \times 1 \times 6) + (4 \times 2 \times 4) + (10 \times 1 \times 0.5) = 30 \times t$$

$$\text{Where, } t = 3.37 \text{ cm XX-axis}$$



4-That is to say: $y = 3.63$ cm as the distance farthest from the neutral axis.

We are above the neutral axis and $M > 0$, so the stress is compression.

Calculation of the moment of inertia relative to the neutral axis (passes through the centre of

gravity):

$$I_1 = \frac{6.2^3}{12} + 6.2 \cdot (2.63)^2 = 87 \text{ cm}^4$$

$$I_2 = \frac{2.4^3}{12} + 2.4 \cdot (0.37)^2 = 11.76 \text{ cm}^4$$

$$I_3 = \frac{10.1^3}{12} + 10.1 \cdot (2.87)^2 = 83.20 \text{ cm}^4$$

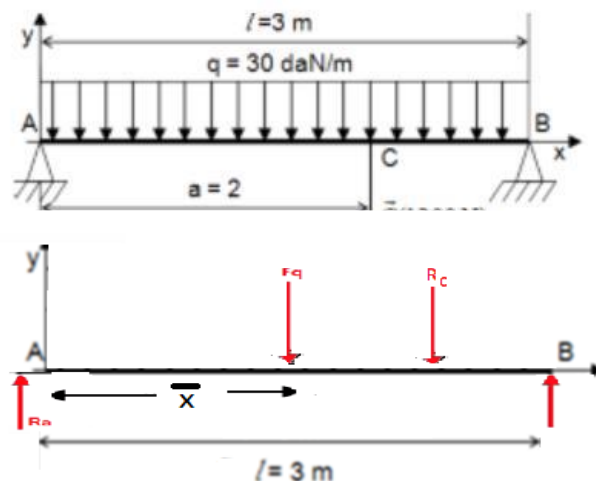
$$I_{\text{totale}} = I_1 + I_2 + I_3 = 87 + 11.76 + 83.20 = 181.96 \text{ cm}^4$$

5- Calculation of the normal stress (in compression):

$$\sigma = \frac{M_{fmax}}{I} \cdot y_G$$

$$\sigma = \frac{3528}{181.96} \times 3.63 = \frac{70.38 \text{ N}}{\text{mm}^2} = 70.38 \text{ MPa} \rightarrow \text{Constraint of compression}$$

Exercise 6



1--Determination of reactions to R_A and R_B support

- Calculation of F_q and x

$$F_q = q \times L = 3 \times 30 \text{ daN} = 90 \text{ daN}$$

$$x = \frac{L}{2} = \frac{3}{2} = 1.5 \text{ m} \rightarrow \text{point of concentration of the distributed force from A}$$

According to the equilibrium equations:

- $\sum F_y = 0 \Rightarrow R_A - F_q - R_c + R_B = 0 \Rightarrow R_A + R_B = F_q + R_c = 900 + 1200 = 2100 \text{ N}$
- $\sum M/A = 0 \Rightarrow (F_q \times x) + (R_c \times a) - (R_B \times l) = 0$
 $\Rightarrow 1.5 F_q - 3 R_B + 2 R_c = 0$
 $\Rightarrow R_B = 1250 \text{ N}$
 $\Rightarrow R_A = 850 \text{ N}$

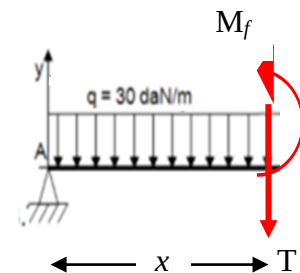
2-Shear effort and bending moment

- $0 < x \leq 2$

- Calculation of F_q and x

$$F_{q'} = q \times L = q \times x = 300x \rightarrow (\text{N})$$

$$\bar{x} = \frac{L}{2} = \frac{x}{2}$$



According to the equilibrium equations, we have:

- $\sum F_y = 0 \Rightarrow R_A - F_{q'} + T = 0 \Rightarrow T = F_{q'} - R_A$
 $\Rightarrow T = 300x - 850 \rightarrow \begin{cases} T(0) = -850 \text{ N} \\ T(2) = -250 \text{ N} \end{cases}$
- $\sum M = 0 \Rightarrow M_f + (F_{q'} \times \frac{x}{2}) - R_A \times x = 0$
 $\Rightarrow M_f = R_A \times x - F_{q'} \times \frac{x}{2} = 850x - 300x \times \frac{x}{2}$
 $\Rightarrow M_f = -150x^2 + 850x \rightarrow \begin{cases} M_f(0) = 0 \\ M_f(2) = 1100 \text{ N.m} \end{cases}$

- $2 < x < 3$

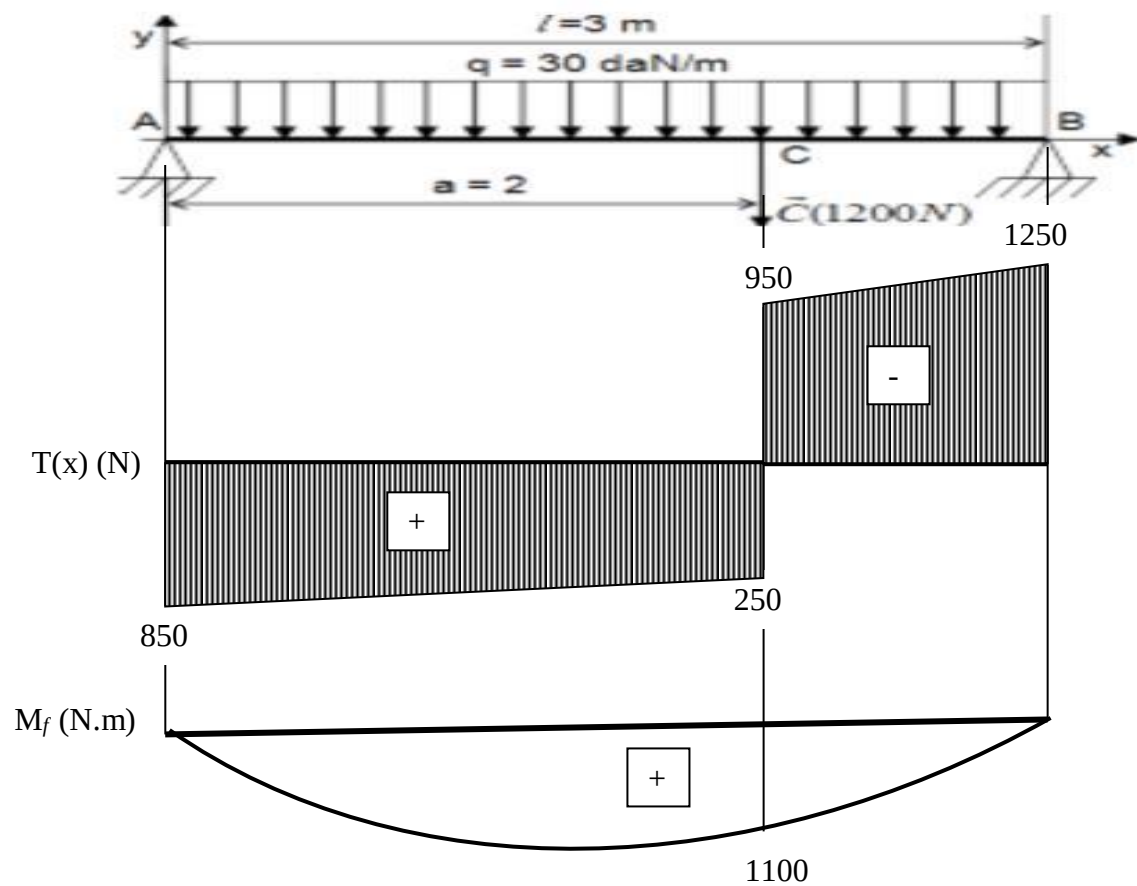
- Calculation of F_q and x

$$F_{q'} = q \times L = q \times x = 300x \rightarrow (\text{N})$$

$$x = \frac{L}{2} = \frac{x}{2}$$

According to the equilibrium equations, we will have:

- $\sum F_y = 0 \Rightarrow R_A - F_{q'} - R_c + T = 0 \Rightarrow T = F_{q'} + R_c - R_A$
 $\Rightarrow T = 300x + 350 \rightarrow \begin{cases} T(2) = 950 \text{ N} \\ T(3) = 1250 \text{ N} \end{cases}$
- $\sum M = 0 \Rightarrow M + (F_{q'} \times \frac{x}{2}) + R_c \times (x - a) - R_A \times x = 0$
 $\Rightarrow M_f = -150x^2 - 350x + 2400 \rightarrow \begin{cases} M_f(2) = 1100 \text{ N.m} \\ M_f(3) = 0 \end{cases}$



3-Normal constraint, due to the bending moment.

$$\sigma_{max} = \frac{M_{mx}}{I_z} \cdot y_{max}$$

$$I_z = \frac{\pi D^4}{64} = \frac{\pi 40^4}{64} = 125,66\text{ mm}^4$$

$$\sigma_{max} = \frac{1100}{125,66} \cdot 20 = 175,08\text{ MPa}$$

$$-175,08\text{ MPa}$$

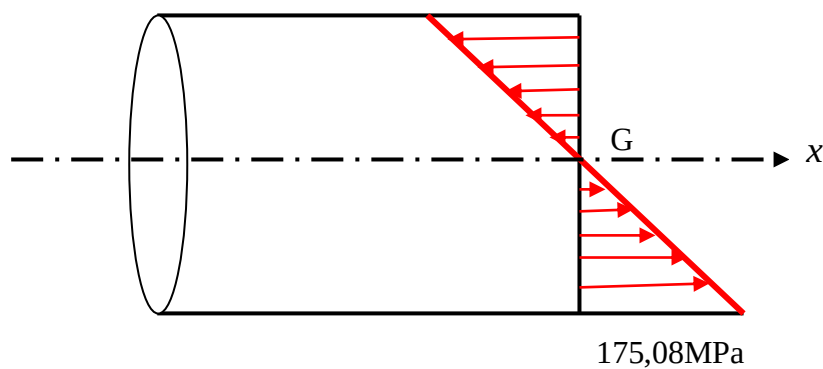
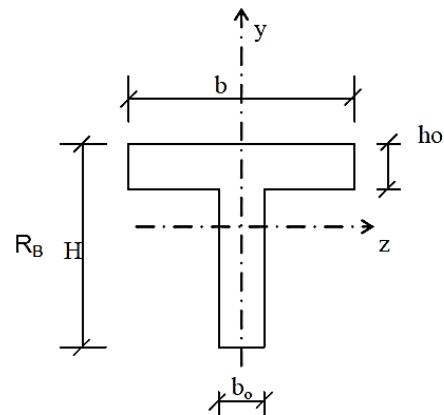
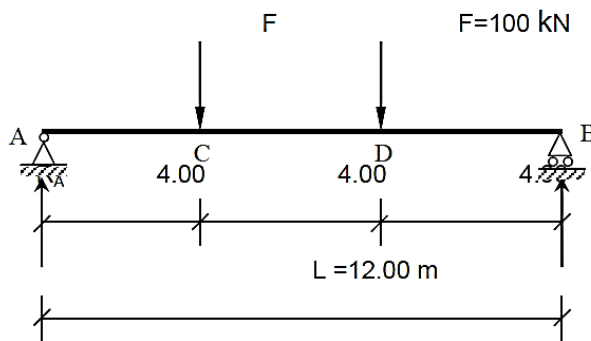


Fig 1.17 Normal stress distribution on the section of study beam.

Exercise 7**1- Internal efforts**

We have, $R_A = R_B = 100 \text{ kN}$ (symetry);

- $0 \leq x \leq 4 \text{ m}$**

$$T_1 = -R_A = -100 \text{ kN}$$

$$M_{f1}(x) = R_A \cdot x = 100x \rightarrow \begin{cases} M_{f1}(0) = 0 \\ M_{f1}(4) = 400 \text{ kN.m} \end{cases}$$

- $4 \leq x \leq 8 \text{ m}$**

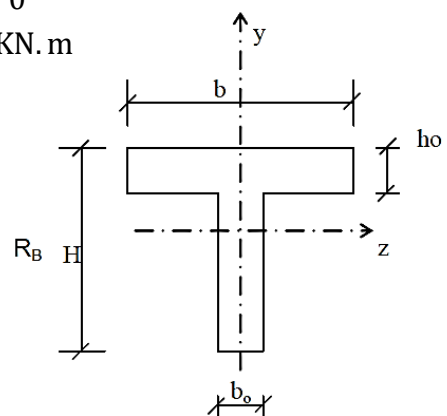
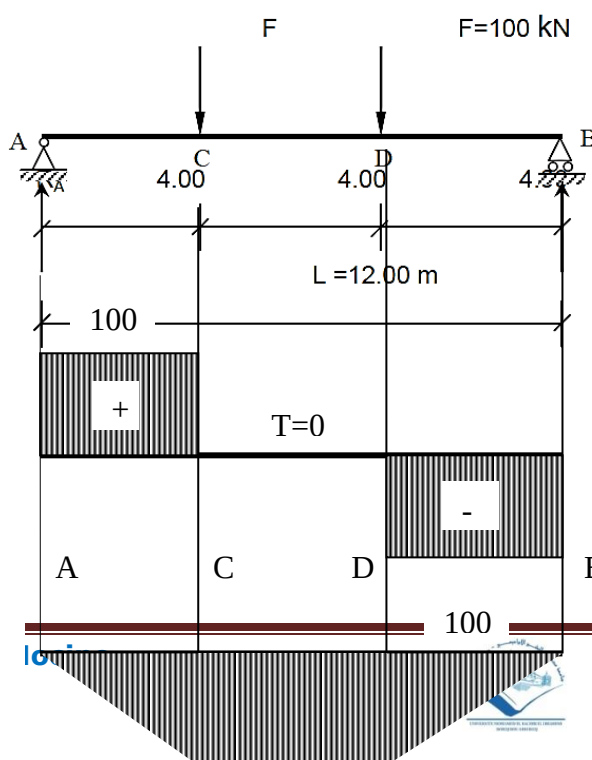
$$T_2 = -R_A + F = -100 + 100 = 0$$

$$M_{f2}(x) = R_A \cdot x - F(x-4) = 400 \rightarrow \begin{cases} M_{f2}(4) = 400 \text{ kN.m} \\ M_{f2}(8) = 400 \text{ kN.m} \end{cases}$$

- $8 \leq x \leq 12 \text{ m}$ (right)**

$$T_3 = +R_B = +100 \text{ kN}$$

$$M_{f3}(x) = R_B \cdot x = 100x \rightarrow \begin{cases} M_{f3}(8) = 800 \\ M_{f3}(12) = 1200 \end{cases}$$

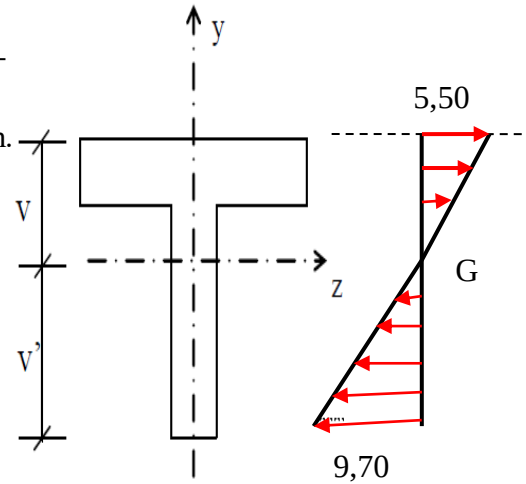




2- σ – diagram

Geometric characteristics of the section:

- Surface: $S = 3800 \text{ cm}^2$
- Position de G: $v = 28.95 \text{ cm}$; $v' = 51.05 \text{ cm}$.
- Inertia: $I_{GZ} = 2122456.17 \text{ cm}^4$



- Normal constraint σ

$$\sigma_{max} = \frac{M_{max}}{I_z} \cdot y_{max}$$

With: $M_{max} = 400 \text{ kN.m}$ (in the most stressed section of the beam)

$$\sigma_{sup} = \frac{400 \cdot 10^6 \cdot 28.95}{2122456.17} = 5.5 \text{ MPa (compression)}$$

$$\sigma_{inf} = \frac{400 \cdot 10^6 \cdot 51.05}{2122456.17} = 9.70 \text{ MPa (traction)}$$

3 – τ – diagram

In section C, the shear force, $T = 100 \text{ kN}$.

$$\tau = \frac{T \cdot S_z}{I_z \cdot b}$$

Section 1-1

$$S_{1-1} = 0 \text{ (} S = 0 \text{)} \rightarrow \tau_{1-1} = 0$$

Section 2-2

$$S_{2-2} = S \times y = 100 \times 20 \times 18.95 = 37900 \text{ cm}^3$$

$$\tau_{2-2} = \frac{T \cdot S_z}{I_z \cdot b} = \frac{100 \cdot 10^3 \cdot 37900}{2122456.17 \cdot 100} = 1.786 \text{ MPa}$$

Section 3-3

$$S_{3-3} = S_{2-2} \times y = 37900 \text{ cm}^3 \rightarrow 100 \times 20 \times 18.95 = 37900 \text{ cm}^3$$

$$\tau_{3-3} = \frac{100 \cdot 10^3 \cdot 37900}{2122456.17 \cdot 30} = 5.6 \text{ MPa}$$

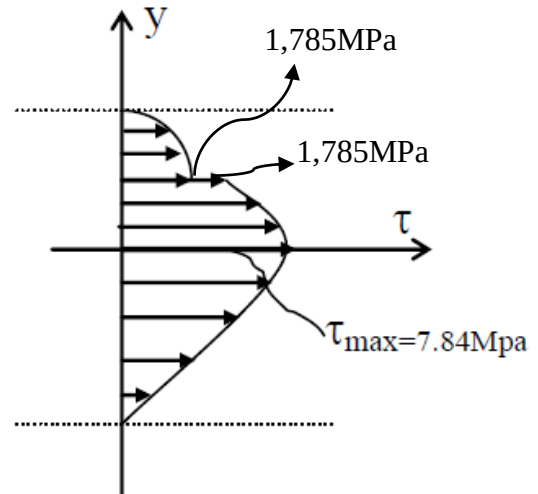
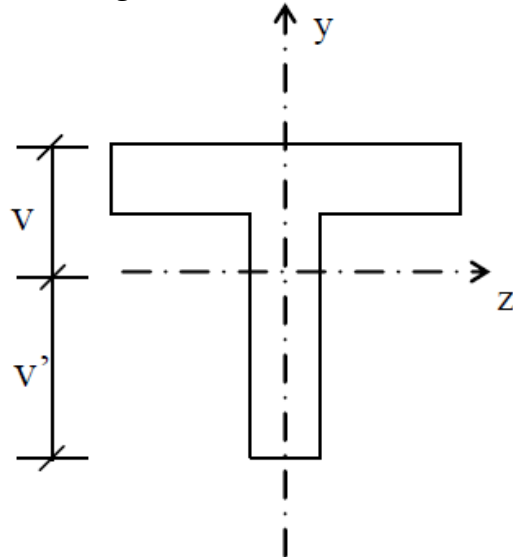
Section 4-4

$$S_{4-4} = S_{3-3} + 3 + 8.95 \times 30 \times \frac{8.95}{2} = 39101.5 \text{ cm}^3 \rightarrow 100 \times 20 \times 18.95 = 37900 \text{ cm}^3$$

$$S_{4-4} = S_{3-3} + 8.95 \times 30 \times 8.95 = 39101.5 \text{ cm}^3$$

$$\tau_{4-4} = \frac{100 \cdot 10^3 \cdot 39101,5}{2122456,17 \cdot 30} = 6,14 \text{ MPa}$$

The diagram will be as follows:



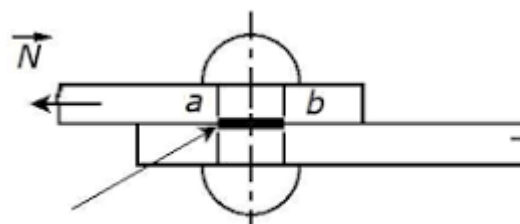
Cisaillement

5.1 Introduction

Any part subjected to two equal forces, with the same line of action but opposite directions, which tend to cause one part of the piece to slide relative to the other and tends to separate the part into two parts C1 and C2. A beam is stressed at simple shear when it is subjected to two directly opposite forces, perpendicular to the middle line, and which tend to shear it.

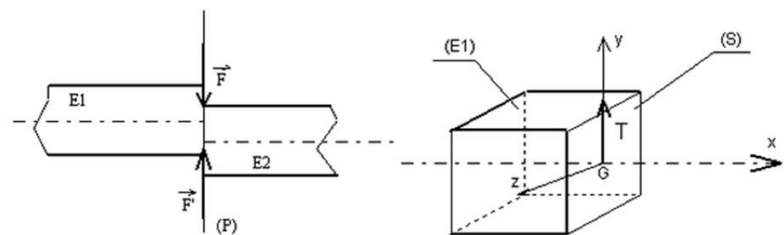
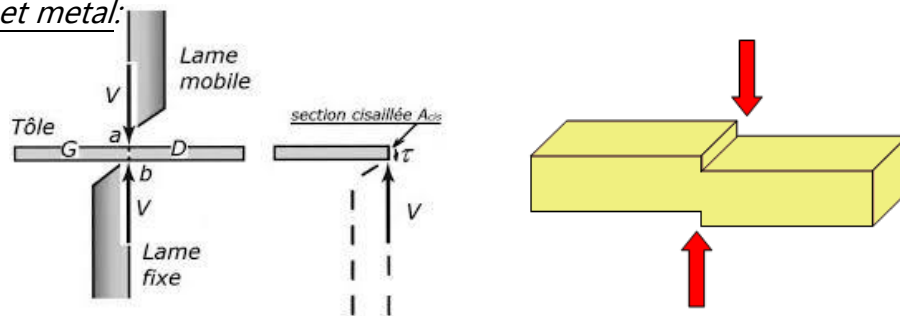
It is concretely manifested by cutting (scissors), breaking assemblies (rivets, bolts), deformation of structures (earthquakes), or slipping layers of wind or viscous fluids.

• *Two plates joined by a rivet: This one will be sheared, because of the force N on the assembly, in a following section ab .*



Sheared section

• Cutting a sheet metal:



• Shear key:

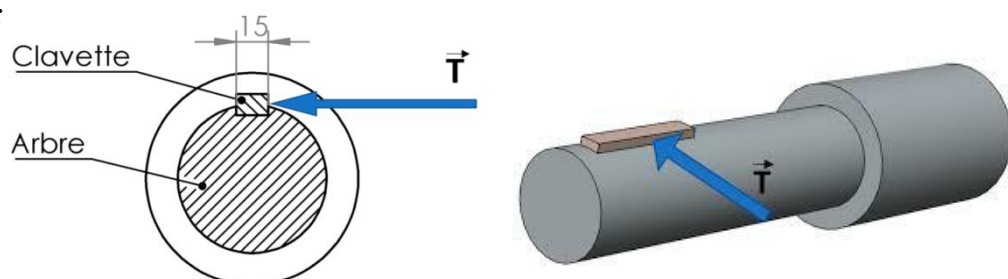


Fig 5.1 Examples of sheared parts mechanisms.

In civil engineering, shear is a critical stress because opposing forces tend to cause one section of the structure to slide relative to the other. It is common in flexed beams under load, geotechnics (soil slippage, subsidence, swelling), structural steel joints (rivets, bolts, welds), in B.A (flexed beams, punching of slabs).

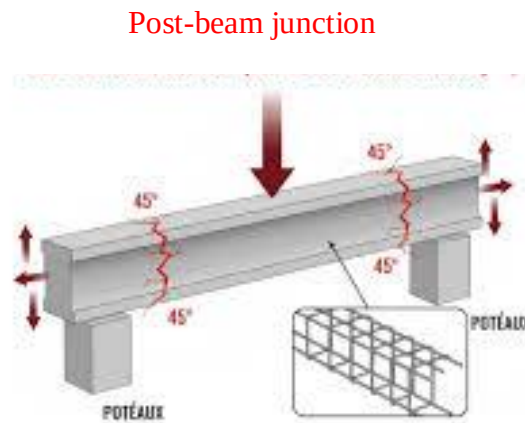


Fig 5.2 Maximum shear at the reinforced concrete supports

5.2 Definition

A beam is sheared if the forces acting on two different sections A1 and A2 reduce to two equal and opposite forces F perpendicular to the neutral axis.

- If the two forces have the same line of action $d=0$, the shear is said to be pure (Fig. 5.1a).
- If the points of application of the forces are slightly offset by a distance d ($d \neq 0$) (Fig. 5.1b), the shear is said to be simple.

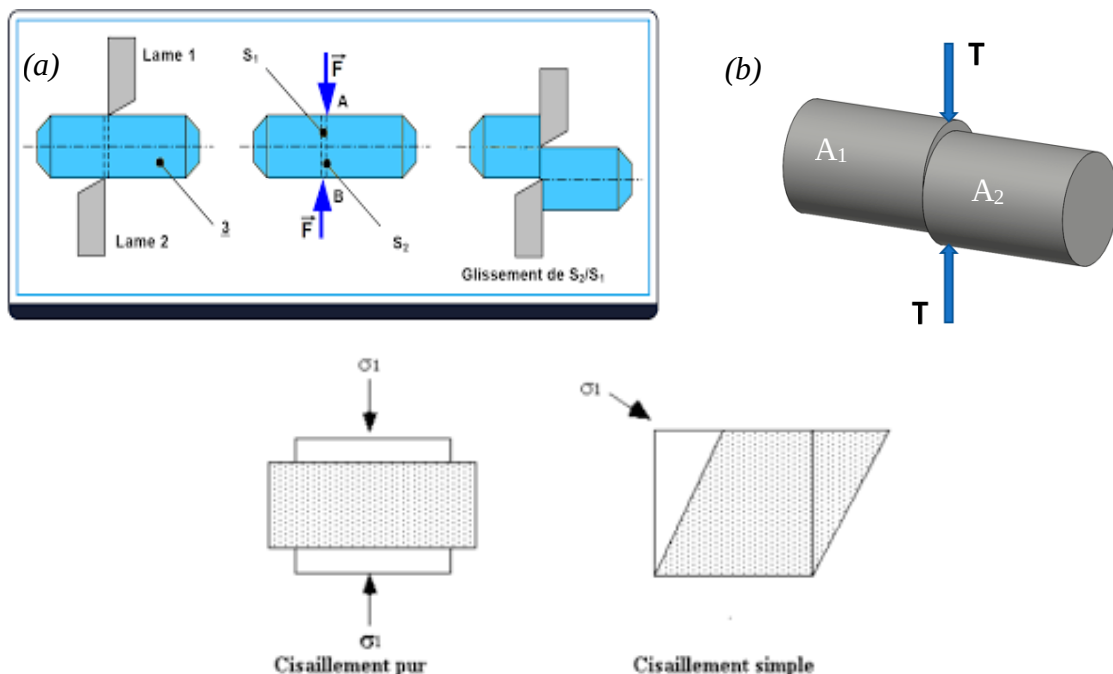


Fig 5.3 Types of shear

A straight section (S) of a beam (P) is stressed at simple shear, if the reduction elements at the center of gravity G of (S) of the torsor of cohesion forces reduce to: $\{\tau_{coh}\}_G = \begin{Bmatrix} 0 & 0 \\ T_y & 0 \\ 0 & 0 \end{Bmatrix}$

5.3 Shear test

Let a recessed beam be submitted in a section SS', as close (infinitely small distance) to the recess as possible Δx , to uniformly distributed forces of resultant P. The upper and lower surfaces of the beam are subject to shear stresses τ . If we assume that these surfaces are free from any other stress than τ , then they are said to be subjected to pure shear. Pure shear is a very special case that is impossible to carry out experimentally. On the other hand, simple shear concerns a section of the beam and not the entire beam.

The beam deforms as the magnitude of the distributed forces increases, inducing transverse shifts Δy of the layers of the beam relative to each other, which is called distortion, slip angle or deviation noted γ (Fig. 5.4).

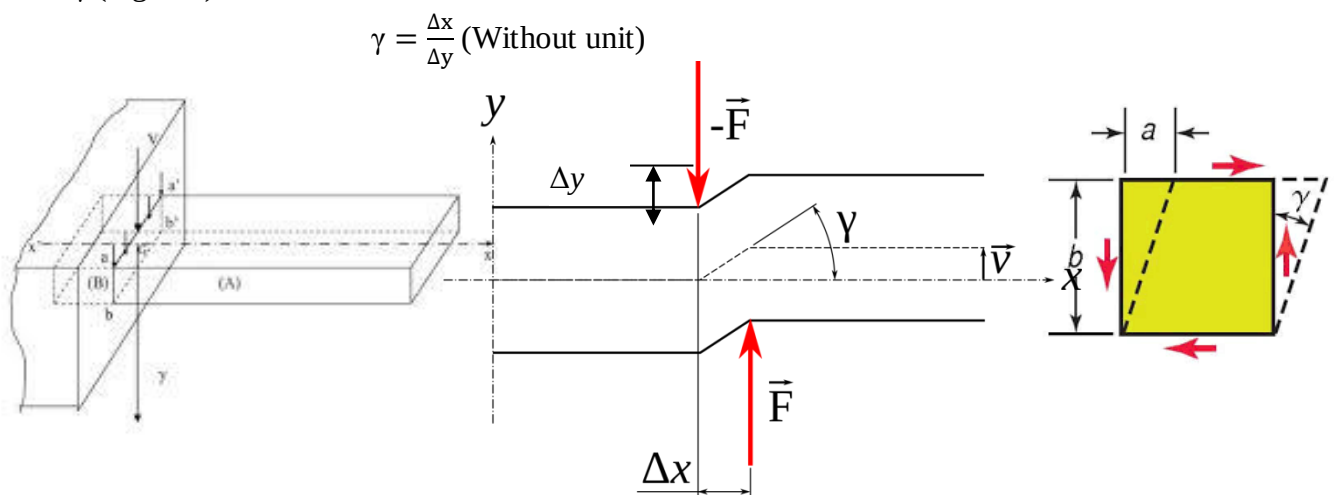


Fig 5.4 Shear test for a straddle beam

According to Navier-Bernoulli's law, the cross-section SS' of area Ω remains planar after deformation and normal to the beam axis, hence the sliding cross-section remains in its own plane. According to Hooke's law, which states the proportionality of stresses to strains, we can write:

$$\sigma = \varepsilon \cdot E$$

→ mathematical law describing the state in tension/compression

In the case of shear, the same principle applies, and we say that the slip angle γ is proportional to the shear stress τ . This is expressed as follows:

$$\tau = \gamma \cdot G$$

With:

$$G = \frac{E}{2(1+\nu)}$$

G: modulus of transverse elasticity, or shear modulus, or Coulomb modulus N/mm² (material characteristic).

ν : Poisson's ratio.

Application

What is the Coulomb modulus G of steel, given that $E = 210000 \text{ N/mm}^2$ and $\nu = 0.30$

$$G = \frac{E}{2(1+\nu)} = \frac{210000}{2(1+0.3)} = 80769 \text{ N/mm}^2$$

Generally, for steel, a first approximation of $G = 80,000 \text{ N/mm}^2$ is used. For other metals, we have:

$$G \approx 0,4.E$$

$$G \approx 0,4 E = 0,4.21000 = 84\,000 \text{ N/mm}^2$$

Note

In reality, the two forces applied in opposite directions generate bending; this is simple shear. When the difference between the forces is small, the phenomenon is called pure shear. If we represent the shear stress τ as a function of the shear angle γ , we obtain a diagram similar in shape to that of pure tension (Fig. 5.5).

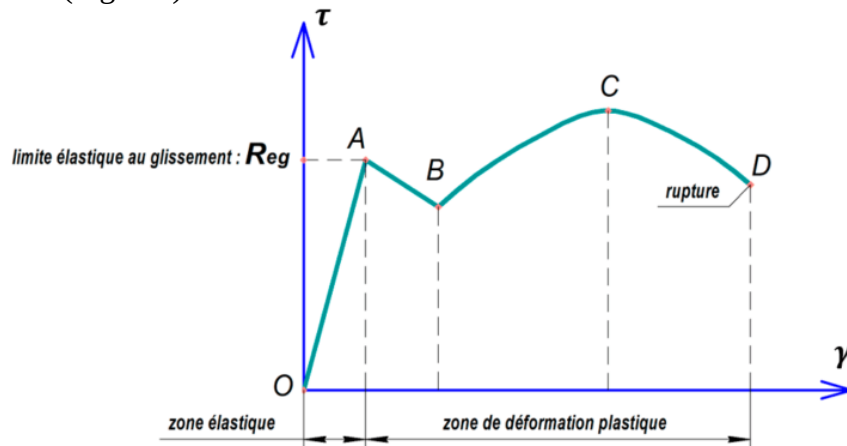


Fig 5.5 Tangential stress diagram τ - distortion γ

Two distinct zones are observed:

- Elastic zone (OA): τ is proportional to the strain γ ; deformations are linear and reversible.
- Plastic zone (ABCD): Beyond τ_e (the material's elastic shear limit), the stress-strain relationship is no longer linear or reversible (plastic deformation zone).

Table 1 E and G values of some materials

Matière	E (Ga)	G (GPa)
Acier	200-210	80-90
Aluminium	70	25-27
Cuivre	110-130	40-45
Verre	60-75	30
Bois	10-15	0,5-1
Béton	0.01-0.1	Très faible
Or	79	27
Argent	72,5	30

5.4 Study of shear stress

In different assemblies of constructions (bolted, riveted, welded, in wood notches, etc.) these elements are subjected in different planes to tension or compression or shearing. The shearing is always followed either by bending, or by traction or compression; this produces tangential stresses τ and normal σ which are so low compared to the tangential stresses that they are neglected.

For a section S beam stressed at simple shear the value of τ is equal to the ratio of the shear force T by section S.

$$\tau_{\text{moy}} = \frac{T}{S}$$

Where,

τ_{moy} : (τ) mean shear tangential stress in N/mm^2 , MPa

T: shear force in N.

S: area of the sheared cross-section (S) of the part in mm^2 .

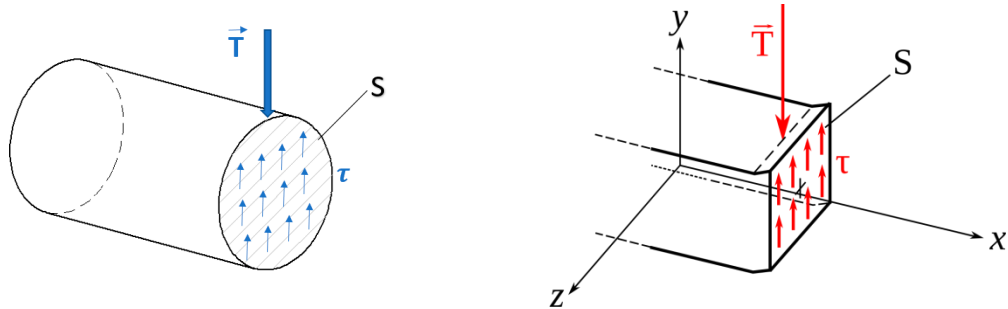


Fig 5.6 Surfaces stressed in shear

5.5 Déformation élastique en cisaillement

We have already seen:
$$\begin{cases} \tau_{\text{moy}} = \frac{T}{S} \\ \tau_{\text{moy}} = \gamma \cdot G \end{cases} \Rightarrow \frac{T}{S} = \gamma \cdot G \Rightarrow \gamma = \frac{T}{G \cdot S}$$

Therefore: since G is a constant, the variation of deformations depends on the shear force T and the sheared area S.

γ : is called relative slip (deformation, distortion).

5.6 Shear strength condition

For a part to work safely in shear, the shear stress τ must be less than the allowable practical shear strength (R_{pg}) (τ_{adm}) (or R_{eg} : elastic shear strength of the material (in MPa).

$$\tau_{adm} = R_{pg} = \frac{R_{eg}}{FS}$$

Where,

FS: Safety Factor

R_{pg} : Practical Shear Strength

The strength condition implies that the actual stress applied (τ) to the part is less than the allowable stress (τ_{adm}):

$$\tau \leq \tau_{adm}$$

Table 2 gives the relationship between the elastic resistance to extension R_e and the elastic resistance to sliding R_{eg} for: steels, alloys, aluminum, and cast irons.

Table 2 Relationship between R _e and R _{eg} for steels, alloys	
Matériaux	Relation entre Re et Reg
Aciers doux, alliages d'aluminium (Re ≤ 270 MPa)	Reg = 0,5 × Re
Aciers mi-durs (320 ≤ Re ≤ 520 MPa)	Reg = 0,7 × Re
Aciers durs, fontes (Re ≥ 600 MPa)	Reg = 0,8 × Re

5.7 Some values of the allowable stress ($\tau_{adm}, \bar{\tau}$)

- **Ductile material:** τ_{adm} is obtained taking into account a safety factor FS.

$$\tau_{adm} = R_{pg} = \frac{R_{eg}}{FS} = \frac{\tau_e}{FS}$$

τ_e : Shear yield strength = R_{eg} (elastic shear strength). For ductile materials, the shear yield strength τ_e is equal to 58% of the tensile yield strength, i.e.:

$$\tau_e = 0,58 \times R_{eg} \approx 0,6 \times R_{eg}$$

$$\tau \leq 0,6 \times \frac{R_{eg}}{FS}$$

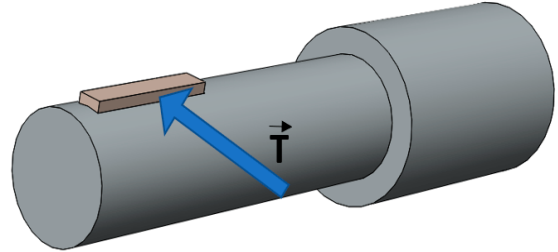
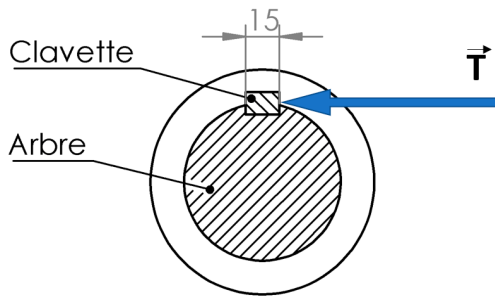
- **Brittle material:** (concrete, grey cast iron, etc.) it can be assumed that the allowable shear stress τ_{adm} will be determined from the ultimate tensile strength R_{break} and is equal to:

$$\tau_{adm} = \frac{R_{break}}{FS}$$

SERIES OF EXERCISES

Exercise 1

A key 85 mm long is subjected to a tangential force of 12,000 N, which puts it under shear stress. Determine the shear stress (tangential stress) experienced by the key?



Solution

1- Determine the section subjected to shear stress

$$S = 85 \times 15 = 1275 \text{ mm}^2$$

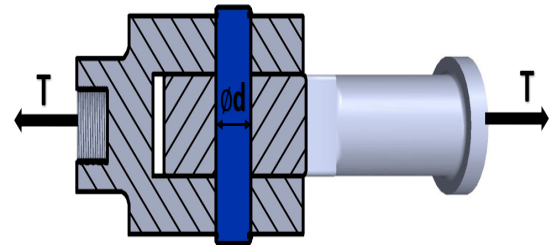
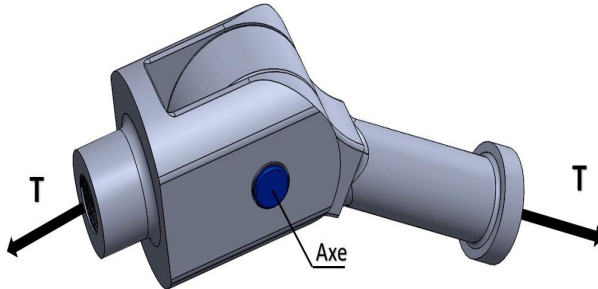
2- Determine the tangential stress

$$\tau_{\text{moy}} = \frac{T}{S} = \frac{12000}{1275} = 9,41 \text{ MPa}$$

Exercise 2

The axis of the joint yoke is subject to a tangential force of 150,000 N. The permissible stress at the shear of the axis is 95 MPa.

Determine the diameter of the axis?



Solution

1- Determination of the stressed section at shear;

$$\tau_{\text{adm}} = \frac{T}{S} = \frac{150000}{95} = 1579 \text{ mm}^2$$

2- Determination of the axis section

$$S_{\text{axe}} = \frac{S}{n}$$

Where,

S : the section subjected to shear stress = 1579 mm^2 ;

n : number of sections stressed at shear = 2;;

$$S_{\text{axe}} = \frac{1579}{2} = 789.5 \text{ mm}^2$$

$$S_{\text{axe}} = \frac{\pi d^2}{4} \Rightarrow d = \sqrt{4 \times 789.5 / \pi} = 31,71 \text{ mm} \approx 32 \text{ mm}$$

Exercise 3

Let us say two transmission shafts constructed from the same steel ($G = 8.104 \text{ MPa}$), the first is solid with a diameter D_1 , and the second is hollow (D : outer diameter, d : inner diameter, knowing that $d=0.8D$)

The torque to be transmitted is 200 N.m and $\tau_p = 100 \text{ MPa}$

- 1- Calculate D_1 .
- 2- Calculate D and d .
- 3- Determine the weight ratio between these two shafts.

Solution

- 1- Let's calculate D_1 .

$$|\tau|_{\max} \leq \tau_{adm} = \tau_p$$

Cylindrical section, so:

$$\tau_{\max} = \frac{16.M_t}{\pi.D^3} \leq \tau_p \rightarrow D_1 \geq \sqrt[3]{\frac{16.M_t}{\pi.\tau_p}} = \sqrt[3]{\frac{16.200}{\pi.100}} = 21,57 \text{ mm}$$

Calculation of D and d

$$|\tau_{\max}| \leq \tau_{adm} = \tau_p$$

$$\text{Cylindrical section, } \rightarrow I_0 = \frac{\pi(D^4 - d^4)}{32} = \frac{\pi.D^4(1 - 0,8^4)}{32}$$

$$\rho_{\max} = \frac{D}{2}$$

$$\tau_{\max} = \frac{16.M_t}{\pi.D^3(1 - 0,8^4)} \leq \tau_p$$

$$\rightarrow D \geq \sqrt[3]{\frac{16.M_t}{\pi.\tau_p(1 - 0,8^4)}}$$

$$D \geq \sqrt[3]{\frac{16.200}{\pi.100.(0,59)}} = 25,8 \text{ mm}$$

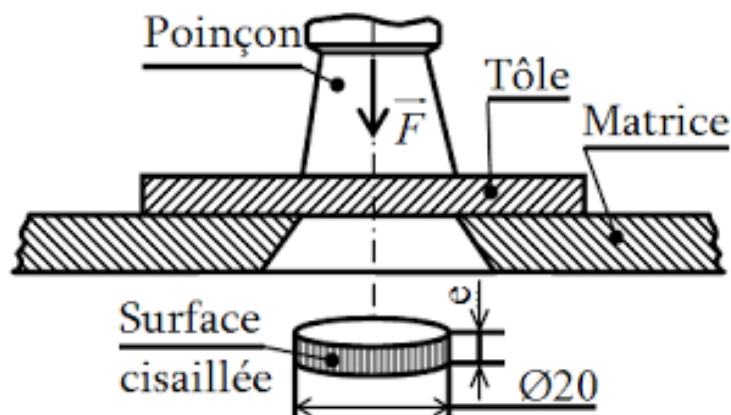
$$\text{So, } d = 0,8 \times D \rightarrow d = 0,8 \times 25,8 = 2,64 \text{ mm}$$

Exercise 4

We want to punch a sheet of thickness $e = 4 \text{ mm}$ in S365 steel ($R_e = 490 \text{ N/mm}^2$).

The hole to be punched will have a diameter of 20 mm .

- 1- Calculate the exerted effort,
- 2- Calculate the compression stress.



Solution

1-Calculation of the exerted effort

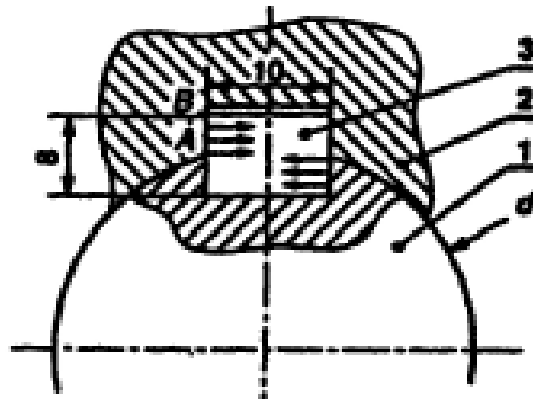
$$\begin{aligned}\tau &> R_g = \frac{R_e}{2} \\ \Rightarrow \frac{F}{S} &> R_g = \frac{R_e}{2} \quad \text{and} \quad S = \pi \cdot d \cdot e \\ \Rightarrow F &> \frac{R_e}{2} \cdot \pi \cdot d \cdot e \Rightarrow F > \frac{490}{2} \cdot \pi \cdot 20.4 \\ \Rightarrow F &> 61,57N\end{aligned}$$

2-Calcul de la contrainte de compression

$$\begin{aligned}\sigma &= \frac{F}{S} \quad \text{Or} \quad S = \frac{\pi \cdot d^2}{4} \\ \Rightarrow \sigma &= \frac{4 \cdot F}{\pi \cdot d^2} = \frac{4 \cdot 61,57}{\pi \cdot 20^2} \\ \Rightarrow \sigma &\approx 196MPa\end{aligned}$$

Exercise 5

A shaft (1) transmits a rotational movement to a core (2) via a key



The shaft with a diameter of 32 mm is made of XC 18 steel, for which $R_e = 265$ MPa. The transmitted torque is $M = 65$ Nm. The key has a transverse dimension of 10 x 8. The key steel is E24 for which $\tau_e = 108$ MPa. The safety coefficient chosen is $FS = 3$.

The maximum allowable pressure on the AB flank of the key-hub contact is $p_m = 30$ MPa.

We give $AB = 4$ mm. It is assumed that the resultant $\|\vec{F}\|$ of the mechanical contact actions on the key flange is perpendicular to the contact surface at the distance $d/2$ from the shaft axis.

Calculate the norm of $\|\vec{F}\|$.

From the resistance condition of non-matting, determine the minimum length of the wedge.

Check the length of the pin from the shear strength condition.

Conclusion.

Solution

1- If M is the transmitted torque;

$$\begin{aligned}M &= \|\vec{F}\| \cdot \frac{d}{2} \Rightarrow \|\vec{F}\| = \frac{2 \cdot M}{d} \\ \Rightarrow \|\vec{F}\| &= \frac{2 \cdot 65}{32 \cdot 10^{-3}} = 4062,5N\end{aligned}$$

2- Let p be the pressure at the key-hub contact; the non-matting condition is:

$$P \leq P_m$$

With: $P = \frac{\|\vec{F}\|}{S_m} \rightarrow S_m: \text{mounting surface}$

$$S_m = AB.l = 4.l$$

$$\frac{\|\vec{F}\|}{4.l} \leq P_m \Rightarrow l > \frac{\|\vec{F}\|}{4.P_m} = \frac{4062,5}{4.30} = 33,85\text{mm}$$

3 - Shear strength condition

$$\tau_{moy} \leq \tau_p$$

$$\tau_p = \frac{\tau_e}{s} = \frac{108}{3} = 36\text{MPa}$$

$$\tau_{moy} = \frac{T}{S_c}$$

$$\Rightarrow T = \|\vec{F}\| \text{ et } S_c = 10.l$$

$$\Rightarrow \frac{\|\vec{F}\|}{10.l} \leq \tau_p \Rightarrow l > \frac{\|\vec{F}\|}{10.\tau_p}$$

$$\Rightarrow l > \frac{4062,5}{10.36} = 11,28\text{mm}$$

Conclusion

As this calculation shows, the non-mating condition leads to choosing a longer key. Even when working in shear, the key must always be calculated based on the non-mating condition.

Exercise 6

What force must be applied to drill a 5 mm thick flat iron with a 40 cm diameter hole punch.

We give $\tau_u = 30\text{kg/mm}^2$.

Solution

Under the effect of P, shear stresses appear in the lateral cylindrical surface with a diameter equal to 40 mm and a length equal to 15 mm; one must therefore have:

$$P \leq \pi.d.e.\tau_u$$

$$P \leq \pi.40.5.30 = 18849,55\text{ kg}$$

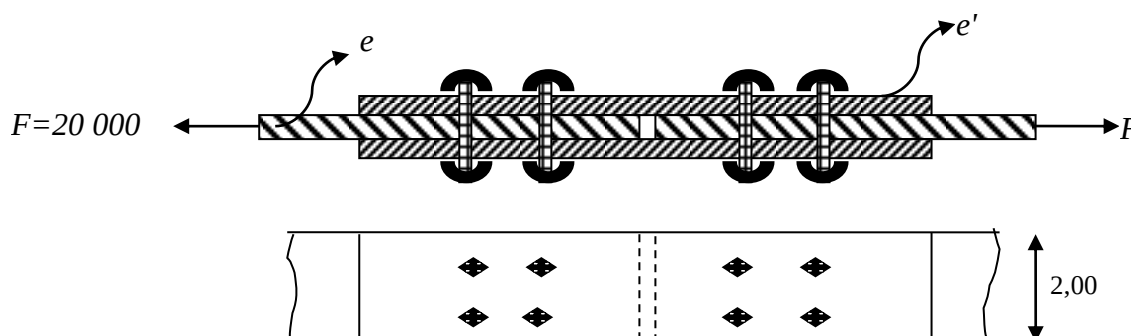
Exercise 7

We want to assemble 2 flat irons (see figure below) using a double joint cover with a length of 200 mm. The tensile effect F is 20,000 kgp.

Determine e, e',

The number and diameter of rivets;

We give: $\tau = 10\text{ kg/mm}^2$; $\sigma_a = 14\text{ kg/mm}^2$.



Solution

1- Calculation of e

To take into account the rivet holes, we take $\tau \leq \bar{\tau} = \frac{20\,000}{200.e} \leq 11$

$$\Rightarrow e > \frac{20\,000}{200.11} = 9,09\text{mm} \approx 9\text{mm}$$

In this case, we choose: $e' = \frac{e}{2} = \frac{9}{2} = 4,5\text{mm}$ et $\varnothing = 18\text{mm}$

Surface of a rivet: $S_r = \frac{\pi.\varnothing^2}{4} = \frac{\pi.18^2}{4} = 254,47\text{mm}^2$

Each rivet has two sheared sections; for the number of rivets, the force of 20,000 kg is balanced by:

$$\frac{F}{2n} = \frac{20\,000}{2n} = \frac{10\,000}{n} \rightarrow \text{effort in a section of rivets.}$$

So, we will have :

$$\tau \leq \bar{\tau} \Rightarrow \frac{10\,000}{10.n} \leq 255. \rightarrow n = \frac{10\,000}{255.10} = 3,92 \cong 4$$

1-Verification of the flat irons

Usable area: $9(200 - 2(18 + 1)) = 1476\text{mm}^2$

The gussets satisfy since:

$$e' = \frac{e}{2} \quad \text{et} \quad F' = \frac{F}{2}$$

2-Verification of the rivets

$$\frac{F}{2.n.\pi.\frac{\varnothing^2}{4}} \leq \bar{\tau}$$

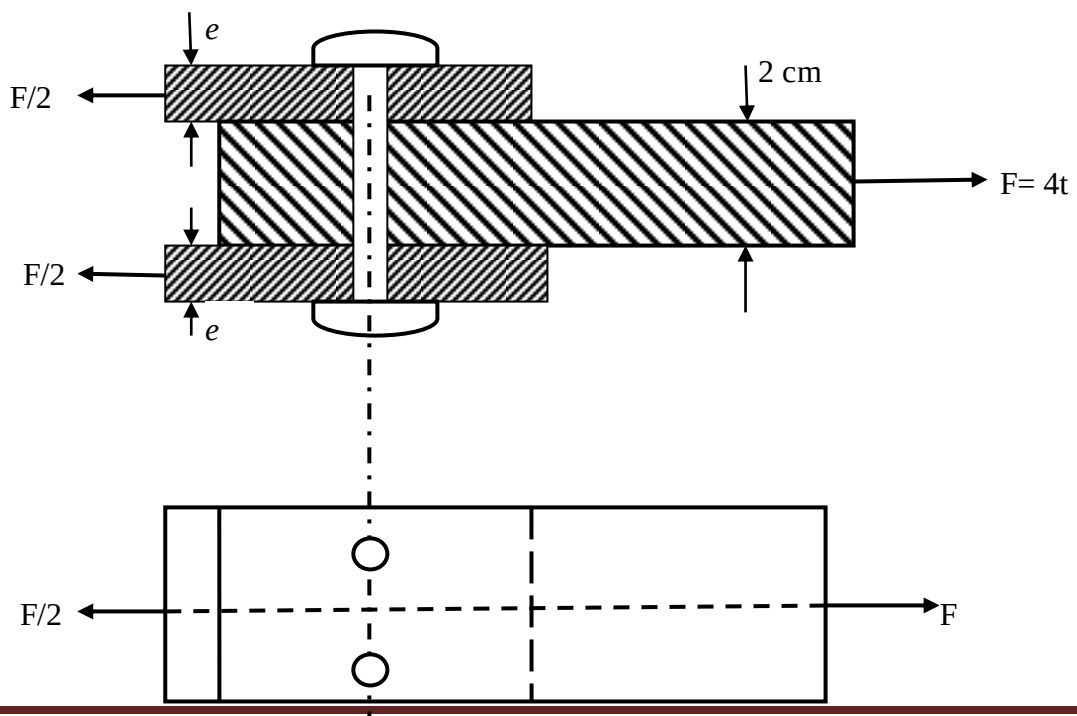
$$\Rightarrow \frac{20\,000}{2.4.\pi.\frac{18^2}{4}} = 9,92\text{kg/mm}^2 < 10\text{kg/mm}^2$$

Exercise 8

Three steel sheets are assembled together by two rivets, each with a diameter of 17mm.

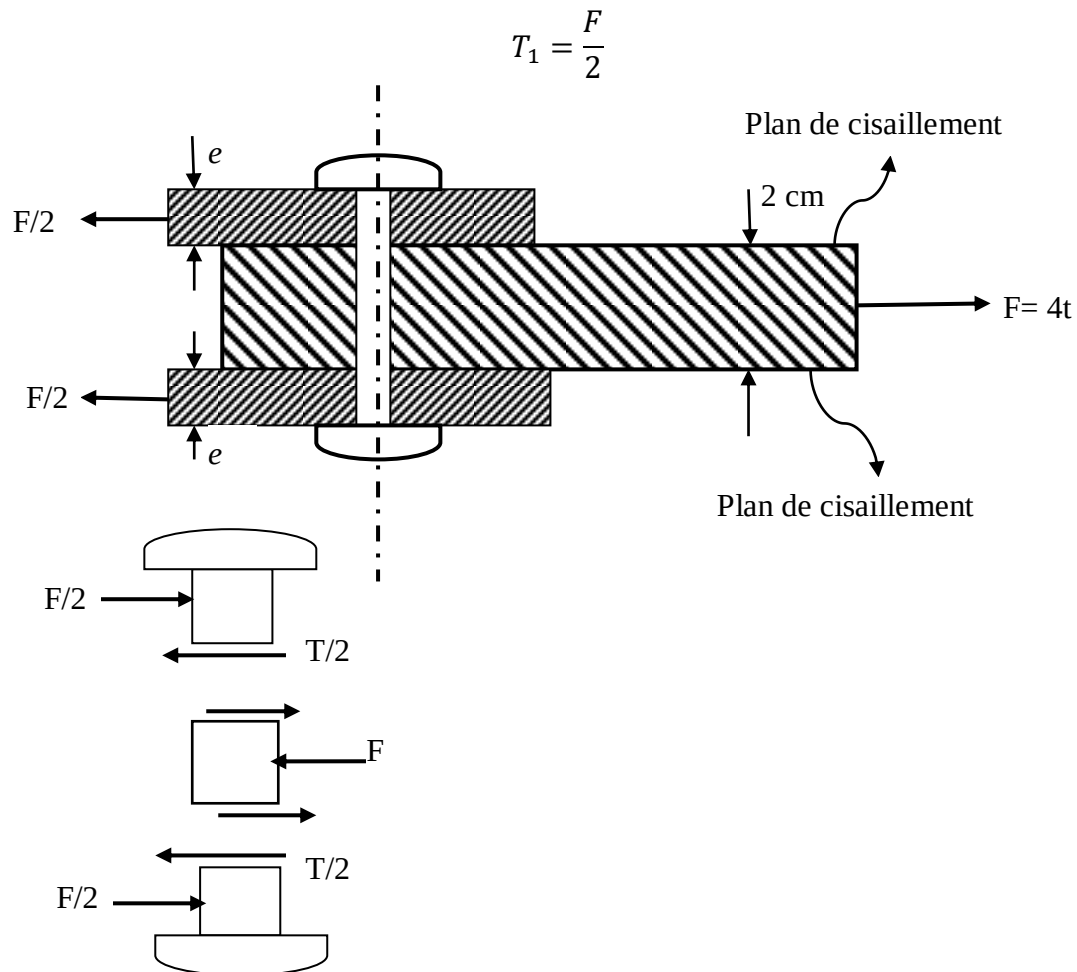
1- Check the strength of the rivets if the allowable shear stresses $\bar{\tau} = 900\text{ kg/cm}^2$.

the minimum thickness of each of the two sheets if $\bar{\sigma} = 1200\text{ kg/cm}^2$.



Solution

- 1- Here we have two shear planes. The shear force (shear) applied to the sheared section at a single shear plane is:



If there are no rivets.

$$T_1 = \frac{F/n}{2}$$

The shear stress on the sheared section (returning to each rivet) is:

$$\tau_1 = \frac{\frac{F/n}{2}}{A_1} \quad A_1 = \frac{\pi \cdot d^2}{4}$$

With the resistance condition:

$$\begin{aligned} \tau &\leq \tau_1 \\ \Rightarrow \tau_1 &= \frac{\frac{F/n}{2}}{A_1} = \frac{\frac{F/n}{2}}{\frac{\pi \cdot d^2}{4}} = \frac{2F}{n \cdot \pi \cdot d^2} \\ \Rightarrow \tau_1 &= \frac{2 \cdot 4000}{2 \cdot \pi \cdot 1,7^2} = 440,6 \text{ kg/cm}^2 < 900 \text{ kg/cm}^2 \end{aligned}$$

The strength condition of the rivets is checked.

- 2-The normal stress in one of the two sheets at the hazardous section is:

$$\begin{aligned} 1- \sigma_{1-1} &= \frac{N}{A_{1-1}} = \frac{F/2}{e(5-2 \cdot 1,7)} \\ \Rightarrow \sigma_{1-1} &= \frac{2000}{1,6 \cdot e} \leq 1200 \end{aligned}$$

$$\Rightarrow e > \frac{2000}{1,6.1200} = 1,04 \text{ cm}$$

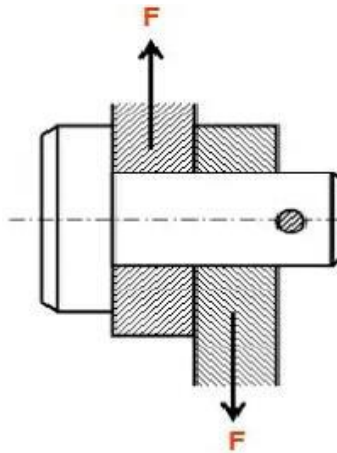
The minimum necessary thickness of the sheets acceptable is $e=1.04$ cm.

Exercise 9

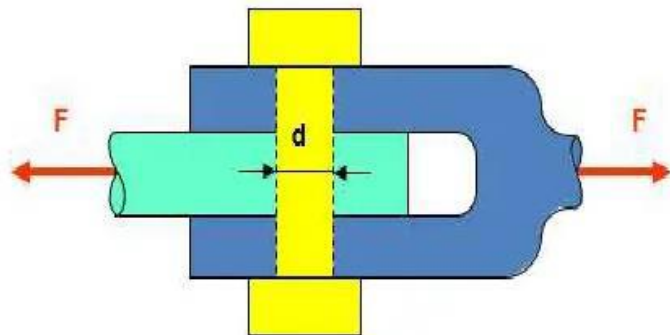
On two joints, the axis must withstand a shear force of 12,000 N. It is made of steel (S275, $R_g = 155$ MPa) and has a transverse modulus of elasticity $G = 8000$ MPa, the safety coefficient $FS = 5$.

➤ Two proposed assemblies:

1- Cantilever mounting



2- Screed assembly



For both cases:

- 1- Calculate the minimum diameter.
- 2- Calculate the shear stress.
- 3- Calculate the angular deformation γ .

Solution

I- Cantilever mounting

1- Determination of the minimum diameter d_1 of the axis:

Knowing that: $R_{eg} = 155$ MPa; $F = 17.000$ N;

We take : $S=5$; $T=F=17.000$ N

$$\tau \leq \bar{\tau} \Rightarrow \frac{F}{A_1} = \frac{4.F}{\pi.d_1^2} \leq \frac{R_{eg}}{5} = \frac{155}{5} 31 \text{ MPa} = 31 \text{ N/mm}^2$$

$$\Rightarrow \frac{4.F}{\pi.d_1^2} \leq 31 \Rightarrow d > \sqrt{\left(\frac{4.F}{31.\pi}\right)} = \sqrt{\left(\frac{4.12000}{31.\pi}\right)} = 22,20 \text{ mm}$$

2- The maximum shear stress $\tau_{\max}$

$$\tau_{\max} = \frac{F}{A} = \frac{12000.4}{\pi.22,20^2} = 31 \text{ MPa}$$

3-the angular deformation γ

$$\tau = \gamma \cdot G \Rightarrow \gamma = \frac{\tau}{G}$$

$$\Rightarrow \gamma = \frac{31}{800} = 0,03875$$

II- Screed assembly

1-Determining the minimum diameter d_1 of the axis:

The force F is absorbed by two parallel cross-sections; total shear area.

Therefore, the force is distributed over two sections:

$$T = F/2 = 6000N.$$

$$S_{T \text{ sheared}} = A_2 = 2A_1$$

$$\tau \leq \bar{\tau} \Rightarrow \frac{F}{A_2} = \frac{4 \cdot F}{\pi \cdot d_2^2} \leq \frac{R_{eg}}{5} = \frac{155}{5} 31 MPa = 31 N/mm^2$$

$$\Rightarrow \frac{4 \cdot F}{\pi \cdot d_2^2} \leq 31 \Rightarrow d_2 > \sqrt{\left(\frac{4 \cdot F}{31 \cdot \pi}\right)} = \sqrt{\left(\frac{4 \cdot 6000}{31 \cdot \pi}\right)} = 15,70 mm$$

→ $d_2 < d_1 \Rightarrow$ A slab-on-grade mounting is better than a cantilevered mounting. For greater shear strength, solutions with multiple shear surfaces are chosen.

2-The maximum shear stress τ_{max}

$$\tau_{max} = \frac{F/2}{A_2} = \frac{6000 \cdot 4}{\pi \cdot 15,7^2} = 30,99 MPa$$

3- Angular deformation γ

$$\tau = \gamma \cdot G \Rightarrow \gamma = \frac{\tau}{G}$$

$$\Rightarrow \gamma = \frac{\tau}{G} = \frac{30,99}{800} = 0,03874$$

Chapter 6

Torsion

6.1 Introduction

The Torsion is the stress experienced by a beam subjected to the action of a couple of opposing forces acting in parallel planes, the reducing element of which is moment acting along the beam's axis. Torsion is expressed as a torsional moment M_t acting along the beam's axis xx' . In torsion, the beam's cross-sections do not obey Bernoulli's principle because they are generally not planar; they undergo warping. If the ends are free, the state is generated only by shear stresses τ and uniform torsion (Fig. 6.1). If warping is prevented, either by a fixed support or supports, or the cross-section varies, the torsional moment M_t induces variable warping (M_t is not constant from one cross-section to another), there is an appearance of normal stresses σ in addition, the beam is in non-uniform torsion (Fig 6.2).

- *Uniform torsion (Saint Venant, free)*: the torsional moment is constant along the length of the beam. The cross-sections rotate relative to each other, but they remain flat or can warp identically along the entire length of the beam. In this case, there are *only shear stresses τ in the section*, for example, circular sections (solid or tubular), massive bars with free ends.

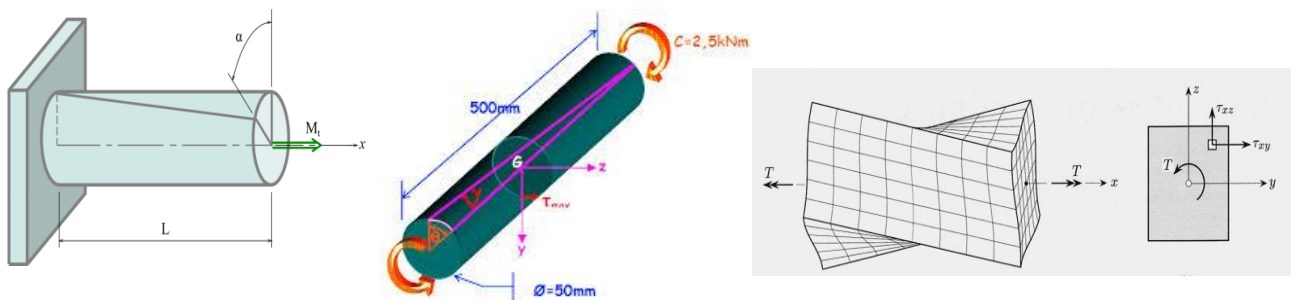


Fig 6.1 Uniform torsion state

- *Non-uniform torsion (torsion prevented)*: Warping is prevented by embedding or support or varies along the beam and consequently the birth of normal stresses σ of traction or compression. It also appears on open thin-walled profiles (such as I, H, T or U). On these shapes, the torsion causes the wings to flex in the opposite direction, and if this deformation is blocked, the torsion becomes non-uniform.

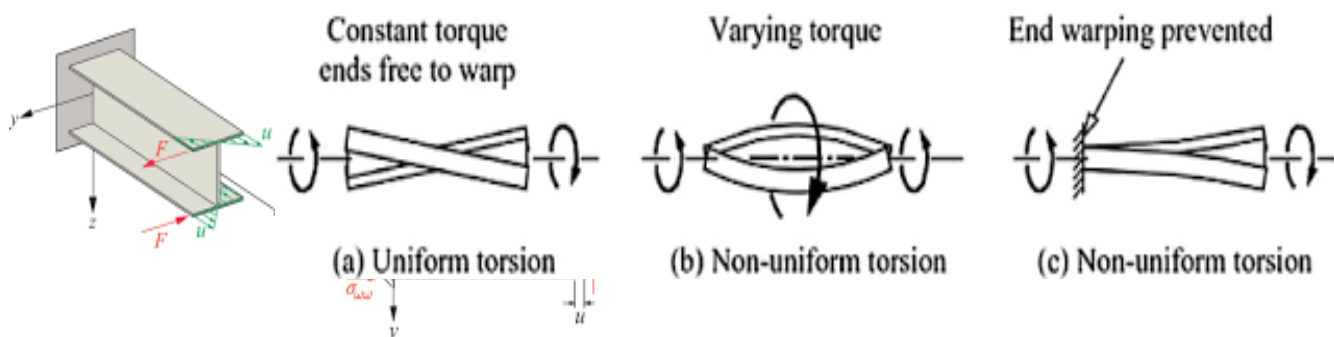


Fig 6.2 Non-Uniform state

6.2 Definition

Let be a cylinder of circular cross-section with radius R and length l , isotropic, with rectilinear axis and negligible own weight, subjected in the plane of each of its butt sections to equal and opposite torques: it is said that the part undergoes a twisting effect.

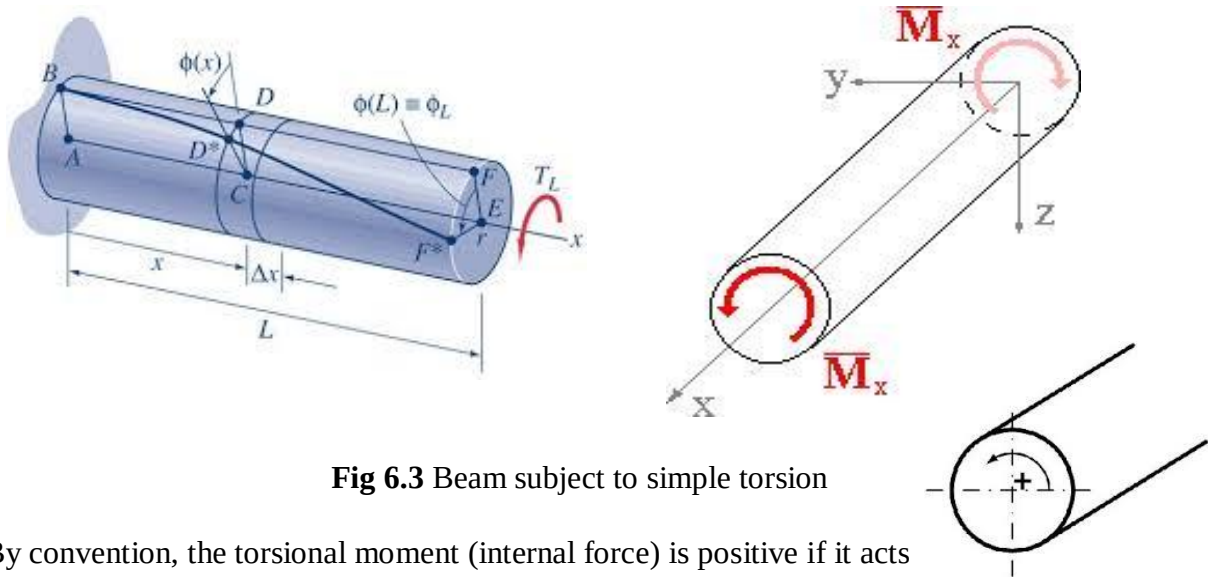


Fig 6.3 Beam subject to simple torsion

• By convention, the torsional moment (internal force) is positive if it acts counter clockwise for an observer looking at the section. The torsional wrench corresponds to a special case of the cohesion wrench where only the moment collinear (torsional moment) with the beam axis is non-zero ($M_t \neq 0$). For a beam with axis $(G, \vec{x})$ (Fig. 6.3), the torsional wrench at the centroid G of a cross-section is written as:

$$\{T\}_{Coh} \begin{Bmatrix} \vec{0} \\ M_t \end{Bmatrix} = \begin{Bmatrix} 0 & M_t \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_{(G, \vec{x}, \vec{y}, \vec{z})}$$

6.3 Experimental torsion study

Consider a circular beam subjected to a constant torsional moment, perfectly fixed at its left end, with cross-section (S_1) and center of gravity G_1 . Let M_1M_2 be a generator before the test. At the free end (the right), with cross-section (S_2) and center of gravity G_2 , a system of external forces is applied, reducing to a moment $\vec{M}G_1$ about the axis (x', x) . The magnitude of this moment is gradually increased while measuring the beam's distortions. The resulting curve has a shape similar to that of the tensile test (Fig. 6.4).

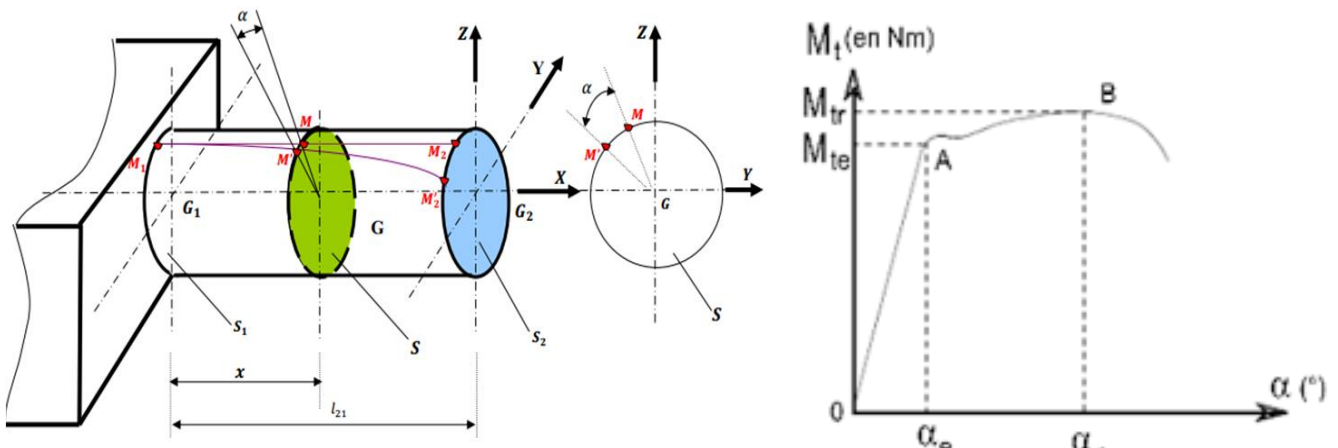


Fig 6.4 Simple torsion test curve

The curve representing M_t as a function of the distortions α has an elastic zone OA and a zone beyond A where the deformations increase rapidly until the beam fails. The test results allow us to conclude that:

- The section, which is flat and normal to the cylinder axis before the test, remains flat and normal to the (x, x') axis after the test; thus, the bar length and the section radius remain unchanged.

- There is no longitudinal deformation (no warping); the sections maintain constant relative distances. The distance between two straight sections is a rotation of angle α (small) around the (x, x') axis, and is proportional to the distance x from the fixed end (Fig. 6.5).

$$d\alpha = C^{te} \cdot dx \Rightarrow C^{te} = \frac{d\alpha}{dx} = \theta \left[\frac{\text{rad}}{\text{mm}} \right]$$

$\rightarrow C^{te}: \text{constant} = \theta \rightarrow \text{Called unit torsion angle}$
 $d\alpha = \text{Angle of rotation of section S in (rad)}$
 $dx = \text{Distance from S to the reference section } S_0 \text{ in (mm)}.$

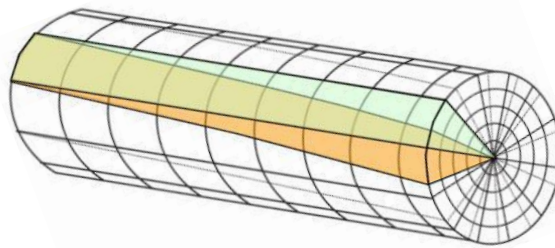


Fig 6.5 No warping of the section of a circular bar

6.4 Similarity of the Hooke's law traction-shear-torsion

Hooke's law models the behavior of solids in the elastic domain subject to constraints. It states that elastic deformation is a linear function of stress. It is a tensile/compressive strain law; however there is a similar law for shear. A further study to show a great analogy between the experimental results of torsion and shear, that is to say, proportionality between the tangential stresses τ and the distortion (deformation) γ which is modeled by the law:

$$\tau_M = G \cdot \gamma_M$$

τ_M : Tangential stress in M (N/mm^2)

γ_M : Distortion due to the torsion moment M_t (rad)

G : The transverse elastic modulus or modulus *Coulomb*, depends on the material (N/mm^2).

$$\text{For steel} \rightarrow G = 8 \cdot 10^4 \text{ N / (mm}^2\text{)}$$

There is a relationship between Yong's modulus and Coulomb's modulus:

$$G = \frac{E}{2(1+\nu)}$$

ν : Poisson's ratio

$$\text{For steel} \rightarrow \nu = 0.27 \div 0.30$$

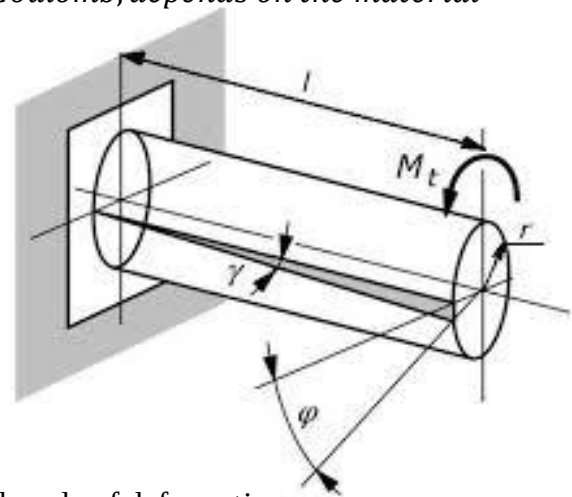


Fig 6.7 Hooke's Law and angle of deformation.

NB

The longitudinal modulus of elasticity E is always greater than the transverse modulus G . In other words, a material will always be less rigid under the effect of a torsional moment than under the effect of a tensile force.

Application

What is the Poisson's ratio of grey cast iron, given that: $E = 90,000 \text{ N/mm}^2$? Given that the Coulomb modulus $G = 36,000 \text{ N/mm}^2$?

Solution

$$\begin{aligned}\text{We know that: } G &= \frac{E}{2(1+\nu)} \\ \Rightarrow \nu &= \frac{E-2G}{2G} = \frac{E}{2G} - 1 \\ \Rightarrow \nu &= \frac{90000}{2 \cdot 36000} - 1 = 0,25\end{aligned}$$

6.4.1 Relationship $\tau - \gamma$ (and angular deformation)

Hooke's law for shear is given by:

$$\tau = G \cdot \gamma \rightarrow (\text{In triangle } M_1MM' \text{ with } \gamma \text{ very small (Fig 6.8)),}$$

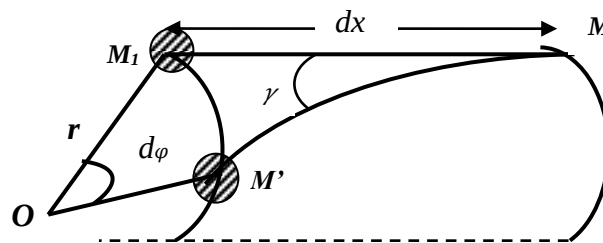


Fig 6.8 Torsion test (simple circular section).

$$\tan \gamma \approx \gamma \text{ (Small deformations)}$$

According to Fig 6.8 we have: $\tan \gamma = \frac{MM'}{M_1M} = \gamma$

$$\frac{d\varphi}{dx} = \theta \rightarrow \text{Relative angle of deformation (in rad)}$$

$$MM' = r \cdot d\varphi \quad \text{et} \quad M_1M = dx$$

$$\Rightarrow \gamma = r \cdot \frac{d\varphi}{dx} = r \cdot \theta$$

$$\frac{d\varphi}{dx} \rightarrow \text{relative rotation analogous to } \frac{\Delta l}{l} \rightarrow \text{Relative longitudinal of deformation.}$$

φ : Angle of rotation between 2 sections.

γ is the resulting distortion on cylindrical surface, similar to the deformation ϵ_x in tension (shear angle).

$$\text{Or:} \quad \tau = G \cdot \gamma = G \cdot r \cdot \frac{d\varphi}{dx} = G \cdot r \cdot \theta$$

If: $r = R \rightarrow$ (That is to say, the lateral surface of the cylinder).

$$\tau = \tau_{max} \Rightarrow \tau_{max} = G \cdot R \cdot \theta \rightarrow \begin{cases} \tau \rightarrow \left[\frac{\text{N}}{\text{mm}^2} \right] \\ G \rightarrow \left[\frac{\text{N}}{\text{mm}^2} \right] \\ \theta \rightarrow \left[\frac{\text{rad}}{\text{mm}} \right] \\ r \rightarrow [\text{mm}] \end{cases}$$

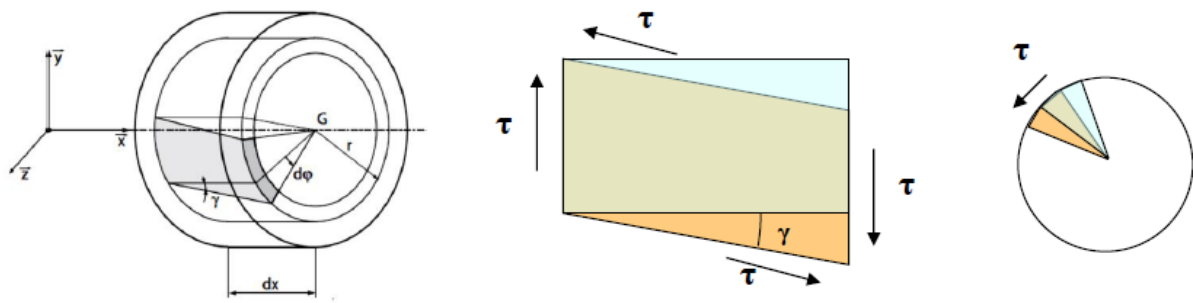


Fig 6.9 Distortion methods for different surfaces.

-Even in tension, there is a slip limit beyond which deformations are irreversible.

-Steel $\rightarrow E = 80\,000\text{ MPa} \rightarrow R_{pg} = 250\text{ MPa}$

-Aluminum $\rightarrow E = 26\,000\text{ MPa} \rightarrow R_{pg} = 200\text{ MPa}$

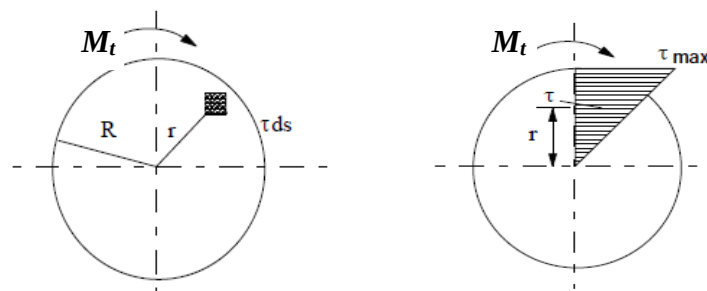


Fig 6.10 Variation of τ as a function of r

6.4.2 Relationship between torsion moment M_t and angle of rotation ϕ

According to the Euler-Bernoulli theory, and according to the small perturbation theorem, the torsion moment M_t the shear stress τ is proportional by the distance r from the torsion axis.

It is maximal on the r periphery and zero in the center.

$$\tau = \frac{M_t}{I_p} \cdot r$$

$$\text{Or, } \tau = G \cdot r \cdot \theta$$

$$\begin{aligned} \Rightarrow M_t &= \frac{\tau \cdot I_p}{r} \\ \Rightarrow M_t &= G \cdot \theta \cdot I_p \end{aligned}$$

Where

- M_t Torsion moment ;
- I_p is the moment of polar inertia of the torsional section, depending on the shape of the section (outer diameter(ext), and inner diameter(int) in the case of a tube).
- The quantity : $W_p = \frac{I_p}{r} \rightarrow$ is called a *torsion resistance module* or *polar resistance module*:

$$\begin{aligned} \tau &= \frac{M_t}{I_p} \cdot r \Rightarrow \tau = \frac{M_t}{I_p/r} \\ \Rightarrow \tau &= \frac{M_t}{W_p} \\ \Rightarrow \tau_{max} &= \frac{M_t}{W_{pmax}} \rightarrow \begin{cases} W_{pmax} = \frac{I_p}{r_{max}} \rightarrow [mm^3] \\ c.\text{à}.d \rightarrow r_{max} = R \rightarrow [mm] \end{cases} \end{aligned}$$

$$\theta = \frac{M_t}{G \cdot I_p} \left[\frac{\text{rad}}{\text{mm}} \right]$$

For drilling probes, shaft of large vessels ...), avoid excessive torsional deformations due to vibrations.

To ensure proper rigidity in the transmission, a limit is generally imposed on the unit angle of torsion:

$$\theta \leq \theta_{\text{limite}} \text{ (Rigidity equation)}$$

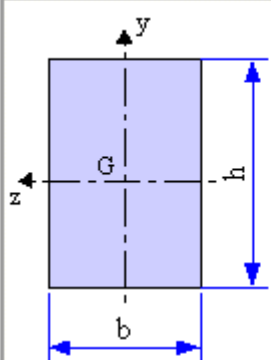
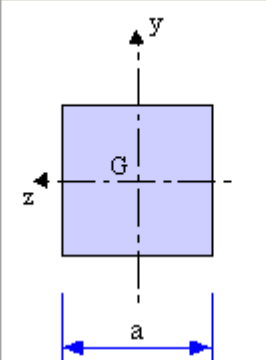
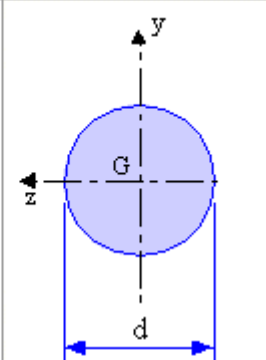
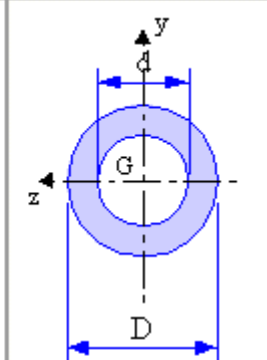
- The angle of rotation (angular deformation) φ of a beam with length l ($x=l$)

$$\begin{aligned} \theta &= \frac{d\varphi}{dx} \\ \varphi &= \int_0^l \frac{M_t}{G \cdot I_p} dx \\ \Rightarrow \varphi &= \frac{M_t \cdot l}{G \cdot I_p} \text{ [rad]} \end{aligned}$$

→ l : is the length of the circular bar [mm].

→ $G I_p$: is called torsional rigidity [N. mm²]

Table 3 some useful polar quadratic moments

				
IGz (mm ⁴)	$\frac{bh^3}{12}$	$\frac{a^4}{12}$	$\frac{\pi d^4}{64}$	$\frac{\pi (D^4 - d^4)}{64}$
I₀ (mm ⁴)	$\frac{bh^3 + hb^3}{12}$	$\frac{a^4}{6}$	$\frac{\pi d^4}{32}$	$\frac{\pi (D^4 - d^4)}{32}$

6.5 Conditions to be verified (proper use)

6.5.1 Strength condition

The maximum stress τ_{max} must always be less than the practical shear stress R_{pg} , by adopting a safety factor FS such that: $R_{pg} = R_e / FS = \tau_{adm}$

Hence the condition of resistance of a part under torsion:

$$\begin{aligned} \tau &\leq \tau_{adm} \\ \Rightarrow \tau_{max} = \tau &= \frac{M_t}{I_p} \cdot R \leq \tau_{adm} \end{aligned}$$

6.5.2 Rigidity condition

Often, the calculation of the dimensions of drive shafts or torsion bars is based more on a deformation condition than a strength condition. Indeed, to ensure rigid transmission and avoid vibrations, the unit torsion angle $\theta_{\max}$ must not exceed a limiting torsion value $\theta_{\lim}$ during operation. Hence the torsional rigidity condition of a part:

$$\theta_{\max} \leq \theta_{\lim} \Leftrightarrow \frac{M_t}{G \cdot I_p} \leq \theta_{\lim}$$

6.5.3 Stress Concentration condition

When using the parts, one must check for any potential problem of stress concentration; if k_t is this coefficient, the resistance condition is written:

$$\tau_{eff\max} \leq R_{pg}$$

Knowing that:

$$\tau_{eff\max} = k_t \cdot \tau_{th\max}$$

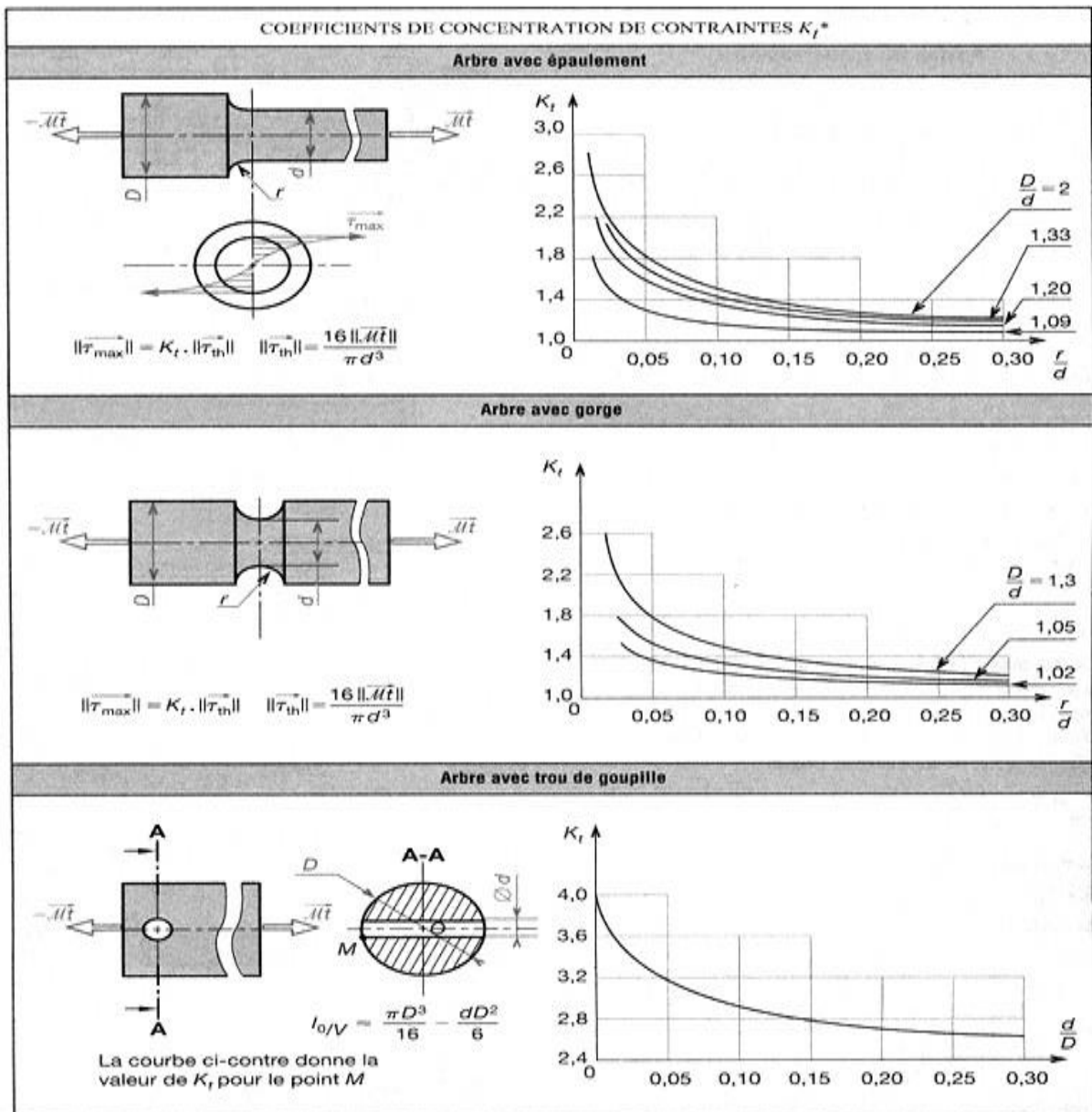


Fig 6.11 Stress concentration coefficient k_t in simple torsion

SERIES OF EXERCISES

Exercise1

Determine the diameter to be given to a shaft transmitting a power of 30 HP, under a speed of 500 tr/min.

Given: $\bar{\sigma} = 75 \text{ daN/mm}^2$; $\bar{\theta} = \frac{1}{4} \text{ degré/m}$; $G=8000 \text{ daN/mm}^2$

Solution

The power $\mathcal{P}$ is equal to the product of the torque $\mathcal{C}$ and the angular velocity $\omega=2\pi n$, where n is the number of revolutions per second; therefore:

$$\mathcal{C} = M_t = \frac{\mathcal{P}}{\omega} = \frac{30 \times 75 \times 60}{2\pi \times 500} = 45 \text{ m.daN} = 45000 \text{ mm.daN}$$

1- Resistance condition

$$\begin{aligned} \tau_{max} &= \frac{M_t}{I_p} \cdot R \leq \tau_{adm} \\ I_p &= \frac{\pi d^4}{32} = \frac{\pi (2R)^4}{32} = \frac{16 \cdot \pi R^4}{32} = \frac{\pi R^4}{2} \\ \tau_{max} &= \frac{2 \cdot M_t}{\pi R^4} \cdot R \leq \tau_{adm} \\ \Rightarrow R^3 &> \frac{2 \cdot M_t}{\pi \cdot \tau_{adm}} \\ \Rightarrow R^3 &> \frac{2 \cdot M_t}{\pi \cdot \tau_{adm}} = \frac{2 \times 45\,000}{500 \cdot \pi} = 3820 \text{ mm}^3 \\ \Rightarrow R &> \sqrt[3]{3820} = 15,63 \text{ mm} \rightarrow d = 32 \text{ mm} \end{aligned}$$

2- Rigidity condition

$$\begin{aligned} \theta_{max} &\leq \theta_{lim} \Leftrightarrow \frac{M_t}{G \cdot I_p} \leq \theta_{lim} \\ \theta_{lim} &= \frac{d\varphi}{dl} = \frac{1}{4} \left(\frac{\pi}{180 \cdot 1000} \right) = 43,63 \cdot 10^{-7} \text{ rd/mm} \\ \Rightarrow \frac{M_t}{G \cdot \frac{\pi d^4}{32}} &\leq \theta_{lim} \Rightarrow \frac{32 \cdot M_t}{G \cdot \pi \cdot d^4} \leq \theta_{lim} \\ \Rightarrow \frac{32 \cdot M_t}{\pi \cdot G \cdot d^4} &\leq \theta_{lim} \Rightarrow d^4 > \frac{32 \cdot M_t}{\pi \cdot G \cdot \theta_{lim}} \\ \Rightarrow d^4 &> \frac{32 \cdot 45\,000 \cdot 10^7}{\pi \cdot 8000 \cdot 43,63} = 13\,132\,197,92 \text{ mm}^4 \\ \Rightarrow d &> \sqrt[4]{13\,132\,197,92} = 60,20 \text{ mm} \\ \text{So, we'll take: } &\max(32 \text{ mm} ; 60,20 \text{ mm}) \\ \Rightarrow d &= 60 \text{ mm} \end{aligned}$$

Exercise 2

Consider a 15 m long boat propeller shaft. The power to be transmitted is 4.5 MW at 350 rpm. The allowable torsional stress of the steel is 80 N/mm².

1- Determine the shaft diameter at this stress, as well as the corresponding angle of twist ($G = 80,000 \text{ N/mm}^2$).

2- Then replace this shaft with a hollow drive shaft to obtain the same angle of twist as before, while reducing the weight by 50%.

Solution

- Calculation of the moment of torsion

$$M_t = \frac{30 \cdot P}{\pi \cdot n} = \frac{30 \times 4,5 \times 10^6}{\pi \times 350} = 1,23 \times 10^5 \text{ N.m}$$

-Research of diameter

Formula for solid trees under stress

$$d > 1,72 \sqrt[3]{\frac{M_t}{\sigma_{adm}}} = 1,72 \sqrt[3]{\frac{1,23 \cdot 10^8}{80}} = 200 \text{ mm}$$

-Finding the angular deformation of the shaft

$$\theta_{max}^{\circ} = \theta^{rad} \times \frac{180}{\pi} = 5,7 \times 10^{-4} / \text{mm} = 0,57^{\circ} / \text{m}$$

- Finding the inner diameter for a 50% weight gain.

In the summary table, we notice that for a weight gain of $k_{g\theta} = 50\%$, you have to : $k_d = 0$

$$d > 1,79 \sqrt[4]{\frac{M_t}{(1-k_d^4)G \cdot \theta_{max}}} = 1,79 \sqrt[4]{\frac{1,23 \cdot 10^8}{(1-0,8^4) \times 80000 \times 0,00001}} = 227 \text{ mm}$$

$$d_i = k_d \times d = 0,8 \times 227 = 180 \text{ mm}$$

With the diameters of 230 mm outside and 180 mm inside, we have a gain of 51%.

Exercise 3

A cylinder is subjected to a torque of $C=2.5 \text{ kN.m}$. The COULOMB modulus of the material is 78 GPa. Calculate:

- 1- the maximum tangential stress in the cylinder.
- 2- the distortion of the generators to rd and $^{\circ}$.
- 3- the angle of rotation of the extreme sections in $^{\circ}$.

Solution

- 1- The torsional moment M_t is constant along the bar.

Consider any section:

$$\tau_{max} = \frac{M_t}{I_p} \cdot R$$

$$I_p = \frac{\pi d^4}{32}$$

$$\tau_{max} = \frac{M_t}{I_p} \cdot R = \frac{2,5 \times 10^6}{\pi} \times \frac{25 \times 32}{50^4} = 101,86 \text{ MPa}$$

- 2- Distortion of generators in rad and in $^{\circ}$.

$$3- \tau = G \cdot \gamma \Rightarrow \gamma = \frac{\tau}{G}$$

$$\Rightarrow \gamma = \frac{101,86}{78 \times 10^3} = 1,305 \times 10^{-3} \text{ rad}$$

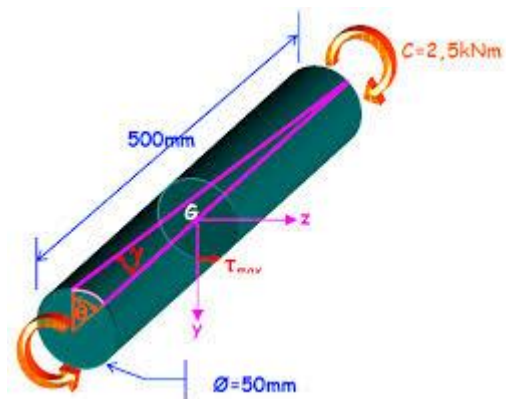
$$\gamma = \frac{1,305 \times 10^{-3} \times 180}{\pi} = 0,075^{\circ}$$

- 3- The angle of rotation of the extreme sections in $^{\circ}$.

$$\varphi_{max} = \frac{M_t \times l}{G \cdot I_p}$$

$$\Rightarrow \varphi_{max} = \frac{2,5 \times 10^6 \times 500 \times 32}{78 \times 10^3 \times \pi \times 50^4}$$

$$\varphi_{max} = 0,0261 \text{ rad} = 1^{\circ} 50'$$



Exercise 4

A hand winch is operated by two men, each exerting a force F of 200 N. What is the side length c of the square to be machined on the shaft for the crank, which has a length L of 400 mm? Assume the allowable shear stress is 60 N/mm².

Solution

-Research of torsion moment

$$M_t = 2F \times L = 2 \times 200 \times 400 = 160\,000 \text{ N}\cdot\text{mm}$$

-Finding the resistance module (for square)

In our case $= b$ (see Table):

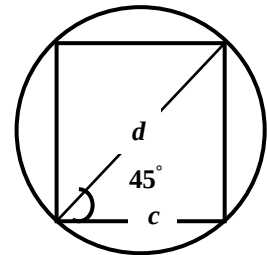
$$w_t = k_2 \times a \times b^2 = 0,2082c^3$$

$$\tau_{adm} > \frac{M_t}{w_t}$$

$$\Rightarrow \tau_{adm} > \frac{M_t}{0,2082c^3} = \sqrt[3]{\frac{M_t}{w_t}}$$

$$\Rightarrow c > \sqrt[3]{\frac{M_t}{0,2082 \times \tau_{adm}}} = \sqrt[3]{\frac{160\,000}{0,2082 \times 60}} = 23,4 \text{ mm}$$

$$\rightarrow c \approx 25 \text{ mm}$$



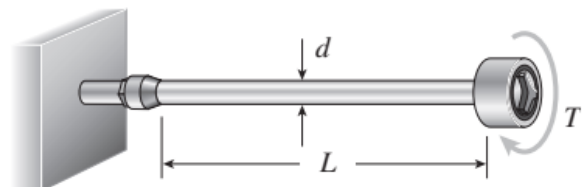
-Finding the permissible diameter of the tree.

Knowing that the diagonal of the square is the diameter of the circumscribed circle, we have

$$d = \frac{c}{\cos 45^\circ} = \frac{25 \times 2}{\sqrt{2}} = 35,36 \text{ mm} \approx 36 \text{ mm}$$

Exercise 6

The steel shaft of a socket wrench has a diameter of 6 mm and a length of 200 mm (see figure). If the allowable shear stress is 60 MPa, what is the maximum allowable torque $M_{\max}$ that can be applied to the wrench? By what angle (in degrees) will the shaft twist under the action of the maximum torque? ($G=78 \text{ GPa}$).

**Solution**

-Calculation of $M_{\max}$

$$\tau_{\max} = \frac{16M_{\max}}{\pi \cdot d^3} \Rightarrow M_{\max} = \frac{\pi \cdot d^3}{16} \tau_{\max}$$

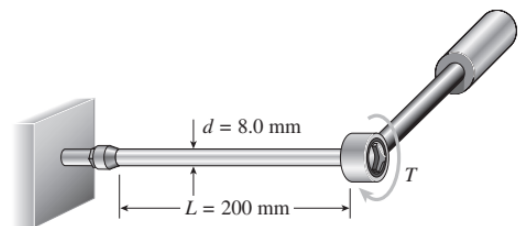
$$\Rightarrow M_{\max} = \frac{\pi \cdot 6^3}{16} 60 = 2544,69 \text{ mm}$$

-Calculation of angle φ (in degree)

$$\varphi = \frac{M_t \cdot l}{G \cdot I_p}$$

$$I_p = \frac{\pi \cdot d^4}{32} = \frac{\pi \cdot 6^4}{32} = 127,23 \text{ mm}^3$$

$$\Rightarrow \varphi = \frac{2544,69 \times 180}{78 \times 10^3 \times 127,23} = 0,0462 \text{ rad} = 2,64^\circ$$



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