COURSE DESTINATED TO STUDENTS OF FIRST YEAR MASTER DYNAMICAL SYSTEMS

INTRODUCTION TO DYNAMICAL SYSTEMS

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Introduction

In this chapter we collect some material which will play a part in the theory to be developed in the subsequent chapters.

1 Preliminary Concepts and Definitions

Definition 1 *We call a differential system a system of the form*

$$\dot{x} = f(x) \quad \text{where } f : E \to \mathbb{R}^n. \tag{1.1}$$

E is an open set of $\mathbb{R}^n$ and $\dot{x} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \ldots \right)$
$f : E \to \mathbb{R}^n$ is called a vector field.

- *If f is a linear map, we say that the system* (1.1) *is linear. Otherwise we say that system* (1.1) *is non-linear.*
- *If $\frac{\partial f}{dt} \equiv 0$ i.e., f does not depend explicitly on t we say that the system* (1.1) *is autonomous and we note $\dot{x} = f(x)$*
- *If f depends explicitly on t we say that the system* (1.1) *is non-autonomous and we note $\dot{x} = f(t, x)$.*

Example 1 *The systems*

- $\dot{x} = x + 1, \;\; \dot{y} = 2x + y$ *is a linear autonomous system.*
- $\dot{x} = x + \ln t, \;\; \dot{y} = 2x + y$ *is a linear non-autonomous system.*
- $\dot{x} = x^2 + 1, \;\; \dot{y} = 2x + y$ *is a nonlinear autonomous system.*
- $\dot{x} = xy + \sin t, \;\; \dot{y} = 2x + y^2 + t$ *is a nonlinear non-autonomous system.*
- $\dot{x} = x + ty + z, \;\; \dot{y} = 2x + y, \dot{z} = \sin x + t$ *is a nonlinear non-autonomous system.*
- $\dot{x} = x + 2z, \;\; \dot{y} = 2x + z, \dot{z} = 1$ *is a linear autonomous system.*

2 Differentiability

The notation of the differentiable function is a generalization to the function of several variables of the differentiable function to the function of a real variable.

Definition 2 *The function* $f:\mathbb{R}^n\to\mathbb{R}^n$ *is differentiable at* $x_0\in\mathbb{R}^n$ *if there exists a linear transformation* $Df(x_o)\in L(\mathbb{R}^n)$ *which satisfies*

$$\lim_{|h|\to 0}\frac{|f(x_0+h)-f(x_0)-Df(x_0)h|}{|h|}=0.$$

The linear transformation $Df(x_0)$ *is called the derivative of* f *en* x_0.

Remark 1 *If* $f:\mathbb{R}^n\to\mathbb{R}^n$ *is differentiable at* $x_0\in\mathbb{R}^n$, *then the partial derivatives,* $\left(\frac{\partial f_i}{\partial x_j}\right)$ $i,j=1,\ldots,n$, *all exist at* x_0 *and for all* $x\in\mathbb{R}^n$,

$$Df(x_0)x=\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}(x_0)\,x_j.$$

Thus, if f *is a differentiable function, the derivative* Df *is given by the* $n\times n$ *Jacobian matrix*

$$Df=\left[\frac{\partial f_i}{\partial x_j}\right].$$

Remark 2 *If* $f:\mathbb{R}^n\to\mathbb{R}^n$ *is differentiable at* a *in* $\mathbb{R}^n$, *then all of its directional derivatives at* a *exist, and for any choice of vector* v *in* $\mathbb{R}^n$ *we have*

$$D_vf(x)=Df(x)v.$$

The left-hand side is a limit, while the right-hand side is a matrix product, with v *treated as a column vector.*

Example 2 *Find the derivative of the function*

$$f(x,y)=\begin{pmatrix}f_1(x,y)\\ f_2(x,y)\end{pmatrix}=\begin{pmatrix}x+xy\\ x^2+y^3\end{pmatrix}.$$

at the point $(1,2)$ *and calculate* $Df\,(1,2)\,X$.

Solution. We have

$$Df\,(x,y)=\begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}=\begin{pmatrix}1+y & x\\ 2x & 3y^2\end{pmatrix},$$

at the point $(1,2)$, we have

$$Df\,(1,2)=\begin{pmatrix}3 & 1\\ 2 & 12\end{pmatrix},$$

and

$$Df\,(1,2)\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3 & 1\\ 2 & 12\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3x+y\\ 2x+12y\end{pmatrix}.$$

Or

$$Df\,(1,2)\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\displaystyle\sum_{j=1}^{n}\frac{\partial f_1}{\partial x_j}(1,2)\,x_j\\ \displaystyle\sum_{j=1}^{n}\frac{\partial f_2}{\partial x_j}(1,2)\,x_j\end{pmatrix}=\begin{pmatrix}\dfrac{\partial f_1}{\partial x}(1,2)\,x+\dfrac{\partial f_1}{\partial y}(1,2)\,y\\ \dfrac{\partial f_2}{\partial x}(1,2)\,x+\dfrac{\partial f_2}{\partial y}(1,2)\,y\end{pmatrix}=\begin{pmatrix}3x+y\\ 2x+12y\end{pmatrix}.$$

3 Continuity

Continuity is then defined as usual:

Definition 3 *Let us assume that V_1 and V_2 are two normalized vector spaces with respective norms $||.||_1$ and $||.||_2$. So*

$$F:V_1\to V_2$$

is continuous in $x_0\in V_1$ if for all $\varepsilon>0$ there exists a $\delta>0$ such that $x\in V_1$ and $||x-x_0||_1<\delta$ implies that

$$||F(x)-F(x_0)||_1<\varepsilon.$$

And F is said to be continuous on the set $E\subset V_1$ if it is continuous at each point $x\in E$. If F is continuous on $E\subset V_1$, we write $F\in C(E)$.

The next theorem, established on p. 219 in Rudin [14].

Theorem 1

- *Suppose that E is an open subset of $\mathbb{R}^n$ and that $f:E\to\mathbb{R}^n$. Then $f\in C^1(E)$ if the partial derivatives, $\left(\dfrac{\partial f_i}{\partial x_j}\right)$ $i,j=1,...,n$, exist and are continuous on E .*
- *Suppose that $f:E\to\mathbb{R}^n$ is differentiable on E. Then if $f\in C^1(E)$ if the derivative $Df:E\to L(\mathbb{R}^n)$ is continuous on E.*

Remark 3

- *A differentiable function f is continuously differentiable if and only if f is of differentiability class C^1*
- *For E an open subset of $\mathbb{R}^n$, the higher order derivatives $D^kf(x_0)$ of a function $f:E\to\mathbb{R}^n$ are defined similarly, and it can be shown that $f\in C^k(E)$ if and only if the*

partial derivatives $\dfrac{\partial^k f_i}{\partial x_{j_1}...\partial x_{j_k}}$ *where* $i, j_1, ..., j_k = 1, ..., n$, *exist and are continuous on* E. *Moreover,* $D^2 f(x_0) : E \times E \to \mathbb{R}^n$ *and for* $(x, y) \in E \times E$

$$D^2 f(x_0)\,(x, y) = \sum_{j_1=1, j_2=1}^{n} \frac{\partial^2 f}{\partial x_{j_1} \partial x_{j_2}} (x_0, y_0)\, x_{j_1} y_{j_2}.$$

Similar formulas hold for $D^k f(x_0) : (E \times ... \times E) \to \mathbb{R}^n$.

- *A function* $f : E \to \mathbb{R}^n$ *is said to be analytic in the open set* $E \subset \mathbb{R}^n$ *if each component* $f_j(x), j = 1, ..., n$ *is analytic in* E, *i.e., say if for* $j = 1, ..., n$ *and* $x_0 \in E$, $f_j(x)$ *has a Taylor series which converges to* $f(x)$ *in a neighborhood of* x_0 *in* E.

4 Lipschitz function

To guarantee uniqueness of solutions, we need a stronger property than continuity; function to be Lipschitz.

Definition 4 *Let* E *be an open subset of* $\mathbb{R}^n$.

- *A function* $f : E \to \mathbb{R}^n$ *is said to satisfy a Lipschitz condition on* E *if there is a positive constant* K *such that for all* $x, y \in E$

$$|f(x) - f(y)| \leqslant K\,|x - y|\,.$$

- *The function* f *is said to be locally Lipschitz on* E *if for each point* $x_0 \in E$ *there is an* $\delta-$*neighborhood* x_0, $N_\delta(x_0) \subset E$ *and a constant* $K_0 > 0$ *such that for all* $x, y \in N_\varepsilon(x_0)$

$$|f(x) - f(y)| \leqslant K_0\,|x - y|\,.$$

By an $\varepsilon-$*neighborhood of a point* $x_0 \in \mathbb{R}^n$, *we mean an open ball of positive radius a; i.e.,*

$$N_\varepsilon(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}.$$

in $\mathbb{R}$

$$N_\varepsilon(x_0) =]x_0 - \varepsilon, x_0 + \varepsilon[\,.$$

Lemma 1 *Let* E *be an open subset of* $\mathbb{R}^n$ *and let* $f : E \to \mathbb{R}^n$. *Then, if* $f \in C^1(E)$, f *is locally Lipschitz on* E.

Proof. Since E is an open subset of $\mathbb{R}^n$, given $x_0 \in E$, there is an $\varepsilon > 0$ such that $N_\varepsilon(x_0) \subset E$. Let $K = \max\limits_{|x-x_0| \le \varepsilon/2} |Df(x)|$, the maximum of the continuous function $Df(x)$ on the compact set $|x - x_0| \le \dfrac{\varepsilon}{2}$. Let N_0 denote the $\dfrac{\varepsilon}{2}$-neighborhood of x_0, $N_{\frac{\varepsilon}{2}}(x_0)$. Then for $x, y \in N_0$, set

$u = y - x$. It follows that $x + su \in N_0$ for $0 \leq s \leq 1$ since N_0 is a convex set. Define the function $F : [0,1] \to \mathbb{R}^n$ by

$$F(s) = f(x + su).$$

Then by the chain rule

$$F'(s) = Df(x + su)u$$

and therefore

$$\begin{aligned} f(y) - f(x) &= F(1) - F(0) = \int_0^1 F'(s)ds \\ &= \int_0^1 Df(x+su)uds. \end{aligned}$$

Then,

$$\begin{aligned} |f(y) - f(x)| &\leq \int_0^1 |Df(x+su)u|\, ds \\ &\leq \int_0^1 \|Df(x+su)\|\, |u|\, ds \\ &\leq K\,|u| = K\,|y-x|\,. \end{aligned}$$

And this proves the lemma.

Remark 4 *Summarizing, we have*

- *differentiable at x_0 $\Rightarrow$Lipchitz continuous at x_0 $\Rightarrow$ continuous at x_0.*
- *The converse implications don't hold, i.e., differentiable at x_0 $\not\Leftarrow$Lipchitz continuous at x_0 $\not\Leftarrow$ continuous at x_0.*
- *For example $f(x) = \sqrt{|x|}$ is continuous at $x = 0$ but not Lipschitz continuous there because its derivative is unbounded as x approaches zero. Also $f(x) = |x|$ is Lipschitz continuous at $x = 0$ but not differentiable there.*

5 Exercises

Exercise 1 *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a class $C^{\mathbf{1}}$ map and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the map defined by*

$$g(u,v) = f\left(\cos u + \sin v, \sin u + \cos v, e^{u-v}\right).$$

1. *Show that g is of the class C^1.*
2. *We assume that f is differentiable at the point $M = (1,1,1)$ of $\mathbb{R}^3$ and that its differential at this point is*

$$Df(M) = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \end{pmatrix}.$$

Determine the differential of g at the point $P = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Exercise 2 *Calculate the derivative of the function*

$$f(x,y)=\begin{pmatrix} x-4x^2y^2 \\ -y+2x^2y+y^2 \end{pmatrix}.$$

1. *Find the zeros of the function i.e. the points* $X_0 \in \mathbb{R}^2$ *where* $f(X_0)=0$.
2. *Calculate* $Df\,(0,1)\,X$ *and* $D^2f\,(0,1)\,X$.

Exercise 3 *Find the largest open subset* $E \subset \mathbb{R}^2$ *for which the following functions are continuously differentiable.*

$$f_1(x,y)=\begin{pmatrix} \dfrac{-x}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{-y}{(x^2+y^2)^{\frac{3}{2}}} \end{pmatrix}, \quad f_2(x,y)=\begin{pmatrix} \sqrt{(x^2+y^2)}+\dfrac{1}{|x+1|} \\ \sqrt{x-1}-\sqrt{y+2} \end{pmatrix}.$$

Exercise 4

1. *Prove that if* $f \in C^1(E)$ *where* E *is a compact convex subset of* $\mathbb{R}^n$, *then* f *satisfies a Lipschitz condition on* E.
2. *Prove that if* f *satisfies a Lipschitz condition on* E, *then* f *is uniformly continuous on* E.
3. *Show that the function* $f(x)=\dfrac{1}{x}$ *is not uniformly continuous on* $E=(0,1)$.
4. *Show that* $f(x)=\dfrac{1}{x}$ *does not satisfy a Lipschitz condition on* $(0,1)$.

Exercise 5 *Prove that if* f *is differentiable at* x_0 *then there exists a* $\delta>0$ *and a* $K_0>0$ *such that for all* $x \in N_\delta(x_0)$

$$|f(x)-f(x_0)| < K_0\,|x-x_0|\,.$$

2

Existence Theory

1 The Fundamental Existence-Uniqueness Theorem

In this section, we establish the fundamental existence-uniqueness theorem for a nonlinear autonomous system of ordinary differential equations

$$\dot{x} = f\left(x\right), \qquad x(0) = x_0$$

under the hypothesis that $f \in C^1(E)$ where E is an open subset of $\mathbb{R}$. Picard's classical method of successive approximations is used to prove this theorem

Definition 5 *Suppose that $f \in C(E)$ where E is an open subset of $\mathbb{R}^n$. Then $x(t)$ is a solution of the differential equation* (1.1) *on an interval I*

- *if $x(t)$ is differentiable on I*
- *and if for all $t \in I, x(t) \in E$ and*

  $$\dot{x}(t) = f(x(t)).$$

  And given $x_0 \in E$, $x(t)$ is a solution of the initial value problem

  $$\dot{x} = f(x), \;\; x(t_0) = x_0$$

  on an interval I
- *if $t_0 \in I, x(t_0) = x_0$*
- *and $x(t)$ is a solution of the differential equation* (1.1) *on the interval I.*

Does such a solution exist? And is it unique?

Notice that the existence of the solution of the elementary differential equation (1.1) is given by

$$x(t) = x(0) + \int_0^t f(s)ds,$$

if $f(t)$ is integrable. And in general, the differential equation (1.1) will have a solution if the function f is continuous. However, continuity of the function f in (1.1) is not sufficient to guarantee uniqueness of the solution. Here is a simple example that demonstrates that uniqueness can indeed fail.

Example 3 *Let* $\dot{x} = |x|\,,\ x(t_0) = x_0$. *Then*

$$x(t) = \begin{cases} x_0 e^{t_0 - t} & \text{if } x_0 < 0, \\ 0 & \text{if } x_0 = 0, \\ x_0 e^{t - t_0} & \text{if } x_0 > 0. \end{cases}$$

Here f *is not differentiable, but it is continuous.*

Example 4 *Let* $\dot{x} = \sqrt{|x|}$, $x(0) = 0$, *we have* $f(x) = \sqrt{|x|}$. *We still have* f *continuous. Solving gives* $2\sqrt{|x|} = t + c$, *thus* $|x| = \dfrac{1}{4}(t+c)^2$. *For* $x(0) = 0$, *we get* $c = 0$, *then*

$$|x| = \frac{1}{4}t^2.$$

However, for $x_0 = 0$ *we have two solutions:* $x(t) \equiv 0$ *and* $|x| = \dfrac{1}{4}t^2$.
Why are these different? Because in the second case, derivatives of $\sqrt{|x|}$ *are not bounded at the origin.*

Let us consider the Cauchy problem

$$\dot{x} = f\,(x)\,, \qquad x(0) = x_0, \tag{2.1}$$

where $f : E \to \mathbb{R}^n$ is a continuous function. In order to find a solution, we will rewrite (2.1) as a fixed point equation. We integrate the differential equation between 0 and t,

$$\int_0^t \dot{x}(s)ds = \int_0^t f(s)ds,$$

then $x(t) - x(0) = \int_0^t f(s)ds$, and we obtain the integral equation

$$x(t) = x(0) + \int_0^t f(s)ds. \tag{2.2}$$

Every solution of (2.1) is thus a solution of (2.2). The converse also holds.

If $x(t)$ is a continuous function that verifies (2.2) on some interval I, then it is automatically of class C^1 and it satisfies (2.1).

The method of successive approximation for the problem (2.1) is called Picard's method of successive approximations.

2.1.1 Picard's method of successive approximations.

This method is based on the fact that $x(t)$ is a solution of the initial value problem (2.1) if and only if $x(t)$ is a continuous function that satisfies the integral equation

$$x(t) = x(0) + \int_0^t f(x\,(s))ds.$$

The successive approximations to the solution of this integral equation are defined by the sequence of functions

$$\begin{cases} u_0(t) = x_0 \\ u_{k+1}(t) = x_0 + \displaystyle\int_0^t f(u_k(s))ds \end{cases} \tag{2.3}$$

for $k = 0, 1, 2, \ldots$. In order to illustrate the mechanics involved in the method of successive approximations, we use the method to solve an elementary linear differential equation.

Example 5 *Solve the initial value problem*

$$\dot{y} = 1 + y(t), \qquad y(0) = 0,$$

by the method of successive approximations.
Let

$$\begin{cases} y_0(t) = 0, \\ y_{k+1}(t) = \displaystyle\int_0^t f(y_k(s))ds, \end{cases}$$

then,

$$y_1(t) = \int_0^t f(y_0(s))ds = \int_0^t ds = t,$$

$$y_2(t) = \int_0^t f(y_1(s))ds = \int_0^t (s+1)\,ds = \frac{1}{2}t(t+2),$$

$$y_3(t) = \int_0^t f(y_2(s))ds = \int_0^t \left(\frac{1}{2}s(s+2)+1\right)ds = \frac{1}{6}t\left(t^2+3t+6\right),$$

$$\vdots$$

$$y_n(t) = t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \ldots \underset{V(0)}{\approx} e^t - 1.$$

In order to show that the successive approximations (2.3) converge to a solution of the initial value problem (2.1) on an interval $I = [-a, a]$, it is first necessary to define linear space $C(I)$ of continuous functions on an interval $I = [-a, a]$. The norm on $C(I)$ is defined as

$$\|u\| = \sup_I |u(t)|.$$

Convergence in this norm is equivalent to uniform convergence.

Definition 6 *Let V be a normed linear space. Then a sequence $\{u_k\} \subset V$ is called a Cauchy sequence if for all $\varepsilon > 0$ there is an N such that $k, m > N$ implies that*

$$\|u_k - u_m\| < \varepsilon.$$

The space V is called complete if every Cauchy sequence in V converges to an element in V.

Remark 5 *For $I = [-a, a]$, then $C(I)$ complete normed linear space.*

We can now prove the fundamental existence-uniqueness theorem for nonlinear systems.

Theorem 2 *(The Fundamental Existence-Uniqueness Theorem). Let E be an open subset of $\mathbb{R}^n$ containing x_0 and assume that $f \in C^1(E)$. Then there exists an $a > 0$ such that the initial value problem (2.1) has a unique solution $x(t)$ on the interval $[-a, a]$.*

Proof. Existence: Since $f \in C^1(E)$, it follows from the lemma 1 that there is an neighborhood $N_\varepsilon(x_0) \subset E$ and a constant $K > 0$ such that for all $x, y \in N_\varepsilon(x_0)$,

$$|f(x) - f(y)| \leq K_0 \, |x - y| \, .$$

Let $b = \dfrac{\varepsilon}{2}$. Then the continuous function $f(x)$ is bounded on the compact set

$$N_0 = \{x \in \mathbb{R}^n : |x - x_0| \leq b\}.$$

Let $M = \max\limits_{x \in N_0} |f(x)|$. Let the successive approximations $u_k(t)$ be defined by (2.3). Then assuming that there exists an $a > 0$ such that $u_k(t)$ is defined and continuous on $[-a, a]$ and satisfies

$$\max_{x \in [-a,a]} |u_k(t) - x_0| \leq b, \tag{2.4}$$

it follows that $f(u_k(t))$ is defined and continuous on $[-a, a]$ and therefore

$$u_{k+1}(t) = x_0 + \int_0^t f(u_k\,(s))ds$$

is defined and continuous on $[-a, a]$ and satisfies

$$\begin{aligned} |u_{k+1}(t) - x_0| &\leq \int_0^t |f(u_k\,(s))| \, ds \\ &\leq \max_{x \in N_0} |f(x)| \int_0^t ds \\ &= Mt \leq Ma, \end{aligned}$$

for all $t \in [-a, a]$. Thus, choosing $0 < a < \dfrac{b}{M}$, it follows by induction that $u_k(t)$ is defined and continuous and satisfies (2.4) for all $t \in [-a, a]$ and $k = 1, 2, 3, ...$
Next, since for all $t \in [-a, a]$ and $k = 0, 1, 2, 3, ..., u_k(t) \in N_0$, it follows from the Lipschitz condition satisfied by f that for all $t \in [-a, a]$

$$\begin{aligned} |u_2(t) - u_1| &\leq \int_0^t |f(u_1\,(s)) - f(u_0\,(s))| \, ds \\ &\leq K \int_0^t |u_1\,(s)) - u_0\,(s))| \\ &\leq Ka \max_{x \in [-a,a]} |u_1(t) - x_0| \\ &\leq Kab. \end{aligned}$$

And then assuming that

$$\max_{x \in [-a,a]} |u_j(t) - u_{j-1}(t)| \leq (Ka)^{j-1} \, b, \tag{2.5}$$

for some integer $j > 2$, it follows that for all $t \in [-a, a]$

$$\begin{aligned} |u_{j+1}(t) - u_j(t)| &\leq \int_0^t |f(u_j(s)) - f(u_{j-1}(s))|\, ds \\ &\leq K \int_0^t |u_j(s)) - u_{j-1}(s))| \\ &\leq Ka \max_{t\in[-a,a]} |u_j(t) - u_{j-1}(t)| \\ &\leq (Ka)^j\, b. \end{aligned}$$

Thus, it follows by induction that (2.5) for some integer $j = 2, 3, \ldots$. Setting $\alpha = Ka$ and choosing $0 < a < \frac{1}{K}$, we see that for $m > k > N$ and $t \in [-a, a]$

$$\begin{aligned} |u_m(t) - u_k(t)| &= \sum_{j=k}^{m-1} |u_{j+1}(t) - u_j(t)| \\ &\leq \sum_{j=N}^{\infty} |u_{j+1}(t) - u_j(t)| \\ &\leq \sum_{j=N}^{\infty} \alpha_j b = \frac{\alpha^N}{1-\alpha} b. \end{aligned}$$

This last quantity approaches zero as $N \to \infty$.
Therefore, for all $\varepsilon > 0$ there exists an N such that $m, k > N$ implies that

$$\|u_m - u_k\| \leq \max_{t\in[-a,a]} |u_m(t) - u_k(t)|,$$

i.e., $\{u_k\}$ is a Cauchy sequence of continuous functions $C([-a, a])$. It follows from the above theorem that $u_k(t)$ converges to a continuous function $u(t)$ uniformly for all $t \in [-a, a]$ as $k \to \infty$. And then taking the limit of both sides of equation (2.2) defining the successive approximations, we see that the continuous function

$$u(t) = \lim_{k\to\infty} u_k(t), \tag{2.6}$$

satisfies the integral equation

$$u(t) = x_0 + \int_0^t f(u(s))ds, \tag{2.7}$$

for all $t \in [-a, a]$. We have used the fact that the integral and the limit can be interchanged since the limit in (2.6) is uniform for all $t \in [-a, a]$. Then, since $u(t)$ is continuous, $f(u(t))$ is continuous, and by the fundamental theorem of calculus, the right-hand side of the integral equation (2.7) is differentiable, and

$$\dot{u}(t) = f(u(t)),$$

for all $t \in [-a, a]$. Furthermore, $u(0) = x_0$ and from (2.4) it follows that $u(t) \in N_\varepsilon(x_0) \subset E$ for all $t \in [-a, a]$. Thus $u(t)$ is a solution of the initial value problem (2.1) on $[-a, a]$. It remains to show that it is the only solution.

Uniqueness: Let $u(t)$ and $v(t)$ be two solutions of the initial value problem (2.1) on $[-a, a]$. Then the continuous function $|u(t) - v(t)|$ achieves its maximum at some point $t_j \in [-a, a]$. It follows that

$$\begin{aligned} \|u-v\| &\leq \max_{t\in[-a,a]} |u(t)-v(t)| \\ &= \left| \int_0^t f(u(s)) - f(v(s)) ds \right| \\ &\leq \int_0^{|t|} |f(u(s)) - f(v(s))| \, ds \\ &\leq K \int_0^{|t|} |u(s)) - v(s))| \\ &\leq Ka \max_{t\in[-a,a]} |u(t)-v(t)| \\ &\leq Ka \|u-v\| . \end{aligned}$$

But $Ka < 1$ and this last inequality can only be satisfied if $\|u-v\| = 0$. Thus, $u(t) = v(t)$ in $[-a,a]$. We have shown that the successive approximations (2.3) converge uniformly to a unique solution of the initial value problem (2.1) on the interval $[-a,a]$ where a is any number satisfying $0 < a < min\left(\frac{b}{M}, \frac{1}{K}\right)$. ■

Remark 6 *Exactly the same method of proof shows that the initial value problem*

$$\dot{x} = f(x), \quad x(t_0) = x_0,$$

has a unique solution on some interval $[t_0 - a, t_0 + a]$.

2 The Maximal Interval of Existence

In this section we show that the initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0,$$

has a unique solution $x(t)$ defined on a maximal interval of existence (α,β). Furthermore, if $\beta < \infty \quad (\alpha > -\infty)$ and if

$$\lim_{t\to\beta^-} x(t) = L \qquad \left(\lim_{t\to\alpha+} x(t) = L\right)$$

exists, then $L \in \dot{E}$, the boundary of E. On the other hand, if the above limit exists and $L \in E$, then $\beta = \infty, f(L) = 0$. The following examples illustrate these ideas.

Example 6 *Consider*

$$\dot{x} = x^2, \quad x(0) = 1.$$

The function $f(x) = x^2 \in C^1(\mathbb{R})$ *and the initial value problem have a unique solution given by*

$$x(t) = \frac{1}{1-t}.$$

The solution is defined on its maximal interval of existence $(\alpha,\beta) = (-\infty, 1)$. *Furthermore,* $\lim_{t\to1^-} x(t) = \infty$.

Example 7 *Consider*

$$\dot{x} = x^2, \quad x(0) = x_0.$$

The function $f(x) = x^2 \in C^1(\mathbb{R})$ *and the initial value problem have a unique solution given by*

$$x(t) = \frac{x_0}{(1 - x_0 t)}.$$

The solution is defined on its maximal interval of existence

$$(\alpha, \beta) = \begin{cases} (-\infty, +\infty) & \text{if } x_0 = 0, \\ \left(-\infty, \dfrac{1}{x_0}\right) & \text{if } x_0 > 0, \\ \left(\dfrac{1}{x_0}, +\infty\right) & \text{if } x_0 < 0. \end{cases}$$

Example 8 *Consider*

$$\dot{x} = \frac{-1}{2x}, \quad x(0) = 1.$$

The solution is given by $x(t) = \sqrt{1 - t}$. *The solution is defined on its maximal interval of existence* $(\alpha, \beta) = (-\infty, 1)$. *The function* $f(x) = \dfrac{-1}{2x} \in C^1(E)$ *where* $E = (0, \infty)$ *and* $\mathring{E} = \{0\}$. *Note that* $\lim\limits_{t \to 1^-} x(t) = 0 \in \mathring{E}$.

Example 9 *The autonomous system*

$$\dot{x} = 1, \quad \dot{y} = -\frac{x}{y}, \quad x(0) = 0,\ y(0) = 1.$$

Thus $f(x_1, x_2) = \left(1, -\dfrac{x}{y}\right)$ *is not continuous when* $y = 0$. *Since the initial value is* $y(0) = 1$,f *satisfies the condition of the theorem on the upper half plane* $E = \{(x, y) : y > 0\}$.
In fact, the solution of this initial value problem is

$$x(t) = t, \quad y(t) = \sqrt{1 - t^2},$$

which is defined for $t \in]-1, 1[$. *As* $t \to 1^-$ *or* $t \to -1^+$, $(x_1(t), x_2(t)) \to (\pm 1, 0)$ *which is on the boundary of* E, $\dot{E} = \{(x, y) : y = 0\}$.

Thus the existence and uniqueness theorem can only guarantee the existence of a solution in a small interval $[t_0 - a, t_0 + a]$ whereas in practice the solution will exist in a much larger interval.

To find the largest interval of existence, we apply the existence and uniqueness theorem successively. Suppose that $f \in C^1(E)$ for some open subset E of $\mathbb{R}^n$ containing x_0. Applying the fundamental existence-uniqueness theorem, we find a_1 and a solution $u_1(t)$ defined on $I_1 = [t_0 - a_1, t_0 + a_1]$.
Now let $t_1 = t_0 + a_1, x_1 = u_1(t_1)$. If x_1 is still in E, we can apply the fundamental existence-uniqueness theorem to the new initial value problem

$$\dot{x} = f(x), \quad x(t_1) = x_1,$$

we get a new a_2 and a solution $u_2(t)$ defined on $I_2 = [t_1 - a_2, t_1 + a_2]$. In $I_1 \cap I_2, u_1$ and u_2 coincide (since both satisfy the differential equation and the initial condition $x(t_1) = x_1$.) Therefore the function

$$u(t) = \begin{cases} u_1(t), & t_0 - a_1 \leq t \leq t_1, \\ u_2(t), & t_1 \leq t \leq t_1 + a_2, \end{cases}$$

also satisfies the original initial value problem. In this way, we can extend the solution to a much larger interval (α, β), where α and β are such that as $t \to \alpha^+$ or $t \to \beta^-$, $x(t)$ approaches the boundary of E.

Lemma 2 ***(Perko [9]).*** *Let E be an open subset of $\mathbb{R}^n$ containing x_0 and suppose $f \in C^1(E)$. Let $u_1(t)$ and $u_2(t)$ be solutions of the initial value problem (2.1) on the intervals I_1 and I_2. Then $0 \in I_1 \cap I_2$ and if I is any open interval containing 0 and contained in $I_1 \cap I_2$, it follows that $u_l(t) = u_2(t)$ for all $t \in I$.*

Theorem 3 *Let E be an open subset of $\mathbb{R}^n$ and assume that $f \in C^1(E)$. Then for each point $x_0 \in E$, there is a maximal interval J on which the initial value problem (2.1) has a unique solution, $x(t)$; i.e., if the initial value problem has a solution $y(t)$ on an interval I then $I \subset J$ and $y(t) = x(t)$ for all $t \in I$. Furthermore, the maximal interval J is open; i.e., $J = (\alpha, \beta)$.*

Proof. By the fundamental existence-uniqueness theorem 2, the initial value problem (2.1) has a unique solution on some open interval $(-a, a)$. Let

$$(\alpha, \beta) = \cup \{I; \quad I \text{ open interval}; \ t_0 \in I; \text{ there exists a solution on } I\}.$$

We define a function $x(t)$ on (α, β) as follows:
Given $t \in (\alpha, \beta)$, t belongs to some open interval I such that (2.1) has a solution $u(t)$ on I
For this given $t \in (\alpha, \beta)$ define $x(t) = u(t)$. Then $x(t)$ is a well-defined function of t since if $t \in I_1 \cap I_2$ where I_1 and I_2 are any two open intervals such that (2.1) has solutions $u_1(t)$ and $u_2(t)$ on I_1 and I_2 respectively, then by the lemma 2; $u_1(t) = u_2(t)$ on the open interval $I_1 \cap I_2$. Also, $x(t)$ is a solution of (2.1); on (α, β) since each point $t \in (\alpha, \beta)$ is contained in some open interval I on which the initial value problem (2.1) has a unique solution $u(t)$ and since $x(t)$ agrees with $u(t)$ on I. The fact that J is open follows from the fact that any solution of (2.1) on an interval $(\alpha, \beta]$ can be uniquely continued to a solution on an interval $(\alpha, \beta + a)$ with $a > 0$.■

Definition 7 *(Maximal interval of existence) The interval (α, β) in Theorem 3 is called the maximal interval of existence of the solution $x(t)$ of the initial value problem (2.1) or simply the maximal interval of existence of the initial value problem (2.1).*

Theorem 4 . *Let E be an open subset of $\mathbb{R}^n$ containing x_0, let $f \in C^1(E)$, and let (α, β) be the maximal interval of existence of the solution $x(t)$ of the initial value problem (2.1). Assume that $\beta < \infty$. Then given any compact set $K \subset E$, there exists a $t \in (\alpha, \beta)$ such that $x(t) \notin K$.*

Proof. Since f is continuous on the compact set K, there is a positive number M such that $|f(x)| \leq M$ for all $x \in K$. Let $x(t)$ be the solution of the initial value problem (2.1) on its maximal interval of existence (α, β) and assume that $\beta < \infty$ and that $x(t) \in K$ for all $t \in (\alpha, \beta)$. We first show that $\lim\limits_{t \to \beta^-} x(t)$ exists. If $\alpha < t_1 < t_2 < \beta$, then

$$|x(t_1) - x(t_2)| \leq \int_{t_1}^{t_2} |f(x(s))|\, ds \leq M\, |t_1 - t_2|\,.$$

Thus as t_1 and t_2 approach β from the left, $|x(t_1) - x(t_2)| \to 0$ which, by the Cauchy criterion for convergence in $\mathbb{R}^n$ implies that $\lim\limits_{t \to \beta^-} x(t)$ exists. Let $x_1 = \lim\limits_{t \to \beta^-} x(t)$. Then $x_1 \in K \subset E$ since K is compact. Next define the function $u(t)$ on $(\alpha, \beta]$ by

$$u(t) = \begin{cases} x(t) & for\ t \in (\alpha, \beta), \\ x_1 & for\ \ t = \beta. \end{cases}$$

Then, $u(t)$ is differentiable on $(\alpha, \beta]$. Indeed,

$$u(t) = x_0 + \int_0^t f(u(s)) ds;$$

which implies that

$$\dot{u}(\beta) = f(u(\beta)),$$

i.e., $u(t)$ is a solution of the initial value problem (2.1) on $(\alpha, \beta]$. The function $u(t)$ is called the continuation of the solution $x(t)$ to $(\alpha, \beta]$. Since $x_1 \in E$, it follows from the fundamental existence-uniqueness theorem that the initial value problem

$$\dot{x} = f(x), \quad x(\beta) = x_1,$$

has a unique solution $x_1(t)$ on some interval $(\beta - a, \beta + a)$. By the lemma 2, $x_1(t) = u(t)$ on $(\beta - a, \beta)$ and $x_1(\beta) = u(\beta) = x_1$. So if we define

$$v(t) = \begin{cases} u(t) & \text{for } t \in (\alpha, \beta]\,, \\ x_1(t) & \text{for } \ t \in (\beta, \beta + a)\,, \end{cases}$$

then $v(t)$ is a solution of the initial value problem (2.1) on $(\alpha, \beta + a)$. But this contradicts the fact that (α, β) is the maximal interval of existence for the initial value problem (2.1). Hence, if $\beta < \infty$, it follows that there exists a $t \in (\alpha, \beta)$ such that $x(t) \notin K$. ■

Corollary 1 . *Let E be an open subset of $\mathbb{R}^n$ and assume that $f \in C^1(E)$ and let (α, β) be the maximal interval of existence of the solution $x(t)$ of the initial value problem* (2.1). *If $\beta < \infty$ $(\alpha > -\infty)$ and if*

$$\lim_{t \to \beta^-} x(t) = L \qquad \left(\lim_{t \to \alpha+} x(t) = L \right),$$

then $L \in \mathring{E}$.

Proof. If $x_1 = \lim\limits_{t \to \beta^-} x(t)$, then the function

$$u(t) = \begin{cases} x(t) & \text{for } t \in [\alpha, \beta)\,, \\ x_1 & \text{for } \ t = \beta, \end{cases}$$

is continuous on $[\alpha, \beta]$. Let K be the image of the compact set $[\alpha, \beta]$ under the continuous map $u(t)$; i.e.,

$$K = \{x \in \mathbb{R}^n : x = u(t) \text{ for some } t \in [\alpha, \beta])\}.$$

Then K is compact. Assume that $x_1 \in E$. Then $K \subset E$ and it follows from theorem 4 that there exists a $t \in (\alpha, \beta)$ such that $x(t) \notin K$. This is a contradiction, and therefore $x_1 \notin E$. But since $x(t) \in E$ for all $t \in [\alpha, \beta)$, it follows that $x_1 = \lim_{t \to \beta^-} x(t) \in E$. Therefore $x_1 \in \bar{E} \setminus E$; i.e., $x_1 \in \mathring{E}$.■

3 The Flow Defined by a Differential Equation

For an autonomous system, we can alternatively define the solution in terms of a flow φ_t:

Definition 8 *Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$. For $x_0 \in E$, let $\varphi(t, x_0)$ be the solution of the initial value problem* (2.1) *defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the set of mappings φ_t defined by*

$$\varphi_t(x_0) = \varphi(t, x_0)$$

is called the flow of the differential equation (1.1) *or the flow defined by the differential equation* (1.1)*; φ_t, is also referred to as the flow of the vector field $f(x)$.*

A function satisfying these properties is a candidate for the solution of $\dot{x} = f(x)$, then it should satisfy

$$\frac{d}{dt}(\varphi_t(x)) = f(\varphi_t(x)).$$

Example 10 *Consider the differential equation*

$$\dot{x} = \frac{1}{x-1},$$

with $f(x) = \dfrac{1}{x-1} \in C^1(E)$ and $E = \{x \in \mathbb{R} : x > 1\}$. The solution of this differential equation and initial condition $x(0) = x_0$ is given by

$$\varphi(t, x_0) = \sqrt{x_0^2 - 2x_0 + 2t + 1} + 1,$$

on its maximal interval of existence $I(x_0) = \left(x_0 - \frac{1}{2}x_0^2 - \frac{1}{2}, \infty\right)$.

Theorem 5 *Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^{1(}E)$. Then Ω is an open subset $\mathbb{R} \times E$ and $\varphi \in C^1(\Omega)$.*

Proof. If $(t_0, x_0) \in \Omega$ and $t_0 > 0$, then according to the definition of the set Ω, the solution $x(t) = \varphi(t, x_0)$ of the initial value problem (2.1) defined on $[0, t_0]$. Thus, as in the proof of theorem 4, the solution $x(t)$ can be extended to an interval $[0, t_0 + \varepsilon]$ for some $\varepsilon > 0$; i.e., $\varphi(t, x_0)$ is defined on the closed interval $[t_0 - \varepsilon, t_0 + \varepsilon]$. It then follows from theorem 2 that there exists a neighborhood of x_0; $N_\delta(x_0)$, such that $\varphi(t, y)$ is defined on $[t_0 - \varepsilon, t_0 + \varepsilon] \times N_\delta(x_0)$; i.e., $(t_0 - \varepsilon, t_0 + \varepsilon) \times N_\delta(x_0) \subset \Omega$. Therefore, Ω is open in $\mathbb{R} \times E$. It follows that $\varphi \in C^1(G)$ where $G = (t_0 - \varepsilon, t_0 + \varepsilon) \times N_\delta(x_0)$. Similar proof holds for $t_0 \leq 0$, and since (t_0, x_0) is an arbitrary point in Ω, it follows that $\varphi \in C^1(\Omega)$. ■

Remark 7 *Theorem 5 can be generalized to show that if $f \in C^r(E)$ with $r > 1$, then $\varphi_t \in C^r(\Omega)$ and that if f is analytic in E, then φ_t is analytic in Ω.*

Theorem 6 . *Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$. Then for all $x_0 \in E$, . if $t \in I(x_0)$ and $s \in I(\varphi_t(x_0))$, it follows that $t + s \in I(x_0)$ and*

$$\varphi_s(\varphi_t(x_0)) = \varphi_{s+t}(x_0).$$

Proof. Suppose that $s > 0, t \in I(x_0)$ and $s \in I(\varphi_t(x_0)$. Let the maximal interval $I(x_0) = (\alpha, \beta)$ and define the function $x : (\alpha, s + t] \to E$ by

$$x(r) = \begin{cases} \varphi(r, x_0) & if\ \alpha < r < t, \\ \varphi(r - t, \varphi_t(x_0)) & if\ t < r < s + t. \end{cases}$$

Then $x(r)$ is a solution of the initial value problem (2.1) on $(\alpha, s + t]$. Hence $s + t \in I(x_0)$ and by uniqueness of solutions

$$\varphi_{s+t}(x_0) = X(s + t) = \varphi(s, \varphi(t, x_0)) = \varphi_s(\varphi_t(x_0).$$

If $s = 0$ the statement of the theorem follows immediately. And if $s < 0$, then we define the function $x : [s + t, \beta) \to E$ by

$$x(r) = \begin{cases} \varphi(r, x_0) & if\ t < r < \beta, \\ \varphi(r - t, \varphi_t(x_0)) & if\ s + t < r < t. \end{cases}$$

Then $x(r)$ is a solution of the initial value problem (2.1) on $[s + t, \beta)$ and the last statement of the theorem follows from the uniqueness of solutions as above. ■

Theorem 7 . *Under the hypotheses of Theorem 5, if $(t, x_0) \in \Omega$ then there exists a neighborhood U of x_0 such that $\{t\} \times U \subset \Omega$ It then follows that the set $V = \varphi_t(U)$ is open in E and that*

- - $\varphi_{-t}(\varphi_t(x)) = x, \forall x \in U,$
- - $\varphi_t(\varphi_{-t}(x)) = x, \forall x \in V.$

Proof. If $(t, x_0) \in \Omega$ then it follows as in the proof of theorem 6 that there exists a neighborhood of $x_0, U = N_\delta(x_0)$, such that $(t - \varepsilon, t + \varepsilon) \times U \subset \Omega$; thus, $\{t\} \times U \subset \Omega$. For $x \in U$, let $y = \varphi_t(x)$ for all $t \in I(x)$. Then $-t \in I(y)$, since the function $h(s) = \varphi(s + t, y)$ is a solution of (1.1) on

$[-t, 0]$ that satisfies $h(-t) = y$; i.e., φ_{-t} is defined on the set $V = \varphi_t(U)$. It then follows from Theorem 5 that

$$\varphi_{-t}(\varphi_t(x)) = \varphi_0(x) = x,$$

for all $x \in U$ and that

$$\varphi_t(\varphi_{-t}(y)) = \varphi_0(y) = y,$$

for all $y \in V$. It remains to prove that V is open. Let $V \subset V^*$ be the maximal subset of E on which φ_{-t} is defined. V^* is open because V^* is open and $\varphi_{-t} : V^* \to E$ is continuous because by Theorem 5, φ_t is continuous. Therefore, the inverse image of the open set U under the continuous map φ_{-t}, i.e., $\varphi_t(U)$, is open in E. Thus, V is open in E. ■

Definition 9 *Let E be an open subset of $\mathbb{R}^n$, let $f \in C^1(E)$, and let $\varphi_t : E \to E$ be the flow of the differential equation* (1.1) *defined for all $t \in \mathbb{R}$. Then a set $S \subset E$ is called invariant with respect to the flow φ_t if $\varphi_t(S) \subset S$ for all $t \in \mathbb{R}_+$ and S is called positively (or negatively) invariant with respect to the flow φ_t if $\varphi_t(S) \subset S$ for all $t > 0$ (or $t < 0$).*

Example 11 *Show that $S = \left\{(x, y) \in \mathbb{R}^2 : y = \dfrac{-x^2}{3}\right\}$ is invariant with respect to the flow φ_t of the system*

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = y + x^2. \end{cases}$$

Let $(x(0), y(0)) = (C_1, C_2)$ .the general solution of $\dot{x} = -x$, is

$$x(t) = x(0)\, e^{-t} = C_1 e^{-t},$$

then, we have $\dot{y} = y + \left(C_1 e^{-t}\right)^2$, the solution of $\dot{y} = y$, is $y = \alpha e^t$, then $\alpha' e^t = \left(C_1 e^{-t}\right)^2$ thus $\alpha' = C_1^2 e^{-3t}$, so $\alpha = -\dfrac{C_1^2}{3} e^{-3t} + a$ and $y = \left(-\dfrac{C_1^2}{3} e^{-3t} + a\right) e^t$,for $y(0) = C_2$, we have $\left(-\dfrac{C_1^2}{3} + a\right) = C_2$, then $a = \dfrac{1}{3} C_1^2 + C_2$, thus

$$y = \left(-\frac{C_1^2}{3} e^{-3t} + \frac{1}{3} C_1^2 + C_2\right) e^t,$$

so

$$\varphi_t(x, y) = \begin{pmatrix} x e^{-t} \\ \left(-\dfrac{x^2}{3} e^{-3t} + \dfrac{x^2}{3} + y\right) e^t \end{pmatrix}.$$

Let $(x, y) \in S$ then $y = \dfrac{-x^2}{3}$, and

$$\varphi_t\left(x, \frac{-x^2}{3}\right) = \begin{pmatrix} x e^{-t} \\ \left(-\dfrac{x^2}{3} e^{-3t} + \dfrac{x^2}{3} + \dfrac{-x^2}{3}\right) e^t \end{pmatrix} = \begin{pmatrix} x e^{-t} \\ -\dfrac{1}{3} x^2 e^{-2t} \end{pmatrix},$$

we have $-\dfrac{1}{3} x^2 e^{-2t} = -\dfrac{1}{3}\left(x e^{-t}\right)$, so $\varphi_t(S) \subset S$.

Definition 10 *Given $x_0 \in E$, define the orbit of x_0 to be the curve*

$$\begin{aligned}\gamma(x_0) &= \{x(t, t_0, x_0) : t \in I(x_0)\} \\ &= \{\varphi_t(x_0) : t \in I(x_0)\}.\end{aligned}$$

This is also called the trajectory through x_0.

Notice that the orbit is a curve in the phase space $E \subset \mathbb{R}^n$, as opposed to the solution trajectory $\{(t, x(t, t_0, x_0)) : t \in I(x_0)\}$ which is a curve in the space-time domain $\mathbb{R}^{n+1}$.

For autonomous flow, we have the following strengthening of the Uniqueness theorem .

Theorem 8 *If $z \in \gamma(x_0)$, then $\gamma(x_0) = \gamma(z)$. Thus, if two orbits intersect, then they are identical.*

Proof. Suppose that $z \in \gamma(x_0)$. This means that $z = x(t_0, 0, x_0)$ for $t_0 \in I(x_0)$, with $x\,(0) = x_0$. Or in terms of the flow, this says that $z = \varphi_{t_0}(x_0) = \varphi_{t_0}(\varphi_0\,(x_0))$.
Using the property of autonomous flow, we have

$$\varphi_t(x_0) = \varphi_{t-t_0}\,(\varphi_{t_0}(x_0)) = \varphi_{t-t_0}(z).$$

This shows that an arbitrary point $\varphi_t(x_0) \in \gamma(x_0)$ belongs to $\gamma(z)$.
Replacing t with $t + t_0$ shows that an arbitrary point $\varphi_t(z) \in \gamma(z)$ belongs to $\gamma(x_0)$. Thus $\gamma(x_0) = \varphi(z)$.■

From the existence and uniqueness theory for general systems, we have that the domain is foliated by the solution trajectories

$$\{(t, x(t, t_0, x_0)) : t \in I(x_0)\}.$$

That is, every point $x_0 = x\,(t_0) \in E$ has a unique trajectory passing through it. This result says that, for autonomous systems, the phase space E is foliated by the orbits. Since $\dot{x} = f(x)$, the orbits are curves in E everywhere tangent to the vector field $f(x)$. They are sometimes also referred to as integral curves. They can be obtained by solving the system

$$\frac{dx_1}{f_1(x)} = \frac{dx_2}{f_2(x)} = ... = \frac{dx_n}{f_n(x)}.$$

Example 12 *Consider the harmonic oscillator*

$$\dot{x} = y, \qquad \dot{y} = -x,$$

The system for the integral curves is $\dfrac{dx}{y} = -\dfrac{dy}{x}$. *Solutions satisfy*

$$x^2 + y^2 = c;$$

and so we confirm that the orbits are concentric circles centred at the origin.

4 Exercise

2.4.1 Corrected exercises

Exercise 6 *Show that the differential equation*

$$\frac{dy}{dt} = -\frac{1}{2y}$$

does not have solution satisfying $y(0) = 0$ *for* $t > 0$.

Solution. Actually, the general solution of this differential equation is

$$y^2 = -t + C,$$

where C is an arbitrary constant. The initial condition implies $C = 0$. Thus, we have $y^2 = -t$, which shows there exists no solution for $t > 0$.

Exercise 7 *Find a region where the differential equation*

$$\dot{x} = x + 3x^{\frac{1}{3}}$$

has a unique solution, i.e., find (x_0, t_0) *such that the solution* $x(t)$ *of the differential equation with* $x(t_0) = x_0$ *is unique in a neighborhood of* (x_0, t_0).

Solution. We will show the following

1. the differential equation has a unique solution with $x(t_0) = x_0$, $x_0 \neq 0$.
2. there are more than one solution satisfying $x(t_0) = 0$.
3. For any given (x_0, t_0), with $x_0 \neq 0$, we choose a small $\delta > 0$ such that $0 \notin [x_0 - \delta, x_0 + \delta]$., we know that the function $f(x) = x + 3x^{\frac{1}{3}}$ satisfies a Lipschitz condition in the region

   $$\mathcal{R} = \{(x,t) : |x - x_0| \leq \delta, \ \ |t - t_0| \leq T\},$$

   where $T > 0$ is any fixed constant. The Fundamental Existence-Uniqueness Theorem, we conclude that the differential equation has a unique solution with $x(t_0) = x_0$, $x_0 \neq 0$.

It is easy to see that $x(t) \equiv 0$ is one solution of the differential equation with $x(t_0) = 0$. We only need to show that there exists another solution that also satisfies $x(t_0) = 0$. Consider the improper integral $\displaystyle\int_0^x \frac{1}{u + 3u^{\frac{1}{3}}} du$. For any $c > 0$, we know that

$$0 < \int_0^x \frac{du}{u + 3u^{\frac{1}{3}}} < \int_0^x \frac{du}{3u^{\frac{1}{3}}} < \frac{1}{2}c^{\frac{2}{3}}.$$

Hence the improper integral converges for $c > 0$. This allows us to define an implicit function $x(t)$ by

$$\int_0^x \frac{du}{u + 3u^{\frac{1}{3}}} = t - t_0.$$

We should have $x(t_0) = 0$, since the last equation becomes an identity when setting $t = t_0$. Obviously, this function $x(t) \not\equiv 0$, otherwise, we will have $t \equiv t_0$, a contradiction. This function $x(t)$ certainly satisfies the differential equation, which can be seen easily by differentiating both sides of the last equation.

Exercise 8 *Let $f(x)$ be a continuous function. Then a function $x(t)$ is a solution of the initial value problem*

$$\frac{dx}{dt} = f\left(x\right), \quad x(a) = c, \tag{2.8}$$

if and only if it is a solution of the integral equation

$$x(t) = c + \int_a^t f(x(s))ds. \tag{2.9}$$

Solution. Let us assume that $x(t)$ is a solution of the initial value problem (2.8). The Fundamental Theorem of Calculus implies that

$$x_k(t) = x_k(a) + \int_a^t \dot{x}_k(s))ds.$$

Using (2.8), we have the integral equation (2.9).
Conversely, if $x(t)$ is a solution of the integral equation (2.9), then $x(a) = c$ and

$$\dot{x}_k(t)) = f\left(x_k(t)\right), \quad k = 1, ..., n.$$

These imply that $x(t)$ satisfies $\dfrac{dx}{dt} = f\left(x\right)$. For a given $f(x)$, if it is defined for all x in $|t-a| \leq T$, and is continuous, then we can define an operator U by

$$U = c + \int_a^t f(x(s))ds. \tag{2.10}$$

For this operator, its domain is

$$\{x(t) : x(t) \text{ is continuous in the interval } |t-a| \leq T\}$$

and its range is

$$\{y(t) : y(t) \text{ is continuously differentiable in the interval } |t-a| \leq T \text{ and } y(a) = c\}.$$

Thus, a solution of the integral equation (2.9) is a fixed point of the operator $U : x = U(x)$.

Exercise 9 *Use the fundamental existence-uniqueness theorem 2 to show that the initial value problem*

$$\dot{x} = 1 + x^2, \quad x(0) = 0,$$

has a unique solution for $|t| \leq \frac{1}{2}$. Determine the region where the true solution is defined by solving this initial value problem. What is the limiting behavior of the true solution as t approaches the end point(s) of the maximal interval of existence?

Solution. Consider the given initial value problem in the region

$$R = \{(x,t) : |x| \leq 1, |t| \leq 1\}.$$

The function $f(x) = 1 + x^2$ satisfies the following Lipschitz condition

$$|f(x) - f(y)| \leq |x+y| \, |x-y| \leq |x-y|,$$

for $|x| \leq 1$ and $|y| \leq 1$. Hence, by the fundamental existence-uniqueness theorem 2, there exists a unique solution of the given initial value problem for $|t| \leq \min\left\{1, \dfrac{1}{M}\right\}$, where

$$M = \sup_{|x|\leq 1, |t| \leq 1} \|f(x)\| = \sup_{|x|\leq 1, |t| \leq 1} \left(1 + x^2\right) = 2.$$

That is, there exists a unique solution for $|t| \leq \dfrac{1}{2}$. The unique true solution of the given initial value problem is

$$x(t) = \tan t,$$

whose maximal interval of existence is $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$. As $t \to \pm\dfrac{\pi}{2}$, $x(t)$ becomes unbounded.

Exercise 10 *Assume that the function $f(x)$ is continuously differentiable in $R = (-\infty, +\infty) \times (a,b)$; and satisfies the inequality*

$$|f(x,t)| \leq A(t)\,|x| + B(t),$$

for some non-negative continuous functions $A(t)$ and $B(t)$. Show that any solution of

$$\dot{x} = f(x), \quad x(t_0) = x_0, \ t_0 \in (a,b),$$

has a maximal interval of existence (a,b).

Solution. Let $x = x(t)$ be a solution of the initial value problem. We only show that it can be extended to the interval $[t_0, b)$. The continuation of the solution to $(a, x_0]$ can be proved similarly. We prove it by contradiction. Suppose that the maximal interval of existence is $[t_0, \beta)$, with $\beta < b$. Select t_1 and t_2 such that

$$t_0 < t_1 < \beta < t_2 < b \ \text{ and } t_2 - t_1 < t_1 - t_0.$$

Denote $T = t_2 - t_1 > 0$. Let A_M and B_M be positive upper bounds of $A(t)$ and $B(t)$ in the interval $[t_0, t_2]$, respectively. Thus, by the condition, we have

$$|f(x)| \leq A_M\,|x| + B_M,$$

for $(x,t) \in (-\infty, +\infty) \times [t_0, t_2]$. We assume A_M is large enough such that $T < \dfrac{1}{A_M}$. Now we will see that the solution $x = x(t)$ can be extended to the interval $[t_0, t_2)$, a contradiction. In fact, since $t_1 \in (t_0, \beta)$ and the solution $x = x(t)$ exists on $[t_0, \beta)$, for any positive number K, the region

$$\mathcal{R}_1 = \{(x,t) : |x - x(t_1)| \leq K, \ \ |t - t_1| \leq T\}$$

is a bounded closed subset of $R = (-\infty, +\infty) \times (a, b)$. In $\mathcal{R}_1$, since

$$|f(x)| \leq A_M |x| + B_M \leq A_M (|x(t_1)| + K) + B_M = M,$$

by the fundamental existence-uniqueness theorem 2, the solution curve $(x(t), t)$ exists and remains in the region

$$\mathcal{R}_2 = \{(x, t) : |x - x(t_1)| \leq K, \ \ |t - t_1| \leq h\},$$

where $h = \min\left\{T, \dfrac{K}{M}\right\}$. Since $\mathcal{R}_2$ is a bounded closed region, then the solution curve $(x(t); t)$ can be extended to the boundary of $\mathcal{R}_2$. That is, the solution exists in $[t_0, t_1 + h)$. Since $\lim\limits_{K \to \infty} \dfrac{K}{M} = \dfrac{1}{A_M} > T$, we know that for sufficiently large K, $h = \min\left\{T, \dfrac{K}{M}\right\} = T$. Thus, the solution $x = x(t)$ can be extended to $[t_0, t_1 + T) = [t_0, t_2)$. This contradiction implies $\beta = b$.

2.4.2 Additional exercises

Exercise 11 1. *Show that the initial value problem*

$$\dot{y} = \frac{1}{1 + y^2}, \quad y(0) = 1,$$

has a unique solution that exists on the whole line.

2. *Show that for* $0 < \alpha < 1$*, the initial value problem*

$$\dot{x}(t) = x^{\alpha}, \quad x(0) = 0$$

has at least two solutions

3. *Show that the function* $x(t) = (3t)^{\frac{1}{3}}$*, which is defined and continuous for all* $t \in \mathbb{R}$*, is a solution of the differential equation*

$$\dot{x} = \frac{1}{x^2},$$

for all $t \neq 0$ *and that it is a solution of the corresponding initial value problem with* $x\left(\dfrac{1}{3}\right) = 1$ *on the interval* $(0, \infty)$.

Exercise 12 *Consider the initial value problem*

$$\ddot{y}(x) + F'(y) = 0, \quad y(x_0) = y_0, \ \dot{y}(x_0) = v_0.$$

(a) *If* $F \in C^2(\mathbb{R})$*, carefully explain why the Fundamental Existence and Uniqueness theorem guarantees that this initial value problem has a unique solution for any point* $(x_0, y_0) \in \mathbb{R}^2$.

(b) *Suppose that* $F(u) > 0, u \in \mathbb{R}$. *Prove that the solution to the initial value problem exists for all* $x \in \mathbb{R}$.

Exercise 13 *Consider*

$$\ddot{y} + q(x)y = 0, \quad y(x_0) = y_0, \ \dot{y}(x_0) = v_0, \quad \textit{where } q \in C[a, b], x_0 \in [a; b].$$

(a) *Carefully explain why this problem has a unique solution.*

(b) *Show that if a solution has a zero in* $[a, b]$ *it must be simple.*

Exercise 14 *Solve the initial value problem*

$$\ddot{x} = -x, \quad x(0) = 0, \quad \dot{x}(0) = 1,$$

by the method of successive approximations.

Exercise 15 *Find the first three successive approximations* $u_1(t), u_2(t)$ *and* $u_3(t)$ *for the initial value problem*

$$\dot{x} = x^2, \quad x(0) = 1.$$

Also, use mathematical induction to show that for all $n > 1$,

$$u_n(t) = 1 + t + ... + t^n + O(t^{n+1}).$$

as $t \to 0$.

Exercise 16 *Under the hypothesis of the Fundamental Existence-Uniqueness Theorem, if* $x(t)$ *is the solution of the initial value problem*

$$\dot{x} = f(x), \quad x(0) = x_0,$$

on an interval I, *prove that the second derivative* $\ddot{x}(t)$ *is continuous on* I.

Exercise 17 *Use the method of successive approximations to show that if the matrix valued function* $A(t)$ *is continuous on* $[-a_0, a_0]$ *then there exists an* $a > 0$ *such that the initial value problem*

$$\dot{x} = Ax, \quad x(0) = I,$$

(where I *is the* $n \times n$ *identity matrix) has a unique fundamental matrix solution* $x(t)$ *on* $[-a, a]$.

Exercise 18 *Find the maximal interval of existence* (α, β) *for the following initial value problems and if* $\alpha > -\infty$ *or* $\beta < \infty$ *discuss the limit of the solution as* $t \to \alpha^+$ *or as* $t \to \beta^-$ *respectively:*

- $\dot{x} = x^2, \quad x(0) = x_0,$
- $\dot{x} = x^2 - 4, \quad x(0) = 0,$
- $\dot{x} = x^2 - 4, \quad x(0) = x_0$
- $\dot{x} = x^3, \quad x(0) = x_0 > 0,$
- $\dot{x} = x^2, \dot{y} = y + \dfrac{1}{x}, \quad x(0) = 1, \quad y(0) = 1,$
- $\dot{x} = \dfrac{1}{2x}, \dot{y} = y^2, \quad x(0) = 1, y(0) = 1,$
- $\dot{x} = \dfrac{1}{2x}, \dot{y} = x, \quad x(0) = 1, y(0) = 1.$

3

Linearization

1 Linearization

Definition 11 *The point* $x_0 \in \mathbb{R}^n$ *is called a fixed point (equilibrium, stationary point, critical point) of* (1.1) *if* $f(x_0) = 0$. *Then* $x = x_0$ *for all time and* $\gamma(x_0) = x_0$.

Definition 12 *The linear system*

$$\dot{x} = Ax \tag{3.1}$$

with the matrix $A = Df(x_0)$ *is called the linearization of* (1.1) *at* x_0.

Remark 8 *Any equilibrium point* X_0 *can be brought back to the origin by a simple change of variable* $\tilde{X} = X - X_0$

If $X_0 = 0$ is an equilibrium point of (1.1), then $f(0) = 0$ and, by Taylor's Theorem,

$$f(X) = Df(0)X + \frac{1}{2}D^2 f(0)X + \ldots$$

It follows that the linear function $Df(0)X$ is a good first approximation to the nonlinear function $f(x)$ near $x = 0$ and it is reasonable to expect that the behavior of the nonlinear system (1.1) near the point $X = 0$ will be approximated by the behavior of its linearization at $X = 0$.

Definition 13 *An equilibrium point* x_0 *called a hyperbolic (or non-degenerate) equilibrium point of* (1.1) *if none of the eigenvalues of the matrix* $Df(x_0)$ *have zero real part.*

Definition 14 *An equilibrium point* x_0 *of* (1.1) *is called*

- *a sink if all of the eigenvalues of the matrix* $Df(x_0)$ *have negative real part;*
- *a source if all of the eigenvalues of* $Df(x_0)$ *have positive real part;*
- *a saddle if it is a hyperbolic equilibrium point and* $Df(x_0)$ *has at least one eigenvalue with a positive real part and at least one with a negative real part.*

Example 13 *Let us classify all of the equilibrium points of the nonlinear system*

$$\begin{cases} \dot{x} = x^2 - 1, \\ \dot{y} = -y + 2x + y^2 - 1. \end{cases}$$

Clearly, $f(X) = 0$ *at* $X_1 = \left(-1, \frac{1}{2}\sqrt{13} + \frac{1}{2}\right)$ *and* $X_2 = \left(-1, \frac{1}{2} - \frac{1}{2}\sqrt{13}\right)$ *and these are the only equilibrium points of this. The derivative*

$$Df(x,y) = \begin{pmatrix} 2x & 0 \\ 2 & -1+2y \end{pmatrix},$$

then,

$$\begin{aligned} Df\left(-1, \frac{1}{2}\sqrt{13} + \frac{1}{2}\right) &= \begin{pmatrix} -2 & 0 \\ 2 & \sqrt{13} \end{pmatrix}, \\ Df\left(-1, \frac{-1}{2}\sqrt{13} + \frac{1}{2}\right) &= \begin{pmatrix} -2 & 0 \\ 2 & -\sqrt{13} \end{pmatrix}. \end{aligned}$$

Thus, $\left(-1, \frac{-1}{2}\sqrt{13} + \frac{1}{2}\right)$ *is a source and* $\left(-1, \frac{1}{2}\sqrt{13} + \frac{1}{2}\right)$ *is a saddle.*

Remark 9

- *The singular point* X_0 *is called non–degenerate if* 0 *is not an eigenvalue.*
- *The singular point* X_0 *is called semi-hyperbolic if exactly one eigenvalue of* $Df(X_0)$ *is equal to* 0*. Hyperbolic and semi-hyperbolic singularities are also said to be elementary singular points.*
- *The singular point* X_0 *is called nilpotent if both eigenvalues of* $Df(X_0)$ *are equal to* 0 *but* $Df(X_0) \equiv 0$.
- *The singular point* X_0 *is called linearly zero if* $Df(X_0) \equiv 0$.
- *The singular point* X_0 *is called a center if there is an open neighborhood consisting, besides the singularity, of periodic orbits.*
- *The singularity is said to be linearly a center if the eigenvalues of* $Df(X_0)$ *are purely imaginary without being zero.*

2 The Hartman-Grobman Theorem

The Hartman-Grobman Theorem is another very important result in the local qualitative theory of ordinary differential equations, this theorem allows us to reduce the study of a dynamical system (1.1) in the neighborhood of a hyperbolic singular point to the study of a linear system (3.1) topologically equivalent to (1.1), in the neighborhood of the origin.

Definition 15 *Two autonomous systems of differential equations such as* (1.1) *and* (3.1) *are said to be topologically equivalent in a neighborhood of the origin or to have the same qualitative structure near the origin if there is a homeomorphism* H *mapping an open set* U *containing the origin onto an open set* V *containing the origin that maps trajectories of* (1.1) *in* U *onto trajectories of* (3.1) *in* V *and preserves their orientation by time in the sense that if a trajectory is directed from* x_1 *to* x_2 *in* U*, then its image is directed from* $H(x_1)$ *to* $H(x_2)$ *in* V*.*
If the homeomorphism H *preserves the parameterization by time, then the systems* (1.1) *and* (3.1) *are said to be topologically conjugate in a neighborhood of the origin.*

Example 14 *Let* A *and* B *be two similar matrices* $(A = PBP^{-1})$*,, then the systems* $\dot{X} = AX$ *and* $\dot{X} = BX$ *are topologically equivalent.*
Indeed

$$\dot{X} = AX = PBP^{-1}X.$$

We choose $Y = P^{-1}X = h(X)$*, so* $X = PY$ *and*

$$\dot{Y} = P^{-1}\dot{X} = P^{-1}PBP^{-1}X = BP^{-1}X = BY,$$

and

$$Y = H(x) = P^{-1}X = P^{-1}e^{At}X_0 = P^{-1}Pe^{Bt}P^{-1}X_0 = e^{Bt}Y_0,$$

thus

$$H(e^{At}X) = P^{-1}e^{At}X = e^{Bt}P^{-1}X = e^{Bt}\left(H(X)\right).$$

So

$$H \circ e^{At} = e^{Bt} \circ H.$$

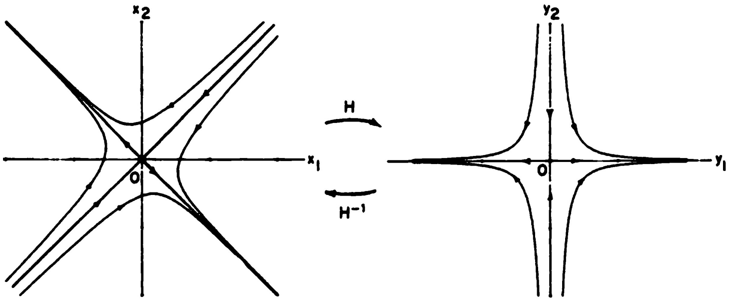
Application

$$A = \begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix},$$

and

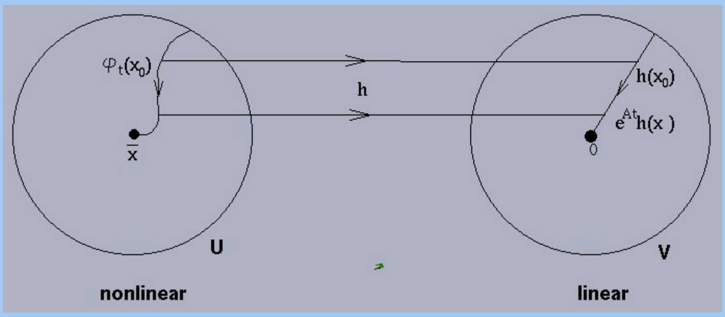
$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ &= PBP^{-1}, \end{aligned}$$

so the base of A *is* $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$*, the base of* B *is the canonical base* $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ *and the homeomorphism* $H(X) = PX$ *is simply a rotation of* 45°*. See the following figure*



Theorem 9 ***(Perko [9]).*** *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let φ_t be the flow of the nonlinear system* (1.1)*. Suppose that $f(0) = 0$ and that the matrix $A = Df(0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x_0 \in U$ and $t \in I_0$*

$$Ho\varphi_t(x_0) = e^{At}H(x_0)$$



i.e., H maps trajectories of (1.1) *near the origin onto trajectories of* (3.1) *near the origin and preserves the parameterization by time.*

For the proof of this Theorem one may consult the book Differential Equations and Dynamical Systems, 3rded by Lawrence Perko [11] at pp. 121-124.

Remark 10 *How to find a homomorphism H such that*

$$Ho\varphi_t(x_0) = e^{At}H(x_0)$$

is difficult. In fact, the Hartman-Grobman Theorem only assures the existence of H. It doesn't tell us any information on how to find H. It is a qualitative property!

3 Exercises

3.3.1 Corrected exercises

Exercise 19 *Show that the dynamical systems*

$$\begin{cases} \dot{r} = -r, \\ \dot{\theta} = 1, \end{cases}, \text{ and } \begin{cases} \dot{\rho} = -2\rho, \\ \dot{\vartheta} = 0, \end{cases},$$

are topologically equivalent with $h(0) = 0$, $h(r, \theta) = (r^2, \theta + \ln r)$ for $r \neq 0$ in polar coordinates, and $\tau(x, t) = t$.

Solution. To show this, integrate the ODEs to get

$$\begin{aligned} \varphi_t^1 &= \varphi_t(r_0, \theta_0) = \begin{pmatrix} r_0 e^{-t} \\ \theta_0 + t \end{pmatrix}, \\ \varphi_t^2 &= \varphi_t(\rho_0, \vartheta_0) = \begin{pmatrix} \rho_0 e^{-2t} \\ \vartheta_0 \end{pmatrix}, \end{aligned}$$

and check

$$\begin{aligned} h\circ\varphi_t^1\left(r_0,\theta_0\right) &= \begin{pmatrix} r_0^2e^{-2t} \\ \theta_0+t+\ln\left(r_0e^{-t}\right) \end{pmatrix} \\ &= \begin{pmatrix} r_0^2e^{-2t} \\ \theta_0+t+\ln\left(r_0\right)-t \end{pmatrix} \\ &= \begin{pmatrix} r_0^2e^{-2t} \\ \theta_0+\ln r_0 \end{pmatrix} = \varphi_t^2\circ h\left(r_0,\theta_0\right). \end{aligned}$$

Exercise 20 *Show that the continuous map $H:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ defined by*

$$H\left(x,y\right)=\begin{pmatrix} x \\ y+\dfrac{x^2}{3} \end{pmatrix}$$

has a continuous inverse $H^{-1}:\mathbb{R}^2\to\mathbb{R}^2$ and that the nonlinear system $\dot{X}=f(X(t))$ with

$$f\left(x,y\right)=\begin{pmatrix} -x \\ y+x^2 \end{pmatrix}$$

is transformed into a linear system $\dot{X}=AX$ with $A=Df(0)$ by H i.e. if $Y=H(X)$, show that $\dot{Y}=AY$.

Solution. Consider the continuous map

$$H\left(x,y\right)=\begin{pmatrix} x \\ y+\dfrac{x^2}{3} \end{pmatrix}$$

which maps $\mathbb{R}^2$ onto $\mathbb{R}^2$. It is not difficult to determine that the inverse of $Y=H(X)$ is given by

$$H^{-1}\left(x,y\right)=\begin{pmatrix} x \\ y-\dfrac{x^2}{3} \end{pmatrix}$$

and that H^{-1} is a continuous mapping of $\mathbb{R}^2$ onto $\mathbb{R}^2$. Furthermore, the mapping H transforms the nonlinear system $\dot{X}=f\left(X\right)$ with

$$f\left(x,y\right)=\begin{pmatrix} -x \\ y+x^2 \end{pmatrix}$$

into the linear system $\dot{Y}=AY$ with

$$A=Df(0,0)=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the sense that if $Y = H(X)$, then

$$\begin{aligned}
\dot{Y} &= \frac{dH}{dt} = \begin{pmatrix} x\,(\dot{x}) \\ (\dot{y}) + 2\frac{x}{3}\dot{x} \end{pmatrix} \\
&= \begin{pmatrix} x\,(-x) \\ (y + x^2) + 2\frac{x}{3}\,(-x) \end{pmatrix} = \begin{pmatrix} -x^2 \\ \frac{1}{3}x^2 + y \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y + \dfrac{x^2}{3} \end{pmatrix} = AY.
\end{aligned}$$

3.3.2 Additional exercises

Exercise 21 *Show that the dynamical systems*

$$\left\{ \begin{array}{l} \dot{r} = 0, \\ \dot{\theta} = 1 \end{array} \right. \quad \textit{and} \quad \left\{ \begin{array}{l} \dot{\rho} = 0, \\ \dot{\vartheta} = \rho + \sin^2 \vartheta, \end{array} \right.$$

are topologically equivalent

Exercise 22 *Consider the system of differential equations*

$$\left\{ \begin{array}{l} \dot{x} = x(1 - x - y), \\ \dot{y} = y(1 - x - y), \end{array} \right.$$

in the (x, y)-plane.

1. *Show that the fixed points of the system are either at the origin or on a line.*
2. *Calculate the Jacobian matrix for a general point (x, y) and deduce that the fixed point at the origin is non-degenerate, whereas the fixed points on the line are not.*
3. *Perform linear stability analysis of the fixed point at the origin. Solve the linearized equation (4). By dividing the two equations, obtain a new equation of the form*

   $$\frac{dy}{dx} = Z(x, y).$$

   Find the general solution to this equation. Explain why your solution is what one would expect given the nature of the fixed point at the origin.

Exercise 23 *Let X be a linear vector field on a neighborhood of $0 \in \mathbb{R}^2$, with $X(0) = 0$, and denote by $\varphi(t, x)$ the flow of X. Show that*

$$D_x \varphi(t, 0) = e^{At}, \quad \textit{with } A = DX(0).$$

Exercise 24 *Show that the continuous map* $H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *defined by*

$$H(x, y, z) = \begin{pmatrix} x \\ y + x^2 \\ z + \dfrac{x^2}{3} \end{pmatrix},$$

has a continuous inverse $H^{-1} : R^3 \to R^3$ *and that the nonlinear system* $\dot{X} = f(X(t))$ *with*

$$f(x, y, z) = \begin{pmatrix} -x \\ -y + x^2 \\ z + x^2 \end{pmatrix}$$

is transformed into a linear system $\dot{X} = AX$ *with* $A = Df(0)$ *by* H *i.e. if* $Y = H(X)$*, show that* $\dot{Y} = AY$*.*

Exercise 25 *Recall that the Hartman Grobman Theorem says that, under certain assumptions, a nonlinear system "looks alike" its linearization. More precisely, the statement of the theorem is as follows:*
Consider a system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

with $f \in C^1(\mathbb{R}^n)$*, and let* $\varphi_t(x)$ *denote its flow. Assume that* x^* *be a hyperbolic equilibrium point. Then there exists a neighborhood* N *of* x^* *such that ' is topologically conjugate to the flow of the corresponding linearized system*

$$\dot{x} = Df(x).$$

This problem will test your knowledge of some concepts and ideas involved in the proof of the theorem.

1. *Define what a hyperbolic equilibrium point is and sketch an example of a possible phase portrait around a non hyperbolic point.*

2. *Describe how the equivalence classes (under linear and hence topological conjugate) for planar linear systems are determined by the corresponding eigenvalues as well as stable, unstable, and center spaces.*

3. *Give a formal definition of topological conjugate, denoting the homeomorphism between the neighborhoods by* H*.*

4. *Restate the Hartman-Grobman theorem using the formal definition of topologically conjugate. In your statement, use* A *to denote the linearization of* f *at* x^**, i.e.* $A = Df(x^*)$*, and use* $\psi_t(x) = e^{At}x$ *to denote the flow of the linearized system.*

5. *The difficulty of the proof lies in the construction of the homeomorphism* H*. But suppose that* H_1 *is a unique homeomorphism satisfying*

$$H_1(x) = (\psi_{-1} \circ H_1 \circ \varphi_1)(x) = e^{-A}(H_1 \circ \varphi_1)(x).$$

Show that the sought homeomorphism H is given by

$$H(x) = \int_0^1 (\psi_{-s} \circ H_1 \circ \varphi_s)(x) ds.$$

You do not need to construct $H_1(x)$.

Exercise 26 *Classify the equilibrium points (as sinks, sources, or saddles) of the nonlinear system $\dot{X} = f(X)$ with $f(X)$ given by*

$$f(x,y) = \begin{pmatrix} x - xy \\ y - x^2 \end{pmatrix}, \quad f(x,y) = \begin{pmatrix} -4y + 2xy - 8 \\ y^2 - x^2 \end{pmatrix},$$

$$f(x,y) = \begin{pmatrix} -x \\ -y + x^2 \\ z + x^2 \end{pmatrix}, \quad f(x,y) = \begin{pmatrix} -y - x \\ kx - y - xz \\ xy - z \end{pmatrix}.$$

4

Invariant manifold

The Hartman-Grobman Theorem states that the flow generated by a smooth vector field in a neighborhood of a hyperbolic equilibrium point is topologically conjugate with the flow generated by its linearization. Hart man's counterexample shows that, in general, the conjugacy cannot be taken to be C^1. However, the Stable Manifold Theorem will tell us that there are important structures for the two flows that can be matched up by smooth changes of variable.

1 The Stable Manifold Theorem

The stable manifold theorem is one of the most important results in the local qualitative theory of ordinary differential equations. The theorem shows that near a hyperbolic equilibrium point x_0, the nonlinear system

$$\dot{x} = f(x) \tag{4.1}$$

has stable and unstable manifolds W^S and W^U tangent at x_0 to the stable and unstable subspaces E_u and E_s of the linearized system is

$$\dot{x} = Ax \tag{4.2}$$

where $A = Df(x_0)$. Furthermore, W^S and W^U are of the same dimensions as E_u and E_s, and if φ_t is the flow of the nonlinear system (4.1), then W^S and W^U positively.

For the linearized system, we can separate the phase space into different domains corresponding to different behaviors in time.

Definition 16 *(Invariant subspaces). The stable, unstable, and centre subspaces of the linearization of f at the fixed point x_0 are the three linear subspaces E_u, E_s, E_c, spanned by the subsets of (possibly generalised) eigenvectors of $A = Df(x_0)$ whose eigenvalues have real parts $< 0, > 0, = 0$ respectively.*

Note that a hyperbolic fixed point has no centre eigenspace. This concept can be extended simply for hyperbolic fixed points into the nonlinear domain.

Definition 17 *-We call a stable (local) manifold in the neighborhood of a hyperbolic equilibrium point x^* of the nonlinear system*

$$\dot{x} = f(x) \tag{4.3}$$

the set W^S which verifies the following properties

- W^S is a tangent at x^* to the stable subspace E_s
- W^S positively invariant for φ_t (φ_t is the flow of (4.3)
- $\lim\limits_{t\to+\infty} \varphi_t(C) = x^*$ for all $C \in W^S$

so we have

$$W^S(x^*) = \left\{ x \in v(x^*) : \lim_{t\to+\infty} \varphi_t(x) = x^* \text{ and } \varphi_t(x) \in v(x^*), \forall t \geq 0 \right\}$$

- We call a stable (local) manifold in the neighborhood of a hyperbolic equilibrium point x^* of the nonlinear system (4.3) the set W^U which verifies the following properties

- W^U is a tangent at x^* to the stable subspace E_u
- W^U negatively invariant for φ_t (φ_t is the flow of (4.3)
- $\lim\limits_{t\to-\infty} \varphi_t(C) = x^*$ for all $C \in W^U$

so we have

$$W^U(x^*) = \left\{ x \in v(x^*) : \lim_{t\to-\infty} \varphi_t(x) = x^* \text{ and } \varphi_t(x) \in v(x^*), \forall t \leq 0 \right\}$$

The local stable (unstable) manifold can be extended to a global invariant manifold W_g^S (resp. W_g^U) by following the flow backwards (forwards) in time from a point in W^S (resp. W^U)

Definition 18 *The global stable and unstable varieties of x^* are defined by*

$$\begin{aligned} W_g^S(x^*) &= \cup_{t\geq 0}\varphi_t\left(W^S\right) \\ W_g^U(x^*) &= \cup_{t\leq 0}\varphi_t\left(W^U\right) \end{aligned}$$

Example 15 *Consider the system*

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = -y + x^2, \quad , \\ \dot{z} = z + x^2, \end{cases} \tag{4.4}$$

find the stable and unstable manifolds and the stable and unstable subspaces of this system in the neighborhood of $(0,0,0)$.

Note that $(0,0,0)$ *is the only equilibrium point of the system*

$$Df(0,0,0) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Eigenvalues $\lambda_{1,2} = \pm 1$.

- $\lambda_1 = -1 < 0$ *is a double eigenvalue, so there exists a 2-dimensional differentiable stable manifold.*
- $\lambda_2 = 1 > 0$ *is a simple eigenvalue, therefore there exists a 1-dimensional differentiable unstable manifold.*

Stable and unstable subspaces:

$$E_\lambda = \left\{ v = (x, y, z) \in \mathbb{R}^3 : Df\,(0,0,0)\,v = \lambda v \right\},$$

we obtain

$$\begin{aligned} E_u &= \langle (0,0,1) \rangle = lax\ z, \\ E_s &= \langle (1,0,0), (0,1,0) \rangle = plan\,(x,y). \end{aligned}$$

The flow of the system:
The solution of $\dot{x} = -x, x(0) = C_1$, *is*

$$x\,(t) = C_1 e^{-t},$$

for the equation $\dot{y} = -y + x^2$ *with* $x\,(t) = C_1 e^{-t}$, *we have* $\dot{y} = -y + \left(C_1 e^{-t}\right)^2$, *the general solution is*

$$y\,(t) = \alpha e^{-t} - C_1^2 e^{-2t},$$

if $y(0) = C_2$, *we obtain* $\alpha = C_2 + C_1^2$, *so*

$$y\,(t) = \left(C_2 + C_1^2\right) e^{-t} - C_1^2 e^{-2t},$$

for $\dot{z} = z + x^2$*with* $x\,(t) = C_1 e^{-t}$ *the general solution is given by*

$$z\,(t) = k e^{t} - \frac{1}{3} C_1^2 e^{-2t},$$

for $z(0) = C_3$ *we get* $k = C_3 + \frac{1}{3} C_1^2$, *then*

$$z\,(t) = \left(C_3 + \frac{1}{3} C_1^2\right) e^{t} - \frac{1}{3} C_1^2 e^{-2t},$$

Then, the flow of the system is:

$$\varphi_t\,(C_1, C_2, C_3) = \begin{pmatrix} C_1 e^{-t} \\ \left(C_2 + C_1^2\right) e^{-t} - C_1^2 e^{-2t} \\ \left(C_3 + \frac{1}{3} C_1^2\right) e^{t} - \frac{1}{3} C_1^2 e^{-2t} \end{pmatrix}.$$

Stable manifold:

$$\lim_{t \to +\infty} \varphi_t\,(x) = (0,0,0) \to C_3 = -\frac{1}{3} C_1^2,$$

so

$$W^S\,(0,0,0) = \left\{ (C_1, C_2, C_3) \in \mathbb{R}^3 : C_3 = -\frac{1}{3} C_1^2 \right\},$$

Unstable manifold

$$\lim_{t \to -\infty} \varphi_t\,(x) = (0,0,0) \to C_3 = -\frac{1}{3} C_1^2,$$

then

$$W^U(0,0,0)=\left\{(C_1,C_2,C_3)\in\mathbb{R}^3 : C_1=C_2=0\right\}.$$

See the following figure

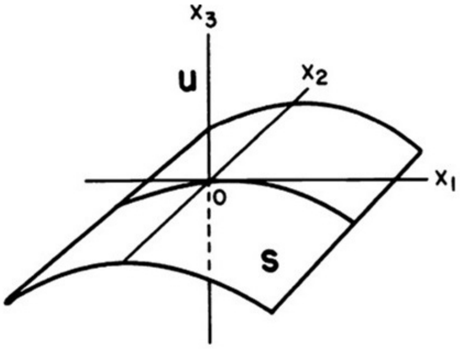


Fig 1 : *The stable and unstable manifolds of system (4.4)*

Theorem 10 ***(Perko [11]).*** *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f\in C^1(E)$, and let φ_t be the flow of the nonlinear system (4.3). Suppose that $f(0)=0$ and that $Df(0)$ has k eigenvalues with negative real parts and $n-k$ eigenvalues with positive real parts. Then there exists a k-dimensional differentiable manifold W^S and there exists an $n-k$-dimensional differentiable manifold W^U.*

Remark 11 **1** *Although the stable manifold theorem and the linearization characterize that $\dot{x}=f(x)$ and $\dot{x}=Df(x_0)x$ have the same stability property near a hyperbolic equilibrium, the stable manifold theorem gives much more information on geometric structures.*

2 *The stable manifold theorem uses a geometric way to characterize the local property near a hyperbolic equilibrium. The linearization uses an analytical way to characterize the local property near a hyperbolic equilibrium.*

2 Exercises

4.2.1 Corrected exercises

Exercise 27 *The flow of the system of differential equations*

$$\begin{cases}\dot{x}=f(x,y),\\ \dot{y}=g(x,y)\end{cases}$$

is given by

$$\varphi_t(x,y)=\begin{pmatrix}\left(x+\frac{1}{5}y^3\right)e^{2t}-\frac{1}{5}y^3e^{-3t}\\ ye^{-t}\end{pmatrix}.$$

1. *Determine the system, i.e., compute $f(x,y)$ and $g(x,y)$.*

2. *Find the equilibria.*

3. *Find the stable and unstable manifolds.*

Solution.

1. By definition of the flow,

$$(f(x,y),g(x,y)) = \frac{d}{dt}\varphi_t(x,y)\bigg|_{t=0} = (2x+y^3,-y).$$

   Therefore, the system is

$$\begin{cases} \dot{x} = 2x+y^3, \\ \dot{y} = -y. \end{cases}$$

2. The only equilibrium is the origin.

3. The stable manifold is the set of all $(x,y) \in \mathbb{R}^2$ such that $\varphi_t(x,y) \to (0,0)$, as $t \to +\infty$. Therefore,

$$W^S(0,0) = \left\{(x,y) : x + \frac{1}{5}y^3 = 0\right\}.$$

   The unstable manifold is the set of points (x,y) such that $\varphi_t(x,y) \to (0,0)$, as $t \to -\infty$. Therefore,

$$W^U(0,0) = \{(x,y) : y = 0\}.$$

Exercise 28 1. *Determine the flow $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$ for the nonlinear system*

$$\begin{cases} \dot{x} = -x+2y^2, \\ \dot{y} = -y, \\ \dot{z} = z-3y^2. \end{cases}$$

2. *Show that $S = \left\{(x,y,z) \in \mathbb{R}^3 : z = y^2\right\}$ is invariant under φ_t.*

3. *Find the local stable manifold and unstable manifold.*

Solution.

1. The solution of $\dot{y} = -y, \;\; y(0) = C_2$, is

$$y(t) = C_2 e^{-t},$$

   for the equation $\dot{x} = -x+2y^2$ with $y(t) = C_1 e^{-t}$, we have $\dot{x} = -x + 2\left(C_2 e^{-t}\right)^2$, the general solution of this last equation is

$$x(t) = \alpha e^{-t} - 2C_2^2 e^{-2t},$$

   we use the initial condition $x(0) = C_1$, we find $\alpha = C_1 + 2C_2^2$, therefore x is of the form

$$x(t) = \left(C_1 + 2C_2^2\right)e^{-t} - 2C_2^2 e^{-2t},$$

for the 3rd equation we have

$$\dot{z} = z - 3\left(C_2 e^{-t}\right)^2,$$

the general solution of this last equation is

$$z(t) = ke^t + C_2^2 e^{-2t},$$

we use the initial condition $z(0) = C_3$, we find $k = C_3 - C_2^2$, so

$$z(t) = \left(C_3 - C_2^2\right)e^t + C_2^2 e^{-2t}$$

and the flow of the system is

$$\varphi_t(C_1, C_2, C_3) = \begin{pmatrix} \left(C_1 + 2C_2^2\right)e^{-t} - 2C_2^2 e^{-2t} \\ C_2 e^{-t} \\ \left(C_3 - C_2^2\right)e^t + C_2^2 e^{-2t} \end{pmatrix}.$$

2. Let $(C_1, C_2, C_3) \in S$, then $C_3 = C_2^2$ and

$$\varphi_t(C_1, C_2, C_3) = \begin{pmatrix} \left(C_1 + 2C_2^2\right)e^{-t} - 2C_2^2 e^{-2t} \\ C_2 e^{-t} \\ C_2^2 e^{-2t} \end{pmatrix} \subset S.$$

3. The stable manifold and the unstable manifold: The origin is a point of equilibrium. The Jacobian for this system

$$Df(0,0,0) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvaluesare $\lambda_1 = \pm 1$. Since $\lambda_1 = -1 < 0$ double eigenvalue, there exists a stable manifold of $dim = 2$ and since $\lambda_2 = 1 > 0$ is a simple eigenvalue, there exists an unstable manifold of $dim = 1$. We have

$$E_\lambda = \left\{v = (x, y, z) \in \mathbb{R}^3 : Df(0,0,0)\, v = \lambda v\right\},$$

the unstable eigenvector is clearly $v_u = (0,0,1)^T$, so the unstable subspace is

$$E_u = \langle(0,0,1)\rangle = lax \text{ z},$$

the stable subspace is

$$E_s = \langle(1,0,0), (0,1,0)\rangle = plan\,(x, y).$$

The points (x, y) on the unstable manifold $W^U(0,0,0)$ satisfy $\varphi_t(x, y) \to 0$ if $t \to -\infty$. Then the form of the first component of $\widetilde{\varphi}$ needs $C_1 + 2C_2^2 = 0$ and $C_2^2 = 0$, thus $C_1 = C_2 = 0$. If we substitute $C_1 = C_2 = 0$ into the second and third components, we find 0. Therefore $W^U(0,0)$ is:

$$W^U(0,0,0) = \left\{(C_1, C_2, C_3) \in \mathbb{R}^3 : C_1 = C_2 = 0\right\},$$

The points (x, y) on the stable manifold $W^S(0,0,0)$ satisfy $\varphi_t(x, y) \to 0$ if $t \to +\infty$, then $C_3 = C_1^2$. So

$$W^S(0,0,0) = \left\{(C_1, C_2, C_3) \in \mathbb{R}^3 : C_3 = C_1^2\right\}.$$

Exercise 29 *1. Determine the flow* $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$ *for the nonlinear system*

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = 3y - x^2, \\ \dot{z} = -z + 2x^2. \end{cases}$$

2. Find the local stable manifold and unstable manifold.

Solution.

1. The solution of $\dot{x} = -x, \;\; x(0) = C_1$, is

$$x(t) = C_1 e^{-t}.$$

   For the equation $\dot{y} = 3y + x^2$ with $x(t) = C_1 e^{-t}$, we have $\dot{y} = 3y + \left(C_1 e^{-t}\right)^2$, the general solution of this last equation is

$$y(t) = \alpha e^{3t} - \frac{1}{5} C_1^2 e^{-2t},$$

   we use the initial condition $y(0) = C_2$, we find $\alpha = C_2 + \frac{1}{5} C_1^2$, therefore, y is of the form

$$y(t) = \left(C_2 + \frac{1}{5} C_1^2\right) e^{3t} - \frac{1}{5} C_1^2 e^{-2t},$$

   for the 3rd equation, we have

$$\dot{z} = -z + 2\left(C_1 e^{-t}\right)^2,$$

   the general solution of this last equation is

$$z(t) = k e^{-t} - 2C_1^2 e^{-2t},$$

   we use the initial condition $z(0) = C_3$, we find $k = C_3 + 2C_1^2$, so

$$z(t) = \left(C_3 + 2C_1^2\right) e^{-t} - 2C_1^2 e^{-2t},$$

   and the flow of the system is

$$\varphi_t(C_1, C_2, C_3) = \begin{pmatrix} C_1 e^{-t} \\ \left(C_2 + \frac{1}{5} C_1^2\right) e^{3t} - \frac{1}{5} C_1^2 e^{-2t} \\ \left(C_3 + 2C_1^2\right) e^{-t} - 2C_1^2 e^{-2t} \end{pmatrix}.$$

2. The stable manifold and the unstable manifold: The origin is a point of equilibrium. The Jacobian for this system

$$Df(0,0,0) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

The eigenvaluesare $\lambda_1 = -1$and $\lambda_2 = 3$. Since $\lambda_1 = -1 < 0$ double eigenvalue, there exists a 2-dimensional differentiable stable manifold, and since $\lambda_2 = 3 > 0$ is a simple eigenvalue, then there exists a 1-dimensional differentiable unstable manifold. We have

$$E_\lambda = \left\{ v = (x, y, z) \in \mathbb{R}^3 : Df\,(0,0,0)\, v = \lambda v \right\},$$

the unstable eigenvector is clearly $v_u = (0,0,1)^T$, so the unstable subspace is

$$E_u = \langle (0,1,0) \rangle = lax\ y,$$

the stable subspace is

$$E_s = \langle (1,0,0)\,, (0,0,1) \rangle = plan\,(x,z)\,,$$

the points (x, y) on the unstable manifold $W^U\,(0,0,0)$ satisfy $\varphi_t(x, y) \to 0$ if $t \to -\infty$. Then the form the first component of $\widetilde{\varphi}$ needs $C_1 = 0$ and from the second component, we get $C_2 = 0$, thus $C_1 = C_2 = 0$. If we substitute $C_1 = C_2 = 0$ into the third component, we find 0. Therefore $W^U\,(0,0)$ is:

$$W^U\,(0,0) = \left\{ (C_1, C_2, C_3) \in \mathbb{R}^3 : C_3 = C_2 = 0 \right\}.$$

The points (x, y) on the stable manifold $W^S\,(0,0,0)$ satisfy $\varphi_t(x, y) \to 0$ if $t \to +\infty$, then

$$\lim_{t \to +\infty} \varphi_t\,(x) = (0,0,0) \to C_2 + \frac{1}{5} C_1^2 = 0,$$

so

$$W^S\,(0,0,0) = \left\{ (C_1, C_2, C_3) \in \mathbb{R}^3 : C_2 = -\frac{1}{5} C_1^2 \right\}.$$

Exercise 30 *1. Determine the flow* $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$ *for the nonlinear system*

$$\begin{cases} \dot{x} = -x + z^2, \\ \dot{y} = 3y + 4z^2, \\ \dot{z} = -z. \end{cases}$$

2). Find the local stable manifold and unstable manifold.

Solution.

1. The solution of $\dot{z} = -z,\ \ z(0) = C_3$, is

$$z\,(t) = C_3 e^{-t},$$

for the equation $\dot{x} = -x + 2y^2$ with $y\,(t) = C_1 e^{-t}$, we have $\dot{x} = -x + \left(C_3 e^{-t}\right)^2$, the general solution of this last equation is

$$x\,(t) = \alpha e^{-t} - C_3^2 e^{-2t},$$

we use the initial condition $x(0) = C_1$, we find $\alpha = C_1 + C_3^2$, therefore, y is of the form

$$x\,(t) = \left(C_1 + C_3^2\right) e^{-t} - C_3^2 e^{-2t},$$

for the second equation, we have

$$\dot{y} = 3y + 4\left(C_3 e^{-t}\right)^2,$$

the general solution of this last equation is

$$y\left(t\right) = ke^{3t} - \frac{4}{5}C_3^2 e^{-2t},$$

we use the initial condition $y(0) = C_2$, we find $k = C_2 + \frac{4}{5}C_3^2$, so

$$y\left(t\right) = \left(C_2 + \frac{4}{5}C_3^2\right)e^t - \frac{4}{5}C_3^2 e^{-2t},$$

and the flow of the system is

$$\varphi_t\left(C_1, C_2, C_3\right) = \left(\begin{array}{c} \left(C_1 + C_3^2\right)e^{-t} - C_3^2 e^{-2t} \\ \left(C_2 + \frac{4}{5}C_3^2\right)e^t - \frac{4}{5}C_3^2 e^{-2t} \\ C_3 e^{-t} \end{array}\right).$$

2. The stable manifold and the unstable manifold: The origin is a point of equilibrium. The Jacobian for this system

$$Df\left(0,0,0\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The eigenvaluesare $\lambda_1 = -1, \lambda_2. = 3$ Since $\lambda_1 = -1 < 0$ double eigenvalue, there exists a 2-dimensional differentiable stable manifold, and since $\lambda_2 = 3 > 0$ is a simple eigenvalue, there exists a 1-dimensional differentiable unstable manifold. We have

$$E_\lambda = \left\{v = (x,y,z) \in \mathbb{R}^3 : Df\left(0,0,0\right)v = \lambda v\right\},$$

the unstable eigenvector is clearly $v_u = (0,0,1)^T$, so the unstable subspace is

$$E_u = \langle(0,1,0)\rangle = lax\ y,$$

the stable subspace is

$$E_s = \langle\left(1,0,0\right), \left(0,0,1\right)\rangle = plan\left(x,z\right),$$

the points (x,y) on the unstable manifold $W^U\left(0,0,0\right)$ satisfy $\varphi_t(x,y) \to 0$ if $t \to -\infty$. Then form the first component of φ_t need $C_1 - C_3^2 = 0$ and $C_3 = 0$. If we substitute $C_1 = C_3 = 0$ into the second and the third components, and we find 0. Therefore $W^U\left(0,0,0\right)$ is:

$$W^U\left(0,0,0\right) = \left\{(C_1, C_2, C_3) \in \mathbb{R}^3 : C_1 = C_3 = 0\right\},$$

The points (x,y) on the stable manifold $W^S\left(0,0,0\right)$ satisfy $\varphi_t(x,y) \to (0,0,0)$ if $t \to +\infty$. then $C_2 + \frac{4}{5}C_3^2 = 0$, so

$$W^S\left(0,0,0\right) = \left\{(C_1, C_2, C_3) \in \mathbb{R}^3 : C_2 + \frac{4}{5}C_3^2 = 0\right\}.$$

Exercise 31 *Consider the skew product system*

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = y + g(x), \end{cases} , \quad \text{with } g(0) = 0, g \in C^1(\mathbb{R}). \tag{4.5}$$

Find the invariant manifolds (stable and unstable manifolds) near the equilibrium point $(0,0)$*, and trace the orbits in the case where* $g(x) = x^2$.

Solution. The origin is an equilibrium point. The Jacobian for this system evaluated at the origin is

$$Df(0,0) = \begin{pmatrix} -1 & 0 \\ g' & 1 \end{pmatrix}.$$

Eigenvalues are $\lambda_{1,2} = \pm 1$, $(0,0)$ is hyperbolic. The unstable eigenvector is clearly $v_u = (0,1)^T$, so the unstable subspace is the y-axis:

$$E_u = \{(x,y) \in \mathbb{R}^2 : x = 0\}.$$

The stable eigenvector is $v_s = (-2, g'(0))^T$, so the stable subspace is a line passing through the origin:

$$E_s = \{(x,y) \in \mathbb{R}^2 : y = -\frac{1}{2} g'(0)\, x\}.$$

The phase portrait for the linearized system looks like (xy-axes, expansion on E_u, the line E_s, contraction of E_s, other trajectories).
The skew product form of (4.5) allows us to integrate these equations exactly $x(t) = x_0 e^{-t}$. Use an integrating factor to solve for $y(t)$

$$y(t) = y_0 e^t + e^t \int_0^t e^{-s} g\left(e^{-s} x_0\right) ds,$$

then

$$\varphi_t(x,y) = \begin{pmatrix} x e^{-t} \\ y e^t + e^t \int_0^t e^{-s} g\left(e^{-s} x\right) ds \end{pmatrix}.$$

Points (x,y) on the unstable manifold U satisfy $\varphi_t(x,y) \to 0$ as $t \to -\infty$. Then the form of the first component of φ_t requires $x = 0$. If we substitute $x = 0$ into the second component and note $g(0) = 0$, we see that any $y \in \mathbb{R}$ is allowed. Therefore, U is the y-axis:

$$W^U(0,0) = \{(x,y) \in \mathbb{R}^2 : x = 0\}.$$

Points (x,y) on the stable manifold S satisfy $\varphi_t(x,y) \to 0$ as $t \to +\infty$. Then the form of the first component of φ_t is compatible with this for any $x \in \mathbb{R}$. From the second component, y must satisfy

$$\lim_{t \to +\infty} \left(y e^t + e^t \int_0^t e^{-s} g\left(e^{-s} x\right) ds \right) = 0.$$

Make the substitution $v = e^{-s}$ in the integral to obtain:

$$\lim_{t \to +\infty} e^t \left(y + \int_{e^{-t}}^1 g(vx)\, ds \right) = 0.$$

Claim $y = -\int_0^1 g(vx)\,dv$, so

$$\begin{aligned}\lim_{t\to+\infty} e^t\left(y+\int_{e^{-t}}^1 g(vx)\,dv\right) &= \lim_{t\to+\infty} e^t\left(-\int_0^1 g(vx)\,dv+\int_{e^{-t}}^1 g(vx)\,dv\right)\\ &= \lim_{t\to+\infty} e^t\left(-\int_0^1 g(vx)\,dv-\int_1^{e^{-t}} g(vx)\,dv\right)\\ &= \lim_{t\to+\infty}\left(-e^t\int_0^{e^{-t}} g(vx)\,dv\right),\end{aligned}$$

since $f(0)=0$ and g is continuous

$$\forall\varepsilon>0, \exists\delta>0 : |vx|<\delta \Rightarrow |g(vx)|<\varepsilon$$

and

$$\left|\int_0^{e^{-t}} g(vx)\right| dv \le \int_0^{e^{-t}} |g(vx)|\,dv < \int_0^{e^{-t}} \varepsilon = e^{-t}\varepsilon,$$

thus

$$e^{-t}\left|\int_0^{e^{-t}} g(vx)\right| dv < \varepsilon.$$

So

$$\lim_{t\to+\infty}\left(-e^t\int_0^{e^{-t}} g(vx)\,dv\right) = 0,$$

and

$$y = -\int_0^1 g(vx)\,dv,$$

and the stable manifold $W^S(0,0)$ is

$$W^S(0,0) = \{(x,y)\in\mathbb{R}^2 : y = -\int_0^1 g(vx)\,dv\},$$

In the case when $f(x)=x^2$, we get

$$\begin{aligned}E_u &= \{(x,y)\in\mathbb{R}^2 : x=0\},\\ E_s &= \{(x,y)\in\mathbb{R}^2 : y=0\},\\ W^S(0,0) &= \{(x,y)\in\mathbb{R}^2 : y=-\frac{1}{3}x^2\},\\ W^U(0,0) &= \{(x,y)\in\mathbb{R}^2 : x=0\}.\end{aligned}$$

See the following figure

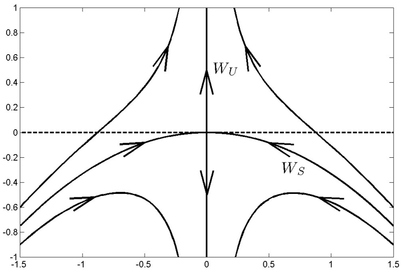


Fig 2 : The stable and unstable manifolds of system (4.5)

Exercise 32 *1) The flow of the autonomous differential system*

$$\begin{cases} \dot{x} = f(x,y), \\ \dot{y} = g(x,y), \end{cases}$$

is given by

$$\varphi_t(x,y) = \begin{pmatrix} x_0 e^{-t} \\ y_0 e^t + e^t \displaystyle\int_0^t e^{-s} h\left(x_0 e^{-s}\right) ds \end{pmatrix}, h \in C^1, h\left(0\right) = 0.$$

Determine the vector field $(f(x,y), g(x,y))^T$
2) Consider the dynamic system of form

$$\begin{cases} \dot{x} = -2y - x + \cos x \sin y + \cos y \sin x, \\ \dot{y} = y - \cos x \sin y - \cos y \sin x. \end{cases} \tag{4.6}$$

(a) *Determine the flow of this system.*

(b) *Look for invariant manifolds (stable and unstable manifolds) near the equilibrium point.*

(c) *Draw the phase portrait of this system.*

Solution.

1. According to the flow properties, we have

$$\frac{d\varphi_t\left(x_0, y_0\right)}{dt} = \begin{pmatrix} f(\varphi_t\left(x_0, y_0\right)) \\ g(\varphi_t\left(x_0, y_0\right)) \end{pmatrix}.$$

Since $\begin{pmatrix} x\left(t\right) \\ y\left(t\right) \end{pmatrix} = \varphi_t\left(x_0, y_0\right)$, then

$$\begin{pmatrix} x\left(t\right) \\ y\left(t\right) \end{pmatrix} = \begin{pmatrix} x_0 e^{-t} \\ y_0 e^t + e^t \displaystyle\int_0^t e^{-s} h\left(x_0 e^{-s}\right) ds \end{pmatrix}.$$

From the first component, we have $x_0 e^{-t} = x$, then $e^{-t} = \dfrac{x}{x_0}$, substituting this into $y = y_0 e^t + e^t \displaystyle\int_0^t e^{-s} h\left(x_0 e^{-s}\right) ds$, we get $y = y_0 e^t + e^t \displaystyle\int_0^t e^{-s} h\left(x_0 e^{-s}\right) ds$, so

$$e^t = \frac{y}{y_0 + \int_0^t e^{-s} h\left(x_0 e^{-s}\right) ds},$$

we replace $e^{-t} = \dfrac{x}{x_0}$ and $e^t = \dfrac{y}{y_0 + \int e^{-s} h\left(x_0 e^{-s}\right) ds}$ in

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{d\varphi_t\left(x_0, y_0\right)}{dt} = \begin{pmatrix} -x_0 e^{-t} \\ y_0 e^t + e^t \displaystyle\int_0^t e^{-s} h\left(x_0 e^{-s}\right) ds + h\left(x_0 e^{-t}\right) \end{pmatrix},$$

we get

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x_0\left(\dfrac{x}{x_0}\right) \\ \dfrac{y}{y_0+\int e^{-s}h\left(x_0e^{-s}\right)ds}\left(y_0+\displaystyle\int_0^t e^{-s}h\left(x_0e^{-s}\right)ds\right)+g\left(x\right) \end{pmatrix},$$

thus

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x \\ y+h\left(x\right) \end{pmatrix}.$$

2. the origin is the only fixed point of the system,

$$\begin{cases} \dot{x} = -2y - x + \cos x \sin y + \cos y \sin x, \\ \dot{Y} = y - \cos x \sin y - \cos y \sin x, \end{cases} \tag{4.7}$$

by the change of variable $x \to X - Y$ and $y \to Y$, we have

$$\begin{cases} \dot{X} - \dot{Y} = \sin X - Y - X, \\ \dot{Y} = Y - \sin\left(X\right). \end{cases}$$

So

$$\begin{cases} \dot{X} = -X, \\ \dot{Y} = Y - \sin\left(X\right), \end{cases}$$

we put $h(X) = -\sin X$ a function of class C^1, and $h(0) = 0$; then, according to 1, the flow of this system is:

$$\begin{aligned} \varphi_t(X_0, Y_0) &= \begin{pmatrix} X\left(t\right) \\ Y\left(t\right) \end{pmatrix} = \begin{pmatrix} X_0e^{-t} \\ Y_0e^t - e^t \displaystyle\int_0^t e^{-s}\sin\left(X_0e^{-s}\right)ds \end{pmatrix} \\ &= \begin{pmatrix} X_0e^{-t} \\ Y_0e^t - e^t\left(\dfrac{1}{X_0}\cos\left(X_0e^{-t}\right) - \dfrac{1}{X_0}\cos\left(X_0\right)\right) \end{pmatrix}, \end{aligned}$$

by the change of variable $X \to x + y$ and $Y \to y$ we have $X_0 \to x_0 + y_0$ and $Y_0 \to y_0$ and

$$\begin{pmatrix} x+y \\ y \end{pmatrix} = \begin{pmatrix} \left(x_0+y_0\right)e^{-t} \\ y_0e^t - e^t\left(\dfrac{\cos\left(\left(x_0+y_0\right)e^{-t}\right)}{x_0+y_0} + \dfrac{\cos\left(x_0+y_0\right)}{x_0+y_0}\right) \end{pmatrix},$$

which implies

$$\begin{pmatrix} x\left(t\right) \\ y\left(t\right) \end{pmatrix} = \begin{pmatrix} \left(x_0+y_0\right)e^{-t} - \left(y_0e^t - e^t\dfrac{\cos\left(\left(x_0+y_0\right)e^{-t}\right)}{x_0+y_0} - \dfrac{\cos\left(x_0+y_0\right)}{x_0+y_0}\right) \\ y_0e^t - e^t\left(\dfrac{\cos\left(\left(x_0+y_0\right)e^{-t}\right)}{x_0+y_0} + \dfrac{\cos\left(x_0+y_0\right)}{x_0+y_0}\right) \end{pmatrix},$$

therefore, the flow of the system (4.6) is

$$\widetilde{\varphi}_t(x_0,y_0)=\begin{pmatrix} (x_0+y_0)\,e^{-t}-e^t\left(y_0-\dfrac{\cos\left((x_0+y_0)\,e^{-t}\right)}{x_0+y_0}-\dfrac{\cos\left(x_0+y_0\right)}{x_0+y_0}\right) \\ y_0e^t-e^t\left(\dfrac{\cos\left((x_0+y_0)\,e^{-t}\right)}{x_0+y_0}-\dfrac{\cos\left(x_0+y_0\right)}{x_0+y_0}\right) \end{pmatrix}.$$

Invariant manifolds: The origin is an equilibrium point of (4.6) The Jacobian for this system at the origin is

$$Df\,(0,0)=\begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}.$$

The eigenvaluesare $\lambda_{1,2}=\pm 1$ so $(0,0)$ is a saddle point. Since $\lambda_1=-1<0$ simple, there exists a stable manifold of $dim=1$ and since $\lambda_2=1>0$ simple eigenvalue, then there exists an unstable manifold of $dim=1$.

3. The unstable eigenvector is clearly $v_u=(1,1)^T$, so the unstable subspace is

$$E_u=\{(x,y)\in\mathbb{R}^2 : x=y\}.$$

The stable eigenvector is $v_s=(1,0)^T$, so the stable subspace is

$$E_s=\left\{(x,y)\in\mathbb{R}^2 : y=0\right\}.$$

The points (x,y) on the unstable manifold $W^U\,(0,0)$ satisfy $\widetilde{\varphi}_t(x,y)\to(0,0)$ if $t\to-\infty$. Then the form of the first component of $\widetilde{\varphi}_t$ need $x_0+y_0=0$. If we substitute $x_0+y_0=0$ into the second component, we find 0. Therefore $W^U\,(0,0)$ is:

$$W^U\,(0,0)=\{(x,y)\in\mathbb{R}^2 : x+y=0\},$$

On the points (x,y); the stable manifold $W^S\,(0,0)$ satisfy $\widetilde{\varphi}_t(x,y)\to(0,0)$ if $t\to+\infty$. then

$$\lim_{t\to+\infty}\left(y_0-\frac{\cos\left((x_0+y_0)\,e^{-t}\right)}{x_0+y_0}-\frac{\cos\left(x_0+y_0\right)}{x_0+y_0}\right)=0,$$

which implies

$$y_0=\frac{1}{x_0+y_0}\left(1-\cos\left(x_0+y_0\right)\right),$$

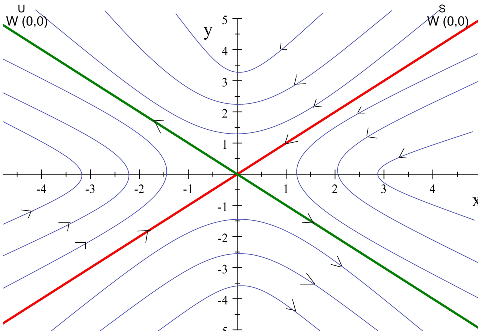
since $(1-\cos\left(x_0+y_0\right))\underset{v(0)}{\simeq}\dfrac{\left(x_0+y_0\right)^2}{2}$, then

$$y_0=\frac{1}{(x_0+y_0)}\left(1-\cos\left((x_0+y_0)\right)\right)\underset{v(0)}{\simeq}\frac{(x_0+y_0)}{2}.$$

Therefore,

$$W^S\,(0,0)=\{(x,y)\in\mathbb{R}^2 : x=y\}.$$

The phase portrait is



Exercise 33 *1) The flow of the autonomous differential system*

$$\begin{cases} \dot{x} = f(x,y), \\ \dot{y} = g(x,y) \end{cases}$$

is given by

$$\varphi_t(x_0,y_0) = \begin{pmatrix} x_0 e^{-t} \\ \left(y_0 + \frac{1}{5}x_0^2\right)e^{3t} - \frac{1}{5}x_0^2 e^{-2t} \end{pmatrix}.$$

Determine the vector field $(f(x,y), g(x,y))^T$.
2) Consider the dynamic system of form

$$\begin{cases} \dot{x} = -x + 4y - 2xy + x^2 + y^2, \\ \dot{y} = 3y - 2xy + x^2 + y^2. \end{cases} \tag{4.8}$$

(a) *Determine the flow of this system.*

(b) *Look for invariant manifolds (stable and unstable manifolds) near the equilibrium point.*

(c) *Draw the phase portrait of this system.*

Solution.

1. According to the flow properties, we have

$$\frac{d\varphi_t(x_0,y_0)}{dt} = \begin{pmatrix} f(\varphi_t(x_0,y_0)) \\ g(\varphi_t(x_0,y_0)) \end{pmatrix}.$$

Since

$$\varphi_t(x_0,y_0) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{-t} \\ \left(y_0 + \frac{1}{5}x_0^2\right)e^{3t} - \frac{1}{5}x_0^2 e^{-2t} \end{pmatrix}.$$

Then, from the first component, we get $e^{-t} = \frac{x}{x_0}$, substituting this in $y = \left(y_0 + \frac{1}{5}x_0^2\right)e^{3t} - \frac{1}{5}x_0^2 e^{-2t}$, we obtain $y = \left(y_0 + \frac{1}{5}x_0^2\right)e^{3t} - \frac{1}{5}x^2$, from this we get

$$e^{3t} = \frac{y + \frac{1}{5}x^2}{\left(y_0 + \frac{1}{5}x_0^2\right)}.$$

We replace $e^{-t} = \frac{x}{x_0}$ and $e^{3t} = \frac{y + \frac{1}{3}x^2}{\left(y_0 + \frac{1}{3}x_0^2\right)}$, in

$$\frac{d\varphi_t(x_0,y_0)}{dt} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x_0 e^{-t} \\ 3\left(y_0 + \frac{1}{5}x_0^2\right)e^{3t} + \frac{1}{5}x_0^2 e^{-2t} \end{pmatrix},$$

we get

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x_0 \left(\dfrac{x}{x_0}\right) \\ 3\left(y_0 + \dfrac{1}{5}x_0^2\right)\left(\dfrac{y+\frac{1}{5}x^2}{y_0+\frac{1}{5}x_0^2}\right) + \dfrac{2}{5}x_0^2\left(\dfrac{x}{x_0}\right)^2 \end{pmatrix},$$

thus

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x \\ x^2+3y \end{pmatrix}.$$

2. The origin is the only fixed point in the system,

$$\begin{cases} \dot{x} = 4y - x + (x-y)^2, \\ \dot{y} = 3y + (x-y)^2. \end{cases}$$

By changing the variable $x \to X + Y$ and $y \to Y$, we get

$$\begin{cases} \dot{X} + \dot{Y} = 4Y - X - Y + X^2, \\ \dot{Y} = 3Y + X^2, \end{cases}$$

then

$$\begin{cases} \dot{X} = -X, \\ \dot{Y} = X^2 + 3Y. \end{cases}$$

According to 1, the flow of this system is

$$\varphi_t(X_0, Y_0) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} X_0 e^{-t} \\ \left(Y_0 + \dfrac{1}{5}X_0^2\right)e^{3t} - \dfrac{1}{5}X_0^2 e^{-2t} \end{pmatrix}.$$

By changing the variable $X \to x - y$ and $Y \to y$, we have $X_0 \to x_0 - y_0$, $Y_0 \to y_0$ and

$$\begin{pmatrix} x-y \\ y \end{pmatrix} = \begin{pmatrix} (x_0 - y_0)e^{-t} \\ \left(y_0 + \dfrac{(x_0-y_0)^2}{5}\right)e^{3t} - \dfrac{(x_0-y_0)^2}{5}e^{-2t} \end{pmatrix}.$$

Which implies

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (x_0 - y_0)e^{-t} + \left(y_0 + \dfrac{(x_0-y_0)^2}{5}\right)e^{3t} - \dfrac{(x_0-y_0)^2}{5}e^{-2t} \\ \left(y_0 + \dfrac{(x_0-y_0)^2}{5}\right)e^{3t} - \dfrac{(x_0-y_0)^2}{5}e^{-2t} \end{pmatrix}.$$

So the flow of the system (4.8) is

$$\widetilde{\varphi}_t(x_0, y_0) = \begin{pmatrix} (x_0 - y_0)e^{-t} + \left(y_0 + \dfrac{(x_0-y_0)^2}{5}\right)e^{3t} - \dfrac{(x_0-y_0)^2}{5}e^{-2t} \\ \left(y_0 + \dfrac{(x_0-y_0)^2}{5}\right)e^{3t} - \dfrac{(x_0-y_0)^2}{5}e^{-2t} \end{pmatrix}.$$

The origin is a point of equilibrium. The Jacobian for this system

$$Df\,(0,0) = \begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}.$$

The eigenvaluesare $\lambda_1 = -1$,$\lambda_2 = 3$ so $(0,0)$ is a saddle point. Since $\lambda_1 = -1 < 0$ simple, there exists a stable manifold of $dim = 1$ and since $\lambda_2 = 3 > 0$ is a simple eigenvalue, there exists an unstable manifold of $dim = 1$.

3. The points (x, y) on the unstable manifold $W^U\,(0,0)$ satisfy $\widetilde{\varphi}_t(x, y) \to 0$ if $t \to -\infty$. Then the form of the first component of $\widetilde{\varphi}$ needs $x_0 - y_0 = 0$. If we substitute $x_0 - y_0 = 0$ into the second component, we find 0. Therefore $W^U\,(0,0)$ is:

$$W^U\,(0,0) = \{(x, y) \in \mathbb{R}^2 : x - y = 0\},$$

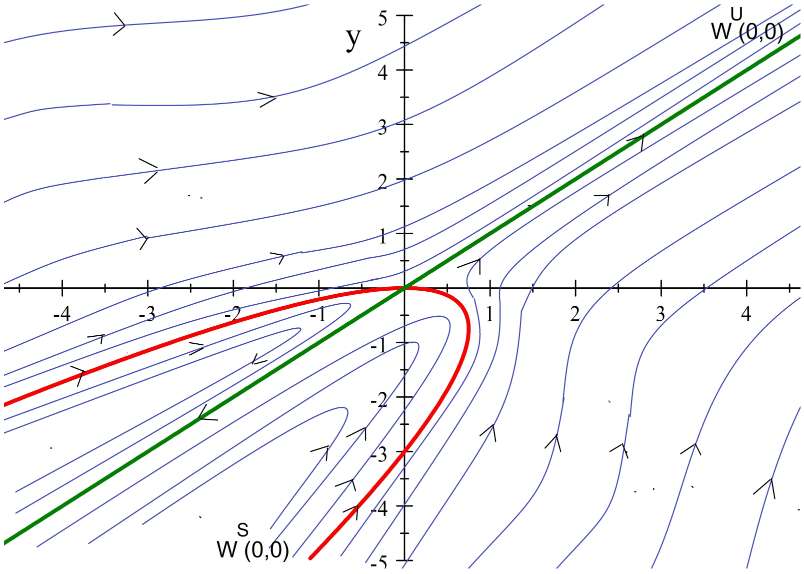
On the points (x, y); the stable manifold $W^S\,(0,0)$ satisfy $\widetilde{\varphi}_t(x, y) \to 0$ if $t \to +\infty$, then

$$\begin{aligned}
\lim_{t\to+\infty} (x_0 - y_0)\,e^{-t} + \left(y_0 + \frac{(x_0 - y_0)^2}{5}\right) e^{3t} - \frac{(x_0 - y_0)^2}{5} e^{-2t} &= 0, \\
\lim_{t\to+\infty} \left(y_0 + \frac{1}{5}\,(x_0 - y_0)^2\right) e^{3t} - \frac{1}{5}\,(x_0 - y_0)^2\, e^{-2t} &= 0,
\end{aligned}$$

therefore, the stable manifold is

$$W^S\,(0,0) = \{(x, y) \in \mathbb{R}^2 : y + \frac{1}{5}\,(x - y)^2 = 0\}.$$

The phase portrait of system is



Exercise 34 *Consider the dynamical eauqtion of the form*

$$\begin{aligned}
\dot{x} &= -x + 2x\left(\frac{1}{2}x^2 - 3xy + 6y^2\right) - 2\left(4y^3 - \frac{1}{2}\right), \qquad (4.9) \\
\dot{y} &= -x + y + x\left(\frac{1}{2}x^2 - 3xy + 6y^2\right) - \left(4y^3 - \frac{1}{2}\right).
\end{aligned}$$

a) Find the equilibria.
b) Determine the flow of this system.
d) Find the stable and unstable manifolds.
e) Sketch the phase portrait by drawing some representative trajectories.

Solution.

a) Equilibrium points are found by solving for x, y in

$$\begin{aligned} -x+2x\left(\frac{1}{2}x^2-3xy+6y^2\right)-2\left(4y^3-\frac{1}{2}\right) &= 0, \\ -x+y+x\left(\frac{1}{2}x^2-3xy+6y^2\right)-\left(4y^3-\frac{1}{2}\right) &= 0, \end{aligned}$$

from the first equation and solving for $x\left(\frac{1}{2}x^2-3xy+6y^2\right)$ gives

$$x\left(\frac{1}{2}x^2-3xy+6y^2\right)=\left(4y^3-\frac{1}{2}\right)+\frac{1}{2}x.$$

Substituting this into the second equation results in $y-\frac{1}{2}x=0$, thus $y=\frac{1}{2}x$. Substituting this into the first equation, we get $1-x=0$. So the solution is: $(x,y)=\left(1,\frac{1}{2}\right)$.

b) By changing the variable $y\to\frac{1}{2}(X-Y)$ and $x\to X$, we have

$$\begin{aligned} \dot{X} &= -X+X\left(X^2-3X(X-Y)+6(X-Y)^2\right)-\left((X-Y)^3-1\right), \\ \frac{1}{2}\left(\dot{X}-\dot{Y}\right) &= -X+\frac{1}{2}(X-Y)+X\left(\frac{1}{2}X^2-3X\left(\frac{1}{2}(X-Y)\right)+6y^2\right) \\ &\quad -\frac{1}{2}\left((X-Y)^3-1\right), \end{aligned}$$

then,

$$\begin{aligned} \dot{X} &= Y^3-X+1, \\ \dot{Y} &= Y. \end{aligned} \tag{4.10}$$

The solution of $\dot{Y}=Y,\ Y(0)=Y_0$, is

$$Y(t)=Y_0e^t.$$

For $\dot{X}=-X+Y^3+1$ with $Y(t)=Y_0e^{2t}$, we have $\dot{X}=-X+Y_0^3e^{3t}+1$. The general solution of $\dot{X}=-X$, is $X(t)=Ce^{-t}$, the general solution of $\dot{X}=-X+Y^3+1$ is given by

$$X(t)=\frac{1}{4}Y_0^3e^{3t}+\alpha e^{-t}+1.$$

Since $X(0)=X_0$, then $\frac{1}{4}Y_0^3+\alpha+1=X_0$, thus $\alpha=-\frac{1}{4}Y_0^3+X_0-1$. So

$$X(t)=\frac{1}{4}Y_0^3e^{3t}+\left(-\frac{1}{4}Y_0^3+X_0-1\right)e^{-t}+1,$$

the flow of the system (4.10) is

$$\widetilde{\varphi}(X_0,Y_0)=\begin{pmatrix}\frac{1}{4}Y_0^3e^{3t}+\left(-\frac{1}{4}Y_0^3+X_0-1\right)e^{-t}+1\\ Y_0e^t\end{pmatrix}.$$

By changing the variable $Y \to (x-2y)$ and $x \to X$, we have $Y_0 \to (x_0-2y_0)$ and $X_0 \to x_0$, we get

$$\begin{pmatrix}x\\(x-2y)\end{pmatrix}=\begin{pmatrix}\frac{1}{4}(x_0-2y_0)^3e^{3t}+\left(-\frac{1}{4}(x_0-2y_0)^3+x_0-1\right)e^{-t}+1\\ (x_0-2y_0)e^t\end{pmatrix},$$

then,

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\begin{pmatrix}\frac{1}{4}e^{3t}(x_0-2y_0)^3-e^{-t}\left(\frac{1}{4}(x_0-2y_0)^3-x_0+1\right)+1\\ \frac{1}{8}e^{3t}(x_0-2y_0)^3-\frac{1}{2}e^t(x_0-2y_0)-\frac{1}{2}e^{-t}\left(\frac{1}{4}(x_0-2y_0)^3-x_0+1\right)+\frac{1}{2}\end{pmatrix}$$

the flow of the system (4.9) is

$$\varphi_t(x_0,y_0)=\begin{pmatrix}\frac{1}{4}e^{3t}(x_0-2y_0)^3-e^{-t}\left(\frac{1}{4}(x_0-2y_0)^3-x_0+1\right)+1\\ \frac{1}{8}e^{3t}(x_0-2y_0)^3-\frac{1}{2}e^t(x_0-2y_0)-\frac{1}{2}e^{-t}\left(\frac{1}{4}(x_0-2y_0)^3-x_0+1\right)+\frac{1}{2}\end{pmatrix}.$$

d) The linearization

$$Df(x,y)=\begin{pmatrix}3x^2-12xy+12y^2-1 & -6(x-2y)^2\\ \frac{3}{2}x^2-6xy+6y^2-1 & -3x^2+12xy-12y^2+1\end{pmatrix},$$

thus

$$Df\left(1,\frac{1}{2}\right)=\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}.$$

The eigenvaluesare $\lambda_1=-1, \lambda_2=1$.So $\left(1,\frac{1}{2}\right)$ is a saddle point. Since $\lambda_1=-1<0$ simple, there exists a stable manifold of dim=1 and since $\lambda_2=1>0$ is a simple eigenvalue, there exists an unstable manifold of $dim=1$.

The stable manifold of (4.9) is the set of all (x,y) such that

$$\lim_{t\to+\infty}\varphi_t(x_0,y_0)=\left(1,\frac{1}{2}\right)\to(x_0-2y_0)=0.$$

Therefore,

$$W^S\left(1,\frac{1}{2}\right)=\left\{(x_0,y_0)\in\mathbb{R}^2 : x_0-2y_0=0\right\}.$$

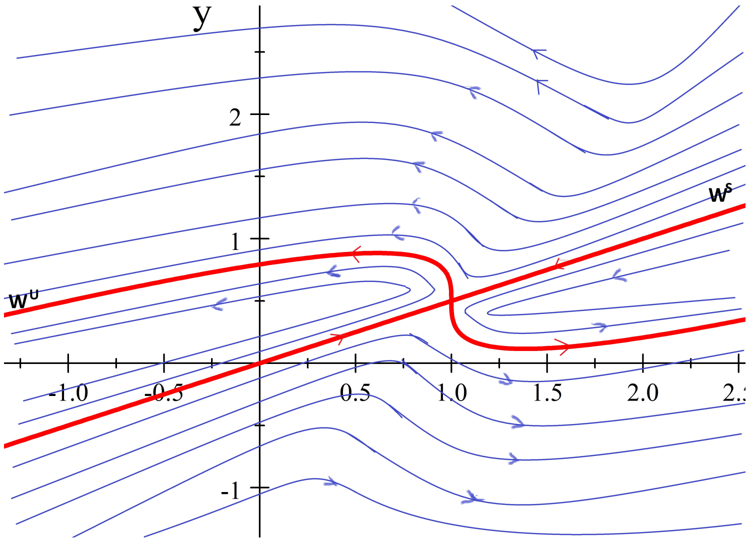
The unstable manifold (4.9) is the set of points (x,y) such that

$$\lim_{t\to-\infty}\varphi_t(x_0,y_0)=\left(1,\frac{1}{2}\right)\to\left(\frac{1}{4}(x_0-2y_0)^3-x_0+1\right)=0.$$

Therefore,

$$W^U\left(1,\frac{1}{2}\right)=\left\{(x_0,y_0)\in\mathbb{R}^2 : \left(\frac{1}{4}(x_0-2y_0)^3-x_0+1\right)=0\right\}.$$

e) The phase portrait is



4.2.2 Additional exercises

Exercise 35 *1. The flow of the autonomous differential system*

$$\begin{cases} \dot{x} = f(x,y), \\ \dot{y} = g(x,y), \end{cases}$$

is given by

$$\varphi_t(x,y) = \begin{pmatrix} \frac{1}{3}e^{2t}y_0^2 + \left(x_0 - \frac{1}{3}y_0^2 - 1\right)e^{-t} + 1 \\ y_0 e^t \end{pmatrix}.$$

Determine the vector field $(f(x,y), g(x,y))^T$.

2. Consider the dynamic system of the form

$$\begin{cases} \dot{x} = (2x+y)^2 - x + 1, \\ \dot{y} = 4x + y - 2(2x+y)^2 - 2. \end{cases} \tag{4.11}$$

(a) *Determine the flow of this system.*

(b) *Look for invariant manifolds (stable and unstable manifolds) near the equilibrium point.*

(c) *Draw the phase portrait of this system.*

5

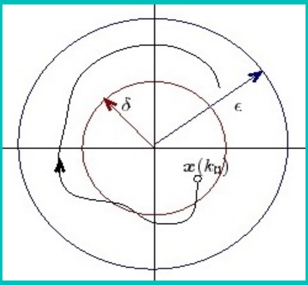
Stability

1 Stability of equilibrium points

A nonlinear system can have several equilibrium positions, which can be stable or unstable. In some situations, equilibrium stability is required, which is defined as follows:

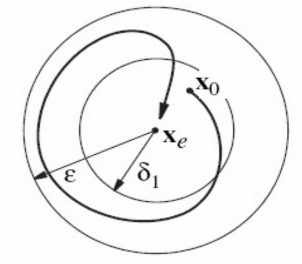
Definition 19 •*An equilibrium point* x^* *of* (4.3) *is stable if given any* $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\|x - x^*\| < \delta \Longrightarrow \|\varphi_t(x) - x^*\| < \varepsilon, \forall t \geq 0$$

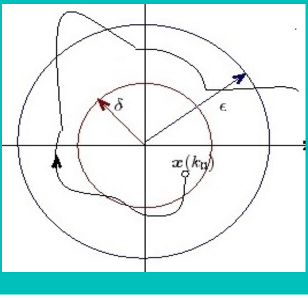


- *An equilibrium point* x^* *of* (4.3) *is is asymptotically stable if it is stable and*

$$\lim_{t \longrightarrow \infty} \|\varphi_t(x) - x^*\| = 0.$$



An equilibrium point x^* *is unstable if it is not stable.*



Example 16 - *A center in $\mathbb{R}^2$ is stable, but not asymptotically stable.*

- *a stable node or focus of a linear system in $\mathbb{R}^2$ is an asymptotically stable equilibrium point,*
- *an unstable node or focus or a saddle of a linear system in $\mathbb{R}^2$ is an unstable equilibrium point.*

Remark 12

- *An equilibrium point x^* of a differential system is said to be asymptotically stable if it is stable and if there exists a neighborhood V of x^* such that any trajectory crossing V converges towards x^* when t tends to infinity.*
- *The notion of asymptotic stability is stronger than the notion of stability.*
- *Asymptotic stability imposes that the limit of the trajectories when $t \to +\infty$ is the equilibrium point, while neutral stability (stable but not asymptotically stable) only imposes that the trajectories remain in a neighborhood of the equilibrium point without necessarily tending towards this point.*

Theorem 11 *(Stability of hyperbolic fixed points).*

- *If x^* is a hyperbolic sink of $A = Df(x^*)$ then it is asymptotically stable.*
- *If x^* is a hyperbolic fixed point with at least one eigenvalue of $A = Df(x^*)$ with $Re(\lambda) > 0$, then it is unstable.*

2 Liapunov Functions

We see that stable equilibrium points that are not asymptotically stable can only occur at nonhyperbolic equilibrium points. But the question as to whether a nonhyperbolic equilibrium point is stable, asymptotically stable, or unstable is a delicate question. The following method, due to Liapunov (in his 1892 doctoral thesis), is very useful in answering this question.

Definition 20 *Let E be an open neighborhood of $\mathbb{R}^n$ containing the origin.*
A real class C^1 function: $V : E \to \mathbb{R}$ is

- *a positive definite function on E if $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$*
- *positive semi-definite function on E if $V(x) \geqslant 0$ for all x*
- *negative definite function on E if $V(0) = 0$ and $V(x) < 0$ if $x \neq 0$*
- *negative semi-definite function on E if $V(x) \leqslant 0$ for all x.*

Remark 13 *The definitions for positive or negative definiteness or semi-definiteness are generalizations of those for quadratic form.*
Here is the definiteness test for planar quadratic form :
Suppose that $Q = Q(x, y)$ *is the quadratic form* $ax^2 + 2bxy + cy^2$*, where* $a, b, c \in \mathbb{R}$*. Then* Q *is*

- *positive definite* $\Longleftrightarrow a, c > 0$ *and* $b^2 < ac$*,*
- *positive semi-definite* $\Longleftrightarrow a, c \geq 0$ *and* $b^2 \leq ac$*,*
- *negative definite* $\Longleftrightarrow a, c < 0$ *and* $b^2 < ac$*,*
- *negative semi-definite* $\Longleftrightarrow a, c \leq 0$ *and* $b^2 \leq ac$*.*

Otherwise, Q *is indefinite.*

Example 17 $V(x, y) = x^2 + y^2$ *is positive definite on* $\mathbb{R}^2$ *and in this case it admits a global minimum at* $(0, 0)$*.*

Remark 14

- *The fact that a function is positive definite (in the sense of definition 20 implies that it admits a minimum at the origin, at least locally.*
- *Level curves associated with a positive definite function* $V(x, y)$*: locations of the points of the plane verifying the equation*

$$V(x, y) = k$$

with $k > 0$ *and* k *small.*

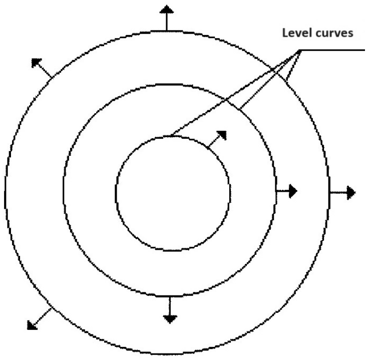


Fig 3 : *The level curves*

- *For a positive definite function, the level curves in the* (x, y) *plane are concentric curves around the origin.*
- *For* $V(x, y) = x^2 + y^2$*, the level curves are concentric circles around the origin.*

Definition 21 $f \in C^1(E), V \in C^1(E)$ *and* φ_t *is the flow of the differential equation* $\dot{x} = f(x)$*, then for* $x \in E$ *the derivative of the function* $V(x)$ *along the solution* $\varphi_t(x)$

$$\dot{V}(x) = \frac{V}{dt}(\varphi_t(x))\bigg|_{t=0} = DV(x)f(x).$$

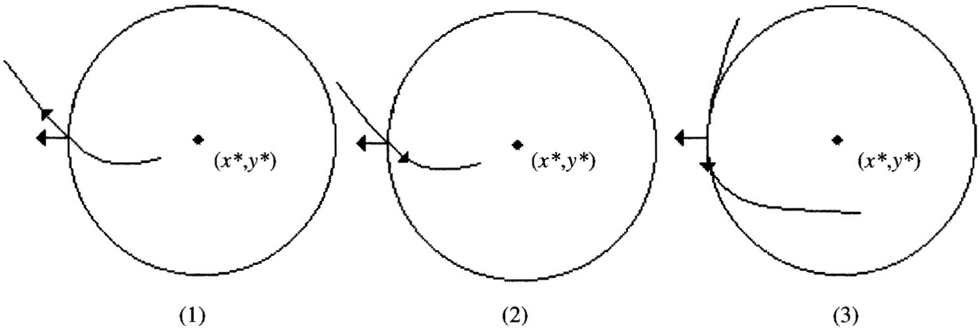
It should also be noted that if V is a positive definite function, then $(0, 0)$ is a minimum of V, and the level curves of V are closed curves in the vicinity of the origin. The speed vector is tangent at any point to the trajectory passing at this point, pointing in the direction of travel of the latter with positive time. The previous dot product can also be written in the following form:

$$\dot{V} = \|\nabla V\| \, \|v\| \cos\theta$$

where θ is the angle made by the gradient vector and the velocity vector. The sign of $\dot{V}$ is therefore that of $\cos\theta$. Three cases can be distinguished:

1. $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, in this case $\cos\theta > 0$ and $\dot{V} > 0$,
2. $-\pi < \theta < -\frac{\pi}{2}$, or $\frac{\pi}{2} < \theta < \pi$ in this case $\cos\theta < 0$ and $\dot{V} < 0$,
3. $\theta = \pm\frac{\pi}{2}$, $\cos\theta = 0$ and $\dot{V} = 0$.

These three cases correspond to three different situations presented in Figure



The three different situations of $\dot{V}$

In case (1), the trajectory cuts the contour curve from the inside to the outside; it is outgoing with respect to the corresponding contour curve. In case (2), the trajectory cuts the contour line from the outside to the inside; it is inbound. Finally, in the last case (3), the trajectory is tangent to the contour curve at this point.

Therefore, it is clear that if on a compact domain D containing the origin, the sign of V is strictly negative everywhere on this domain, then the trajectories are always inward with respect to the contour lines that surround the equilibrium and finally approach it. We can therefore conclude that the asymptotic stability of balance.

On the other hand, if the sign of $\dot{V}$ is strictly positive on a compact domain containing the equilibrium point, the trajectories are everywhere outgoing, and we must expect the origin to be unstable.

Definition 22 *We define a Liapunov function* $V : \mathbb{R}^m \to \mathbb{R}$ *as follows :*

- V *and all its partial derivatives* $\dfrac{\partial V}{\partial x_i}$ *are continuous ;*
- V *is positive definite; that is,* $V(0) = 0$ *and* $V(x) > 0$ *for* $x \neq 0$ *in some neighborhood* $\{x : \|x\| \leq k\}$ *of the origin.*

A Lyapunov function V *for the system (4.7) is said to be*

- *strong if the derivative* $\dot{V}$ *is negative definite; that is,* $V(0) = 0$ *and* $\dot{V}(x) < 0$ *for* $x \neq 0$ *such that* $\|x\| \leq k$.
- *weak if the derivative* $\dot{V}$ *is negative semi-definite; that is,* $V(0) = 0$ *and* $\dot{V}(x) \leq 0$ *for* $x \neq 0$ *such that* $\|x\| \leq k$.

The next theorem, proved in [11].

Theorem 12 *(Liapunov). Let* E *be an open subset of* $\mathbb{R}^n$ *containing* x^*, *let* U *be a neighborhood of* x^* *and let* $U_0 = U \diagdown x^*$. *Suppose that* $f \in C^1(E)$ *and that* $f(x^*) = 0$. *Let us further assume that there exists a real valued function* $V \in C^1(E)$ *satisfying* $V(x^*) = 0$ *and* $V(x) > 0$ *if* $x \neq x^*$. *So*

- *If the derivative of* V *along the orbits is negative in* U_0, *i.e.*

$$\dot{V}(x) = \frac{d}{dt} V(\varphi_t(x)) \Big|_{t=0} = \nabla V.f(x) \leq 0, \forall x \in U_0.$$

  Then, x^* *is stable.*

- *if the derivative of* V *along the orbits is strictly negative*

$$\dot{V}(x) = \nabla V.f(x) < 0, \forall x \in U_0$$

  then x^* *is asymptotically stable.*

- *if the derivative of* V *along the orbits is positive,*

$$\dot{V}(x) = \nabla V.f(x) > 0, \forall x \in U_0$$

  then, x^* *is unstable.*

Example 18 *Consider the system*

$$\begin{cases} \dot{x} = (x-2)^3, \\ \dot{y} = y^3. \end{cases} \tag{5.1}$$

Note that $(2, 0)$ *is an equilibrium point of this system.*
We make the change of variable $u = x - 2, y = v$, *therefore the system*

$$\dot{u} = u^3, \quad \dot{v} = v^3. \tag{5.2}$$

We have

$$Df\,(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The origin is a non hyperbolic equilibrium point of this system and the G-H theorem does not apply, we use the Liapunov function.

$$V(x,y) = x^2 + y^2$$

is a positive definite function i.e., $V(0,0) = 0$ *and* $V(x,y) > 0$ *if* $(x,y) \neq (0,0)$ *and we have*

$$\dot{V}\,(x) = \nabla V.f\,(u,v) = \begin{pmatrix} 2u & 2v \end{pmatrix} \begin{pmatrix} u^3 \\ v^3 \end{pmatrix} = 2u^4 + 2v^4 > 0$$

on $\mathbb{R}^2 \setminus \{(0,0)\}$, *The origin is therefore a point of unstable equilibrium of the system* (5.2) *and consequently* $(2,0)$ *is a point of unstable equilibrium of the system* (5.1).

Example 19 *Give sufficient conditions on the parameters of the Lorenz equations*

$$\begin{cases} \dot{x} = \delta\,(x-y)\,, \\ \dot{y} = rx - y - xz, \\ \dot{z} = -bz + xy, \end{cases}$$

so that all the orbits converge towards the point $(0,0,0)$ *(the origin is said to be globally asymptotically stable).*
Let

$$V(X) = C_1x^2 + C_2y^2 + C_3z^2$$

with C_1, C_2 *and* C_3 *being positive constants.*
By calculating $V(x) = DV(x)f(x)$, *we find*

$$\begin{aligned} \dot{V}(X) &= \begin{pmatrix} 2C_1x & 2C_2y & 2C_3z \end{pmatrix} \begin{pmatrix} \delta\,(x-y) \\ rx - y - xz \\ -bz + xy \end{pmatrix} \\ &= 2\delta C_1x^2 + (2C_3 - 2C_2)\,xyz + (2rC_2 - 2\delta C_1)\,xy + (-2C_2)\,y^2 + (-2bC_3)\,z^2. \end{aligned}$$

If we take

$$\begin{aligned} (2C_3 - 2C_2) &= 0, \\ (2rC_2 - 2\delta C_1) &= 0, \\ \delta &< 0, \end{aligned}$$

then, we obtain

$$r = \delta\frac{C_1}{C_2}, C_3 = C_2 > 0, b > 0,$$

so

$$V(X) = C_1x^2 + C_2y^2 + C_2z^2$$

and

$$\dot{V}(X) = 2\delta C_1x^2 - 2C_2y^2 - 2bC_3z^2 < 0$$

on $\mathbb{R}^3 \setminus \{(0,0,0)\}$, *then, the origin is asymptotically stable.*

Remark 15 *Consider the second-order differential equation*

$$\ddot{x} + q(x) = 0$$

where the continuous function $q(x)$ *satisfies* $xq(x) > 0$ *for* $x \neq 0$. *This differential equation can be written as the system*

$$\dot{x} = x, \qquad \dot{y} = -q(x).$$

The total energy of the system

$$V(x) = \frac{y^2}{2} + \int_0^x q(s)ds$$

(which is the sum of the kinetic energy $\frac{1}{2}\dot{x}^2$ *and the potential energy) serves as a Liapunov function for this system.*

$$\dot{V}(x) = q(x)y + y\left(-q(x)\right) = 0.$$

The solution curves are given by $V(x) = c$*; i.e., the energy is constant on the solution curves or trajectories of this system; and the origin is a stable equilibrium point.*

3 Exercises

5.3.1 Corrected exercises

Exercise 36 *Determine the stability of the critical points of*

$$\begin{cases} \dot{x} = y(x+1), \\ \dot{y} = x(1+y^3). \end{cases}$$

Solution. There are two critical points for this system: $(0,0)$ and $(-1,-1)$. At $(0,0)$, the linearization is

$$\dot{x} = y, \qquad \dot{y} = x.$$

The eigenvalues of the coefficient matrix are ±1. Hence the origin is a saddle point for the linearized system, the origin is unstable.

Near the other critical point $(-1,-1)$, we shift it to the origin by making

$$x = -1 + u, \qquad y = -1 + v.$$

Then (u, v) satisfies the following system

$$\begin{cases} \dot{u} = (v-1)\,u \\ \dot{v} = (u-1)\left(1 + (v-1)^3\right). \end{cases}$$

The linearization at $(0,0)$ of this system is

$$\dot{u} = -u, \qquad \dot{v} = -3v,$$

whose eigenvalues of the coefficient matrix are -1 and -3. The nonlinear system is strictly stable at $(0,0)$. So the original system is strictly stable at $(-1,-1)$.

Exercise 37 *Show that for the differential system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\dfrac{1}{m}(f(x, y)y + \lambda x), \end{cases}$$

the critical point $(0, 0)$ *is stable where* $m > 0, \lambda > 0$ *and* $f(x, y) \geq 0$ *in a neighborhood of the origin.*

Solution. In fact, take

$$V(x, y) = \frac{1}{2}(\lambda x^2 + my^2).$$

Then

$$\dot{V} = \lambda xy - my\frac{1}{m}\left(f(x, y)y + \lambda x\right) = -my^2 f(x, y) \leq 0.$$

By Liapunov theorem, the origin is stable.

Exercise 38 *Consider the following system of ordinary differential equations*

$$\begin{cases} \dot{x} = \ -x + y\,(1 + x)\,, \\ \dot{y} = x\,(1 - x) - y\left(1 + y^2\right), \end{cases}$$

1. Find the (unique) equilibrium point (x^*, y^*) *of this system.*
2. Linearize the system about (x^*, y^*)*. Can you reach any conclusions about the stability of* (x^*, y^*)*?*
3. Write down the Liapunov theorem that gives sufficient conditions for the stability of an equilibrium point.
4. Apply the previous theorem to show that the equilibrium solution (x^*, y^*) *is, indeed, stable.*

Solution.

1. The fixed point is $(x^*, y^*) = (0, 0)$.

2. We compute the Jacobian for the above system at the fixed point:

   $$J(x; y) = \begin{pmatrix} -1 + y & 1 + x \\ 1 - 2x & -1 - 3y^2 \end{pmatrix}.$$

   For $(x^*, y^*) = (0, 0), \lambda_{1,2} = -2, 0, v_1 = (1, 1), v_2 = (1, -1)$. Stable in one direction, but marginal in the other.

3. **Liapunov stability theorem:** Consider the system $\dot{x} = F(x)$ with a fixed point at the origin. If there exists a real valued function $V(x)$ in a neighborhood $\mathcal{N}(0)$ such that:

   i) the partial derivatives $\dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial y}$ exist and are continuous,

   ii) the function $V(x)$ is positive definite,

   iii) $\dfrac{dV}{dt}$ is negative semi-definite (definite),

   then the origin is a stable (asymptotically stable) fixed point.

4. by showing that the function $V(x,y) = ax^2 + 2y^2, a > 0$, condition $V(0) = 0$ is satisfied, since when $x = 0$ and $y = 0$, then $V(x,y) = 0$.

   Condition (1) is also satisfied since both terms are positive if we choose $a > 0$. This makes $V(x,y) > 0$ for non zero x, y. We now need to check the third condition. This condition is always the hardest one to check. The orbital derivative $\dot{V}(x,y)$ is

$$\begin{aligned} \dot{V}(x,y) &= 2ax\dot{x} + 4y\dot{y} \\ &= 2ax(-x + y + yx) + 4y(x - y - x^2 - y^3) \\ &= \left(-2ax^2 - 4y^4 - 4y^2\right) + \left(2axy + 2ax^2y - 4yx^2 + 4xy\right) \end{aligned}$$

   We see that if we choose $a \geq 0$ then the first term above, which is $\left(-2ax^2 - 4y^4 - 4y^2\right)$ is always negative (or negative semidefinite for $x = 0, x = 0$) and can not be positive.

   For $a = 2$

$$\dot{V}(x,y) = 8xy - 4y^2 - 4x^2 - 4y^4 = -4(x-y)^2 - 4y^4 < 0 \quad if \ (x,y) \neq (0,0).$$

   Therefore, $(x^*, y^*) = (0,0)$ is asymptotically stable.

Exercise 39 *Consider the dynamical system of the form*

$$\ddot{y} + \dot{y} + y^3 = 0.$$

(i) Show that the origin is the only fixed point of the system.
(ii) State the linearization theorem and judge whether it is possible to draw conclusions from it concerning the stability of the fixed point.
(iii) State the Liapunov stability theorem and deduce from it the stability properties for the fixed point.

Solution. Defining $\dot{y} = x, x = x$, we obtain

$$\dot{x} = -x - y^3, \qquad \dot{y} = x.$$

By substituting $(0,0)$ into the right-hand sides of the equations, one immediately sees that it is an equilibrium. Setting the time derivatives to zero and multiplying the first equation by y and the second by x and subtracting the results, we get

$$x\,(x) - y\left(-x - y^3\right) = 0 \Longrightarrow x^2 + xy + y^4 = 0$$

for any equilibrium. It follows that the origin is the only equilibrium.
We compute the Jacobian for the above system at the fixed point:

$$J(0;0) = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$$

we have $\det J(0;0) = 0 \Longrightarrow$ non-simple linearization.
The linearization theorem can not be applied since the system is non-simple.

Liapunov stability theorem: Consider the system $\dot{x} = F(x)$ with a fixed point at the origin. If there exists a real valued function $V(x)$ in a neighborhood $\mathcal{N}(0)$ such that:

i) the partial derivatives $\dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial y}$ exist and are continuous,

ii) the function $V(x)$ is positive definite,

iii) $\dfrac{dV}{dt}$ is negative semi-definite (definite),

then the origin is a stable (asymptotically stable) fixed point.
By showing that the function

$$V(x,y) = 2x^2 + 2xy + y^2 + y^4 > 0 \text{ in } \mathbb{R}^2 \setminus \{0\}$$

the partial derivatives $\dfrac{\partial V}{\partial x} = 4x + 2y, \dfrac{\partial V}{\partial y} = 4y^3 + 2y + 2x$ exist and are continuous, and the function $V(x)$ is positive definite,

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dx}\frac{dx}{dt} + \frac{dV}{dy}\frac{dy}{dt} \\ &= -2x^2 - 2y^4 < 0 \qquad\qquad if\ (x,y) \neq (0,0). \end{aligned}$$

origin is an asymptotically stable fixed point .

Exercise 40 *Construct a strong Liapunov function for the following regular system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -2x - 3(1+y^2)y, \end{cases}$$

and justify your result.

Solution. To construct a strong Liapunov function, we first try to make a linear equivalent transform

$$\begin{pmatrix} x \\ y \end{pmatrix} = K \begin{pmatrix} u \\ v \end{pmatrix}$$

where K is the matrix formed by two linearly independent eigenvectors of the coefficient matrix:

$$K = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} &= K^{-1}\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = K^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} \\ &= K^{-1}AK\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \end{aligned}$$

For the latter system, the function $u^2 + v^2$ is a strong Liapunov function. Since the inverse transform gives $u = -x - y$ and $v = 2x + y$, thus, we have a strong Liapunov function of the original nonlinear system:

$$V(x,y) = (-x-y)^2 + (2x+y)^2 = 5x^2 + 6xy + 2y^2.$$

To see that it is indeed a strong Liapunov function, we only need to check that

$$\begin{aligned} \dot{V}(x,y) &= (10x+6y)y + (6x+4y)\left(-2x - 3(1+y^2)y\right) \\ &= -2\left(6x^2 + 8xy + 3y^2 + 9xy^3 + 6y^4\right) \\ &\leq -\left(6x^2 + 8xy + 3y^2\right) < 0. \end{aligned}$$

Here the inequalities hold since $9xy^3 + 6y^4$ is of higher order than 2 and

$$\left(6x^2 + 8xy + 3y^2\right) = (-x-y)^2 + (2x+y)^2 + (x+y)^2$$

is positive definite.

Exercise 41 *We consider the trivial differential system $\dot{X}(t) = 0$, where $X(t) \in \mathbb{R}^2$. Determine equilibria and their stability (unstable, stable, asymptotically stable).*

Solution. The system is written as $\dot{X}(t) = F(X(t))$ where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is the zero function. By definition, equilibria are points X of $\mathbb{R}^2$ such that $F(X) = 0$. So here all points of $\mathbb{R}^2$ are equilibria. Furthermore, they are all stable, and none is asymptotically stable.
Indeed, let X^* be an equilibrium and $(J; X(.))$ a maximal solution. Note that $J = \mathbb{R}$ and $X(t) = X(0)$ for all $t \in \mathbb{R}$. Stability: for all $\varepsilon > 0$, there exists $\alpha > 0$ such that if $\|X(0) - X^*\| < \alpha$ then for all $t \geq 0$, $\|X(t) - X^*\| < \varepsilon$ Just take $\alpha = \varepsilon$. So X is stable. Absence of asymptotic stability: i.e. $\varepsilon > 0$. There exists $X_0 \neq X^*$ such that

$$\|X_0 - X^*\| < \varepsilon.$$

If $X(0) = X_0$ then for all $t \in \mathbb{R}$,

$$\|X(t) - X^*\| = \|X_0 - X^*\| \neq 0.$$

In particular, we do not have $\|X(t) - X^*\| \to 0$. So X is not attractive and therefore not asymptotically stable.

5.3.2 Additional exercises

Exercise 42 *Let the equation*

$$\ddot{x} + b\,(\dot{x})^3 + x = 0.$$

a) *Reduce this equation to a system of first order differential equations.*

b) *Linearize the systems in the vicinity of the equilibrium point, conclude.*

c) *Study the stability in the vicinity of the equilibrium point, using the Liapunov function.*

Exercise 43 *Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined in polar coordinates $r \geq 0, 0 \leq \theta < 2\pi$ by*

$$f(r, \theta) = (r^2, \theta - \sin\theta).$$

1. *Show that $(r, \theta) = (0, 0), (r, \theta) = (1, 0), (r, \theta) = (1, \pi)$ are the only fixed points of the map.*
2. *Show that the fixed point $(1, 0)$ is a saddle point. Determine the stability of the other two fixed points and classify them as sink, source, or saddle point.*
3. *Show that the unstable manifold of the saddle point $(1, 0)$ is the set of points*

   $$U = \{(t, 0) : t > 0\}.$$

   Show that all orbits starting outside the unit circle are attracted towards infinity, and that all orbits starting inside the unit circle are attracted towards the origin. Given this information, describe (without providing a proof) the stable manifold of the saddle point $(1, 0)$.
4. *Sketch the system in the (x, y)-plane $(x = r\cos\theta, y = r\sin\theta)$, indicating the three fixed points, the unstable and stable manifold of the saddle point $(1, 0)$ and the general direction of orbits inside and outside the unit circle.*

Exercise 44 1. *Show that the critical point* $(0,0)$ *is stable to the following regular system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -2x - 3(1+y^2)y, \end{cases}$$

by Liapunov function.

2. *Show that the critical point* $(0,0)$ *is attractive to the system.*

Exercise 45 *Consider the dynamical system of the form*

$$\begin{cases} \dot{x} = -x - y^3, \\ \dot{y} = x. \end{cases}$$

1. *Show that the origin is the only fixed point of the system.*
2. *State the linearization theorem and judge whether it is possible to draw conclusions from it concerning the stability of the fixed point.*
3. *State the Liapunov stability theorem and deduce from it the stability properties for the fixed point by showing that the function*

$$V(x,y) = 2x^2 + 2xy^2 + y^2 + y^4$$

is a strong Liapunov function for the above system.

Exercise 46 *Consider the following second order differential equation*

$$\ddot{x} + \dot{x} + x - \dot{x}x^2 = 0.$$

1. *Find a suitable transformation of variables that changes this equation into a system of two first order differential equations.*
2. *Determine all fixed points of the system. State the linearization theorem and judge whether it is possible to draw conclusions from it concerning the stability of the fixed points and the qualitative behavior of the system near the fixed points.*
3. *State the Liapunov stability theorem and deduce from it the stability properties for the fixed point first by showing that the function*

$$V(x,y) = x^2 + y^2$$

is a weak Liapunov function for the above system. Use the extension of the Lyapunov theorem to draw a stronger conclusion. Determine the domain of stability.

Exercise 47 *Consider the following dynamical system*

$$\begin{cases} \dot{x} = -8x - xy^2 - 3y^3, \\ \dot{y} = 2xy^2 + 2x^2y. \end{cases}$$

1. *Determine all fixed points of the system. State the linearization theorem and judge whether it is possible to draw conclusions from it concerning the stability of the fixed points and the qualitative behavior of the system near the fixed points.*

2. *State the Liapunov stability theorem and deduce from it the stability properties for the fixed point at the origin by showing that the function*

$$V(x,y) = 2x^2 + 3y^2$$

is a weak Liapunov function for the above system.

3. *State the extension of the Liapunov stability theorem and argue that the origin is an asymptotically stable fixed point.*

4. *Show that the domain of Liapunov stability is bounded by the ellipse*

$$2x^2 + 3y^2 = 12.$$

Exercise 48 *Consider the dynamical system of the form*

$$\begin{cases} \dot{x} = -3x - \dfrac{1}{2}y^3 + xy^2, \\ \dot{y} = 2xy^2 + 2yx^2. \end{cases}$$

1. *State the Liapunov stability theorem and the definitions for a weak and strong Liapunov function. Show that the function*

$$V(x,y) = 8x^2 + 2y^2$$

is a weak Liapunov function for the system specified above. Deduce from this the stability properties of the fixed point.

2. *Determine the length of the major and minor axis of the ellipse which confines the maximal domain of stability.*

3. *State the extension of the Liapunov stability theorem and conclude from it that the fixed point is asymptotically stable.*

Exercise 49 *Consider the dynamical system of the form*

$$\begin{cases} \dot{x} = xy^2 - 9x - 16y^3, \\ \dot{y} = 4xy^2 + 2yx^2. \end{cases}$$

1. *State the Liapunov stability theorem and the definitions for a weak and strong Liapunov function. Show that the function*

$$V(x,y) = 4x^2 + 16y^2$$

is a weak Liapunov function for the dynamical system specified above. Deduce from this the stability properties of the fixed point.

2. *Determine the length of the major and minor axis of the ellipse which confines the maximal domain of stability.*
3. *state the extension of the Liapunov stability theorem and conclude from it that the fixed point is asymptotically stable.*

Exercise 50 *Find the complete solution to the differential equation*

$$\dot{x} = ax(1-x), \quad x(0) = x_0.$$

Assuming $a > 0$*, which are the stationary point and are they (asymptotically) stable? Are they exponentially (un)stable ?*

Exercise 51 *Consider the following ODE on the first (on-negative) quadrant of* $\mathbb{R}^2$*:*

$$\begin{cases} \dot{x} = a_1 x - a_2 xy, \\ \dot{y} = a_2 xy - a_3 y, \end{cases} \tag{5.3}$$

where $a_1, a_2, a_3 > 0$*.*

1. *Find the equilibrium points and their types (sink, saddle, source, center) of* (5.3)*.*
2. *Show that*
$$L(x;y) = a_2(x+y) - a_1 - a_3 - a_3 \ln \frac{a_2 x}{a_3} - a_1 \ln \frac{a_2 y}{a_1}$$
is a Liapunov function (but never strict). Hence sketch the phase portrait of (5.3)*.*
3. *Using the change of coordinates* $u = \ln \frac{a_2 y}{a_1}, v = \ln \frac{a_2 x}{a_3}$*, show that* (5.3) *is in fact a Hamiltonian system.*

6

Critical Points in R^2

1 Saddles, Nodes, Foci, and Centers

In this section we let $X = (x, y)^T$, and

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y), \end{cases} \tag{6.1}$$

If we let $r^2 = x^2 + y^2$ and $\theta = \arctan \frac{y}{x}$, then we have

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ \dot{\theta} &= \frac{\left(\frac{y}{x}\right)'}{\left(1 + \frac{y}{x}\right)^2} = \frac{\dot{y}x - \dot{x}y}{x^2 + y^2} = \frac{\dot{y}x - \dot{x}y}{r^2}, \end{aligned}$$

so

$$r^2\dot{\theta} = \dot{y}x - \dot{x}y.$$

It follows that for $r > 0$, the non-linear system (6.1) can be written in terms of polar coordinates as

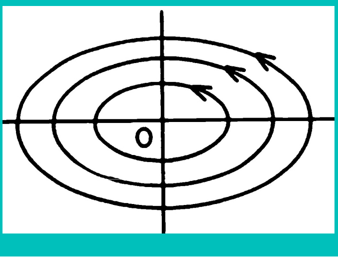
$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} = r\cos\theta P\left(r\cos\theta, r\sin\theta\right) + r\sin\theta Q\left(r\cos\theta, r\sin\theta\right) \\ r^2\dot{\theta} &= \dot{y}x - \dot{x}y = r\cos\theta Q\left(r\cos\theta, r\sin\theta\right) - r\sin\theta P\left(r\cos\theta, r\sin\theta\right). \end{aligned}$$

or as

$$\frac{dr}{d\theta} = \frac{r\left(\cos\theta P\left(r\cos\theta, r\sin\theta\right) + \sin\theta Q\left(r\cos\theta, r\sin\theta\right)\right)}{\cos\theta Q\left(r\cos\theta, r\sin\theta\right) - \sin\theta P\left(r\cos\theta, r\sin\theta\right)}.$$

Writing the system of differential equations (6.1) in polar coordinates will often reveal the nature of the equilibrium point or critical point at the origin.

Definition 23 *The origin is called a center for the non-linear system* (6.1) *if there exists a* $\delta > 0$ *such that every solution curve of* (6.1) *in the deleted neighborhood* $N_\delta(0) - \{0\}$ *is a closed curve with* 0 *in its interior.*



Example 20 *Write the system*

$$\begin{cases} \dot{x} = -y - xy, \\ \dot{y} = x + x^2, \end{cases}$$

in polar coordinates. For $r > 0$ we have

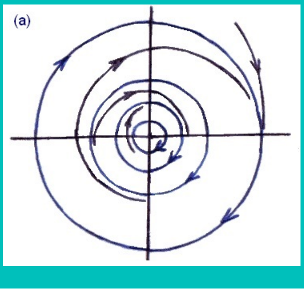
$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{-xy - x^2y + xy + x^2y}{r} = 0,$$

and

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x^2 + x^3 + y^2 + xy^2}{r^2} = 1 + x > 0,$$

for $x > -1$.
Thus, along any trajectory of this system in the half plane $x > -1, r(t)$ is constant and $\theta(t)$ increases without bound as $t \to \infty$, then the origin is a center for this non-linear system.

Definition 24 *The origin is called a center-focus for (6.1) if there exists a sequence of closed solution curves Γ_n, with Γ_{n+1} in the interior of Γ_n such that $\Gamma_n \to 0$ as $n \to \infty$ and such that every trajectory between Γ_n and Γ_{n+1} spirals towards Γ_n or Γ_{n+1} as $t \to \pm\infty$.*



Example 21 *Consider the system*

$$\begin{cases} \dot{x} = -y + x\sqrt{x^2 + y^2}\sin\left(\dfrac{1}{\sqrt{x^2 + y^2}}\right), \\ \dot{y} = x + y\sqrt{x^2 + y^2}\sin\left(\dfrac{1}{\sqrt{x^2 + y^2}}\right). \end{cases}$$

In polar coordinates, for $r > 0$, we have

$$\dot{r} = r^2 \sin\frac{1}{r^2}, \qquad \dot{\theta} = 1.$$

For $r > 0$ with $\dot{r} = 0$ at $r = 0$. Clearly, $\dot{r} = 0$ for $r = \frac{1}{n\pi}$; i.e., each of the circles $r = \frac{1}{n\pi}$ is a trajectory of this system. Furthermore, for $n\pi < \frac{1}{r} < (n+1)\pi$, we have
$\dot{r} < 0$ if n is odd and,
$\dot{r} > 0$ if n is even; i.e.,
the trajectories between the circles $r = \frac{1}{n\pi}$ spiral inward or outward to one of these circles.
Thus, we see that the origin is a center-focus for this nonlinear system according to definition 24 above.

Definition 25 *The origin is called*

- *a stable focus for* (6.1) *if there exists a* $\delta > 0$ *such that for* $0 < r_0 < \delta$ *and* $\theta \in \mathbb{R}$, $r(t, r_0, \theta_0) \to 0$ *and* $|\theta(t, r_0, \theta_0)| \to +\infty$ *as* $t \to +\infty$.
- *It is called an unstable focus if* $r(t, r_0, \theta_0) \to 0$ *and* $|\theta(t, r_0, \theta_0)| \to +\infty$ *as* $t \to -\infty$.

Any trajectory of (6.1) *which satisfies* $r(t) \to 0$ *and* $|\theta(t)| \to \infty$ *as* $\to \pm\infty$ *is said to spiral towards the origin as* $t \to \pm\infty$

Example 22 *Consider the system*

$$\begin{cases} \dot{x} = -y + x^3 + xy^2, \\ \dot{y} = x + y^3 + x^2 y. \end{cases}$$

In polar coordinates, we have

$$\dot{r} = r^3, \quad \dot{\theta} = 1,$$

the solution of this equation with $r(0) = 0, \quad \theta(0) = \theta_0$, *is given by*

$$\begin{aligned} r(t) &= \frac{r_0}{\sqrt{1 - 2tr_0^2}}, \quad t < \frac{1}{2r_0^2} \\ \theta\,(t) &= t + \theta_0 \end{aligned}$$

we have $r(t) \to 0$ *and* $|\theta\,(t)| \to \infty$ *as* $t \to -\infty$, *therefore, the origin is an unstable focus for this nonlinear system.*

Example 23 *Consider the system*

$$\begin{cases} \dot{x} = -y - x^3 - xy^2, \\ \dot{y} = x - y^3 - x^2 y, \end{cases}$$

In polar coordinates, for $r > 0$, *we have*

$$\dot{r} = -r^3, \qquad \dot{\theta} = 1.$$

The solution of this system with $r(0) = 0, \quad \theta(0) = \theta_0$, *is given by*

$$\begin{aligned} r(t) &= \frac{r_0}{\sqrt{1 + 2tr_0^2}}, \quad t > -\frac{1}{2r_0^2}, \\ \theta\,(t) &= t + \theta_0, \end{aligned}$$

we have $r(t) \to 0$ *and* $|\theta\,(t)| \to \infty$ *as* $t \to +\infty$. *Then the origin is a stable focus for this nonlinear system.*

Definition 26 *The origin is called*

- *a stable node for* (6.1) *if there exists a* $\delta > 0$ *such that for* $0 < r_0 < \delta$ *and* $\theta \in \mathbb{R}$, $r(t, r_0, \theta_0) \to 0$ *as* $t \to \infty$ *and* $\lim_{t\to\infty} \theta(t, r_0, \theta_0)$ *exists; i.e., each trajectory in a deleted neighborhood of the* $t \to \infty$ *origin approaches the origin along a well-defined tangent line as* $t \to \infty$.
- *an unstable node if* $r(t, r_0, \theta_0) \to 0$ *as* $t \to \infty$ *and* $\lim_{t\to\infty} \theta(t, r_0, \theta_0)$ *exists for all* $r_0 \in (0, \delta)$ *and* $\theta \in \mathbb{R}$.
- *a proper node for* (6.1) *if it is a node and if every ray through the origin is tangent to some trajectory of* (6.1).

Example 24 *Consider the system*

$$\begin{cases} \dot{x} = x - xy, \\ \dot{y} = y + x^2, \end{cases}$$

In polar coordinates, for $r > 0$, *we have*

$$\dot{r} = r, \qquad \dot{\theta} = r\cos\theta$$

the solution of $\dot{r} = r$ *with* $r(0) = 0$, *is given by*

$$r(t) = r_0 e^t,$$

and the solution of $\dot{\theta} = r\cos\theta$, *is given by*

$$\dot{\theta} = r\cos\theta \Rightarrow \theta(t) = \arcsin\frac{C_3 e^{2r_0 e^t} - 1}{C_3 e^{2r_0 e^t} + 1},$$

since $\theta(0) = \theta_0$, *then*

$$\theta_0 = \arcsin\frac{C_3 e^{2r_0} - 1}{C_3 e^{2r_0} + 1} \Rightarrow C_3 = \left(\frac{(\sin\theta_0 + 1)}{e^{2r_0}(1 - \sin\theta_0)}\right).$$

Thus

$$\begin{aligned} r(t) &= r_0 e^t, \\ \theta(t) &= \arcsin\frac{\left(\frac{(\sin\theta_0+1)}{e^{2r_0}(1-\sin\theta_0)}\right) e^{2r_0 e^t} - 1}{\left(\frac{(\sin\theta_0+1)}{e^{2r_0}(1-\sin\theta_0)}\right) e^{2r_0 e^t} + 1}, \end{aligned}$$

we have $r(t) \to 0$ *and* $|\theta(t)| \to \arcsin\dfrac{\left(\frac{(\sin\theta_0+1)}{e^{2r_0}(1-\sin\theta_0)}\right) - 1}{\left(\frac{(\sin\theta_0+1)}{e^{2r_0}(1-\sin\theta_0)}\right) + 1} =$ *exist as* $t \to -\infty$. *Then the origin is an unstable node for this nonlinear system.*

Definition 27 *The origin is a (topological) saddle for* (6.1) *if there exists two trajectories* Γ_1, *and* Γ_2 *which approach* 0 *as* $t \to +\infty$ *and two trajectories* Γ_3 *and* Γ_4 *which approach 0 as* $t \to -\infty$ *and if there exists a* $\delta > 0$ *such that all other trajectories which start in the deleted neighborhood of the origin* $N_\delta(0) - \{0\}$ *leave* $N_\delta(0)$ *as* $t \to \pm\infty$. *The special trajectories* $\Gamma_1, ..., \Gamma_4$ *are called separatrices.*

Definition 28 *Bendixson).Let* E *be an open subset of* $\mathbb{R}^2$ *containing the origin, and let If* $f \in C^1(E)$. *If the origin is an isolated critical point of* (6.1), *then either every neighborhood of the origin contains a closed solution curve with* 0 *in its interior, or there exists a trajectory approaching* 0 *as* $t \to \pm\infty$.

The next theorem follows immediately from the Stable Manifold Theorem and the Hartman-Grobman Theorem

Theorem 13 *Suppose that* E *is an open subset of* $\mathbb{R}^2$ *containing the origin and that* $f \in C^1(E)$. *If the origin is a hyperbolic equilibrium point of the nonlinear system* (6.1), *then the origin is a (topological) saddle for* (6.1) *if and only if the origin is a saddle for the linear system*

$$\dot{x} = Ax, \tag{6.2}$$

with $A = Df(0)$.

. The next theorem, proved in [1]

Theorem 14 *Let* E *be an open subset of* $\mathbb{R}^2$ *containing the origin and let* $f \in C^2(E)$. *Suppose that the origin is a hyperbolic critical point of* (6.1). *Then the origin is*

- *a stable (or unstable) node for the nonlinear system* (6.1) *if and only if it is a stable (or unstable) node for the linear system* (6.2) *with* $A = Df(0)$.
- *a stable (or unstable) focus for the nonlinear system* (6.1) *if and only if it is a stable (or unstable) focus for the linear system* (6.2) *with* $A = Df(0)$.

Theorem 15 ***(Perko [11]).*** *Let* E *be an open subset of* $\mathbb{R}^2$ *containing the origin, and let* $f \in C^1(E)$ *with* $f(0) = 0$. *Suppose that the origin is a center for the linear system* (6.2) *with* $A = Df(0)$. *Then the origin is either a center, a center-focus or a focus for the nonlinear system* (6.1).

Corollary 2 ***(Perko [11]).*** *Let* E *be an open subset of* $\mathbb{R}^2$ *containing the origin, and let* f *be analytic in* E *with* $f(0) = 0$. *Suppose that the origin is a center for the linear system* (6.2) *with* $A = Df(0)$. *Then the origin is either a center or a focus for the nonlinear system* (6.1).

Yet another approach is to look for symmetries in the differential equations. The easiest symmetries to see are symmetries with respect to the x and y axes.

Definition 29 *The system* (6.1) *is said to be symmetric with respect to the x-axis if it is invariant under the transformation* $(t, y) \to (-t, -y)$*; it is said to be symmetric with respect to the y-axis if it is invariant under the transformation* $(t, x) \to (-t, -x)$.

Example 25 *The system*

$$\begin{cases} \dot{x} = -y - x^4, \\ \dot{y} = x + x^3 y, \end{cases}$$

is symmetric with respect to the x-axis because

$$\begin{cases} (-x\,(-t))' = x\,(-t)' = -y\,(-t) + (-x(-t))^4, \\ (y\,(-t))' = -y\,(-t)' = (-x\,(-t)) + (-x\,(-t))^3\, y\,(-t)\,. \end{cases}$$

Thus

$$\begin{cases} x\,(-t)' = -y\,(-t) + x\,(-t)^4\,, \\ y\,(-t)' = (x\,(-t)) + (x\,(-t))^3\, y\,(-t)\,. \end{cases}$$

So

$$\dot{x} = -y + x^2, \;\; \dot{y} = x - x^5 y.$$

Theorem 16 *Let E be an open subset of $\mathbb{R}^2$ containing the origin, and let $f \in C^1(E)$ with $f(0) = 0$. If the nonlinear system* (6.1) *is symmetric with respect to the x-axis or the y-axis, and if the origin is a center for the linear system* (6.2) *with $A = Df(0)$, then the origin is a center for the nonlinear system* (6.1).

Proof. By theorem 15, any trajectory of (6.1) in $N_\delta(0)$ which crosses the positive x-axis will also cross the negative x-axis. If the system (6.1) is symmetric with respect to the x-axis, then the trajectories of (6.1) in $N_\delta(0)$ will be symmetric with respect to the x-axis, and hence all trajectories of (6.1) in $N_\delta(0)$ will be closed; i.e., the origin will be a center for (6.1). ■

2

Nonhyperbolic Critical Points in $\mathbf{R}^2$

We assume that the origin is an isolated critical point of the planar system

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y), \end{cases} \tag{6.3}$$

where P and Q are analytic in a neighborhood of the origin.

Definition 30

- *The semi-hyperbolic are the singular points having a unique eigenvalue equal to zero.*
- *The nilpotent singular points have both eigenvalues zero, but their linear part is not identically zero.*

In this section we give some results for the case where the matrix A has one or two zero eigenvalues, but $A \neq 0$.

Definition 31

- *A sector which is topologically equivalent to the sector shown in Figure 4(a) is called a hyperbolic sector.*
- *A sector which is topologically equivalent to the sector shown in Figure 4(b) is called a parabolic sector.*
- *And a sector which is topologically equivalent to the sector shown in Figure 4(c) is called an elliptic sector.*

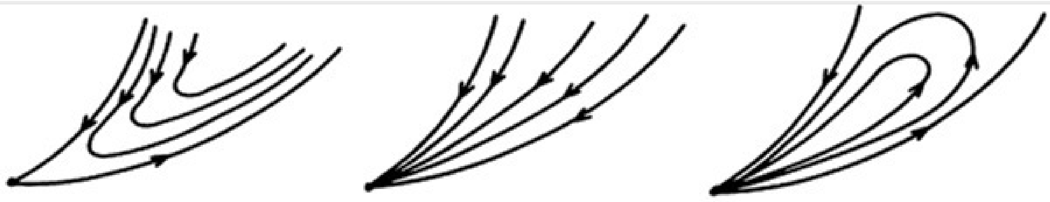


Figure 4. (a) A hyperbolic sector. (b) A parabolic sector. (c) An elliptic sector.

Definition 32 *A neighborhood of the origin consists of*

- *one elliptic sector,*
- *one hyperbolic , sector,*
- *two parabolic sectors and*
- *four separatrices.*

This type of critical point is called a critical point with an elliptic domain

Example 26 *The system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^3 + 4xy, \end{cases} \tag{6.4}$$

has an elliptic sector at the origin. Every trajectory which approaches the origin does so tangent to the x-axis. The phase portrait for this system is shown in the following figure

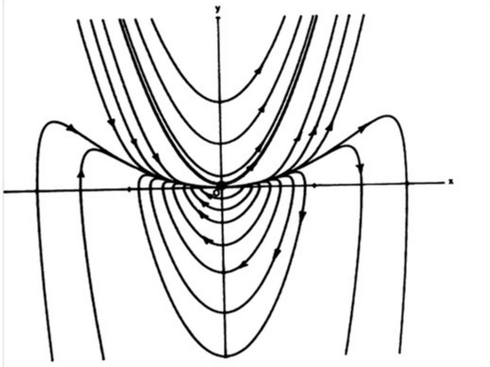


Figure 2. A critical point with an elliptic domain at the origin.

Fig 5 : *The phase portrait of (6.4)*

Another type of nonhyperbolic critical point for a planar system is a saddle-node

Definition 33 *A saddle-node consists of*

- *two hyperbolic sectors and*
- *one parabolic sector*

(as well as three separatrices and the critical point itself).

Example 27 *The system*

$$\begin{cases} \dot{x} = x^2, \\ \dot{y} = y, \end{cases} \tag{6.5}$$

has a saddle-node at the origin, this system is easy to discuss since it can be solved explicitly for

$$x(t) = \frac{1}{\frac{1}{x_0} - t}, \qquad y(t) = y_0 e^t,$$

The phase portrait for this system is shown in the following figure.

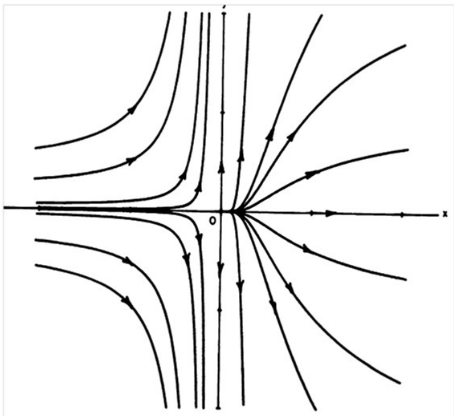


Figure 3. A saddle-node at the origin.

Fig 6 : *The phase portrait of (6.5)*

One other type of behavior that can occur at a nonhyperbolic critical point is Cusp.

Definition 34 *A neighborhood of the origin consists of*

- *two hyperbolic sectors and*
- *two separatrices.*

This type of critical point is called a cusp

Example 28 *The system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x^2, \end{cases} \tag{6.6}$$

has a Cusp. The phase portrait for this system is shown in the following figure.

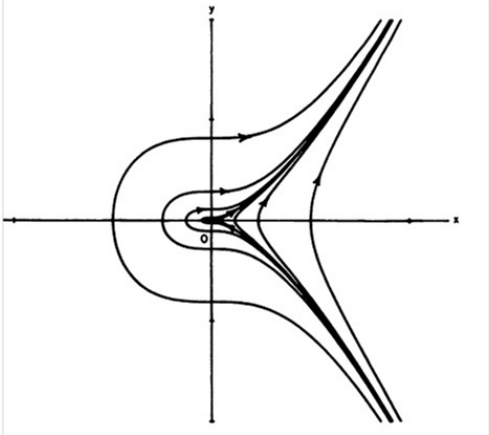


Figure 4. A cusp at the origin.

Fig 7 : The phase portrait of (6.6)

We first consider the case when the matrix A has one zero eigenvalue, i.e., when $\det A = 0$, but $trA \neq 0$. In this case, the system (6.3) can be put into the form

$$\begin{cases} \dot{x} = p_2(x, y), \\ \dot{y} = y + q_2(x, y), \end{cases} \tag{6.7}$$

where p_2 and q_2 are analytic in a neighborhood of the origin and have expansions that begin with second-degree terms in x and y.

The following theorem is proved on p. 340 in [1].

Theorem 17 *Let the origin be an isolated critical point for the analytic system* (6.7). *Let* $y = \varphi(x)$ *be the solution of the equation* $y + q_2(x, y) = 0$ *in a neighborhood of the origin and let the expansion of the function i,*$\psi(x) = p_2(x, \varphi(x))$ *in a neighborhood of* $x = 0$ *has the form* $\psi(x) = a_m x^m + ...$ *where* $m \geq 2$ *and* $a_m \neq 0$. *Then*

1. *for* m *odd and* $a_m > 0$, *the origin is an unstable node,*
2. *for* m *odd* $a_m < 0$, *the origin is a (topological) saddle and*
3. *for* m *even, the origin is a saddle-node.*

Example 29 *For the system*

$$\begin{cases} \dot{x} = x^2 + y^2, \\ \dot{y} = y - x^2, \end{cases}$$

we have $y - x^2 = 0$ *then* $y = x^2$ *thus* $\varphi(x) = x^2$ *and* $\psi(x) = p_2(x, \varphi(x)) = x^2 + x^4$ *we have* $m = 2$ *even. So the origin is a saddle-node.*

Next consider the case when A has two zero eigenvalues, i.e., $\det A = 0$, $trA = 0$, but $A \neq 0$. In this case, the system (6.3) can be put in the "normal" form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= a_k x^k[1 + h(x)] + b_n x^n y[1 + g(x)] + y^2 R(x, y), \end{aligned} \tag{6.8}$$

where $h(x), g(x)$ and $R(x, y)$ are analytic in a neighborhood of the origin, $h(0) = g(0) = 0, k > 2, a_k \neq 0$ and $n \geq 1$.

The next two theorems are proved on pp. 357-362 in [1].

Theorem 18 *Let* $k = 2m + 1$ *with* $m \geq 1$ *in* (6.8) *and let* $\lambda = b_n + 4(m + 1)a_k$. *Then*

(a) *if* $a_k > 0$, *the origin is a (topological) saddle.*

(b) *If* $a_k < 0$, *the origin is*

1. *a focus or a center if* $b_n = 0$ *and also if* $b_n \neq 0$ *and* $n > m$ *or if* $n = m$ *and* $\lambda < 0$,
2. *a node if bn* $\neq 0$, n *is an even number and* $n < m$ *and also if* $b_n \neq 0$, n *is an even number,* $n = m$ *and* $\lambda > 0$ *and*
3. *a critical point with an elliptic domain if* $b_n \neq 0$, n *is an odd number and* $n < m$ *and also if* $b_n \neq 0$, n *is an odd number,* $n = m$ *and* $\lambda >$ *0.*

Theorem 19 *Let* $k = 2m$ *with* $m \geq 1$ *in* (6.8). *Then the origin is*

1. *a cusp if* $b_n = 0$ *and also if* $b_n \neq 0$ *and* $n > m$ *and*
2. *a saddle-node if* $b_n \neq 0$ *and* $n < m$.

3 Exercises

6.3.1 Corrected exercises

Exercise 52 *Determine the nature of the critical points of the following nonlinear systems*

1. $\dot{x} = -x - \dfrac{y}{\ln\sqrt{x^2+y^2}}, \quad \dot{y} = -y + \dfrac{x}{\ln\sqrt{x^2+y^2}},$
2. $\dot{x} = -y - x^3 - xy^2, \quad \dot{y} = x - y^3 - x^2y,$
3. $\dot{x} = -2x\left(x^2+y^2\right)^2 + 3y, \quad \dot{y} = -2y\left(x^2+y^2\right)^2 - 3x.$

Solution.

1. For the system

$$\begin{cases} \dot{x} = -x - \dfrac{y}{\ln\sqrt{x^2+y^2}}, \\ \dot{y} = -y + \dfrac{x}{\ln\sqrt{x^2+y^2}}, \end{cases}$$

we have $f \notin C^1(\mathbb{R})$. In polar coordinates, we have

$$\dot{r} = -r, \qquad \dot{\theta} = \frac{1}{\ln r},$$

The solution with $r(0) = 0, \ \theta(0) = \theta_0$ is

$$r(t) = r_0 e^{-t}, \qquad \theta(t) = \theta_0 - \ln\left(1 - \frac{t}{\ln r_0}\right),$$

in the neighborhood 0 we have $r_0 < 1$, so $t \in \left]\ln r_0, +\infty\right[$. We only have $r(t) \to 0$ and $|\theta(t)| \to \infty$ if $t \to +\infty$, therefore the origin is a stable focus for this nonlinear system.

2. For the system

$$\begin{cases} \dot{x} = -by - a\left(x^3 + xy^2\right), \\ \dot{y} = bx - a\left(y^3 + x^2y\right), \end{cases}$$

Note that $(0,0)$ is an equilibrium point of the system, and we have $(0,0)$ is the center for the linear system associated $\dot{x} = -by, \dot{y} = bx$. Then the origin is either a center or a focus for the nonlinear system.

In polar coordinates, we have

$$\dot{r} = -ar^3, \quad \dot{\theta} = b.$$

The solution of this system with $r(0) = r_0, \theta(0) = \theta_0$, is

$$r(t) = \frac{r_0}{\sqrt{1+2atr_0^2}}, \quad \theta(t) = bt + \theta_0, \quad t > -\frac{1}{2r_0^2}$$

If $a > 0$ the solution is defined in $D_r = \left]-\dfrac{1}{2ar_0^2}, +\infty\right[$, thus $r(t) \to 0$ and $|\theta(t)| \to \infty$, if $t \to +\infty$. Therefore, the origin is a stable focus for this nonlinear system.

If $a < 0$ the solution is defined in $D_r = \left]-\infty, -\dfrac{1}{2ar_0^2}\right[$, thus $r(t) \to 0$ and $|\theta(t)| \to \infty$, as $t \to -\infty$. Therefore, the origin is an unstable focus for this nonlinear system.

3. For the system

$$\begin{cases} \dot{x} = \ -2x\left(x^2+y^2\right)^2+3y, \\ \dot{y} = -2y\left(x^2+y^2\right)^2-3x, \end{cases}$$

Note that $(0,0)$ is an equilibrium point of the system, and $(0,0)$ is a center for the linear system associated $\dot{x}=-y, \dot{y}=x$. Then the origin is either a center or a focus for the nonlinear system.

In polar coordinates, we have

$$\dot{r}=-2r^5, \qquad \dot{\theta}=-3.$$

The solution of this system with $r(0)=r_0, \theta(0)=\theta_0$, is

$$r(t)=\left(\frac{r_0^4}{8tr_0^4+1}\right)^{\frac{1}{4}}, \qquad \theta\left(t\right)=-3t+\theta_0.$$

The solution is defined in $t>-\dfrac{1}{8r_0^4}$, then $r(t)\to 0$ and $|\theta\left(t\right)|\to\infty$ as $t\to+\infty$, therefore the origin is a stable focus for this nonlinear system.

Exercise 53 *Determine the nature of the critical points of the following nonlinear systems*

1. $\dot{x}=y, \quad \dot{y}=-x+x^3y+xy^2,$
2. $\dot{x}=-y+x^2, \ \ \dot{y}=x-x^5y.$

Solution.

1. $(0,0)$ is a unique equilibrium point of the system. Linearization of the system in the neighbourhood of $(0,0)$ is

$$\dot{x}=y, \qquad \dot{y}=-x.$$

So $(0,0)$ is a center for the linear system. Then the origin is either a center or a focus for the nonlinear system. But as

$$\begin{cases} \left(-x\left(-t\right)\right)'=x\left(-t\right)'=y\left(-t\right), \\ \left(y\left(-t\right)\right)'=-y\left(-t\right)'=-\left(-x\left(-t\right)\right)+\left(-x\left(-t\right)\right)^3y+\left(-x\left(-t\right)\right)y\left(-t\right)^2. \end{cases}$$

Thus

$$\begin{cases} x\left(-t\right)'=y\left(-t\right), \\ y\left(-t\right)'=-\left(x\left(-t\right)\right)+\left(x\left(-t\right)\right)^3y+\left(x\left(-t\right)\right)y\left(-t\right)^2. \end{cases}$$

So

$$\dot{x}=y, \qquad \dot{y}=-x+x^3y+xy^2.$$

Therefore, the nonlinear system is symmetric with respect to y-axis (is invariant under the transformation $(t,x)\to(-t,-x)$). So the origin is a center for the nonlinear system.

2. Note that $(0,0)$ is a unique equilibrium point of the system. Linearization of the system in the neighbourhood of $(0,0)$ is

$$\dot{x}=y, \qquad \dot{y}=-x.$$

So $(0,0)$ is a center for the linear system. Then the origin is either a center or a focus for the nonlinear system. But as

$$\begin{cases} (-x(-t))' = x(-t)' = -y(-t) + (-x(-t))^2, \\ (y(-t))' = -y(-t)' = (-x(-t)) + (-x(-t))^5 y(-t). \end{cases}$$

Thus

$$\begin{cases} x(-t)' = y(-t) + x(-t)^2, \\ y(-t)' = (x(-t)) + (x(-t))^5 y. \end{cases}$$

So

$$\dot{x} = -y + x^2, \quad \dot{y} = x - x^5 y.$$

Therefore, the nonlinear system is symmetric with respect to the y-axis (is invariant under the transformation $(t,x) \to (-t,-x)$), then the origin is a center for the nonlinear system.

Exercise 54 *Determine the nature of the critical points of the following nonlinear systems*

1. $\dot{x} = y^2 - x^4, \quad \dot{y} = y - shx,$
2. $\dot{x} = y^2 - x^4, \quad \dot{y} = y - \sin x,$
3. $x' = y^2 - x^4, \quad y' = y - x^2 + \arctan x.$

Solution.

1. For the system

$$\dot{x} = y^2 - x^4, \qquad \dot{y} = y - shx,$$

we have $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. So $\det A = 0$, and $trA \neq 0$ then $(0,0)$ is a non-hyperbolic equilibrium point and the system has the form

$$\begin{cases} \dot{x} = p_2(x,y), \\ \dot{y} = y + q_2(x,y). \end{cases} \tag{6.9}$$

Let $y = \varphi(x) = shx = x + \frac{x^3}{3}$, development limited of shx is given by $shx = x + \frac{x^3}{3}$,thus

$$\psi(x) = p_2(x, \varphi(x)) = \left(x + \frac{x^3}{3}\right)^2 - x^4 = \frac{1}{9}x^6 - \frac{1}{3}x^4 + x^2,$$

since $m = 2$ even so the origin is a saddle node.

2. For the system

$$\dot{x} = y^2 - x^4, \quad \dot{y} = y - \sin x,$$

so $\det A = 0$, and $trA \neq 0$ then $(0,0)$ is a non-hyperbolic equilibrium point, and the system has the form (6.9), we have $y = \varphi(x) = \sin x$ and $\psi(x) = p_2(x, \varphi(x)) = (\sin x)^2 - x^4$, the development limited of the function $\psi(x)$ is

$$\psi(x) = (\sin x)^2 - x^4 = \left(x - \frac{x^3}{6}\right)^2 - x^4 = \frac{1}{36}x^6 - \frac{4}{3}x^4 + x^2,$$

since $m = 2$ even so the origin is a saddle node.

3. For the system
$$\dot{x} = y^2 - x^4, \qquad \dot{y} = y - x^2 + \arctan x,$$
so $\det A = 0$, and $trA \neq 0$ then $(0,0)$ is a non-hyperbolic equilibrium point, and the system has the form (6.9), we have $y = \varphi(x) = x^2 - \arctan x$ and $\psi(x) = p_2(x, \varphi(x)) = \left(x^2 - \arctan x\right)^2 - x^4$, the development of the function $\arctan x = x - \dfrac{x^3}{3}$, so
$$\begin{aligned}\psi(x) &= \left(x^2 - \arctan x\right)^2 - x^4 = \left(x^2 - \left(x - \frac{x^3}{3}\right)\right)^2 - x^4 \\ &= \frac{1}{9}x^6 + \frac{2}{3}x^5 - \frac{2}{3}x^4 - 2x^3 + x^2,\end{aligned}$$
since $m = 2$ even so, the origin is a saddle node.

Exercise 55 *Determine the nature of the critical points of the following nonlinear systems*

1. $\dot{x} = y, \quad \dot{y} = -x^3 - 2x^2y.$
2. $\dot{x} = x + y, \quad \dot{y} = -y - x + x^2 + 2x^3 + 2x^2y,$
3. $\dot{x} = y, \quad \dot{y} = x^5 + 2xy,$
4. $\dot{x} = y, \quad \dot{y} = x^2 - x^3y,$
5. $\dot{x} = y, \quad \dot{y} = x^2 + 2x^2y + xy^2.$

Solution.

1. For the system
$$\dot{x} = y, \qquad \dot{y} = -x^3 - 2x^2y.$$
We have $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, thus $\det A = 0$, and $trA = 0$; therefore $(0,0)$ is a non-hyperbolic equilibrium point, and the system is of "normal" form
$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= a_k x^k\left(1 + h(x)\right) + b_n x^n y\left(1 + g(x)\right) + y^2 R(x,y),\end{aligned} \tag{6.10}$$
we have $k = 3 = 2 \times 1 + 1$ thus $m = 1, a_k = -1, b_n = -2, n = 2$, since , $a_k = -1 < 0, b_n = -2 \neq 0, n = 2 > m = 1$, then the origin is a focus or a center.

2. For the system
$$\begin{cases} \dot{x} = x + y, \\ \dot{y} = -y - x + x^2 + 2x^3 + 2x^2y, \end{cases}$$
we have $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, thus $\det A = 0$, and $trA = 0$; therefore $(0,0)$ is a non-hyperbolic equilibrium point. By introducing the change of variables $z = x + y \rightarrow y = (z - x)$, we obtain $\dot{z} = \dot{x} + \dot{y} \rightarrow \dot{y} = \dot{z} - \dot{x}$ and
$$\begin{cases} \dot{x} = z, \\ \dot{z} - \dot{x} = -\left(z - x\right) - x + x^2 + 2x^3 + 2x^2\left(z - x\right). \end{cases}$$

Thus

$$\dot{x} = z, \qquad \dot{z} = x^2 + 2x^2 z.$$

And the system is of "normal" form (6.10), we have $k = 2 = 2 \times 1$, so $m = 1, a_k = 1, b_n = 2, n = 2$, since , $b_n = 2 \neq 0, n = 2 > m = 1$, then the origin is a cusp.

3. For the system

$$\dot{x} = y, \qquad \dot{y} = x^5 + 2xy,$$

thus $\det A = 0$, and $trA = 0$; therefore, $(0,0)$ is a non-hyperbolic equilibrium point, and the system is of "normal" form (6.10), we have $k = 2m+1, m = 2, n = 3, b_n = 2 \neq 0$ and $n < m$, therefore $(0,0)$ is a critical point with an elliptic domain.

4. For the system

$$\dot{x} = y, \qquad \dot{y} = x^2 - x^3 y,$$

thus $\det A = 0$, and $trA = 0$; therefore, $(0,0)$ is a non-hyperbolic equilibrium point, and the system is of "normal" form (6.10), we have $k = 2m, m = 1, n = 3, b_n = -1 \neq 0$ and $n > m$, then the origin is a cusp.

5. For the system

$$\dot{x} = y, \qquad \dot{y} = x^2 + 2x^2 y + xy^2.$$

Thus $\det A = 0$, and $trA = 0$; therefore, $(0,0)$ is a non-hyperbolic equilibrium point, and the system is of "normal" form (6.10), we have $k = 2m, m = 1, n = 2, b_n = 2 \neq 0$ and $n > m$, then the origin is a cusp.

6.3.2 Additional exercises

Exercise 56 *Write the following systems in polar coordinates and determine if the origin is a center, a stable focus, or an unstable focus.*

- $\dot{x} = -y + xy^2, \;\; \dot{y} = x + y^3,$
- $\dot{x} = -y + x^5, \dot{y} = x + y^5,$
- $\dot{x} = x - y, \;\; \dot{y} = x - y.$

Exercise 57 *Let the non-linear system*

$$\begin{cases} \dot{x} = -y + (x-1)\sqrt{x^2 - 2x + y^2 + 1}\,\sin\dfrac{1}{\sqrt{x^2 - 2x + y^2 + 1}}, \\ \dot{y} = x + y\sqrt{x^2 - 2x + y^2 + 1}\,\sin\dfrac{1}{\sqrt{x^2 - 2x + y^2 + 1}} - 1. \end{cases} \tag{6.11}$$

Show that the equilibrium point of system (6.11) *is a center-focus for this system but is a center for the linear system associated with the neighborhood of this point.*

Exercise 58 I *Let the non-linear system*

$$\begin{cases} \dot{x} = \; -2x\left(x^2 + y^2\right)^2 + 3y, \\ \dot{y} = -2y\left(x^2 + y^2\right)^2 - 3x. \end{cases} \tag{6.12}$$

1. *Study the stability in the neighborhood* $X^* = (0,0)$*, using the Liapunov function.*
2. *Show that the origin is a stable focus for system* (6.12) *but a center for the linear system associated with the vicinity of this point.*

II *Let the non-linear system:*

$$\begin{cases} \dot{x} = ax + y - x\left(y^2 + x^2\right)^3, \\ \dot{y} = -x + ay - y\left(y^2 + x^2\right)^3. \end{cases} \tag{6.13}$$

1. *Linearize this system in the neighborhood of point* $X^* = (0,0)$.
2. *What can we conclude about the stability of the equilibrium point* $(0,0)$ *for* $a > 0; a = 0$ *and* $a < 0$*?*
3. *For* $a < 0$ *Study the stability in the neighborhood* $X^* = (0,0)$*, using the Liapunov function.*
4. *For* $a = 0$*, show that the origin is a focus for system* (6.13) *but a center for the linear system associated with the neighborhood of this point.*

Exercise 59 *Let the non-linear system*

$$\begin{cases} \dot{x} = ax + 3y - cx^3 - cxy^2, \\ \dot{y} = -3x + 2ay - cx^2y - cy^3, \end{cases} , \quad a, c \in \mathbb{R}. \tag{6.14}$$

1. *Linearize this system in the neighborhood of point* $X^* = (0,0)$.
2. *What can we conclude about the stability of the equilibrium point* $(0,0)$ *for* $a = 0$ *and* $a \neq 0$*?*
3. *For* $ac < 0$ *study the stability in the neighborhood* $X^* = (0,0)$*, using the Liapunov function.*
4. *For* $a = 0$*, show that the origin is a focus (stable or unstable) for the system* (6.14).

Exercise 60 *Write the following systems in polar coordinates and determine if the origin is a center, a stable focus, or an unstable focus.*

1. $\dot{x} = x - y, \ \ \dot{y} = x + y,$
2. $\dot{x} = -y + xy^2, \ \ \dot{y} = x + y^3,$
3. $\dot{x} = -y + x^5, \ \ \dot{y} = x + y^5.$

Exercise 61 *Determine the nature of the critical points of the following nonlinear systems*

1. $\dot{x} = y - x^4y, \ \ \dot{y} = -3x - x^2 + y^2x,$
2. $\dot{x} = -x - y, \ \ \dot{y} = x + y - x^2 + y^3,$
3. $\dot{x} = x + y, \ \ \dot{y} = -x - y - x^3 + y^3,$
4. $\dot{x} = y, \ \ \dot{y} = x^2 - x^3y,$
5. $\dot{x} = y^2 - x^4, \ \ \dot{y} = y - \cos x + 1.$

Chapter

7

Center Manifold Theory

The Hartman-Grobman Theorem therefore completely solves the problem of determining the stability and qualitative behavior in a neighborhood of a hyperbolic critical point of a nonlinear system. In this section, we present the Local Center Manifold Theorem, which generalizes the stable manifold theorem to nonhyperbolic equilibrium points and shows that the qualitative behavior in a neighborhood of a nonhyperbolic critical point x_0 of the nonlinear system is determined by its behavior on the center manifold near x_0. The center manifold of a dynamical system is based upon an equilibrium point of that system

$$\dot{x} = f\left(x\right). \tag{7.1}$$

where $f \in E$ and $f(0) = 0$.

The system (7.1) can be written in diagonal form

$$\begin{aligned} \dot{x} &= Cx + F(x,y,z), \\ \dot{y} &= Py + G(x,y,z), \\ \dot{z} &= Qz + H(x,y,z), \end{aligned} \tag{7.2}$$

where $(x,y,z) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u$, $F(0) = G(0) = H(0) = 0$, and $DF(0) = DG(0) = DH(0) = 0$ and

- the square matrix C has c eigenvalues with zero real parts,
- the square matrix P has s eigenvalues with negative real parts,
- the square matrix Q has u eigenvalues with positive real parts

Definition 35
- *The local center manifold of* (7.1) *at* 0,

$$W^C(0) = \{(x,y) \in \mathbb{R}^c \times \mathbb{R}^s : y = h(x) \text{ for } |x| < \delta\} \tag{7.3}$$

for some $\delta > 0$, *where* $h \in C^r(N_\delta(0))$, $h(0) = 0$, *and* $Dh(0) = 0$ *since* $W^c(0)$ *is tangent to the center subspace* E_c *at the origin.*

- *The local center manifold of* (7.1) *is defined by*

$$W_g^C\left(x^*\right) = \cup_{t\geq 0}\varphi_t\left(W^C\right)$$

Remark 16 *A center manifold of the equilibrium then consists of those nearby orbits that neither decay nor grow exponentially quickly.*

Theorem 20 *(The Local Center Manifold Theorem).Let $f \in C^r(E)$, where E is an open subset of $\mathbb{R}^n$ containing the origin and $r > 1$. Suppose that $f(0) = 0$ and that $Df(0)$ has c eigenvalues with zero real parts and s eigenvalues with negative real parts, where $c + s = n$. The system (7.2) then can be written in diagonal form*

$$\begin{aligned} \dot{x} &= Cx + F(x, y) \\ \mathring{y} &= Py + G(x, y), \end{aligned}$$

where $(x, y) \in \mathbb{R}^c \times \mathbb{R}^s$, C is a square matrix with c eigenvalues having zero real parts, P is a square matrix with s eigenvalues with negative mat parts, and $F(0) = G(0) = 0, DF(O) = DG(O) = 0$; furthermore, there exists a $\delta > 0$ and a function $h \in C^r(N_\delta(0))$ that defines the local center manifold (7.3) and satisfies

$$Dh(x)[Cx + F(x, h(x))] - Ph(x) - G(x, h(x)) = 0 \tag{7.4}$$

for $|x| < \delta$; and the flow on the center manifold $W^c(0)$ is defined by the system of differential equations

$$\dot{x} = Cx + F(x, h(x)) \tag{7.5}$$

for all $x \in \mathbb{R}^c$ with $|x| < \delta$.

The proof of the center manifold theorem is a bit harder and more tedious. We will not give the proof. It may consult the textbook Elements of Differentiable Dynamics and Bifurcation Theory, Academic Press, New York, 1989 at p.32, by D. Ruelle [15] or J. Carr, Applications of Center manifold Theory, Springer-Verlag, New York, 1981 [3].

Remark 17

- *The center manifold approach treats the case where equilibrium is not hyperbolic, but it is much more complicated.*
- *The centre manifold need not be unique.*
- *If f and g are $C^k, k > 2$, the centre manifold is also C^k, however, if F and G are analytic, the centre manifold need not be analytic.*

Example 30 *Consider the system*

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = -y - x^2. \end{cases}$$

The linear normal form (based on the linearization at the origin) has the constant matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = diag(0, -1).$$

Then the eigenpairs are $\lambda_1 = 0, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \lambda_2 = -1, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Since the matrix A is already in normal form, no coordinate transformation is needed.
Now the centre manifold takes the form

$$y = h(x) = ax^2 + bx^3 + cx^4 + O(x^5). \tag{7.6}$$

There is no constant term because the centre manifold passes through the origin; there is no linear term because this manifold should be tangent to e_1 *(or equivalently* E^c*, the centre We can determine the coefficients by comparing two ways for calculating* $\dot{y}$*. Directly from the* $\dot{y}$ *equation of the system*

$$\dot{y} = -y - x^2 = -(ax^2 + bx^3 + cx^4) - x^2 + O(x^5). \tag{7.7}$$

On the other hand, differentiating (7.6) gives $\dot{y} = \dfrac{dh(x)}{dt} = \dot{x}h'(x)$*, i.e.,*

$$x(ax^2 + bx^3 + cx^4 + \cdots)(2ax + 3bx2 + 4cx3 + \cdots) = 2ax4 + \cdots. \tag{7.8}$$

Equating coefficients of x^2, x^3 *and* x^4 *in (7.7) and (7.8) gives*

$$-a - 1 = 0, -b = 0, -c = 2a^2,$$

i.e., $a = -1, b = 0$ *and* $c = -2$*. Thus the centre manifold is*

$$y = -x^2 - 2x^4 + O(x^5)$$

and the dynamics on the centre manifold is

$$\dot{x} = xh(x) = -x^3 - 2x^5 + O(x^7).$$

Thus $\dot{x} < 0$ *if* $x > 0$ *and* $\dot{x} > 0$ *if* $x < 0$*. So the origin is stable, and the solutions look like a stable node, but the motion onto the centre manifold in the y-direction is much faster than the motion on the centre manifold.*

Remark 18 *If you try higher order terms, you get*

$$y = -x^2 - 2x^4 - 12x^6 - 112x^8 - 1360x^{10} - 19872x^{12} + \cdots.$$

The fast increase of the coefficients implies that this approximation is valid only in a small neighbourhood of the origin.

Example 31 *Consider the system*

$$\begin{cases} \dot{x} = y - x + xy, \\ \dot{y} = x - y - x^2. \end{cases}$$

First, we have to convert the linearized system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

into normal form. It is easy to calculate the eigenpairs of A*,*

$$\lambda_1 = 0, e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda_2 = -2, e_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Let

$$P = (e_1, e_2)^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

the change of variable from (x, y) *to* (u, v) *(in normal form) is*

$$\begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{x+y}{2} \\ \dfrac{x-y}{2} \end{pmatrix},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = P^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = (e_1, e_2) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u-v \\ u+v \end{pmatrix}.$$

The new system in (u, v) *is*

$$\dot{u} = uv - v^2, \qquad \dot{v} = -2v + uv - u^2.$$

The centre manifold is parametrized by

$$v = h(u) = au^2 + bu^3 + cu_4 + ...,$$

then

$$\begin{aligned} \dot{v} &= (2au + 3bu^2 + 4cu^3 +)\dot{u} \\ &= (2au + 3bu^2 + 4cu^3 + ...)u(au^2 + bu^3 + cu^4 + ...\ ) - (au^2 + bu^3 + cu^4 + ...\ )^2 \\ &= 2a^2u^4 + \cdots \end{aligned}$$

and on the other hand

$$\begin{aligned} \dot{v} &= -2v + uv - u^2 \\ &= -2(au^2 + bu^3 + cu^4 + ...) + u(au^2 + bu^3 + cu^4 + ...\ ) - u^2 \\ &= (2a+1)u^2 + (2b-a)u^3 + (3c-b)u^4 + ... \end{aligned}$$

Comparing the coefficients of u^2, u^3 *and* u^4 *of the two expressions of* $\dot{v}$*, we get*

$$a = -\frac{1}{2}, b = -\frac{1}{4}, c = -\frac{3}{8},$$

or

$$v = -\frac{1}{2}u^2 - \frac{1}{4}u^3 - \frac{3}{8}u^4 + ...$$

The dynamics on the centre manifold is

$$\dot{u} = uv - v^2 = -\frac{1}{2}u^3 - \frac{1}{4}u^4 - \frac{3}{8}u^5 + ...$$

which is stable if u *is small. Going back to the original coordinates, the centre manifold is approximately*

$$y - x = -\frac{1}{4}(x+y)^2 - \frac{1}{16}(x+y)^3 - \frac{3}{64}(x+y)^4 + ...$$

Example 32 *Consider the following system with* $c = 2$ *and* $s = 1$*:*

$$\begin{cases} \dot{x}_1 = x_1 y - x_1 x_2^2, \\ \dot{x}_2 = x_2 y - x_2 x_1, \\ \dot{y} = -y + x_1 + x_2. \end{cases}$$

In this example, we have $C = 0,\ P = [-1]$,

$$F(x,y) = \begin{pmatrix} x_1 y - x_1 x_2^2 \\ x_2 y - x_2 x_1 \end{pmatrix}$$

and

$$G(x,y) = x_1 + x_2.$$

We substitute the expansions

$$h(x) = ax_1 + bx_1x_2 + cx_2 + 0(|x|^3)$$

and

$$Dh(x) = [2ax_1 + bx_2, bx_1 + 2cx_2] + O(|x|^2)$$

into equation (7.4) *to obtain*

$$\begin{gathered} (2ax_1 + bx_2)[x_1(ax_1 + bx_1x_2 + cx_2) - x_1x_2] + \\ (bx_1 + 2cx_2)]x_2(ax_1 + bx_1x_2 + cx_2) - x_2x_1] + \\ (ax_1 + bx_1x_2 + cx_2) - (x_1 + x_2) + 0(|x|^3) = 0. \end{gathered}$$

Since this is an identity for all x_1, x_2 *with* $|x| < \delta$, *we obtain* $a = 1, b = 0, c = 1, \ldots.$ *Thus,*

$$h(x_1, x_2) = x_1 + x_2 + 0(|x|^3).$$

Substituting this result into equation (7.5) *then yields*

$$\begin{aligned} x_1 &= x_1^3 + 0(|x|^4), \\ x_2 &= x_2^3 + 0(|x|^4), \end{aligned}$$

on the center manifold $W^c(0)$ *near the origin.*

Example 33 *For our last example, we consider the system*

$$\begin{cases} \dot{x} = y + z, \\ \dot{y} = z + x^2, \\ \dot{z} = -z + y + xz. \end{cases}$$

by the change of variable $y + z = Y$, *we get* $y = Y - z$. *This yields the following system in diagonal form*

$$\begin{cases} \dot{x} = Y, \\ \dot{Y} - \dot{z} = z + x^2, \\ \dot{z} = -z + (Y - z)^2 + xz. \end{cases}$$

thus

$$\begin{cases} \dot{x} = Y, \\ \dot{Y} = x^2 + (Y - z)^2 + xz, \\ \dot{z} = -z + (Y - z)^2 + xz. \end{cases}$$

with $c = 2, s = 1, P = [-1]$,

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F(x,y) = \begin{pmatrix} 0 \\ x^2 + (y-z)^2 + xz \end{pmatrix},$$

$$G(x,y) = (y-z)^2 + xz.$$

Let us substitute the expansions for $h(x)$ *and* $Dh(x)$ *into equation (7.4) to obtain*

$$\begin{gathered}(2ax+by)y + (bx+2cy)[x + (y-z)^2 + xz] + (ax^2 + bxy + cy^2) \\ -(y - ax - bxy - cy^2)^2 - x(ax^2 + bxy + cy^2) + O(|X|^3 = 0.\end{gathered}$$

Since this is an identity for all x, y *with* $|X| < \delta$, *we obtain* $a = 0, b = 0, c = 1, \ \dots \ ,$ *i.e.,*

$$h(x) = y^2 + O(|X|^3).$$

Substituting this result into equation (7.5) then yields

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x^2 + y^2 + O(|X|^3), \end{cases}$$

on the center manifold $W^c(0)$ *near the origin.*

1 Exercises

7.1.1 Corrected exercises

Exercise 62 *Find the approximation for the flow on the local center manifold for the system*

$$\begin{cases} \dot{x} = xy + x^3, \\ \dot{y} = -y - 2x^2, \end{cases} \tag{7.9}$$

and investigate its local phase portrait in the neighbourhood of the origin.

Solution. The Jacobian obtained by linearization is

$$Df\,(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

its eigenvalues are 0 and -1. Thus the origin is not a hyperbolic equilibrium, therefore, the linearization does not determine the local phase portrait. Since there is a negative eigenvalue, the system has a one-dimensional stable manifold, along which the trajectories tend to the origin. At the end of this section, it will be shown that there is an invariant center manifold that is tangential to the center subspace belonging to the eigenvalue 0, and the behavior of the trajectories in this manifold can easily be determined by investigating the phase portrait of a one-dimensional system the stable subspace E_s is determined by $(0,1)$, the center subspace E_c

is given by $(1, 0)$. The approximation of the function determining the center manifold can be given in the form

$$h(x) = a_2x^2 + a_3x^3 + ...,$$

because the center manifold is tangential to the center subspace at the origin, it can be given as a function of x and $h(0) = 0 = h'(0)$. The invariance of the manifold implies $y(t) = h(x(t))$, the derivative of which yields

$$\dot{y}(t) = h'(x(t))\dot{x}(t).$$

Substituting the derivatives of x and y from the differential equations into this equation one obtains

$$-h\left(x\right) - 2x^2 = h'\left(x\right)\left(xh\left(x\right) + x^3\right).$$

Using the power series expansion of the coefficients of the corresponding terms are equal on the left and right hand sides. Using the coefficients of the quadratic term x^2 on both sides, we get $-a_2 - 2 = 0$, yielding $a_2 = -2$. Hence the function defining the center manifold can be given as

$$h(x) = -2x^2 + O(x^3).$$

Thus the approximation of the center manifold up to the second degree can be given as $h(x) = -2x^2$. Substituting $h(x) = -2x^2 + O(x^3)$ into the first equation of the reduced system

$$\dot{x} = x(-2x^2 + a_3x^3 + ...) + x^3 = -x^3 + O(x^4).$$

The local phase portrait at the origin does not depend on the terms of $O(x^4)$, because they do not have influence on the direction field. In this one-dimensional system the trajectories tend to the origin, hence along the center manifold the trajectories tend to the origin. Since the other eigenvalue of the system is negative, all solutions in a neighbourhood of the origin tend to the origin, thus according to the center manifold reduction the origin is asymptotically stable. The phase portrait is shown in the following figure.

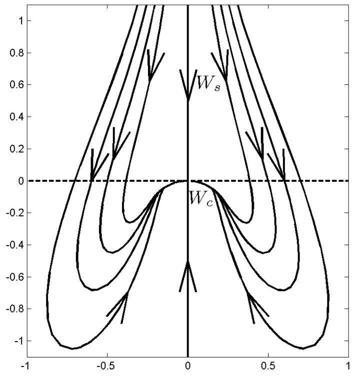


Fig 8 : The stable and center manifolds of system (7.9)

7.1.2 Additional exercises

Exercise 63 *Find stable, unstable and center subspaces E_s, E_u and E_c for the linear systems given by the following matrices:*

$$\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} -3 & 6 \\ 2 & -4 \end{pmatrix}.$$

Exercise 64 *Compute a reduction to the centre manifold up to order* 5*, i.e., an error of size* $O(|x|^6)$*, for the system*

$$\begin{cases} \dot{x} = xy - x^3 - xy^2, \\ \dot{y} = -y + x^3 + x^2 y, \end{cases}$$

near the origin. Compute a dynamical equation on the manifold. Is the origin stable?

Exercise 65 *Compute a reduction to the centre manifold for the system*

$$\begin{cases} \dot{x} = 10(y - x), \\ \dot{y} = x - y - xz, \\ \dot{z} = xy - \dfrac{8}{3} z. \end{cases}$$

8

Gradient and Hamiltonian systems

1 Hamiltonian systems

Definition 36 *Let E be an open subset of $\mathbb{R}^{2n}$ and let $H \in C^2(E)$ where $H = H(x,y)$ with $x, y \in \mathbb{R}$. A system of the form*

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x} \tag{8.1}$$

where

$$\frac{\partial H}{\partial x} = \left(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \dots, \frac{\partial H}{\partial x_n}\right)^T, \qquad \frac{\partial H}{\partial y} = \left(\frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}, \dots, \frac{\partial H}{\partial y_n}\right)^T$$

is called a Hamiltonian system with n degrees of freedom on E.

Definition 37 *Such a function H is known as the Hamiltonian function.*

Example 34 *The Hamiltonian function*

$$H(X,Y) = H(x_1, x_2, y_1, y_2) = \frac{1}{2}\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right)$$

is the energy function for

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = -x_1, \\ \dot{y}_2 = -x_2. \end{cases}$$

Theorem 21 *(Conservation of Energy). The total energy $H(x,y)$ of the Hamiltonian system* (8.1) *remains constant along trajectories of* (8.1)*.*

Proof. The total derivative of the Hamiltonian function $H(x,y)$ along a trajectory $(x(t), y(t))$ of (8.1)

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} = 0.$$

Thus, $H(x,y)$ is constant along any solution curve of (8.1) and the trajectories of (8.1) lie on the surfaces $H(x,y) = \text{constant}$.

Remark 19 *In real systems, there is often a conserved quantity, although not necessarily energy.*
If the system is Hamiltonian, the orbits lie in the level sets of H, locally of dimension one less than that of the phase space X (i.e. of co-dimension 1).
In general it is hard to know if H even exists and is usually difficult to find when it does.

Definition 38 *A critical point of the system*

$$\dot{x} = f(x) \tag{8.2}$$

at which $Df(x_0)$ has no zero eigenvalues is called a nondegenerate critical point of the system, otherwise, it is called a degenerate critical point of the system.

Note that any nondegenerate critical point of a planar system is either a hyperbolic critical point of the system or a center of the linearized system.

Theorem 22 *Any nondegenerate critical point of an analytic Hamiltonian system* (8.1) *is either a (topological) saddle or a center; furthermore, (x_0, y_0) is a (topological) saddle for* (8.1) *if it is a saddle of the Hamiltonian function $H(x, y)$ and a strict local maximum or minimum of the function $H(x, y)$ is a center for* (8.1).

Proof. We assume that the critical point is at the origin. Thus, $H_x(0,0) = H_y(0,0) = 0$ and the linearization of (8.2) at the origin is

$$\dot{x} = Ax \tag{8.3}$$

where

$$A = \begin{pmatrix} H_{yx}(0,0) & H_{yy}(0,0) \\ -H_{xx}(0,0) & -H_{xy}(0,0) \end{pmatrix}.$$

We see that $trA = 0$ and that $\det A = H_{xx}(0)H_{yy}(0) - H_{xy}^2(0)$. Thus, the critical point at the origin is a saddle of the function $H(x, y)$

- if $\det A < 0$ if it is a saddle for the linear system (8.3) if it is a (topological) saddle for the Hamiltonian system (8.2) according to Theorem 13.

- if $trA = 0$ and $\det A > 0$, the origin is a center for the linear system (8.3). And then, according to the 2, the origin is either a center or a focus for (8.2).

Thus, if the nondegenerate critical point $(0,0)$ is a strict local maximum or minimum of the function $H(x, y)$, then $\det A > 0$ and, according to the above lemma, the origin is not a focus for (8.2); i.e., the origin is a center for the Hamiltonian system (8.2).■

2 Newtonian system

One particular type of Hamiltonian system with one degree of freedom is the Newtonian system with one degree of freedom, $\ddot{x} = f(x)$ where $f \in C^1(a,b)$. This differential equation can be written as a system in $\mathbb{R}^2$:

$$\dot{x} = y, \qquad \dot{y} = f(x). \tag{8.4}$$

The total energy for this system $H(x,y) = T(y) + U(x)$ where $T(y) = \frac{y^2}{2}$ is the kinetic energy and

$$U(x) = \int_{x_0}^{x} f(s)ds$$

is the potential energy. With this definition of $H(x,y)$ we see that the Newtonian system (8.4) can be written as a Hamiltonian system.

Theorem 23 *The critical points of the Newtonian system* (8.4) *all lie on the x-axis. The point $(x_0, 0)$ is a critical point of the Newtonian system* (8.4) *iff it is a critical point of the function $U(x)$, i.e., a zero of the function $f(x)$.*

- *If $(x_0, 0)$ is a strict local maximum of the analytic function $U(x)$, it is a saddle for* (8.4) *.*
- *If $(x_0, 0)$ is a strict local minimum of the analytic function $U(x)$, it is a center for* (8.4) *.*
- *If $(x_0, 0)$ is a horizontal inflection point of the function $U(x)$, it is a cusp for the system* (8.4) *.*

And finally, the phase portrait of (8.4) *is symmetric with respect to the x-axis.*

Example 35 *consider*

$$\dot{x} = y, \qquad \dot{y} = x^3 - x, \tag{8.5}$$

we have

$$H(x;y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4.$$

Compute $\frac{dH}{dt}$ along the flow.

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = y(x - x^3) + y(x^3 - x) = 0.$$

So H does not vary along orbits of the flow. Look at sets $\{H = c\}$. Draw the fixed points $y = 0, x = 0, 1$. Calculate H at the fixed points: $H(0,0) = 0, H(1,0) = \frac{1}{4}$. Draw the level sets, remembering techniques for phase portraits. We know the behaviour of the flow:

1. *At the fixed points x^* are such that $f(x^*) = 0$.*

2. *All orbits of the flow must lie within a set of type* $\{H = c\}$, *i.e.,* $\dfrac{dH}{dt} = 0$.

3. *For* $0 < c < \dfrac{1}{4}$ *the sets look like ellipses. Contains no fixed points anywhere on the curve* $|f(x)|$ *not zero anywhere* $\Rightarrow |f(x)|$ *attains a positive minimum on the curve* $\Rightarrow$ *velocity is bounded away from* 0 *on the curve* $\Rightarrow$ *periodic orbit of flow with finite period* Tc.

4. $c = \dfrac{1}{4}$ *uses similar arguments; note that the fixed points are in this set. There is a heteroclinic orbit between the fixed points* $(1, 0)$ *and* $(-1, 0)$, *so any orbit starting on one of these heteroclinic orbits will tend to the appropriate fixed points as* $t \to \pm\infty$. *See the following figure.*

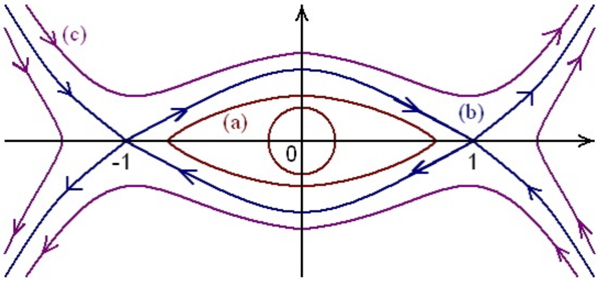


Fig 9 : *The phase portrait of (8.5)*

The level sets $H = c$ *show the fixed points, heteroclinic orbits, and periodic orbits, enabling us to get a good idea of the flow. In (a),* $c < \dfrac{1}{4}$, *in (b)* $c = \dfrac{1}{4}$ *while in (c),* $c > \dfrac{1}{4}$.

Orbits not on the heteroclinic orbits will tend to $+\infty$ *or* $-\infty$ *as* $t \to -\infty$ *or the fixed points as* $t \to -\infty$. *Observe that we have learned a lot of information about the flow with solving the equations.*

Example 36 *Pendulum* $\ddot{x} = -\sin(\theta)$. *Write* $p = \dot{\theta}$ *to get*

$$\dot{\theta} = p, \qquad \dot{p} = -\sin(\theta)$$

Consider

$$H(p, \theta) = \frac{1}{2}p^2 - \cos(\theta).$$

thus

$$\frac{dH}{dt} = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial \theta}\dot{\theta} = -p\sin(\theta) + p\sin(\theta) = 0.$$

So H *is constant on all orbits. Sets* $\{H = c\}$. *There are fixed points at* $(0, 2k)$ *and* $(0, 2k + 1)$ *where* $k \in \mathbb{Z}$, *with periodic orbits around* $(0, 2k)$ *and homoclinic orbits joining* $(0, (2k+1))$. *See* *the following figure.*

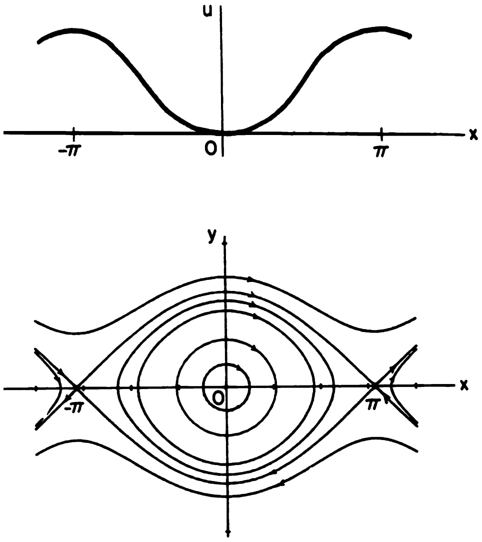


Fig 10 : *The phase portrait for the undamped pendulum*

3 Gradient systems

Definition 39 *Let* $V : U \to \mathbb{R}$ *(*U *is an open set of* $\mathbb{R}^n$*) be a function of class* $C^{2.}$ *A gradient system on* $U \subset \mathbb{R}^n$ *is a differential equation of the form*

$$\dot{x} = -\nabla V(x) \tag{8.6}$$

where $\nabla V(x) = \left(\dfrac{\partial V}{\partial x_1}, \dfrac{\partial V}{\partial x_2}, \ldots, \dfrac{\partial V}{\partial x_n}\right)^T$. *(The negative sign is a traditional convention). The equilibrium points for* (8.6) *are the critical points of* V *, i.e., the points* a *for which* $\nabla V(a) = 0$

Consider the level sets of the function V , $V^{-1}(c) = \{x; V(x) = c\}$. If $x \in V^{-1}(c)$ is a regular point, i.e., if $\nabla V(x) \neq 0$, then, by the implicit function Theorem, locally near x, $V^{-1}(c)$ is a smooth hypersurface surface of dimension $n-1$. For example, if $n = 2$, the level sets are smooth curves.

We summarize the properties of the gradient systems in

Proposition 1 *1 Let $V : U \to \mathbb{R}^n$ (U an open set in $\mathbb{R}^n$) be of class C^2 and let*

$$\dot{x} = -\nabla V(x) \tag{8.7}$$

be a gradient system.

1. *If x is a regular point of the level curve $V^{-1}(c)$, then the solution curve $x(t)$ is perpendicular to the level surface $V^{-1}(c)$.*
2. *If a is an isolated minimum of V , then it a is an asymptotically stable critical point.*
3. *If a is an isolated minimum of V or a saddle point of V , then a is an unstable critical point.*

Proof. Let y be a vector which is tangent to the level surface $V^{-1}(c)$ at the point x.

1. For any curve $\gamma(t)$ in the level set $V^{-1}(c)$ with $\gamma(0) = x$ and $\gamma'(0) = y$ we have

   $$0 = \frac{d}{dt} V((t))|_{t=0} = \langle \nabla V(x), y \rangle , \tag{8.8}$$

   and so $\nabla V(x)$ is perpendicular to any tangent vector to the level set $V^{-1}(c)$ at all regular points of V. This proves 1.

2. If $x(t)$ is a solution of (8.6), then we have

   $$\frac{d}{dt} V((t)) = - \langle \nabla V(x), \nabla V(x) \rangle . \tag{8.9}$$

   If a is an isolated minimum of V, then consider the Liapunov function $W(x) = V(x) - V(a)$. We have $\dot{W}(x) < 0$ in a neighborhood of a and so, by the Stability Theorem of Liapunov, a is an asymptotically stable equilibrium point. This proves 2.

3. If a is an isolated minimum of V or a saddle point of V , we consider the function $W(x) = V(a) - V(x)$. If a is a isolated minimum $W(x)$ is a Liapunov function and if a is a saddle point. In both cases we have $\dot{W}(x) > 0$ in a neighborhood of a, and thus, by the Stability Theorem of Liapunov, a is unstable.■

Remark 20 • *Note that the equilibrium points or critical points of the gradient system (8.7) correspond to the critical points of the function $V(x)$ where grad $V(x) = 0$.*

- *Points where $\nabla V(x) \neq 0$ are called regular points of the function $V(x)$.*
- *At regular points of $V(x)$, the gradient vector grad $V(x)$ is perpendicular to the level surface $V(x)$ =constant through the point.*
- *And it is easy to show that at a critical point x_0 of $V(x)$, which is a strict local minimum of $V(x)$, the function $V(x) - V(x_0)$ is a strict Liapunov function for the system (8.7) in some neighborhood of x_0.*

Once again, for planar gradient systems, we can be very specific about the nature of the critical points of the system:

Theorem 24 . *Any nondegenerate critical point of an analytic gradient system* (8.7) *on* $\mathbb{R}^2$ *is either a saddle or a node; furthermore, if* (x_0, y_0) *is a saddle of the function* $V(x, y)$*, it is a saddle of* (8.7) *and if* (x_0, y_0) *is a strict local maximum or minimum of the function* $V(x, y)$*, it is respectively an unstable or a stable node for* (8.7).

One last topic, which shows that there is an interesting relationship between gradient and Hamiltonian systems, is considered in this section. We only give the details for planar systems.

Definition 40 *Consider the planar system*

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y). \tag{8.10}$$

The system orthogonal to (8.10) *is defined as the system*

$$\dot{x} = Q(x, y), \qquad \dot{y} = -P(x, y). \tag{8.11}$$

Remark 21 (8.10) *and* (8.11) *have the same critical points and at regular points, trajectories of* (8.10) *are orthogonal to the trajectories of* (8.11)*. Furthermore,*

- *centers of* (8.10) *correspond to nodes of* (8.11),
- *saddles of* (8.10) *correspond to saddles of* (8.11), *and*
- *foci of* (8.10) *correspond to foci of* (8.11).

Also, if (8.10) *is a Hamiltonian system with* $P = H_y$ *and* $Q = -H_x$*, then* (8.11) *is a gradient system and conversely.*

Theorem 25 *The system* (8.10) *is a Hamiltonian system if the system* (8.11) *orthogonal to* (8.10) *is a gradient system.*

In higher dimensions, we have that if (8.1) is a Hamiltonian system with n degrees of freedom, then the system

$$\dot{x} = -\frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x} \tag{8.12}$$

orthogonal to (8.1) is a gradient system in $\mathbb{R}^{2n}$ and the trajectories of the gradient system (8.12) cross the surfaces $H(x, y) =$ constant orthogonally. In example 37 if we take $H(x, y) = V(x, y)$, then the figure of example 37 illustrates the orthogonality of the trajectories of the Hamiltonian and gradient flows, the Hamiltonian flow swirling clockwise.

Example 37 *Let* $V(x, y) = x^2(x-1)^2 + y^2$*. The gradient system has the form*

$$\begin{cases} \dot{x} = -4x(x-1)\left(x - \dfrac{1}{2}\right) - 2y, \\ \dot{y} = -2y. \end{cases} \tag{8.13}$$

There are critical points at $(0,0)$, $\left(\frac{1}{2},0\right)$ *and* $(1,0)$. *It follows from theorem 24 that* $(0,0)$ *and* $(1,0)$ *are stable nodes and that* $\left(\frac{1}{2},0\right)$ *is a saddle for this system. The level curves* $V(x,y)=$*constant and the trajectories of this system are shown in the following figure*

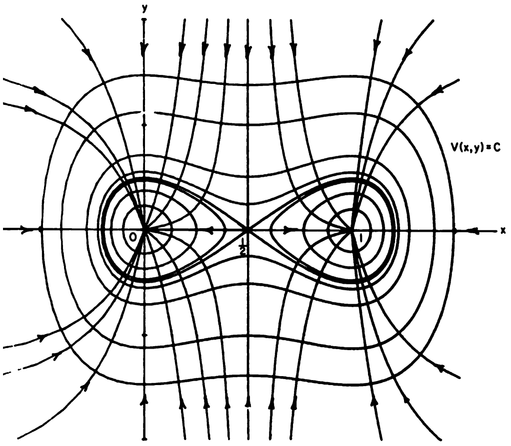


Fig 11 : *The level curves (closed curves) and the trajectories of the gradient system in Example 37*

4 Exercises

8.4.1 Corrected exercises

Exercise 66 **(i)** *What is meant by a Hamiltonian system in two-dimensions, equations of motion and an autonomous system?*

(ii) *Provide a criterium that can be used to decide whether a system is Hamiltonian or not. Employ the criterium to determine the value of* μ *such that the system*

$$\begin{cases} \dot{x} = 3x^2y^2 + 2x + 5y, \\ \dot{y} = -\mu x^2y^3 - 2y + \sin(x^5) \end{cases}$$

becomes a Hamiltonian system.

(iii) *Provide a definition for a potential system. Derive the Hamiltonian function for the dynamical system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -2x + \dfrac{20x}{1+x^2}, \end{cases} \tag{8.14}$$

and confirm that it is a potential system. Fix the potential in such a way that its value at zero is zero.

(iv) *Compute all fixed points of the system in (iii) and determine their nature by making use of the fact that the system is a potential system.*

(v) *Sketch the potential for the system in (iii) and compute the equation for the separatrices by making use of the fact that the Hamiltonian is conserved along a trajectory. Draw the phase portrait for the system. Include the separatrices and some representative trajectories. Include the direction of time on the trajectories and provide a reasoning for your choices. Explain in which areas of your diagram the motion is bounded.*

Solution.

i) Def.: A system of differential equations on $\mathbb{R}^2$ is said to be a Hamiltonian system with one degree of freedom if there exists a twice continuously differentiable function $H(x,y)$ such that

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}. \tag{8.15}$$

The equations (8.15) are said to be the equations of motion corresponding to the Hamiltonian H. When H does not depend explicitly on the time t, i.e. it is of the form $H(x(t),y(t))$ and not $H(x(t),y(t),t)$, the system is called autonomous.

ii) A dynamical system

$$\dot{x} = F_1(x,y), \qquad \dot{y} = F_2(x,y),$$

is a Hamiltonian system if and only if

$$div(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0.$$

We compute

$$div(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 3\mu x^2y^2 + 2 - 6x^2y^22 - 2 = 0.$$

Therefore, the system is a Hamiltonian system when $\mu = 2$.

iii) Def.: A Hamiltonian system which is of the form

$$H(x,y) = \frac{1}{2}y^2 + V(x),$$

where $V(x)$ is a function which only depends on x and not y, is called a potential system with potential (function) $V(x)$.

From the definition in (i) follows

$$\dot{x} = \frac{\partial H}{\partial y} = y \Longrightarrow H(x,y) = \frac{1}{2}y^2 + f(x),$$

thus

$$\dot{y} = -\frac{\partial H}{\partial x} = -2x + \frac{20x}{1+x^2} \Longrightarrow H(x,y) = x^2 - 10\ln\left(1+x^2\right) + h(y),$$

Therefore

$$H(x,y) = \frac{1}{2}y^2 + x^2 - 10\ln\left(1+x^2\right) + c.$$

such that the potential is

$$V(x) = x^2 - 10\ln\left(1+x^2\right) + c.$$

From $V(0) = 0$ follows $c = 0$.

iv) The fixed points for the Hamiltonian system described by

$$H(x,y)=\frac{1}{2}y^2+V(x),$$

are located at the points $(a_k,0)$ with $k=1,2,3,...$, where the a_k are stationary points of the potential $V(x)$. If $V(a_k)$ is a minimum, then the point $(a_k,0)$ is a centre and on the other hand, $V(a_k)$ is a maximum, the point $(a_k,0)$ is a saddle point. We compute the stationary points from

$$\dot{V}(x)=\frac{d}{dx}\left(x^2-10\ln\left(1+x^2\right)\right)=\frac{2x\left(x^2-9\right)}{x^2+1}=0,$$

for $x=0,\pm3$. Furthermore,

$$\ddot{V}(x)=\frac{d}{dx}\left(\frac{2x\left(x^2-9\right)}{x^2+1}\right)=\frac{2\left(x^4+12x^2-9\right)}{\left(x^2+1\right)^2},$$

and therefore

$$\ddot{V}(0)=-18\Rightarrow x=0 \text{ is a minimum of } V(x)\Rightarrow (0,0) \text{ is a saddle point,}$$
$$\ddot{V}(\pm3)=\frac{18}{5}\Rightarrow x=\pm3 \text{ are maxima of } V(x)\Rightarrow (\pm3,0) \text{ are centres.}$$

v) The separatrix crosses the saddle point, i.e., $H(0,0)=0$ is conserved on the separatrix. From $\frac{1}{2}y^2+x^2-10\ln\left(1+x^2\right)=0$, the equation for the separatrix is therefore

$$y=\pm\sqrt{2}\sqrt{10\ln\left(x^2+1\right)-x^2}.$$

The direction of time follows from $\dot{x}>0$ for $y>0$ and $\dot{x}<0$ for $y<0$. All trajectories are bounded. We assemble all the information in the following figure.

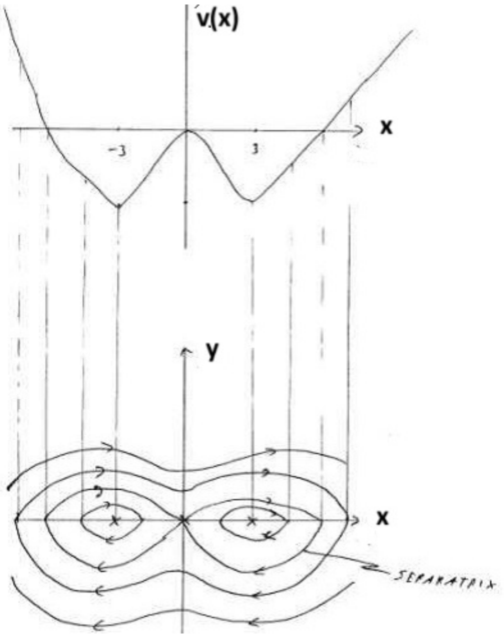


Fig 12 : The phase portrait of system (8.14)

Exercise 67 *Consider the two dynamical systems of the form*

$$i)\begin{cases}\dot{x}=y,\\ \dot{y}=-x^4+x,\end{cases}\quad , \qquad ii)\begin{cases}\dot{x}=x^2+y,\\ \dot{y}=x^3+2y.\end{cases}$$

(i) *Provide an argument (without explicit proof) which confirms that the first system is a set of equations of motion for a Hamiltonian system, whereas the second is not.*

(ii) *Find the Hamiltonian function and confirm that the system is also a potential system. Determine the potential and sketch it. Exploit the fact that the system is a potential system to deduce the position and the nature of the fixed points.*

(iii) *Sketch the phase portrait by drawing some representative trajectories. Determine the equation for the separatrices, include it in the phase portrait, and indicate the regions where* $\dot{x}>0$ *and* $\dot{y}>0$.

Solution.

i) A dynamical system

$$\begin{cases}\dot{x}=F_1(x,y),\\ \dot{y}=F_2(x,y),\end{cases}$$

is a Hamiltonian system if and only if

$$div(F)=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}=0.$$

For the first system, we have

$$div(F)=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}=0.$$

Then the first system is Hamiltonian.

For the second system, we have

$$div(F)=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}=2x+2\neq 0.$$

Then the second system is not Hamiltonian.

ii) From the definition in (i) follows

$$\dot{x}=\frac{\partial H}{\partial y}=y\Longrightarrow H(x,y)=\frac{1}{2}y^2+f(x),$$

so

$$\dot{y}=-\frac{\partial H}{\partial x}=-x^4+x\Longrightarrow H(x,y)=\frac{1}{5}x^5-\frac{1}{2}x^2+h(y),$$

thus

$$H(x,y)=\frac{1}{2}y^2+\frac{1}{5}x^5-\frac{1}{2}x^2,$$

such that the potential is

$$V(x)=\frac{1}{5}x^5-\frac{1}{2}x^2.$$

The fixed points

$$\dot{V}(x) = \frac{d}{dx}\left(\frac{1}{5}x^5 - \frac{1}{2}x^2\right) = x^4 - x = 0,$$

for $x = 0, 1$, then the fixed points are $(0,0), (1,0)$

$$\ddot{V}(x) = \frac{d}{dx}\left(x^4 - x\right) = 4x^3 - 1$$

and therefore, $\ddot{V}(0) = -1 \Rightarrow x = 0$ is a minimum of $V(x) \Rightarrow (0,0)$ is a saddle point, $\ddot{V}(1) = 3 \Rightarrow x = 1$ are maxima of $V(x) \Rightarrow (1,0)$ are centres.

iii) The phase portrait.

From $\frac{1}{2}y^2 + \frac{1}{5}x^5 - \frac{1}{2}x^2 = 0$, we get

$$y = \pm\frac{1}{5}\sqrt{5}x\sqrt{5 - 2x^3}.$$

The direction of time follows from $\dot{x} > 0$ for $y > 0$ and $\dot{y} > 0$ for $x > 0$ or $x < 1$.

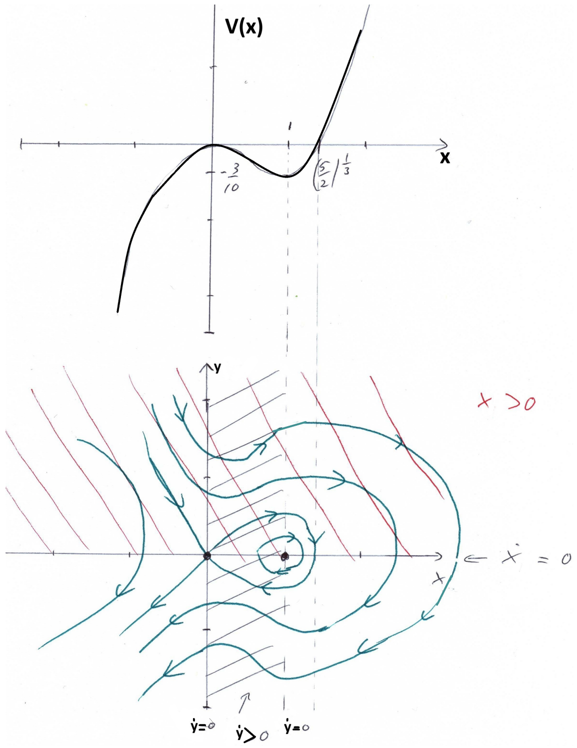


Fig 13 : The phase portrait of the first system (i)

Exercise 68 **(i)** *What is meant by a Hamiltonian system in two-dimensions, equations of motion and an autonomous system?*

(ii) *) Prove that a two dimensional Hamiltonian system can only have two types of nondegenerate fixed points, that is a saddle points or centres.*

(iii) *Provide a criterium that can be used to decide whether a system is Hamiltonian or not. Employ the criterium to determine the value of* α*,* β *and* γ *such that the system*

$$\begin{cases} \dot{x} = \alpha x^2 y^2 e^{(\alpha-1)x^2} + 2x + 5y + \sin y, \\ \dot{y} = \alpha^2 x^3 y^3 e^{(\alpha-1)x^2} + \beta x y^3 e^{(\alpha-1)x^2} + \gamma y, \end{cases}$$

becomes a Hamiltonian system.

(iv) *Provide a definition for a potential system. Derive the Hamiltonian function for the dynamical system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x^2 + \dfrac{5}{(3-x)(2+x)}, \end{cases}$$

and confirm that it is a potential system. Fix the potential in such a way that its value at zero is zero.

Solution.

i) Def.: A system of differential equations on $\mathbb{R}^2$ is said to be a Hamiltonian system with one degree of freedom if there exists a twice continuously differentiable function $H(x,y)$ such that

$$\dot{x} = \frac{\partial H}{\partial y} \text{ and } \dot{y} = -\frac{\partial H}{\partial x}. \tag{8.16}$$

The equations are said to be the equations of motion corresponding to the Hamiltonian H. When H does not depend explicitly on the time t, i.e. it is of the form $H(x(t), y(t))$ and not $H(x(t), y(t), t)$, the system is called autonomous.

ii) Any nondegenerate fixed point of a Hamiltonian system is either a saddle point or a centre.

Proof: We compute the Jacobian matrix to

$$A = \begin{pmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix}.$$

The eigenvalues are then obtained from

$$\det(A - \lambda I) = (H_{yx} - \lambda)(-H_{yx} - \lambda) + H_{xx}H_{yy} = 0,$$

such that

$$\lambda^2 = -H_{xx}H_{yy} + H_{yx}^2.$$

When the fixed point is nondegenerate we only have the two possibilities

$-H_{xx}H_{yy} + H_{yx}^2 > 0 \equiv$ real eigenvalues of opposite sign $\equiv$ saddle point

$-H_{xx}H_{yy} + H_{yx}^2 < 0 \equiv$ purely imaginary eigenvalues $\equiv$ centre

which is what we wanted to prove.

iii) A dynamical system

$$\dot{x} = F_1(x,y), \qquad \dot{y} = F_2(x,y),$$

is a Hamiltonian system if and only if

$$div(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0.$$

For the first system, we have that the first system is Hamiltonian.

$$div(F) = \alpha e^{x^2(\alpha-1)}\left(5\alpha-2\right)x^3y^2 + e^{x^2(\alpha-1)}\left(2\alpha+3\beta\right)xy^2 + (\gamma+2) = 0.$$

Therefore, the system is a Hamiltonian system when $\alpha = \frac{2}{5}, \beta = \frac{-4}{15}, \gamma = -2$ or $\alpha = 0, \beta = 0, \gamma = -2$.

iv) Def.: A Hamiltonian system which is of the form

$$H(x,y) = \frac{1}{2}y^2 + V(x),$$

where $V(x)$ is a function which only depends on x and not y is called a potential system with potential (function) $V(x)$.

From the definition in (i) follows

$$\dot{x} = \frac{\partial H}{\partial y} = y \Longrightarrow H(x,y) = \frac{1}{2}y^2 + f(x),$$

so

$$\dot{y} = -\frac{\partial H}{\partial x} = x^2 + \frac{5}{(3-x)(2+x)} \Longrightarrow H(x,y) = \frac{1}{3}x^3 + \ln\frac{2+x}{3-x} + h(y),$$

with $f(x)$ and $h(y)$ some arbitrary functions. Therefore

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{3}x^3 + \ln\frac{2+x}{3-x} + c;$$

such that the potential is

$$V(x) = \frac{1}{3}x^3 + \ln\frac{2+x}{3-x} + c.$$

From $V(0) = 0$ follows $c = -\ln\left(\frac{2}{3}\right)$.

Exercise 69 **(i)** *Provide a definition for a Hamiltonian system in two dimensions.*

(ii) *Prove that any nondegenerate fixed point of a Hamiltonian system is either a saddle point or a centre. Provide a simple criterium involving derivatives of the Hamiltonian, which allows use to decide which type of fixed point is realized.*

(iii) *Provide a definition for a potential system. Use the fact that the following system*

$$H(x,y) = \frac{1}{2}y^2 + \kappa e^{-x}\sin x, \qquad \kappa \in \mathbb{R}$$

is a potential system to find and classify all its fixed points.

(iv) *For a potential system derive the expression*

$$T = 2\int_{\alpha}^{\beta} \frac{dx}{\sqrt{2\left(E - V(x)\right)}}$$

for the period of a motion around a centre. Explain in your derivation the meaning of the constants α, β, *and* E.

(v) *Given the initial condition* $x = 2^{3/4}, y = 2$ *compute the period* T *for the potential system*

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{4}x^4.$$

Hint: You may use the integral

$$\int_0^1 \frac{dx}{1-x^4} = \sqrt{\pi}\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right).$$

Draw the phase portrait of the system.

$$(a) \qquad \dot{x} = -\gamma e^{-x}\left(\cos x - \sin x\right), \qquad \dot{y} = -y.$$

Solution.

i) Def.: A system of differential equations on $\mathbb{R}^2$ is said to be a Hamiltonian system with one degree of freedom if there exists a twice continuously differentiable function $H(x,y)$ such that

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}.$$

The equations are said to be the equations of motion corresponding to the Hamiltonian H. When H does not depend explicitly on the time t, i.e. it is of the form $H(x(t), y(t))$ and not $H(x(t), y(t), t)$, the system is called autonomous.

ii) Any nondegenerate fixed point of a Hamiltonian system is either a saddle point or a centre.

We compute the Jacobian matrix to

$$A = \begin{pmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix}.$$

The eigenvalues are then obtained from

$$\det(A - \lambda I) = (H_{yx} - \lambda)(-H_{yx} - \lambda) + H_{xx}H_{yy} = 0,$$

such that

$$\lambda^2 = -H_{xx}H_{yy} + H_{yx}^2.$$

When the fixed point is nondegenerate we only have the two possibilities

$-H_{xx}H_{yy} + H_{yx}^2 > 0 \equiv$ real eigenvalues of opposite sign $\equiv$ saddle point

$-H_{xx}H_{yy} + H_{yx}^2 < 0 \equiv$ purely imaginary eigenvalues $\equiv$ centre

iii) Def.: A Hamiltonian system which is of the form

$$H(x,y) = \frac{1}{2}y^2 + V(x),$$

where $V(x)$ is a function which only depends on x and not y, is called a potential system with potential (function) $V(x)$.

-For the system

$$(b) \qquad \dot{x} = y, \qquad \dot{y} = -\gamma e^{-x}\left(\cos x - \sin x\right)$$

under consideration we have

$$\begin{aligned} V(x) &= \gamma e^{-x}\sin x, \\ V'(x) &= \gamma e^{-x}(\cos x - \sin x), \\ V''(x) &= -2\gamma e^{-x}\cos x. \end{aligned}$$

The stationary points from $V(x)$ are at

$$x^{(n)} = \frac{\pi}{4} + n\pi, \qquad n \in \mathbb{N}.$$

Therefore,

$$V(x^{(n)}) = -2\gamma e^{-(\frac{\pi}{4}+n\pi)}\cos(\frac{\pi}{4}+n\pi) = (-1)^{n+1}\sqrt{2}\gamma e^{-(\frac{\pi}{4}+n\pi)}$$

for $\gamma \in \mathbb{R}^+, n$ even: $V(x^{(n)}) < 0$ maximum at $x^{(n)}$ saddle point at $(x^{(n)}, 0)$,

for $\gamma \in \mathbb{R}^+, n$ odd: $V(x^{(n)}) > 0$ minimum at $x^{(n)}$ centre at $(x^{(n)}, 0)$,

for $\gamma \in \mathbb{R}^-, n$ even: $V(x^{(n)}) > 0$ minimum at $x^{(n)}$ centre at $(x^{(n)}, 0)$,

for $\gamma \in \mathbb{R}^-, n$ odd: $V(x^{(n)}) < 0$ maximum at $x^{(n)}$ saddle point at $(x^{(n)}, 0)$.

iv) For a potential system, we have

$$H(x,y) = \frac{1}{2}y^2 + V(x).$$

Since $H(x,y)$ is conserved along a trajectory, i.e. $H(x,y) = E = const$, we can write

$$E = \frac{1}{2}y^2 + V(x) \Longrightarrow y = \pm\sqrt{2\left(E - V\left(x\right)\right)}.$$

For the period T we have to integrate along a trajectory

$$\begin{aligned} T &= \int_C dt = \int_C \frac{dx}{x} = \int_C \frac{dx}{y} \\ &= \int_\alpha^\beta \frac{dx}{\sqrt{2\left(E - V(x)\right)}} + \int_\beta^\alpha \frac{dx}{-\sqrt{2\left(E - V(x)\right)}} \\ &= \int_\alpha^\beta \frac{2dx}{\sqrt{2\left(E - V(x)\right)}}, \end{aligned}$$

where α, β are the turning point, i.e., the values for x when $y = 0$. Therefore,

$$T = 2\int_\alpha^\beta \frac{dx}{\sqrt{2\left(E - V(x)\right)}}.$$

v) First compute the constant E from

$$H(2^{3/4},2)=\frac{1}{2}2^2+\frac{1}{4}2^3=4=E.$$

The turning points result from solving

$$E=4=H(x_t,0)=\frac{1}{4}x_t^4\Longrightarrow 4=\frac{1}{4}x_t^4\Longrightarrow x_t=\pm 2.$$

Therefore

$$\begin{aligned} T &= 2\int_{-2}^{2}\frac{dx}{\sqrt{2\left(4-\frac{x^4}{4}\right)}}=4\int_{0}^{2}\frac{dx}{\sqrt{2\left(4-\frac{x^4}{4}\right)}}\\ &= 4\int_0^1\frac{2dx}{\sqrt{8\left(1-x^4\right)}}=\sqrt{8}\int_0^1\frac{dx}{\sqrt{(1-x^4)}}. \end{aligned}$$

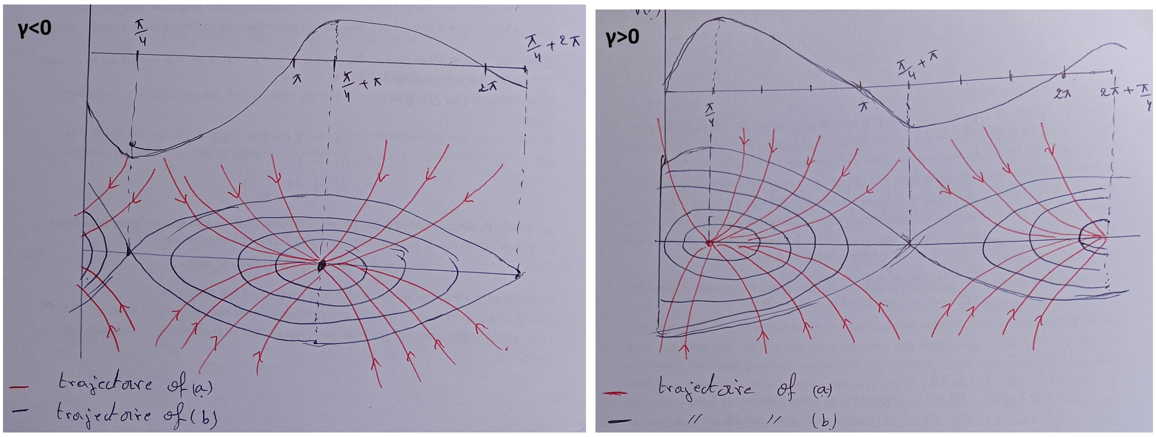


Fig 13 : The phase portrait of systems (a) and (b) in the case when $\gamma<0.$ and $\gamma>0$.

8.4.2 Additional exercises

Exercise 70 *Consider the two dynamical systems of the form*

$$a)\left\{\begin{array}{l}\dot{x}=y,\\ \dot{y}=-2\cos(2x).\end{array}\right.,\qquad b)\left\{\begin{array}{l}\dot{x}=2x^3+\cos(y),\\ \dot{y}=3x+\cos(y).\end{array}\right.$$

(i) *Show that the system a) is a set of equations of motion for a Hamiltonian system, whereas the system b) is not.*

(ii) *For the system a) derive the Hamiltonian function and confirm that the system is also a potential system. Set the ground state energy to zero.*

(iii) *Find all fixed points of the system a) in the range* $0\leq x\leq 2\pi$ *and determine their nature.*

(iv) *Determine the equation for the separatrices of the system a) and sketch the phase portrait in the range* $0\leq x\leq 2\pi$ *by drawing some representative trajectories. Include the seperatrix in your phase portrait and indicate the area of bounded motion.*

Exercise 71 *Consider the two dynamical systems*

$$a)\left\{\begin{array}{l}\dot{x}=y,\\ \dot{y}=x+x^2.\end{array}\right.,\qquad b)\left\{\begin{array}{l}\dot{x}=y-y^2+x^2,\\ \dot{y}=-x-\lambda xy.\end{array}\right.$$

(i) *Provide a criterium which can be used to establish whether a system is Hamiltonian or not. Use this criterium to show that the system (a) is a set of equations of motion for a Hamiltonian system. Determine the value for the parameter* λ *such that the system (b) is also a Hamiltonian system.*

(ii) *Prove that a Hamiltonian system is preserved along trajectories. What does this imply for the form of the trajectories?*

(iii) *For the system a) derive the Hamiltonian function and confirm that the system is also a potential system. Fix the potential such that it vanishes at the origin.*

(iv) *Find all fixed points of the system a) and determine their nature by exploiting the fact that the system is a potential system.*

(v) *Sketch the potential, determine the equation for the separatrices of the system a) and sketch the phase portrait by drawing some representative trajectories. Include the separatrix in your phase portrait and indicate the area of bounded motion. Provide a reasoning for the choice of direction on the trajectories.*

Exercise 72 *Consider the two dynamical systems*

$$a)\begin{cases}\dot{x}=y,\\ \dot{y}=-x^3+x.\end{cases},\qquad b)\begin{cases}\dot{x}=x^3+\kappa x\cos(y),\\ \dot{y}=-3x^2y+2\sin(y).\end{cases}$$

(i) *Show that the system a) is a set of equations of motion for a Hamiltonian system. Determine the value for the parameter* κ *such that the system b) also becomes a Hamiltonian system.*

(ii) *For the system a) derive the Hamiltonian function and confirm that the system is also a potential system. Set the minimum of the potential to zero.*

(iii) *Find all fixed points of the system a) and determine their nature.*

(iv) *Sketch the potential, determine the equation for the separatrices of the system a) and sketch the phase portrait by drawing some representative trajectories. Include the separatrix in your phase portrait and indicate the area of bounded motion. Provide a reason for the choice of direction on the trajectories.*

Exercise 73

(i) *Provide a criterium which can be used to establish whether a dynamical system with one degree of freedom is Hamiltonian or not.*

(ii) *Compute the equations of motion corresponding to the Hamiltonian*

$$H(x,y)=\frac{1}{2}y^2+\frac{x}{1+x^4}.$$

(iii) *Exploit the fact that the Hamiltonian is a potential system to find and classify all fixed points for the dynamical system computed in (ii).*

(iv) *Sketch a graph of the potential and use it to construct a phase portrait for the dynamical system computed in (ii). Compute the separatrices and include them in your phase portrait.*

(v) *Identify the region in phase space in which the motion is periodic and find the period of such a motion. You may leave your answer in the form of a definite integral.*

8

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