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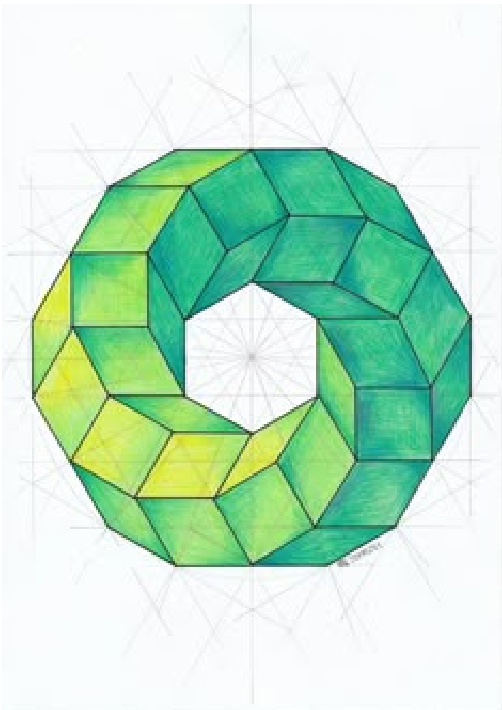
Lecture Notes

Analysis III



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Preface

This course material is designed to support second-year undergraduate mathematics students enrolled in the third-semester fundamental course **Analysis 3**. The course spans a total of **90 hours**, including lectures, tutorials, and practical sessions. It carries a **coefficient of 4** and grants **7 credits** as part of the mathematics degree program.

To follow this course effectively, it is recommended to have completed and understood the concepts covered in **Analysis 1** and **Analysis 2**. These prior courses provide the essential foundation needed to grasp the more advanced concepts presented here, such as function series, generalized integrals, and series expansions.

The main objective of this course is to equip students with both theoretical knowledge and practical skills to tackle more advanced topics in mathematical analysis. Students will explore various modes of convergence for sequences and series of functions, including pointwise and uniform convergence, and understand their implications for continuity, differentiability, and integrability. The course also delves into the development of functions as power series, allowing for a deeper understanding of analytic functions and their properties. A significant portion of the material is dedicated to Fourier series, their convergence, and their applications, especially in solving partial differential equations and studying periodic phenomena. Additionally, students will study generalized integrals, learning the criteria for convergence and the broader context they offer beyond classical integrals. The course concludes with the study of functions defined by integrals, focusing on their continuity, differentiability, and structural properties through classical analysis techniques.

This document is structured to provide a balanced approach between theory and practice. Each chapter gradually introduces fundamental results, accompanied by rigorous proofs and illustrative examples. At the end of each section, numerous exercises are proposed, with detailed solutions provided to help students practice and assess their understanding. Active participation in tutorials is highly encouraged, as well as asking questions and exploring additional readings to deepen the comprehension of key concepts.

We hope this material will be a valuable companion throughout your study of advanced analysis. Whether for consolidating knowledge, preparing for exams, or simply discovering the elegance of mathematical analysis, we wish you a rewarding and inspiring learning experience.

I Sequences

1. A pre-requisite: Sequences

1.1 Limit and convergence

Definition 1.1.1
We say that a sequence (u_n) **converges to the real number** ℓ (or tends to the real number ℓ) if:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |u_n - \ell| \leq \varepsilon.$$

A sequence that does not converge is called a **divergent sequence**.

Theorem 1.1.1
If (u_n) converges to ℓ_1 and if (u_n) converges to ℓ_2, then $\ell_1 = \ell_2$. This real number is then called the **limit** of the sequence (u_n) and is denoted by

$$\lim_{n \to +\infty} u_n = \ell.$$

Proposition 1.1.2
Every convergent sequence is bounded.

R

- We say that a sequence (u_n) **diverges to** $+\infty$ if:

  $$\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, u_n \geq M.$$

  We denote this by $\lim_{n \to +\infty} u_n = +\infty$.
- We say that a sequence (u_n) **diverges to** $-\infty$ if:

  $$\forall m \in \mathbb{R}, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, u_n \leq m.$$

  We denote this by $\lim_{n \to +\infty} u_n = -\infty$.

Theorem 1.1.3 — Squeeze Theorem.
If $(u_n), (v_n)$, and (w_n) are three sequences such that, from a certain index onward,

$$u_n \leq v_n \leq w_n,$$

and (u_n) and (w_n) converge to the same limit ℓ, then (v_n) is convergent with limit ℓ.

Theorem 1.1.4 — Theorem of Divergence by Lower Bound.
If $(u_n),(v_n)$ are two sequences such that, from a certain index onward,

$$u_n \leq v_n$$

and (u_n) diverges to $+\infty$, then (v_n) diverges to $+\infty$.

Proposition 1.1.5
If $\lim_{n\to+\infty} u_n = a$ and if $\lim_{x\to a} f(x) = b$, then $(f(u_n))$ tends to b.

1.2 Limits and Order

Proposition 1.2.1 — Preservation of Inequalities upon Taking Limits.
If $(u_n),(v_n)$ are two sequences converging to ℓ_1 and ℓ_2 respectively, and if from a certain index onward, $u_n \leq v_n$, then $\ell_1 \leq \ell_2$.

Proposition 1.2.2
If (u_n) is a sequence converging to $\ell > 0$, then from a certain index onward, $u_n > 0$.

Theorem 1.2.3 — Monotone Convergence Theorem.
A monotone increasing sequence converges if and only if it is bounded. If it is not bounded, then it tends to $+\infty$.

Definition 1.2.1
Two sequences of real numbers (u_n) and (v_n) are said to be **adjacent** if one is increasing, the other is decreasing, and $(v_n - u_n)$ tends to 0.

Theorem 1.2.4 — Convergence of Adjacent Sequences.
Two adjacent sequences converge to the same limit.

Theorem 1.2.5 — Nested Interval Theorem.
Let (I_n) be a sequence of intervals in $\mathbb{R}$, $I_n = [a_n, b_n]$. We assume these intervals are nested, meaning that for every integer n, $I_{n+1} \subset I_n$. Then there exists a real number x belonging to all I_n. Moreover, if the sequence $(b_n - a_n)$ tends to 0, then $\bigcap_n I_n = \{x\}$.

1.3 Subsequences

If (u_n) is a sequence, a **subsequence** of (u_n) (or **extracted sequence** of (u_n)) is any sequence of the form $(u_{\phi(n)})$, where $\phi : \mathbb{N} \to \mathbb{N}$ is strictly increasing.

Proposition 1.3.1
If (u_n) is a sequence converging to ℓ, then every subsequence of (u_n) also converges to ℓ.

Conversely, if (u_{2n}) and (u_{2n+1}) converge to the **same** limit ℓ, then (u_n) converges to ℓ.

Theorem 1.3.2 — Bolzano-Weierstrass Theorem.
From any bounded sequence of real or complex numbers, a convergent subsequence can be extracted.

1.4 Cauchy Sequences

- A sequence (u_n) is called a Cauchy sequence if, for every $\varepsilon > 0$, there exists an integer N such that, for all $p,q \geq N$, we have $|u_p - u_q| \leq \varepsilon$.

- A sequence of real or complex numbers is a Cauchy sequence if and only if it is convergent.
- In other words, $\mathbb{R}$ and $\mathbb{C}$ are **complete** metric spaces.

1.5 Arithmetic-Geometric Sequences

A sequence (u_n) is an **arithmetic-geometric sequence** if there exist two numbers a and b such that $u_{n+1} = au_n + b$ for every integer n. Generally, we require $a \neq 1$ and $b \neq 0$ to avoid having an arithmetic or geometric sequence.

We then seek ℓ, the solution to the equation

$$\ell = a\ell + b,$$

and then study the sequence (v_n) defined by

$$v_n = u_n - \ell.$$

It can be easily shown that the sequence (v_n) is a geometric sequence with ratio a. We then study (v_n) to understand the behavior of (u_n).

Second-Order Linear Recurrence Sequences

A sequence (u_n) is called a **second-order linear recurrence sequence** if there exist two numbers a and b such that, for every integer n, we have

$$u_{n+2} = au_{n+1} + bu_n.$$

We study these sequences by introducing the characteristic equation

$$r^2 = ar + b$$

and analyzing sequences that satisfy such a recurrence relation based on the roots of this characteristic equation.

- **First case:** The characteristic equation has two distinct real roots, r_1 and r_2. There exist two real numbers λ and μ such that, for every integer n, we have

  $$u_n = \lambda r_1^n + \mu r_2^n.$$

  The real numbers λ and μ can be determined from the values of u_0 and u_1.
- **Second case:** The characteristic equation has a double root r. There exist two real numbers λ and μ such that, for every integer n, we have

  $$u_n = \lambda r^n + \mu n r^n.$$

- **Third case:** The characteristic equation has two complex conjugate roots, of the form $re^{i\alpha}$ and $re^{-i\alpha}$. There exist two real numbers λ and μ such that, for every integer n, we have

  $$u_n = \lambda r^n \cos(n\alpha) + \mu r^n \sin(n\alpha).$$

1.6 Arithmetic and Geometric Sequences

Definition 1.6.1
- A sequence (u_n) is an **arithmetic sequence** with common difference r if, for every $n \in \mathbb{N}$, $u_{n+1} = u_n + r$.
- A sequence (u_n) is a **geometric sequence** with common ratio q if, for every $n \in \mathbb{N}$, $u_{n+1} = qu_n$.

Proposition 1.6.1
If (u_n) is an arithmetic sequence with common difference r, then

- For every $n \in \mathbb{N}$, $u_n = u_0 + nr$;
- If for $n \in \mathbb{N}$, we denote $S_n = u_0 + \cdots + u_n$, then

$$S_n = (n+1) \times \frac{u_0 + u_n}{2} = \text{number of terms} \times \frac{\text{first term} + \text{last term}}{2}.$$

Proposition 1.6.2
If (u_n) is a geometric sequence with common ratio q, then

- For every $n \in \mathbb{N}$, $u_n = q^n u_0$;
- If for $n \in \mathbb{N}$, we denote $S_n = u_0 + \cdots + u_n$, and if $q \neq 1$, then

$$S_n = \frac{u_0 - u_{n+1}}{1-q} = \frac{\text{first term} - \text{term after the last}}{1-q}.$$

The behavior of a geometric sequence is given by the formula for its general term and the following result.

Theorem 1.6.3
Let $q \in \mathbb{R}$. Then the sequence (q^n)

- Tends to $+\infty$ if $q > 1$;
- Is constant and equal to 1 if $q = 1$;
- Tends to 0 if $q \in]-1, 1[$;
- Takes the values $+1$ and -1 alternately if $q = -1$. In particular, it diverges;
- Takes positive and negative values alternately if $q < -1$, with $(|q|^n)$ tending to $+\infty$. In particular, (q^n) diverges.

1.7 Domination, Negligibility, and Equivalence Relations

Let (u_n) and (v_n) be two sequences of real numbers. We assume that (v_n) does not vanish from a certain rank onward.

- We say that (u_n) is **dominated** by (v_n) if the sequence $\left(\frac{u_n}{v_n}\right)$ is bounded. In other words, if there exists a real number M and an integer n_0 such that for all $n \geq n_0$, we have $|u_n| \leq M|v_n|$. We write

$$u_n = O(v_n).$$

- We say that (u_n) is **negligible** in front of (v_n) if the sequence $\left(\frac{u_n}{v_n}\right)$ tends to 0. We write

$$u_n = o(v_n).$$

- We say that (u_n) is **equivalent** to (v_n) if the sequence $\left(\frac{u_n}{v_n}\right)$ tends to 1. We write

$$u_n \sim v_n.$$

- We have $u_n \sim v_n$ if and only if $u_n - v_n = o(v_n)$ if and only if $u_n - v_n = o(u_n)$.
- If two sequences (u_n) and (v_n) are equivalent, then they have the same sign from a certain rank onward.
- If two sequences (u_n) and (v_n) are equivalent, then one converges if and only if the other converges. In this case, their limits are equal.

1.7.1 Calculation Rules for Equivalents

Let (u_n), (v_n), (x_n), and (y_n) be four sequences:

- If $u_n \sim v_n$ and $x_n \sim y_n$, then $u_n x_n \sim v_n y_n$.
- If $u_n \sim v_n$ and $x_n \sim y_n$, then $\frac{u_n}{x_n} \sim \frac{v_n}{y_n}$.
- If $u_n \sim v_n$ and $p \in \mathbb{Z}$, then $u_n^p \sim v_n^p$.

1.7.2 Calculation Rules for the Negligibility Relation

Let (u_n), (v_n), and (w_n) be three sequences:

- If $u_n = o(w_n)$ and $v_n = o(w_n)$, then $\alpha u_n + \beta v_n = o(w_n)$.
- If $u_n = o(v_n)$ and $v_n = o(w_n)$, then $u_n = o(w_n)$.
- If $u_n = o(w_n)$, then $u_n v_n = o(w_n v_n)$.

1.8 Decimal Expansion of a Real Number

Let x be a real number. The **decimal expansion** of x is any representation of $x - \lfloor x \rfloor$ in the form

$$x - \lfloor x \rfloor = \sum_{n=1}^{+\infty} \frac{a_n}{10^n}$$

where, for all $n \geq 1$, a_n is an integer in $\{0, \dots, 9\}$. This relation is sometimes written as:

$$x = \lfloor x \rfloor + \overline{0.a_1 a_2 \dots a_n \dots}.$$

Theorem 1.8.1

- Every real number x has a decimal expansion. To find it, consider the sequence (x_n) of decimal approximations from below to within 10^{-n} of x, defined by

$$x_n = \frac{\lfloor 10^n x \rfloor}{10^n}.$$

  Then the sequence (a_n) defined by $a_n = 10^n(x_n - x_{n-1})$ gives a decimal expansion of x.
- If x is not a decimal, it has a unique decimal expansion. If x is a decimal number, it has exactly two decimal expansions. The first, given by the previous method, has all a_n equal to zero from a certain point onward. This expansion is called the **proper decimal expansion** of x. The second is obtained from the first: if N is the largest integer such that $a_N \neq 0$, then by replacing a_N with $a_N - 1$, and finishing with nines, we get another decimal expansion of x. This expansion is called the **improper decimal expansion** of x.
- A number x is rational if and only if its (proper) decimal expansion is periodic (from a certain point onward).

II Series

2. Infinite series

In this chapter, we will focus on sums with an infinite number of terms. For example, what could be the value of the following infinite sum:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots \quad = \quad ?$$

Let's take also this example. Imagine an ant walking at a speed of 1 centimeter per second on a 1-kilometer rope. However, the ant, being tired, slows down to $\frac{1}{2}$ cm/s after one second. After one more second, its speed slows down to $\frac{1}{3}$ cm/s, and so on. Do you think the ant will reach the other end of the rope?

We can solve this puzzle by considering it as a numerical series. This chapter will teach you what a numerical series is, how to calculate a numerical series, and other essential information.

The problem of the ant describes walking at a speed of 1 cm/s on a 1 km rope, with its speed decreasing over time. After each second, its speed reduces to $\frac{1}{n}$ cm/s in the n-th second. Step-by-step reasoning:

1. Initial speed and distance covered: The ant starts with a speed of 1 cm/s and covers 1 cm in the first second.
2. Speed after the first second: After one second, the ant's speed decreases to $\frac{1}{2}$ cm/s. In the second second, it covers $\frac{1}{2}$ cm.
3. Speed after two seconds: After another second, its speed decreases to $\frac{1}{3}$ cm/s, covering $\frac{1}{3}$ cm in the third second.
4. Speed pattern: The speed follows the pattern $\frac{1}{n}$ cm/s in the n-th second. The distance covered in each second is $\frac{1}{n}$ cm.
5. Total distance covered after infinite time: The total distance covered by the ant after n seconds is the sum of distances covered each second:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

   This is the *harmonic series*, that we will see later that it diverges. In other words, the total distance covered grows without bound as time progresses.

Since the harmonic series diverges, the ant will eventually cover any distance, no matter how large, given enough time. Thus, *the ant will reach the other end of the rope*, but it will take an infinite amount of time. The ant slows down but never completely stops, continuously making progress.

In short, while the ant will reach the other end, it will take forever to do so!

An **infinite series** is the sum of the terms of an infinite sequence. To understand this concept, consider the series:

$$S = u_1 + u_2 + u_3 + u_4 + \cdots$$

where $u_1, u_2, u_3, \ldots$ are the terms of the sequence. The main question is whether this sum converges to a finite value or diverges.

For an infinite series to **converge**, the sequence of its partial sums must approach a finite limit. The n-th partial sum is given by:

$$S_n = u_1 + u_2 + \cdots + u_n$$

The series converges if:

$$\lim_{n\to\infty} S_n = S$$

where S is a finite number.

■ **Example 2.1 — Geometric Series.**
Consider the geometric series:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Here, each term is half of the previous term, so the common ratio r is $\frac{1}{2}$. This series is an example of a geometric series, where each term u_n can be expressed as:

$$u_n = \left(\frac{1}{2}\right)^{n-1}$$

The sum S of an infinite geometric series with first term u_1 and common ratio r (where $|r| < 1$) is given by:

$$S = \frac{u_1}{1-r}$$

In our case, $u_1 = 1$ and $r = \frac{1}{2}$, so the sum of the series is:

$$S = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

Thus, the infinite series:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

converges to 2. ■

Infinite series have a wide range of applications in mathematics, engineering, and science, such as solving differential equations, modeling physical phenomena, and performing numerical analysis.

2.1 Generalities

We call a *series with general term* u_n the sequence $(S_n)_{n\geq 0}$ where for all $n \geq 0$,

$$S_n = \sum_{k=0}^{n} u_k.$$

We denote this sequence by $\sum u_k$, and S_n is called the *partial sum of order n* of the series $\sum u_k$.

We say that the series $\sum u_n$ *converges* if the sequence of its partial sums $(S_n)_{n\geq 0}$ is convergent. We say that it *diverges* otherwise. In the case of convergence, we write

$$\sum_{k=0}^{+\infty} u_k = \lim_{n\to+\infty} S_n.$$

The real or complex number $\sum_{k=0}^{+\infty} u_n$ is called the *sum* of the series $\sum u_k$.

Notation. A series can be written in various ways, and of course with different symbols for the index:

$$\sum_{i=0}^{+\infty} u_i \qquad \sum_{n\in\mathbb{N}} u_n \qquad \sum_{k\geq 0} u_k \qquad \sum u_k.$$

In our case, we will differentiate between any series $\sum_{k\geq 0} u_k$, and we will reserve the notation $\sum_{k=0}^{+\infty} u_k$ for a convergent series or its sum.

2.1.1 Geometric Series

Proposition 2.1.1

Let $q \in \mathbb{C}$. The geometric series $\sum_{k\geq 0} q^k$ is convergent if and only if $|q| < 1$. We then have

$$\sum_{k=0}^{+\infty} q^k = 1 + q + q^2 + q^3 + \cdots = \frac{1}{1-q}.$$

Proof. Consider

$$S_n = 1 + q + q^2 + q^3 + \cdots + q^n.$$

- We immediately dismiss the case $q = 1$, for which $S_n = n+1$. In this case, $S_n \to +\infty$, and the series diverges.
- Let $q \neq 1$ and multiply S_n by $1-q$:

  $$(1-q)S_n = (1 + q + q^2 + \cdots + q^n) - (q + q^2 + \cdots + q^{n+1}) = 1 - q^{n+1}.$$

  Thus,

  $$S_n = \frac{1-q^{n+1}}{1-q}.$$

  If $|q| < 1$, then $q^n \to 0$, hence $q^{n+1} \to 0$, and thus $S_n \to \frac{1}{1-q}$. In this case, the series $\sum_{k\geq 0} q^k$ converges. If $|q| \geq 1$, the sequence (q^n) does not have a finite limit (it may tend towards $+\infty$, for instance, if $q = 2$, or be divergent, for example, if $q = -1$). Therefore, if $|q| \geq 1$, the sequence (S_n) does not have a finite limit, so the series $\sum_{k\geq 0} q^k$ diverges.

■

Example 2.2

1. Geometric series with ratio $q = \frac{1}{2}$: $\sum_{k=0}^{+\infty} \frac{1}{2^k} = \frac{1}{1-\frac{1}{2}} = 2$. This solves Zeno's paradox: the arrow indeed reaches the wall!

2. Geometric series with ratio $q = \frac{1}{3}$, starting with the first term $\frac{1}{3^3}$. By adjusting for the series starting at $k = 0$ and subtracting the initial terms:

$$\sum_{k=3}^{+\infty} \frac{1}{3^k} = \sum_{k=0}^{+\infty} \frac{1}{3^k} - 1 - \frac{1}{3} - \frac{1}{9} = \frac{1}{1-\frac{1}{3}} - \frac{13}{9} = \frac{3}{2} - \frac{13}{9} = \frac{1}{18}.$$

3. Calculating a sum starting from $k = 0$ is purely conventional. We can always perform an index shift to bring it back to a sum starting from 0. Another way to compute the same series $\sum_{k=3}^{+\infty} \frac{1}{3^k}$ as above is by making the index shift $n = k - 3$ (so $k = n + 3$):

$$\sum_{k=3}^{+\infty} \frac{1}{3^k} = \sum_{n=0}^{+\infty} \frac{1}{3^{n+3}} = \sum_{n=0}^{+\infty} \frac{1}{3^3} \frac{1}{3^n} = \frac{1}{27} \sum_{n=0}^{+\infty} \frac{1}{3^n} = \frac{1}{27} \frac{1}{1-\frac{1}{3}} = \frac{1}{18}.$$

4. $\displaystyle\sum_{k=0}^{+\infty} (-1)^k \left(\frac{1}{2}\right)^{2k} = \sum_{k=0}^{+\infty} \left(-\frac{1}{4}\right)^k = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}.$

■

2.2 Convergent Series

2.2.1 Remainder

The convergence of a series does not depend on its first terms: changing a finite number of terms in a series does not affect its nature, whether convergent or divergent. However, if the series is convergent, its sum is obviously altered.

A practical way to study the convergence of a series is to study its remainder: *the remainder of order n* of a convergent series $\sum_{k=0}^{+\infty} u_k$ is:

$$R_n = u_{n+1} + u_{n+2} + \cdots = \sum_{k=n+1}^{+\infty} u_k$$

Proposition 2.2.1
If a series is convergent, then $S = S_n + R_n$ (for all $n \geq 0$) and $\lim_{n\to+\infty} R_n = 0$.

Proof.
- $S = \sum_{k=0}^{+\infty} u_k = \sum_{k=0}^{n} u_k + \sum_{k=n+1}^{+\infty} u_k = S_n + R_n$.
- Therefore, $R_n = S - S_n \to S - S = 0$ as $n \to +\infty$.

■

2.2.2 Sequences and Series

There is no difference between the study of sequences and series. One can easily switch from one to the other.

First, recall that to a series $\sum_{k\geq 0} u_k$, we associate the partial sum $S_n = \sum_{k=0}^{n} u_k$, and by definition, the series is convergent if the sequence $(S_n)_{n\geq 0}$ converges.

Conversely, if we want to study a sequence $(v_k)_{k\geq 0}$, we can use the following result:

Proposition 2.2.2
A *telescoping sum* is a series of the form:

$$\sum_{k\geq 0} (v_{k+1} - v_k).$$

This series is convergent if and only if $\ell := \lim_{k\to+\infty} v_k$ exists, and in this case, we have:

$$\sum_{k=0}^{+\infty} (v_{k+1} - v_k) = \ell - v_0.$$

Proof.

$$\begin{aligned} S_n &= \sum_{k=0}^{n}(v_{k+1}-v_k) \\ &= (v_1-v_0)+(v_2-v_1)+(v_3-v_2)+\cdots+(v_{n+1}-v_n) \\ &= -v_0+v_1-v_1+v_2-v_2+\cdots+v_n-v_n+v_{n+1} \\ &= v_{n+1}-v_0 \end{aligned}$$

■

Here is a very important example.

■ Example 2.3

The series

$$\sum_{k=0}^{+\infty}\frac{1}{(k+1)(k+2)} = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$$

is convergent and has the value 1. Indeed, it can be written as a telescoping sum, and more precisely, the partial sum satisfies:

$$S_n = \sum_{k=0}^{n}\frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right) = 1-\frac{1}{n+2} \to 1 \quad \text{as } n\to+\infty.$$

By a change of index, we also have that the series

$$\sum_{k=1}^{+\infty}\frac{1}{k(k+1)} \quad \text{and} \quad \sum_{k=2}^{+\infty}\frac{1}{k(k-1)}$$

are convergent and have the same sum of 1. ■

2.2.3 The Term of a Convergent Series Tends to 0

Theorem 2.2.3

If the series $\sum_{k\geq 0} u_k$ converges, then the sequence of general terms $(u_k)_{k\geq 0}$ tends to 0.

Proof. The key point is that the general term can be derived from the partial sums by the formula:

$$u_n = S_n - S_{n-1}.$$

For all $n \geq 0$, let $S_n = \sum_{k=0}^{n} u_k$. For all $n \geq 1$, we have $u_n = S_n - S_{n-1}$. If $\sum_{k\geq 0} u_k$ converges, the sequence $(S_n)_{n\geq 0}$ converges to the sum S of the series. The same holds for the sequence $(S_{n-1})_{n\geq 1}$. By the linearity of the limit, the sequence (u_n) tends to $S-S=0$. ■

The contra-positive of this result is often used:

Corollary 2.2.4

A series whose general term does not tend to 0 cannot converge.

■ Example 2.4

1. Consider the series of general term $u_n = \frac{n}{n+1}$. We have $\lim_{n\to\infty}\frac{n}{n+1} = 1$, then the series diverges.
2. Lets consider now $\frac{1}{n^2+n}$. We observe that

$$\frac{1}{k^2+k} = \frac{1}{k} - \frac{1}{k+1}.$$

By summing these equalities for $k = 1 \dots n$,

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right).$$

This is a telescopic sum, it means :

$$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

As $\frac{1}{n+1}$ tends to zero when n tends to infinity. So S_n tends to 1 as $n \to \infty$ and we remark that $\lim_{n\to+\infty} u_n = 0$.

■

■ Example 2.5
For example, the series $\sum_{k\geq 1}\left(1+\frac{1}{k}\right)$ and $\sum_{k\geq 1} k^2$ are divergent.

More interestingly, the series $\sum u_k$ with general term:

$$u_k = \begin{cases} 1 & \text{if } k = 2^\ell \quad \text{for some } \ell \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

diverges. Indeed, even though the terms equal to 1 are rare, there are still infinitely many of them! ■

2.2.4 Linearity

Proposition 2.2.5
Let $\sum_{k=0}^{+\infty} a_k$ and $\sum_{k=0}^{+\infty} b_k$ be two convergent series with respective sums A and B, and let $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$). Then the series $\sum_{k=0}^{+\infty}(\lambda a_k + \mu b_k)$ is convergent with sum $\lambda A + \mu B$. We have:

$$\sum_{k=0}^{+\infty}(\lambda a_k + \mu b_k) = \lambda \sum_{k=0}^{+\infty} a_k + \mu \sum_{k=0}^{+\infty} b_k.$$

Proof. $A_n = \sum_{k=0}^{n} a_k \to A \in \mathbb{C}$ and $B_n = \sum_{k=0}^{n} b_k \to B \in \mathbb{C}$. Therefore,

$$\sum_{k=0}^{n}(\lambda a_k + \mu b_k) = \lambda \sum_{k=0}^{n} a_k + \mu \sum_{k=0}^{n} b_k = \lambda A_n + \mu B_n \to \lambda A + \mu B.$$

■

■ Example 2.6
For example

$$\sum_{k=0}^{+\infty}\left(\frac{1}{2^k} + \frac{5}{3^k}\right) = \sum_{k=0}^{+\infty} \frac{1}{2^k} + 5\sum_{k=0}^{+\infty} \frac{1}{3^k} = \frac{1}{1-\frac{1}{2}} + 5\frac{1}{1-\frac{1}{3}} = 2 + 5 \cdot \frac{3}{2} = \frac{19}{2}.$$

■

As an application to series with complex terms, convergence is equivalent to the convergence of the real and imaginary parts:

Proposition 2.2.6
Let $(u_k)_{k\geq 0}$ be a sequence of complex numbers. For all k, let $u_k = a_k + ib_k$, where a_k is the real part of u_k and b_k the imaginary part. The series $\sum u_k$ converges if and only if the two series $\sum a_k$ and $\sum b_k$ converge. If this is the case, we have:

$$\sum_{k=0}^{+\infty} u_k = \sum_{k=0}^{+\infty} a_k + i \sum_{k=0}^{+\infty} b_k.$$

■ **Example 2.7**
Consider the geometric series $\sum_{k\geq 0} r^k$, where $r = \rho e^{i\theta}$ is a complex number with modulus $\rho < 1$ and argument θ.

Since the modulus of r is strictly less than 1, the series converges and

$$\sum_{k=0}^{+\infty} r^k = \frac{1}{1-r}.$$

Moreover, by de Moivre's formula, $r^k = \rho^k e^{ik\theta}$. The real and imaginary parts of r^k are

$$a_k = \rho^k \cos(k\theta) \quad \text{and} \quad b_k = \rho^k \sin(k\theta).$$

From the previous proposition, we deduce that:

$$\sum_{k=0}^{+\infty} a_k = \Re\left(\sum_{k=0}^{+\infty} r^k\right) = \Re\left(\frac{1}{1-r}\right), \quad \sum_{k=0}^{+\infty} b_k = \Im\left(\sum_{k=0}^{+\infty} r^k\right) = \Im\left(\frac{1}{1-r}\right).$$

The calculation gives:

$$\sum_{k=0}^{+\infty} \rho^k \cos(k\theta) = \frac{1-\rho\cos\theta}{1+\rho^2-2\rho\cos\theta}, \quad \sum_{k=0}^{+\infty} \rho^k \sin(k\theta) = \frac{\rho\sin\theta}{1+\rho^2-2\rho\cos\theta}.$$

■

2.2.5 Cauchy Criterion

R There exist series $\sum_{k\geq 0} u_k$ such that $\lim_{k\to+\infty} u_k = 0$, but $\sum_{k\geq 0} u_k$ diverges. The most classic example is the *harmonic series*: *The series* $\sum_{k\geq 1} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ *diverges.*
More precisely, we have $\lim_{n\to+\infty} S_n = +\infty$. However, we have $u_k = \frac{1}{k} \to 0$ (as $k \to +\infty$).

To show that the series diverges, we will use the Cauchy criterion.
Recall that a sequence (s_n) of real (or complex) numbers converges if and only if it is a Cauchy sequence, that is:

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall m,n \geq n_0 \qquad |s_n - s_m| < \varepsilon$$

For series, this gives us:

Theorem 2.2.7 — Cauchy Criterion.
A series $\sum_{k=0}^{+\infty} u_k$ converges if and only if

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall m,n \geq n_0 \qquad \left|u_n + \cdots + u_m\right| < \varepsilon\ .$$

It is also formulated as follows:

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall m,n \geq n_0 \qquad \left|\sum_{k=n}^{m} u_k\right| < \varepsilon$$

or equivalently:

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad \forall p \in \mathbb{N} \qquad \left|u_n + \cdots + u_{n+p}\right| < \varepsilon$$

Proof. The proof is simply that the sequence (S_n) of partial sums converges if and only if it is a Cauchy sequence. Then, it is enough to note that

$$\left|S_m - S_{n-1}\right| = \left|u_n + \cdots + u_m\right|.$$

■

Let us return to the harmonic series $\sum_{k\geq 1}\frac{1}{k}$. The partial sum is $S_n = \sum_{k=1}^{n}\frac{1}{k}$. We calculate the difference between two partial sums, keeping the terms between $n+1$ (which plays the role of n) and $2n$ (which plays the role of m):

$$S_{2n} - S_n = \frac{1}{n+1} + \cdots + \frac{1}{2n} \geq \frac{n}{2n} = \frac{1}{2}$$

The sequence of partial sums is not a Cauchy sequence (because $\frac{1}{2}$ is not smaller than, for example, $\varepsilon = \frac{1}{4}$), therefore the series does not converge.

If we wish to conclude the demonstration without directly using the Cauchy criterion, we can proceed by contradiction. Suppose $S_n \to \ell \in \mathbb{R}$ (as $n \to +\infty$). Then we also have $S_{2n} \to \ell$ (as $n \to +\infty$), and thus $S_{2n} - S_n \to \ell - \ell = 0$. This contradicts the inequality $S_{2n} - S_n \geq \frac{1}{2}$.

We conclude with a deeper study of the harmonic series.

Proposition 2.2.8

For the harmonic series $\sum\limits_{k\geq 1}\frac{1}{k}$ and its partial sum $S_n = \sum\limits_{k=1}^{n}\frac{1}{k}$, we have

$$\lim_{n\to+\infty} S_n = +\infty.$$

Proof. Let $M > 0$. We choose $m \in \mathbb{N}$ such that $m \geq 2M$. Then for $n \geq 2^m$ we have:

$$\begin{aligned}
S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^m} + \cdots + \frac{1}{n} \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots + \left(\frac{1}{2^{m-1}+1} + \cdots + \frac{1}{2^m}\right) \\
&\geq 1 + \frac{1}{2} + 2\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + 8\cdot\frac{1}{16} + \cdots + 2^{m-1}\cdot\frac{1}{2^m} \\
&= 1 + m\cdot\frac{1}{2} \geq M
\end{aligned}$$

The trick is to group the terms. In each parentheses, there are successively $2, 4, 8, \ldots$ terms, up to

$$2^m - (2^{m-1} + 1) + 1 = 2^m - 2^{m-1} = 2^{m-1} \quad \text{terms.}$$

Thus, for any $M > 0$, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, we have $S_n \geq M$; therefore, (S_n) tends to $+\infty$. Hence, the harmonic series diverges. ■

2.3 Series with Positive Terms

Series with non-negative terms behave like increasing sequences and are therefore easier to study. If the sequence (u_n) is a sequence of positive real numbers, then the sequence (S_n) is increasing. From this, we deduce the following results.

Theorem 2.3.1
A series with positive terms converges if and only if the sequence of its partial sums is bounded.

R

- Since $\sum(-u_n)$ and $\sum u_n$ have the same nature, we can also apply the results to series with negative terms.
- Since $\sum_{n\geq N} u_n$ and $\sum_{n\geq 0} u_n$ have the same nature, we can apply the results even if the first terms do not have a constant sign.

In summary, we will state the theorems for series with positive terms, but they remain valid if the terms are all real and have the same sign from a certain index onwards.

What is the general method for determining the nature of a series with positive terms? We compare it with simple classical series using the following comparison theorem.

Theorem 2.3.2 — Comparison Theorem.
Let $\sum u_k$ and $\sum v_k$ be two series with non-negative terms. Suppose there exists $k_0 \geq 0$ such that for all $k \geq k_0$, $u_k \leq v_k$.

- If $\sum v_k$ converges, then $\sum u_k$ converges.
- If $\sum u_k$ diverges, then $\sum v_k$ diverges.

Proof. As we have observed, convergence does not depend on the initial terms. Without loss of generality, we can assume $k_0 = 0$. Let $S_n = u_0 + \cdots + u_n$ and $S'_n = v_0 + \cdots + v_n$. The sequences (S_n) and (S'_n) are increasing, and furthermore, for all $n \geq 0$, $S_n \leq S'_n$. If the series $\sum v_k$ converges, then the sequence (S'_n) converges. Let S' be its limit. The sequence (S_n) is increasing and bounded by S', so it converges, and thus the series $\sum u_k$ also converges. Conversely, if the series $\sum u_k$ diverges, then the sequence (S_n) tends towards $+\infty$, and the same is true for the sequence (S'_n), and thus the series $\sum v_k$ diverges as well. ■

■ **Example 2.8**

- Consider the series $\sum \frac{1}{n2^n}$, which has positive terms. We have $\frac{1}{n2^n} \leq \frac{1}{2^n}$, and $\sum \frac{1}{2^n}$ is a geometric series with ratio $\frac{1}{2} < 1$, hence it is convergent. Therefore, $\sum \frac{1}{n2^n}$ is also a convergent series.
- Consider the series $\sum \frac{1}{\sqrt{n}}$, which has positive terms. We have $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$ for $n \geq 1$, and since $\sum \frac{1}{n}$ is a divergent series, $\sum \frac{1}{\sqrt{n}}$ is also a divergent series.

■

Corollary 2.3.3 — Equivalence.
Let (u_n) and (v_n) be two sequences of positive real numbers such that $u_n \sim v_n$. Then $\sum_n u_n$ converges if and only if $\sum_n v_n$ converges.

■ **Example 2.9**
Consider the series $\sum \sin\left(\frac{1}{3n}\right)$, which has positive terms. Since $\frac{1}{3n}$ tends to 0 as n tends to infinity, and since $\sin(x) \sim x$ near zero, we have $\sin\left(\frac{1}{3n}\right) \sim \frac{1}{3n}$. Now, the series $\sum \frac{1}{3n}$ is a convergent geometric series, so the series $\sum \sin\left(\frac{1}{3n}\right)$ is also convergent. ■

To apply these results, we need reference series. We have already studied the convergence of geometric series. We will soon study the convergence of series $\sum_{n\geq 1} \frac{1}{n^\alpha}$.

2.4 Comparison Between Series and Integral

This section establishes the connection between series and improper integrals. It is an essential link between two mathematical objects that are, in the end, quite similar. For this part, one needs to be familiar with improper integrals $\int_0^{+\infty} f(t)\, \mathrm{d}t$.

2.4.1 Series/Integral Comparison Theorem

Theorem 2.4.1
Let $f : [0, +\infty[\to [0, +\infty[$ be a decreasing function. Then, the series $\sum_{k \geq 0} f(k)$ (where the general term is $u_k = f(k)$) and the improper integral $\int_0^{+\infty} f(t)\, \mathrm{d}t$ are of the same nature.

"Of the same nature" means that the series and the integral from the theorem are either both convergent or both divergent.

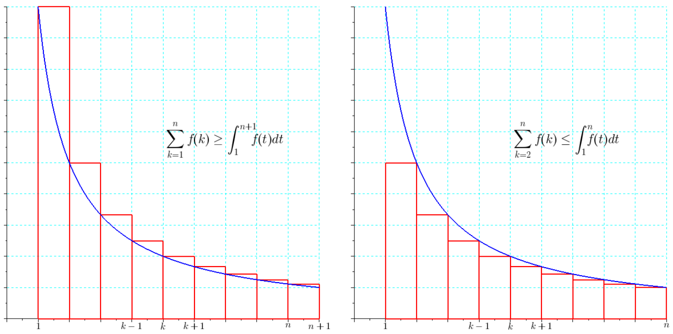
Note! It is important that f is positive and decreasing.

2.4.2 Proof

The easiest way is to understand the diagram well and rework the proof whenever needed.

Proof. Let $k \in \mathbb{N}$. Since f is decreasing, for $k \leq t \leq k+1$, we have $f(k+1) \leq f(t) \leq f(k)$ (note the order). By integrating over the interval $[k, k+1]$ of length 1, we get:

$$f(k+1) \leq \int_k^{k+1} f(t)\, \mathrm{d}t \leq f(k)$$



In the diagram, this inequality means that the area under the curve, between the abscissas k and $k+1$, is between the area of the green rectangle with height $f(k+1)$ and base 1 and the area of the blue rectangle with height $f(k)$ and the same base 1.

We sum these inequalities for k varying from 0 to $n-1$:

$$\sum_{k=0}^{n-1} f(k+1) \leq \sum_{k=0}^{n-1} \int_k^{k+1} f(t)\, \mathrm{d}t \leq \sum_{k=0}^{n-1} f(k).$$

Thus:

$$u_1 + \cdots + u_n \leq \int_0^n f(t)\, \mathrm{d}t \leq u_0 + \cdots + u_{n-1}.$$

The series $\sum u_k$ converges and has a sum S if and only if the sequence of partial sums converges to S. If this is the case, $\int_0^n f(t)\, \mathrm{d}t$ is bounded by S, and since $\int_0^x f(t)\, \mathrm{d}t$ is an increasing function of x (due to the positivity of f), the integral converges. Conversely, if the integral converges, then $\int_0^n f(t)\, \mathrm{d}t$ is bounded, and so is the sequence of partial sums, and the series converges. ■

Let $f : [0, +\infty[\to \mathbb{R}$ be a piecewise continuous function. We are interested in series of the form $\sum f(n)$. When f is monotonic, we can bound $f(n)$ using the method of rectangles. Specifically, we have:

if f is increasing, then for all $n \geq 1$, $\int_{n-1}^{n} f(t)\,dt \leq f(n) \leq \int_n^{n+1} f(t)\,dt$,
if f is decreasing, then for all $n \geq 1$, $\int_n^{n+1} f(t)\,dt \leq f(n) \leq \int_{n-1}^{n} f(t)\,dt$.

By summing these inequalities, we obtain bounds for the partial sums and remainders of the series.

2.4.3 Riemann Series

The comparison theorem (Theorem 2.3.2) and the theorem of asymptotics (Theorem **??**) allow us to reduce the study of series with positive terms to a catalog of series whose convergence is known. In this catalog, we find the Riemann series and Bertrand series.

Let us begin with the *Riemann series* $\sum_{k\geq 1}\frac{1}{k^\alpha}$, where $\alpha > 0$ is a real number.

Proposition 2.4.2

If $\quad \alpha > 1 \quad$ then $\quad \sum_{k=1}^{+\infty}\frac{1}{k^\alpha} \quad$ converges.

If $\quad 0 < \alpha \leq 1 \quad$ then $\quad \sum_{k\geq 1}\frac{1}{k^\alpha} \quad$ diverges.

Proof. In Theorem 2.4.1, nothing requires starting from 0: for $m \in \mathbb{N}$, the series $\sum_{k\geq m} f(k)$ and the improper integral $\int_m^{+\infty} f(t)\,\mathrm{d}t$ are of the same nature.

We apply it to $f : [1,+\infty[\to [0,+\infty[$ defined by $f(t) = \frac{1}{t^\alpha}$. For $\alpha > 0$, this is a decreasing and positive function. We can apply Theorem 2.4.1.

We know that:

$$\int_1^x \frac{1}{t^\alpha}\,\mathrm{d}t = \begin{cases} \dfrac{1}{1-\alpha}(x^{1-\alpha}-1) & \text{if } \alpha \neq 1 \\ \ln(x) & \text{if } \alpha = 1 \end{cases}$$

For $\alpha > 1$, $\int_1^{+\infty}\frac{1}{t^\alpha}\,\mathrm{d}t$ is convergent, so the series $\sum_{k=1}^{+\infty}\frac{1}{k^\alpha}$ converges.

For $0 < \alpha \leq 1$, $\int_1^{+\infty}\frac{1}{t^\alpha}\,\mathrm{d}t$ is divergent, so the series $\sum_{k\geq 1}\frac{1}{k^\alpha}$ diverges. ■

2.4.4 Bertrand Series

A more sophisticated family of series are the *Bertrand series*: $\sum_{k\geq 2}\frac{1}{k^\alpha(\ln k)^\beta}$ where $\alpha > 0$ and $\beta \in \mathbb{R}$.

Proposition 2.4.3

Let the Bertrand series be

$$\sum_{k\geq 2}\frac{1}{k^\alpha(\ln k)^\beta}.$$

If $\quad \alpha > 1 \quad$ then it converges. $\qquad$ If $\quad 0 < \alpha < 1 \quad$ then it diverges.

If $\quad \alpha = 1 \quad$ and $\quad \begin{cases} \beta > 1 & \text{then it converges.} \\ \beta \leq 1 & \text{then it diverges.} \end{cases}$

Proof. The proof is the same as for the Riemann series. For example, in the case $\alpha = 1$:

$$\int_2^x \frac{1}{t(\ln t)^\beta}\,\mathrm{d}t = \begin{cases} \dfrac{1}{1-\beta}\left((\ln x)^{1-\beta} - (\ln 2)^{1-\beta}\right) & \text{if } \beta \neq 1 \\ \ln(\ln x) - \ln(\ln 2) & \text{if } \beta = 1 \end{cases}$$

■

2.4.5 Applications

In particular, we find the following results:

1. $\sum \frac{1}{k^2}$ converges (take $\alpha = 2$),
2. whereas $\sum \frac{1}{k}$ diverges (take $\alpha = 1$).

Let us conclude with two examples of using asymptotics with the Riemann and Bertrand series.

Example 2.10

1. Is the series
$$\sum_{k\geq 1} \ln\left(1+\frac{1}{\sqrt{k^3}}\right)$$
convergent?
Since
$$\ln\left(1+\frac{1}{\sqrt{k^3}}\right) \quad \sim \quad \frac{1}{\sqrt{k^3}}$$
and the Riemann series $\sum \frac{1}{\sqrt{k^3}} = \sum \frac{1}{k^{\frac{3}{2}}}$ converges (because $\frac{3}{2} > 1$), by the theorem of asymptotics, the series $\sum_{k=1}^{+\infty} \ln\left(1+\frac{1}{\sqrt{k^3}}\right)$ also converges.
2. Is the series
$$\sum_{k\geq 1} \frac{1-\cos\left(\frac{1}{k\sqrt{\ln k}}\right)}{\sin\left(\frac{1}{k}\right)}$$
convergent?
We seek an asymptotic equivalent of the general term (which is positive):
$$\frac{1-\cos\left(\frac{1}{k\sqrt{\ln k}}\right)}{\sin\left(\frac{1}{k}\right)} \quad \sim \quad \frac{1}{2k\ln k}$$
Now, the Bertrand series $\sum \frac{1}{k\ln k}$ diverges, so our series also diverges.

■

2.5 Quotient Rule of d'Alembert

The quotient rule of d'Alembert is an effective way to determine whether a series of real numbers converges or not.

Theorem 2.5.1 — Quotient Rule of d'Alembert.
Let $\sum u_k$ be a series with positive terms whose general term are non-zero real number.

1. If there exists a constant $0 < q < 1$ and an integer k_0 such that, for all $k \geq k_0$,
$$\frac{u_{k+1}}{u_k} \leq q < 1, \quad \text{then} \quad \sum u_k \quad \text{converges.}$$
2. If there exists an integer k_0 such that, for all $k \geq k_0$,
$$\frac{u_{k+1}}{u_k} \geq 1, \quad \text{then} \quad \sum u_k \quad \text{diverges.}$$

Here is a direct and most commonly used application for series with strictly positive real terms:

Corollary 2.5.2 — Quotient Rule of d'Alembert.
Let $\sum u_k$ be a series with strictly positive terms such that $\frac{u_{k+1}}{u_k}$ converges to ℓ.

1. If $\ell < 1$, then $\sum u_k$ converges.
2. If $\ell > 1$, then $\sum u_k$ diverges.
3. If $\ell = 1$, one cannot generally conclude.

Proof. First, recall that the geometric series $\sum q^k$ converges if $|q| < 1$, and diverges otherwise.

In the first case of the theorem, the assumption $\frac{u_{k+1}}{u_k} \leq q$ implies $u_{k_0+1} \leq u_{k_0} q$, then $u_{k_0+2} \leq u_{k_0} q^2$. By induction, for all $k \geq k_0$,
$$u_k \leq u_{k_0} q^{-k_0} \cdot q^k = c \cdot q^k,$$

where c is a constant. Since $0<q<1$, the series $\sum q^k$ converges, and by the comparison theorem 2.3.2, the series $\sum u_k$ converges.

If $\frac{u_{k+1}}{u_k} \geq 1$, the sequence (u_k) is increasing, so it cannot tend to zero, and the series diverges. ■

Example 2.11

1. For all fixed $x \in \mathbb{R}$, the *exponential series*

$$\sum_{k=0}^{+\infty} \frac{x^k}{k!} \quad \text{converges.}$$

Indeed, for $u_k = \frac{x^k}{k!}$, we have

$$\left|\frac{u_{k+1}}{u_k}\right| = \frac{\left|\frac{x^{k+1}}{(k+1)!}\right|}{\left|\frac{x^k}{k!}\right|} = \frac{|x|}{k+1} \to 0 \quad \text{as } k \to +\infty.$$

Since the limit is $\ell = 0 < 1$, by the quotient rule of d'Alembert, the series is absolutely convergent, and hence convergent. By definition, the sum is $\exp(x)$:

$$\exp(x) = \sum_{k=0}^{+\infty} \frac{x^k}{k!}.$$

2. $\sum_{k\geq 0} \frac{k!}{1\cdot 3\cdots(2k-1)}$ converges because $\frac{u_{k+1}}{u_k} = \frac{k+1}{2k+1}$ tends to $\frac{1}{2} < 1$.
3. $\sum_{k\geq 0} \frac{(2k)!}{(k!)^2}$ diverges because $\frac{u_{k+1}}{u_k} = \frac{(2k+1)(2k+2)}{(k+1)^2}$ tends to $4 > 1$.

■

2.6 Cauchy's Root Test

Theorem 2.6.1 — Cauchy's Root Test.
Let $\sum u_k$ be a series with positive terms of real numbers.

1. If there exists a constant $0<q<1$ and an integer k_0 such that, for all $k \geq k_0$,

$$\sqrt[k]{u_k} \leq q < 1, \quad \text{then} \quad \sum u_k \quad \text{converges.}$$

2. If there exists an integer k_0 such that, for all $k \geq k_0$,

$$\sqrt[k]{u_k} \geq 1, \quad \text{then} \quad \sum u_k \quad \text{diverges.}$$

Most of the time, you will apply this to a series with strictly positive terms.

Corollary 2.6.2 — Cauchy's Root Test.
Let $\sum u_k$ be a series with positive terms, such that $\sqrt[k]{u_k}$ converges to ℓ.

1. If $\ell < 1$, then $\sum u_k$ converges.
2. If $\ell > 1$, then $\sum u_k$ diverges.
3. If $\ell = 1$, one cannot generally conclude.

In practice, it is important to handle k-th roots well:

$$\sqrt[k]{u_k} = (u_k)^{\frac{1}{k}} = \exp\left(\frac{1}{k}\ln u_k\right).$$

Proof. Recall that the nature of the series does not depend on its first terms. In the first case of the theorem, $\sqrt[k]{u_k} \le q$ implies $u_k \le q^k$. Since $0 < q < 1$, the series $\sum q^k$ converges, and by the comparison theorem 2.3.2, we conclude that $\sum u_k$ converges.

In the second case, $\sqrt[k]{u_k} \ge 1$, so $u_k \ge 1$. The general term does not tend to zero, so the series diverges.

Finally, for the last point in the corollary, consider $u_k = \frac{1}{k}$ and $v_k = \frac{1}{k^2}$. We have $\sqrt[k]{u_k} \to 1$ as well as $\sqrt[k]{v_k} \to 1$. However, $\sum u_k$ diverges while $\sum v_k$ converges. ■

■ **Example 2.12**

1. For example,

$$\sum \left(\frac{2k+1}{3k+4}\right)^k \quad \text{converges,}$$

   because $\sqrt[k]{u_k} = \frac{2k+1}{3k+4}$ tends to $\frac{2}{3} < 1$.
2. However,

$$\sum \frac{2^k}{k^\alpha} \quad \text{diverges,}$$

   for any $\alpha > 0$. Indeed,

$$\sqrt[k]{u_k} = \frac{\sqrt[k]{2^k}}{\left(\sqrt[k]{k}\right)^\alpha} = \frac{2}{\left(k^{\frac{1}{k}}\right)^\alpha} = \frac{2}{\left(\exp\left(\frac{1}{k}\ln k\right)\right)^\alpha} \to 2 > 1.$$

■

■ **Example 2.13**
Determine all $z \in \mathbb{C}$ such that the series

$$\sum_{k\ge1} \left(1+\frac{1}{k}\right)^{k^2} z^k$$

is absolutely convergent.

Let $u_k = \left(1+\frac{1}{k}\right)^{k^2} z^k$. We have

$$\sqrt[k]{|u_k|} = \left(1+\frac{1}{k}\right)^k |z| \to e|z|.$$

This limit satisfies $e|z| < 1$ if and only if $|z| < \frac{1}{e}$.

- If $|z| < \frac{1}{e}$, the series $\sum u_k$ is absolutely convergent.
- If $|z| > \frac{1}{e}$, for sufficiently large k, we have $\sqrt[k]{|u_k|} > 1$, so the series diverges.
- If $|z| = \frac{1}{e}$, Cauchy's Root Test does not allow us to conclude. We study the general term by hand:

$$|u_k| = \left(1+\frac{1}{k}\right)^{k^2} \left(\frac{1}{e}\right)^k.$$

  Then

$$\ln|u_k| = k^2 \ln\left(1+\frac{1}{k}\right) + k\ln\frac{1}{e}.$$

  After some calculations, we find

$$\ln|u_k| \to -\frac{1}{2},$$

  which implies $|u_k| \to e^{-\frac{1}{2}} \ne 0$. Thus $\sum |u_k|$ diverges.

■

2.6.1 D'Alembert vs Cauchy

This section may be skipped during an initial reading.

We will compare d'Alembert's quotient rule with Cauchy's root test. We will see that Cauchy's root test is more powerful than d'Alembert's quotient rule. However, in practice, d'Alembert's quotient rule remains the most commonly used.

Proposition 2.6.3
Let (u_k) be a sequence with strictly positive terms.

$$\text{If } \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \ell \quad \text{then} \quad \lim_{k\to+\infty} \sqrt[k]{u_k} = \ell.$$

In other words, if we can apply d'Alembert's quotient rule, we can also apply Cauchy's root test.

Proof. For all $\varepsilon > 0$, there exists k_0 such that, for all $k \geq k_0$,

$$\ell - \varepsilon < \frac{u_{k+1}}{u_k} < \ell + \varepsilon.$$

By induction, we deduce:

$$u_{k_0}(\ell-\varepsilon)^{k-k_0} \leq u_k \leq u_{k_0}(\ell+\varepsilon)^{k-k_0}.$$

Taking the k-th root of both sides:

$$\lim_{k\to+\infty} \sqrt[k]{u_{k_0}(\ell-\varepsilon)^{k-k_0}} = \ell - \varepsilon,$$

and

$$\lim_{k\to+\infty} \sqrt[k]{u_{k_0}(\ell+\varepsilon)^{k-k_0}} = \ell + \varepsilon.$$

Thus, for $k > k_1$, we have

$$\ell - 2\varepsilon < \sqrt[k]{u_k} < \ell + 2\varepsilon,$$

which gives the result. ■

2.7 Alternating Series

There is another type of series that is easy to study: *alternating series*. These are series where the sign of the general term changes with each term.

■ Example 2.14
The series $\sum \frac{(-1)^n}{n^\alpha}$ is convergent if and only if $\alpha > 0$. ■

2.7.1 Leibniz Criterion

Let $(u_k)_{k\geq 0}$ be a sequence such that $u_k \geq 0$. The series $\sum_{k\geq 0}(-1)^k u_k$ is called an *alternating series*.

We have the following convergence criterion, which is extremely easy to verify:

Theorem 2.7.1 — Leibniz Criterion.
Suppose that $(u_k)_{k\geq 0}$ is a sequence that satisfies the following:

1. $u_k \geq 0$ for all $k \geq 0$,
2. the sequence (u_k) is decreasing,
3. and $\lim_{k\to+\infty} u_k = 0$.

Then, the alternating series $\sum_{k=0}^{+\infty}(-1)^k u_k$ converges.

Proof. We proceed by examining two adjacent sequences.

- The sequence (S_{2n+1}) is increasing because $S_{2n+1} - S_{2n-1} = u_{2n} - u_{2n+1} \geq 0$.
- The sequence (S_{2n}) is decreasing because $S_{2n} - S_{2n-2} = u_{2n} - u_{2n-1} \leq 0$.
- $S_{2n} \geq S_{2n+1}$ because $S_{2n+1} - S_{2n} = -u_{2n+1} \leq 0$.
- Finally, $S_{2n+1} - S_{2n}$ tends to 0 since $S_{2n+1} - S_{2n} = -u_{2n+1} \to 0$ as $n \to +\infty$.

As a result, (S_{2n+1}) and (S_{2n}) both converge and converge to the same limit S. Thus, (S_n) converges to S.

Additionally, we have shown that $S_{2n+1} \leq S \leq S_{2n}$ for all n.

Furthermore,

$$0 \geq R_{2n} = S - S_{2n} \geq S_{2n+1} - S_{2n} = -u_{2n+1}$$

and

$$0 \leq R_{2n+1} = S - S_{2n+1} \leq S_{2n+2} - S_{2n+1} = u_{2n+2}.$$

Therefore, regardless of the parity of n, we have

$$|R_n| = |S - S_n| \leq u_{n+1}.$$

■

■ **Example 2.15**
The *alternating harmonic series*

$$\sum_{k=0}^{+\infty} (-1)^k \frac{1}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges. Indeed, setting $u_k = \frac{1}{k+1}$, we have:

1. $u_k \geq 0$,
2. (u_k) is a decreasing sequence,
3. and (u_k) tends to 0.

By Leibniz's criterion (Theorem 2.7.1), the alternating series $\sum_{k=0}^{+\infty} (-1)^k \frac{1}{k+1}$ converges. ■

2.7.2 Remainder

Not only does Leibniz's criterion prove the convergence of the series $\sum_{k=0}^{+\infty} (-1)^k u_k$, but the proof also provides two additional important results: an estimate of the sum and a bound on the remainder.

Corollary 2.7.2
Let $\sum_{k=0}^{+\infty} (-1)^k u_k$ be an alternating series satisfying the conditions of Theorem 2.7.1. Let S be the sum of the series, and let (S_n) denote the sequence of partial sums.

1. The sum S satisfies the bounds:

$$S_1 \leq S_3 \leq S_5 \leq \cdots \leq S_{2n+1} \leq \cdots \leq S \leq \cdots \leq S_{2n} \leq \cdots \leq S_4 \leq S_2 \leq S_0.$$

2. Moreover, if $R_n = S - S_n = \sum_{k=n+1}^{+\infty} (-1)^k u_k$ is the remainder after n terms, then:

$$|R_n| \leq u_{n+1}.$$

For an alternating series, the rate of convergence is therefore dictated by how quickly the sequence (u_k) decreases to 0. This decrease can sometimes be quite slow.

■ **Example 2.16**
For example, we saw that the alternating harmonic series $\sum_{k=0}^{+\infty} \frac{(-1)^k}{k+1}$ converges; let S be its sum. The partial sums are $S_0 = 1$, $S_1 = 1 - \frac{1}{2}$, $S_2 = 1 - \frac{1}{2} + \frac{1}{3}$, $S_3 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$, $S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$, ... The bounds from the corollary give:

$$1-\frac{1}{2} \leq 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4} \leq \cdots \leq S_{2n+1} \leq \cdots \leq S \leq \cdots \leq S_{2n} \leq \cdots \leq 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5} \leq 1-\frac{1}{2}+\frac{1}{3} \leq 1$$

Thus,

$$S_3 = \frac{35}{60} \approx 0.58333\ldots \leq S \leq S_4 = \frac{47}{60} \approx 0.78333\ldots$$

If we calculate further, for $n = 200$ we get:

$$S_{201} \approx 0.69067\ldots \leq S \leq S_{200} \approx 0.69562\ldots$$

This gives the first two decimal places of $S \approx 0.69\ldots$

Furthermore, we have an estimate for the error, using the inequality $|R_n| \leq u_{n+1}$. We find that the error in approximating S by S_{200} is: $|S - S_{200}| = |R_{200}| \leq u_{201} = \frac{1}{202} < 5 \cdot 10^{-3}$.

In fact, you will later learn that $S = \ln 2 \approx 0.69314\ldots$ ■

2.7.3 Counterexample

We conclude with two warnings:

1. We cannot drop the condition that (u_k) is decreasing in Leibniz's criterion.
2. It is not possible to replace u_k by an asymptotic equivalent at infinity in Theorem 2.7.1, because the property of being decreasing is not preserved by asymptotic equivalence.

■ **Example 2.17**
Here are two alternating series:

$$\sum_{k\geq 2} \frac{(-1)^k}{\sqrt{k}} \quad \text{converges,} \qquad \sum_{k\geq 2} \frac{(-1)^k}{\sqrt{k}+(-1)^k} \quad \text{diverges.}$$

■

2.7.4 Abel's Summation Theorem

Abel's summation theorem applies to certain convergent series that are not absolutely convergent. It is a theorem that applies to series of the form $\sum a_k b_k$ and is stronger than the Leibniz criterion for alternating series, but it is also more challenging to implement.

Theorem 2.7.3 — Abel's Summation Theorem.
Let $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ be two sequences such that:

1. The sequence $(a_k)_{k\geq 0}$ is a decreasing sequence of positive real numbers that tends to 0.
2. The partial sums of the sequence $(b_k)_{k\geq 0}$ are bounded:

$$\exists M \quad \forall n \in \mathbb{N} \qquad \left| b_0 + \cdots + b_n \right| \leq M.$$

Then the series $\sum_{k\geq 0} a_k b_k$ converges.

Leibniz's criterion concerning alternating series is a special case: indeed, if $b_k = (-1)^k$ then $\left|\sum_{k=0}^{n} b_k\right| \leq 1$. Therefore, if (a_k) is a positive, decreasing sequence that tends to 0, then $\sum a_k b_k$ converges.

Proof. The idea of the proof is to make a change in the summation, similar to integration by parts. For any $n \geq 0$, let $B_n = b_0 + \cdots + b_n$. By hypothesis, the sequence (B_n) is bounded. We write the partial sums of the series $\sum a_k b_k$ in the following form:

$$\begin{aligned} S_n &= a_0 b_0 + a_1 b_1 + \cdots + a_{n-1} b_{n-1} + a_n b_n \\ &= a_0 B_0 + a_1 (B_1 - B_0) + \cdots + a_{n-1}(B_{n-1} - B_{n-2}) + a_n (B_n - B_{n-1}) \\ &= B_0(a_0 - a_1) + B_1(a_1 - a_2) + \cdots + B_{n-1}(a_{n-1} - a_n) + B_n a_n \,. \end{aligned}$$

Since (B_n) is bounded, and a_n tends to 0, the last term $B_n a_n$ tends to 0. We will show that the series $\sum B_k (a_k - a_{k+1})$ is absolutely convergent. Indeed,

$$\left|B_k(a_k - a_{k+1})\right| = \left|B_k\right|(a_k - a_{k+1}) \leq M(a_k - a_{k+1}) \,,$$

because the sequence (a_k) is a sequence of positive, decreasing real numbers, and $|B_k|$ is bounded by M. Now

$$M(a_0 - a_1) + \cdots + M(a_n - a_{n+1}) = M(a_0 - a_{n+1}) \,,$$

which tends to Ma_0 since (a_k) tends to 0. The series $\sum M(a_k - a_{k+1})$ converges, so the series $\sum \left|B_k(a_k - a_{k+1})\right|$ also converges, by the comparison theorem **??**. Thus, the series $\sum B_k(a_k - a_{k+1})$ is convergent, which means that the sequence (S_n) is convergent, proving that the series $\sum a_k b_k$ converges. ■

2.7.5 Fourier Series

The most common application is the case where $b_k = e^{ik\theta}$.

Corollary 2.7.4
Let θ be a real number such that $\theta \neq 2n\pi$ (for any $n \in \mathbb{Z}$). Let (a_k) be a sequence of positive, decreasing real numbers tending to 0. Then the *Fourier series*: $\sum a_k e^{ik\theta}$ $\quad \sum a_k \cos(k\theta)$ $\quad \sum a_k \sin(k\theta)$ converge

Proof. To apply Abel's summation theorem (theorem 2.7.3) with $b_k = e^{ik\theta}$, we need to verify that the partial sums of the sequence $(e^{ik\theta})$ are bounded. Since $e^{ik\theta} = (e^{i\theta})^k$, and by hypothesis $e^{i\theta}$ is different from 1, we have the sum of a geometric series:

$$\left|1 + e^{i\theta} + \cdots + e^{ik\theta}\right| = \left|\frac{1 - e^{i(k+1)\theta}}{1 - e^{i\theta}}\right| \leq \left|\frac{2}{1 - e^{i\theta}}\right| \,.$$

Thus, the result follows.

Since $\sum a_k e^{ik\theta} = \sum a_k \cos(k\theta) + i \sum a_k \sin(k\theta)$, the convergence of the series $\sum a_k \cos(k\theta)$ and $\sum a_k \sin(k\theta)$ is a direct consequence of proposition 2.2.6. ■

2.7.6 Absolutely Convergent Series

Definition 2.7.1 A series $\sum_{k\geq 0} u_k$ of real (or complex) numbers is said to be absolutely convergent if the series $\sum_{k\geq 0} |u_k|$ is convergent.

■ **Example 2.18**

1. For example, the series $\sum_{k\geq 1} \frac{\cos k}{k^2}$ is absolutely convergent. For $u_k = \frac{\cos k}{k^2}$, we have $|u_k| \leq \frac{1}{k^2}$. Since the series $\sum_{k\geq 1} \frac{1}{k^2}$ converges, the series $\sum_{k\geq 1} |u_k|$ also converges.
2. The alternating harmonic series $\sum_{k=0}^{+\infty} \frac{(-1)^k}{k+1}$ is not absolutely convergent. For $v_k = \frac{(-1)^k}{k+1}$, the series $\sum_{k\geq 0} |v_k| = \sum_{k\geq 0} \frac{1}{k+1}$ diverges.

■

A series, such as the alternating harmonic series, which is convergent but not absolutely convergent, is called a conditionally convergent series.

Being absolutely convergent is a stronger condition than being convergent:

Theorem 2.7.5
Every absolutely convergent series is convergent.

Proof. We will use the Cauchy criterion. Let $\sum u_k$ be an absolutely convergent series. Since the series $\sum |u_k|$ converges, the sequence of remainders (R'_n), where $R'_n = \sum_{k=n+1}^{+\infty} |u_k|$, tends to 0. Therefore, in particular, it is a Cauchy sequence.

Let $\varepsilon > 0$ be fixed. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $p \geq 0$:

$$|u_n| + |u_{n+1}| + \cdots + |u_{n+p}| < \varepsilon.$$

Thus, for $n \geq n_0$ and $p \geq 0$, we have:

$$\left|u_n + u_{n+1} + \cdots + u_{n+p}\right| \leq |u_n| + |u_{n+1}| + \cdots + |u_{n+p}| < \varepsilon.$$

Therefore, by the Cauchy criterion (Theorem 2.2.7), the series $\sum u_k$ is convergent. ■

2.8 Products of Two Series

This section, dedicated to the product of two series, can be skipped during a first reading.

For a product of sums, there are several ways to arrange the terms once the product is expanded. In the case of a finite sum, the order of terms is not important, but for a series, it is essential. We choose to group terms based on the indices, as follows:

$$(a_0 + a_1)(b_0 + b_1) = \underbrace{a_0 b_0}_{\text{sum of indices}=0} + \underbrace{a_0 b_1 + a_1 b_0}_{\text{sum of indices}=1} + \underbrace{a_1 b_1}_{\text{sum of indices}=2}$$

$$\begin{aligned}(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = {} & \underbrace{a_0 b_0}_{\text{sum of indices}=0} + \underbrace{a_0 b_1 + a_1 b_0}_{\text{sum of indices}=1} \\ & + \underbrace{a_0 b_2 + a_1 b_1 + a_2 b_0}_{\text{sum of indices}=2} + \underbrace{a_1 b_2 + a_2 b_1}_{\text{sum of indices}=3} + \underbrace{a_2 b_2}_{\text{sum of indices}=4}\end{aligned}$$

More generally, here are different ways to write a product of two sums:

$$\left(\sum_{i=0}^{n} a_i\right)\left(\sum_{j=0}^{n} b_j\right) = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i b_j = \sum_{0 \leq k \leq 2n}\ \sum_{i+j=k} a_i b_j = \sum_{0 \leq k \leq 2n}\ \sum_{0 \leq i \leq k} a_i b_{k-i}.$$

The last two forms correspond to our decomposition based on the sum of indices.

2.8.1 The Cauchy Product

Definition 2.8.1
Let $\sum_{i \geq 0} a_i$ and $\sum_{j \geq 0} b_j$ be two series. We call the product series or Cauchy product the series $\sum_{k \geq 0} c_k$ where $c_k = \displaystyle\sum_{i=0}^{k} a_i b_{k-i}$.

Another way to write the coefficient c_k is:

$$c_k = \sum_{i+j=k} a_i b_j$$

Theorem 2.8.1
If the series $\sum_{i=0}^{+\infty} a_i$ and $\sum_{j=0}^{+\infty} b_j$ of real (or complex) numbers are absolutely convergent, then the product series

$$\sum_{k=0}^{+\infty} c_k = \sum_{k=0}^{+\infty}\left(\sum_{i=0}^{k} a_i b_{k-i}\right)$$

is absolutely convergent, and we have:

$$\sum_{k=0}^{+\infty} c_k = \left(\sum_{i=0}^{+\infty} a_i\right) \times \left(\sum_{j=0}^{+\infty} b_j\right).$$

Proof. **Notations.**

- $S_n = a_0 + \cdots + a_n$, $S_n \to S$,
- $T_n = b_0 + \cdots + b_n$, $T_n \to T$,
- $P_n = c_0 + \cdots + c_n$.

We need to show that $P_n \to S \cdot T$.

Case 1: $a_k \geq 0, b_k \geq 0$ $(\forall k)$.

In this case $c_k \geq 0$ and we have

$$P_n \leq S_n \cdot T_n \leq S \cdot T.$$

The sequence (P_n) is increasing and bounded, thus convergent: $P_n \to P$.

Also, we have

$$P_n \leq S_n \cdot T_n \leq P_{2n}.$$

Thus, as $n \to +\infty$, we have: $P \leq S \cdot T \leq P$. So $P_n \to S \cdot T$.

Case 2: $a_k \in \mathbb{C}, b_k \in \mathbb{C}$ $(\forall k)$.

Let:

- $S'_n = |a_0| + \cdots + |a_n|$, $S'_n \to S'$,
- $T'_n = |b_0| + \cdots + |b_n|$, $T'_n \to T'$,
- $P'_n = c'_0 + \cdots + c'_n$ where $c'_k = \sum_{i=0}^{k} |a_i b_{k-i}|$.

From the first case, $P'_n \to P'$ with $P' = S' \cdot T'$. Thus,

$$|S_n \cdot T_n - P_n| = \Big|\sum_{\substack{0 \leq i,j \leq n \\ i+j>n}} a_i b_j\Big| \leq \sum_{\substack{0 \leq i,j \leq n \\ i+j>n}} |a_i b_j| = S'_n \cdot T'_n - P'_n \to S' \cdot T' - P' = 0.$$

Thus $P_n = S_n \cdot T_n - (S_n \cdot T_n - P_n) \to S \cdot T - 0 = S \cdot T$.

So the series $\sum c_k$ is convergent, and its sum is $S \cdot T$. Moreover, $|c_k| \leq c'_k$. The convergence of $\sum c'_k$ thus implies the absolute convergence of $\sum c_k$. ■

▪ **Example 2.19** Let $\sum_{i=0}^{+\infty} a_i$ be an absolutely convergent series and let $\sum_{j=0}^{+\infty} b_j$ be the series defined by $b_j = \frac{1}{2^j}$. The series $\sum b_j$ is absolutely convergent.

Let

$$c_k = \sum_{i=0}^{k} a_i b_{k-i} = \sum_{i=0}^{k} a_i \times \frac{1}{2^{k-i}}.$$

Then the series $\sum c_k$ converges absolutely and

$$\sum_{k=0}^{+\infty} c_k = \left(\sum_{i=0}^{+\infty} a_i\right) \times \left(\sum_{j=0}^{+\infty} b_j\right) = 2\sum_{i=0}^{+\infty} a_i.$$

▪

If the series $\sum a_i$ and $\sum b_j$ are not absolutely convergent, but only convergent, then the Cauchy series can be divergent.

■ Example 2.20 — Counterexample.
Let $a_i = b_i = \frac{(-1)^i}{\sqrt{i+1}}, i \geq 0$. Then $\sum a_i$ and $\sum b_j$ converge by the Leibniz criterion, but are not absolutely convergent. We have

$$c_k = \sum_{i=0}^{k} a_i b_{k-i} = \sum_{i=0}^{k} \frac{(-1)^i}{\sqrt{i+1}} \frac{(-1)^{k-i}}{\sqrt{k-i+1}} = (-1)^k \sum_{i=0}^{k} \frac{1}{\sqrt{(i+1)(k-i+1)}}.$$

Now, for $x \in \mathbb{R}$, $(x+1)(k-x+1) = -x^2 + (k+1)^2 - (2k+1)(x+1)$.

Thus, we have:

$$\frac{1}{\sqrt{(i+1)(k-i+1)}} \sim \frac{1}{k+1}, \text{ as } i \to k/2.$$

Then, the term $\sum_{i=0}^{k} \frac{1}{\sqrt{(i+1)(k-i+1)}}$ behaves asymptotically like:

$$\sum_{i=0}^{k} \frac{1}{k+1} \sim 1.$$

Since $(-1)^k$ oscillates between -1 and 1, the sequence (c_k) does not tend to zero, leading to the divergence of the series $\sum c_k$. ■

R The absolute convergence of the product series is crucial. When the series $\sum a_i$ and $\sum b_j$ are not absolutely convergent, the Cauchy product can fail to converge. The above example highlights how oscillatory terms can accumulate and prevent convergence.

2.9 Permutation of Terms

This section, which is dedicated to the permutation of terms, can be skipped during a first reading.

Theorem 2.9.1
Let $\sum_{k=0}^{+\infty} u_k$ be an absolutely convergent series with sum S. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection of the set of indices. Then, the series $\sum_{k=0}^{+\infty} u_{\sigma(k)}$ converges and

$$\sum_{k=0}^{+\infty} u_{\sigma(k)} = S.$$

Remark: The condition of absolute convergence is essential. For a convergent series that is not absolutely convergent, it is possible to rearrange the terms to obtain any desired value!

As an example of a permutation, we can rearrange the terms $u_0, u_1, u_2, u_3, \dots$ by taking two even-indexed terms followed by one odd-indexed term, which results in:

$$u_0, u_2, u_1, u_4, u_6, u_3, u_8, u_{10}, u_5, \dots$$

However, it is *not allowed* to group all even-indexed terms first and then the odd-indexed terms:

$$u_0, u_2, u_4, \dots, u_{2k}, \dots, u_1, u_3, \dots, u_{2k+1}, \dots$$

Proof. By hypothesis, $\sum_{k=0}^{+\infty} |u_k|$ converges. According to the Cauchy criterion:

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \qquad \sum_{n=n_0+1}^{+\infty} |u_k| < \varepsilon.$$

Let $S = \sum_{k=0}^{+\infty} u_k$. Fix $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$ such that $\{0,1,2,\dots,n_0\} \subset \{\sigma(0),\sigma(1),\dots,\sigma(k_0)\}$. For $n \geq k_0$, we have:

$$\left| S - \sum_{k=0}^{n} u_{\sigma(k)} \right| \leq \left| S - \sum_{k=0}^{n_0} u_k \right| + \left| \sum_{k=0}^{n_0} u_k - \sum_{k=0}^{n} u_{\sigma(k)} \right|.$$

For the first term, we have:

$$\left| S - \sum_{k=0}^{n_0} u_k \right| = \left| \sum_{k=n_0+1}^{+\infty} u_k \right| \leq \sum_{k=n_0+1}^{+\infty} |u_k| \leq \varepsilon.$$

For the second term:

$$\left| \sum_{k=0}^{n} u_{\sigma(k)} - \sum_{k=0}^{n_0} u_k \right| = \left| \sum_{k \in \{\sigma(0),\dots,\sigma(n)\} \setminus \{0,\dots,n_0\}} u_k \right| \leq \sum_{k \in \{\sigma(0),\dots,\sigma(n)\} \setminus \{0,\dots,n_0\}} |u_k| \leq \sum_{k > n_0} |u_k| = \sum_{k=n_0+1}^{+\infty} |u_k| \leq \varepsilon.$$

This proves that $\left| S - \sum_{k=0}^{n} u_{\sigma(k)} \right| \leq 2\varepsilon$, which gives the result. ■

■ **Example 2.21**
In this example we try to understand that if a series is not absolutely convergent, some surprising phenomena can occur.

Recall that the alternating harmonic series converges:

$$\sum_{k=0}^{+\infty} (-1)^k \frac{1}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Let S be the sum of this series. (In fact, $S = \ln 2$.)

Now, if we regroup the terms of this series into groups of 3 and simplify, we find that the resulting sum is half of the original sum! Here is how this rearrangement looks:

$$\begin{aligned}
&\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \cdots \\
=&\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) + \cdots \\
=&\frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) \\
=&\frac{1}{2}S
\end{aligned}$$

Consider now rearranging the series into groups of 4 instead:

$$\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}\right) + \cdots$$

Each group sums to a value smaller than the corresponding sum of the series without rearrangement, leading to a sum different from S. This shows how rearrangement of terms affects the sum when the series is not absolutely convergent.

■

2.10 Exercises and problems

Exercise 2.1

1. Decompose the following expression into partial fractions $u_n = \frac{1}{n^2+n}$ for all $n \geq 1$. Then, calculate the sum $\sum_{n\geq 1} \frac{1}{n^2+n}$.
   What is the nature of the series with general term?
2. Study and calculate the sum, in case of convergence, of the following series:

$$\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right) \quad , \sum_{n\geq 2} \ln\left(1-\frac{1}{n}\right) \quad , \sum_{n\geq 1} \ln\left(1+\frac{(-1)^n}{n}\right).$$

Exercise 2.2

1. Decompose the following expression into partial fractions $u_n = \frac{1}{n^2+n}$ for all $n \geq 1$. Then, calculate the sum $\sum_{n\geq 1} \frac{1}{n^2+n}$.
   What is the nature of the series with general term?
2. Study and calculate the sum, in case of convergence, of the following series:

$$\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right) \quad , \sum_{n\geq 1} \ln\left(1-\frac{1}{n}\right) \quad , \sum_{n\geq 1} \ln\left(1+\frac{(-1)}{n}\right).$$

Exercise 2.3
Study the convergence of the series with general terms:

$$1.\ u_n = \ln\left(\frac{n^2+n+1}{n^2+n-1}\right) \qquad 2.\ u_n = \frac{1}{n+(-1)^n\sqrt{n}} \qquad 3.\ u_n = \ln\left(\frac{2}{\pi}\arctan\frac{n^2+1}{n}\right)$$

Exercise 2.4
Calculate the sums of the following series after verifying their convergence:

$$1.\ \sum_{n=0}^{+\infty} \frac{n+1}{3^n} \qquad 2.\ \sum_{n=3}^{+\infty} \frac{2n-1}{n^3-4n} \qquad 3.\ \sum_{n=2}^{+\infty} \ln\left(1+\frac{(-1)^n}{n}\right).$$

Exercise 2.5
Study the convergence of the following series $\sum u_n$:

1. $u_n = \frac{n}{n^3+1}$	2. $u_n = \frac{\sqrt{n}}{n^2+\sqrt{n}}$	3. $u_n = n\sin(1/n)$
4. $u_n = \frac{1}{\sqrt{n}}\ln\left(1+\frac{1}{\sqrt{n}}\right)$	5. $u_n = \frac{(-1)^n+n}{n^2+1}$	6. $u_n = \frac{1}{n!}$
7. $u_n = \frac{3^n+n^4}{5^n-2^n}$	8. $u_n = \frac{n+1}{2^n+8}$	9. $u_n = \frac{1}{\ln(n^2+1)}$

Exercise 2.6

Consider the series $\sum_{n\geq 1} \frac{(-1)^k}{k}$, and denote, for $n \geq 1$,

$$S_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}, \ u_n = S_{2n}, \ v_n = S_{2n+1}.$$

1. Is the series absolutely convergent?
2. Prove that the sequences (u_n) and (v_n) are adjacent.
3. Conclude that the series is convergent.

■

Exercise 2.7

Study the nature of the following series $\sum u_n$:

$$1.\ u_n = \frac{\sin n^2}{n^2} \qquad 2.\ u_n = \frac{(-1)^n \ln n}{n} \qquad 3.\ u_n = \frac{\cos(n^2\pi)}{n \ln n}$$

■

Exercise 2.8

Consider two complex sequences (u_n) and (v_n). We are interested in the convergence of the series $\sum_n u_n v_n$. For $n \geq 1$, let $s_n = \sum_{k=0}^{n} u_k$.

1. Show that for all $(p,q) \in \mathbb{N}^2$ such that $p \leq q$, we have:

$$\sum_{k=p}^{q} u_k v_k = s_q v_q - s_{p-1} v_p + \sum_{k=p}^{q-1} s_k (v_k - v_{k+1}).$$

2. Show that if the sequence (s_n) is bounded, and the sequence (v_n) takes values in $\mathbb{R}^+$, is decreasing, and tends to zero, then $\sum_n u_n v_n$ is convergent.
3. Show that the series $\sum_{n\geq 1} \frac{\sin(n\theta)}{\sqrt{n}}$ converges for all $\theta \in \mathbb{R}$.

■

Exercise 2.9

Fix $\alpha > 0$ and define $u_n = \sum_{p=n}^{+\infty} \frac{(-1)^p}{p^\alpha}$. The goal of the exercise is to prove that the series with general term u_n converges.

1. Let $n \geq 1$ be fixed. Define

$$v_p = \frac{1}{(p+n)^\alpha} - \frac{1}{(p+n+1)^\alpha}.$$

   Prove that the sequence (v_p) decreases to 0. Deduce the convergence of $\sum_{p=0}^{+\infty} (-1)^p v_p$. What is the sign of its sum?
2. Using the alternating series criterion, prove that the series with general term (u_n) converges.

■

3. Sequences and series of functions

Imagine we're looking at an infinite set of functions, each one slightly different from the last, like snapshots of a moving object. Let's take a sequence of functions where each function is defined as $f_n(x) = \frac{x}{n}$. For each value of n, this function gives a different curve, but they all have a similar shape — a straight line starting from the origin with a slope that depends on n.

As n increases (e.g., $n = 1, 2, 3, \ldots$), these lines get flatter and flatter, getting closer and closer to the x-axis. This is an example of a sequence of functions "converging" to the zero function $f(x) = 0$ as $n \to \infty$. Here, we can observe how the functions change and "approach" another function as n grows.

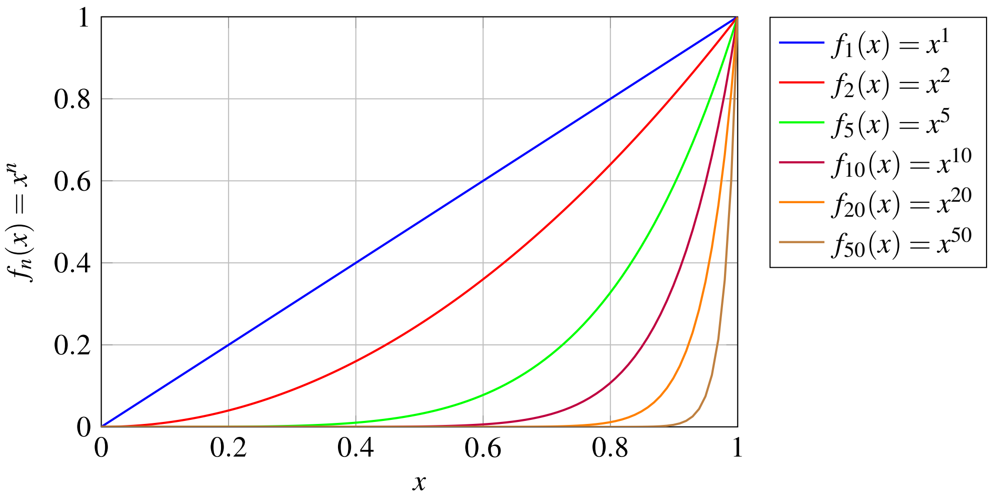
Now, let's consider a series of functions — a sum of infinitely many terms. A classic example is the *geometric series of functions*:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

When we sum this series, it converges to the function $f(x) = \frac{1}{1-x}$ for $|x| < 1$. This means that, in a certain range, infinitely adding powers of x together actually "builds" another function, providing a different way to represent $\frac{1}{1-x}$.

These sequences and series of functions are powerful because they allow us to break down and approximate complex functions using simpler ones — a strategy that underpins Fourier series, Taylor series, and even solutions to differential equations. Each sequence or series of functions offers a new way to approximate and understand functions that might otherwise be too complex to analyze directly.

Through examples like these, we see how sequences and series of functions become essential tools in mathematics, helping us approximate, analyze, and simplify a wide range of problems.



In mathematics, **sequences** and **series of functions** extend the concept of sequences and series from numbers to functions. These concepts are fundamental in calculus and analysis, particularly in understanding uniform convergence, integration, and differentiation of function sequences.

- **Sequence of Functions**: A sequence of functions $\{f_n(x)\}$ is a list of functions

  $$f_1(x), f_2(x), f_3(x), \ldots$$

  where each function is defined on a common domain. As n increases, we examine how the sequence $f_n(x)$ behaves or converges at each point x.
- **Series of Functions**: A series of functions is the sum of a sequence of functions, written as:

  $$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \ldots$$

  The goal is often to determine whether the series converges to a function $S(x)$ and under what conditions this convergence occurs.

A well-known example of a series of functions is the **Taylor series** for the exponential function e^x. The Taylor series of e^x around $x = 0$ is given by:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Here, each term $f_n(x) = \frac{x^n}{n!}$ represents a function in the series. The series sums these functions:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

- As we add more terms in this series, the partial sums $S_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}$ get closer to e^x. This process demonstrates **pointwise convergence**—the sequence of functions $S_N(x)$ converges to e^x for each fixed value of x.
- If we consider the series over a closed interval, we might also examine **uniform convergence**, which concerns whether the series converges uniformly to e^x.

Understanding sequences and series of functions is crucial for approximating functions, solving differential equations, and working with power series and Fourier series. The Taylor series for e^x illustrates how a function can be represented as an infinite sum of simpler functions, which is useful in both theoretical analysis and practical computation.

3.1 Simple (Pointwise) Convergence, Uniform Convergence

Definition 3.1.1
Let A be a subset of $\mathbb{R}$; let (f_n) be a sequence of functions from A to $\mathbb{R}$ and $f : A \to \mathbb{R}$.

We say that (f_n) *converges pointwise* to f on A if:

$$\forall \varepsilon > 0, \forall x \in A, \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, |f_n(x) - f(x)| \leq \varepsilon.$$

We say that (f_n) *converges uniformly* to f on A if:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall x \in A, \forall n \geq n_0, |f_n(x) - f(x)| \leq \varepsilon.$$

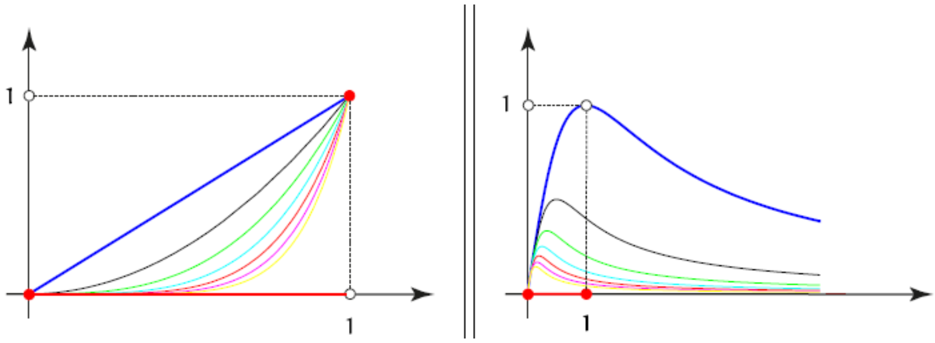


Figure 3.1: Pointwise convergence

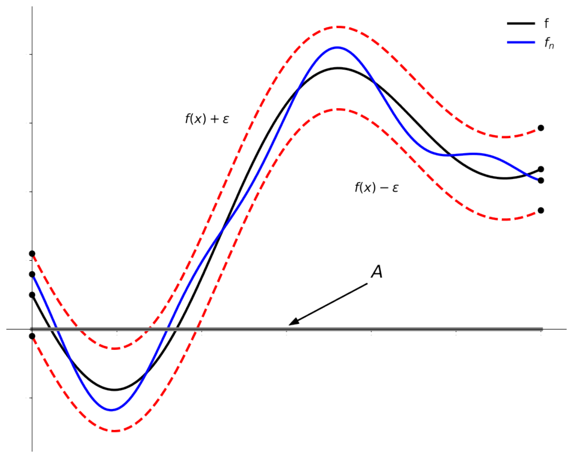


Figure 3.2: Uniform convergence

R Pointwise convergence means that for each $x \in A$, the sequence of real numbers $(f_n(x))$ converges to $f(x)$. Uniform convergence further requires that the convergence happens at the same rate. If all functions f_n and f are bounded, then (f_n) converges uniformly to f on A if and only if $(\|f_n - f\|_{A,\infty})$ tends to 0, where

$$\|g\|_{\infty,A} = \sup\{|g(x)|; x \in A\}.$$

■ **Example 3.1**

1. Consider the sequence of functions $f_n(x) = \frac{1}{1+nx}$ defined on the interval $[0,1]$.
   - For each $x \in [0,1]$, observe that

   $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx} = 0.$$

   Therefore, $\{f_n(x)\}$ converges **pointwise** to the zero function $f(x) = 0$ on $[0,1]$.

- To check for **uniform convergence**, consider the difference $|f_n(x)-0| = |f_n(x)| = \frac{1}{1+nx}$. We observe that

$$\sup_{x\in[0,1]} |f_n(x)-0| = \frac{1}{1+n} \to 0 \quad \text{as } n\to\infty.$$

  Thus, $f_n(x)$ converges uniformly to $f(x)=0$ on $[0,1]$.

Hence, $f_n(x) = \frac{1}{1+nx}$ is an example of a sequence of functions that converges **both pointwise and uniformly** to the zero function on $[0,1]$.

2. Now, consider the sequence of functions $g_n(x) = x^n \sin\left(\frac{1}{x}\right)$ defined on $(0,1]$.
   - For each $x\in(0,1]$,

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} x^n \sin\left(\frac{1}{x}\right) = 0,$$

     since $x^n \to 0$ as $n\to\infty$ for any $x\in(0,1)$. Thus, the sequence $\{g_n(x)\}$ converges **pointwise** to the zero function $g(x)=0$ on $(0,1]$.
   - To determine if the convergence is **uniform**, examine $|g_n(x)-0| = |g_n(x)| = |x^n \sin\left(\frac{1}{x}\right)|$. For values of x close to zero, the oscillations of $\sin\left(\frac{1}{x}\right)$ prevent the expression $|x^n \sin\left(\frac{1}{x}\right)|$ from uniformly approaching zero across the entire interval. Therefore,

$$\sup_{x\in(0,1]} |g_n(x)-0| \not\to 0 \quad \text{as } n\to\infty.$$

     The convergence is thus **not uniform**.

   So, $g_n(x) = x^n \sin\left(\frac{1}{x}\right)$ is an example of a sequence of functions that converges **pointwise but not uniformly** to the zero function on $(0,1]$.

■

3.2 Preserved Properties

Let I be an interval of $\mathbb{R}$, (f_n) a sequence of functions from I to $\mathbb{R}$, and $f : I \to \mathbb{R}$. (Simple (Pointwise) convergence preserves properties related to order: for example, if (f_n) converges uniformly to f on I and if all f_n are increasing, then f is increasing. If all f_n are convex, then f is convex. However, regularity properties are not preserved by pointwise convergence.

Theorem 3.2.1

Continuity Suppose all f_n are continuous at $a\in I$ and that (f_n) converges uniformly to f on I. Then f is continuous at a.
In particular, if all f_n are continuous on I, then f is continuous on I.

Limit/Integral Permutation Suppose $I=[a,b]$ is a segment, all functions f_n are continuous, and (f_n) converges uniformly to f on I. Then

$$\lim_{n\to+\infty} \int_a^b f_n(t)\,dt = \int_a^b \lim_{n\to+\infty} f_n(t)\,dt = \int_a^b f(t)\,dt.$$

Differentiability Suppose all functions f_n are of class $\mathscr{C}^1$ and there exists $g : I\to\mathbb{R}$ such that:
1. (f_n) converges pointwise to f on I.
2. The sequence of functions (f_n') converges uniformly to g on every segment contained in I.

Then the function f is of class $\mathscr{C}^1$ and $f' = g$.

Character $\mathscr{C}^k$ Suppose all functions f_n are of class $\mathscr{C}^k$ and there exist functions $g_j : I\to\mathbb{R}$, $0\le j\le k$, such that:
1. For all $j=0,\dots,k-1$, $(f_n^{(j)})$ converges pointwise to g_j on I.
2. $(f_n^{(k)})$ converges uniformly to g_k on all segments contained in I.

Then g_0 is of class $\mathscr{C}^k$ on I and for all $j \leq k$, $g_0^{(j)} = g_j$.

Limit Interchange Theorem Suppose $I = [a, b[$ and that (f_n) converges uniformly to f on I. Further, suppose each function f_n has a limit ℓ_n at b. Then the sequence (ℓ_n) converges to a limit ℓ, f has a limit at b, and

$$\lim_{x \to b} f(x) = \ell.$$

This theorem is often applied with $b = +\infty$.

Example 3.2

Consider the sequence (f_n) defined on by $f_n(x) = (2xn + (-1)^n x^2)/n$. For fixed $x \in$ we have

$$\lim_n f_n(x) = \lim_n \frac{2xn + (-1)^n x^2}{n} = 2x.$$

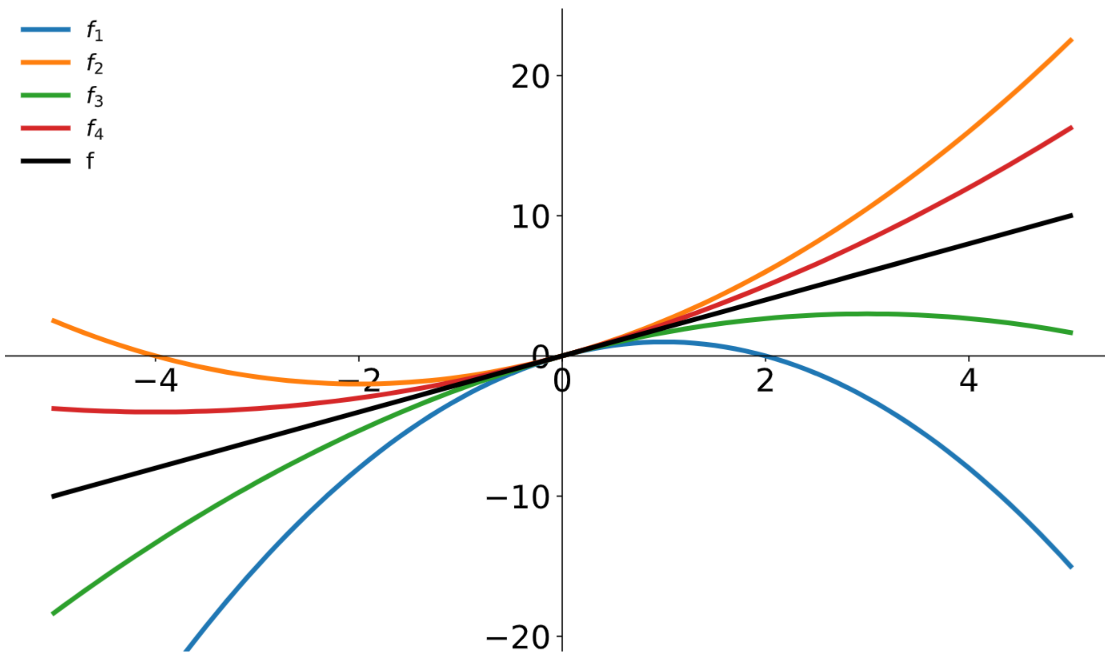
Hence, (f_n) converges pointwise to $f(x) = 2x$ on . In Figure 8.1, we graph $f_n(x)$ for the values $n = 1, 2, 3, 4$ and the function $f(x) = 2x$. Notice that $f_n'(x) = (2n + 2(-1)^n x)/n$ and therefore $\lim_n f_n'(x) = 2$, and for the limit function $f(x) = 2x$ we have $f'(x) = 2$. Hence, the sequence of derivatives (f_n') converges pointwise to f'. Also, after some basic computations,

$$\begin{aligned}\int_{-1}^{1} f_n(x)\,dx &= \int_{-1}^{1} \frac{2xn + (-1)^n x^2}{n}\,dx \\ &= \frac{2(-1)^n}{3n}\end{aligned}$$

and therefore

$$\begin{aligned}\lim_n \int_{-1}^{1} f_n(x)\,dx &= \lim_n \frac{2(-1)^n}{3n} \\ &= 0.\end{aligned}$$

On the other hand it is clear that $\int_{-1}^{1} f(x)\,dx = 0$.



■

Example 3.3

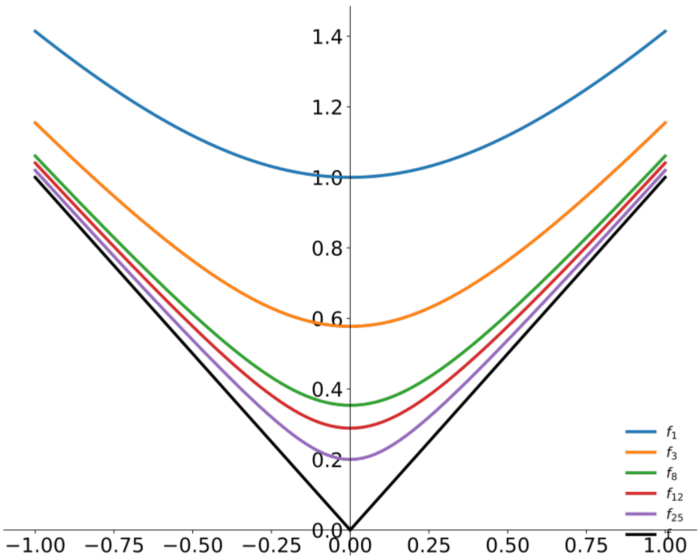
Consider the sequence (f_n) defined on $A=[-1,1]$ by $f_n(x)=\sqrt{\frac{nx^2+1}{n}}$. For fixed $x\in A$ we have

$$\begin{aligned}\lim_n f_n(x) &= \lim_n \sqrt{\frac{nx^2+1}{n}}\\ &= \lim_n \sqrt{x^2+\frac{1}{n}}\\ &= \sqrt{x^2}\\ &= |x|.\end{aligned}$$

Hence, (f_n) converges pointwise on A to the function $f(x)=|x|$. Notice that each function f_n is continuous on A and the pointwise limit f is also continuous. After some basic calculations we find that

$$f_n'(x)=\frac{x}{\sqrt{\frac{nx^2+1}{n}}}$$

and $f_n'(x)$ exists for each $x\in[-1,1]$, in other words, f_n is differentiable on A. However, $f(x)=|x|$ is not differentiable on A since f does not have a derivative at $x=0$.



So, we observe here the weakness of pointwise convergence, namely, that if (f_n) is a sequence of differentiable functions on A and (f_n) converges pointwise to f on A then f is not necessarily differentiable on A. ■

■ Example 3.4

Let's take the sequence (f_n) on $A=[0,1]$ defined by $f_n(x)=2nxe^{-nx^2}$. For fixed $x\in[0,1]$ we find that

$$\lim_n f_n(x)=\lim_n \frac{2nx}{e^{nx^2}}=0.$$

Hence, (f_n) converges pointwise to $f(x)=0$ on A. Consider

$$\begin{aligned}\int_0^1 f_n(x)\,dx &= \int_0^1 2nxe^{-nx^2}\,dx\\ &= -e^{-nx^2}\Big|_0^1\\ &= 1-e^{-n}\end{aligned}$$

and therefore

$$\lim_n \int_0^1 f_n(x)=\lim_n(1-e^{-n})=1.$$

On the other hand, $\int_0^1 f(x)\,dx = 0$. Therefore,

$$\int_0^1 f(x)\,dx \neq \lim_n \int_0^1 f_n(x)\,dx$$

or another way to write this is

$$\lim_n \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_n f_n(x)\,dx.$$

■

R In the example above, we see that the pointwise limit of a sequence of functions does not necessarily preserve the properties of continuity or differentiability. Also, highlights that combining the operations of integration and limits can lead to unexpected or surprising outcomes. In particular, it shows that the interchange of the limit operation with integration is not always valid.

3.3 Infinite Series of Functions

An infinite series of functions is a formal sum of the form $\sum_{n=1}^{\infty} f_n(x)$, where each f_n is a function defined on a common domain. The study of such series concerns the convergence of the partial sums and the behavior of the resulting limit function, particularly with respect to continuity, differentiability, and integrability.

Definition 3.3.1
Let A be a non-empty subset of . An infinite series of functions on A is a series of the form $\sum_{n=1}^{\infty} f_n(x)$ for each $x \in A$ where (f_n) is a sequence of functions on A. The sequence of partial sums generated by the series $\sum f_n$ is the sequence of functions (s_n) on A defined as $s_n(x) = f_1(x) + \cdots + f_n(x)$ for each $x \in A$.

- We say that the series of functions $\sum_{n\geq 0} f_n$ *converges pointwise* to S on A if the sequence of its partial sums $S_n(x) = \sum_{k=1}^{n} f_k(x)$ converges pointwise to S on A.
- We say that the series of functions $\sum_{n\geq 0} f_n$ *converges uniformly* to S on A if the sequence of its partial sums $S_n(x) = \sum_{k=1}^{n} f_k(x)$ converges uniformly to S on A.

R If the series of functions $\sum_{n\geq 0} f_n$ converges pointwise on A, for $n \in \mathbb{N}$, we introduce its remainder of order n defined on A by $R_n(x) = \sum_{k=n+1}^{+\infty} f_k(x)$. Saying that the series $\sum_{n\geq 0} f_n$ converges uniformly on A is equivalent to saying that the sequence of remainders (R_n) converges uniformly to 0 on A.

Proposition 3.3.1
We say that the series of functions $\sum_{n\geq 0} f_n$ *converges normally* on A if each function f_n is bounded on A and if the numerical series $\sum_{n\geq 0} \|u_n\|_{\infty,A}$ is convergent.

We now state a useful theorem for normal and uniform convergence of infinite series of functions.

Theorem 3.3.2 — Weierstrass M-Test.
Let (f_n) be a sequence of functions on A and suppose that there exists a sequence of non-negative numbers (M_n) such that $|f_n(x)| \leq M_n$ for all $x \in A$, and all $n \in \mathbb{N}$. If $\sum M_n$ converges then $\sum f_n$ converges uniformly on A.

Proof. Let $\varepsilon > 0$ be arbitrary. Let $t_n = \sum_{k=1}^{n} M_k$ be the sequence of partial sums of the series $\sum M_n$. By assumption, (t_n) converges and thus (t_n) is a Cauchy sequence. Hence, there exists $K \in \mathbb{N}$ such that $|t_m - t_n| < \varepsilon$ for all $m > n \geq K$. Let (s_n) be the sequence of partial sums of $\sum f_n$. Then if $m > n \geq K$ then for all $x \in A$ we have

$$\begin{aligned} |s_m(x) - s_n(x)| &= |f_m(x) + f_{m-1}(x) + \cdots + f_{n+1}(x)| \\ &\leq |f_m(x)| + |f_{m-1}(x)| + \cdots + |f_{n+1}(x)| \\ &\leq M_m + M_{m-1} + \cdots + M_{n+1} \\ &= |t_m - t_n| \\ &< \varepsilon. \end{aligned}$$

Hence, the sequence (s_n) satisfies the Cauchy Criterion for uniform convergence and the proof is complete. ■

Corollary 3.3.3
If $\sum_n f_n$ converges normally on A, then it converges uniformly.

■ **Example 3.5**
Consider the series $\sum_{n=1}^{\infty} \frac{n\sin(nx)}{e^n}$. We have

For any $x \in$ it holds that

$$\left|\frac{n\sin(nx)}{e^n}\right| \leq \frac{n}{e^n}.$$

A straightforward application of the Ratio test shows that $\sum_{n=1}^{\infty} \frac{n}{e^n}$ is a convergent series. Hence, by the M-Test, the given series converges uniformly on $A = \mathbb{R}$. ■

3.4 Theorems Related to Sequences of Functions

The theorems related to sequences of functions remain true in this new context. They now have the following statements:

Theorem 3.4.1

Continuity Suppose all u_n are continuous at $a \in I$ and that $\sum_{n\geq 0} u_n$ converges uniformly to S on I. Then S is continuous at a.
In particular, if all u_n are continuous on I, then S is continuous on I.

Sum/Integral Permutation Suppose $I = [a,b]$ is a segment, all functions u_n are continuous, and $\sum_n u_n$ converges uniformly on $[a,b]$. Then the series $\sum_{n\geq 0} \int_a^b u_n(t)dt$ converges and we have

$$\int_a^b \sum_{n=0}^{+\infty} u_n(t)dt = \sum_{n=0}^{+\infty} \int_a^b u_n(t)dt.$$

Differentiability Suppose all functions u_n are of class $\mathscr{C}^1$ and that
1. $\sum_{n\geq 0} u_n$ converges simply on I.
2. $\sum_{n\geq 0} u'_n$ converges uniformly on every segment contained in I.

Then the function $S : x \mapsto \sum_{n=0}^{+\infty} u_n(x)$ is of class $\mathscr{C}^1$ and $S'(x) = \sum_{n=0}^{+\infty} u'_n(x)$.

Character $\mathscr{C}^k$ Suppose all functions u_n are of class $\mathscr{C}^k$ and that
1. For all $j = 0, \ldots, k-1$, $\sum_{n\geq 0} u_n^{(j)}$ converges simply on I;
2. $\sum_{n\geq 0} u_n^{(k)}$ converges uniformly on all segments contained in I.

Then the function $S : x \mapsto \sum_{n=0}^{+\infty} u_n(x)$ is of class $\mathscr{C}^k$ and for all $j = 0, \ldots, k$, $S^{(j)}(x) = \sum_{n=0}^{+\infty} u_n^{(j)}(x)$.

Limit Interchange Theorem Suppose $I = [a,b[$ and that $\sum_{n\geq 0} u_n$ converges uniformly to S on I. Further, suppose each function u_n has a limit ℓ_n at b. Then the series $\sum_{n\geq 0} \ell_n$ converges, S has a limit at b, and

$$\lim_{x\to b} S(x) = \sum_{n=0}^{+\infty} \ell_n.$$

In other words, under the previous assumptions,

$$\lim_{x\to b} \sum_{n=0}^{+\infty} u_n(x) = \sum_{n=0}^{+\infty} \lim_{x\to b} u_n(x).$$

Example 3.6

We have shown in the Example 3.5 that the series $\sum_{n=1}^{\infty} \frac{n\sin(nx)}{e^n}$ is uniformly convergent. Now, we propose to prove that for any $x \in$ it holds that

$$\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{n\sin(nx)}{e^n} \right) = \frac{2e}{e^2-1}$$

since we have uniform convergence, then

$$\begin{aligned} \int_0^{\pi} \sum_{n=1}^{\infty} \frac{n\sin(nx)}{e^n}\, dx &= \sum_{n=1}^{\infty} \int_0^{\pi} \frac{n\sin(nx)}{e^n}\, dx \\ &= \sum_{n=1}^{\infty} -\frac{\cos(nx)}{e^n}\Big|_0^{\pi} \\ &= \sum_{n=1}^{\infty} \left[\left(\tfrac{1}{e}\right)^n - \left(\tfrac{-1}{e}\right)^n \right] \\ &= \left(\tfrac{1}{1-1/e} - 1 \right) - \left(\tfrac{1}{1+1/e} - 1 \right) \\ &= \frac{2e}{e^2-1} \end{aligned}$$

■

3.5 Exercises

Exercise 3.1

For $x \geq 0$ and $n \geq 1$, let $f_n(x) = \frac{n}{1+n(1+x)}$.

1. Show that the sequence of functions $(f_n)_{n\geq 1}$ converges pointwise on $[0,+\infty[$ to a function f, which should be specified.
2. Prove that the convergence is actually uniform on $[0,+\infty[$.

■

Exercise 3.2

Define, for $n \geq 1$ and $x \in]0,1]$, $f_n(x) = nx^n \ln(x)$ and $f_n(0) = 0$.

1. Show that (f_n) converges pointwise on $[0,1]$ to a function f, which should be specified. Let $g = f - f_n$.
2. Study the variations of g.
3. Deduce that the convergence of (f_n) to f is not uniform on $[0,1]$.
4. Let $a \in [0,1[$. Noting that there exists $n_0 \in \mathbb{N}$ such that $e^{-1/n} \geq a$ for all $n \geq n_0$, prove that the sequence (f_n) converges uniformly to f on $[0,a]$.

■

Exercise 3.3
Let (f_n) be the sequence of functions defined on $[0,1]$ by $f_n(x) = n^2x(1-nx)$ if $x \in [0,1/n]$ and $f_n(x) = 0$ otherwise.

1. Study the pointwise limit of the sequence (f_n).
2. Compute $\int_0^1 f_n(t)\,dt$. Is there uniform convergence on $[0,1]$?
3. Study the uniform convergence on $[a,1]$ for $a \in]0,1]$.

■

Exercise 3.4
For $x \in I = [0,1]$, $a \in \mathbb{R}$, and $n \geq 1$, let $f_n(x) = n^a x^n(1-x)$.

1. Study the pointwise convergence on I of the series with general term f_n. Let S denote the sum of the series.
2. Study the normal convergence on I of the series with general term f_n.
3. In this question, assume $a = 0$. Compute S on $[0,1[$. Deduce that the convergence is not uniform on $[0,1]$.
4. Assume $a > 0$. Show that the convergence is not uniform on I.

■

Exercise 3.5
Let $f_n(x) = (-1)^n \ln\left(1 + \frac{x}{n(1+x)}\right)$, defined for $x \geq 0$ and $n \geq 1$.

1. Show that the series $\sum_{n\geq 1} f_n$ converges pointwise on $\mathbb{R}_+$.
2. Show that the series $\sum_{n\geq 1} f_n$ converges uniformly on $\mathbb{R}_+$.
3. Is the convergence normal on $\mathbb{R}_+$?

■

Exercise 3.6
Consider the series of functions $\sum_{n\geq 2} f_n$, where $f_n(x) = \frac{xe^{-nx}}{\ln n}$.

1. Show that $\sum_{n\geq 2} f_n$ converges pointwise on $\mathbb{R}_+$.
2. Show that the convergence is not normal on $\mathbb{R}_+$.
3. For $x \in \mathbb{R}_+$, let $R_n(x) = \sum_{k\geq n+1} u_k(x)$. Prove that, for all $x > 0$,

$$0 \leq R_n(x) \leq \frac{xe^{-x}}{\ln(n+1)(1-e^{-x})},$$

and deduce that the series converges uniformly on $\mathbb{R}_+$.

■

4. Power series

Power series play a fundamental role in mathematical analysis, serving as a versatile tool for representing functions and solving complex problems across various fields. These series are particularly valuable because they allow us to express transcendental functions, such as the exponential, logarithmic, and trigonometric functions, in terms of infinite sums of polynomials. This approach not only simplifies calculations but also provides deep insights into the properties of these functions. Power series can approximate functions to any desired accuracy within their radius of convergence, making them indispensable in both theoretical analysis and practical applications, including physics, engineering, and computer science.

4.1 Power Series - Radius of Convergence

Definition 4.1.1
A *power series* is any series of functions of the form $\sum_n a_n z^n$ (centerd at 0) where (a_n) is a sequence of complex numbers and $z \in \mathbb{C}$.

R We can also define The power series centered at c where $c \in \mathbb{R}$ by

$$\sum_{n=0}^{\infty} a_n (z-c)^n.$$

■ **Example 4.1**

1. The geometric series converges for $|x| < 1$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \quad |x| < 1.$$

2. The power series representation of the exponential function e^x converges for all $x \in \mathbb{R}$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad x \in \mathbb{R}.$$

3. The power series for the sine function converges for all $x \in \mathbb{R}$:

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \quad x \in \mathbb{R}.$$

4. The power series for $\ln(x)$ centered at $x = 1$ converges for $0 < x \leq 2$:

$$\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots, \quad 0 < x \leq 2.$$

■

Theorem 4.1.1 — Abel's Lemma.
If the sequence $(a_n z_0^n)$ is bounded, then for all $z \in \mathbb{C}$ with $|z| < |z_0|$, the series $\sum_n a_n z^n$ is absolutely convergent.

Theorem 4.1.2
Let

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

be a power series. There exists $0 \leq R \leq \infty$ such that the series converges absolutely for $0 \leq |x-c| < R$ and diverges for $|x-c| > R$. Furthermore, if $0 \leq \rho < R$, then the power series converges uniformly on the interval $|x-c| \leq \rho$, and the sum of the series is continuous in $|x-c| < R$.

Proof. Assume without loss of generality that $c = 0$ (otherwise, replace x by $x - c$). Suppose the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for some $x_0 \in \mathbb{R}$ with $x_0 \neq 0$. Then its terms converge to zero, so they are bounded, and there exists $M \geq 0$ such that

$$|a_n x_0^n| \leq M \quad \text{for } n = 0, 1, 2, \ldots.$$

If $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left| \frac{x}{x_0} \right|^n \leq M r^n, \quad r = \left| \frac{x}{x_0} \right| < 1.$$

Comparing the power series with the convergent geometric series $\sum M r^n$, we see that $\sum a_n x^n$ is absolutely convergent. Thus, if the power series converges for some $x_0 \in \mathbb{R}$, then it converges absolutely for every $x \in \mathbb{R}$ with $|x| < |x_0|$.

Let

$$R = \sup\{|x| \geq 0 : \sum a_n x^n \text{ converges}\}.$$

If $R = 0$, then the series converges only for $x = 0$. If $R > 0$, then the series converges absolutely for every $x \in \mathbb{R}$ with $|x| < R$, because it converges for some $x_0 \in \mathbb{R}$ with $|x| < |x_0| < R$. Moreover, the definition of R implies that the series diverges for every $x \in \mathbb{R}$ with $|x| > R$. If $R = \infty$, then the series converges for all $x \in \mathbb{R}$.

Finally, let $0 \leq \rho < R$ and suppose $|x| \leq \rho$. Choose $\sigma > 0$ such that $\rho < \sigma < R$. Then $\sum |a_n \sigma^n|$ converges, so $|a_n \sigma^n| \leq M$, and therefore

$$|a_n x^n| = |a_n \sigma^n| \left| \frac{x}{\sigma} \right|^n \leq M r^n,$$

where $r = \rho/\sigma < 1$. Since $\sum M r^n < \infty$, the M-test implies that the series converges uniformly on $|x| \leq \rho$, and then it follows that the sum is continuous on $|x| \leq \rho$. Since this holds for every $0 \leq \rho < R$, the sum is continuous in $|x| < R$. ■

The following definition therefore makes sense for every power series.

Definition 4.1.2
If the power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

converges for $|x-c| < R$ and diverges for $|x-c| > R$, then $0 \leq R \leq \infty$ is called the *radius of convergence* of the power series.

The *radius of convergence* of the power series is defined as

$$R = \sup \rho \geq 0 : (a_n \rho^n) \text{ is bounded} \in \mathbb{R}_+ \cup +\infty.$$

Proposition 4.1.3
Let $\sum_n a_n z^n$ be a power series with radius of convergence R. Then, for all $z \in \mathbb{C}$,

- if $|z| < R$, the series $\sum_n a_n z^n$ converges absolutely;
- if $|z| > R$, the series $\sum_n a_n z^n$ diverges grossly (its general term does not tend to 0);
- if $|z| = R$, then we cannot conclude in general.

The open disk $D(0,R)$ is called the *open disk of convergence* of the power series.

Corollary 4.1.4 — Normal Convergence.
Let $\sum_n a_n z^n$ be a power series with radius of convergence $R > 0$ and let $r \in (0,R)$. Then the series $\sum_n a_n z^n$ converges normally on the closed disk $D(0,r)$. In particular, the sum of the power series is continuous on its open disk of convergence.

To calculate the radius of convergence of a power series, we often use d'Alembert's ratio test for series, which states:

Theorem 4.1.5 — D'Alembert's Ratio Test.
Let (u_n) be a sequence of strictly positive real numbers. If u_{n+1}/u_n tends to ℓ, then

- if $\ell > 1$, the series $\sum_n u_n$ diverges grossly;
- if $\ell < 1$, the series $\sum_n u_n$ converges absolutely.

Suppose that $a_n \neq 0$ for all sufficiently large n and the limit

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists or diverges to infinity. Then the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has radius of convergence R.

When we apply this rule to a power series $\sum_n a_n z^n$ by setting $u_n = |a_n z^n|$, we obtain that if $|a_{n+1}|/|a_n|$ converges to ℓ, then the radius of convergence of the power series is $1/\ell$.

■ Example 4.2
The geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

has radius of convergence

$$R = \lim_{n\to\infty} \frac{1}{1} = 1.$$

Thus, it converges for $|x| < 1$, to $\frac{1}{1-x}$, and diverges for $|x| > 1$. ■

■ Example 4.3
The series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

has radius of convergence

$$R = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$$

■

Theorem 4.1.6 — Hadamard. The radius of convergence R of the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

is given by

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}},$$

where $R = 0$ if the lim sup diverges to ∞, and $R = \infty$ if the lim sup is 0.

Proof. Let

$$r = \limsup_{n\to\infty} |a_n (x-c)^n|^{1/n} = |x-c| \limsup_{n\to\infty} |a_n|^{1/n}.$$

By the root test, the series converges if $0 \leq r < 1$, or $|x-c| < R$, and diverges if $1 < r \leq \infty$, or $|x-c| > R$, which proves the result. ■

4.2 Operations on Power Series

Consider $\sum_n a_n z^n$ and $\sum_n b_n z^n$ two power series with respective radii of convergence R_a and R_b.

- **Comparison of radius of convergence**: if $a_n = O(b_n)$, then $R_a \geq R_b$. In particular, if $a_n \sim b_n$, then $R_a = R_b$.
- **Radius of convergence of the derived series**: the radius of convergence of $\sum_n n a_n z^{n-1}$ is equal to the radius of convergence of $\sum_n a_n z^n$.
- **Sum of two power series**: the radius of convergence of the sum series $\sum_n (a_n + b_n) z^n$ satisfies $R \geq \min(R_a, R_b)$. Moreover, for all $z \in \mathbb{C}$ with $|z| < \min(R_a, R_b)$, we have

$$\sum_{n\geq 0} (a_n + b_n) z^n = \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} b_n z^n.$$

We call the *product power series* of $\sum_n a_n z^n$ and $\sum_n b_n z^n$ the power series $\sum_n c_n z^n$ with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Proposition 4.2.1
The radius of convergence R of the product series $\sum_n c_n z^n$ of $\sum_n a_n z^n$ and $\sum_n b_n z^n$ satisfies $R \geq \min(R_a, R_b)$. Moreover, for all $z \in \mathbb{C}$ with $|z| < \min(R_a, R_b)$, we have

$$\sum_{n\geq 0} c_n z^n = \left(\sum_{n\geq 0} a_n z^n\right) \times \left(\sum_{n\geq 0} b_n z^n\right).$$

4.3 Regularity, Case of Real Variable

We now focus on the case where the variable can only take real values, and we will now denote power series as $\sum_n a_n x^n$. We are interested in the regularity of the power series within its interval of convergence $]-R, R[$.

Theorem 4.3.1 — Integration of a Power Series.
Let $f(x) = \sum_{n\geq 0} a_n x^n$ be a power series with radius of convergence $R > 0$ and let F be a primitive of f. Then, for all $x \in]-R,R[$,

$$F(x) = F(0) + \sum_{n\geq 0} \frac{a_n}{n+1} x^{n+1}.$$

■ **Example 4.4**
For $|x| < 1$ we have $\frac{1}{1+x} = \sum_{n\geq 0}(-1)^n x^n$. Then, for all $x \in]-1,1[$ we get

$$\begin{aligned} \int_0^x \frac{1}{1+t}\,\mathrm{d}t &= \int_0^x \sum_{n\geq 0}(-1)^n t^n\,\mathrm{d}t \\ \ln(x+1) &= \sum_{n\geq 0}(-1)^n \int_0^x t^n\,\mathrm{d}t = \sum_{n\geq 0}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n. \end{aligned}$$

■

Theorem 4.3.2 — Term-by-Term Differentiation.
Let $f(x) = \sum_{n\geq 0} a_n x^n$ be a power series with radius of convergence $R > 0$. Then f is of class $\mathscr{C}^\infty$ on $]-R,R[$. Moreover, for all $x \in]-R,R[$ and all $k \geq 0$, we have

$$f^{(k)}(x) = \sum_{n\geq k} n(n-1)\cdots(n-k+1)a_n x^{n-k}.$$

■ **Example 4.5**
We are going to find a power series representation for the following function and determine its radius of convergence. $\frac{1}{(1-x)^2}$
To do this problem let's notice that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

Then since we've got a power series representation for $\frac{1}{1-x}$, all that we'll need to do is differentiate that power series to get a power series representation for $g(x)$.

$$\begin{aligned} \frac{1}{(1-x)^2} &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\ &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) \\ &= \sum_{n=1}^{\infty} n x^{n-1} \end{aligned}$$

Then since the original power series had a radius of convergence of $R = 1$ the derivative, and hence $g(x)$, will also have a radius of convergence of $R = 1$. ■

Theorem 4.3.3 — Expression of Coefficients of a Power Series.
Let $f(x) = \sum_{n\geq 0} a_n x^n$ be a power series with radius of convergence $R > 0$. Then, for all $n \geq 0$,

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Corollary 4.3.4
If $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$ coincide on a neighborhood of 0, then for all $n \geq 0$, $a_n = b_n$.

4.4 Power Series Expansions

Let I be an interval containing 0 and $f : I \to \mathbb{R}$. We say that f has a power series expansion at 0 if there exists $r > 0$ and a sequence (a_n) such that, for all $x \in (-r, r)$, we have $f(x) = \sum_{n\geq 0} a_n x^n$. In particular, a function with a power series expansion at 0 is of class $\mathscr{C}^\infty$ in a neighborhood of 0. A linear combination of functions with power series expansions has a power series expansion. The product of two functions with power series expansions has a power series expansion. The same is true for the derivative or a primitive of a function with a power series expansion.

Taylor series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

If we set $a = 0$, so we are talking about the Taylor Series about $x = 0$, we call the series a Maclaurin Series for $f(x)$ or,

Maclaurin Series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \end{aligned}$$

Before working any examples of Taylor Series we first need to address the assumption that a Taylor Series will in fact exist for a given function. Let's start out with some notation and definitions that we'll need.

To determine a condition that must be true in order for a Taylor series to exist for a function let's first define the n^{th} degree Taylor polynomial of $f(x)$ as,

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Note that this really is a polynomial of degree at most n. If we were to write out the sum without the summation notation this would clearly be an n^{th} degree polynomial. We'll see a nice application of Taylor polynomials in the next section.

Notice as well that for the full Taylor Series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the n^{th} degree Taylor polynomial is just the partial sum for the series.

Next, the remainder is defined to be,

$$R_n(x) = f(x) - T_n(x)$$

So, the remainder is really just the error between the function $f(x)$ and the n^{th} degree Taylor polynomial for a given n.

With this definition note that we can then write the function as,

$$f(x) = T_n(x) + R_n(x)$$

We now have the following Theorem.

Theorem 4.4.1
Suppose that $f(x) = T_n(x) + R_n(x)$. Then if,

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$ then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

on $|x-a| < R$.

In general, showing that

$$\lim_{n\to\infty} R_n(x) = 0$$

is a somewhat difficult process and so we will be assuming that this can be done for some R in all of the examples that we'll be looking at.

Corollary 4.4.2
Let I be an interval containing 0 and $f : I \to \mathbb{R}$. If f has a power series expansion at 0, then there exists $r > 0$ such that, for all $x \in (-r,r)$,

$$f(x) = \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}x^n.$$

■ **Example 4.6**

1. Find the Taylor Series for $f(x) = \mathbf{e}^x$ about $x = 0$.
   This is actually one of the easier Taylor Series that we'll be asked to compute. To find the Taylor Series for a function we will need to determine a general formula for $f^{(n)}(a)$. This is one of the few functions where this is easy to do right from the start.
   To get a formula for $f^{(n)}(0)$ all we need to do is recognize that,
   $$f^{(n)}(x) = \mathbf{e}^x \qquad n = 0,1,2,3,\ldots$$
   and so,
   $$f^{(n)}(0) = \mathbf{e}^0 = 1 \qquad n = 0,1,2,3,\ldots$$
   Therefore, the Taylor series for $f(x) = \mathbf{e}^x$ about $x = 0$ is,
   $$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$
2. Find the Taylor Series for $f(x) = \cos(x)$ about $x = 0$.
   First, we'll need to take some derivatives of the function and evaluate them at $x = 0$.

$$\begin{aligned}
f^{(0)}(x) &= \cos x & f^{(0)}(0) &= 1\\
f^{(1)}(x) &= -\sin x & f^{(1)}(0) &= 0\\
f^{(2)}(x) &= -\cos x & f^{(2)}(0) &= -1\\
f^{(3)}(x) &= \sin x & f^{(3)}(0) &= 0\\
f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1\\
f^{(5)}(x) &= -\sin x & f^{(5)}(0) &= 0\\
f^{(6)}(x) &= -\cos x & f^{(6)}(0) &= -1\\
&\vdots & &\vdots
\end{aligned}$$

In this example, unlike the previous ones, there is not an easy formula for either the general derivative or the evaluation of the derivative. However, there is a clear pattern to the evaluations. So, let's plug what we've got into the Taylor series and see what we get,

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n\\
&= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots\\
&= \underbrace{1}_{n=0} + \underbrace{0}_{n=1} - \underbrace{\frac{1}{2!}}_{n=2}x^2 + \underbrace{0}_{n=3} + \underbrace{\frac{1}{4!}}_{n=4}x^4 + \underbrace{0}_{n=5} - \underbrace{\frac{1}{6!}}_{n=6}x^6 + \cdots
\end{aligned}$$

So, we only pick up terms with even powers on the x's. This doesn't really help us to get a general formula for the Taylor Series. However, let's drop the zeroes and "renumber" the terms as follows to see what we can get.

$$\cos x = \underbrace{1}_{n=0} - \underbrace{\frac{1}{2!}}_{n=1}x^2 + \underbrace{\frac{1}{4!}}_{n=2}x^4 - \underbrace{\frac{1}{6!}}_{n=3}x^6 + \cdots$$

By renumbering the terms as we did we can actually come up with a general formula for the Taylor Series and here it is,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

■

4.5 Common Power Series Expansions

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!}, \qquad R = +\infty$$

$$\cos x = \sum_{n\geq 0} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad R = +\infty$$

$$\sin x = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad R = +\infty$$

$$\cosh x = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!}, \qquad R = +\infty$$

$$\sinh x = \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!}, \qquad R = +\infty$$

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n, \qquad R = 1$$

$$\frac{1}{1+x} = \sum_{n\geq 0} (-1)^n x^n, \qquad R = 1$$

$$\ln(1+x) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n, \qquad R = 1$$

$$\arctan(x) = \sum_{n\geq 0} \frac{(-1)^n}{2n+1} x^{2n+1}, \qquad R = 1$$

$$(1+x)^\alpha = 1 + \sum_{n\geq 1} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n, \qquad R = 1$$

4.6 Exercises

Exercise 4.1

Determine the radius of convergence of the following power series:

1. $\sum_n \frac{1}{\sqrt{n}} x^n$ **2.** $\sum_n \frac{n!}{(2n)!} x^n$ **3.** $\sum_{n\geq 1} \frac{n!}{2^{2n}\sqrt{(2n)!}} x^n$

4. $\sum_n (\ln n) x^n$ **5.** $\sum_n \frac{\sqrt{n} x^{2n}}{2^n+1}$ **6.** $\sum_n (2+ni) z^n$

7. $\sum_n \frac{(1+i)^n z^{3n}}{n \cdot 2^n}$ **8.** $\sum_{n\geq 1} \ln\left(1+\sin\frac{1}{n}\right) x^n$ **9.** $\sum_{n\geq 1} \left(\exp(1/n)-1\right) x^n$

■

Exercise 4.2

Expand the following functions into power series around 0. Specify the radius of convergence of the obtained power series.

1. $\ln(1+2x^2)$ **2.** $\dfrac{1}{a-x}$ with $a \neq 0$

3. $\ln(a+x)$ with $a > 0$ **4.** $\dfrac{e^x}{1-x}$

5. $\ln(1+x-2x^2)$ **6.** $(4+x^2)^{-3/2}$

■

Exercise 4.3

Consider the power series $f(x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n(2n+1)} x^{2n+1}$.

1. What is its radius of convergence, denoted R? Is there convergence at the endpoints of the interval of definition?

2. On which interval is the function f a priori continuous? Prove that it is actually continuous on $[-R,R]$.
3. Express the sum of the derived series on $]-R,R[$ using standard functions. Deduce an expression for f on $]-R,R[$.
4. Calculate $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n(2n+1)}$.

■

Exercise 4.4
Consider the power series $f(x) = \sum_{n=2}^{+\infty} \frac{(-1)^n}{n(n-1)} x^n$.

1. Determine the domain of definition of f.
2. Prove that f is continuous on its domain of definition.
3. Express f', then f, using standard functions on the interval $]-1,1[$.
4. Deduce from the previous questions the value of $\sum_{n\geq 2} \frac{(-1)^n}{n(n-1)}$.

■

5. Fourier series

The Fourier series is a powerful mathematical tool used to analyze periodic functions by decomposing them into simpler components—sines and cosines. This technique, named after Joseph Fourier, finds applications in diverse fields such as signal processing, heat transfer, and vibrations. It provides an elegant way to study functions in both time and frequency domains.

5.1 Trigonometric series

Definition 5.1.1
A *trigonometric series* is any series of the form:

$$\sum_{n=0}^{\infty} \left(a_n \cos(n\omega x) + b_n \sin(n\omega x)\right), \tag{5.1}$$

where $x \in \mathbb{R}$, $\omega > 0$, $a_n, b_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

R

(a) The general term $S_n = a_n \cos(n\omega x) + b_n \sin(n\omega x)$ of the trigonometric series is periodic with a period $T = \frac{2\pi}{n\omega}$.
(b) If the series (5.1) converges to $S(x)$, the function $S(x)$ is periodic with a period $T = \frac{2\pi}{\omega}$.

For all $x \in \mathbb{R}$, we have the inequality:

$$|a_n \cos(n\omega x) + b_n \sin(n\omega x)| \leq |a_n| + |b_n|.$$

This leads to the following result.

Proposition 5.1.1
If the numerical series

$$\sum_{n=0}^{\infty} |a_n| \quad \text{and} \quad \sum_{n=0}^{\infty} |b_n|$$

converge, then the trigonometric series

$$\sum_{n=0}^{\infty}(a_n\cos(n\omega x)+b_n\sin(n\omega x))$$

is absolutely convergent for all $x\in\mathbb{R}$ and the sum function is continuous on $\mathbb{R}$.

Example 5.1

Consider the series

$$\sum_{n=1}^{\infty}\frac{\cos(n\omega x)}{n^2}.$$

Thus, $a_n=\frac{1}{n^2}$ and $b_n=0$. We have:

$$|a_n\cos(n\omega x)+b_n\sin(n\omega x)|\leq|a_n|+|b_n|=\frac{1}{n^2}.$$

Now,

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

is a convergent Riemann series since $p=2>1$. Therefore, the trigonometric series

$$\sum_{n=1}^{\infty}\frac{\cos(n\omega x)}{n^2}$$

is absolutely convergent on $\mathbb{R}$. ■

The next proposition gives us another way to see the convergence of such series, which is based on Abel criterion.

Proposition 5.1.2

If the numerical sequences $(a_n)_n$ and $(b_n)_n$ are positive and decreasing to 0, then, by Abel's test, the trigonometric series

$$\sum_{k=0}^{\infty}(a_n\cos(n\omega x)+b_n\sin(n\omega x))$$

is convergent for all $x\neq\frac{2k\pi}{\omega}$, where $k\in\mathbb{Z}$.

5.1.1 Calculating the Coefficients of the Trigonometric Series: Real Case

Let us consider the conditions for uniform convergence of the trigonometric series (5.1) to $S(x)$.

- *Calculation of* a_0. Assume that the series is integrable term by term over any interval $I=[\beta,\beta+T]$. We have:

$$\int_{\beta}^{\beta+T}S(x)\,dx=\int_{\beta}^{\beta+T}a_0\,dx+\sum_{n=1}^{\infty}\int_{\beta}^{\beta+T}(a_n\cos(n\omega x)+b_n\sin(n\omega x))\,dx.$$

  Given $\omega=\frac{2\pi}{T}$, for all $n\geq1$, we find:

$$\int_{\beta}^{\beta+T}\cos(n\omega x)\,dx=0\quad\text{and}\quad\int_{\beta}^{\beta+T}\sin(n\omega x)\,dx=0.$$

  Since $\int_{\beta}^{\beta+T}a_0\,dx=Ta_0$, it follows that:

$$a_0=\frac{1}{T}\int_{\beta}^{\beta+T}S(x)\,dx.$$

- *Calculation of Other Coefficients.* Using uniform convergence, we obtain:

$$a_n=\frac{2}{T}\int_{\beta}^{\beta+T}S(x)\cos(n\omega x)\,dx,$$

$$b_n=\frac{2}{T}\int_{\beta}^{\beta+T}S(x)\sin(n\omega x)\,dx,\quad n\geq1.$$

Auxiliary Integrals

To compute the coefficients, we use the following auxiliary integrals where n and k are strictly positive integers:

$$\int_{\beta}^{\beta+T} \cos(k\omega x)\cos(n\omega x)\,dx = \begin{cases} 0 & \text{if } k \neq n, \\ \frac{T}{2} & \text{if } k = n, \end{cases}$$

$$\int_{\beta}^{\beta+T} \sin(k\omega x)\sin(n\omega x)\,dx = \begin{cases} 0 & \text{if } k \neq n, \\ \frac{T}{2} & \text{if } k = n, \end{cases}$$

$$\int_{\beta}^{\beta+T} \cos(k\omega x)\sin(n\omega x)\,dx = 0.$$

- *Conclusion.* For all $\beta \in \mathbb{R}$, we have:

$$a_0 = \frac{1}{T}\int_{\beta}^{\beta+T} S(x)\,dx,$$

$$a_n = \frac{2}{T}\int_{\beta}^{\beta+T} S(x)\cos(n\omega x)\,dx, \quad n \geq 1,$$

$$b_n = \frac{2}{T}\int_{\beta}^{\beta+T} S(x)\sin(n\omega x)\,dx, \quad n \geq 1.$$

In particular, for $\beta = -\pi$, we have the following result.

5.2 Fourier coefficients

Determination of the coefficients $a_0, a_1, a_2, \ldots$ and $b_1, b_2, b_3, \ldots$ makes use of orthogonality relations for sine and cosine. We first define the widely used Kronecker delta δ_{nm} as:

$$\delta_{nm} = \begin{cases} 1 & \text{if } n = m; \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

The orthogonality relations for n and m positive integers are then given with compact notation as the integration formulas:

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = L\delta_{nm}, \tag{5.3}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = L\delta_{nm}, \tag{5.4}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0. \tag{5.5}$$

To illustrate the integration technique used to obtain these results, we derive (5.4) assuming that n and m are positive integers with $n \neq m$. Changing variables to $\xi = \pi x/L$, we obtain:

$$\begin{aligned}\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx &= \frac{L}{\pi}\int_{-\pi}^{\pi} \sin(m\xi)\sin(n\xi)d\xi \\ &= \frac{L}{2\pi}\int_{-\pi}^{\pi} [\cos((m-n)\xi) - \cos((m+n)\xi)]d\xi \\ &= \frac{L}{2\pi}\left[\frac{1}{m-n}\sin((m-n)\xi) - \frac{1}{m+n}\sin((m+n)\xi)\right]_{-\pi}^{\pi} \\ &= 0.\end{aligned}$$

For $m = n$, however:

$$\begin{aligned}\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx &= \frac{L}{\pi}\int_{-\pi}^{\pi} \sin^2(n\xi)d\xi \\ &= \frac{L}{2\pi}\int_{-\pi}^{\pi}(1-\cos(2n\xi))d\xi \\ &= \frac{L}{2\pi}\left[\xi - \frac{1}{2n}\sin 2n\xi\right]_{-\pi}^{\pi} \\ &= L.\end{aligned}$$

Integration formulas given by (5.3) and (5.5) can be similarly derived.

To determine the coefficient a_n, we multiply both sides of (**??**) by $\cos(n\pi x/L)$ with n a nonnegative integer, and change the dummy summation variable from n to m. Integrating over x from $-L$ to L and assuming that the integration can be done term by term in the infinite sum, we obtain:

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx &= \frac{a_0}{2}\int_{-L}^{L}\cos\frac{n\pi x}{L}dx \\ &+ \sum_{m=1}^{\infty}\left\{a_m\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx + b_m\int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx\right\}.\end{aligned}$$

If $n = 0$, then the second and third integrals on the right-hand side are zero and the first integral is $2L$ so that the right-hand side becomes La_0. If n is a positive integer, then the first and third integrals on the right-hand side are zero, and the second integral is $L\delta_{nm}$. For this case, we have:

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx &= \sum_{m=1}^{\infty} La_m\delta_{nm} \\ &= La_n,\end{aligned}$$

where all the terms in the summation except $m = n$ are zero by virtue of the Kronecker delta. We therefore obtain for $n = 0, 1, 2, \ldots$:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx. \tag{5.6}$$

To determine the coefficients b_1, b_2, $b_3, \ldots$, we multiply both sides of (**??**) by $\sin(n\pi x/L)$, with n a positive integer, and again change the dummy summation variable from n to m. Integrating, we obtain:

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx &= \frac{a_0}{2}\int_{-L}^{L}\sin\frac{n\pi x}{L}dx \\ &+ \sum_{m=1}^{\infty}\left\{a_m\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx + b_m\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx\right\}.\end{aligned}$$

Here, the first and second integrals on the right-hand side are zero, and the third integral is $L\delta_{nm}$ so that:

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx &= \sum_{m=1}^{\infty} Lb_m\delta_{nm} \\ &= Lb_n.\end{aligned}$$

Hence, for $n = 1, 2, 3, \ldots$:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \tag{5.7}$$

Our results for the Fourier series of a function $f(x)$ with period $2L$ are thus given by (**??**), (5.6) and (5.7).

Definition 5.2.1
Let $L>0$ be a fixed number and f be a periodic function with period $2L$, defined on $(-L,L)$. The Fourier series of f is a way of expanding the function f into an infinite series involving sines and cosines:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right), \tag{5.8}$$

where a_0, a_n, and b_n are called the *Fourier coefficients* of $f(x)$, and are given by the formulas

$$a_0=\frac{1}{L}\int_{-L}^{L} f(x)\,dx,\quad a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\quad b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

R If $L=\pi$, then the Fourier series is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right),$$

where the coefficients are computed as:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

R

- To find a Fourier series, it is sufficient to calculate the integrals that give the coefficients a_0, a_n, and b_n and plug them into the big series formula, equation (5.8) above.
- Typically, $f(x)$ will be piecewise defined.
- Big advantage that Fourier series have over Taylor series: the function $f(x)$ can have discontinuities!
- Useful identities for Fourier series: if n is an integer, then
  - $\sin(n\pi)=0$
  - $\cos(n\pi)=(-1)^n=\begin{cases}1 & n \text{ even}\\ -1 & n \text{ odd}\end{cases}$

  e.g., $\sin(\pi)=\sin(2\pi)=\sin(3\pi)=\sin(20\pi)=0$ e.g., $\cos(\pi)=\cos(3\pi)=\cos(5\pi)=-1$; but $\cos(0\pi)=\cos(2\pi)=\cos(4\pi)=1$
- If $f(x)$ is an even function, then the formulas for the coefficients simplify. Specifically, since $f(x)$ is even, $f(x)\sin\left(\frac{nx}{\pi}\right)$ is an odd function, and thus

  $$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{f(x)}_{\text{even}}\underbrace{\sin\left(\frac{nx}{\pi}\right)}_{\text{odd}}dx=0$$

  Therefore, for even functions, you can automatically conclude (no computations necessary!) that the b_n coefficients are all 0.
- If $f(x)$ is odd, then we get two freebies:

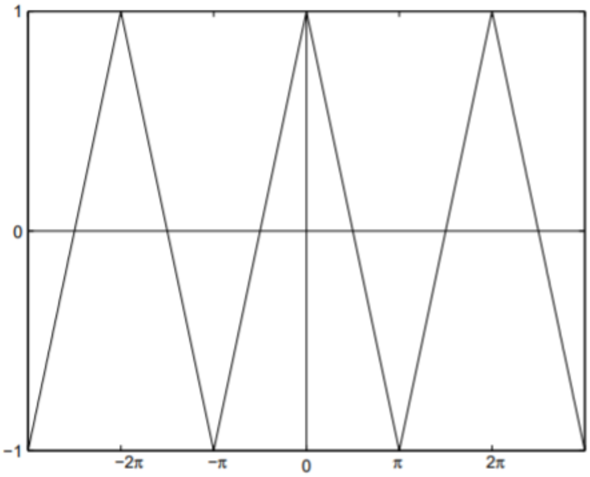
  $$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{f(x)}_{\text{odd}}dx=0$$

  $$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{f(x)}_{\text{odd}}\underbrace{\cos\left(\frac{nx}{\pi}\right)}_{\text{even}}dx=0$$

R In general, your function may be neither even nor odd. In those cases, you should use the original formulas for computing Fourier coefficients, given in equation (5.8).

■ **Example 5.2**

1. Let's determine the Fourier coefficients of the even triangle function represented by



The triangle function depicted in Figure is an even function of x with period 2π (i.e., $L = \pi$). Its definition on $0 < x < \pi$ is given by

$$f(x) = 1 - \frac{2x}{\pi}.$$

Because f is even the coefficient $b_n = 0$ for all n, thus it can be represented by the Fourier cosine series given. The coefficient a_0 is

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} f(x)dx \\ &= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx \\ &= \frac{2}{\pi}\left[x-\frac{x^2}{\pi}\right]_0^{\pi} = 0 \end{aligned}$$

Note that $a_0/2$ is the average value of f over one period, and it is obvious from Fig. that this value is zero. The coefficients for $n > 0$ are

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx \\ &= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos(nx)dx \\ &= \frac{2}{\pi}\int_0^{\pi}\cos(nx)dx - \frac{4}{\pi^2}\int_0^{\pi} x\cos(nx)dx \\ &= \frac{2}{n\pi}\sin(nx)]_0^{\pi} - \frac{4}{\pi^2}\left\{\left[\frac{x}{n}\sin(nx)\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)dx\right\} \\ &= \frac{4}{n\pi^2}\int_0^{\pi}\sin(nx)dx \\ &= -\frac{4}{n^2\pi^2}\cos(nx)]_0^{\pi} \\ &= \frac{4}{n^2\pi^2}(1-\cos(n\pi)). \end{aligned}$$

Since

$$\cos(n\pi) = \begin{cases} -1, & \text{if } n \text{ odd;} \\ 1, & \text{if } n \text{ even;} \end{cases}$$

we have

$$a_n = \begin{cases} 8/(n^2\pi^2), & \text{if } n \text{ odd;} \\ 0, & \text{if } n \text{ even.} \end{cases}$$

The Fourier cosine series for the triangle function is therefore given by

$$f(x) = \frac{8}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right).$$

Convergence of this series is rapid. As an interesting aside, evaluation of this series at $x = 0$, using $f(0) = 1$, yields an infinite series for $\pi^2/8$:

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

2. Let's determine the Fourier coefficients of the $2\pi-$ periodic function f, defined for $x \in [0, 2\pi[$ by $f(x) = x^2$.
   First, let's be careful about the following fact: just because the function is defined by x^2 on $[0, 2\pi[$ doesn't mean the function is even. Its definition by 2π-periodicity means that f is not equal to x^2 on $]-2\pi, 0]$. In particular, we cannot say that the coefficients b_n are zero. For the calculation of the Fourier coefficients, we first have:
   $$a_0(f) = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{4\pi^2}{3}.$$
   Furthermore, by performing two integrations by parts, we have:
   $$\begin{aligned} a_n(f) &= \frac{1}{\pi}\int_0^{2\pi} x^2\cos(nx)dx \\ &= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n}\right]_0^{2\pi} - \frac{2}{\pi n}\int_0^{2\pi} x\sin(nx)dx \\ &= \frac{2}{\pi n}\left[x\frac{\cos nx}{n}\right]_0^{2\pi} - \frac{2}{\pi n^2}\int_0^{2\pi}\cos(nx)dx \\ &= \frac{4}{n^2}. \end{aligned}$$
   Similarly, we find $b_n = \dfrac{-4\pi}{n}$. Therefore, the Fourier series of f is:
   $$S(f,x) = \frac{4\pi^2}{3} + \sum_{n\geq 1}\left(\frac{4}{n^2}\cos(nx) - \frac{4\pi}{n}\sin(nx)\right).$$

■

5.3 Fourier Series

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous and 2π-periodic function. We define the *exponential Fourier coefficients* of f as the sequence $(c_n(f))_{n\in\mathbb{Z}}$ given by

$$c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt.$$

We define the *trigonometric Fourier coefficients* of f as the two sequences $(a_n(f))n \geq 0$ and $(b_n(f))n \geq 1$ given by

$$\begin{aligned} a_0(f) &= \frac{1}{2\pi}\int_0^{2\pi} f(t)dt \\ a_n(f) &= \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)dt, \quad n \geq 1 \\ b_n(f) &= \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)dt, \quad n \geq 1. \end{aligned}$$

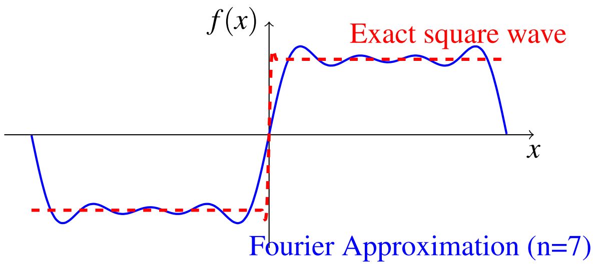
The *Fourier series* of f is the series of functions

$$S(t) = \sum_{n=-\infty}^{+\infty} c_n(f)e^{int} = a_0(f) + \sum_{n\geq 1}\big(a_n(f)\cos(nt) + b_n(f)\sin(nt)\big).$$

The partial sums of this series are denoted by

$$S_n(f)(t) = \sum_{k=-n}^{n} c_k(f)e^{ikt} = a_0(f) + \sum_{k=1}^{n}\big(a_k(f)\cos(kt) + b_k(f)\sin(kt)\big).$$

Below is a comparison of a square wave with its Fourier series approximation:



■ **Example 5.3**
We want to determine the Fourier series for $f(x) = L - x$ on $-L \leq x \leq L$.

So, let's go ahead and just run through formulas for the coefficients.

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} L - x\,dx = L$$

$$\begin{aligned}
A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L}(L-x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{L}\left(\frac{L}{n^2\pi^2}\right)\left(n\pi(L-x)\sin\left(\frac{n\pi x}{L}\right) - L\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{L} \\
&= \frac{1}{L}\left(\frac{L}{n^2\pi^2}\right)(-2n\pi L\sin(-n\pi)) = 0, \qquad n = 1,2,3,\ldots
\end{aligned}$$

$$\begin{aligned}
B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{L}\left(-\frac{L}{n^2\pi^2}\right)\left[L\sin\left(\frac{n\pi x}{L}\right) - n\pi(x-L)\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{L} \\
&= \frac{1}{L}\left[\frac{L^2}{n^2\pi^2}(2n\pi\cos(n\pi) - 2\sin(n\pi))\right] = \frac{2L(-1)^n}{n\pi}, \qquad n = 1,2,3,\ldots
\end{aligned}$$

Note that in this case we had $A_0 \neq 0$ and $A_n = 0$, $n = 1,2,3,\ldots$ This will happen on occasion so don't get excited about this kind of thing when it happens.

The Fourier series is then,

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) \\
&= A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) = L + \sum_{n=1}^{\infty}\frac{2L(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

■

5.4 Theorems of Pointwise Convergence

Definition 5.4.1
we say that $f(x)$ is piecewise smooth if the function can be broken into distinct pieces and on each piece both the function and its derivative, $f'(x)$, are continuous. A piecewise smooth function may not be continuous everywhere however the only discontinuities that are allowed are a finite number of jump discontinuities.

Theorem 5.4.1 — Dirichlet Theorem.
Let f be a piecewise $\mathscr{C}^1$ and 2π-periodic function. Then, for all $x \in \mathbb{R}$, $S_n(f)(x)$ converges to

$$\frac{f(x^+)+f(x^-)}{2}$$

where $f(x^+)$ (resp. $f(x^-)$) denotes the right (left) limit of f at x.
In particular, if f is continuous then $S_n(f)(x)$ converges to f.

Theorem 5.4.2 — Normal Convergence Theorem.
Let f be a continuous, piecewise $\mathscr{C}^1$ and 2π-periodic function. Then the Fourier series of f converges normally to f.

■ Example 5.4
Let's consider the function,

$$f(x) = \begin{cases} L & \text{if } -L \le x \le 0 \\ 2x & \text{if } 0 \le x \le L \end{cases}$$

Let's find the Fourier series for this function. We have

$$\begin{aligned} A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\left[\int_{-L}^{0} f(x)\,dx + \int_0^L f(x)\,dx\right] \\ &= \frac{1}{2L}\left[\int_{-L}^{0} L\,dx + \int_0^L 2x\,dx\right] = \frac{1}{2L}\left[L^2+L^2\right] = L \end{aligned}$$

$$\begin{aligned} A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx &= \frac{1}{L}\left[\int_{-L}^{0} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + \int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right] \\ &= \frac{1}{L}\left[\int_{-L}^{0} L\cos\left(\frac{n\pi x}{L}\right)dx + \int_0^L 2x\cos\left(\frac{n\pi x}{L}\right)dx\right] \end{aligned}$$

At this point it will probably be easier to do each of these individually.</p>

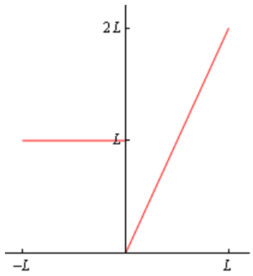
$$\int_{-L}^{0} L\cos\left(\frac{n\pi x}{L}\right)dx = \left(\frac{L^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{0} = \frac{L^2}{n\pi}\sin(n\pi) = 0$$

$$\begin{aligned} \int_0^L 2x\cos\left(\frac{n\pi x}{L}\right)dx &= \left(\frac{2L}{n^2\pi^2}\right)\left(L\cos\left(\frac{n\pi x}{L}\right) + n\pi x\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\ &= \left(\frac{2L}{n^2\pi^2}\right)\left(L\cos(n\pi) + n\pi L\sin(n\pi) - L\cos(0)\right) \\ &= \left(\frac{2L^2}{n^2\pi^2}\right)\left((-1)^n - 1\right) \end{aligned}$$

So, by replacing all of this together we get,

$$\begin{aligned} A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[0 + \left(\frac{2L^2}{n^2\pi^2}\right)\left((-1)^n-1\right)\right] \\ &= \frac{2L}{n^2\pi^2}\left((-1)^n-1\right), \qquad n = 1,2,3,\ldots \end{aligned}$$

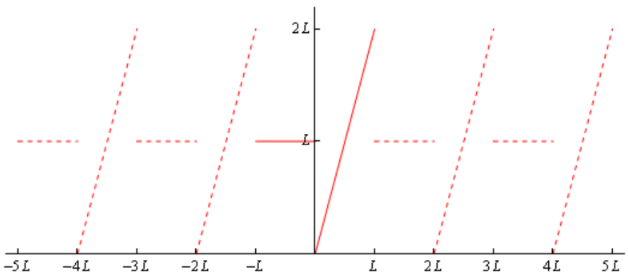
Here is a sketch of this function on the interval on which it is defined, i.e. $-L \le x \le L$.



This function has a jump discontinuity at $x = 0$ because $f(0^-) = L \neq 0 = f(0^+)$ and note that on the intervals $-L \leq x \leq 0$ and $0 \leq x \leq L$ both the function and its derivative are continuous. This is therefore an example of a piecewise smooth function. Note that the function itself is not continuous at $x = 0$ but because this point of discontinuity is a jump discontinuity the function is still piecewise smooth.

The last term we need to define is that of periodic extension. Given a function, $f(x)$, defined on some interval, we'll be using $-L \leq x \leq L$ exclusively here, the periodic extension of this function is the new function we get by taking the graph of the function on the given interval and then repeating that graph to the right and left of the graph of the original function on the given interval.

It is probably best to see an example of a periodic extension at this point to help make the words above a little clearer. Here is a sketch of the period extension of the function we looked at above,



The original function is the solid line in the range $-L \leq x \leq L$. We then got the periodic extension of this by picking this piece up and copying it every interval of length $2L$ to the right and left of the original graph. This is shown with the two sets of dashed lines to either side of the original graph.

Note that the resulting function that we get from defining the periodic extension is in fact a new periodic function that is equal to the original function on $-L \leq x \leq L$.

With these definitions out of the way we can now proceed to talk a little bit about the convergence of Fourier series. We will start off with the convergence of a Fourier series and once we have that taken care of the convergence of Fourier Sine/Cosine series will follow as a direct consequence. Here then is the theorem giving the convergence of a Fourier series.

■

5.5 Theorems of Mean Square Convergence

Theorem 5.5.1 — Parseval's Theorem.
If f is piecewise continuous and 2π-periodic, then

$$\|f - S_n(f)\|_2 \to 0.$$

In particular, we have

$$\frac{1}{2\pi}\int 0^{2\pi}|f(t)|^2dt = \sum_{n\in\mathbb{Z}}|c_n(f)|^2$$
$$= |a_0(f)|^2 + \frac{1}{2}\sum_{n\geq 1}\left(|a_n(f)|^2+|b_n(f)|^2\right).$$

■ **Example 5.5**

In the example 5.2, we found the Fourier series for the $2\pi-$ periodic function f, defined for $x\in[0,2\pi[$ by $f(x)=x^2$. Its series is

$$S(f,x) = \frac{4\pi^2}{3}+\sum_{n\geq 1}\left(\frac{4}{n^2}\cos(nx)-\frac{4\pi}{n}\sin(nx)\right).$$

This function is piecewise continuous, then by Dirichlet theorem we have

$$\frac{1}{2}\left(f(x^+)+f(x^-)\right) = \frac{4\pi^2}{3}+\sum_{n\geq 1}\left(\frac{4}{n^2}\cos(nx)-\frac{4\pi}{n}\sin(nx)\right).$$

We want to find $\sum_{n\geq 1}\frac{1}{n^4}$ which is can be calculated by applying Parseval's theorem. We thus have:

$$\frac{1}{2\pi}\int_0^{2\pi}x^4dx = \left(\frac{4\pi^2}{3}\right)^2+\frac{1}{2}\sum_{n\geq 1}\left(\frac{16}{n^4}+\frac{16\pi^2}{n^2}\right).$$

Using the fact that $\frac{1}{2\pi}\int_0^{2\pi}x^4dx=\frac{16\pi^4}{5}$, and reinjecting the previous calculation, we find that

$$\sum_{n\geq 1}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

■

5.6 Exercises

Exercise 5.1

Determine the Fourier series (sine and cosine terms) for the following functions:

1. f is 2π-periodic, defined by $f(x)=x$ for $-\pi\leq x<\pi$.
2. A square wave function: f is 2π-periodic, defined by $f(x)=1$ if $x\in[0,\pi[$, and $f(x)=-1$ if $x\in[-\pi,0[$.
3. A function with L-periodicity, where $L>0$, defined by $f(x)=|x|$ for $x\in[-L/2,L/2]$.

■

Exercise 5.2

Determine the Fourier series of the periodic function of period 2π defined by $f(x)=x^2$ for $-\pi\leq x\leq\pi$.

Deduce the sums of the series $\sum_{n\geq 1}\frac{1}{n^2}$, $\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^2}$, $\sum_{n\geq 1}\frac{1}{n^4}$. ■

Exercise 5.3

Let f be the 2π-periodic function such that $f(x)=e^x$ if $x\in[-\pi,\pi[$. Determine the Fourier series of f. Deduce the values of the following sums: $\sum_{n\geq 1}\frac{1}{n^2+1}$ and $\sum_{n\geq 1}\frac{(-1)^n}{n^2+1}$. ■

Exercise 5.4

Let f be the periodic function of period 2 satisfying $f(x)=x-x^3$ for all $x\in]-1,1]$.

1. Determine the Fourier coefficients of f.

2. Deduce the sum of the series $\sum_{p=0}^{+\infty} \frac{(-1)^p}{(2p+1)^3}$.

Exercise 5.5 The goal of this exercise is to determine whether the differential equation (E)

$$y'' + e^{it}y = 0$$

admits 2π-periodic solutions.

1. (a) Show that the trigonometric series $\sum_{n\geq 0} \frac{1}{(n!)^2} e^{int}$ converges uniformly on $\mathbb{R}$ to a function f that is 2π-periodic.
   (b) Show that the function f is of class C^2 and satisfies the equation (E).
2. Let $g : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic solution of class C^2 of (E). Denote by

$$\sum_{n\in\mathbb{Z}} c_n(g)e^{int} \quad \text{and} \quad \sum_{n\in\mathbb{Z}} c_n(g'')e^{int}$$

   the Fourier series of g and g'', respectively.
   (a) Express $c_n(g'')$ in terms of $c_n(g)$.
   (b) Using the fact that g is a solution of (E), express $c_n(g'')$ in terms of $c_{n-1}(g)$.
   (c) Deduce that the set of 2π-periodic solutions of (E) is the one-dimensional vector space spanned by the function f.
3. Does (E) have any solutions that are not 2π-periodic?

III

Integrals

6. Improper Integrals

While the definite integral is initially introduced as the area under a curve between two finite points, its applications extend far beyond these simple geometric interpretations. We encounter numerous situations where the region of integration is unbounded, necessitating a broader definition of the integral. These scenarios fall into two primary categories:

Infinite Intervals of Integration:

• Infinite Upper Bound: Consider the integral $\int_1^\infty \frac{1}{x^2}\,dx$. Here, the interval of integration extends to infinity. Geometrically, we seek to calculate the area under the curve $y = \frac{1}{x^2}$ from $x = 1$ that extends indefinitely to the right.

• Infinite Lower Bound: Similarly, the integral $\int_{-\infty}^0 e^x\,dx$ represents the area under the curve $y = e^x$ over an interval extending indefinitely to the left.

• Infinite Interval on Both Sides: The integral $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$ covers the entire real line.

Unbounded Integrands within the Interval of Integration:

• Singularity at an Endpoint: Consider the integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. The function $\frac{1}{\sqrt{x}}$ tends to infinity as x approaches 0. Thus, we have a singularity at the lower endpoint of the interval.

• Singularity within the Interval: It is also possible to have a singularity at a point within the interval of integration.

To calculate an improper integral, we utilize the concept of limits. For example, for the integral $\int_1^\infty \frac{1}{x^2}\,dx$, we consider the sequence of integrals $\int_1^b \frac{1}{x^2}\,dx$ as b approaches infinity. If this sequence of numbers has a finite limit as b approaches infinity, then we say the improper integral converges, and its value is equal to this limit. Otherwise, we say it diverges.

Improper integrals have numerous applications in mathematics, physics, and engineering. They enable us to: Calculate probabilities associated with continuous random variables. Model physical phenomena such as radioactive decay or gravitational force. Analyze signals in signal processing.

6.1 Convergence of improper integrals

Definition 6.1.1

1. Let f be a continuous function on $[a, +\infty[$. We say that the integral $\int_a^{+\infty} f(t)\,\mathrm{d}t$ *converges* if the

limit, as x tends to $+\infty$, of the primitive $\int_a^x f(t)\,\mathrm{d}t$ exists and is finite. If this is the case, we define:

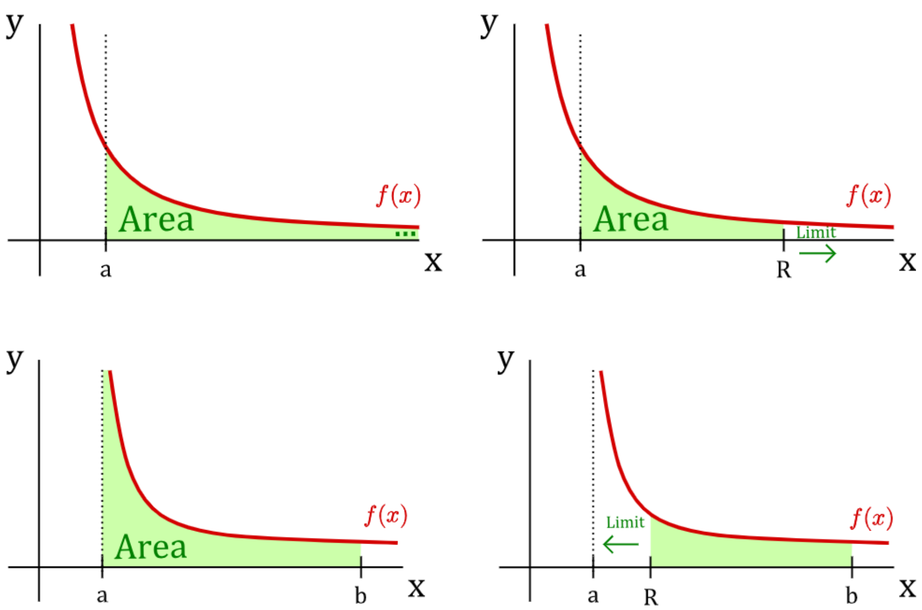
$$\int_a^{+\infty} f(t)\,\mathrm{d}t = \lim_{x\to+\infty}\int_a^x f(t)\,\mathrm{d}t\,. \tag{6.1}$$

Otherwise, we say that the integral *diverges*.

2. Let f be a continuous function on $]a,b]$. We say that the integral $\int_a^b f(t)\,\mathrm{d}t$ *converges* if the right-hand limit, as x tends to a, of $\int_x^b f(t)\,\mathrm{d}t$ exists and is finite. If this is the case, we define:

$$\int_a^{b} f(t)\,\mathrm{d}t = \lim_{x\to a^+}\int_x^b f(t)\,\mathrm{d}t\,. \tag{6.2}$$

Otherwise, we say that the integral *diverges*.



R

- Convergence thus means a finite limit. Divergence means either that there is no limit or that the limit is infinite.
- Note that the second definition is consistent with the integral of a function that is continuous on the entire interval $[a,b]$ (instead of $]a,b]$). We know that the primitive $\int_x^b f(t)\,\mathrm{d}t$ is a continuous function. Consequently, the usual integral $\int_a^b f(t)\,\mathrm{d}t$ is also the limit of $\int_x^b f(t)\,\mathrm{d}t$ (as $x\to a^+$). In this case, the two integrals coincide.
- By splitting and by change of variable $t\mapsto -t$, we reduce the problem to integrals of two types. Integral over $[a,+\infty[$ or integral over $]a,b]$, with the function unbounded at a.

R When we can find an antiderivative (primitive) F(x) of the function to be integrated (for example, $F(x)=\int_a^x f(t)\,\mathrm{d}t$), studying the convergence of the integral boils down to calculating the limit of F(x).

■ **Example 6.1**

1. The integral

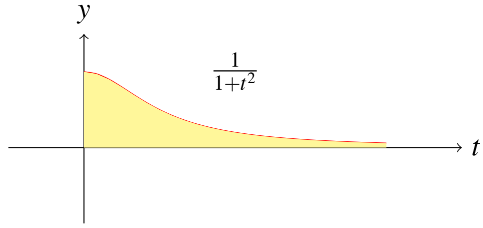
$$\int_0^{+\infty}\frac{1}{1+t^2}\,\mathrm{d}t$$

converges.
Indeed,

$$\int_0^x \frac{1}{1+t^2}\,\mathrm{d}t = \Big[\arctan t\Big]_0^x = \arctan x \qquad\text{and}\qquad \lim_{x\to+\infty}\arctan x = \frac{\pi}{2}\,.$$

We can write:

$$\int_0^{+\infty} \frac{1}{1+t^2}\,\mathrm{d}t = \Big[\arctan t\Big]_0^{+\infty} = \frac{\pi}{2},$$



This demonstrates that the domain under the curve is unbounded, yet its area is finite.

2. The integral

$$\int_0^1 \ln t\,\mathrm{d}t \qquad \text{converges.}$$

Indeed,

$$\int_x^1 \ln t\,\mathrm{d}t = \Big[t\ln t - t\Big]_x^1 = x - x\ln x - 1 \qquad \text{and} \qquad \lim_{x\to 0^+}(x - x\ln x - 1) = -1\,.$$

We can write:

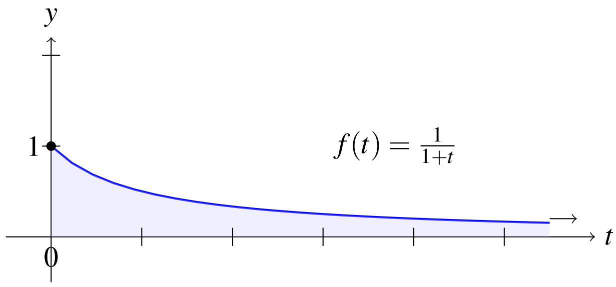
$$\int_0^1 \ln t\,\mathrm{d}t = \Big[t\ln t - t\Big]_0^1 = -1\,.$$

3. However, the integral

$$\int_0^{+\infty} \frac{1}{1+t}\,\mathrm{d}t \qquad \text{diverges.}$$

Indeed,

$$\int_0^x \frac{1}{1+t}\,\mathrm{d}t = \Big[\ln(1+t)\Big]_0^x = \ln(1+x) \qquad \text{and} \qquad \lim_{x\to+\infty} \ln(1+x) = +\infty\,.$$



■

6.1.1 Properties

When it converges, this new integral satisfies the same properties as the usual Riemann integral.

- *Chasles' Relation.* Let $f : [a,+\infty[\to \mathbb{R}$ be a continuous function, and let $a' \in [a,+\infty[$. Then the improper integrals $\int_a^{+\infty} f(t)\,\mathrm{d}t$ and $\int_{a'}^{+\infty} f(t)\,\mathrm{d}t$ are of the same nature. If they converge, then

$$\int_a^{+\infty} f(t)\,\mathrm{d}t = \int_a^{a'} f(t)\,\mathrm{d}t + \int_{a'}^{+\infty} f(t)\,\mathrm{d}t.$$

- *Linearity.* Let f and g be two continuous functions on $[a,+\infty[$, and let λ, μ be two real numbers. If the integrals $\int_a^{+\infty} f(t)\,\mathrm{d}t$ and $\int_a^{+\infty} g(t)\,\mathrm{d}t$ converge, then $\int_a^{+\infty} \big(\lambda f(t) + \mu g(t)\big)\,\mathrm{d}t$ also converges, and

$$\int_a^{+\infty} \big(\lambda f(t) + \mu g(t)\big)\,\mathrm{d}t = \lambda \int_a^{+\infty} f(t)\,\mathrm{d}t + \mu \int_a^{+\infty} g(t)\,\mathrm{d}t\,.$$

R If such integral $\int_a^{+\infty}(f+g)$ converges, doesn't mean that $\int_a^{+\infty} f$ and $\int_a^{+\infty} g$ converges. Indeed, the two improper integrals

$$\int_1^{+\infty} x+\frac{1}{x^2}\,\mathrm{d}x \quad \text{and} \quad \int_1^{+\infty} x\,\mathrm{d}x,$$

are divergent, while

$$\int_1^{+\infty} (x+\frac{1}{x^2}-x)\,\mathrm{d}x = \int_1^{+\infty} \frac{1}{x^2}\,\mathrm{d}x$$

is convergent.

- *Positivity of the Integral.* Let $f,g:[a,+\infty[\to\mathbb{R}$ be continuous functions with convergent integrals.

$$\text{If} \quad f\le g \quad \text{then} \quad \int_a^{+\infty} f(t)\,\mathrm{d}t \le \int_a^{+\infty} g(t)\,\mathrm{d}t.$$

  In particular

$$\text{If} \quad f\ge 0 \quad \text{then} \quad \int_a^{+\infty} f(t)\,\mathrm{d}t \ge 0.$$

- *More than one uncertain point.* Doubly improper integrals, where both endpoints of the interval are improper, can be addressed by breaking the problem into two separate integrals, each with only one improper endpoint.
  Let Let $f:]a,b[\to\mathbb{R}$ $(a,b$ may be infinite)be a continuous function. The integral $\int_a^b f(t)\,\mathrm{d}t$ is said to *converge* if there exists $c\in]a,b[$ such that both improper integrals $\int_a^c f(t)\,\mathrm{d}t$ and $\int_c^b f(t)\,\mathrm{d}t$ converge. The value of this doubly improper integral is then given by:

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t.$$

  By the Chasles relation, the nature and value of this doubly improper integral do not depend on the choice of c, provided $a<c<b$.

■ **Example 6.2**

1. We want to determine whether $\int_{-\infty}^{\infty} x\sin(x^2)\,dx$ is convergent or divergent.
   We must compute both $\int_0^{\infty} x\sin(x^2)\,dx$ and $\int_{-\infty}^{0} x\sin(x^2)\,dx$. Note that we don't have to split the integral up at 0, any finite value a will work. First we compute the indefinite integral. Let $u=x^2$, then $du=2x\,dx$ and hence,

$$\int x\sin(x^2)\,dx = \frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(x^2)+C$$

   Using the definition of improper integral gives:

$$\begin{aligned}\int_0^{\infty} x\sin(x^2)\,dx = \lim_{A\to\infty}\int_0^A x\sin(x^2)\,dx &= \lim_{A\to\infty}\left[-\frac{1}{2}\cos(x^2)\right]\Big|_0^A \\ &= -\frac{1}{2}\lim_{A\to\infty}\cos(A^2)+\frac{1}{2}\end{aligned}$$

   This limit does not exist since $\cos x$ between -1 and $+1$. In particular, $\cos x$ does not approach any particular value as x gets larger and larger. Thus, $\int_0^{\infty} x\sin(x^2)\,dx$ diverges, and hence, the integral $\int_{-\infty}^{\infty} x\sin(x^2)\,dx$ diverges.
2. Does the following integral converge?

$$\int_{-\infty}^{+\infty} \frac{t\,\mathrm{d}t}{(1+t^2)^2}$$

   Choose $c=2$ arbitrarily. The question reduces to determining whether the two integrals

$$\int_{-\infty}^{2} \frac{t\,\mathrm{d}t}{(1+t^2)^2} \quad \text{and} \quad \int_2^{+\infty} \frac{t\,\mathrm{d}t}{(1+t^2)^2}$$

converge.
Using the fact that a primitive of $\frac{t}{(1+t^2)^2}$ is $-\frac{1}{2}\frac{1}{1+t^2}$, we compute:

$$\int_x^2 \frac{t\,dt}{(1+t^2)^2} = -\frac{1}{2}\left[\frac{1}{1+t^2}\right]_x^2 = -\frac{1}{2}\left(\frac{1}{5} - \frac{1}{1+x^2}\right) \to -\frac{1}{10} \quad \text{as } x \to -\infty.$$

Thus, $\int_{-\infty}^2 \frac{t\,dt}{(1+t^2)^2}$ converges and equals $-\frac{1}{10}$.
Similarly,

$$\int_2^x \frac{t\,dt}{(1+t^2)^2} = -\frac{1}{2}\left[\frac{1}{1+t^2}\right]_2^x = -\frac{1}{2}\left(\frac{1}{1+x^2} - \frac{1}{5}\right) \to +\frac{1}{10} \quad \text{as } x \to +\infty.$$

Hence, $\int_2^{+\infty} \frac{t\,dt}{(1+t^2)^2}$ converges and equals $+\frac{1}{10}$.
Therefore, $\int_{-\infty}^{+\infty} \frac{t\,dt}{(1+t^2)^2}$ converges and equals:

$$-\frac{1}{10} + \frac{1}{10} = 0.$$

This result is unsurprising because the function is odd. Repeat the calculations with a different choice of c and verify that the result remains the same.

■

6.1.2 General convergence criteria

p-Integrals (Riemann integrals)

Integrals of the form $\frac{1}{x^p}$ arise in the study of series. These integrals may be considered either as improper integrals with an infinite limit of integration, $\int_a^\infty \frac{1}{x^p}\,dx$, or as improper integrals with a discontinuity at $x = 0$, $\int_0^a \frac{1}{x^p}\,dx$. In asymptotic analysis, it is important to determine the conditions under which these integrals converge or diverge.

p-Test for Infinite Limit

Theorem 6.1.1 — p-Test for Infinite Limit.
For $a > 0$:

i. If $p > 1$, then $\int_a^\infty \frac{1}{x^p}\,dx$ converges.
ii. If $p \leq 1$, then $\int_a^\infty \frac{1}{x^p}\,dx$ diverges.

Proof. i. If $p > 1$, we have:

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{R\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_a^R = \lim_{R\to\infty}\left(\frac{R^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}\right) = \frac{a^{1-p}}{p-1}.$$

ii. If $p \leq 1$, the resulting limit becomes infinite, and the integral diverges.

■

p-Test for Discontinuity

Theorem 6.1.2 — p-Test for Discontinuity.
For $a > 0$:

i. If $p < 1$, then $\int_0^a \frac{1}{x^p}\,dx$ converges.
ii. If $p \geq 1$, then $\int_0^a \frac{1}{x^p}\,dx$ diverges.

Proof. i. If $p < 1$, we calculate:

$$\int_0^a \frac{1}{x^p}\,dx = \lim_{R\to 0^+} \left.\frac{x^{1-p}}{1-p}\right|_R^a = \lim_{R\to 0^+}\left(\frac{a^{1-p}}{1-p} - \frac{R^{1-p}}{1-p}\right) = \frac{a^{1-p}}{1-p}.$$

ii. If $p \geq 1$, the resulting limit becomes infinite, and the integral diverges.

■

■ Example 6.3

Determine if the following integrals are convergent or divergent.

$$(1)\ \int_1^{\infty} \frac{1}{x^3}\,dx, \qquad (2)\ \int_0^5 \frac{1}{x^4}\,dx.$$

This is a p-integral with an infinite upper limit of integration and $p = 3 > 1$. Therefore, by the p-Test for Infinite Limit, $\int_1^{\infty} \frac{1}{x^3}\,dx$ converges.

$\int_0^5 \frac{1}{x^4}\,dx$ as a p-integral with a discontinuity at $x = 0$ and $p = 4 \geq 1$. Thus, by the p-Test for Discontinuity, the integral diverges. ■

Bertrand Integrals

A *Bertrand integral* is defined as

$$\int_2^{+\infty} \frac{1}{t\,(\ln t)^{\beta}}\,\mathrm{d}t \quad \text{where} \quad \beta \in \mathbb{R}.$$

By basic calculus we obtain

$$\int_2^{+\infty} \frac{1}{t\,(\ln t)^{\beta}}\,\mathrm{d}t = \begin{cases} \lim\limits_{x\to+\infty} \left[\frac{1}{-\beta+1}(\ln t)^{-\beta+1}\right]_2^x & \text{if } \beta \neq 1, \\ \lim\limits_{x\to+\infty} \left[\ln(\ln t)\right]_2^x & \text{if } \beta = 1. \end{cases}$$

From this, we can deduce the nature of Bertrand integrals

$$\bullet\text{If } \beta > 1, \quad \text{then} \quad \int_2^{+\infty} \frac{1}{t\,(\ln t)^{\beta}}\,\mathrm{d}t \quad \text{converges.}$$

$$\bullet\text{If } \beta \leq 1, \quad \text{then} \quad \int_2^{+\infty} \frac{1}{t\,(\ln t)^{\beta}}\,\mathrm{d}t \quad \text{diverges.}$$

Here is an example of an application:

■ Example 6.4

Does the integral

$$\int_2^{+\infty} \sqrt{t^2+3t}\,\ln\left(\cos\frac{1}{t}\right)\,\sin^2\left(\frac{1}{\ln t}\right)\,\mathrm{d}t \qquad \text{converge?}$$

The uncertain point is $+\infty$. To answer the question, let's compute an asymptotic equivalent of the function near $+\infty$. We have:

$$\sqrt{t^2+3t} = t\sqrt{1+\frac{3}{t}} \underset{+\infty}{\sim} t$$

$$\ln\left(\cos\frac{1}{t}\right) = \ln\left(1-\frac{1}{2t^2}+o\left(\frac{1}{t^2}\right)\right) \underset{+\infty}{\sim} -\frac{1}{2t^2}$$

$$\sin^2\left(\frac{1}{\ln t}\right) \underset{+\infty}{\sim} \left(\frac{1}{\ln t}\right)^2$$

Thus, an equivalent of the function near $+\infty$ is:

$$\sqrt{t^2+3t}\,\ln\left(\cos\frac{1}{t}\right)\,\sin^2\left(\frac{1}{\ln t}\right) \underset{+\infty}{\sim} -\frac{1}{2t\,(\ln t)^2}$$

Notice that in this equivalence, both functions are negative near $+\infty$. According to the theorem 6.1.4, both associated integrals have the same nature. But since the Bertrand integral

$$\int_2^{+\infty} \frac{1}{t\,(\ln t)^2}\,\mathrm{d}t$$

converges, our initial integral is also convergent. ■

Comparison test

The following test helps determine the convergence or divergence of improper integrals that are difficult to compute by comparing them to simpler ones. While stated for the interval $[a,\infty)$, similar versions apply to other improper integrals.

Theorem 6.1.3 — Comparison Test for Improper Integrals.
Assume that $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^\infty f(x)\,dx$ **converges**, then $\int_a^\infty g(x)\,dx$ also **converges**.
2. If $\int_a^\infty g(x)\,dx$ **diverges**, then $\int_a^\infty f(x)\,dx$ also **diverges**.

Informally

- (i) If $f(x)$ is larger than $g(x)$, and the area under $f(x)$ is finite (converges), then the area under $g(x)$ must also be finite (converges).
- (ii) If $f(x)$ is larger than $g(x)$, and the area under $g(x)$ is infinite (diverges), then the area under $f(x)$ must also be infinite (diverges).

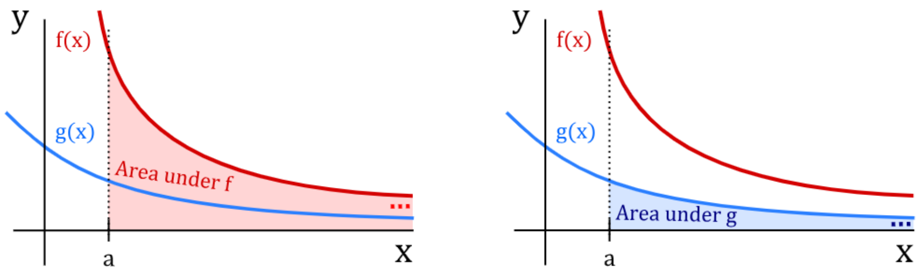


Figure 6.1: Illustration of the Comparison Test for Improper Integrals.

Proof. As we have observed, the convergence of the integrals does not depend on the left endpoint of the interval. Thus, we can limit ourselves to studying $\int_A^x f(t)\,\mathrm{d}t$ and $\int_A^x g(t)\,\mathrm{d}t$.

Using the positivity of the integral, we find that for all $x \geq A$,

$$\int_A^x f(t)\,\mathrm{d}t \leq \int_A^x g(t)\,\mathrm{d}t.$$

If $\int_A^{+\infty} g(t)\,\mathrm{d}t$ converges, then $\int_A^x f(t)\,\mathrm{d}t$ is an increasing function bounded above by $\int_A^{+\infty} g(t)\,\mathrm{d}t$, and hence it also converges. Conversely, if $\int_A^x f(t)\,\mathrm{d}t$ tends to $+\infty$, then $\int_A^x g(t)\,\mathrm{d}t$ also tends to $+\infty$. ■

■ Example 6.5
We want to use the Comparison Test to prove convergence of the improper integral $\int_2^\infty \frac{\cos^2 x}{x^2}\,dx$.

Observe that $0 \leq \cos^2 x \leq 1$, which implies:

$$0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}.$$

Let $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{\cos^2 x}{x^2}$. Clearly, $f(x) \geq g(x) \geq 0$.

It is straightforward to verify that $\int_2^\infty \frac{1}{x^2}\,dx$ converges. Thus, by the Comparison Test, $\int_2^\infty \frac{\cos^2 x}{x^2}\,dx$ also converges. ■

Theorem of Equivalents

We can replace the function to be integrated with an equivalent function in the neighborhood of $+\infty$ to study the convergence of an integral.

Theorem 6.1.4 — Theorem of Equivalents.
Let f and g be two continuous and strictly positive functions on $[a,+\infty[$. Suppose they are equivalent in the neighborhood of $+\infty$, that is:

$$\lim_{t\to+\infty}\frac{f(t)}{g(t)}=1.$$

Then the integral $\int_a^{+\infty} f(t)\,\mathrm{d}t$ converges if and only if $\int_a^{+\infty} g(t)\,\mathrm{d}t$ converges.

R It is essential that f and g are positive functions.
The equivalence of f and g in the neighborhood of $+\infty$ is denoted by:

$$f(t)\underset{+\infty}{\sim} g(t).$$

Proof. To say that two functions are equivalent in the neighborhood of $+\infty$ means that their ratio tends to 1, or equivalently:

$$\forall\varepsilon>0\quad \exists A>a\quad \forall t>A\qquad \left|\frac{f(t)}{g(t)}-1\right|<\varepsilon,$$

which can also be expressed as:

$$\forall\varepsilon>0\quad \exists A>a\quad \forall t>A\qquad (1-\varepsilon)g(t)<f(t)<(1+\varepsilon)g(t).$$

Fix $\varepsilon<1$ and apply the comparison theorem (**??**) on the interval $[A,+\infty[$. If the integral $\int_A^{+\infty} f(t)\,\mathrm{d}t$ converges, then the integral $\int_A^{+\infty}(1-\varepsilon)g(t)\,\mathrm{d}t$ also converges, and hence by linearity, so does $\int_A^{+\infty} g(t)\,\mathrm{d}t$.
Conversely, if $\int_A^{+\infty} f(t)\,\mathrm{d}t$ diverges, then $\int_A^{+\infty}(1+\varepsilon)g(t)\,\mathrm{d}t$ diverges, and therefore $\int_A^{+\infty} g(t)\,\mathrm{d}t$ also diverges. ■

Example 6.6
Does the integral

$$\int_1^{+\infty}\frac{t^5+3t+1}{t^3+4}e^{-t}\,\mathrm{d}t\quad \text{converge?}$$

Since

$$\frac{t^5+3t+1}{t^3+4}e^{-t}\underset{+\infty}{\sim} t^2e^{-t},$$

and we have already shown that the integral $\int_1^{+\infty} t^2e^{-t}\,\mathrm{d}t$ converges, it follows that our integral also converges.
■

6.1.3 Cauchy's rule

Theorem 6.1.5 — Cauchy Criterion.
Let $f:[a,+\infty[\to\mathbb{R}$ be a continuous function. The improper integral $\int_a^{+\infty} f(t)\,\mathrm{d}t$ converges if and only if

$$\forall\varepsilon>0\quad \exists M\geq a\qquad \left(u,v\geq M\implies \left|\int_u^v f(t)\,\mathrm{d}t\right|<\varepsilon\right).$$

Proof. Necessary condition. We assume that the integral $\displaystyle\int_a^A f(t)\,dt$ is convergent. This means that the function F defined by $F(x)=\displaystyle\int_a^x f(t)\,dt$ has a finite limit, which we denote by L, as $x\to A$. Therefore, for all $\varepsilon>0$, there exists M_ε such that the inequalities $M_\varepsilon<x<A$ imply $|F(x)-\mathscr{L}|<\varepsilon$.

Let $\varepsilon > 0$. Hence, there exists $M_\varepsilon \in [a,A[$ such that:

$$\forall u, \forall v; M_\varepsilon < u < v < A \implies |F(u) - \mathcal{L}| < \varepsilon \text{ and } |\mathcal{F}(v) - \mathcal{L}| < \varepsilon.$$

By the triangle inequality:

$$|\mathcal{F}(v) - \mathcal{F}(u)| < 2\varepsilon \implies \left| \int_u^v f(t)\,dt \right| < 2\varepsilon.$$

Thus, we have shown:

$$\forall \varepsilon > 0, \exists M_\varepsilon \in [a,\omega[, \forall u, \forall v, M_\varepsilon < u < v < \omega \implies \left| \int_u^v f(t)\,dt \right| < 2\varepsilon.$$

Sufficient condition. We assume the condition is fulfilled. Consider a sequence (x_n) of points in $[a,A[$, such that $\lim\limits_{n\to+\infty} x_n = \omega$. Let $\varepsilon > 0$. There exists N_ε such that, for $n > N_\varepsilon$, we have $x_n > M_\varepsilon$. The inequalities $p > m > N_\varepsilon$ then imply:

$$|F(x_p) - F(x_n)| = \left| \int_{x_n}^{x_p} f(t)\,dt \right| < \varepsilon.$$

Thus, we have shown:

$$\forall \varepsilon > 0, \exists N_\varepsilon, \forall p, \forall m, N_\varepsilon < m < p \implies \left| \int_{x_n}^{x_p} f(t)\,dt \right| < \varepsilon.$$

This means that the sequence $F(x_n))$ is a Cauchy sequence. Hence, it converges in $\mathbb{R}$. According to the previous theorem, the improper integral $\int_a^A f(t)\,dt$ is convergent. ■

6.1.4 Absolute convergence and semi-convergence

Definition 6.1.2 — Absolute convergence.

Let f be a continuous function on $[a,+\infty[$. We say that $\int_a^{+\infty} f(t)\,\mathrm{d}t$ is *absolutely convergent* if $\int_a^{+\infty} |f(t)|\,\mathrm{d}t$ converges.

The following theorem is often used to establish the convergence of an integral. However, it does not provide the value of the integral.

Theorem 6.1.6

If the integral $\int_a^{+\infty} f(t)\,\mathrm{d}t$ is absolutely convergent, then it is convergent.

In other words, absolute convergence is a stronger condition than simple convergence.

Proof. This result follows from the Cauchy criterion (Theorem 6.1.5) applied first to $|f|$ and then to f.

$$\text{Since } \int_a^{+\infty} |f(t)|\,\mathrm{d}t$$

converges, the direct implication of the Cauchy criterion ensures that:

$$\forall \varepsilon > 0 \quad \exists M \geq a \quad \text{such that} \quad u, v \geq M \implies \int_u^v |f(t)|\,\mathrm{d}t < \varepsilon.$$

Moreover, since

$$\left|\int_u^v f(t)\,\mathrm{d}t\right| \leq \int_u^v |f(t)|\,\mathrm{d}t < \varepsilon,$$

the reverse implication of the Cauchy criterion implies that $\int_a^{+\infty} f(t)\,\mathrm{d}t$ converges. ■

■ Example 6.7

For instance, $\int_1^{+\infty} \frac{\sin t}{t^2}\,\mathrm{d}t$ is absolutely convergent, and therefore convergent. Indeed, for any t, $\frac{|\sin t|}{t^2} \leq \frac{1}{t^2}$. Now, the Riemann integral $\int_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t$ is convergent. Thus, by the comparison theorem, the result follows. ■

Definition 6.1.3 — Semi-convergence.

An integral

$$\int_a^{+\infty} f(t)\,\mathrm{d}t$$

is *semi-convergent* if it is convergent but not absolutely convergent.

■ Example 6.8

$$\int_1^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t \quad \text{is semi-convergent.}$$

We will prove that it is convergent but not absolutely convergent.

1. **The integral is convergent.**
   To show this, let us use integration by parts (with $u' = \sin t$ and $v = \frac{1}{t}$):

$$\int_1^x \frac{\sin t}{t}\,\mathrm{d}t = \left[\frac{-\cos t}{t}\right]_1^x - \int_1^x \frac{\cos t}{t^2}\,\mathrm{d}t.$$

   Let us analyze the two terms:
   - The term $\left[\frac{-\cos t}{t}\right]_1^x = -\frac{\cos x}{x} + \cos 1$. Since the function $\frac{\cos x}{x}$ tends to 0 as $x \to +\infty$ (because $\cos x$ is bounded and $\frac{1}{x}$ tends to 0), the term $\left[\frac{-\cos t}{t}\right]_1^x$ has a finite limit (equal to $\cos 1$).
   - For the second term, note that

$$\int_1^{+\infty} \frac{\cos t}{t^2}\,\mathrm{d}t$$

     is absolutely convergent. Indeed, $\frac{|\cos t|}{t^2} \leq \frac{1}{t^2}$, and the Riemann integral

$$\int_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t$$

     converges. Hence, $\int_1^{+\infty} \frac{\cos t}{t^2}\,\mathrm{d}t$ converges, meaning that

$$\int_1^x \frac{\cos t}{t^2}\,\mathrm{d}t$$

     also has a finite limit.

   Conclusion: $\int_1^x \frac{\sin t}{t}\,\mathrm{d}t$ has a finite limit as $x \to +\infty$. Therefore, by definition,

$$\int_1^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$$

   converges.

2. **The integral is not absolutely convergent.**
Here is a way to verify this. Since $\left|\sin t\right| \leq 1$ for all t, we have:

$$\frac{|\sin t|}{t} \geq \frac{\sin^2 t}{t} = \frac{1-\cos(2t)}{2t}.$$

Applying integration by parts to $\frac{\cos(2t)}{t}$ (with $u' = \cos(2t)$ and $v = \frac{1}{t}$), we get:

$$\int_1^x \frac{1-\cos(2t)}{2t}\,\mathrm{d}t = \frac{1}{2}\Big[\ln t\Big]_1^x - \frac{1}{4}\left[\frac{\sin(2t)}{t}\right]_1^x - \frac{1}{4}\int_1^x \frac{\sin(2t)}{t^2}\,\mathrm{d}t.$$

Now, $\int_1^{+\infty} \frac{\sin(2t)}{t^2}\,\mathrm{d}t$ converges absolutely. Among the three terms in the above sum, the last two converge, but the first term diverges to $+\infty$.
Thus, the integral diverges, and by the comparison theorem, the integral

$$\int_1^{+\infty} \frac{|\sin t|}{t}\,\mathrm{d}t$$

also diverges.

■

6.1.5 Dirichlet's rule, Abel's rule

To demonstrate that an integral converges when it is not absolutely convergent, we can use the following theorem.

Theorem 6.1.7 — Abel's Theorem. Let f be a $\mathscr{C}^1$ function on $[a,+\infty[$, positive, decreasing, and tending to zero as $x \to +\infty$. Let g be a continuous function on $[a,+\infty[$, such that the primitive $\int_a^x g(t)\,\mathrm{d}t$ is bounded. Then the integral

$$\int_a^{+\infty} f(t)\,g(t)\,\mathrm{d}t \quad \text{converges.}$$

Using $f(t) = \frac{1}{t}$ and $g(t) = \sin t$, we recover the result that the integral

$$\int_1^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$$

converges.

Proof. This theorem generalizes the preceding Example 6.8. For all $x \geq a$, define $G(x) = \int_a^x g(t)\,\mathrm{d}t$. By hypothesis, G is bounded, so there exists M such that $|G(x)| \leq M$ for all x. Now, perform integration by parts:

$$\int_a^x f(t)\,g(t)\,\mathrm{d}t = \Big[f(t)\,G(t)\Big]_a^x - \int_a^x f'(t)\,G(t)\,\mathrm{d}t.$$

Since G is bounded and $f(t) \to 0$ as $t \to +\infty$, the first term in brackets converges. Let us now show that the second term also converges by verifying that

$$\int_a^{+\infty} f'(t)\,G(t)\,\mathrm{d}t \quad \text{is absolutely convergent.}$$

We have:

$$\big|f'(t)\,G(t)\big| = \big|f'(t)\big|\,\big|G(t)\big| \leq \big(-f'(t)\big)\,M,$$

because f is decreasing (hence $f'(t) \leq 0$) and $|G|$ is bounded by M. By the comparison theorem (**??**), it suffices to show that $\int_a^{+\infty} -f'(t)\,\mathrm{d}t$ converges. Indeed:

$$\int_a^x -f'(t)\,\mathrm{d}t = f(a) - f(x), \quad \text{and} \quad \lim_{x\to+\infty}\big(f(a)-f(x)\big) = f(a).$$

Thus, the second term also converges, completing the proof. ■

■ **Example 6.9**
As an application, if α is a strictly positive real number, and k is a positive odd integer, then the integral

$$\int_1^{+\infty} \frac{\sin^k(t)}{t^\alpha}\,\mathrm{d}t \quad \text{converges.}$$

Note that this integral is absolutely convergent only for $\alpha > 1$. We verify that the hypotheses of Theorem 6.1.7 are satisfied for $f(t) = \frac{1}{t^\alpha}$ and $g(t) = \sin^k(t)$. To ensure that the primitive of $\sin^k(t)$ is bounded, recall the linearization, which transforms $\sin^k(t)$ into a linear combination of $\sin(\ell t)$, for $\ell = 1, \ldots, k$, whose primitives are always bounded. ■

6.2 Improper integrals and series

Improper integrals and infinite series are closely related through their interpretation as sums:

Improper Integrals as Continuous Sums

An improper integral $\int_a^\infty f(x)\,dx$ can be viewed as the continuous analog of an infinite series $\sum_{n=a}^\infty f(n)$, where $f(x)$ is evaluated at discrete points. For example:

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \text{is analogous to} \quad \sum_{n=1}^\infty \frac{1}{n^2}.$$

Integral Test for Convergence

Theorem 6.2.1 — Integral Test.
If $f(x)$ is positive, continuous, and decreasing for $x \geq 1$, then the series $\sum_{n=1}^\infty f(n)$ converges if and only if the improper integral $\int_1^\infty f(x)\,dx$ converges.

Proof. Consider the series $\sum_{n=1}^\infty f(n)$. By approximating the series with the integral of $f(x)$ over intervals $[n, n+1]$, we can show that:

$$\int_1^\infty f(x)\,dx \quad \text{and} \quad \sum_{n=1}^\infty f(n)$$

share the same convergence behavior. ■

■ **Example 6.10**
Test the convergence of $\sum_{n=1}^\infty \frac{1}{n^p}$ for $p > 1$.

Solution: Use the integral $\int_1^\infty \frac{1}{x^p}\,dx$. For $p > 1$, the integral evaluates to:

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1},$$

which converges. Hence, the series $\sum_{n=1}^\infty \frac{1}{n^p}$ also converges for $p > 1$. ■

Error Estimation

The remainder of a series $\sum_{n=N}^\infty f(n)$ can be approximated by the corresponding integral $\int_N^\infty f(x)\,dx$. For example:

$$R_N = \sum_{n=N}^\infty \frac{1}{n^2} \approx \int_N^\infty \frac{1}{x^2}\,dx = \frac{1}{N}.$$

6.3 Cauchy's principal value

The **Cauchy Principal Value** (denoted as p.v.) is a technique for assigning finite values to certain improper integrals that are otherwise undefined due to singularities or infinite limits. It is widely used in applications such as Fourier analysis, Hilbert transforms, and physics problems involving Green's functions or oscillatory integrals.

Definition 6.3.1

- Let $f(x)$ be a function that is not integrable on $[a,b]$ due to a singularity at $x=c$, where $a<c<b$. The **Cauchy Principal Value** is defined as:

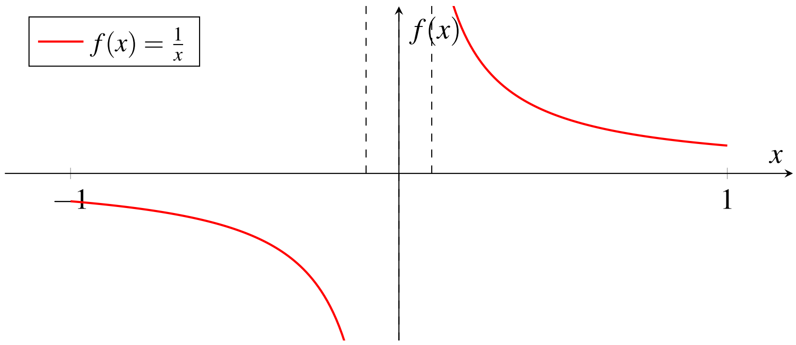
$$\text{p.v.} \int_a^b f(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0^+}\left(\int_a^{c-\varepsilon} f(x)\,\mathrm{d}x + \int_{c+\varepsilon}^b f(x)\,\mathrm{d}x\right),$$

  if the limit exists.
- For improper integrals with infinite bounds, the principal value is defined as:

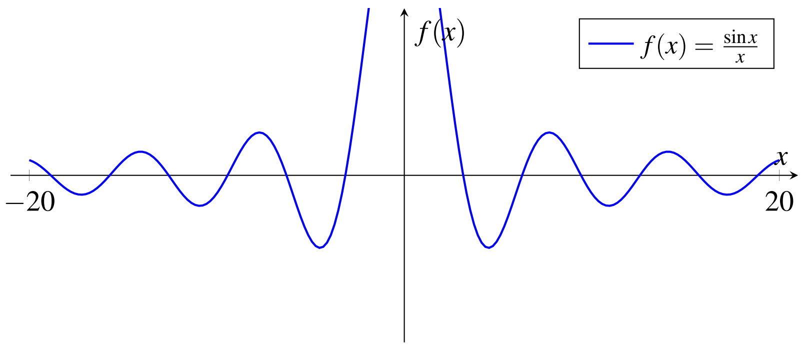
$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x,$$

  if the symmetric limit exists.

The following figures illustrate how the principal value is computed for both cases.
For the integral $\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x$, we exclude a symmetric region around $x=0$:



The principal value removes the singularity by symmetrically integrating around $x=0$.
For oscillatory integrals such as $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x$, we consider symmetric limits:



The integral oscillates and does not converge in the standard sense, but the symmetric limit exists.

1. **Symmetry**: If $f(x)$ is an odd function, then:
$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 0.$$
2. **Linearity**: For two functions $f(x)$ and $g(x)$ and constants c_1, c_2,
$$\text{p.v.} \int_a^b \big(c_1 f(x) + c_2 g(x)\big)\,\mathrm{d}x = c_1 \text{p.v.} \int_a^b f(x)\,\mathrm{d}x + c_2 \text{p.v.} \int_a^b g(x)\,\mathrm{d}x.$$
3. **Relation to Ordinary Convergence**: If the standard integral converges, then the principal value coincides with it.

Example 6.11

1. *Finite Singularity* Consider:
$$\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x.$$
The integral is undefined at $x = 0$, but the principal value is:
$$\text{p.v.} \int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x = \lim_{\varepsilon \to 0^+} \left(\int_{-1}^{-\varepsilon} \frac{1}{x}\,\mathrm{d}x + \int_{\varepsilon}^{1} \frac{1}{x}\,\mathrm{d}x \right).$$
After computation, the logarithmic divergences cancel, and we obtain:
$$\text{p.v.} \int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x = 0.$$
2. *Infinite Bounds* Consider:
$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x.$$
The standard integral diverges due to oscillations, but the principal value is:
$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

■

6.4 Mean value formulas, Second mean value theorem

The mean value theorem for the improper integrals can be easily extended from whose of proper integrals.

6.4.1 Mean Value Theorem for Improper Integrals

Theorem 6.4.1 — Mean Value Theorem for Improper Integrals. Let $f(x)$ and $g(x)$ be continuous on $[a,\infty)$, with $g(x) \geq 0$ on this interval, and suppose the improper integral $\int_a^\infty g(x)\,dx$ converges. If $f(x)$ is bounded on $[a,\infty)$, then there exists some $c \in [a,\infty)$ such that:
$$\int_a^\infty f(x)g(x)\,dx = f(c)\int_a^\infty g(x)\,dx.$$

Proof. Since $g(x) \geq 0$ and $\int_a^\infty g(x)\,dx$ converges, $g(x)$ can be interpreted as a weight function. By the continuity of $f(x)$ and $g(x)$ and the boundedness of $f(x)$ on $[a,\infty)$, consider the function:
$$F(c) = f(c)\int_a^\infty g(x)\,dx.$$
Define:
$$I = \int_a^\infty f(x)g(x)\,dx.$$
By the intermediate value property, there exists $c \in [a,\infty)$ such that $F(c) = I$. ■

■ **Example 6.12**
Let $f(x) = \frac{1}{x+1}$ and $g(x) = e^{-x}$ for $x \geq 0$. Compute:

$$\int_0^\infty \frac{1}{x+1} e^{-x}\, dx.$$

Solution: We know $\int_0^\infty e^{-x}\, dx = 1$. Thus, there exists $c \in [0, \infty)$ such that:

$$\int_0^\infty \frac{1}{x+1} e^{-x}\, dx = \frac{1}{c+1} \cdot 1.$$

Approximating numerically, $c \approx 0.567$. ■

6.4.2 Second Mean Value Theorem for Improper Integrals

Theorem 6.4.2 — Second Mean Value Theorem for Improper Integrals.
Let $f(x)$ and $g(x)$ be continuous on $[a, b)$, where b may be infinite. Assume $f(x)$ is monotonic on $[a, b)$ and $g(x) \geq 0$ on $[a, b)$. Then there exists a point $c \in [a, b)$ such that:

$$\int_a^b f(x)g(x)\, dx = f(a) \int_a^c g(x)\, dx + f(c) \int_c^b g(x)\, dx.$$

Proof. By the monotonicity of $f(x)$, let $f(x)$ be decreasing without loss of generality. Partition $[a, b)$ into $[a, c]$ and $[c, b)$ for some $c \in [a, b)$. Then:

$$\int_a^b f(x)g(x)\, dx = \int_a^c f(x)g(x)\, dx + \int_c^b f(x)g(x)\, dx.$$

Using the mean value property for definite integrals, choose c such that the equality holds with $f(c)$ appropriately weighting $\int_c^b g(x)\, dx$. The details follow from monotonicity. ■

■ **Example 6.13**
Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ for $x \geq 1$. Compute:

$$\int_1^\infty \frac{1}{x} \cdot \frac{1}{x^2}\, dx.$$

Solution: The integral converges as:

$$\int_1^\infty \frac{1}{x^3}\, dx = \left[-\frac{1}{2x^2} \right]_1^\infty = \frac{1}{2}.$$

By the second mean value theorem, $c \in [1, \infty)$ exists such that:

$$\frac{1}{c} \cdot \frac{1}{2} = \int_1^\infty \frac{1}{x^3}\, dx.$$

Numerically, $c \approx 1.5$. ■

6.5 Practical methods for calculating certain generalized integrals

6.5.1 Integration by Parts

Theorem 6.5.1
Let u and v be two functions of class $\mathscr{C}^1$ on the interval $[a, +\infty[$. Suppose that $\lim_{t \to +\infty} u(t)v(t)$ exists and is finite. Then the integrals $\displaystyle\int_a^{+\infty} u(t)\, v'(t)\, \mathrm{d}t$ and $\displaystyle\int_a^{+\infty} u'(t)\, v(t)\, \mathrm{d}t$ are of the same nature. In case of convergence, we have:

$$\int_a^{+\infty} u(t)\, v'(t)\, \mathrm{d}t = [uv]_a^{+\infty} - \int_a^{+\infty} u'(t)\, v(t)\, \mathrm{d}t$$

We recall that $\left[uv\right]_a^{+\infty} = \lim_{t\to+\infty}(uv)(t) - (uv)(a)$.

It is better not to directly apply the theorem but to prove it each time by performing integration by parts on the interval $[a,x]$ and ensuring that the terms have finite limits as $x \to +\infty$.

Proof. This follows from the usual formula for integration by parts:

$$\int_a^x u(t)\,v'(t)\,\mathrm{d}t = \left[uv\right]_a^x - \int_a^x u'(t)\,v(t)\,\mathrm{d}t$$

noting that by hypothesis, the bracket term has a finite limit as $x \to +\infty$. ■

Example 6.14

Let $\lambda > 0$. What is the expectation of the exponential distribution:

$$\int_0^{+\infty} \lambda t e^{-\lambda t}\,\mathrm{d}t \quad ?$$

We perform integration by parts with $u = \lambda t$, $v' = e^{-\lambda t}$. Thus, $u' = \lambda$ and $v = \frac{-1}{\lambda}e^{-\lambda t}$. Hence:

$$\begin{aligned}
\int_0^x \lambda t e^{-\lambda t}\,\mathrm{d}t &= \int_0^x u(t)\,v'(t)\,\mathrm{d}t \\
&= \left[uv\right]_0^x - \int_0^x u'(t)\,v(t)\,\mathrm{d}t \\
&= \left[\lambda t \cdot \frac{-1}{\lambda}e^{-\lambda t}\right]_0^x - \int_0^x \lambda \cdot \frac{-1}{\lambda}e^{-\lambda t}\,\mathrm{d}t \\
&= -xe^{-\lambda x} + \int_0^x e^{-\lambda t}\,\mathrm{d}t \\
&= -xe^{-\lambda x} + \left[\frac{-1}{\lambda}e^{-\lambda t}\right]_0^x \\
&= -xe^{-\lambda x} - \frac{1}{\lambda}\left(e^{-\lambda x} - 1\right) \\
&\longrightarrow \frac{1}{\lambda} \qquad \text{as } x \to +\infty
\end{aligned}$$

Thus, the integral converges and

$$\int_0^{+\infty} \lambda t e^{-\lambda t}\,\mathrm{d}t = \frac{1}{\lambda}.$$

■

6.5.2 Change of Variables

Theorem 6.5.2

Let f be a function defined on the interval $I = [a,+\infty[$. Let $J = [\alpha,\beta[$ be an interval with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ or $\beta = +\infty$. Let $\varphi : J \to I$ be a diffeomorphism of class $\mathscr{C}^1$. The integrals $\int_a^{+\infty} f(x)\,\mathrm{d}x$ and $\int_\alpha^\beta f\big(\varphi(t)\big)\cdot\varphi'(t)\,\mathrm{d}t$ are of the same nature. In case of convergence, we have:

$$\int_a^{+\infty} f(x)\,\mathrm{d}x = \int_\alpha^\beta f\big(\varphi(t)\big)\cdot\varphi'(t)\,\mathrm{d}t$$

The proof is identical to the usual change of variables technique. Again, it is better not to directly apply the theorem but to perform a standard change of variable on the interval $[a,x]$ and analyze the limits as $x \to +\infty$.

We recall that $\varphi : J \to I$ is a *diffeomorphism of class* $\mathscr{C}^1$ if φ is a $\mathscr{C}^1$ bijection whose inverse is also $\mathscr{C}^1$.

The following example is particularly interesting: the function $f(t) = \sin(t^2)$ has a convergent integral but does not tend to 0 as $t \to +\infty$. This contrasts with the case of series, where the general term of a convergent series always tends to 0.

■ **Example 6.15**
The Fresnel integral:

$$\int_1^{+\infty} \sin(t^2)\,\mathrm{d}t \quad \text{converges.}$$

We perform the change of variable $u = t^2$, which gives $t = \sqrt{u}$, $\mathrm{d}t = \frac{\mathrm{d}u}{2\sqrt{u}}$. $\varphi : u \mapsto t = \sqrt{u}$ is a diffeomorphism between $u \in [1, x^2]$ and $t \in [1, x]$. Hence:

$$\int_1^x \sin(t^2)\,\mathrm{d}t = \int_1^{x^2} \sin(u)\,\frac{\mathrm{d}u}{2\sqrt{u}}.$$

By Abel's theorem, $\int_1^{+\infty} \frac{\sin u}{\sqrt{u}}\,\mathrm{d}u$ converges, which implies that $\int_1^{x^2} \sin(u)\,\frac{\mathrm{d}u}{2\sqrt{u}}$ has a finite limit as $x \to +\infty$, proving that $\int_1^x \sin(t^2)\,\mathrm{d}t$ also has a finite limit. Conclusion: $\int_1^{+\infty} \sin(t^2)\,\mathrm{d}t$ converges. ■

■ **Example 6.16**
Let us calculate the value of the two integrals:

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin t)\,\mathrm{d}t \qquad J = \int_0^{\frac{\pi}{2}} \ln(\cos t)\,\mathrm{d}t.$$

1. **The integral I converges.**
   The uncertain point is at $t = 0$. Since $\sin t \underset{0^+}{\sim} t$, $\ln t \leq \frac{1}{\sqrt{t}}$ (for small t), and the integral $\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{t}}\,\mathrm{d}t$ converges, then $\int_0^{\frac{\pi}{2}} \ln t\,\mathrm{d}t$ converges, implying that I converges.
2. **Verification that $I = J$.**
   Perform the change of variable $t = \frac{\pi}{2} - u$. We have $\mathrm{d}t = -\mathrm{d}u$ and a diffeomorphism between $t \in [x, \frac{\pi}{2}]$ and $u \in [\frac{\pi}{2} - x, 0]$. Hence:

$$\int_x^{\frac{\pi}{2}} \ln(\sin t)\,\mathrm{d}t = \int_{\frac{\pi}{2}-x}^{0} \ln\Big(\sin\Big(\frac{\pi}{2} - u\Big)\Big)\,\big(-\mathrm{d}u\big) = \int_0^{\frac{\pi}{2}-x} \ln(\cos u)\,\mathrm{d}u.$$

   Thus, as $x \to 0$, this proves $I = J$ (and in particular, J converges).
3. **Calculation of $I + J$.**

$$\begin{aligned}
I + J &= \int_0^{\frac{\pi}{2}} \ln(\sin t)\,\mathrm{d}t + \int_0^{\frac{\pi}{2}} \ln(\cos t)\,\mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \big(\ln(\sin t) + \ln(\cos t)\big)\,\mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \ln(\sin t \cdot \cos t)\,\mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \ln\big(\tfrac{1}{2}\sin(2t)\big)\,\mathrm{d}t \\
&= -\frac{\pi}{2}\ln 2 + \int_0^{\frac{\pi}{2}} \ln\big(\sin(2t)\big)\,\mathrm{d}t
\end{aligned}$$

   Since $I = J$, we have:

$$2I = -\frac{\pi}{2}\ln 2 + K.$$

It remains to evaluate $K = \int_0^{\frac{\pi}{2}} \ln\big(\sin(2t)\big)\,\mathrm{d}t$:

$$
\begin{aligned}
K &= \int_0^{\frac{\pi}{2}} \ln\big(\sin(2t)\big)\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{\pi} \ln(\sin u)\,\mathrm{d}u \qquad \text{(change of variable } u = 2t\text{)} \\
&= \frac{1}{2}I + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \ln(\sin u)\,\mathrm{d}u \\
&= \frac{1}{2}I + \frac{1}{2}\int_{\frac{\pi}{2}}^{0} \ln\big(\sin(\pi - v)\big)\,(-\mathrm{d}v) \qquad \text{(change of variable } v = \pi - u\text{)} \\
&= \frac{1}{2}I + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin v)\,\mathrm{d}v \\
&= \frac{1}{2}I + \frac{1}{2}I \\
&= I
\end{aligned}
$$

4. **Conclusion.**
Thus, since $2I = -\frac{\pi}{2}\ln 2 + K$ and $K = I$, we find:

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin t)\,\mathrm{d}t = -\frac{\pi}{2}\ln 2$$

and $J = I$.

■

6.6 Exercises

Exercise 6.1
Are the following improper integrals convergent?

1. $\int_0^1 \ln t\,dt$ **2.** $\int_0^{+\infty} e^{-t^2}\,dt$

3. $\int_0^{+\infty} x(\sin x)e^{-x}\,dx$ **4.** $\int_0^{+\infty} (\ln t)e^{-t}\,dt$

5. $\int_0^1 \frac{dt}{(1-t)\sqrt{t}}$

■

Exercise 6.2
Are the following improper integrals convergent?

1. $\int_0^{+\infty} \frac{dt}{e^t - 1}$ **2.** $\int_0^{+\infty} \frac{te^{-\sqrt{t}}}{1+t^2}\,dt$

3. $\int_0^1 \cos^2\left(\frac{1}{t}\right)dt$

■

Exercise 6.3

Are the following improper integrals convergent?

$$\textbf{1.}\ \int_0^{+\infty} \frac{\ln t}{t^2+1}dt \qquad \textbf{2.}\ \int_1^{+\infty} \frac{\sqrt{\ln x}}{(x-1)\sqrt{x}}dx$$
$$\textbf{3.}\ \int_1^{+\infty} e^{-\sqrt{\ln t}}dt$$

■

Exercise 6.4

1. Show that the improper integrals $\int_1^{+\infty} \frac{\sin t}{t}dt$ and $\int_1^{+\infty} \frac{\cos t}{t}dt$ are convergent.
   We want to prove that the function $\frac{\sin t}{t}$ is not integrable, meaning that $\int_1^{+\infty} \left|\frac{\sin t}{t}\right| dt$ diverges.
2. Method 1. Prove that, for all $t \in \mathbb{R}$, $|\sin t| \geq \frac{1-\cos 2t}{2}$. Deduce the result from this.
3. Method 2. Prove that, for all $k \in \mathbb{N}$,
$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin t|}{t}dt \geq \frac{1}{(k+1)\pi} \int_0^{\pi} |\sin t| dt.$$
   Then find the result again.

■

Exercise 6.5

1. Show that the integral $\int_0^{+\infty} \frac{\ln t}{1+t^2}dt$ converges.
2. Then, using the change of variables $u = \frac{1}{t}$, show that $\int_0^{+\infty} \frac{\ln t}{1+t^2}dt = 0$.
3. Let $a > 0$. Calculate the integral $\int_0^{+\infty} \frac{\ln t}{a^2+t^2}dt$.

■

Exercise 6.6
he goal of this exercise is to prove the following relation:

$$\int_0^1 \frac{\ln t}{t^2-1}dt = \lim_{n\to+\infty} \sum_{k=0}^{n} \frac{1}{(2k+1)^2}.$$

1. Prove the convergence of the integral.
2. Show that, for all integers $k \geq 0$, the integral $I_k = \int_0^1 t^k \ln t dt$ converges, and then calculate I_k.
3. Show that, for all integers $n \geq 1$,
$$\sum_{k=0}^{n} \frac{1}{(2k+1)^2} = \int_0^1 \frac{\ln t}{t^2-1}dt - \int_0^1 \frac{t^{2n+2}\ln t}{t^2-1}dt.$$
4. Prove that the function $t \mapsto \frac{t^2 \ln t}{t^2-1}$ can be extended by continuity at 0 and 1. Deduce that there exists a constant $M > 0$, which we will not try to calculate, such that, for all $t \in]0,1[$,
$$\left|\frac{t^2 \ln t}{t^2-1}\right| \leq M.$$
5. Deduce that $\lim_{n\to+\infty} \int_0^1 \frac{t^{2n+2}\ln t}{t^2-1}dt = 0$, and then the desired relation.

■

Exercise 6.7

The goal of this exercise is to compute the value of $I = \int_0^{+\infty} \frac{\sin t}{t} dt$. For each integer n, we define

$$I_n = \int_0^{\pi/2} \frac{\sin\big((2n+1)t\big)}{\sin t} dt \text{ and } J_n = \int_0^{\pi/2} \frac{\sin\big((2n+1)t\big)}{t} dt.$$

1. Justify that, for all $n \geq 0$, I_n and J_n are well-defined.
2. Show that, for all $n \geq 1$, $I_n - I_{n-1} = 0$. Deduce the value of I_n.
3. Let $\phi : [0, \pi/2] \to \mathbb{R}$ be a continuously differentiable function. Show, using integration by parts, that $\int_0^{\pi/2} \phi(t) \sin\big((2n+1)t\big) dt$ tends to 0 as n tends to infinity.
4. Prove that the function $t \mapsto \frac{1}{t} - \frac{1}{\sin t}$ can be extended to a continuously differentiable function on $[0, \pi/2]$.
5. Deduce that $J_n - I_n$ tends to 0 as n tends to infinity.
6. Prove, using a change of variables, that J_n tends to I as n tends to infinity.
7. Deduce the value of I.

■

Exercise 6.8

1. Compute p.v. $\int_{-2}^{2} \frac{x}{x^2-1} \, dx$.
2. Show that p.v. $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} \, dx = 0$.
3. Prove that p.v. $\int_{-\infty}^{\infty} \frac{\sin(ax)}{x} \, dx = \pi$ for $a > 0$.

■

7. Functions defined by integrals

Functions defined by integrals are fundamental in mathematical analysis and have wide applications in physics, engineering, and probability. This chapter explores the continuity, differentiability, and properties of such functions. We also study uniform convergence, generalized integrals, and special functions like Euler's Gamma and Beta functions.

7.1 Function Defined by an Integral

Let $f:(x,t)\mapsto f(x,t)$ be a function of two variables, x and t. We consider x as a parameter and $t\in[a,b]$ as a variable of integration. This allows us to define

$$F(x)=\int_a^b f(x,t)\,\mathrm{d}t\ .$$

For a fixed x, for $F(x)$ to exist, it is sufficient that the partial function $t\mapsto f(x,t)$ is continuous on $[a,b]$. However, this does not guarantee the continuity of the function F. We provide sufficient conditions for F to be continuous and then differentiable.

7.1.1 Continuity

Theorem 7.1.1
Let I be an interval of $\mathbb{R}$ and $J=[a,b]$ a closed bounded interval. Let f be a continuous function on $I\times J$ with values in $\mathbb{R}$ (or $\mathbb{C}$). Then the function F defined for all $x\in I$ by

$$F(x)=\int_a^b f(x,t)\,\mathrm{d}t$$

is continuous on I.

■ **Example 7.1**
Let

$$F(x)=\int_0^\pi \sin(x+t)\cdot e^{xt^2}\,\mathrm{d}t,$$

defined for $x\in I=\mathbb{R}$. The function $(x,t)\mapsto f(x,t)=\sin(x+t)\cdot e^{xt^2}$ is continuous on $\mathbb{R}\times[0,\pi]$, so the function $x\mapsto F(x)$ is continuous on $\mathbb{R}$.

We calculate that $F(0)=\int_0^\pi \sin(t)\cdot 1\,\mathrm{d}t = \Big[-\cos(t)\Big]_0^\pi = 2$. Even though we do not have a general formula for $F(x)$, we deduce from continuity that $F(x)\to F(0)=2$ when $x\to 0$. ■

The proofs in this section use uniform continuity, which is the subject of the next section.

Proof. Let x_0 be a point in I. By restricting the interval to $I\cap[x_0-\alpha, x_0+\alpha]$, we assume that I is a closed bounded interval. Heine's theorem (theorem **??**) then applies to the function f on $I\times J$: it is therefore uniformly continuous. In particular, for any $\varepsilon>0$, there exists $\delta>0$ such that, *for all* $t\in J$,

$$|x-x_0|<\delta \implies \big|f(x,t)-f(x_0,t)\big| \le \frac{\varepsilon}{b-a}\,.$$

In this case,

$$\begin{aligned}\big|F(x)-F(x_0)\big| &= \left|\int_a^b \big(f(x,t)-f(x_0,t)\big)\,\mathrm{d}t\right| \\ &\le \int_a^b \big|f(x,t)-f(x_0,t)\big|\,\mathrm{d}t \\ &\le (b-a)\frac{\varepsilon}{b-a} = \varepsilon\,.\end{aligned}$$

Thus, F is continuous at x_0. ■

7.1.2 Differentiability

Theorem 7.1.2

Let I be an interval of $\mathbb{R}$ and $J=[a,b]$ a closed bounded interval. Assume that:

- $(x,t)\mapsto f(x,t)$ is a continuous function on $I\times J$ (with values in $\mathbb{R}$ or $\mathbb{C}$),
- the partial derivative $(x,t)\mapsto \frac{\partial f}{\partial x}(x,t)$ exists and is continuous on $I\times J$.

Then the function F defined for all $x\in I$ by $F(x)=\displaystyle\int_a^b f(x,t)\,\mathrm{d}t$ is of class $\mathscr{C}^1$ on I and:

$$F'(x)=\int_a^b \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t\,.$$

A mnemonic abbreviation for the interchange of derivative and integral is:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b = \int_a^b \frac{\partial}{\partial x}$$

■ Example 7.2

Let us study $F(x)=\int_0^1 \frac{\mathrm{d}t}{x^2+t^2}$ for $x\in]0,+\infty[$. Let $f(x,t)=\frac{1}{x^2+t^2}$. Then:

- f is continuous on $]0,+\infty[\times[0,1]$,
- $\frac{\partial f}{\partial x}(x,t)=\frac{-2x}{(x^2+t^2)^2}$ is continuous on $]0,+\infty[\times[0,1]$.

Therefore, we have

$$F'(x)=\int_0^1 \frac{-2x}{(x^2+t^2)^2}\mathrm{d}t.$$

For this example, we can explicitly compute $F(x)$:

- $F(x)=\dfrac{1}{x^2}\displaystyle\int_0^1 \frac{\mathrm{d}t}{1+\left(\frac{t}{x}\right)^2} = \frac{1}{x}\left[\arctan\frac{t}{x}\right]_{t=0}^{t=1} = \frac{1}{x}\arctan\frac{1}{x}.$
- $F'(x)=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{x}\arctan\dfrac{1}{x}\right) = -\dfrac{1}{x^2}\arctan\dfrac{1}{x}-\dfrac{1}{x^3}\dfrac{1}{1+x^{-2}}.$
- This proves $\displaystyle\int_0^1 \frac{-2x}{(x^2+t^2)^2}\mathrm{d}t = -\frac{1}{x^2}\arctan\frac{1}{x}-\frac{1}{x(1+x^2)}.$

■

Proof. Let $x_0 \in I$. To simplify the notation, we assume that x_0 is not an endpoint of I. We need to show that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in [x_0 - \delta, x_0 + \delta]$ and $x \neq x_0$:

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - \int_a^b \frac{\partial f}{\partial x}(x_0, t)\, \mathrm{d}t \right| \leq \varepsilon \,.$$

Let us write:

$$\begin{aligned} \left| F(x) - F(x_0) - (x - x_0) \int_a^b \frac{\partial f}{\partial x}(x_0, t)\, \mathrm{d}t \right| &= \left| \int_a^b \left(f(x,t) - f(x_0,t) - (x - x_0) \frac{\partial f}{\partial x}(x_0, t) \right) \mathrm{d}t \right| \\ &\leq \int_a^b \left| f(x,t) - f(x_0,t) - (x - x_0) \frac{\partial f}{\partial x}(x_0, t) \right| \mathrm{d}t \,. \end{aligned}$$

By the mean value theorem, for any $t \in [a,b]$, there exists x_1 strictly between x_0 and x such that

$$f(x,t) - f(x_0,t) = (x - x_0) \frac{\partial f}{\partial x}(x_1, t) \,.$$

Fix $\alpha > 0$ such that $[x_0 - \alpha, x_0 + \alpha]$ is contained in I: the partial derivative $\partial f / \partial x$ is uniformly continuous on $[x_0 - \alpha, x_0 + \alpha] \times [a,b]$, by Heine's theorem (theorem **??**). Therefore, there exists $\delta > 0$ such that, for all x satisfying $|x - x_0| < \delta$ *and for all* $t \in [a,b]$,

$$\left| \frac{\partial f}{\partial x}(x,t) - \frac{\partial f}{\partial x}(x_0,t) \right| < \frac{\varepsilon}{b-a} \,.$$

If $|x - x_0| < \delta$, then any x_1 strictly between x_0 and x also satisfies $|x_1 - x_0| < \delta$, so:

$$\left| \frac{\partial f}{\partial x}(x_1,t) - \frac{\partial f}{\partial x}(x_0,t) \right| < \frac{\varepsilon}{b-a} \,.$$

Substituting into the above expression, we obtain:

$$\begin{aligned} \left| f(x,t) - f(x_0,t) - (x - x_0) \frac{\partial f}{\partial x}(x_0, t) \right| &= \left| (x - x_0) \frac{\partial f}{\partial x}(x_1, t) - (x - x_0) \frac{\partial f}{\partial x}(x_0, t) \right| \\ &\leq |x - x_0| \frac{\varepsilon}{b-a} \,. \end{aligned}$$

It remains to integrate with respect to t between a and b:

$$\left| F(x) - F(x_0) - (x - x_0) \int_a^b \frac{\partial f}{\partial x}(x_0, t)\, \mathrm{d}t \right| \leq \int_a^b |x - x_0| \frac{\varepsilon}{b-a}\, \mathrm{d}t = |x - x_0| \varepsilon \,,$$

from which the result follows by dividing by $|x - x_0|$. ■

■ **Example 7.3**
Let us compute the *Gaussian integral*:

$$\int_0^{+\infty} e^{-t^2}\, \mathrm{d}t = \frac{\sqrt{\pi}}{2}$$

Let us define, for $x \in I = [0, +\infty[$:

$$F(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}\, \mathrm{d}t \qquad G(x) = \left(\int_0^x e^{-t^2}\, \mathrm{d}t \right)^2 \qquad H(x) = F(x) + G(x)$$

1. **Study of $F(x)$.**
   Let $f(x,t) = \frac{e^{-x^2(t^2+1)}}{t^2+1}$. Note that:

- f is a continuous function on $[0,+\infty[\times[0,1]$,
- $\frac{\partial f}{\partial x}(x,t) = -2xe^{-x^2(t^2+1)}$ is also continuous.

Therefore, by Theorem 7.1.2, F is continuous, differentiable, and

$$F'(x) = \int_0^1 \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t = -2\int_0^1 xe^{-x^2(t^2+1)}\,\mathrm{d}t = -2xe^{-x^2}\int_0^1 e^{-x^2t^2}\,\mathrm{d}t.$$

2. **Study of $G(x)$.**
G is not strictly an integral depending on a parameter. If we let $G_0(x) = \int_0^x e^{-t^2}\,\mathrm{d}t$, then G_0 is simply a primitive of $x \mapsto e^{-x^2}$ and $G(x) = G_0(x)^2$. Since $G_0'(x) = e^{-x^2}$ (the derivative of a primitive is the function itself), we have:

$$G'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(G_0(x)^2\right) = 2G_0'(x)G_0(x) = 2e^{-x^2}\int_0^x e^{-t^2}\,\mathrm{d}t = 2xe^{-x^2}\int_0^1 e^{-x^2u^2}\,\mathrm{d}u$$

For the last equality, we used the change of variable $t = xu$ (so $\mathrm{d}t = x\mathrm{d}u$, $u = \frac{t}{x}$, and u varies from 0 to 1 as t varies from 0 to x).

3. **Study of $H(x)$.**
From our previous calculations, we find $H'(x) = F'(x) + G'(x) = 0$ for all $x \in [0,+\infty[$. This means that the function H is constant. Now,

$$H(0) = F(0) + G(0) = \int_0^1 \frac{1}{t^2+1}\,\mathrm{d}t \;+\; 0 = \Big[\arctan t\Big]_0^1 = \frac{\pi}{4}.$$

Therefore, H is the constant function equal to $\frac{\pi}{4}$.

4. **Limit of $H(x)$ as $x \to +\infty$.**
   - As $x \to +\infty$, $G(x) \to \left(\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t\right)^2$.
   - And $F(x) \to 0$ because

$$|F(x)| = \left|\int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}\,\mathrm{d}t\right| \le \int_0^1 e^{-x^2}\,\mathrm{d}t = e^{-x^2}\int_0^1 1\cdot\,\mathrm{d}t = e^{-x^2} \to 0.$$

   - Thus, $H(x) = F(x) + G(x) \to \left(\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t\right)^2$.

5. **Conclusion.**
H is a constant function: $H(x) = \frac{\pi}{4}$, so its limit as $x \to +\infty$ is also $\frac{\pi}{4}$. However, we have computed this limit in another way, which proves:

$$\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

■

7.1.3 Fubini's Theorem

Theorem 7.1.3 — Fubini's Theorem.
Let $I = [\alpha,\beta]$ and $J = [a,b]$ be two closed bounded intervals. Let f be a continuous function on $I \times J$, with values in $\mathbb{R}$ (or $\mathbb{C}$). Then the function F defined for all $x \in I$ by

$$F(x) = \int_a^b f(x,t)\,\mathrm{d}t$$

is integrable on I, and

$$\int_\alpha^\beta F(x)\,\mathrm{d}x = \int_\alpha^\beta\left(\int_a^b f(x,t)\,\mathrm{d}t\right)\mathrm{d}x = \int_a^b\left(\int_\alpha^\beta f(x,t)\,\mathrm{d}x\right)\mathrm{d}t\,.$$

In short, the order of integration can be interchanged:

$$\int_\alpha^\beta\int_a^b = \int_a^b\int_\alpha^\beta$$

■ Example 7.4
Let's compute:

$$I=\int_0^{\pi}\left(\int_0^1 (t\sin x+2x)\,\mathrm{d}t\right)\mathrm{d}x$$

First method. We first integrate with respect to t, then with respect to x:

$$I=\int_{x=0}^{x=\pi}\left(\int_{t=0}^{t=1}(t\sin x+2x)\,\mathrm{d}t\right)\mathrm{d}x=\int_{x=0}^{x=\pi}\left[\frac{t^2}{2}\sin x+2xt\right]_{t=0}^{t=1}\mathrm{d}x$$
$$=\int_{x=0}^{x=\pi}\left(\frac{\sin x}{2}+2x\right)\mathrm{d}x=\left[\frac{-\cos x}{2}+x^2\right]_{x=0}^{x=\pi}=\pi^2+1$$

Second method. We use Fubini's theorem, which states that we can first integrate with respect to x, then with respect to t:

$$I=\int_{x=0}^{x=\pi}\left(\int_{t=0}^{t=1}(t\sin x+2x)\,\mathrm{d}t\right)\mathrm{d}x=\int_{t=0}^{t=1}\left(\int_{x=0}^{x=\pi}(t\sin x+2x)\,\mathrm{d}x\right)\mathrm{d}t \qquad \text{by Fubini}$$
$$=\int_{t=0}^{t=1}\left[-t\cos x+x^2\right]_{x=0}^{x=\pi}\mathrm{d}t=\int_{t=0}^{t=1}(2t+\pi^2)\,\mathrm{d}t=\left[t^2+\pi^2 t\right]_{t=0}^{t=1}=\pi^2+1$$

■

7.2 Improper Integrals Depending on a Parameter

7.2.1 Uniform Convergence of an Improper Integral With Respect to a Parameter

Basic Definition and Examples

Suppose that the improper integral

$$F(y)=\int_a^{\omega} f(x,y)\,dx$$

over the interval $[a,\omega]$ converges for each value $y\in Y$. For definiteness, we shall assume that the integral Eq. 4 has only one singularity and that it involves the upper limit of the integration (that is, either $\omega=+\infty$ or the function f is unbounded as a function of x in a neighborhood of ω).

Definition 7.2.1
We say that the improper integral Eq. 4 depending on the parameter $y\in Y$ converges uniformly on the set $E\subset Y$ if for every $\varepsilon>0$ there exists a neighborhood $U_{[a,\omega)}(\omega)$ of ω in the set $[a,\omega)$ such that the estimate

$$\left|\int_b^{\omega} f(x,y)\,dx\right|<\varepsilon$$

for the remainder of the integral Eq. 4 holds for every $b\in U_{[a,\omega)}(\omega)$ and every $y\in E$.

If we introduce the notation

$$F_b(y)=\int_a^{b} f(x,y)\,dx$$

for a proper integral approximating the improper integral of Eq. 4, the basic definition of this section can be restated in a different form equivalent to the previous one: uniform convergence of the integral of Eq. 4 on the set $E\subset Y$ by definition means that

$$F_b(y)\rightrightarrows F(y) \text{ on } E \text{ as } b\to\omega, b\in[a,\omega).$$

■ Example 7.5
The integral

$$\int_1^{+\infty} \frac{dx}{x^2+y^2}$$

converges uniformly on the entire set $\mathbb{R}$ of values of the parameter $y \in \mathbb{R}$. ■

■ Example 7.6
The integral

$$\int_0^{+\infty} e^{-xy}\,dx$$

converges only when $y > 0$. Moreover, it converges uniformly on every set $\{y \in \mathbb{R} \mid y \geq y_0 \geq 0\}$. ■

■ Example 7.7
Let us show that each of the integrals

$$\Phi(x) = \int_0^{+\infty} x^{\alpha} y^{\alpha+\beta+1} e^{-(1+x)y}\,dy$$

and

$$F(y) = \int_0^{+\infty} x^{\alpha} y^{\alpha+\beta+1} e^{-(1+x)y}\,dx$$

in which α, β are fixed positive numbers, converges uniformly on the set of non-negative values of the parameter. ■

The Cauchy Criterion for Uniform Convergence of an Integral

Proposition 7.2.1 — Cauchy Criterion.
A necessary and sufficient condition for the improper integral of Eq. 4 depending on parameter $y \in Y$ to converge uniformly on a set $E \subset Y$ is that for every $\varepsilon > 0$ there exists a neighborhood $U_{[a,\omega)}$ of the point ω such that

$$\left| \int_{b_1}^{b_2} f(x,y)\,dx \right| < \varepsilon$$

for every $b_1, b_2 \in U_{[a,\omega)}$ and every $y \in E$.

Corollary 7.2.2
If the function f in the integral of Eq. 4 is continuous on the set $[a,\omega) \times [c,d]$ and the integral of Eq. 4 converges for every $y \in]c,d[$ but diverges for $y = c$ or $y = d$, then it converges non-uniformly on the interval $]c,d[$ and also on any set $E \subset]c,d[$ whose closure contains the point of divergence.

■ Example 7.8
The integral

$$\int_0^{+\infty} e^{-tx^2}\,dx$$

converges for $t > 0$ and diverges at $t = 0$, hence it demonstrably converges non-uniformly on every set of positive numbers having 0 as a limit point. ■

Sufficient Conditions for Uniform Convergence of an Improper Integral Depending on a Parameter

Proposition 7.2.3 — Weierstrass Test.
Suppose the functions $f(x,y)$ and $g(x,y)$ are integrable with respect to x on every closed interval $[a,b] \subset [a,\omega)$

for each value of $y \in Y$. If the inequality $|f(x,y)| \le g(x,y)$ holds for each value of $y \in Y$ and every $x \in [a,\omega)$ and the integral

$$\int_a^{\omega} g(x,y)\,dx$$

converges uniformly on Y, then the integral

$$\int_0^{\omega} f(x,y)\,dx$$

converges absolutely for each $y \in Y$ and uniformly on Y.

The most frequently encountered case of Proposition 2 occurs when the function g is independent of the parameter y. It is this case in which Proposition 2 is usually called the Weierstrass M-test for uniform convergence of an integral.

■ Example 7.9
The integral

$$\int_0^{\infty} \frac{\cos(\alpha x)}{1+x^2}\,dx$$

converges uniformly on the whole set $\mathbb{R}$ of the parameter α, since

$$\left|\frac{\cos(\alpha x)}{1+x^2}\right| \le \frac{1}{1+x^2},$$

and the integral

$$\int_0^{\infty} \frac{dx}{1+x^2}$$

converges. ■

Proposition 7.2.4 — Abel-Dirichlet Test.
Assume that the function $f(x,y)$ and $g(x,y)$ are integrable with respect to x at each $y \in Y$ on every closed interval $[a,b] \subset [a,\omega)$. A sufficient condition for uniform convergence of the integral

$$\int_a^{\omega} (f \cdot g)\,dx$$

on the set Y is that one of the following two pairs of conditions holds:

1. Either there exists a constant $M \in \mathbb{R}$ such that

$$\left|\int_a^{b} f(x,y)\,dx\right| < M$$

for any $b \in [a,\omega)$ and any $y \in Y$, and for each $y \in Y$ the function $g(x,y)$ is monotonic with respect to x on the interval $[a,\omega)$ and $g(x,y) \rightrightarrows 0$ on Y as $x \to \omega$, $x \in [a,\omega)$, or
2. The integral

$$\int_a^{\omega} f(x,y)\,dx$$

converges uniformly on the set Y, and for each $y \in Y$ the function $g(x,y)$ is monotonic with respect to x on the interval $[a,\omega)$ and there exists a constant $M \in \mathbb{R}$ such that

$$|g(x,y)| < M$$

for every $x \in [a,\omega)$ and every $y \in Y$.

Applying the second mean-value theorem for the integral, we have

$$\int_{b_1}^{b_2}(f\cdot g)(x,y)\,dx = g(b_1,y)\int_{b_1}^{\xi} f(x,y)\,dx + g(b_2,y)\int_{\xi}^{b_2} f(x,y)\,dx.$$

■ **Example 7.10**
The integral

$$\int_0^{\infty} \frac{\sin x}{x} e^{-xy}\,dx$$

converges uniformly on the set $\{y \in \mathbb{R} \mid y \geq 0\}$. ■

■ **Example 7.11**
The integral

$$\int_0^{\infty} \frac{\sin(xy)}{x}\,dx$$

converges uniformly on the set $\{y \in \mathbb{R} \mid y \geq y_0 > 0\}$ and not uniformly convergent on the set $\{y \in \mathbb{R} \mid y > 0\}$.
■

■ **Example 7.12**
The integral

$$\int_0^{+\infty} \frac{\cos(x^2)}{x^p}\,dx$$

converges uniformly on each $p \in [\alpha, \beta] \subset (-1, 1)$. ■

7.3 Limiting Passage under the Sign of an Improper Integral and Continuity of an Improper Integral Depending on a Parameter

Proposition 7.3.1
Let $f(x,y)$ be a family of functions depending on a parameter $y \in Y$ that are integrable, possibly in the improper sense, on the interval $a \leq x \leq \omega$, and let $\mathscr{B}_Y$ be a base in Y. If

1. for every $b \in [a, \omega)$,

$$f(x,y) \rightrightarrows \varphi(x) \text{ on } [a,b] \text{ over the base } \mathscr{B}_Y,$$

2. the integral $\int_a^{\omega} f(x,y)\,dx$ converges uniformly on Y,

then the limit function φ is improperly integrable on $[a, \omega)$ and the following equality holds:

$$\lim_{\mathscr{B}_Y} \int_a^{\omega} f(x,y)\,dx = \int_a^{\omega} \varphi\,dx.$$

Proposition 7.3.2
If

1. the function $f(x,y)$ is continuous on the set

$$\{(x,y) \in \mathbb{R}^2 \mid a \leq x < \omega, c \leq y \leq d\},$$

2. the integral $F(y) = \int_a^{\omega} f(x,y)\,dx$ converges uniformly on $[c,d]$,

then the function $F(y)$ is continuous on $[c,d]$.

Proposition 7.3.3
Suppose $f(x,y)$ is continuous on $[a,+\infty) \times [c,d]$, and the integral $\int_a^{\infty} f(x,y)\,dx$ converges uniformly on $[c,d]$, then we have

$$\int_c^d dy \int_a^{+\infty} f(x,y)\,dx = \int_a^{+\infty} dx \int_c^d f(x,y)\,dy.$$

Proposition 7.3.4
Suppose $f(x,y)$, $f_y(x,y)$ are continuous on $[a,+\infty)\times[c,d]$, for each $y\in[c,d]$ the integral $\int_a^{+\infty} f(x,y)\,dx$ converges. Furthermore, the integral $\int_a^{+\infty} f_y(x,y)\,dx$ is uniformly convergent. Then we have

$$\frac{d}{dy}\int_a^{+\infty} f(x,y)\,dx = \int_a^{+\infty} \frac{\partial}{\partial y} f(x,y)\,dx.$$

7.4 The Eulerian Integrals

In this section and the next, we shall illustrate the application of the theory developed above to some specific integrals of importance in analysis that depend on a parameter.

Following Legendre, we define the Eulerian integrals of first and second kinds respectively as the two special functions that follow:

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx,$$

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}\,dx.$$

The first of these is called the beta function, and the second the gamma function.

7.4.1 The Beta Function

Domain of Definition

A necessary and sufficient condition for the convergence of the integral of the beta function at the lower limit is that $\alpha > 0$. Similarly, convergence at 1 occurs if and only if $\beta > 0$. Thus, the beta function is defined when both of the following conditions hold simultaneously:

$$\alpha > 0 \text{ and } \beta > 0.$$

Symmetry

We can verify that:

$$B(\alpha,\beta) = B(\beta,\alpha).$$

The Reduction Formula

If $\alpha > 1$, the following equality holds:

$$B(\alpha,\beta) = \frac{\alpha-1}{\alpha+\beta-1}B(\alpha-1,\beta).$$

We can now write the reduction formula:

$$B(\alpha,\beta) = \frac{\alpha-1}{\alpha+\beta-1}B(\alpha,\beta-1).$$

It can be seen immediately from the definition of the beta function that

$$B(\alpha,1) = \frac{1}{\alpha},$$

and so for $n\in\mathbb{N}$ we obtain

$$B(\alpha,n) = \frac{(n-1)!}{\alpha(\alpha+1)\cdots(\alpha+n-1)}.$$

In particular, for $m,n\in\mathbb{N}$,

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}.$$

Other Forms of Representation of the Beta Function

1. One form for the beta function is

$$B(\alpha,\beta) = 2\int_0^{\frac{\pi}{2}} \cos^{2p-1}\varphi \sin^{2q-1}\varphi \, d\varphi.$$

In particular,

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \pi.$$

2. The other form for the beta function is

$$B(\alpha,\beta) = \int_0^{+\infty} \frac{t^{\beta-1}}{(1+t)^{\alpha+\beta}} dt.$$

7.4.2 The Gamma Function

Domain of Definition

The Gamma Function is:

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx.$$

It can be seen from the definition that the integral defining the gamma function converges at zero only for $\alpha > 0$, while it converges at infinity for all values of $\alpha \in \mathbb{R}$, due to the presence of the rapidly decreasing factor e^{-x}. Thus, the gamma function is defined for $\alpha > 0$.

Smoothness and the Formula for the Derivatives

The gamma function is infinitely differentiable, and

$$\Gamma^{(n)}(\alpha) = \int_0^{+\infty} x^{\alpha-1} \ln^n x e^{-x} dx.$$

The Reduction Formula

The relation

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

holds. It is known as the reduction formula for the gamma function. Since $\Gamma(1) = 1$, we conclude that for $n \in \mathbb{N}$,

$$\Gamma(n+1) = n!.$$

Thus, the gamma function turns out to be closely connected with the number-theoretic function $n!$.

The Euler-Gauss Formula

This is usually given by the following equality:

$$\Gamma(\alpha) = \lim_{n\to\infty} n^{\alpha} \frac{(n-1)!}{\alpha(\alpha+1)\cdots(\alpha+n-1)}.$$

The Complement Formula

For $0 < \alpha < 1$, the values α and $1-\alpha$ of the argument of the gamma function are mutually complementary, so that the equality

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)} \quad (0 < \alpha < 1)$$

holds. It follows in particular from Eq. 14 that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

We observe that

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} x^{-\frac{1}{2}} e^{-x}\,dx = 2\int_0^{+\infty} e^{-u^2}\,du = \sqrt{\pi}.$$

Connection Between the Beta and Gamma Function

The connection between the beta and gamma function is

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

■ **Example 7.13**
Find the result of

$$I = \int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x\,dx.$$

■

■ **Example 7.14**
Find the result of

$$\int_0^1 x^8 \sqrt{1-x^3}\,dx.$$

■

■ **Example 7.15**
Suppose $\alpha > -1$, find the results of the following integrals

$$\int_0^{\frac{\pi}{2}} \sin^\alpha x\,dx = \int_0^{\frac{\pi}{2}} \cos^\alpha x\,dx.$$

Furthermore, find the volume of the n-dimensional sphere of the form

$$B_n = \{(x_1, x_2, \dots, x_n) \mid x_1^2 + x_2^2 + \cdots + x_n^2 \le R^2\}.$$

■

7.5 Exercises

Exercise 7.1

- Show that

$$F : x \mapsto \int_0^{+\infty} \frac{\sin(x^2t^2)}{1+t^2}\,dt$$

  is well-defined and continuous on $\mathbb{R}$.
- Show that

$$G : x \mapsto \int_0^{+\infty} \sin(x^2t^2) e^{-xt}\,dt$$

is well-defined and continuous on $]0,+\infty[$.

■

Exercise 7.2

- Determine the domain of definition D of

$$F : x \mapsto \int_0^{+\infty} \frac{e^{-tx}}{1+t^2}\,dt.$$

- Show that F is of class $\mathscr{C}^\infty$ on $D \setminus \{0\}$.

■

Exercise 7.3
For $x \in \mathbb{R}$, define

$$F(x) = \int_0^{+\infty} \frac{\sin(xt)}{t} e^{-t}\,dt.$$

1. Justify that F is well-defined on $\mathbb{R}$.
2. Justify that F is $\mathscr{C}^1$ and give an expression for $F'(x)$ for all $x \in \mathbb{R}$.
3. Compute $F'(x)$.
4. Deduce a simplified expression for $F(x)$.

■

Exercise 7.4

Define

$$f(x) = \int_0^1 \frac{t^{x-1}}{1+t}\,dt.$$

1. Determine the domain of definition of f.
2. Prove that f is continuous on its domain of definition.
3. Compute $f(x)+f(x+1)$ for all $x>0$.
4. Deduce an asymptotic equivalent for f at 0.
5. Determine the limit of f at $+\infty$.

■

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