People's Democratic Republic Of Algeria

Ministry of Higher Education and Scientific Research

Mohamed El Bachir El Ibrahimi University
Bordj Bou Arreridj

Mathematics Department, Faculty of Mathematics and Computer Science

Mathematical Analysis and Applications Laboratory



Distributions

Course & Exercises

Destined to Master 1
Mathematical Analysis and Applications

Dr. Debbiche Hanene

Academic Year: 2025-2026

Contents

Introduction 3

1 Spaces of test functions

1.1 Notations 5

1.2 Lebesgue's spaces 7

1.3 Function spaces $C^k(\Omega)$ $(0 \leq k \leq \infty)$ . . 9

1.4 Function spaces $C_c^k(\Omega)$ $(0 \leq k \leq \infty)$. 11

2 Distributions

2.1 Definitions and examples 29

2.2 Convergence in the sense of distributions. 36

2.3 Operations on distributions 40

2.3.1 Addition and multiplication by a constant 40

2.3.2 Translation 40

2.3.3 Multiplication by a $C^\infty-$functions 41

2.3.4 Differentiation in the sense of distributions 43

2.4 Homogeneous distributions 48

2.5 Even and odd distributions 51

3 Support and convolution of distributions

3.1 Support of a distribution 70

3.1.1 Distributions with compact support 73

3.2 Convolution of distributions 74

3.2.1 Tensor product 74

3.2.2 Convolution 79

3.3 Application to differential equations 85

Bibliography 100

Introduction

Distributions (also known as generalized functions) extend the classical notion of functions to describe objects that may not have pointwise values but still possess well-defined derivatives of all orders.
Distributions are considered a great mathematical achievement of the twentieth century. They were introduced by Laurent Schwartz in the 1940s to rigorously model singular objects such as the Dirac delta (named after Paul Dirac, 1902–1984) and the Heaviside function (named after Oliver Heaviside, 1850–1925).

Consider the Dirac delta as an example. Physicists originally defined it as follows

$$\delta(x) = \begin{cases} +\infty & \text{for } x = 0, \\ 0 & \text{for } x \neq 0, \end{cases} \quad \text{with} \quad \int_{-\infty}^{+\infty} \delta(x)\, dx = 1. \tag{1}$$

However, this informal definition leads to mathematical contradictions. For instance, the square of this function is not well-defined, contradicting the properties of ordinary functions. Moreover, since $\delta(x)$ is zero everywhere except at the origin, it vanishes almost everywhere. According to Lebesgue integration theory, its integral should therefore be zero, contradicting (1). This $\delta(x)$ cannot be a function.
Dirac's intuitive definition relied on physical reasoning but lacked mathematical rigor. Laurent Schwartz resolved this issue by introducing distributions as continuous linear functionals on the space of smooth, compactly supported functions (infinitely differentiable functions with compact support). The Dirac delta is thus defined not as a pointwise function, but as a distribution acting on test functions φ via $\langle \delta_0, \varphi \rangle = \varphi(0)$. This framework allows distributions to generalize functions and provides a consistent way to handle differentiation, even for singular objects.

Distributions are crucial in physics and engineering, where discontinuous phenomena (e.g., impulses, shocks) naturally lead to differential equations whose solutions are distributions

rather than ordinary functions. The core idea of distribution theory is to define these generalized objects by their action on test functions, enabling a unified treatment of differentiation and integration in contexts where classical methods fail.

The manuscript is structured into three chapters:
Chapter 1 introduces the test functions and examines their properties.
Chapter 2 presents the notion of distributions. It covers their definition, convergence and some operations (addition, multiplication by smooth functions, translation, and derivatives...). Finally, we present the homogeneous, even, and odd distributions.
Chapter 3 is devoted to the support of distributions, the convolution product and application to differential equations.

This course is intended for Master's students in mathematics L.M.D "Mathematical Analysis and Applications". It provides them with the fundamental definitions and concepts necessary to become familiar with the calculus of distributions, such as the notions of differentiation, limits of sequences, and other operations on distributions. Through carefully selected examples, students will be able to appreciate the distinction between the classical behaviour of ordinary functions and the generalized behaviour of distributions.

Chapter 1

Spaces of test functions

In this chapter, we introduce one such class of functions: the class of test functions. These functions play several distinct and equally important roles. Their importance is particularly linked to the fact that the space of test functions contains "enough functions" for it to be dense in many function spaces.

1.1 Notations

Throughout this course we will take Ω is a non-empty open subset of $\mathbb{R}^n$ and that the underlying field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

1. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$, we denote by $\|x\|$ the Euclidean norm of x, i.e.,

$$\|x\| := \sqrt{\sum_{i=1}^{n} x_i^2}.$$

2. For $\varepsilon > 0$ and $x_0 \in \mathbb{R}^n$ (or $\mathbb{C}^n$), we denote by $B(x_0, \varepsilon)$ the open Euclidean ball around x_0 with radius ε, i.e.,

$$B(x_0, \varepsilon) := \{x : \|x - x_0\| < \varepsilon\},$$

and we denote by $\overline{B(x_0, \varepsilon)}$ the closed Euclidean ball around x_0 with radius ε, i.e,

$$\overline{B(x_0, \varepsilon)} = \{x : \|x - x_0\| \leq \varepsilon\}.$$

3. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be a multi-index of length (or order) $|\alpha| = \sum_{i=1}^{n} \alpha_i$. For multi-indices α, β, we define:

   (i) $\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

(ii) $\beta \leq \alpha \iff \beta_i \leq \alpha_i \quad \forall i = 1, \ldots, n.$

(iii) $\alpha - \beta := (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$, if $\beta \leq \alpha$.

(iv) For $x \in \mathbb{R}^n$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

(v) $\alpha! := \prod_{i=1}^{n} \alpha_i! = \alpha_1! \cdots \alpha_n!$.

(vi) $\binom{\alpha}{\beta} := \prod_{i=1}^{n} \binom{\alpha_i}{\beta_i} = \prod_{i=1}^{n} \frac{\alpha_i!}{(\alpha_i - \beta_i)!\,\beta_i!} = \frac{\alpha!}{(\alpha-\beta)!\,\beta!}$.

(vii) For all $x, y \in \mathbb{R}^n$, we have

$$(x+y)^\alpha = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} x^{\alpha-\beta} y^\beta.$$

For $u : \Omega \to \mathbb{K}$, we write

$$\partial^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

If $\alpha = e_i = (0, 0, \ldots, 1, \ldots, 0)$ (where $\alpha_j = \delta_{ij}$, the Kronecker delta), we write

$$\partial^\alpha u = \partial_i u = \frac{\partial u}{\partial x_i}.$$

In particular, if $\alpha = (0, \cdots, 0)$, $\partial^\alpha u = u$.

For functions of a single variable we also use the following shorter notation for the α^{th} order derivative $u^{(\alpha)} = \dfrac{d^\alpha u}{dx^\alpha}$ where in this case $\alpha \in \mathbb{N}$.

Example 1.1. (i) *If $x = (x_1, x_2) \in \mathbb{R}^2$ and $\alpha = (2, 3) \in \mathbb{N}^2$ we write*

$$x^\alpha = x_1^2 x_2^3.$$

(ii) *If $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\alpha = (1, 3, 0) \in \mathbb{N}^3$ we write*

$$x^\alpha = x_1 x_2^3.$$

(iii) *Let P be a polynomial of degree at most $m \in \mathbb{N}$*

$$P(x) = \sum_{|\alpha| \leq m} c_\alpha x^\alpha \quad \forall x \in \mathbb{R}^n,$$

with $\alpha \in \mathbb{N}^n$, $c_\alpha \in \mathbb{C}$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

(iv) *Let* $u : \mathbb{R}^2 \to \mathbb{R}$, $u(x_1, x_2) = 2x_1^2 x_2^3$ *and* $\alpha = (2, 1) \in \mathbb{N}^2$ *with* $|\alpha| = 3$, *we have*

$$\begin{aligned} \partial^\alpha u(x_1, x_2) &= \frac{\partial^3 u(x_1, x_2)}{\partial x_1^2 \partial x_2} = \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial}{\partial x_2}(2x_1^2 x_2^3)\right) \\ &= \frac{\partial^2}{\partial x_1^2}\left(6x_1^2 x_2^2\right) = 12x_2^2. \end{aligned}$$

1.2 Lebesgue's spaces

We denote by $L^p(\Omega)$ with $p \in [1, \infty)$ the Lebesgue space

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R};\ u \text{ is measurable, } \int_\Omega |u(x)|^p dx < \infty \right\}.$$

It is a Banach space with respect to the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

For $p = 2$, it is a Hilbert space with the inner product

$$(u, v) = \int_\Omega u(x) v(x) dx.$$

For $p = \infty$, we set

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R};\ u \text{ is measurable, } \operatorname{ess\,sup}_{x \in \Omega} |u(x)| < \infty \right\}.$$

It is a Banach space with respect to the norm

$$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|.$$

Finally, we define the space of locally integrable functions

$$L^1_{\text{loc}}(\Omega) = \left\{ u : \Omega \to \mathbb{R};\ u \text{ is measurable, } \int_K |u(x)| dx < \infty \, \forall K \subset \Omega \text{ compact} \right\}.$$

Example 1.2. 1. *The function* $\frac{1}{x} \in L^1_{loc}(0, \infty)$, *but* $\frac{1}{x} \notin L^1(0, \infty)$ *and* $\frac{1}{x} \notin L^1(-1, 1)$. *Indeed, for any compact* $K \subset (0, \infty)$, *there exists* $a, b > 0$ *such that* $K \subset [a, b]$. *Then*

$$\int_K \left| \frac{1}{x} \right| dx \leq \int_a^b \frac{1}{x} dx = \ln(b) - \ln(a) < \infty.$$

Thus, $\frac{1}{x} \in L^1_{loc}(0,\infty)$. *However*

$$\int_0^\infty \left|\frac{1}{x}\right| dx = \int_0^1 \frac{1}{x}\, dx + \int_1^\infty \frac{1}{x}\, dx = \infty.$$

So $\frac{1}{x} \notin L^1(0,\infty)$.
Consider the compact $K = [-\frac{1}{2}, \frac{1}{2}] \subset (-1,1)$. *Then*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{1}{x}\right| dx = 2\int_0^{\frac{1}{2}} \frac{1}{x}\, dx = \infty.$$

So $\frac{1}{x} \notin L^1_{loc}(-1,1)$, *and in particular* $\frac{1}{x} \notin L^1(-1,1)$.

2. *The function* $f(x) = \ln|x| \in L^1_{loc}(\mathbb{R})$. *Indeed, the function* $x \mapsto \ln|x|$ *is continuous on* $\mathbb{R}^*$ *and has a primitive on* $\mathbb{R}^*_+$ *and* $\mathbb{R}^*_-$. *Therefore, it is integrable on every compact set contained in either of the two intervals* $\mathbb{R}^*_+$ *and* $\mathbb{R}^*_-$. *The remaining case is when the compact set* K *of* $\mathbb{R}$ *contains* 0. *We have*

$$\int_K |\ln|x||\, dx \le \int_{-a}^{a} |\ln|x||\, dx = \lim_{\epsilon\to 0^+} \int_{a\ge|x|>\epsilon} |\ln|x||\, dx, \text{ with } 0<\epsilon<a<1.$$

Then

$$\int_{a\ge|x|>\epsilon} |\ln|x||\, dx = 2(a - a\ln(a) + \epsilon\ln(\epsilon) - \epsilon),$$

and

$$\int_{|x|<a} |\ln(|x|)|\, dx = \lim_{\epsilon\to 0^+} \int_{r\ge|x|>\epsilon} |\ln|x||\, dx = 2(a - a\ln(a)).$$

Thus $f \in\in L^1_{loc}(\mathbb{R})$.

Theorem 1.1. (Lebesgue's dominated convergence theorem) *If* $(u_j)_{j\in\mathbb{N}}$ *is a sequence of Lebesgue integrable functions on* Ω *that converge almost everywhere to a limit function* u *on* Ω *and if there exists a Lebesgue integrable function* v *on* Ω *such that*

$$|u_j(x)| \le v(x) \quad \text{almost everywhere on} \quad \Omega,$$

and for all $j \in \mathbb{N}$ *then* u *is Lebesgue integrable on* Ω *and*

$$\lim_{j\to+\infty} \int_\Omega u_j(x)\, dx = \int_\Omega \lim_{j\to+\infty} u_j(x)\, dx = \int_\Omega u(x)\, dx.$$

Definition 1.1. (Convolution Product) *Let* $f, g \in L^1(\mathbb{R}^n)$. *The convolution product of* f *and* g, *denoted by* $f * g$, *is the function defined by*

$$(f*g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\, dy.$$

Proposition 1.1. *The convolution product is commutative and associative.*

1.3 Function spaces $C^k(\Omega)$ $(0 \leq k \leq \infty)$

Definition 1.2. *We write $C(\Omega, \mathbb{K})$ or $C(\Omega)$ for the set of continuous functions $\Omega \to \mathbb{K}$. We will also write $C^0(\Omega) = C(\Omega)$ and*

$$\|u\|_{C(\Omega)} = \sup_{x\in\Omega} |u(x)|, \quad u \in C(\Omega).$$

For $k \in \mathbb{N}$ we write $C^k(\Omega, \mathbb{K})$ or $C^k(\Omega)$ for the k-times continuously differentiable functions from Ω to $\mathbb{K}$, and

$$C^k(\Omega) = \Big\{u \in C^{k-1}(\Omega), \quad \frac{\partial u}{\partial x_i} \in C^{k-1}(\Omega), \quad i = 1, \cdots, n\Big\}, \quad k \geq 1.$$

In general, we will write C^k instead of $C^k(\Omega)$.
$C^\infty(\Omega, \mathbb{K})$ or $C^\infty(\Omega)$ is the set of ∞-times continuously differentiable functions from Ω to $\mathbb{K}$ i.e.,

$$C^\infty(\Omega) = \cap_{k\in\mathbb{N}} C^k(\Omega).$$

A function $u : \Omega \to \mathbb{K}$ is called smooth if it is in $C^\infty(\Omega)$ and we denote by $\mathcal{E}(\Omega) = C^\infty(\Omega)$.

Example 1.3. *The funtion*

$$u(x) = \begin{cases} x & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x \leq 0, \end{cases}$$

is continuos, but not differentiable at $x = 0$, so it is of class $C^0(\mathbb{R})$.
The function $u(x) = e^x$ and all polynomials are $C^\infty(\mathbb{R})$.

Proposition 1.2. *If $u, v \in C^k(\Omega)$ and $\lambda \in \mathbb{K}$ then $u + v$, λu, $u\,v$ and $\dfrac{1}{u}$ (if $u(x) \neq 0$, for all $x \in \Omega$) belong to $C^k(\Omega)$.*

Theorem 1.2. *If $u \in C(\Omega)$ then $u \in L^1_{loc}(\Omega)$.*

Proof. We show that if u is a continuous function then it is locally integrable. Indeed,

1. u is measurable function because it is continuous.
2. We have u is continuous on Ω, so it is also continuous on any compact K of Ω and since a continuous function on a compact set is bounded. Hence

$$\exists C > 0, \quad |u(x)| \leq C, \quad \forall x \in K,$$

and since the compact K has finite measure (denoted $\text{mes}(K) < \infty$), we have

$$\int_K |u(x)|\, dx \leq C\ \text{mes}(K) < \infty.$$

□

Now, the Leibniz formula expresses the derivative on $\alpha-$th order of the product of two functions.

Theorem 1.3. (Leibniz's rule) *If $k \in \mathbb{N}$, $u, v \in C^k(\Omega)$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, then*

$$\partial^\alpha (u\, v) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta u\ \partial^{\alpha-\beta} v.$$

In one dimension ($n = 1$), the first derivative is described by the well known formula

$$(u\, v)' = u'\, v + u\, v'.$$

For $\alpha \in \mathbb{N}$, we can write the general formula for the derivative of $\alpha-$th order of the product of functions $(u\, v)$ as follows

$$(u\, v)^{(\alpha)} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} u^{(\beta)}\ v^{(\alpha-\beta)}.$$

Example 1.4. *Let $\alpha = (1, 1) \in \mathbb{N}^2$. We have*

$$\begin{aligned}
\partial^\alpha (u\, v) &= \partial^{(1,1)} (u\, v) = \frac{\partial^2 (u\, v)}{\partial x_1 \partial x_2} \\
&= \frac{\partial}{\partial x_1} \Big(\frac{\partial (u\, v)}{\partial x_2} \Big) \\
&= \frac{\partial}{\partial x_1} \Big(u\, \frac{\partial v}{\partial x_2} + v\, \frac{\partial u}{\partial x_2} \Big) \\
&= u\, \frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_2} + v\, \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial v}{\partial x_1} \frac{\partial u}{\partial x_2}.
\end{aligned}$$

Using Leibniz's rule, we obtain

$$\begin{aligned}
\partial^\alpha (u\, v) &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta u\, \partial^{\alpha-\beta} v \\
&= \sum_{\beta_1=0}^{1} \sum_{\beta_2=0}^{1} \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} \partial^{(\beta_1,\beta_2)} u\, \partial^{(\alpha_1-\beta_1, \alpha_2-\beta_2)} v,
\end{aligned}$$

then

$$\begin{aligned}
\partial^\alpha(u\,v) &= \sum_{\beta_1=0}^{1}\left[\binom{1}{\beta_1}\binom{1}{0}\partial^{(\beta_1,0)}u\,\partial^{(1-\beta_1,1)}v + \binom{1}{\beta_1}\binom{1}{1}\partial^{(\beta_1,1)}u\,\partial^{(1-\beta_1,0)}v\right]\\
&= \sum_{\beta_1=0}^{1}\left[\frac{1!}{\beta_1!(1-\beta_1)!}\partial^{(\beta_1,0)}u\,\partial^{(1-\beta_1,1)}v + \frac{1!}{\beta_1!(1-\beta_1)!}\partial^{(\beta_1,1)}u\,\partial^{(1-\beta_1,0)}v\right]\\
&= \partial^{(0,0)}u\,\partial^{(1,1)}v + \partial^{(0,1)}u\,\partial^{(1,0)}v + \partial^{(1,0)}u\,\partial^{(0,1)}v + \partial^{(1,1)}u\,\partial^{(0,0)}v\\
&= u\,\frac{\partial^2 v}{\partial x_1\partial x_2} + \frac{\partial u}{\partial x_2}\,\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_2} + v\,\frac{\partial^2 u}{\partial x_1\partial x_2}.
\end{aligned}$$

Definition 1.3. (Convergence in $C^k(\Omega)$) *Let $(u_j)_{j\in\mathbb{N}}$ be a sequence and $u \in C^k(\Omega)$ (in $\mathcal{E}(\Omega)$. resp).*

$$u_j \xrightarrow{j\to+\infty} u \quad in \quad C^k(\Omega) \quad\Longleftrightarrow\quad \begin{aligned}&\forall K \subset \Omega \text{ compact}, \forall \alpha \in \mathbb{N}^n,\ |\alpha| \le k,\\ &\partial^\alpha u_j \to \partial^\alpha u \text{ uniformly on } K,\end{aligned}$$

(i.e. $\|\partial^\alpha u_j - \partial^\alpha u\|_{C(K)} = \|\partial^\alpha u_j - \partial^\alpha u\|_\infty = \sup_{x\in K}|\partial^\alpha u_j(x) - \partial^\alpha u(x)| \to 0$).
(resp. $u_j \xrightarrow{j\to+\infty} u$ in $\mathcal{E}(\Omega) \iff \forall K \subset \Omega$ compact, $\forall \alpha \in \mathbb{N}^n$, $\partial^\alpha u_j \to \partial^\alpha u$ uniformly on K).

1.4 Function spaces $C_c^k(\Omega)$ $(0 \le k \le \infty)$

Definition 1.4. (Support) *Let $u \in C(\Omega)$. We define the support of u as*

$$\operatorname{supp}(u) := \overline{\{x \in \Omega \mid u(x) \neq 0\}}.$$

Example 1.5. *Define $u : \mathbb{R} \to \mathbb{R}$ by*

$$u(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1,\\ 0 & \text{if } |x| \ge 1.\end{cases}$$

Then $\operatorname{supp}(u) = [-1, 1]$.

Proposition 1.3. (Properties of the support) *Let $u, v \in C(\Omega)$, we have*

1. $u = 0 \iff \operatorname{supp}(u) = \emptyset$.
2. $\operatorname{supp}(u)$ *is closed in* Ω.
3. $\operatorname{supp}(uv) \subset \operatorname{supp}(u) \cap \operatorname{supp}(v)$.
4. $(\operatorname{supp}(u))^c$ *is the largest open subset of* Ω *where* u *vanishes.*

5. *For all* $\alpha \in \mathbb{N}^n$, $\text{supp}(\partial^\alpha u) \subset \text{supp}(u)$.

Definition 1.5. *For* $k \in \mathbb{N} \cup \{\infty\}$, *we define*

$$C_c^k(\Omega) = \{u \in C^k(\Omega),\ \text{supp}(u)\ \textit{is compact}\},$$

(i.e., zero outside a compact set).
The space of test functions on Ω *is* $C_c^\infty(\Omega) = \mathcal{D}(\Omega)$.
If K *is a compact set of* Ω, $C_c^k(K)$ *denotes the set of elements* u *of* $C_c^k(\Omega)$ *such that* $\text{supp}(u) \subset K$. *We denote by*

$$C_c^\infty(K) = \mathcal{D}(K).$$

Example 1.6. *Consider the function*

$$h(x) = \begin{cases} e^{\frac{1}{t}} & \textit{if } t < 0, \\ 0 & \textit{if } t \geq 0. \end{cases}$$

It is clear that $supp(h) = \mathbb{R}_-$ *and that* $h \in C^\infty(\mathbb{R} \setminus \{0\})$. *On the other hand, we can show that*

$$h^{(m)}(t) = P_m\left(\frac{1}{t}\right) e^{\frac{1}{t}},$$

where P_m *is a sequence of polynomials. We deduce that*

$$\lim_{x \to 0^-} h^{(m)}(t) = 0 \quad \textit{for all } m \in \mathbb{N},$$

and therefore $h \in C^\infty(\mathbb{R})$. *From the function* h, *we can construct a function* $\psi \in \mathcal{D}(\mathbb{R})$. *Indeed, we consider*

$$\psi(x) = \begin{cases} e^{-\frac{b-a}{(b-x)(x-a)}}, & \textit{if} \quad x \in]a, b[, \\ 0, & \textit{otherwise.} \end{cases}$$

We observe that $\psi(x) = h(a - x)h(x - b)$ *for all* $x \in \mathbb{R}$, *then,* $\psi \in C^\infty(\mathbb{R})$. *Since* $\text{supp}(\psi) = [a, b]$ *we deduce that* $\psi \in \mathcal{D}(\mathbb{R})$.
We can also construct a function $\rho \in \mathcal{D}(\mathbb{R}^n)$. *Let*

$$\rho(x) = \begin{cases} e^{\frac{-1}{(1-\|x\|^2)}}, & \|x\| < 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Smoothness of ρ *follows from the chain rule, noting that* $\rho(x) = h(\|x\|^2 - 1)$ *where* h *is as defined above and* $\text{supp}(\rho) = \overline{B(0,1)}$ *then* $\rho \in \mathcal{D}(\mathbb{R}^n)$.

Proposition 1.4. 1. *If* $\varphi \in \mathcal{D}(\Omega)$, *then* $\frac{\partial \varphi}{\partial x_i} \in \mathcal{D}(\Omega)$, $\forall i = 1, \cdots, n$.

2. *If* $\varphi, \psi \in \mathcal{D}(\Omega)$ *and* $a, b \in \mathbb{K}$. *Then,* $a\varphi + b\psi \in \mathcal{D}(\Omega)$. *Therefore,* $\mathcal{D}(\Omega)$ *is a linear space.*

Proof.

1. It is immediate, as $\frac{\partial\varphi}{\partial x_i}$ is $C^\infty(\Omega)$ because of the definition of φ. Moreover, if $x \notin \text{supp}(\varphi)$, since the complementary of the support is an open set, there exists a neighbourhood of x in which $\varphi = 0$. Thus, in that neighbourhood, $\frac{\partial\varphi}{\partial x_i} = 0$ and $\frac{\partial\varphi}{\partial x_i}(x) = 0$, which implies that $x \notin \text{supp}(\frac{\partial\varphi}{\partial x_i})$. In other words, $\text{supp}(\frac{\partial\varphi}{\partial x_i}) \subseteq \text{supp}(\varphi)$, since $\text{supp}(\varphi)$ is compact then $\text{supp}(\frac{\partial\varphi}{\partial x_i})$ is compact. Then $\frac{\partial\varphi}{\partial x_i} \in \mathcal{D}(\Omega)$, $\forall i = 1, \cdots, n$.

2. Let $A = \text{supp}(\varphi)$ and $B = \text{supp}(\psi)$. Then,

$$a\varphi(x) + b\psi(x) = 0, \quad \forall x \notin A \cup B.$$

This, together with the fact of $A \cup B$ being closed, implies that $\text{supp}(a\varphi + b\psi) \subseteq A \cup B$. Moreover, as A and B are bounded, $\text{supp}\,(a\varphi + b\psi)$ is bounded too. Also, the fact that both φ and ψ are in $C^\infty(\Omega)$ implies $a\varphi + b\psi \in C^\infty(\Omega)$. Finally, being working with functions with values in a field, every property for a linear space holds trivially.

□

Proposition 1.5. *Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\tau \in \mathbb{R}$, $a \in \mathbb{R} \setminus \{0\}$, and $g \in C^\infty(\mathbb{R}^n)$. Then the functions*

$$\varphi(x + \tau), \quad \varphi(-x), \quad \varphi(ax) \quad \textit{and} \quad g\,\varphi \in \mathcal{D}(\mathbb{R}^n).$$

Before defining distributions corresponding to test functions, we must first discuss a notion of convergence in $\mathcal{D}(\Omega)$.

Definition 1.6. (Convergence in $\mathcal{D}(\Omega)$) *Let $(\varphi_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. We say $\varphi_j \to \varphi$ in $\mathcal{D}(\Omega)$ if*

1. *There exists a compact set $K \subset \Omega$ such that* $\text{supp}(\varphi), \text{supp}(\varphi_j) \subset K, \quad \forall j \in \mathbb{N}$.

2. *For each multi-index $\alpha \in \mathbb{N}^n$*

$$\sup_K |\partial^\alpha(\varphi_j - \varphi)| \to 0,$$

(ie, $\partial^\alpha\varphi_j \to \partial^\alpha\varphi$ uniformly on K).

Example 1.7. 1. *Let $\phi \in \mathcal{D}(\mathbb{R})$. Let us define, for $j \in \mathbb{N}$,*

$$\forall x \in \mathbb{R}, \quad \varphi_j(x) = \phi\left(x + \frac{1}{j+1}\right) - \phi(x).$$

Then $\varphi_j \to 0$ in $\mathcal{D}(\mathbb{R})$. Indeed, if $\operatorname{supp}(\phi) \subset [-a, a]$, $a > 0$, then for all j, we have $\operatorname{supp}(\varphi_j) \subset [-a-1, a+1]$. Moreover, by the mean value theorem, for all $\alpha \in \mathbb{N}$,

$$\left|\phi^{(\alpha)}\left(x + \frac{1}{j+1}\right) - \phi^{(\alpha)}(x)\right| \leq \frac{1}{j+1}\|\phi^{(\alpha+1)}\|_\infty \to 0$$

as $j \to +\infty$.

2. *The sequence $\varphi_j(x) = \frac{1}{j}\rho(x)$ with ρ is defined in Example 1.6 belongs to $\mathcal{D}(\mathbb{R}^n)$ and converges to zero in $\mathcal{D}(\mathbb{R}^n)$ as $j \to +\infty$.*
The sequence $\varphi_j(x) = \rho(\frac{x}{j}) \in \mathcal{D}(\mathbb{R}^n)$ does not converge in $\mathcal{D}(\mathbb{R}^n)$ when $j \to +\infty$ because there cannot exist a compact K outside of which all the $\varphi_j(x)$ vanish.

Remark 1.1. 1. *We have $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$. If $(\varphi_j)_{j\in\mathbb{N}}$ is a sequence in $\mathcal{D}(\Omega)$ tends to φ in $\mathcal{D}(\Omega)$, then $\varphi_j \to \varphi$ in $\mathcal{E}(\Omega)$. The inclusion map $i : \mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ is continuous ($\mathcal{D}(\Omega) \hookrightarrow \mathcal{E}(\Omega)$). Moreover, $\mathcal{D}(\Omega)$ is dense in $\mathcal{E}(\Omega)$.*

2. *Convergence in $\mathcal{D}(\Omega)$ differs from convergence in $\mathcal{E}(\Omega)$. Indeed, let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\varphi(0) \neq 0$. We consider the sequence of functions $\varphi_j(x) = \varphi\left(\frac{x}{j}\right)$. We show that the sequence φ_j converge to $\varphi(0)$ in $\mathcal{E}(\mathbb{R}^n)$. Let K be a compact subset of $\mathbb{R}^n$ such that $K \subset B(0,\varepsilon)$. Then, if $x \in K$, we have $\frac{x}{j} \in B(0,\varepsilon)$*

   (i) *If $\alpha = 0$, then*
   $$|\varphi_j(x) - \varphi(0)| = \left|\varphi\left(\frac{x}{j}\right) - \varphi(0)\right| \to 0,$$
   uniformly on K as $j \to +\infty$ (by uniform continuity of φ on the compact set $\overline{B(0,\varepsilon)}$).

   (ii) *If $|\alpha| \geq 1$, then $\partial^\alpha\varphi(0) = 0$ and*
   $$\partial^\alpha(\varphi_j(x)) = \left(\frac{1}{j}\right)^{|\alpha|}\partial^\alpha\varphi\left(\frac{x}{j}\right).$$
   It follows that
   $$\begin{aligned}|\partial^\alpha(\varphi_j(x)) - \partial^\alpha(\varphi(0))| &= \left(\frac{1}{j}\right)^{|\alpha|}\left|\partial^\alpha\varphi\left(\frac{x}{j}\right)\right| \\ &\leq \left(\frac{1}{j}\right)^{|\alpha|}\|\partial^\alpha\varphi\|_{C(\overline{B(0,\varepsilon)}))},\end{aligned}$$
   then
   $$\lim_{j\to+\infty}\|\partial^\alpha\varphi_j - \partial^\alpha(\varphi(0))\|_{C(K)} = 0.$$
   So, the sequence $(\varphi_j)_j$ converges to $\varphi(0)$ in $\mathcal{E}(\mathbb{R}^n)$. However, $\varphi(0) \notin \mathcal{D}(\mathbb{R}^n)$.

We take another example. Let $\psi \in \mathcal{D}(\mathbb{R})$ with $\psi \neq 0$. We consider the sequence

$$\varphi_j(x) = \frac{1}{j}\psi(x-j), j = 1, 2, \cdots$$

This converges to 0 in $\mathcal{E}(\mathbb{R})$. For any compact K et $\alpha \in \mathbb{N}$, we have

$$\sup_{x\in K} |\varphi_j^{(\alpha)}(x)| \to 0 \quad as \quad j \to +\infty.$$

Moreover, $supp(\varphi_j) \cap K = \emptyset$ for j sufficiently large, i.e., the support of φ_j eventually leaves any compact set. The sequence $(\varphi_j)_j$ does not converge in $\mathcal{D}(\mathbb{R})$. We have

$$\sup_{x\in \mathbb{R}} |\varphi_j^{(\alpha)}(x)| \to 0 \quad as \quad j \to +\infty,$$

but there is no compact set which contains the support of φ_j for all j.

Definition 1.7. (Regularizing sequences) *A sequence $(\rho_j)_{j\geq 1} \subset \mathcal{D}(\mathbb{R}^n)$ is called a mollifier (or regularizing sequence) in $\mathcal{D}(\mathbb{R}^n)$ if it satisfies the following properties*

(i) *For all $j \geq 1$: $\rho_j \geq 0$.*

(ii) $\displaystyle\int_{\mathbb{R}^n} \rho_j(x)\, dx = 1.$

(iii) $\mathrm{supp}(\rho_j) \subset B(0, \varepsilon_j)$ *with* $\lim\limits_{j\to\infty} \varepsilon_j = 0.$

Construct a regularizing sequence

Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ be non-negative and not identically zero. Then there exists $a > 0$ such that $\mathrm{supp}(\psi) \subset B(0, a)$. We put

$$\phi(x) = \frac{\psi(x)}{\displaystyle\int_{\mathbb{R}^n} \psi(x)dx}.$$

Note that $\displaystyle\int_{\mathbb{R}^n} \phi(x)\, dx = 1$ and $\phi(x) \geq 0$ for all x. Define

$$\varphi_j(x) = \frac{1}{r_j^n}\phi\left(\frac{x}{r_j}\right),$$

where $\lim\limits_{j\to+\infty} r_j = 0$ and $r_j > 0$ for all $j \in \mathbb{N}$. We have

(i) $\forall j \in \mathbb{N}, \varphi_j \geq 0,$

(ii) $\displaystyle\int_{\mathbb{R}^n} \varphi_j(x)\,dx = \frac{1}{r_j^n}\int_{\mathbb{R}^n} \phi\left(\frac{x}{r_j}\right)\,dx,$
using the substitution $y = \frac{x}{r_j}$, we get $dx = r_j^n\,dy$, so

$$\int_{\mathbb{R}^n} \varphi_j(x)\,dx = \frac{1}{r_j^n}\int_{\mathbb{R}^n} \phi(y) r_j^n\,dy = \int_{\mathbb{R}^n} \phi(y)\,dy = 1.$$

(iii) Since $\mathrm{supp}(\phi) \subset B(0,a)$, we have $\mathrm{supp}(\varphi_j) \subset B(0, r_j a)$, and $\lim\limits_{j\to+\infty} r_j = 0$.

It is enough to put $\varepsilon_j = r_j a$ in the definition we deduce that the sequence (φ_j) is a regularizing sequence.

Theorem 1.4. (Exhaustive sequences) *Let Ω be an open set in $\mathbb{R}^n$. There exists an exhaustive sequence of compact sets K_j covering Ω. That is, a sequence satisfying*

1. $\forall j \in \mathbb{N}$, $K_j \subset \Omega$.
2. $\forall j \in \mathbb{N}$, $K_j \subset \overset{\circ}{K}_{j+1}$ *(and we write generally $K_j \subset\subset K_{j+1}$).*
3. $\Omega = \bigcup_{j\in\mathbb{N}} K_j$.

The adjective "exhaustive" meaning "complete" was used to emphasize that the sequence of compact sets $(K_j)_j$ covers the entire open set Ω.

Theorem 1.5. (Partition of unity) *Let K be a compact subset of Ω, and let Ω_i, $i = 1, 2, \ldots, k$, be a finite covering of K, i.e.,*

$$K \subset \bigcup_{i=1}^{k} \Omega_i,$$

then there exist functions $\varphi_i \in \mathcal{D}(\Omega_i)$ such that $0 \le \varphi_i \le 1$ for $i = 1, 2, \ldots, k$, and

$$\sum_{i=1}^{k} \varphi_i = 1 \quad \textit{in a neighborhood of } K.$$

Let us now consider two open sets Ω_1, Ω_2 in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$. We denote by $\mathcal{D}(\Omega_1) \otimes \mathcal{D}(\Omega_2)$ the subspace of $\mathcal{D}(\Omega_1 \times \Omega_2)$ consisting of functions that are finite linear combinations of terms of the form $u(x_1)v(x_2)$ where $u \in \mathcal{D}(\Omega_1)$ and $v \in \mathcal{D}(\Omega_2)$.

Proposition 1.6. *The space $\mathcal{D}(\Omega_1) \otimes \mathcal{D}(\Omega_2)$ is dense in $\mathcal{D}(\Omega_1 \times \Omega_2)$.*

Proof. Let $\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$. Suppose that $\mathrm{supp}(\varphi)$ is contained in $K = K_1 \times K_2$ where K_i is a compact subset of $\Omega_i, i = 1, 2$. There exists a sequence $(P_j)_{j\in\mathbb{N}}$ of polynomials of (x_1, x_2)

such that, for every α in $\mathbb{N}^{n_1+n_2}$, $(\partial^\alpha P_j)$ converges uniformly on K to $\partial^\alpha\varphi$. Let $\theta_i \in \mathcal{D}(\Omega_i)$, with $\theta_i = 1$ on $K_i, i = 1, 2$. Define

$$\varphi_j(x_1, x_2) = \theta_1(x_1)\theta_2(x_2)P_j(x_1, x_2) = \sum_{|\alpha_1+\alpha_2|\leq m_j} a_{j,\alpha_1,\alpha_2}\theta_1(x_1)x_1^{\alpha_1}\theta_2(x_2)x_2^{\alpha_2},$$

where $a_{j,\alpha_1,\alpha_2} \in \mathbb{C}$. Then φ_j belongs to $\mathcal{D}(\Omega_1) \otimes \mathcal{D}(\Omega_2)$ and for every $\alpha \in \mathbb{N}^{n_1+n_2}$, $(\partial^\alpha\varphi_j)$ converges uniformly to $\partial^\alpha\varphi$ on $K_1 \times K_2$. Moreover, since $\operatorname{supp}(\varphi_j) \subset \operatorname{supp}(\theta_1) \times \operatorname{supp}(\theta_2)$, which is a fixed compact set, the sequence (φ_j) converges to φ in $\mathcal{D}(\Omega_1 \times \Omega_2)$. □

Now, we consider the following theorem underlies many of the important applications of convolutions on $\mathbb{R}^n$. If h is any function on $\mathbb{R}^n$ and $\varepsilon > 0$, we set

$$h_\varepsilon(x) = \varepsilon^{-n}h(\varepsilon^{-1}x).$$

We observe that if $h \in L^1(\mathbb{R}^n)$ then $\displaystyle\int_{\mathbb{R}^n} h_\varepsilon$ is independent of ε

$$\int_{\mathbb{R}^n} h_\varepsilon(x)\,dx = \int_{\mathbb{R}^n} h(\varepsilon^{-1}x)\varepsilon^{-n}\,dx = \int_{\mathbb{R}^n} h(y)\,dy.$$

Theorem 1.6. *Suppose $h \in L^1(\mathbb{R}^n)$ and $\displaystyle\int_{\mathbb{R}^n} h(x)\,dx = A$.*

(i) *If $f \in L^p(\mathbb{R}^n)(1 \leq p < \infty)$, then $f * h_\varepsilon \to Af$ in the $L^p(\mathbb{R}^n)$ norm as $\varepsilon \to 0$.*

(ii) *If f is bounded and uniformly continuous, then $f * h_\varepsilon \to Af$ uniformly as $\varepsilon \to 0$.*

(iii) *If $f \in L^\infty(\mathbb{R}^n)$ and f is continuous on an open set U, then $f * h_\varepsilon \to Af$ uniformly on compact subsets of U as $\varepsilon \to 0$.*

Theorem 1.7. *Let $f \in C_c^k(\mathbb{R}^n)$ and $g \in L^1_{loc}(\mathbb{R}^n)$ $(k \in \mathbb{N})$. Then*

$$f * g \in C^k(\mathbb{R}^n) \quad \text{and} \quad \partial^\alpha(f * g) = (\partial^\alpha f) * g \quad \text{for} \quad |\alpha| \leq k.$$

In particular, if $f \in \mathcal{D}(\mathbb{R}^n)$ and $g \in L^1_{loc}(\mathbb{R}^n)$, then $f * g \in C^\infty(\mathbb{R}^n)$.

Theorem 1.8. 1. *$\mathcal{D}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for all $1 \leq p < +\infty$.*

2. *$\mathcal{D}(\Omega)$ is dense in $C^k(\Omega)$ for all $k \in \mathbb{N}$ and in $L^p(\Omega)$ for all $1 \leq p < +\infty$.*

Now, we consider the following application of the density theorem 1.8

Lemma 1.1. (Dubois-Reymond) *Let $u \in L^1_{loc}(\Omega)$. We assume that for all $\varphi \in \mathcal{D}(\Omega)$,*

$$\int_\Omega u(x)\varphi(x)\,dx = 0.$$

Then $u = 0$ almost everywhere.

Proof. We write that Ω is a countable union of open sets $\omega_n \subset \Omega$ such that $\overline{\omega_n}$ is a compact subset of Ω. It suffices to take

$$\omega_n = \left\{ x \in \Omega : |x| \leq n \text{ and } d(x, \Omega^c) > \frac{1}{n} \right\}.$$

It is therefore sufficient to show that $u = 0$ a.e (ie, almost everywhere) on an open set $\omega \subset \Omega$ such that $\overline{\omega}$ is a compact subset of Ω.

Let $g = u|_\omega \in L^1(\omega)$. Let $\varepsilon > 0$. By Theorem 1.8, there exists $\psi_\varepsilon \in \mathcal{D}(\omega)$ such that

$$\|g - \psi_\varepsilon\|_{L^1(\omega)} \leq \varepsilon.$$

For all $\varphi \in \mathcal{D}(\omega)$, since $\displaystyle\int_\omega g\,\varphi\,dx = \int_\Omega u\,\varphi\,dx$ we have

$$\int_\omega \psi_\varepsilon \varphi\,dx = \int_\omega g\varphi\,dx + \int_\omega (\psi_\varepsilon - g)\varphi\,dx = 0 + \int_\omega (\psi_\varepsilon - g)\varphi\,dx.$$

Hence

$$\left| \int_\omega \psi_\varepsilon \varphi\,dx \right| \leq \int_\omega |\psi_\varepsilon - g||\varphi|\,dx \leq \varepsilon \|\varphi\|_\infty.$$

Let $\delta > 0$. Take φ as follows

$$\varphi = \frac{\overline{\psi}_\varepsilon}{\sqrt{\delta^2 + |\psi_\varepsilon|^2}} \in \mathcal{D}(\omega).$$

We have $|\varphi| \leq 1$ and

$$\psi_\varepsilon \varphi = \frac{|\psi_\varepsilon|^2}{\sqrt{\delta^2 + |\psi_\varepsilon|^2}}.$$

Then,

$$\forall \delta \geq 0, \quad \int_\omega \frac{|\psi_\varepsilon|^2}{\sqrt{\delta^2 + |\psi_\varepsilon|^2}}\,dx \leq \varepsilon.$$

By applying the dominated convergence theorem, we can pass to the limit when $\delta \to 0$ to obtain

$$\int_\omega |\psi_\varepsilon|\,dx \leq \varepsilon.$$

Therefore,

$$\|g\|_{L^1(\omega)} \leq \|g - \psi_\varepsilon\|_{L^1(\omega)} + \|\psi_\varepsilon\|_{L^1(\omega)} \leq \varepsilon + \varepsilon = 2\varepsilon.$$

Thus $g = 0$ a.e on ω, and by countable additivity, u is null almost everywhere on Ω. □

Exercises

Exercise 1.1. *Determine whether the following statements are true or false. Justify your answers.*

1. *It is always possible to construct a test function $\psi \in \mathcal{D}(-1,1)$ such that $\psi'(x) \geq 0$ for all $x \in (-1,1)$.*
2. *The functions $\varphi(x) = e^{-x^2}$ and $1_{\mathbb{Q}}$ are test functions.*
3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x\varphi(x)$ with $\varphi \in \mathcal{D}(\mathbb{R})$. Then $f \in \mathcal{D}(\mathbb{R})$.*
4. *If $\varphi \in \mathcal{D}(\Omega)$, then $\partial^\alpha \varphi \in \mathcal{D}(\Omega)$ for all $\alpha \in \mathbb{N}^n$.*

Exercise 1.2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\varphi(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

1. *Verify that $\varphi \in \mathcal{D}(\mathbb{R})$.*
2. *We set $\psi(x) = \displaystyle\int_{-\infty}^{x} \varphi(y)\,dy$. Is $\psi \in C^\infty(\mathbb{R})$? Determine its support.*
3. *Let $C = \dfrac{1}{\displaystyle\int_{\mathbb{R}} \varphi(x)\,dx}$ and define φ_j by*

$$\varphi_j(x) = Cj\varphi(jx) \quad \text{for all } j \in \mathbb{N} \setminus \{0\}.$$

Show that φ_j is a regularizing sequence.

Exercise 1.3. **I)** *Let $\psi \in C^\infty(\mathbb{R})$ and $(\varphi_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathbb{R})$ that converges to φ in $\mathcal{D}(\mathbb{R})$. Show that*

$$\psi\varphi_j \xrightarrow{j\to+\infty} \psi\varphi,$$

and

$$\varphi_j^{(\alpha)} \xrightarrow{j\to+\infty} \varphi^{(\alpha)}, \quad \forall \alpha \in \mathbb{N},$$

in $\mathcal{D}(\mathbb{R})$.

II) Let $\varphi \in \mathcal{D}(\mathbb{R})$. For $t \in \mathbb{R} \setminus \{0\}$, we define

$$\varphi_t(x) = \frac{\varphi(x+t) - \varphi(x)}{t}.$$

1. Show that $\varphi_t \in \mathcal{D}(\mathbb{R})$ for $t \neq 0$.
2. Show that φ_t converges in $\mathcal{D}(\mathbb{R})$ as $t \to 0$ to a function that we will determine.

Exercise 1.4. *Let $\varphi \in \mathcal{D}(\mathbb{R})$. Study the convergence in $\mathcal{D}(\mathbb{R})$ of the following sequences*

$$\psi_j(x) = \frac{1}{j}\varphi(x), \quad \eta_j(x) = \frac{1}{j}\varphi(jx).$$

Exercise 1.5. *Let $\varphi \in \mathcal{D}(\mathbb{R})$ such that*

$$0 \leq \varphi(x) \leq 1, \quad \textit{and} \quad \textit{supp}(\varphi) \subset]1, 2[.$$

Define $\varphi_j(x) = e^{-j}\varphi(jx), \quad j \geq 1$. Show that $\varphi_j \in \mathcal{D}(\mathbb{R})$ for all $j \geq 1$ and study its convergence in $\mathcal{D}(\mathbb{R})$.

Solutions of the exercises

Exercise 1.1

1. False. The condition $\psi' \geq 0$ means that ψ is increasing. Note that $\psi(-1) = \psi(1) = 0$ this is only verified if $\psi = 0$.

2. The functions $\varphi(x) = e^{-x^2}$ is a test function (false) because $\text{supp}(\varphi) = \mathbb{R}$ which is not compact.
   The function $1_{\mathbb{Q}}$ is a test function (false). Indeed, the indicator function of the rational numbers $\mathbb{Q}$ is defined by
   $$\mathbf{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$
   We have
   $$\text{supp}(\mathbf{1}_{\mathbb{Q}}) = \overline{\{x \in \mathbb{R} : \mathbf{1}_{\mathbb{Q}}(x) \neq 0\}} = \overline{\mathbb{Q}} = \mathbb{R}.$$
   Then $\mathbf{1}_{\mathbb{Q}}$ is not a test function.
   We can say that $\mathbf{1}_{\mathbb{Q}}$ is not a test function on $\mathbb{R}$ because it is not continuous.

3. True. We set $g(x) = x$ then $f = g\,\varphi$. we have $g \in C^\infty(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$ so $g\,\varphi \in C^\infty(\mathbb{R})$. Since
   $$\text{supp}(g\,\varphi) \subset \text{supp}(g) \cap \text{supp}(\varphi),$$
   we obtain $\text{supp}(g\,\varphi) \subset \text{supp}(\varphi)$ and since $\text{supp}(\varphi)$ is compact we deduce that the support of $g\,\varphi$ is also compact then $g\,\varphi \in \mathcal{D}(\mathbb{R})$.

4. True. Since $\varphi \in C^\infty(\Omega)$ then $\partial^\alpha \varphi \in C^\infty(\Omega)$. Moreover, from the observation that for all $\alpha \in \mathbb{N}^n$, $\text{supp}(\partial^\alpha \varphi) \subset \text{supp}(\varphi)$. Since $\text{supp}(\varphi)$ is compact, $\text{supp}(\partial^\alpha \varphi)$ must be as well, and thus $\partial^\alpha \varphi \in \mathcal{D}(\Omega)$ for all $\alpha \in \mathbb{N}^n$.

Exercise 1.2

1. $\varphi \in C^\infty(\mathbb{R})$ because it is composed of two functions of class $C^\infty(\mathbb{R})$ and $supp(\varphi) = [-1, 1]$ is compact, we deduce that $\varphi \in \mathcal{D}(\mathbb{R})$ (see details in example 1.6).

2. We have $\psi' = \varphi \in C^\infty(\mathbb{R})$ then $\psi \in C^\infty(\mathbb{R})$.
   Now, we determine the support of ψ, We distinguish the following cases
   (a) If $x < -1$ we have $\varphi(x) = 0$ then $\psi(x) = 0$.
   (b) If $-1 < x < 1$,
   $$\psi(x) = \int_{-\infty}^{x} \varphi(y)dy = \int_{-1}^{x} \varphi(y)dy > 0.$$

(c) If $x \geq 1$,

$$\psi(x) = \int_{-\infty}^{x} \varphi(y)\,dy = \int_{-1}^{x} \varphi(y)\,dy = \int_{-1}^{1} \varphi(y)\,dy + \int_{1}^{x} \varphi(y)\,dy = \int_{-1}^{1} \varphi(y)\,dy > 0.$$

We deduce that $\text{supp}(\psi) = [-1, +\infty[$ which is not compact so ψ is not a test function.

3. Let $C = \dfrac{1}{\displaystyle\int_{\mathbb{R}} \varphi(x)\,dx}$ and define φ_j by

$$\varphi_j(x) = Cj\varphi(jx) \quad \text{for all } j \in \mathbb{N} \setminus \{0\}.$$

Show that φ_j is a regularizing sequence.
Recall that a sequence (φ_j) is called a mollifier (or regularizing sequence) if $\varphi_j \in \mathcal{D}(\mathbb{R})$ and for all $j \in \mathbb{N} \setminus \{0\}$, we have

$$\varphi_j \geq 0, \quad \int_{\mathbb{R}} \varphi_j(x)\,dx = 1, \quad \text{supp}(\varphi_j) \subset [-\varepsilon_j, \varepsilon_j],$$

with $\lim_{j\to+\infty} \varepsilon_j = 0$. Since $\varphi \in \mathcal{D}(\mathbb{R})$, it follows that $(\varphi_j) \subset \mathcal{D}(\mathbb{R})$. Let $j \in \mathbb{N} \setminus \{0\}$. We have $\varphi \geq 0$, and thus $\varphi_j \geq 0$. We set $y = jx$ and by using the definition of C, we obtain

$$\int_{\mathbb{R}} \varphi_j(x)\,dx = Cj \int_{\mathbb{R}} \varphi(jx)\,dx = C \int_{\mathbb{R}} \varphi(y)\,dy = 1.$$

The support of φ is $[-1, 1]$. Since $jx \in [-1, 1]$ then $x \in [-\frac{1}{j}, \frac{1}{j}]$, the support of φ_j is $[-\frac{1}{j}, \frac{1}{j}]$, and we clearly have $\lim_{j\to+\infty} \frac{1}{j} = 0$. Therefore, the sequence (φ_j) is a mollifier (or regularizing sequence).

Exercise 1.3

I) Since $\varphi_j \to \varphi$ in $\mathcal{D}(\mathbb{R})$, there exists a compact set $K \subset \mathbb{R}$ such that

$$\text{supp}(\varphi_j), \text{supp}(\varphi) \subset K.$$

We have

$$\text{supp}(\psi\varphi_j) \subset \text{supp}(\psi) \cap \text{supp}(\varphi_j) \subset \text{supp}(\varphi_j) \subset K,$$

and

$$\text{supp}(\psi\varphi) \subset \text{supp}(\psi) \cap \text{supp}(\varphi) \subset \text{supp}(\varphi) \subset K.$$

We need to show that for $\alpha \in \mathbb{N}$, the sequence of derivatives $(\psi\varphi_j)^{(\alpha)}$ converges uni-

formly to $(\psi\varphi)^{(\alpha)}$. By the Leibniz rule for derivatives of products we get

$$(\psi\varphi_j)^{(\alpha)} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \psi^{(\alpha-\beta)} \varphi_j^{(\beta)},$$

and

$$(\psi\varphi)^{(\alpha)} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \psi^{(\alpha-\beta)} \varphi^{(\beta)}.$$

Since $\varphi_j \to \varphi$ in $\mathcal{D}(\mathbb{R})$, for each β, $\varphi_j^{(\beta)} \to \varphi^{(\beta)}$ uniformly on $\mathbb{R}$. Since $\psi \in C^\infty(\mathbb{R})$, so each derivative $\psi^{(\alpha-\beta)}$ is continuous so it is bounded on the compact set K. Therefore,

$$\psi^{(\alpha-\beta)} \varphi_j^{(\beta)} \to \psi^{(\alpha-\beta)} \varphi^{(\beta)} \quad \text{uniformly on } K.$$

Since the sum is finite, thus

$$(\psi\varphi_j)^{(\alpha)} \to (\psi\varphi)^{(\alpha)} \quad \text{uniformly on } K.$$

Then

$$\psi\varphi_j \xrightarrow{j\to+\infty} \psi\varphi,$$

in $\mathcal{D}(\mathbb{R})$.
Fix $\alpha \in \mathbb{N}$, then for all $j, \beta \in \mathbb{N}$ we have $\mathrm{supp}(\varphi_{\mathrm{j}}^{(\alpha)}) \subset \mathrm{supp}(\varphi_{\mathrm{j}}) \subset \mathrm{K}$ and $\varphi_j^{(\alpha)} = \dfrac{d^\alpha \varphi_j}{dx^\alpha}$. So

$$(\varphi_j^{(\alpha)})^{(\beta)} = \frac{d^\beta \varphi_j^{(\alpha)}}{dx^\beta} = \frac{d^{\alpha+\beta}\varphi_j}{dx^{\alpha+\beta}} = \varphi_j^{(\alpha+\beta)},$$

converges uniformly to $(\varphi^{(\alpha)})^{(\beta)} = \varphi^{(\alpha+\beta)}$ and since $\alpha \in \mathbb{N}$ was chosen arbitrarily, this holds for all $\alpha \in \mathbb{N}$ and thus

$$\varphi_j^{(\alpha)} \xrightarrow{j\to+\infty} \varphi^{(\alpha)}, \quad \forall \alpha \in \mathbb{N},$$

in $\mathcal{D}(\mathbb{R})$.

II) 1. We need to show that $\varphi_t \in \mathcal{D}(\mathbb{R})$. Indeed, since $\varphi \in C^\infty(\mathbb{R})$, the function $x \mapsto \varphi(x+t)$ is also in $C^\infty(\mathbb{R})$ (as a composition of functions in $C^\infty(\mathbb{R})$). Therefore, φ_t is a linear combination of functions in $C^\infty(\mathbb{R})$ and is itself $C^\infty(\mathbb{R})$.
Let $\mathrm{supp}(\varphi) \subset [-M, M]$ for some $M > 0$. Then $\mathrm{supp}(\varphi_t) \subset [-M - |t|, M + |t|]$ which is compact. Therefore, $\varphi_t \in \mathcal{D}(\mathbb{R})$.

2. Recall the Taylor-Lagrange's inequality: Let $f : I \to \mathbb{R}$ and $a, b \in I$. Suppose that f is n times differentiable on the closed interval $[a, b]$, and $n+1$ times differentiable

on the open interval $]a, b[$. Then there exists $c \in [a, b]$ such that

$$f(b) - \sum_{k=0}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) = \frac{f^{(n+1)}(c)(b-a)^{n+1}}{(n+1)!}.$$

We can deduce Taylor-Lagrange's inequality : under the same hypotheses, if there exists $C \in \mathbb{R}_+$ such that $|f^{(n+1)}(y)| \leq C$ for all $y \in [a, b]$, then

$$\left| f(b) - \sum_{k=0}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) \right| \leq \frac{C(b-a)^{n+1}}{(n+1)!}.$$

Now, if φ_t converges in $\mathcal{D}(\mathbb{R})$ as $t \to 0$ to a function ψ, then it converges uniformly and hence pointwise to ψ. We must therefore have $\psi = \varphi'$. Let $M \geq 0$ such that $\text{supp}(\varphi) \subset [-M, M]$. Since we are interested in the case $t \to 0$, we can restrict ourselves to $t \in [-1, 1]$. For all $t \in [-1, 1]$, we have $\text{supp}(\varphi_t) \subset [-M-1, M+1]$, so the support condition is satisfied.
Let $\alpha \in \mathbb{N}$. For all $x \in \mathbb{R}$ and all $t \in [-1, 1]$,

$$\left| \varphi_t^{(\alpha)}(x) - (\varphi')^{(\alpha)}(x) \right| = \frac{1}{|t|} \left| \varphi^{(\alpha)}(x+t) - \varphi^{(\alpha)}(x) - t\varphi^{(\alpha+1)}(x) \right| \leq \frac{1}{|t|} \frac{t^2}{2} \left\| \varphi^{(\alpha+2)} \right\|_\infty,$$

where we applied Taylor-Lagrange's inequality between x and $x+t$ to the function $\varphi^{(\alpha)} \in \mathcal{D}(\mathbb{R})$. Therefore,

$$\left\| \varphi_t^{(\alpha)} - \varphi^{(\alpha+1)} \right\|_\infty \leq |t| \left\| \varphi^{(\alpha+2)} \right\|_\infty \underset{t \to 0}{\longrightarrow} 0.$$

Finally, we obtain $\varphi_t \underset{t \to 0}{\longrightarrow} \varphi'$ in $\mathcal{D}(\mathbb{R})$.

Exercise 1.4 Let $\varphi \in \mathcal{D}(\mathbb{R})$ with $\text{supp}(\varphi) \subset K = [-a, a]$, where $a > 0$.
Convergence of the sequence (ψ_j) in $\mathcal{D}(\mathbb{R})$
Study the pointwise convergence of ψ_j. Let x be fixed in $\mathbb{R}$. We have

$$\lim_{j \to +\infty} \psi_j(x) = \lim_{j \to +\infty} \frac{1}{j} \varphi(x) = 0.$$

Then ψ_j converges pointwise to the function $\psi(x) = 0$.
We note that $\text{supp}(\psi_j) = \text{supp}(\varphi)$. We have

(i) $\text{supp}(\psi_j) \subset K$.

(ii) $\text{supp}(\psi) = \emptyset \subset K$.

(iii) Let $\alpha \in \mathbb{N}$, we have

$$\lim_{j\to+\infty} \sup_{x\in K} |\psi_j^{(\alpha)}(x) - \psi^{(\alpha)}(x)| = \lim_{j\to+\infty} \frac{1}{j} \sup_{x\in K} |\varphi^{(\alpha)}(x)| = 0.$$

Thus $\psi_j \to 0$ in $\mathcal{D}(\mathbb{R})$.

Convergence of the sequence (η_j) in $\mathcal{D}(\mathbb{R})$

Study the pointwise convergence. Let $M = \sup_{y\in\mathbb{R}} |\varphi(y)|$. For fixed $x \in \mathbb{R}$ we have

$$0 \leq \lim_{j\to+\infty} |\eta_j(x)| \leq \lim_{j\to+\infty} \frac{M}{j} = 0.$$

So, η_j converges pointwise to the function $\eta = 0$. We have

(i) $\operatorname{supp}(\eta_j) \subset [-\frac{a}{j}, \frac{a}{j}] \subset K$.

(ii) $\operatorname{supp}(\eta) = \emptyset \subset K$.

(iii) Let $\alpha \in \mathbb{N}$, we have

$$\lim_{j\to+\infty} \sup_{x\in K} |\eta_j^{(\alpha)}(x) - \eta^{(\alpha)}(x)| = \lim_{j\to+\infty} j^{\alpha-1} \sup_{x\in K} |\varphi^{(\alpha)}(jx)| = +\infty, \quad \forall \alpha > 1.$$

Therefore, we conclude that η_j does not converge in $\mathcal{D}(\mathbb{R})$.

Exercise 1.5 The function $\varphi_j \in C^\infty(\mathbb{R})$ (as a composition of the two functions φ and $x \mapsto jx$, both infinitely differentiable on $\mathbb{R}$). Moreover, since $\operatorname{supp}(\varphi) \subset [1, 2]$, we have

$$\begin{aligned}
\operatorname{supp}(\varphi_j) &= \overline{\{x \in \mathbb{R} \mid \varphi_j(x) \neq 0\}} \\
&= \overline{\{x \in \mathbb{R} \mid \varphi(jx) \neq 0\}} \\
&\subset \overline{\{x \in \mathbb{R} \mid jx \in \operatorname{supp}\varphi\}} \\
&\subset \overline{\{x \in \mathbb{R} \mid jx \in]1, 2[\}} \\
&= \left[\frac{1}{j}, \frac{2}{j}\right],
\end{aligned}$$

and thus $\varphi_j \in \mathcal{D}(\mathbb{R})$.

It is easy to see that for all $\alpha \in \mathbb{N}$, we have

$$\varphi_j^{(\alpha)}(x) = e^{-j} j^\alpha \varphi^{(\alpha)}(jx).$$

Let us show that $\varphi_j \to 0$ in $\mathcal{D}(\mathbb{R})$. We have

$$\operatorname{supp}(\varphi_j) \subset \left[\frac{1}{j}, \frac{2}{j}\right] \subset [1, 2] = K, \quad \forall j \in \mathbb{N}.$$

Moreover,

$$\begin{aligned}
\left\|\varphi_j^{(\alpha)} - 0\right\|_{C(K)} &= \left\|\varphi_j^{(\alpha)}\right\|_{C\left(\left[\frac{1}{j},\frac{2}{j}\right]\right)} \\
&= \sup_{x\in\left[\frac{1}{j},\frac{2}{j}\right]} \left|\varphi_j^{(\alpha)}(x)\right| \\
&= e^{-j} j^{\alpha} \sup_{x\in\left[\frac{1}{j},\frac{2}{j}\right]} \left|\varphi^{(\alpha)}(jx)\right| \\
&\leq e^{-j} j^{\alpha} \sup_{y\in[1,2]} \left|\varphi^{(\alpha)}(y)\right| \\
&\leq e^{-j} j^{\alpha} \sup_{y\in K} \left|\varphi^{(\alpha)}(y)\right| \underset{j\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

Since this holds for all $\alpha \in \mathbb{N}$ and all supports are contained in the fixed compact set $K = [1, 2]$ we conclude that $\varphi_j \to 0$ in $\mathcal{D}(\mathbb{R})$.

Additional exercises

Exercise 1.6. *Let $f, g \in L^1(\mathbb{R}^n)$. Show that $f * g \in L^1(\mathbb{R}^n)$ and*

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}.$$

Exercise 1.7. *Let $f \in L^1(\mathbb{R})$. Consider the mapping*

$$Sf : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \displaystyle\int_0^1 f(x-y).\,dy \end{array}$$

1. *Show that if f is continuous with compact support, then Sf is continuous.*
2. *Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of continuous functions with compact support that converges to f in $L^1(\mathbb{R})$.*
   (i) *Show that $(Sf_j)_{j\in\mathbb{N}}$ converges to Sf uniformly on $\mathbb{R}$.*
   (ii) *Deduce that Sf is continuous on $\mathbb{R}$.*

Exercise 1.8. *Let $\psi \in \mathcal{D}(\mathbb{R})$. Determine a necessary and sufficient condition on ψ for there to exist $\varphi \in \mathcal{D}(\mathbb{R})$ such that*

$$\varphi' + \varphi = \psi.$$

Exercise 1.9. *I) Show that if $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $x \in \mathbb{R}$ then $\tau_x\phi \in \mathcal{D}(\mathbb{R}^n)$ where τ_x is the translation operator defined by*

$$\tau_x\phi(y) = \phi(y - x), \quad \forall y \in \mathbb{R}^n.$$

II) Suppose $\varphi \in \mathcal{E}(\mathbb{R}^n)$. Let $(x_j)_{j\geq 1} \subset \mathbb{R}^n$ be a sequence with $x_j \to 0$. Show that

$$\tau_{x_j}\varphi \to \varphi, \qquad \text{as } j \to \infty,$$

in $\mathcal{E}(\mathbb{R}^n)$.

Exercise 1.10. *Let $\psi \in \mathcal{D}(\mathbb{R})$. Determine a necessary and sufficient condition on ψ for there to exist $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi' = \psi$.*

Exercise 1.11. *Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $h \in \mathbb{R}^n \setminus \{0\}$. For $t \in \mathbb{R} \setminus \{0\}$, we define*

$$\varphi_t(x) = \frac{\varphi(x+th) - \varphi(x)}{t}.$$

1. *Show that $\varphi_t \in \mathcal{D}(\mathbb{R}^n)$ for $t \neq 0$.*
2. *Show that φ_t converges in $\mathcal{D}(\mathbb{R}^n)$ as $t \to 0$ to a function that we will determine.*

Exercise 1.12. *Let (φ_j) be a sequence of functions in $\mathcal{D}(\mathbb{R})$ defined by*

$$\varphi_j(t) = \begin{cases} \frac{1}{2^j} \exp\left(-\frac{1}{1-\frac{|t|^2}{j^2}}\right) & \text{if } |t| < j, \\ 0 & \text{otherwise} \end{cases}$$

1. *Show that, for each $\alpha \in \mathbb{N}$, the sequence of functions $(\varphi_j^{(\alpha)})$ converges uniformly on every compact set to a function $\varphi \in \mathcal{D}(\mathbb{R})$ that should be specified.*
2. *Do we have convergence in $\mathcal{D}(\mathbb{R})$?*

Chapter 2

Distributions

The theory of distributions is a field of mathematics that extends the concept of functions to include objects called distributions. It is particularly useful in the analysis and solution of differential equations, where classical functions may not suffice. The theory of distributions enables the definition of derivatives for functions that are not necessarily differentiable in the classical sense.

2.1 Definitions and examples

Definition 2.1. *A distribution $T \in \mathcal{D}'(\Omega)$ is a linear functional on the space of test functions $T : \mathcal{D}(\Omega) \to \mathbb{K}$, which is continuous,*

$$\varphi_j \to \varphi \quad in \;\; \mathcal{D}(\Omega) \Rightarrow T(\varphi_j) \to T(\varphi) \quad in \; \mathbb{K}. \tag{2.1}$$

Remark 2.1. (i) *We often write $\langle T, \varphi \rangle$ instead of $T(\varphi)$.*

(ii) *By the linearity of T, the condition (2.1) is equivalent to*

$$\varphi_j \to 0 \quad in \;\; \mathcal{D}(\Omega) \Rightarrow T(\varphi_j) \to 0 \quad in \; \mathbb{K}.$$

Theorem 2.1. *Let $T : \mathcal{D}(\Omega) \to \mathbb{K}$ be a linear map. The following are equivalent*

1. $T \in \mathcal{D}'(\Omega)$.
2. *For each compact $K \subset \Omega$, there exists a constant $C > 0$ and $m \in \mathbb{N}$ such that*

$$|\langle T, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \sup_{x \in K} |\partial^\alpha \varphi(x)|, \quad for \; all \; \varphi \in \mathcal{D}(K). \tag{2.2}$$

Proof. (1 ⇒ 2) By contradiction: Assume that we have $T \in \mathcal{D}'(\Omega)$ but there exists $K \subset \Omega$ such that for all $m \in \mathbb{N}$

$$|\langle T, \varphi_m \rangle| > m \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{C(K)}.$$

for some $\varphi_m \in \mathcal{D}(K)$ with $\|\partial^\alpha \varphi_m\|_{C(K)} = \sup_{x \in K} |\partial^\alpha \varphi(x)|$ and $\varphi_m \neq 0$.
Thus

$$0 < \|\varphi_m\|_{C(K)} \leq \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{C(K)}.$$

Let

$$\psi_m(x) = \frac{\varphi_m(x)}{m \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{C(K)}} \quad \text{for} \quad x \in \Omega.$$

Then $\psi_m \in \mathcal{D}(K)$ and for any $\beta \in \mathbb{N}^n$ and $|\beta| \leq m$ we have

$$\|\partial^\beta \psi_m\|_{C(K)} \leq \frac{\|\partial^\beta \varphi_m\|_{C(K)}}{m \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{C(K)}} \leq \frac{1}{m}.$$

Hence $\psi_m \to 0$ in $\mathcal{D}(\Omega)$ as $m \to \infty$). On the other hand, we obtain

$$|\langle T, \psi_m \rangle| = \frac{|\langle T, \varphi_m \rangle|}{m \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_m\|_{C(K)}} > 1,$$

therefore $(\langle T, \psi_m \rangle)_m$ does not tend to zero in $\mathbb{K}$. This contradicts the fact that $T \in \mathcal{D}'(\Omega)$.
(2 ⇒ 1) Let $\varphi_j \to 0$ in $\mathcal{D}(\Omega)$ with $\text{supp}(\varphi_j) \subset K$ for all j. Choose $C > 0$ and $m \in \mathbb{N}$ according to (2.2). Then we have

$$|\langle T, \varphi_j \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi_j\|_{C(K)} \to 0 \quad j \to \infty,$$

hence $T \in \mathcal{D}'(\Omega)$. □

Let's look the important examples

Example 2.1. 1. *The Dirac distribution at a point $x_0 \in \Omega$*

$$\langle \delta_{x_0}, \varphi \rangle = \varphi(x_0) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Linearity of δ_{x_0}: For all $\lambda, \gamma \in \mathbb{K}$ and all $\varphi, \phi \in \mathcal{D}(\Omega)$, we have

$$\langle \delta_{x_0}, \lambda\varphi + \gamma\phi \rangle = \lambda\varphi(x_0) + \gamma\phi(x_0) = \lambda\langle \delta_{x_0}, \varphi \rangle + \gamma\langle \delta_{x_0}, \phi \rangle,$$

Moreover, for $\varphi \in \mathcal{D}(K)$, we have

$$|\langle \delta_{x_0}, \varphi \rangle| = |\varphi(x_0)| \leq \|\varphi\|_{C(K)},$$

where $\|\varphi\|_{C(K)} = \sup_{x\in K} |\varphi(x)|$ *then* $\delta_{x_0} \in \mathcal{D}'(\Omega)$ *(the estimate* (2.2) *holds with* $m = 0$ *and* $C = 1$*).*

2. *Let* $f \in L^1_{loc}(\Omega)$ *and define* $T_f : \mathcal{D}(\Omega) \to \mathbb{K}$ *by*

$$\langle T_f, \varphi\rangle = \int_\Omega f(x)\varphi(x)\,dx, \quad \forall\varphi \in \mathcal{D}(\Omega).$$

Clearly T_f *is linear; we show that it also satisfies* (2.2)*: Let* $K \subset \Omega$ *be a compact set and* $\varphi \in \mathcal{D}(K)$*, then*

$$\begin{aligned} |\langle T_f, \varphi\rangle| &= \left|\int_K f(x)\varphi(x)\,dx\right| \\ &\leq \|\varphi\|_{C(K)} \int_K |f(x)|\,dx \\ &= \|f\|_{L^1(K)}\|\varphi\|_{C(K)}. \end{aligned}$$

Thus we obtain (2.2) *with* $m = 0$ *and* $C = \|f\|_{L^1(K)}$*.*

3. **The Heaviside Function:** *Let* H *denote (the* L^∞*-class of) a function on* $\mathbb{R}$ *with*

$$H(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

We define a linear functional on $\mathcal{D}(\mathbb{R})$*, also denoted by* H*, by setting*

$$\langle H, \varphi\rangle = \int_{\mathbb{R}} H(x)\varphi(x)\,dx = \int_0^{+\infty} \varphi(x)\,dx, \quad \forall\varphi \in \mathcal{D}(\mathbb{R}).$$

If $K \subset \mathbb{R}$*, we have*

$$|\langle H, \varphi\rangle| \leq mes(K)\|\varphi\|_{L^\infty(K)},$$

which proves (2.2) *with* $m = 0$ *and* $C = mes(K)$*. Hence* $H \in \mathcal{D}'(\mathbb{R})$*.*
In the plane, we call the Heaviside function the function H *defined by*

$$H(x, y) = \begin{cases} 1 & \text{if} \quad x > 0, \quad y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We define a linear functional on $\mathcal{D}(\mathbb{R}^2)$ *as follows*

$$\langle H, \varphi\rangle = \int_{\mathbb{R}^2} H(x, y)\,\varphi(x, y)\,dx\,dy = \int_0^{+\infty}\int_0^{+\infty} \varphi(x, y)\,dx\,dy, \quad \forall\varphi \in \mathcal{D}(\mathbb{R}^2).$$

We have $H \in \mathcal{D}'(\mathbb{R}^2)$*.*

In general, the Heaviside function $H : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$H(x) = \begin{cases} 1 & \text{if} \quad x_1 > 0,\ x_2 > 0, \cdots x_n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We define a linear functional on $\mathcal{D}(\mathbb{R}^n)$ by

$$\langle H, \varphi \rangle = \int_{\mathbb{R}^n} H(x)\, \varphi(x)\, dx_1 \cdots dx_n = \int_0^{+\infty} \cdots \int_0^{+\infty} \varphi(x)\, dx_1 \cdots dx_n, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

4. *In general, we cannot define a distribution through the expression*

$$\langle T, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)\, dx,$$

if the function is not locally integrable. This is the case for $f(x) = \frac{1}{x}$, as it is not integrable around the origin in $\mathbb{R}$. However, we can define an associated distribution in the following way. Let $T : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ be defined by

$$\langle T, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx. \tag{2.3}$$

This expression defines a distribution and it is called the Cauchy principal value of $\frac{1}{x}$. We denote it as

$$T = PV\left(\frac{1}{x}\right).$$

To prove that it is a distribution, we must ensure that the limit exists. For this, a change of variable allows us to write

$$\begin{aligned} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx &= \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}\, dx \\ &= \int_{\varepsilon}^{\infty} \frac{\varphi(x) - \varphi(-x)}{x}\, dx. \end{aligned}$$

Now, know that φ has compact support, we can find a constant $a > 0$ such that the support of φ is contained in $[-a, a]$. Thus, the limit (2.3) becomes

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{a} \frac{\varphi(x) - \varphi(-x)}{x}\, dx.$$

Using the characteristic function χ, we have

$$\left\langle PV\left(\frac{1}{x}\right), \varphi \right\rangle = \lim_{n \to +\infty} \int_0^{\infty} \chi_{[\frac{1}{n}, a]}(x) \frac{\varphi(x) - \varphi(-x)}{x}\, dx.$$

For any test function $\varphi \in C^\infty(\mathbb{R})$, we know from the mean value theorem that for all $x > 0$, there exists $t \in (-x, x)$ such that

$$\varphi(x) - \varphi(-x) = 2x\varphi'(t).$$

Moreover, $\varphi \in \mathcal{D}(\mathbb{R})$ implies that $\varphi' \in \mathcal{D}(\mathbb{R})$, so it is bounded, and thus

$$\frac{|\varphi(x) - \varphi(-x)|}{x} \leq 2\|\varphi'\|_\infty.$$

Thus, we can write

$$\left|\chi_{[\frac{1}{n},a]}(x)\frac{\varphi(x) - \varphi(-x)}{x}\right| \leq 2\|\varphi'\|_\infty \chi_{[0,a]}(x),$$

and the integral remains bounded by an integrable function on $\mathbb{R}$. By the dominated convergence theorem, we have

$$\begin{aligned}\left\langle PV\left(\frac{1}{x}\right), \varphi \right\rangle &= \int_0^\infty \lim_{n\to\infty} \chi_{[\frac{1}{n},a]}(x)\frac{\varphi(x) - \varphi(-x)}{x}\, dx \\ &= \int_0^a \frac{\varphi(x) - \varphi(-x)}{x}\, dx. \end{aligned} \tag{2.4}$$

Finally, the integral is finite because the singularity at $x = 0$ represents a removable discontinuity. Indeed, we have

$$\lim_{x\to 0} \frac{\varphi(x) - \varphi(-x)}{x} = 2\varphi'(0).$$

After seeing the functionl is well defined, linearity is obtained using expression (2.4) and the linearity of integral. To verify continuity, consider a sequence $(\varphi_j)_j$ converging to φ in $\mathcal{D}(\mathbb{R})$. This means that their supports are contained in a compact subset $K \subset \mathbb{R}$, and thus for some sufficiently large $a > 0$, $K \subset [-a, a]$. Therfore,

$$\left\langle PV\left(\frac{1}{x}\right), \varphi_j - \varphi \right\rangle = \int_0^a \frac{\varphi_j(x) - \varphi(x) - (\varphi_j(-x) - \varphi(-x))}{x}\, dx, \quad \forall j \in \mathbb{N}.$$

Again, using the same reasoning, we have

$$\frac{\varphi_j(x) - \varphi(x) - (\varphi_j(-x) - \varphi(-x))}{x} \leq 2\|\varphi_j' - \varphi'\|_\infty.$$

Since $\varphi_j' \to \varphi'$ uniformly, there exists a constant $M > 0$ such that

$$\|\varphi_j' - \varphi'\|_\infty \leq M, \quad \forall j \in \mathbb{N}.$$

Thus, by the dominated convergence theorem, we conclude that

$$\lim_{j\to\infty}\left\langle PV\left(\frac{1}{x}\right),\varphi_j-\varphi\right\rangle=0.$$

This proves that $PV(\frac{1}{x})$ is continuous. Then $PV(\frac{1}{x})\in\mathcal{D}'(\mathbb{R})$.

Definition 2.2. (Regular Distribution) *A distribution $T\in\mathcal{D}'(\Omega)$ is called regular, if there exists $f\in L^1_{loc}(\Omega)$ such that*

$$\langle T,\varphi\rangle=\int_\Omega f(x)\varphi(x)\,dx=\langle T_f,\varphi\rangle\quad\forall\varphi\in\mathcal{D}(\Omega).$$

The lemma of Dubois-Reymond 1.1 allows us to identify T_f with the function f in a unique way. The map $f\mapsto T_f$ is an injection of $L^1_{loc}(\Omega)$ into $\mathcal{D}'(\Omega)$.
Conventionally, one abuses notation by identifying T_f with f, provided no confusion can arise, and thus the pairing between T_f and φ is often written

$$\langle f,\varphi\rangle=\langle T_f,\varphi\rangle,\quad\forall\varphi\in\mathcal{D}(\Omega).$$

Remark 2.2. *The Dirac distribution δ_{x_0} is not a regular distribution. Suppose the contrary, that is, there exists a function $f\in L^1_{loc}(\Omega)$ with $\delta_{x_0}=T_f$.*
Choose $\varphi\in\mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}(\varphi)\subset\overline{B(0,1)}$, $\varphi(0)=1$, and define $\varphi_j(x)=\varphi(j(x-x_0))$ $(j\in\mathbb{N}^*)$. Then $supp(\varphi_j)\subset\overline{B(x_0,\frac{1}{j})}$, $\varphi_j(x_0)=1$, and we have

$$\begin{aligned}1=|\langle\delta_{x_0},\varphi_j\rangle| &\le \int_{\overline{B(x_0,\frac{1}{j})}}|f(x)||\varphi_j(x)|\,dx\\ &\le \int_{\overline{B(x_0,\frac{1}{j})}}|f(x)||\varphi(j(x-x_0))|\,dx\\ &\le \|\varphi\|_{L^\infty}\int_{\overline{B(x_0,\frac{1}{j})}}|f(x)|\,dx.\end{aligned}$$

Now, the function $|f(x)|\mathbf{1}_{\overline{B(x_0,1)}}(x)$ is integrable (since $f\in L^1_{loc}$) and dominates $|f(x)|\mathbf{1}_{\overline{B(x_0,\frac{1}{j})}}(x)$, which tends to 0 almost everywhere. By Lebesgues dominated convergence theorem we obtain

$$\int_{\overline{B(x_0,\frac{1}{j})}}|f(x)|\,dx\to 0\quad(j\to\infty).$$

Therefore, $1\le 0$, a contradiction.

Definition 2.3. (Distributions of finite order) *A distribution $T\in\mathcal{D}'(\Omega)$ is said to be of*

finite order, if in the estimate (2.2)*, the integer* m *may be chosen uniformly for all* K*, i.e.,*

$$\exists m \in \mathbb{N},\ \forall K \subset \Omega,\ \exists C > 0 :\ |\langle T, \varphi\rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{C(K)} \quad \forall \varphi \in \mathcal{D}(K).$$

The least such $m \in \mathbb{N}$ *satisfying the above is called the order of the distribution.*

Example 2.2. 1. *A regular distribution and* δ_{x_0} *are of order* 0 *(see Example* 2.1*).*

2. *There exist distributions that are not of finite order. Let* T *be the linear form on* $\mathcal{D}(\mathbb{R})$ *defined by*

$$\forall \varphi \in \mathcal{D}(\mathbb{R}), \quad \langle T, \varphi\rangle = \sum_{j=0}^{+\infty} \varphi^{(j)}(j).$$

Then T *is a distribution on* $\mathbb{R}$ *of infinite order. Let* $[-a, a] \subset \mathbb{R}$ *and* $\varphi \in \mathcal{D}(\mathbb{R})$*, with* $\operatorname{supp}(\varphi) \subset [-a, a]$*. Let* $p_0 = E(a) + 1$*, where* $E(a)$ *is the integer part of* a*. We have*

$$|\langle T, \varphi\rangle| = \left|\sum_{j=0}^{+\infty} \varphi^{(j)}(j)\right| = \left|\sum_{j=0}^{p_0} \varphi^{(j)}(j)\right| \leq \sum_{j=0}^{p_0} \left|\varphi^{(j)}(j)\right| \leq \sum_{j=0}^{p_0} \|\varphi^{(j)}(j)\|_\infty.$$

Thus $T \in \mathcal{D}'(\mathbb{R})$*. By contradiction, we assume that* T *is of finite order* m*. Let* $\psi_0 \in \mathcal{D}(]-\frac{1}{2}, \frac{1}{2}[)$*, equal to* 1 *on* $[-\frac{1}{4}, \frac{1}{4}]$ *and positive. Let* $\lambda > 1$*. Define*

$$\psi(x) = \frac{x^{m+1}}{(m+1)!}\psi_0(x), \quad \textit{for} \quad x \in \mathbb{R},$$

and $\varphi(x) = \psi(\lambda(x - (m+1)))$*. Consider the compact set* $K = [m + \frac{1}{2}, m + \frac{3}{2}] \subset \mathbb{R}$*. Since* $\lambda > 1$*,* φ *is supported in* K *and is* C^∞*.*
By Leibniz's formula, $\psi^{(m+1)}(0) = \psi_0(0) = 1$*. Furthermore, since* $\operatorname{supp}(\varphi) \subset K$*, we have*

$$\langle T, \varphi\rangle = \varphi^{(m+1)}(m+1) = \lambda^{m+1}\psi^{(m+1)}(0) = \lambda^{m+1}.$$

For $j \leq m$*,*

$$\forall x \in \mathbb{R}, \quad \left|\varphi^{(j)}(x)\right| \leq \lambda^j \sup_{x \in \mathbb{R}} \left|\psi^{(j)}(x)\right| \leq \lambda^m \|\psi^{(j)}\|_\infty.$$

Now, T *is assumed to be of order* m*, so for* $K = [m + \frac{1}{2}, m + \frac{3}{2}]$*, there exists* $C > 0$

$$|\langle T, \varphi\rangle| \leq C \sum_{j=0}^{m} \|\varphi^{(j)}\|_\infty,$$

hence

$$\lambda^{m+1} \leq C \sum_{j=0}^{m} \lambda^j \|\psi^{(j)}\|_\infty \leq C \left(\sum_{j=0}^{m} \|\psi^{(j)}\|_\infty\right) \lambda^m.$$

This leads to a contradiction as $\lambda \to +\infty$, since

$$\lambda \leq C \sum_{j=0}^{m} \|\psi^{(j)}\|_\infty < \infty,$$

is impossible. Thus T cannot be of finite order.

2.2 Convergence in the sense of distributions.

The notion of convergence of distributions is also called weak convergence in the following sense

Definition 2.4. (Sequential convergence in $\mathcal{D}'(\Omega)$) *Let (T_j) be a sequence in $\mathcal{D}'(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. We say that (T_j) converges to T in $\mathcal{D}'(\Omega)$, denoted $T_j \to T$ as $j \to +\infty$, if*

$$\lim_{j\to+\infty} \langle T_j, \varphi\rangle = \langle T, \varphi\rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Example 2.3. 1. *Let (T_j) be a sequence of distributions defined by*

$$\langle T_j, \varphi\rangle = \int_{|x|>\frac{1}{j}} \frac{\varphi(x)}{x}\, dx, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

It is clear that $T_j \in \mathcal{D}'(\mathbb{R})$ since $T_j = T_{f_j}$ where

$$f_j(x) = \begin{cases} \frac{1}{x} & \text{if } |x| > \frac{1}{j}, \\ 0 & \text{if } |x| \leq \frac{1}{j}, \end{cases}$$

is a locally integrable function on $\mathbb{R}$.
The sequence $(T_j)_{n\in\mathbb{N}^*}$ converges in $\mathcal{D}'(\mathbb{R})$. Indeed, let $\varphi \in \mathcal{D}(\mathbb{R})$ and $\text{supp}(\varphi) \subset [-a, a]$, $a > 0$. Then

$$\begin{aligned} \langle T_j, \varphi\rangle &= \int_{\frac{1}{j}<|x|<a} \frac{\varphi(x)}{x}\, dx \\ &= \int_{-a}^{-\frac{1}{j}} \frac{\varphi(x)}{x}\, dx + \int_{\frac{1}{j}}^{a} \frac{\varphi(x)}{x}\, dx \\ &= \int_{\frac{1}{j}}^{a} \frac{\varphi(x) - \varphi(-x)}{x}\, dx \to \langle PV\big(\frac{1}{x}\big), \varphi\rangle \quad \text{as} \quad j \to +\infty. \end{aligned}$$

2. **Convergence of regular distributions**: *we consider* $f_j \in C(\mathbb{R})$ *given by*

$$f_j(x) = e^{ijx} \quad x \in \mathbb{R},\ j \in \mathbb{N}.$$

The sequence $(f_j(x))_{j\in\mathbb{N}}$ *converges as* $j \to +\infty$ *iff* $x \in 2\pi\mathbb{Z}$, *but* (f_j) *converges to* 0 *in* $\mathcal{D}'(\mathbb{R})$. *Indeed, let* $\varphi \in \mathcal{D}(\mathbb{R})$ *we have*

$$\langle f_j, \varphi\rangle = \int_{\mathbb{R}} e^{ijx}\varphi(x)dx,$$

by integration by parts we get

$$\langle f_j, \varphi\rangle = -\frac{1}{ij}\int_{\mathbb{R}} e^{ijx}\varphi'(x)dx.$$

Then

$$\begin{aligned} |\langle f_j, \varphi\rangle| &\leq \frac{1}{ij}\int_{\mathbb{R}} |e^{ijx}||\varphi'(x)|\,dx \\ &\leq \frac{1}{ij}\int_{\mathbb{R}} |\varphi'(x)|\,dx \to 0 \quad as \quad j \to +\infty. \end{aligned}$$

Remark 2.3. *When a sequence of distributions is defined by locally integrable functions, it is important to distinguish between pointwise (classical) convergence and convergence in the sense of distributions.*

The following theorem characterizes the sequences of functions for which the pointwise convergence coincides with convergence in the sense of distributions.

Theorem 2.2. *Let* $(f_j)_j$ *be a sequence of functions in* $L^1_{loc}(\Omega)$ *converging to* f *almost everywhere in* Ω, *and suppose that* $|f_j| \leq g$ *with* $g \in L^1_{loc}(\Omega)$. *Then*

$$f_j \longrightarrow f \qquad in\ \mathcal{D}'(\Omega).$$

Proof. Let $\varphi \in \mathcal{D}(\Omega)$ and set $K = \mathrm{supp}(\varphi)$. Then $f_j\,\varphi$ converges to $f\varphi$ almost everywhere and $|f_j\varphi| \leq |g||\varphi|$ for almost every x in K. Using the dominated convergence theorem, we deduce that

$$\begin{aligned} \lim_{j\to+\infty}\langle T_{f_j}, \varphi\rangle &= \lim_{j\to+\infty}\int_\Omega f_j\,\varphi\,dx = \lim_{j\to+\infty}\int_K f_j\,\varphi\,dx \\ &= \int_K f\varphi\,dx = \int_\Omega f\,\varphi\,dx = \langle T_f, \varphi\rangle, \end{aligned}$$

which completes the proof. □

Example 2.4. (i) *To see that almost everywhere convergence of a sequence of locally integrable functions does not imply convergence in $\mathcal{D}'(\Omega)$, consider the sequence*

$$f_j(x) = \begin{cases} j^2 & \text{if } |x| < \frac{1}{j} \\ 0 & \text{if } |x| \geq \frac{1}{j}. \end{cases}$$

This sequence converges to 0 almost everywhere. But if we choose $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi = 1$ on $(-1, 1)$, it follows that

$$\begin{aligned} \lim_{j\to+\infty} \langle f_j, \varphi \rangle &= \lim_{j\to+\infty} \int_{-\frac{1}{j}}^{\frac{1}{j}} f_j(x)\varphi(x)\,dx \\ &= \lim_{j\to+\infty} \int_{-\frac{1}{j}}^{\frac{1}{j}} j^2\,dx \\ &= \lim_{j\to+\infty} 2j = +\infty. \end{aligned}$$

(ii) *Conversely, convergence in the sense of distributions does not imply pointwise convergence, as can be seen by considering the sequence of functions $h_j(x) = \sin(jx)$. Indeed, for all $\varphi \in \mathcal{D}(\mathbb{R})$ we have*

$$\begin{aligned} \lim_{j\to+\infty} \langle h_j, \varphi \rangle &= \lim_{j\to+\infty} \int_{\mathbb{R}} h_j(x)\varphi(x)\,dx \\ &= \lim_{j\to+\infty} -\frac{1}{j} \int_{\mathbb{R}} \cos(jx)\varphi'(x)\,dx = 0. \end{aligned}$$

It is clear that this sequence does not converge pointwise.

(iii) *Finally, observe that even when a sequence of functions converges almost everywhere and converges in the sense of distributions, the two limits may be different. Indeed, we consider*

$$F_j(x) = \begin{cases} j & \text{if } |x| < \frac{1}{2j} \\ \\ 0 & \text{if } |x| \geq \frac{1}{2j}. \end{cases}$$

We have $\displaystyle\int_{\mathbb{R}} F_j(x)\,dx = 1$ and F_j converges to zero almost everywhere in $\mathbb{R}$. On the other hand, for all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned} \langle F_j, \phi \rangle &= \int_{-\frac{1}{2j}}^{\frac{1}{2j}} j\varphi(x)\,dx \\ &= \varphi(0) + \int_{-\frac{1}{2j}}^{\frac{1}{2j}} j\left(\varphi(x) - \varphi(0)\right)\,dx. \end{aligned}$$

Since

$$|\varphi(x)-\varphi(0)| = \left|\int_0^x \varphi'(s)ds\right| \le |x| \sup_{x\in\mathbb{R}} |\varphi'(x)|,$$

it follows that

$$\begin{aligned}\left|\int_{-\frac{1}{2j}}^{\frac{1}{2j}} j(\varphi(x)-\varphi(0))\,dx\right| &\le \sup_{x\in\mathbb{R}}|\varphi'(x)| \int_{-\frac{1}{2j}}^{\frac{1}{2j}} j|x|\,dx \\ &= \frac{1}{4j}\sup_{x\in\mathbb{R}}|\varphi'(x)| \to 0 \quad \text{as } j\to+\infty,\end{aligned}$$

hence

$$\lim_{j\to\infty}\langle F_j,\varphi\rangle = \varphi(0) = \langle\delta_0,\varphi\rangle.$$

In other words, $F_j \to \delta_0$ in the sense of distributions.

First, we shall use a tilde to denote the reflection of a function in the origin $\tilde{\varphi}(x) = \varphi(-x)$.

Theorem 2.3. *Let h be a positive function and integrable on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} h(x)\,dx = 1$. Let $h_\varepsilon(x) = \varepsilon^{-n}h\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$. Then $h_\varepsilon \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0$.*

Proof. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ by Theorem 1.6 we have

$$\begin{aligned}\lim_{\epsilon\to0}\langle h_\epsilon,\varphi\rangle &= \lim_{\epsilon\to0}\int_{\mathbb{R}^n} h_\epsilon(x)\varphi(x)\,dx \\ &= \lim_{\epsilon\to0}\int_{\mathbb{R}^n} h_\epsilon(x)\tilde{\varphi}(0-x)\,dx \\ &= \lim_{\epsilon\to0} h_\epsilon * \tilde{\varphi}(0) \\ &= \tilde{\varphi}(0) \\ &= \varphi(0) \\ &= \langle\delta_0,\varphi\rangle.\end{aligned}$$

Thus $h_\varepsilon \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0$. □

Example 2.5. *Let us consider the equality $\int_{\mathbb{R}} \frac{1}{1+x^2}\,dx = \pi$ and define the function $h(x) = \frac{1}{\pi(1+x^2)}$. We will use the previous theorem to construct the sequence of functions*

$$h_\varepsilon(x) = \frac{1}{\varepsilon}\left(\frac{1}{\pi\left(1+\left(\frac{x}{\varepsilon}\right)^2\right)}\right) = \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}.$$

This sequence of functions $h_\varepsilon(x)$ converges to the distribution δ_0 (the Dirac distribution at the point 0) in the sense of distributions as $\varepsilon \to 0$.

Theorem 2.4. *Suppose* $(T_j) \subset \mathcal{D}'(\Omega)$ *and define*

$$\langle T, \varphi \rangle = \lim_{j \to \infty} \langle T_j, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Then T *is a disribution.*

Remark 2.4.
1. *In the statement of the Theorem* 2.3*, we can replace* $\mathbb{R}^n$ *with any open set* Ω *containing the origin.*
2. *The previous result in Theorem* 2.4 *is quite strong and unusual in the classical framework. To convince ourselves of this, consider, for example, a sequence of continuous functions* $(f_j)_j$ *such that, for every* x*,* $f_j(x)$ *converges. It is clear that the limit is not necessarily continuous.*

2.3 Operations on distributions

Several operations defined for functions can be extended in the same way for distributions.

2.3.1 Addition and multiplication by a constant

Addition: The sum of two distributions T and S in $\mathcal{D}'(\Omega)$ is defined by

$$\langle T + S, \varphi \rangle = \langle T, \varphi \rangle + \langle S, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

$T + S$ is indeed a distribution because it defines a linear and continuous functional on $\mathcal{D}(\Omega)$.

Multiplication by a constant: The multiplication of a distribution by a constant $\lambda \in \mathbb{K}$ is defined by

$$\langle \lambda T, \varphi \rangle = \langle T, \lambda \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

It is obvious that this is a distribution (ie. $\lambda T \in \mathcal{D}'(\Omega)$).

Remark 2.5. *It is clear that, from the above, we have just provided the space* $\mathcal{D}'(\Omega)$ *with a vector space structure on* $\mathbb{K}$*.*

2.3.2 Translation

If f is locally integrable in $\mathbb{R}^n$ and $a \in \mathbb{R}^n$, then the translation by a of f is the function $\tau_a f$ defined by $\tau_a f(x) = f(x - a)$. The regular distribution associated $\tau_a f$ verifies for all

$\varphi \in \mathcal{D}(\mathbb{R})$

$$\begin{aligned}\langle \tau_a T_f, \varphi\rangle &= \langle T_{\tau_a f}, \varphi\rangle = \int_{R^n} f(x-a)\varphi(x)\,dx \\ &= \int_{R^n} f(y)\varphi(y+a)\,dy = \langle T_f, \tau_{-a}\varphi\rangle.\end{aligned}$$

This can be extended to distributions in the following way

Definition 2.5. *The translation of T by $a \in \mathbb{R}^n$ is the distribution $\tau_a T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{K}$ defined by*

$$\tau_a T(\varphi) = \langle \tau_a T, \varphi\rangle = \langle T, \tau_{-a}\varphi\rangle\,.$$

Example 2.6. *The translation of δ_{x_0} by $a \in \mathbb{R}^n$ is δ_{x_0+a} (ie. $\tau_a\delta_{x_0} = \delta_{x_0+a}$). Indeed, we have*

$$\langle \tau_a\delta_{x_0}, \varphi\rangle = \langle \delta_{x_0}, \tau_{-a}\varphi\rangle = \tau_{-a}\varphi(x_0) = \varphi(x_0+a) = \langle \delta_{x_0+a}, \varphi\rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Remark 2.6. *It is possible to define translation on $\mathcal{D}'(\Omega)$. For $a \in \mathbb{R}^n$, consider the open set*

$$\Omega + a = \{x + a, \quad x \in \Omega\}.$$

For any $\varphi \in \mathcal{D}(\Omega + a)$, observe that $\tau_{-a}\varphi$ in $\mathcal{D}(\Omega)$. For $T \in \mathcal{D}'(\Omega)$, we can define $\tau_a T \in \mathcal{D}'(\Omega + a)$ by

$$\langle \tau_a T, \varphi\rangle = \langle T, \tau_{-a}\varphi\rangle\,, \quad \forall \varphi \in \mathcal{D}(\Omega + a).$$

2.3.3 Multiplication by a $C^\infty-$functions

Theorem 2.5. (and definition) *Let $\psi \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. The linear form ψT on $\mathcal{D}(\Omega)$ defined by*

$$\langle \psi T, \varphi\rangle = \langle T, \psi\varphi\rangle, \quad \forall \varphi \in \mathcal{D}(\Omega), \tag{2.5}$$

is a distribution.

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$ and $\lambda_1, \lambda_2 \in \mathbb{K}$, we have

$$\begin{aligned}\langle \psi T, \lambda_1\varphi_1 + \lambda_2\varphi_2\rangle &= \langle T, \psi(\lambda_1\varphi_1 + \lambda_2\varphi_2)\rangle \\ &= \lambda_1\langle T, \psi\varphi_1\rangle + \lambda_2\langle T, \psi\varphi_2\rangle \\ &= \lambda_1\langle \psi T, \varphi_1\rangle + \lambda_2\langle \psi T, \varphi_2\rangle.\end{aligned}$$

then ψT is linear.
Continuitry: we have $\varphi_j \to \varphi$ as $j \to +\infty$ in $\mathcal{D}(\Omega)$ then $\psi\varphi_j \to \psi\varphi$ in $\mathcal{D}(\Omega)$ as $j \to +\infty$.

Since $T \in \mathcal{D}'(\Omega)$ and from the formula (2.5) we obtain

$$\lim_{j\to+\infty} \langle \psi T, \varphi_j \rangle = \lim_{j\to+\infty} \langle T, \psi\varphi_j \rangle = \langle T, \psi\varphi \rangle = \langle \psi T, \varphi \rangle.$$

□

Example 2.7. 1. *If $\psi \in C^\infty(\Omega)$ and if $f \in L^1_{loc}(\Omega)$, we have $\psi T_f = T_{\psi f}$.*

2. *If $\psi \in C^\infty(\Omega)$ and $x_0 \in \Omega$ we have $\psi\delta_{x_0} = \psi(x_0)\delta_{x_0}$. In particular on $\mathbb{R}$, $x\delta_0 = 0$.*

Proposition 2.1. *For $T, S \in \mathcal{D}'(\Omega)$ and $\psi_1, \psi_2 \in C^\infty(\Omega)$ we have*

$$(\psi_1 + \psi_2)T = \psi_1 T + \psi_2 T, \quad \psi_1\psi_2 T = \psi_1(\psi_2 T), \quad \psi_1(T+S) = \psi_1 T + \psi_1 S.$$

Proposition 2.2. *Let $T \in \mathcal{D}'(\mathbb{R})$, we have equivalence between*

1. $xT = 0$.
2. $T = c\delta_0$, *where* $c \in \mathbb{C}$.

Proof. We have already seen in the above example that 2 implies 1. Let us prove the converse. For all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle xT, \varphi \rangle = \langle T, x\varphi \rangle = 0.$$

Thus, T vanishes on all functions of the form $x\varphi$ where $\varphi \in \mathcal{D}(\mathbb{R})$. Equivalently

$$\psi = x\varphi, \quad \varphi \in \mathcal{D}(\mathbb{R}) \iff \psi \in \mathcal{D}(\mathbb{R}) \text{ and } \psi(0) = 0. \tag{2.6}$$

The implication $\Rightarrow$ is evident. Conversely, if $\psi \in \mathcal{D}(\mathbb{R})$, $\mathrm{supp}(\psi) \subset [-a, a]$, and $\psi(0) = 0$, Taylor's formula allows us to write

$$\psi(x) = \psi(0) + x\int_0^1 \psi'(tx)dt = x\varphi(x).$$

The function φ is C^∞ and if $|x| > a$, so we have $\psi(x) = 0$ hence $\varphi(x) = 0$. ie. $\varphi \in \mathcal{D}(\mathbb{R})$. From (2.6), we deduce that T vanishes on all functions $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi(0) = 0$.
Fix a function $\chi \in \mathcal{D}(\mathbb{R})$, $\chi = 1$ for $|x| \leq 1$. Let $\phi \in \mathcal{D}(\mathbb{R})$ be arbitrary. We set $\psi = \phi - \phi(0)\chi$. Then $\psi(0) = 0$, so $\langle T, \psi \rangle = 0$. Consequently,

$$\langle T, \phi \rangle = \phi(0)\langle T, \chi \rangle = c\phi(0),$$

with $c = \langle T, \chi \rangle$. This proves 2. □

2.3.4 Differentiation in the sense of distributions

One of the strengths of distributions is that we can naturally define their derivatives. This notion generalizes the usual derivative for those distributions that come from smooth functions. The idea behind the definition below comes from integration by parts: suppose a distribution f is given by a smooth function, and let φ be a test function. Then

$$\langle f', \varphi \rangle = \int_\Omega f'(x)\varphi(x)\,dx = -\int_\Omega f(x)\varphi'(x)\,dx, \tag{2.7}$$

by integration by parts (the boundary terms vanish).

Definition 2.6. *Let $T \in \mathcal{D}'(\Omega)$ and $1 \le i \le n$. We define the partial derivative $\dfrac{\partial T}{\partial x_i} = \partial_i T$ of T by*

$$\langle \partial_i T, \varphi \rangle = -\langle T, \partial_i \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{2.8}$$

If $n = 1$, we denote the derivative by T' instead of $\partial_1 T$.

Remark 2.7. *The definition of $\partial_i T : \mathcal{D}(\Omega) \to \mathbb{K}$ is clearly linear. It is also sequentially continuous, since $\varphi_j \to \varphi$ (as $j \to +\infty$) in $\mathcal{D}(\Omega)$ implies $\partial_i \varphi_j \to \partial_i \varphi$ in $\mathcal{D}(\Omega)$. Hence*

$$\langle \partial_i T, \varphi_j \rangle = -\langle T, \partial_i \varphi_j \rangle \xrightarrow{j \to +\infty} -\langle T, \partial_i \varphi \rangle = \langle \partial_i T, \varphi \rangle.$$

Thus, we obtain a map $\partial_i : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$.

Example 2.8. 1. *By the integration by parts formula (2.7), if f is a smooth function, then its usual derivative and distributional derivative agree.*

2. *We write the derivative of δ_0 with respect to variable x_i where $i \in \{1, \cdots, n\}$, we get*

$$\left\langle \frac{\partial \delta_0}{\partial x_i}, \varphi \right\rangle = -\left\langle \delta_0, \frac{\partial \varphi}{\partial x_i} \right\rangle = -\frac{\partial \varphi}{\partial x_i}(0), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

In particulier if $n = 1$, the derivative of δ_0 is the distribution given by

$$\langle \delta_0', \varphi \rangle = -\varphi'(0).$$

3. **Derivative of the Heaviside function***: Let H be the Heaviside function on $\mathbb{R}$. Then*

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_0^{+\infty} \varphi'(x)\,dx = -\big[\varphi(x)\big]_0^{+\infty} = \varphi(0).$$

Thus, $H' = \delta_0$, where δ_0 is the Dirac delta distribution.

Since the classical derivative of the Heaviside function is zero almost everywhere ($H'(x) = 0$ a.e), this example shows that a function's distributional derivative does not necessarily coincide with its pointwise derivative defined almost everywhere. Therefore, for the distributional derivative to be guaranteed to coincide with the pointwise (classical) derivative, the function must be differentiable at every point in its domain.

4. *Consider the discontinuous but locally integrable function*

$$f(x) = \begin{cases} x^2 + x & \text{if } x < 1, \\ e^{-5x} & \text{if } x > 1. \end{cases}$$

Then

$$\langle T_f', \varphi \rangle = -\langle T_f, \varphi' \rangle = -\int_{-\infty}^{1} (x^2 + x)\varphi'(x)\,dx - \int_{1}^{\infty} e^{-5x}\varphi'(x)\,dx \quad \forall \varphi \in \mathcal{D}(\mathbb{R}),$$

and

$$\begin{aligned} -\int_{-\infty}^{1} (x^2 + x)\varphi'(x)\,dx &= -(x^2 + x)\varphi(x)\big|_{x=-\infty}^{x=1} + \int_{-\infty}^{1} (2x + 1)\varphi(x)\,dx \\ &= -2\varphi(1) + \int_{-\infty}^{1} (2x + 1)\varphi(x)\,dx \end{aligned}$$

$$\begin{aligned} -\int_{1}^{\infty} e^{-5x}\varphi'(x)\,dx &= -e^{-5x}\varphi(x)\big|_{x=1}^{x=\infty} + \int_{1}^{\infty} -5e^{-5x}\varphi(x)\,dx \\ &= e^{-5}\varphi(1) - \int_{1}^{\infty} 5e^{-5x}\varphi(x)\,dx. \end{aligned}$$

It follows that

$$\begin{aligned} \langle T_f', \varphi \rangle &= -2\varphi(1) + \int_{-\infty}^{1} (2x + 1)\varphi(x)\,dx + e^{-5}\varphi(1) - \int_{1}^{\infty} 5e^{-5x}\varphi(x)\,dx \\ &= [e^{-5} - 2]\varphi(1) + \int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx, \end{aligned}$$

where

$$f'(x) = \begin{cases} 2x + 1 & \text{if } x < 1, \\ -5e^{-5x} & \text{if } x > 1. \end{cases}$$

5. *The derivative of* $\ln|x|$ *in the sense of distributions exists and is given by* $PV\left(\frac{1}{x}\right)$. *Indeed, let* $\varphi \in \mathcal{D}(\mathbb{R})$ *and* $a > 0$ *such that* $\operatorname{supp}(\varphi) \subset [-\mathrm{a}, \mathrm{a}]$. *Let us denote* T *the*

distribution associated with the function $\ln|x|$. *Then*

$$\begin{aligned}\langle T', \varphi\rangle = -\langle T, \varphi'\rangle &= -\int_{-a}^{a} \varphi'(x)\ln|x|dx \\ &= -\lim_{\varepsilon\to 0}\left(\int_{-a}^{-\varepsilon} \varphi'(x)\ln(-x)dx + \int_{\varepsilon}^{a} \varphi'(x)\ln(x)dx\right).\end{aligned}$$

But

$$\int_{\varepsilon}^{a} \varphi'(x)\ln(x)dx = [\varphi(x)\ln(x)]_{\varepsilon}^{a} - \int_{\varepsilon}^{a} \frac{\varphi(x)}{x}dx = -\varphi(\varepsilon)\ln(\varepsilon) - \int_{\varepsilon}^{a} \frac{\varphi(x)}{x}dx,$$

and

$$\int_{-a}^{-\varepsilon} \varphi'(x)\ln(-x)dx = [\varphi(x)\ln(-x)]_{-a}^{-\varepsilon} - \int_{-a}^{-\varepsilon} \frac{\varphi(x)}{x}dx = \varphi(-\varepsilon)\ln(\varepsilon) - \int_{-a}^{-\varepsilon} \frac{\varphi(x)}{x}dx.$$

Now

$$|\varphi(-\varepsilon) - \varphi(\varepsilon))| \leq 2\varepsilon \sup_{\mathbb{R}} |\varphi'(x)|,$$

we conclude that

$$\langle T', \varphi\rangle = \lim_{\varepsilon\to 0}\left(\int_{-a}^{-\varepsilon} \frac{\varphi(x)}{x}dx + \int_{\varepsilon}^{a} \frac{\varphi(x)}{x}dx\right) = \left\langle PV\left(\frac{1}{x}\right), \varphi\right\rangle.$$

Definition 2.7. *Let* $T \in \mathcal{D}'(\Omega)$. *For any multi-index* α, *the distributional derivative* $\partial^\alpha T \in \mathcal{D}'(\Omega)$ *is defined as*

$$\langle \partial^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, \partial^\alpha \varphi\rangle, \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{2.9}$$

Note that the right-hand side of (2.9) is well defined for any multi-index α and which represents a continuous linear functional in $\mathcal{D}(\Omega)$ i.e. a distribution on Ω.

Example 2.9. *If* $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, *then*

$$\langle \partial^\alpha \delta_0, \varphi\rangle = (-1)^{|\alpha|}\partial^\alpha \varphi(0), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Proposition 2.3. *Differentiation in* $\mathcal{D}'(\Omega)$ *is an operation that is*

(i) *linear, i.e.,*

$$\partial^\alpha(aT + bS) = a\partial^\alpha T + b\partial^\alpha S, \quad \forall T, S \in \mathcal{D}'(\Omega) \quad , \forall a, b \in \mathbb{K}, \text{ and } \forall \alpha \in \mathbb{N}^n,$$

(ii) *continuous, i.e., if* $(T_j)_{j\in\mathbb{N}}$ *converges to* T *in* $\mathcal{D}'(\Omega)$, *then* $(\partial^\alpha T_j)_{j\in\mathbb{N}}$ *converges to* $\partial^\alpha T$ *in* $\mathcal{D}'(\Omega)$, *for all* $\alpha \in \mathbb{N}^n$.

Proof.

(i) Let $T, S \in \mathcal{D}'(\Omega)$, $a, b \in \mathbb{K}$, $\alpha \in \mathbb{N}^n$, and $\varphi \in \mathcal{D}(\Omega)$. Then

$$\begin{aligned}\langle \partial^\alpha (aT + bS), \phi \rangle = (-1)^{|\alpha|} \langle aT + bS, \partial^\alpha \phi \rangle &= (-1)^{|\alpha|} \left(a\langle T, \partial^\alpha \varphi \rangle + b\langle S, \partial^\alpha \varphi \rangle \right) \\ &= a(-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle + b(-1)^{|\alpha|} \langle S, \partial^\alpha \varphi \rangle \\ &= \langle a\partial^\alpha T + b\partial^\alpha S, \varphi \rangle.\end{aligned}$$

(ii) By definition of the convergence of $(T_j)_{j\in\mathbb{N}}$ to T in $\mathcal{D}'(\Omega)$, we have

$$\lim_{j\to+\infty} \langle T_j, \varphi \rangle = \langle T, \varphi \rangle, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

Note that if $\varphi \in \mathcal{D}(\Omega)$ then $\partial^\alpha \varphi \in \mathcal{D}(\Omega)$ for all $\alpha \in \mathbb{N}^n$. Therefore

$$\lim_{j\to+\infty} \langle T_j, \partial^\alpha \varphi \rangle = \langle T, \partial^\alpha \varphi \rangle, \qquad \forall \varphi \in \mathcal{D}(\Omega),$$

or equivalently

$$\lim_{j\to+\infty} \langle \partial^\alpha T_j, \varphi \rangle = \langle \partial^\alpha T, \varphi \rangle, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

□

Theorem 2.6. *Let T be a distribution. Then, for every $i, j \in \{1, \cdots, n\}$,*

$$\frac{\partial^2 T}{\partial x_i \partial x_j} = \frac{\partial^2 T}{\partial x_j \partial x_i}.$$

Proof. It is an immediate consequence of the fact that test functions are infinitely smooth, thus Schwarz's theorem holds for them. □

Jump Formula

Proposition 2.4. *Let f be a locally integrable function on $\mathbb{R}$ such that $\frac{df}{dx}$ exists in the classical sense except at a finite number of points $a_1 < a_2 < \cdots < a_n$, where $f(a_i^+)$ and $f(a_i^-)$ exist. Then, the derivative in the sense of distributions is given by*

$$T_f' = T_{f'} + \sum_{i=1}^{n} \big(f(a_i^+) - f(a_i^-) \big) \delta_{a_i}, \tag{2.10}$$

where $f(a_i^+), f(a_i^-)$ are the right and left limits of f at a_i respectively.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$. We have

$$\begin{aligned}\langle T'_f, \varphi\rangle &= -\langle T_f, \varphi'\rangle \\ &= -\int_{\mathbb{R}} f(x)\varphi'(x)\,dx \\ &= -\int_{-\infty}^{a_1} f(x)\varphi'(x)\,dx - \sum_{i=1}^{n-1}\int_{a_i}^{a_{i+1}} f(x)\varphi'(x)\,dx - \int_{a_n}^{+\infty} f(x)\varphi'(x)\,dx.\end{aligned}$$

Using integration by parts, we obtain

$$\langle T'_f, \varphi\rangle = \sum_{i=1}^{n}\big(f(a_i^+) - f(a_i^-)\big)\varphi(a_i) + \int_{\mathbb{R}} f'(x)\varphi(x)\,dx$$

$$= \left\langle \sum_{i=1}^{n}\big(f(a_i^+) - f(a_i^-)\big)\delta_{a_i} + T_{f'}, \varphi \right\rangle.$$

. □

Example 2.10. *We take the example* 2.8. *By using* (2.10) *we obtain*

$$\langle T'_f, \varphi\rangle = \langle T_{f'}, \varphi\rangle + \langle [f(1^+) - f(1^-)]\delta_1, \varphi\rangle,$$

with $f(1^+) = \lim\limits_{\substack{x\to 1\\ >}} f(x) = e^{-5}$ *and* $f(1^-) = \lim\limits_{\substack{x\to 1\\ <}} f(x) = 2$. *Then*

$$\langle T'_f, \varphi\rangle = \int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx + [e^{-5} - 2]\varphi(1),$$

where

$$f'(x) = \begin{cases} 2x+1 & \text{if } x < 1, \\ -5e^{-5x} & \text{if } x > 1. \end{cases}$$

Remark 2.8. *The result also holds in the case of a countable number of discontinuity points* $(a_i)_{i\geq 1}$. *In this case, we have*

$$T'_f = T_{f'} + \sum_{i=1}^{+\infty}\big(f(a_i^+) - f(a_i^-)\big)\delta_{a_i}.$$

The derivation of a product ψT

Theorem 2.7. *Let* $\psi \in C^\infty(\Omega)$, $T \in \mathcal{D}'(\Omega)$ *and* $1 \leq i \leq n$. *Then we have*

$$\partial_i(\psi T) = (\partial_i\psi)T + \psi\partial_i T. \tag{2.11}$$

Proof. For all $\varphi \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned}
\langle \partial_i(\psi T), \varphi \rangle &= -\langle \psi T, \partial_i \varphi \rangle \\
&= -\langle T, \psi \partial_i \varphi \rangle \\
&= -\langle T, \partial_i(\psi\varphi) - \varphi \partial_i \psi \rangle \\
&= -\langle T, \partial_i(\psi\varphi) \rangle + \langle T, \varphi \partial_i \psi \rangle \\
&= \langle \partial_i T, \psi\varphi \rangle + \langle T, \varphi \partial_i \psi \rangle \\
&= \langle \psi \partial_i T, \varphi \rangle + \langle (\partial_i \psi) T, \varphi \rangle \\
&= \langle \psi \partial_i T + (\partial_i \psi) T \ , \varphi \rangle.
\end{aligned}$$

□

Example 2.11. *By* (2.11)*, we have*

$$\partial_i(\psi\delta_0) = (\partial_i\psi)\delta_0 + \psi\partial_i\delta_0 \implies \psi(0)\partial_i\delta_0 = (\partial_i\psi)(0)\delta_0 + \psi\partial_i\delta_0,$$

(because $\psi\delta_0 = \psi(0)\delta_0$). Hence,

$$\psi\partial_i\delta_0 = \psi(0)\partial_i\delta_0 - (\partial_i\psi)(0)\delta_0 \quad in \quad \mathcal{D}'(\mathbb{R}^n).$$

For $n = 1$, we have

$$\psi\delta_0' = \psi(0)\delta_0' - \psi'(0)\delta_0 \quad in \quad \mathcal{D}'(\mathbb{R}).$$

In particulier,

$$x\delta_0' + \delta_0 = 0, \quad x^m\delta_0' = 0, \quad \forall m > 1.$$

Theorem 2.8. (Leibniz Rule) *Let $T \in \mathcal{D}'(\Omega)$ and $\psi \in C^\infty(\Omega)$. Then for any $\alpha \in \mathbb{N}^n$, the following formula holds in $\mathcal{D}'(\Omega)$*

$$\partial^\alpha(\psi T) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\partial^\beta \psi) \partial^{\alpha-\beta} T. \tag{2.12}$$

2.4 Homogeneous distributions

Definition 2.8. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $p \in \mathbb{R}$. We say that T is homogeneous of degree p if for all $\lambda > 0$, we have*

$$\langle T, \varphi_\lambda \rangle = \lambda^{-n-p} \langle T, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n),$$

where φ_λ is the function $x \mapsto \varphi(\lambda x)$.

Example 2.12. 1. *The delta Dirac δ_0 is homogenous distribution. Indeed, we have*

$$\langle \delta_0, \varphi_\lambda \rangle = \varphi(0) = \langle \delta_0, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

This shows that δ_0 is homogeneous of degree $-n$.

2. *The principal value is homogenous distribution. Indeed,*

$$\begin{aligned} \langle PV\big(\frac{1}{x}\big), \varphi_\lambda \rangle &= \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(\lambda x)}{x}\, dx \\ &= \lim_{\varepsilon \to 0} \int_{|\lambda u| \geq \varepsilon} \frac{\varphi(u)}{u}\, du \\ &= \langle PV\big(\frac{1}{x}\big), \varphi \rangle. \end{aligned}$$

We deduce that $PV\big(\frac{1}{x}\big)$ is homogeneous of degree -1.

Remark 2.9. (i) *If $T = T_f$ is given by a continuous function, this definition coincides with the classical notion of a homogeneous function of degree p, i.e., $f(\lambda x) = \lambda^p f(x)$ for $x \in \mathbb{R}^n$ and $\lambda > 0$. Indeed, if f is homogeneous in this sense, then for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have*

$$\begin{aligned} \langle T_f, \varphi_\lambda \rangle = \langle f, \varphi_\lambda \rangle &= \int_{\mathbb{R}^n} f(x)\, \varphi(\lambda x)\, dx \\ &= \lambda^{-n} \int_{\mathbb{R}^n} f\left(\frac{x}{\lambda}\right) \varphi(x)\, dx \\ &= \lambda^{-n-p} \int_{\mathbb{R}^n} f(x) \varphi(x)\, dx \\ &= \lambda^{-n-p} \langle f, \varphi \rangle. \end{aligned}$$

Conversely, the reverse implication also holds.

(ii) *If T is a homogeneous distribution of degree $p \in \mathbb{R}$, then $\frac{\partial T}{\partial x_i}$ is homogeneous of degree $p-1$. Indeed, the distribution $S = \frac{\partial T}{\partial x_i}$ is homogeneous of degree $p-1$ if*

$$\langle S, \phi_\lambda \rangle = \lambda^{-n-(p-1)} \langle S, \phi \rangle,$$

where $\phi_\lambda(x) = \phi(\lambda x)$, and n is the dimension of the space.
Let T be a homogeneous distribution of degree p, i.e., for any test function φ and $\lambda > 0$

$$\langle T, \varphi_\lambda \rangle = \lambda^{-n-p} \langle T, \varphi \rangle.$$

By definition of the derivative in the sense of distributions

$$\langle S, \phi \rangle = - \left\langle T, \frac{\partial \phi}{\partial x_i} \right\rangle .$$

Let $\phi_\lambda(x) = \phi(\lambda x)$, *we have*

$$\langle S, \phi_\lambda \rangle = - \left\langle T, \frac{\partial \phi_\lambda}{\partial x_i} \right\rangle .$$

Now, compute the derivative

$$\frac{\partial \phi_\lambda}{\partial x_i}(x) = \frac{\partial}{\partial x_i} \left[\phi(\lambda x) \right] = \lambda \frac{\partial \phi(\lambda x)}{\partial x_i} = \lambda \left(\frac{\partial \phi}{\partial x_i} \right)_\lambda (x).$$

Thus,

$$\langle S, \phi_\lambda \rangle = -\lambda \left\langle T, \left(\frac{\partial \phi}{\partial x_i} \right)_\lambda \right\rangle .$$

Since T *is homogeneous of degree* p, *we have*

$$\langle T, \left(\frac{\partial \phi}{\partial x_i} \right)_\lambda \rangle = \lambda^{-n-p} \left\langle T, \frac{\partial \phi}{\partial x_i} \right\rangle .$$

So

$$\langle S, \phi_\lambda \rangle = -\lambda^{-n-p+1} \left\langle T, \frac{\partial \phi}{\partial x_i} \right\rangle .$$

By the definition of S

$$\langle S, \phi \rangle = - \left\langle T, \frac{\partial \phi}{\partial x_i} \right\rangle .$$

Therefore,

$$\langle S, \phi_\lambda \rangle = \lambda^{-n-p+1} \langle S, \phi \rangle = \lambda^{-n-(p-1)} \langle S, \phi \rangle.$$

Hence $S = \frac{\partial T}{\partial x_i}$ *is homogeneous of degree* $p - 1$.

(iii) *If* ψ *is a homogeneous function of class* C^∞ *of degree* p_1 *and* T *a homogeneous distribution of degree* p_2 *then* ψT *is homogeneous of degree* $p_1 + p_2$. *Indeed, let* ψ *be a homogeneous function of degree* p_1, *i.e.* $\psi(\lambda x) = \lambda^{p_1} \psi(x)$ *for* $\lambda > 0$ *and* $x \in \Omega$.
By definition, for any test function φ *and* $\lambda > 0$ *we have*

$$\langle \psi T, \varphi_\lambda \rangle = \langle T, \psi \varphi_\lambda \rangle,$$

where

$$\psi \varphi_\lambda(x) = \psi(x) \varphi(\lambda x) = \frac{1}{\lambda^{p_1}} \psi(\lambda x) \varphi(\lambda x) = \lambda^{-p_1} \psi_\lambda(x) \varphi_\lambda(x).$$

Then

$$\langle \psi T, \varphi_\lambda \rangle = \langle T, \lambda^{-p_1} \psi_\lambda \varphi_\lambda \rangle,$$

since T is homogeneous distribution of degree p_2 we get

$$\begin{aligned} \langle \psi T, \varphi_\lambda \rangle &= \lambda^{-p_1} \lambda^{-n-p_2} \langle T, \psi\varphi \rangle \\ &= \lambda^{-n-p_1-p_2} \langle T, \psi\varphi \rangle \\ &= \lambda^{-n-(p_1+p_2)} \langle \psi T, \varphi \rangle. \end{aligned}$$

Theorem 2.9. *A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree $p \in \mathbb{R}$ if and only if*

$$\sum_{i=1}^{n} x_i \frac{\partial T}{\partial x_i} = pT.$$

2.5 Even and odd distributions

If T_f is a regular distribution defined by a locally integrable function f, we have

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx = \int_{\mathbb{R}^n} f(-x)\varphi(-x)\, dx.$$

Let us define $\tilde{\varphi}(x) = \varphi(-x)$. If f is even, it follows that

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\tilde{\varphi}(x)\, dx = \langle T_f, \tilde{\varphi} \rangle,$$

and if f is odd, we have

$$\langle T_f, \varphi \rangle = -\int_{\mathbb{R}^n} f(x)\tilde{\varphi}(x)\, dx = -\langle T_f, \tilde{\varphi} \rangle.$$

The parity of a distribution is defined as follows

Definition 2.9. *Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and let $\tilde{\varphi}(x) = \varphi(-x)$. A distribution T is said to be even if*

$$\langle T, \varphi \rangle = \langle T, \tilde{\varphi} \rangle,$$

and it is said to be odd if

$$\langle T, \varphi \rangle = -\langle T, \tilde{\varphi} \rangle.$$

Remark 2.10. *Any distribution can be decomposed as the sum of an even and an odd distribution, which are called respectively the even part (or the even component) and the odd part*

(or the odd component) of the distribution, and are defined by

$$\langle T_{even}, \varphi \rangle = \frac{\langle T, \varphi \rangle + \langle T, \tilde{\varphi} \rangle}{2}, \quad \langle T_{odd}, \varphi \rangle = \frac{\langle T, \varphi \rangle - \langle T, \tilde{\varphi} \rangle}{2}.$$

It is straightforward to verify that T_{even} is even, T_{odd} is odd, and $T = T_{even} + T_{odd}$. This decomposition is unique.

Example 2.13. 1. *The Dirac distribution δ_0 is even because $\tilde{\varphi}(0) = \varphi(0)$.*

2. *The distribution δ_{x_0}, for $x_0 \neq 0$, can be decomposed into the sum of an even distribution and an odd distribution*
$$\delta_{x_0} = \frac{\delta_{x_0} + \delta_{-x_0}}{2} + \frac{\delta_{x_0} - \delta_{-x_0}}{2}.$$

3. *The principal value is odd distribution. Indeed, we have*

$$\begin{aligned}
\left\langle PV\left(\frac{1}{x}\right), \varphi \right\rangle &= \int_0^\infty \frac{\varphi(x) - \varphi(-x)}{x}\, dx \\
&= \int_0^\infty \frac{\tilde{\varphi}(-x) - \tilde{\varphi}(x)}{x}\, dx \\
&= -\int_0^\infty \frac{\tilde{\varphi}(x) - \tilde{\varphi}(-x)}{x}\, dx \\
&= -\left\langle PV\left(\frac{1}{x}\right), \tilde{\varphi} \right\rangle.
\end{aligned}$$

Exercises

Exercise 2.1. *Are the maps T_1, T_2, T_3 and $Fp\left(\frac{1}{x^2}\right)$ (finite part) defined for all $\varphi \in \mathcal{D}(\mathbb{R})$ by*

$$\langle T_1, \varphi\rangle = 1 + \int_0^1 \varphi(t), dt, \quad \langle T_2, \varphi\rangle = \varphi(0) \int_{-\infty}^{+\infty} \varphi(t), dt,$$

$$\langle T_3, \varphi\rangle = \sum_{k\geq 1} \frac{1}{k}\left(\varphi\left(\frac{1}{k}\right) - \varphi(0)\right), \quad \langle T_4, \varphi\rangle = \sum_{\kappa=0}^{n} \varphi^{(\kappa)}(0),$$

and

$$\left\langle \mathrm{Fp}\left(\frac{1}{x^2}\right), \varphi \right\rangle = \lim_{\varepsilon\to 0^+} \left[\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2}\, dx - 2\frac{\varphi(0)}{\varepsilon}\right],$$

distributions?

Exercise 2.2. *Let $PV\left(\frac{1}{x}\right)$ be the form defined by*

$$\langle VP\left(\frac{1}{x}\right), \varphi\rangle = \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx = \int_0^{+\infty} \frac{\varphi(x) - \varphi(-x)}{x}\, dx, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

Show that $PV\left(\frac{1}{x}\right) \in \mathcal{D}'(\mathbb{R})$ and it is of order 1.

Exercise 2.3.

1. *For all $j \geq 1$. Calculate the limits of $(T_j)_{j\geq 1}$ defined by*

$$T_j = j(\delta_{\frac{1}{j}} - \delta_{\frac{-1}{j}}) \quad \textit{in} \quad \mathcal{D}'(\mathbb{R})$$

2. *Calculate the limits in $\mathcal{D}'(\mathbb{R})$ of the following sequence of functions*

$$h_j(x) = \frac{j}{1 + (jx)^2}.$$

3. *Let $f_j(x) = j^{\frac{3}{2}} x e^{-jx^2}$. Show that*

$$\langle f_j, \varphi\rangle = \frac{\sqrt{j}}{2} \int_{\mathbb{R}} e^{-jx^2} \varphi'(x)\, dx, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

Prove that f_j converges in $\mathcal{D}'(\mathbb{R})$. $\Big($Note that $\int_{\mathbb{R}} e^{-t^2}\,dt = \sqrt{\pi}\Big)$.

Exercise 2.4. (i) *Let $g \in C^1([a,b])$. Prove that*

$$\lim_{\gamma\to+\infty} \int_a^b \cos(\gamma x) g(x), dx = 0.$$

(ii) *Let T_j be the distribution associated with $\sin^2(jx)$. Study the convergence in $\mathcal{D}'(\mathbb{R})$ of the sequence (T_j).* **(Hint:** $\sin^2(x) = \frac{1-\cos(2x)}{2}$**)**.

(iii) *Let S_j be a distribution associated with $j\sin(jx)H$. Study the convergence in $\mathcal{D}'(\mathbb{R})$ of the sequence (S_j).*

Exercise 2.5. ***(Translations and differentiation).*** *Let $T \in \mathcal{D}'(\mathbb{R})$. Show that*

$$\frac{1}{a}(T - \tau_a T) \xrightarrow{\mathcal{D}'(\mathbb{R})} T' \quad \text{as } a \to 0.$$

Exercise 2.6. 1. *Calculate the derivative of the distribution $PV\big(\frac{1}{x}\big)$ in $\mathcal{D}'(\mathbb{R})$.*

2. *Let $f(x) = |x^2 - 4x + 3|$ be a function on $\mathbb{R}$. This function defines a distribution denoted by T_f). Calculate $(T_f)' - T_{f'}$ and $(T_f)'' - T_{f''}$* **(using the jump formula)**.

3. *Let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by $F(x) = \frac{1}{2}|x|$. - Compute F'' in the sense of distributions.*

4. *Let T be defined by*

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^2), \quad \langle T, \varphi\rangle = \int_{\mathbb{R}} \varphi(x,-x), dx.$$

- Show that $T \in \mathcal{D}'(\mathbb{R}^2)$ and calculate the formula $\frac{\partial T}{\partial x} - \frac{\partial T}{\partial y}$.

Exercise 2.7. *Let $\psi \in C^\infty(\mathbb{R})$ and let $F = \psi H$, where H is the Heaviside function. Show that for all $k \in \mathbb{N}^*$, we have*

$$F^{(k)} = \psi^{(k)} H + \sum_{i=0}^{k-1} \psi^{(k-1-i)}(0)\delta_0^{(i)} \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Exercise 2.8. ***Some differential equations.*** *Solve the following differential equations in the space of distributions*

$$T' + xT = \delta_0, \quad T' + T = H, \quad xT' + T = 0.$$

Exercise 2.9. *I) Prove that*

$$< T', \varphi_\lambda >= -\lambda < T, \psi_\lambda >, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}),$$

where $\varphi_\lambda(x) = \varphi(\lambda x)$ *and* $\psi_\lambda(x) = \varphi'(\lambda x)$. *Deduce that if* T *is homogeneous of degree* p *then* T *is homogeneous of degree* $p - 1$.
II) Prove that the distribution $\partial^\alpha \delta_0$ *is homogeneous of degree* $-n - |\alpha|$ *with* $\alpha \in \mathbb{N}^n$.

Solutions of the exercises

Exercise 2.1 The maps T_1 and T_2 do not define distributions because they are not linear. It is easy to verify that T_3 is linear.

Continuity: for every compact set $K \subset \mathbb{R}$ and every $\varphi \in \mathcal{D}(K)$, by the mean value theorem we have for each $k \geq 1$

$$\left|\frac{1}{k}\left(\varphi\left(\frac{1}{k}\right) - \varphi(0)\right)\right| \leq \frac{1}{k}\left(\frac{1}{k} - 0\right) \sup_{x\in K} |\varphi'(x)| \leq \frac{1}{k^2} \sup_{x\in K} |\varphi'(x)|.$$

Since the series $\sum_{k\geq 1} \frac{1}{k^2}$ converges, it follows that the series

$$\sum_{k\geq 1} \frac{1}{k}\left(\varphi\left(\frac{1}{k}\right) - \varphi(0)\right),$$

converges absolutely. Moreover, we have

$$|\langle T_3, \varphi\rangle| \leq C \sup_{x\in K} |\varphi'(x)|,$$

where $C = \displaystyle\sum_{k\geq 1} \frac{1}{k^2}$. This shows that T_3 is a distribution of order at most 1.

It is easy to verify that T_4 is linear. Let $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\varphi) \subset [-a, a]$, $a > 0$. We have

$$|\langle T_4, \varphi\rangle| \leq \sum_{\kappa=0}^{n} \left|\varphi^{(\kappa)}(0)\right| \leq \sum_{\kappa=0}^{n} \sup_{x\in[-a,a]} |\varphi^{(\kappa)}(x)|.$$

Then T_4 is continuous and therefore $T_4 \in \mathcal{D}'(\mathbb{R})$.

Now, we prove that $\operatorname{Fp}\left(\dfrac{1}{x^2}\right)$ is a distribution. Indeed, the linearity is clear.

Let $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\varphi) \subset [-a, a]$, where $a > 0$. Define

$$u_\varepsilon = \int_{-a}^{-\varepsilon} \frac{\varphi(x)}{x^2} dx + \int_{\varepsilon}^{a} \frac{\varphi(x)}{x^2} dx - 2\frac{\varphi(0)}{\varepsilon}.$$

Write the Taylor expansion of order 2 for φ around zero

$$\varphi(x) = \varphi(0) + x\varphi'(0) + \frac{x^2}{2}\varphi''(\xi_x), \quad \xi_x \in (0, x).$$
$$\varphi(x) = \varphi(0) + x\varphi'(0) + \frac{x^2}{2}\varphi''(\eta_x), \quad \eta_x \in (x, 0).$$

Therefore,

$$u_\varepsilon = \int_{-a}^{-\varepsilon} \frac{\varphi(0)}{x^2}dx + \int_{-a}^{-\varepsilon} \frac{\varphi'(0)}{x}dx + \frac{1}{2}\int_{-a}^{-\varepsilon} \varphi''(\eta_x)dx + \int_{\varepsilon}^{a} \frac{\varphi(0)}{x^2}dx + \int_{\varepsilon}^{a} \frac{\varphi'(0)}{x}dx + \frac{1}{2}\int_{\varepsilon}^{a} \varphi''(\xi_x)dx - 2\frac{\varphi(0)}{\varepsilon}.$$

Since $x \mapsto \frac{1}{x^2}$ is even

$$\int_{-a}^{-\varepsilon} \frac{\varphi(0)}{x^2}dx = \int_{\varepsilon}^{a} \frac{\varphi(0)}{x^2}dx.$$

Since $x \mapsto \frac{1}{x}$ is odd

$$\int_{-a}^{-\varepsilon} \frac{\varphi'(0)}{x}dx = -\int_{\varepsilon}^{a} \frac{\varphi'(0)}{x}dx.$$

So, we obtain

$$u_\varepsilon = 2\int_{\varepsilon}^{a} \frac{\varphi(0)}{x^2}dx + \frac{1}{2}\int_{-a}^{-\varepsilon} \varphi''(\eta_x)dx + \frac{1}{2}\int_{\varepsilon}^{a} \varphi''(\xi_x)dx - 2\frac{\varphi(0)}{\varepsilon}.$$

Moreover, we have

$$2\int_{\varepsilon}^{a} \frac{\varphi(0)}{x^2}dx = 2\varphi(0)\left[-\frac{1}{x}\right]_{\varepsilon}^{a} = 2\varphi(0)\left(-\frac{1}{a} + \frac{1}{\varepsilon}\right).$$

Therefore,

$$u_\varepsilon = 2\varphi(0)\left(-\frac{1}{a} + \frac{1}{\varepsilon}\right) + \frac{1}{2}\int_{-a}^{-\varepsilon} \varphi''(\eta_x)dx + \frac{1}{2}\int_{\varepsilon}^{a} \varphi''(\xi_x)dx - 2\frac{\varphi(0)}{\varepsilon}.$$

Hence,

$$u_\varepsilon = -\frac{2\varphi(0)}{a} + \frac{1}{2}\int_{-a}^{-\varepsilon} \varphi''(\eta_x)dx + \frac{1}{2}\int_{\varepsilon}^{a} \varphi''(\xi_x)dx.$$

Taking the limit as $\varepsilon \to 0$ we obtain

$$\begin{aligned}\left\langle \mathrm{Fp}\left(\frac{1}{x^2}\right), \varphi \right\rangle &= -\frac{2\varphi(0)}{a} + \frac{1}{2}\lim_{\varepsilon\to 0}\left\{\int_{-a}^{-\varepsilon} \varphi''(\eta_x)dx + \int_{\varepsilon}^{a} \varphi''(\xi_x)dx\right\} \\ &= -\frac{2\varphi(0)}{a} + \frac{1}{2}\left\{\int_{-a}^{0} \varphi''(\eta_x)dx + \int_{0}^{a} \varphi''(\xi_x)dx\right\}.\end{aligned}$$

From the above expression, we have

$$\left|\left\langle \mathrm{Fp}\left(\frac{1}{\mathrm{x}^2}\right), \varphi \right\rangle\right| \leq 2\frac{|\varphi(0)|}{a} + \frac{1}{2}\left|\int_{-a}^{0} \varphi''(\eta_x)dx\right| + \frac{1}{2}\left|\int_{0}^{a} \varphi''(\xi_x)dx\right|.$$

Since $|\varphi(0)| \leq \|\varphi\|_\infty$ and $|\varphi''(\eta_x)| \leq \|\varphi''\|_\infty$ we get

$$\left|\left\langle \mathrm{Fp}\left(\frac{1}{\mathrm{x}^2}\right), \varphi \right\rangle\right| \leq \frac{2}{a} \sup_{x\in[-a,a]} |\varphi(x)| + a \sup_{x\in[-a,a]} |\varphi''(x)|.$$

Then $\mathrm{Fp}\left(\dfrac{1}{\mathrm{x}^2}\right)$ is a distribution of order at most 2.

Exercise 2.2 It is clear that $\mathrm{PV}\left(\frac{1}{x}\right)$ is linear. Indeed, for all $\gamma_1, \gamma_2 \in \mathbb{R}$ and all $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R})$ we have

$$\begin{aligned}
\left\langle \mathrm{PV}\left(\frac{1}{x}\right), \gamma_1\varphi_1 + \gamma_2\varphi_2 \right\rangle &= \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} \frac{\gamma_1\varphi_1(x) + \gamma_2\varphi_2(x)}{x}\, dx \\
&= \gamma_1 \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} \frac{\varphi_1(x)}{x}\, dx + \gamma_2 \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} \frac{\varphi_2(x)}{x}\, dx \\
&= \gamma_1 \left\langle \mathrm{PV}\left(\frac{1}{x}\right), \varphi_1 \right\rangle + \gamma_2 \left\langle \mathrm{PV}\left(\frac{1}{x}\right), \varphi_2 \right\rangle.
\end{aligned}$$

Continuity: For every compact set $K \subset \mathbb{R}$, there exists $a > 0$ such that $K \subset [-a, a]$ and for all $\varphi \in \mathcal{D}(K)$

$$\begin{aligned}
\left|\left\langle \mathrm{PV}\left(\frac{1}{x}\right), \varphi \right\rangle\right| &= \left| \int_0^a \frac{\varphi(x) - \varphi(-x)}{x} dx \right| \\
&= \left| \int_0^a \int_{-1}^1 \varphi'(tx) dt dx \right| \\
&\leq \int_0^a \int_{-1}^1 |\varphi'(tx)| dt dx \\
&\leq 2a \sup_{x\in K} |\varphi'(x)|.
\end{aligned}$$

Therefore, $\mathrm{PV}\left(\frac{1}{x}\right)$ is distribution of order ≤ 1. To show that the order of the distribution $\mathrm{PV}\left(\frac{1}{x}\right)$ is $m = 1$, it suffices to prove that the order is not 0. That is, we need to show that there exists a compact set $K \subset \mathbb{R}$ such that for every constant $C > 0$, we have

$$\exists \varphi \in \mathcal{D}(K), \quad \left|\left\langle \mathrm{PV}\left(\frac{1}{x}\right), \varphi \right\rangle\right| > C\|\varphi\|_\infty.$$

Consider a sequence of test functions $(\varphi_j) \in \mathcal{D}(\mathbb{R})$ defined as follows

$$\mathrm{supp}(\varphi_j) \subset [-2, 2], \quad 0 \leq \varphi_j \leq 1,$$

$$\varphi_j(x) = 1, \ \ \forall x \in \left[\frac{1}{j}, 1\right],$$

and

$$\varphi_j(x) = 0 \ \ \forall x \in \left]-2, \frac{1}{2j}\right[.$$

We have $\|\varphi_j\|_\infty = 1$ and on the other hand we have

$$\begin{aligned}
\left\langle \mathrm{PV}\left(\frac{1}{x}\right), \varphi_j \right\rangle &= \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\varphi_j(x)}{x}\, dx \\
&= \lim_{\varepsilon\to 0} \left[\int_{-2}^{-\varepsilon} \frac{\varphi_j(x)}{x}\, dx + \int_{\varepsilon}^{2} \frac{\varphi_j(x)}{x}\, dx\right] \\
&= \int_{\frac{1}{2j}}^{\frac{1}{j}} \frac{\varphi_j(x)}{x}\, dx + \int_{\frac{1}{j}}^{1} \frac{\varphi_j(x)}{x}\, dx + \int_{1}^{2} \frac{\varphi_j(x)}{x}\, dx \\
&\geq \int_{\frac{1}{j}}^{1} \frac{1}{x}\, dx = \ln(j).
\end{aligned}$$

Therefore $\lim_{j\to+\infty} \left\langle \mathrm{PV}\left(\frac{1}{x}\right), \varphi_j \right\rangle = +\infty$. We deduce that the order of $\mathrm{PV}\left(\frac{1}{x}\right)$ is not 0. So it is of order 1.

Exercise 2.3

1. Let $\varphi \in \mathcal{D}(\mathbb{R})$. Then

$$\begin{aligned}
\langle T_j, \phi\rangle &= j\left\langle \delta_{\frac{1}{j}}, \varphi\right\rangle - j\left\langle \delta_{-\frac{1}{j}}, \varphi\right\rangle \\
&= j\left(\varphi\left(\frac{1}{j}\right) - \varphi\left(-\frac{1}{j}\right)\right) \\
&= \frac{\varphi\left(\frac{1}{j}\right) - \varphi\left(-\frac{1}{j}\right)}{\frac{1}{j}} \\
&= \frac{\varphi\left(\frac{1}{j}\right) - \varphi(0)}{\frac{1}{j}} + \frac{\varphi\left(-\frac{1}{j}\right) - \varphi(0)}{-\frac{1}{j}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{j\to+\infty} \langle T_j, \varphi\rangle &= \lim_{j\to+\infty} \frac{\varphi\left(\frac{1}{j}\right) - \varphi(0)}{\frac{1}{j}} + \lim_{j\to+\infty} \frac{\varphi\left(-\frac{1}{j}\right) - \varphi(0)}{-\frac{1}{j}} \\
&= \varphi'(0) + \varphi'(0) \\
&= 2\varphi'(0) \\
&= 2\langle \delta_0, \phi'\rangle \\
&= -2\langle \delta_0', \varphi\rangle,
\end{aligned}$$

i.e. $\lim_{j\to+\infty} T_j = -2\delta_0' \quad \text{in } \mathcal{D}'(\mathbb{R})$.

2. For all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle h_j, \varphi \rangle = \int_{\mathbb{R}} \frac{j}{1+(jx)^2} \varphi(x)\, dx,$$

we set $s = jx$ we obtain

$$\begin{aligned} \langle h_j, \varphi \rangle &= \int_{\mathbb{R}} \frac{j}{1+s^2} \varphi\left(\frac{s}{j}\right) \frac{1}{j}\, ds \\ &= \int_{\mathbb{R}} \frac{1}{1+s^2} \varphi\left(\frac{s}{j}\right) ds. \end{aligned}$$

On the other hand, it is clear that

$$\frac{1}{1+s^2} \varphi\left(\frac{s}{j}\right) \longrightarrow \frac{1}{1+s^2} \varphi(0) \quad \text{a.e.}$$

and that

$$\left| \frac{1}{1+s^2} \varphi\left(\frac{s}{j}\right) \right| \leq C \frac{1}{1+s^2} \leq C,$$

where $C = \sup_{t \in \text{supp}(\varphi)} |\varphi(t)|$.
Using the dominated convergence theorem, it follows that

$$\begin{aligned} \lim_{j \to +\infty} \langle h_j, \varphi \rangle &= \lim_{j \to +\infty} \int_{\mathbb{R}} \frac{1}{1+s^2} \varphi\left(\frac{s}{j}\right) ds \\ &= \int_{\mathbb{R}} \lim_{j \to +\infty} \frac{1}{1+s^2} \varphi\left(\frac{s}{j}\right) ds \\ &= \varphi(0) \int_{\mathbb{R}} \frac{1}{1+s^2}\, ds \\ &= \pi \varphi(0) \\ &= \pi \langle \delta_0, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}). \end{aligned}$$

In other words,

$$\lim_{j \to +\infty} h_j = \pi \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

3. Let $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\text{supp}(\varphi) \subset [-a, a]$. By using a simple integration by parts we obtain

$$\begin{aligned} \langle f_j, \varphi \rangle &= j^{\frac{3}{2}} \int_{-a}^{a} x e^{-jx^2} \phi(x)\, dx \\ &= \left[-\frac{\sqrt{j}}{2} e^{-jx^2} \varphi(x) \right]_{-a}^{a} + \frac{\sqrt{j}}{2} \int_{-a}^{a} e^{-jx^2} \varphi'(x)\, dx \\ &= \frac{\sqrt{j}}{2} \int_{-a}^{a} e^{-jx^2} \varphi'(x) dx = \frac{\sqrt{j}}{2} \int_{\mathbb{R}} e^{-jx^2} \varphi'(x)\, dx. \end{aligned}$$

We set $s = \sqrt{j}x$ we get

$$\langle f_j, \varphi \rangle = \frac{1}{2} \int_{\mathbb{R}} e^{-s^2} \varphi' \left(\frac{s}{\sqrt{j}} \right) ds \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

On the other hand, it is clear that

$$e^{-s^2} \varphi \left(\frac{s}{\sqrt{j}} \right) \longrightarrow e^{-s^2} \varphi(0) \quad \text{a.e.}$$

and that

$$\left| e^{-s^2} \varphi \left(\frac{s}{\sqrt{j}} \right) \right| \leq C e^{-s^2} \leq C \quad \text{where } C = \sup_{t \in \text{supp}(\varphi)} |\varphi(t)|.$$

Using the dominated convergence theorem, it follows that

$$\begin{aligned}
\lim_{j \to +\infty} \langle f_j, \varphi \rangle &= \lim_{j \to +\infty} \frac{1}{2} \int_{\mathbb{R}} e^{-s^2} \varphi' \left(\frac{s}{\sqrt{j}} \right) ds \\
&= \frac{1}{2} \int_{\mathbb{R}} \lim_{j \to +\infty} e^{-s^2} \varphi' \left(\frac{s}{\sqrt{j}} \right) ds \\
&= \frac{1}{2} \int_{\mathbb{R}} e^{-s^2} \varphi'(0) \, ds = \frac{1}{2} \varphi'(0) \int_{\mathbb{R}} e^{-s^2} \, ds \\
&= \frac{\sqrt{\pi}}{2} \varphi'(0) = \frac{\sqrt{\pi}}{2} \langle \delta_0, \varphi' \rangle \\
&= -\frac{\sqrt{\pi}}{2} \langle \delta_0', \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).
\end{aligned}$$

In other words

$$\lim_{j \to +\infty} f_j = -\frac{\sqrt{\pi}}{2} \delta_0' \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Exercise 2.4

(i) By using an integration by parts we get

$$\int_a^b \cos(\gamma x) g(x) \, dx = \left[\frac{1}{\gamma} \sin(\gamma x) g(x) \right]_a^b - \frac{1}{\gamma} \int_a^b \sin(\gamma x) g'(x) dx.$$

But

$$\left[\frac{1}{\gamma} \sin(\gamma x) g(x) \right]_a^b = \frac{1}{\gamma} \left(\sin(\gamma b) g(b) - \sin(\lambda a) g(a) \right),$$

and this tends to 0 when $\gamma \to +\infty$, since $\sin(\gamma a)$ and $\sin(\gamma b)$ are bounded. Similarly,

$$\left| \frac{1}{\gamma} \int_a^b \sin(\gamma x) g'(x) \, dx \right| \leq \frac{1}{\lambda} \int_a^b \|g'\|_\infty \leq \frac{(b-a)\|g'\|_\infty}{\gamma}.$$

(ii) We reduce to the previous question using the following trigonometric identity

$$\sin^2(jx) = \frac{1}{2} - \frac{\cos(2jx)}{2}.$$

We then have, for all $\varphi \in \mathcal{D}(\mathbb{R})$ with $\text{supp}(\varphi) \subset [\text{a}, \text{b}]$,

$$\langle T_j, \varphi \rangle = \int_{\mathbb{R}} \sin^2(jx)\varphi(x)\,dx = \frac{1}{2}\int_{\mathbb{R}} \varphi(x)\,dx - \frac{1}{2}\int_a^b \cos(2jx)\varphi(x)\,dx.$$

The integral tends to 0 according to the result of the previous question and therefore the sequence (T_j) tends to the distribution associated with the constant function equal to $\frac{1}{2}$.

(iii) We integrate by parts we get

$$\begin{aligned}
\langle S_j, \varphi \rangle &= \int_0^{+\infty} j\sin(jx)\varphi(x)\,dx \\
&= [-\cos(jx)\varphi(x)]_0^{+\infty} + \int_0^{+\infty} \cos(jx)\varphi'(x)\,dx \\
&= \varphi(0) + \int_0^{+\infty} \cos(jx)\varphi'(x)\,dx.
\end{aligned}$$

According to the result of the first question (we actually only integrate over a bounded interval since φ has compact support), the last integral tends to 0. Thus, we have proved that S_j converges to δ_0.

Exercise 2.5 Let $\varphi \in \mathcal{D}(\mathbb{R})$, for all $a \in \mathbb{R}^*$,

$$\begin{aligned}
\left\langle \frac{T - \tau_a T}{a}, \varphi \right\rangle &= \frac{1}{a}(\langle T, \varphi \rangle - \langle \tau_a T, \varphi \rangle) \\
&= \frac{1}{a}(\langle T, \varphi \rangle - \langle T, \tau_{-a}\varphi \rangle) \\
&= -\left\langle T, \frac{\tau_{-a}\varphi - \varphi}{a} \right\rangle.
\end{aligned}$$

We have $\dfrac{\tau_{-a}\varphi - \varphi}{a}\colon x \mapsto \frac{\varphi(x+a)-\varphi(x)}{a}$. From exercise 1.4, the sequence φ_a converges to φ' in $\mathcal{D}(\mathbb{R})$ when $a \to 0$. By continuity of T on $\mathcal{D}(\mathbb{R})$, we get

$$\left\langle \frac{T - \tau_a T}{a}, \varphi \right\rangle = -\langle T, \varphi_a \rangle \xrightarrow{a\to 0} -\langle T, \varphi' \rangle = \langle T', \varphi \rangle,$$

hence the result.

Exercise 2.6

1. For all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned}\left\langle \left(\mathrm{PV}\left(\frac{1}{x}\right)\right)', \varphi \right\rangle &= -\left\langle \mathrm{PV}\left(\frac{1}{x}\right), \varphi' \right\rangle, \\ &= -\lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\varphi'(x)}{x}\, dx, \\ &= -\lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} \frac{\varphi'(x)}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{\varphi'(x)}{x}\, dx \right),\end{aligned}$$

   By an integration by parts, we obtain

$$\begin{aligned}\left\langle \left(\mathrm{PV}\left(\frac{1}{x}\right)\right)', \varphi \right\rangle &= -\lim_{\varepsilon \to 0} \left(-\frac{\varphi(-\varepsilon) + \varphi(\varepsilon)}{\varepsilon} + \int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2} dx \right) \\ &= -\lim_{\varepsilon \to 0} \left(-\frac{2\varphi(0) + O(\varepsilon^2)}{\varepsilon} + \int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2} dx \right),\end{aligned}$$

   (because $\varphi(-x) + \varphi(x) - 2\varphi(0) = O(x^2)$ in the neighborhood of 0). Hence

$$\left\langle \left(\mathrm{PV}\left(\frac{1}{x}\right)\right)', \varphi \right\rangle = -\left\langle \mathrm{Fp}\left(\frac{1}{\mathrm{x}^2}\right), \varphi \right\rangle,$$

   (ie. $\left(\mathrm{PV}\left(\frac{1}{x}\right)\right)' = -\mathrm{Fp}\left(\frac{1}{\mathrm{x}^2}\right)$).

2. The function f is continuous (and therefore defines a distribution), and consequently has no jumps. Thus

$$(T_f)' - T_{f'} = 0.$$

   We observe that f can be written as

$$f(x) = \begin{cases} -(x-1)(x-3) & \text{if} \quad x \in]1,3[, \\ (x-1)(x-3) & \text{outside this interval.} \end{cases}$$

   We have

$$f'(x) = \begin{cases} -2x+4 & \text{if} \quad x \in]1,3[, \\ 2x-4 & \text{outside this interval,} \end{cases}$$

   Moreover, we have $a_1 = 1$, $a_2 = 3$ and

$$\begin{aligned} f'(1^+) - f'(1^-) &= 4, \\ f'(3^+) - f'(3^-) &= 4. \end{aligned}$$

Therefore, according to the jump formula

$$\begin{aligned}(T_f)'' - T_{f''} &= ((T_f)')' - T_{f''} \\ &= (T_{f'})' - T_{f''} \\ &= T_{f''} + \sum_{i=1}^{2} \big(f(a_i^+) - f(a_i^-)\big)\delta_{a_i} - T_{f''} \\ &= (f'(1^+) - f'(1^-)\delta_1 + (f'(3^+) - f'(3^-))\delta_3 \\ &= 4(\delta_1 + \delta_3).\end{aligned}$$

3. Let us first compute the derivative of F in the sense of distributions. Let $\varphi \in \mathcal{D}(\mathbb{R})$. Then

$$\begin{aligned}\langle T'_F, \varphi\rangle &= -\langle T_F, \varphi'\rangle \\ &= -\frac{1}{2}\int_{\mathbb{R}} |x|\varphi'(x)\,dx \\ &= \frac{1}{2}\int_{-\infty}^{0} x\varphi'(x)\,dx - \frac{1}{2}\int_{0}^{+\infty} x\varphi'(x)\,dx.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}\langle (T_F)', \varphi\rangle &= -\frac{1}{2}\int_{-\infty}^{0} \varphi(x)\,dx + \frac{1}{2}\int_{0}^{+\infty} \varphi(x)\,dx \\ &= \int_{\mathbb{R}} g(x)\varphi(x)\,dx \\ &= \langle T_g, \varphi\rangle,\end{aligned}$$

where

$$g(x) = \begin{cases} -\frac{1}{2} & \text{if} \quad x \in]-\infty, 0[, \\ \frac{1}{2} & \text{if} \quad x \in]0, +\infty[, \end{cases}$$

(ie. $(T_F)' = T_{F'}$ with $F' = g$).
By the jump formula we have (in the sense of distributions)

$$(T_F)'' = ((T_F)')' = (T_{F'})' = (F'(0^+) - F'(0^-))\delta_0 = \delta_0.$$

Therefore, in the sense of distributions, we have $(T_F)'' = \delta_0$.

4. Let K be a compact subset of $\mathbb{R}^2$. Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with support contained in K. Let us set $K_1 = \{x \in \mathbb{R};\ (x, -x) \in K\}$ which is compact since K is compact.

$$|\langle T, \phi\rangle| \leq \left(\int_{K_1} dx\right) \sup_{(x,y)\in K} |\varphi(x, y)|.$$

Thus, T is a distribution on $\mathbb{R}^2$ of order 0.
We set $\psi(x) = \varphi \circ f(x)$ where $f(x) = (f_1(x), f_2(x)) = (x, -x)$, by differentiating we find for each $t \in \mathbb{R}$

$$\begin{aligned}
\psi'(a)t &= d\psi_a(t) \\
&= d\varphi_{f(a)} \circ df_a(t) \\
&= d\varphi_{f(a)}((f_1'(a), f_2'(a))t) \\
&= d\varphi_{f(a)}(t, -t) \\
&= \frac{\partial \varphi}{\partial x}(f(a))t - \frac{\partial \varphi}{\partial y}(f(a))t.
\end{aligned}$$

Considering $t = 1$ we obtain

$$\forall s \in \mathbb{R}, \psi'(s) = \frac{\partial \varphi}{\partial x}(s, -s) - \frac{\partial \varphi}{\partial y}\varphi(s, -s).$$

Hence to compute $\frac{\partial T}{\partial x} - \frac{\partial T}{\partial y}$ we write for every $\varphi \in \mathcal{D}(\mathbb{R}^2)$

$$\begin{aligned}
\left\langle \frac{\partial T}{\partial x} - \frac{\partial T}{\partial y}, \varphi \right\rangle &= -\left\langle T, \frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y} \right\rangle \\
&= -\int_{-\infty}^{+\infty} \left(\frac{\partial \varphi}{\partial x}(s, -s) - \frac{\partial \varphi}{\partial y}\varphi(s, -s) \right) ds \\
&= -\int_{-\infty}^{+\infty} \psi'(s)\, ds \\
&= \psi(-\infty) - \psi(+\infty) \\
&= \varphi(-\infty, +\infty) - \varphi(+\infty, -\infty) \\
&= 0 - 0 = 0.
\end{aligned}$$

Thus

$$\frac{\partial T}{\partial x} - \frac{\partial T}{\partial y} = 0.$$

Exercise 2.7 We have

$$F' = (\psi H)' = \psi' H + \psi H' = \psi' H + \psi \delta_0 = \psi' H + \psi(0)\delta_0 = \psi' H + \sum_{i=0}^{0} \psi^{(i)}(0)\delta_0^{(i)},$$

then the property is true for $k = 1$.
Suppose that the property is true at order k and let us show that it remains true at order

$k+1$. Using the definition of $F^{(k+1)}$ and the recurrence hypothesis, we obtain

$$\begin{aligned} F^{(k+1)} &= \left(F^{(k)}\right)' \\ &= \left(\psi^{(k)}H + \sum_{i=0}^{k-1}\psi^{(k-1-i)}(0)\delta_0^{(i)}\right)' \\ &= \psi^{(k+1)}H + \psi^{(k)}H' + \sum_{i=0}^{k-1}\psi^{(k-1-i)}(0)\delta_0^{(i+1)} \\ &= \psi^{(k+1)}H + \psi^{(k)}\delta_0 + \sum_{m=1}^{k}\psi^{(k-m)}(0)\delta_0^{(m)} \end{aligned}$$

hence

$$F^{(k+1)} = \psi^{(k+1)}H + \sum_{m=0}^{k}\psi^{(k-m)}(0)\delta_0^{(m)}.$$

Exercise 2.8 We multiply $T' + xT = \delta_0$ by $e^{\frac{x^2}{2}}$ we get

$$e^{\frac{x^2}{2}}(T' + xT) = e^{\frac{x^2}{2}}\delta_0,$$

then $\frac{d}{dx}(e^{\frac{x^2}{2}}T) = \delta_0$ (since $e^{\frac{x^2}{2}}\delta_0 = \delta_0$). Thus $e^{\frac{x^2}{2}}T = H(x) + c$ (ie. $T = (H(x)+c)e^{\frac{-x^2}{2}}$).
$T' + T = H \Leftrightarrow e^x(T'+T) = e^xH(x) \Leftrightarrow \frac{d}{dx}(e^xT) = e^xH(x) \Leftrightarrow e^xT = \int_{-\infty}^{x} H(t)e^t\,dt + c$. Since $\int_{-\infty}^{x} H(t)e^t\,dt = 0$ for $x < 0$, $\quad e^x - 1$ for $x \geq 0$ we get

$$T = (H(x)(e^x - 1) + c)e^{-x}.$$

The equation $xT' + T = 0$ is equivalent to $(xT)' = 0$. Therefore there exists $c_1 \in \mathbb{R}$ such that $xT = c_1$.
We have $x\,\mathrm{PV}\left(\frac{1}{x}\right) = 1$. Indeed, for all $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\left\langle x\,\mathrm{PV}\left(\frac{1}{x}\right), \varphi \right\rangle = \left\langle \mathrm{PV}\left(\frac{1}{x}\right), x\varphi \right\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{x\varphi(x)}{x}dx = \int_{\mathbb{R}}\varphi(x)dx = \langle 1, \varphi\rangle.$$

Therefore $x\,\mathrm{PV}\left(\frac{1}{x}\right) = 1$ in $\mathcal{D}'(\mathbb{R})$.
We deduce that $xT = c_1\left(x\,\mathrm{PV}\left(\frac{1}{x}\right)\right)$ then

$$x\left(T - c_1\,\mathrm{PV}\left(\frac{1}{x}\right)\right) = 0.$$

Using the proposition 2.2 we get

$$T - c_1\,\mathrm{PV}\left(\frac{1}{x}\right) = c\delta_0,$$

where c is an arbitrary constant. Therefore the solutions of $xT = c_1$ take the form

$$c_1 \,\mathrm{PV}\left(\frac{1}{x}\right) + c\delta_0.$$

Exercise 2.9 I) We have

$$(\varphi_\lambda(x))' = (\varphi(\lambda x))' = \lambda\varphi'(\lambda x) = \lambda(\varphi')_\lambda(x),$$

then

$$\langle T', \varphi_\lambda\rangle = -\langle T, (\varphi_\lambda)'\rangle = -\lambda\langle T, (\varphi')_\lambda\rangle.$$

Consequently if T is homogeneous of degree p, then

$$\begin{aligned}\langle T', \varphi_\lambda\rangle &= -\lambda\langle T, (\varphi')_\lambda\rangle = -\lambda\lambda^{-1-p}\langle T, \varphi'\rangle \\ &= -\lambda^{-p}\langle T, \varphi'\rangle = \lambda^{-p}\langle T', \varphi\rangle = \lambda^{-1-(p-1)}\langle T', \varphi\rangle,\end{aligned}$$

i.e. T' is homogeneous of degree $p-1$.
II) The distribution $\partial^\alpha\delta$ is homogeneous of degree $-n-|\alpha|$. Indeed, for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\begin{aligned}\langle \partial^\alpha\delta, \varphi_\lambda\rangle &= (-1)^{|\alpha|}\langle \delta, \partial^\alpha\varphi_\lambda\rangle = (-1)^{|\alpha|}\langle \delta, (\partial^\alpha\varphi)_\lambda\rangle \\ &= (-1)^{|\alpha|}\lambda^{|\alpha|}\partial^\alpha\varphi(0) = (-1)^{|\alpha|}\lambda^{|\alpha|}\langle \delta, \partial^\alpha\varphi\rangle \\ &= \lambda^{|\alpha|}\langle \partial^\alpha\delta, \varphi\rangle = \lambda^{-n-(-n-|\alpha|)}\langle \partial^\alpha\delta, \varphi\rangle.\end{aligned}$$

Additional exercises

Exercise 2.10. *Let T be a linear form defined by*

$$\langle T, \varphi \rangle = \int_{\mathbb{R}} \varphi\left(t^2\right)\, dt \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

Show that

$$\langle T, \varphi \rangle = \int_{\mathbb{R}} \frac{H(x)}{\sqrt{|x|}} \varphi(x)\, dx \quad \forall \varphi \in \mathcal{D}(\mathbb{R}),$$

where H is the Heaviside function.
Deduce that T defines a distribution on $\mathbb{R}$. What is its order?

Exercise 2.11. - *Calculate in $\mathcal{D}'(\mathbb{R})$ the following products*

$$x\, PV\left(\frac{1}{x}\right), \quad x\, \mathrm{Fp}\left(\frac{1}{\mathrm{x}^2}\right) \quad \textit{and} \quad \mathrm{x}^2\, \mathrm{Fp}\left(\frac{1}{\mathrm{x}^2}\right).$$

Calculate products $(xPV(\frac{1}{x})))\delta_0$ and $PV(\frac{1}{x})(x\delta_0)$. What can we deduce?

Exercise 2.12. *Calculate the derivative in the sense of distributions of the following functions*

$$x_+ = max(0, x) = \begin{cases} x & si \quad x \geq 0, \\ 0 & si \quad x < 0. \end{cases} \qquad f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \sqrt{x} & \text{if } x > 0. \end{cases}$$

Exercise 2.13. 1. *Show that in the sense of distributions,*

$$(H(x) \ln x)' = \mathrm{Fp}\left(\frac{H(x)}{x}\right),$$

where

$$\left\langle \mathrm{Fp}\left(\frac{H(x)}{x}\right), \varphi \right\rangle = \lim_{\varepsilon \to 0} \left(\int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x} dx + \varphi(0) \ln \varepsilon \right), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

2. *Show that, for all $m \in \mathbb{N}$, we have $x\delta_0^{(m)} = -m\delta_0^{(m-1)}$.*
Show that a distribution $T \in \mathcal{D}(\mathbb{R})$ satisfies the relation $x^m T = 0$ if and only if $T = c_0\delta_0 + c_1\delta_0 + \cdots + c_{m-1}\delta_0^{(m-1)}$, where c_j, $0 \leq j \leq m-1$, are constants.

Exercise 2.14. *I) Show that*

1) *for all* $n \in \mathbb{N}$

$$\frac{d^n}{dx^n}(e^x H) = e^x H + \sum_{k=0}^{n-1} \frac{d^k}{dx^k}\delta_0,$$

2) *for all* $n \in \mathbb{N}^*$

$$\frac{1}{(n-1)!}\frac{d^n}{dx^n}(x^{n-1}H) = \delta_0.$$

II) Show that, for all $n \in \mathbb{N}$, *we have* $x\delta_0^{(n)} = -n\delta_0^{(n-1)}$. *Deduce that*

$$x^m \delta_0^{(n)} = \begin{cases} (-1)^m\, n(n-1)\cdots(n-m+1)\delta_0^{(n-m)} & \text{if} \quad 1 \le m \le n, \\ 0 & \text{if} \quad m > n. \end{cases}$$

Exercise 2.15. *Determine the limits in* $\mathcal{D}'(\mathbb{R})$ *as* $j \to +\infty$ *of the following sequences*

1) $f_j(x) = \frac{j}{\sqrt{\pi}} e^{-j^2x^2}$ *and* $F_j(x) = \displaystyle\int_{-\infty}^{x} f_j(t)\, dt$.

2) $T_j = j^2(\delta_{\frac{1}{j}} + \delta_{\frac{-1}{j}} - 2\delta_0)$.

Exercise 2.16. *For* $j \in \mathbb{N}$, *we set*

$$f_j(x) = H(x)\, h_j(x) \quad \text{with } h_j(x) = \int_0^j \frac{e^{-xt}}{1+t^2}\, dt.$$

(i) *Show that*

$$f_j'' = h_j'' H + h_j(0)\, \delta_0' + h_j'(0)\, \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Deduce that

$$f_j'' + f_j = \frac{1 - e^{-jx}}{x} H + \arctan(j)\, \delta_0' - \frac{1}{2}\ln(1+j^2)\, \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

(ii) *Let* $f(x) = H(x)\, h(x) \quad$ *with* $h(x) = \displaystyle\int_0^{+\infty} \frac{e^{-xt}}{1+t^2}\, dt$. *Show that for all* $j \in \mathbb{N}$ *and all* $x \in \mathbb{R}$, *we have*

$$0 \le f(x) - f_j(x) = H(x) \int_j^{+\infty} \frac{e^{-xt}}{1+t^2}\, dt \le \frac{\pi}{2} - \arctan(j).$$

(iii) *Deduce that* $f_j \xrightarrow[j\to+\infty]{} f \quad$ *in* $\mathcal{D}'(\mathbb{R})$ *and show that* $f_j'' \xrightarrow[j\to+\infty]{} f'' \quad$ *in* $\mathcal{D}'(\mathbb{R})$, *and by using the result from question* (i),

$$f'' + f = \frac{\pi}{2}\, \delta_0' + \lim_{j\to+\infty} \left(\frac{1 - e^{-jx}}{x} H - \frac{1}{2}\ln(1+j^2)\, \delta_0 \right) \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Chapter 3

Support and convolution of distributions

In this chapter, we will introduce for distributions a notion of support, the convolution product and application to differential equations.

3.1 Support of a distribution

Definition 3.1. *Let $T \in \mathcal{D}'(\Omega)$ and ω be an open set contained in Ω. We say that T vanishes in ω if*

$$\langle T, \varphi \rangle = 0, \quad \forall \varphi \in \mathcal{D}(\omega).$$

Lemma 3.1. *Let $(\omega_i)_{i\in I}$ be a family of open sets of Ω and $\omega = \cup_{i\in I}\omega_i$: If $T \in \mathcal{D}'(\Omega)$ vanishes in each ω_i then T vanishes in ω.*

Proof. Let $\varphi \in \mathcal{D}(\omega)$, let $K = \text{supp}(\varphi)$. Since $K \subset \bigcup_{i\in I}\omega_i$, the Borel-Lebesgue property implies that there exists a finite subset of indices $J \subset I$ such that $K \subset \bigcup_{i\in J}\omega_i$. Let $(\chi_i)_{i\in J}$ be a partition of unity relative to the covering $(\omega_i)_{i\in J}$ of K, i.e., $\chi_i \in \mathcal{D}(\omega_i)$, and $\sum_{i\in J}\chi_i = 1$ for $x \in \bigcup_{i\in J}\omega_i$. Since $\text{supp}(\varphi) = K$ we have

$$\varphi(x) = \sum_{i\in J} \chi_i(x)\varphi(x),$$

and thus

$$\langle T, \varphi \rangle = \sum_{i\in J} \langle T, \chi_i\varphi \rangle,$$

but $\chi_i \in \mathcal{D}(\omega_i)$ and T$= 0$ in ω_i, so $\langle T, \chi_i\varphi \rangle = 0$, and $\langle T, \varphi \rangle = 0$, i.e., $T = 0$ in ω. □

This lemma shows that for any distribution $T \in \mathcal{D}'(\Omega)$, there exists a largest open set where T vanishes: it is the union of all open sets where $T = 0$. We introduce the following definition

Definition 3.2. *For $T \in \mathcal{D}'(\Omega)$, the support of T, denoted by* $\mathrm{supp}(T)$*, is the complement of the largest open set on which T vanishes.*

Proposition 3.1. 1. *$x_0 \notin \mathrm{supp}(T) \Longleftrightarrow \exists \mathcal{V}_{x_0}$ neighborhood of x_0 such that*

$$\langle T, \varphi \rangle = 0, \quad \forall \varphi \in \mathcal{V}_{x_0}.$$

2. *$x_0 \in \mathrm{supp}(T) \Longleftrightarrow \forall \mathcal{V}_{x_0}$ neighborhoods of x_0, $\exists \varphi \in \mathcal{V}_{x_0}$ such that $\langle T, \varphi \rangle \neq 0$.*

3. *$\mathrm{supp}(T) \subset F \Longleftrightarrow T = 0$ dans F^c.*

4. *$T = 0 \Longleftrightarrow \mathrm{supp}(T) = \emptyset$.*

Example 3.1. 1. *Let $f \in C(\Omega)$, we have*

$$\mathrm{supp}(T_f) = \mathrm{supp}(f).$$

Indeed, if $x_0 \notin \mathrm{supp}(f)$, there exists $\mathcal{V}_{x_0}$ such that $f(x) = 0$ for all $x \in \mathcal{V}_{x_0}$. Then, if $\varphi \in \mathcal{D}(\mathcal{V}_{x_0})$ $f(x)\varphi(x) = 0$ in Ω and therefore $\langle T_f, \varphi \rangle = 0$ i.e. $x_0 \notin \mathrm{supp}(f)$.
Conversely, if $x_0 \notin \mathrm{supp}(T_f)$, it exists $\mathcal{V}_{x_0}$ such that

$$\langle T_f, \varphi \rangle = \int_\Omega f(x)\varphi(x)\, dx = 0, \quad \forall \varphi \in \mathcal{D}(\mathcal{V}_{x_0}),$$

which implies that $f(x) = 0$ almost everywhere (therefore everywhere since f is continuous) in $\mathcal{V}_{x_0}$ then $x_0 \notin \mathrm{supp}(f)$.
This result can be generalized to the case of a function $f \in L^1_{loc}(\Omega)$.
In particular, the support of the distribution T_H associated with the Heaviside function is $\mathbb{R}_+$.

2. *Let a function $f : \mathbb{R} \to \mathbb{R}$ be defined as*

$$f(x) = 0, \quad x \in \mathbb{R} \backslash \{x_1\}.$$

This function is piecewise continous and its support is

$$\mathrm{supp}(f) = \overline{\{x \in \mathbb{R}, \quad f(x) \neq 0\}} = \overline{\{x_1\}} = \{x_1\}.$$

The support of the distribution $T_f \in \mathcal{D}'(\mathbb{R})$ generated by the function f, is the empty set $\emptyset$ (ie, $\mathrm{supp}(T_f) = \emptyset$). Indeed, for all $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)\, dx,$$

hence $T_f = 0$ on $\mathbb{R}$.

3. *Let Ω be an open set of $\mathbb{R}^n$ and let $x_0 \in \Omega$. Then*

$$\mathrm{supp}(\delta_{x_0}) = \{x_0\}.$$

Indeed, let us show that the largest open set on which δ_{x_0} vanishes is equal to $\Omega\backslash\{x_0\}$. Let us first note that $\Omega\backslash\{x_0\}$ is an open set such that δ_{x_0} vanishes. Indeed, if $\varphi \in \mathcal{D}(\Omega\backslash\{x_0\})$, then $\varphi(x_0) = 0$ and therefore $\langle\delta_{x_0}, \varphi\rangle = 0$. Suppose then that $\Omega\backslash\{x_0\}$ is not the largest open set on which δ_{x_0} vanishes. It follows that Ω is the largest open set such that δ_{x_0} vanishes and, consequently, $\delta_{x_0} = 0$. Which is absurd.

Proposition 3.2. (Support and operations) *Let $T, S \in \mathcal{D}'(\Omega)$. We have*

1. $T_{|\Omega\backslash\mathrm{supp}(T)} = 0$.
2. $\mathrm{supp}(S + T) \subset \mathrm{supp}(S) + \mathrm{supp}(T)$.
3. *For all multi-index $\alpha \in \mathbb{N}^n$, we have*

$$\mathrm{supp}(\partial^\alpha T) \subset \mathrm{supp}(T).$$

Theorem 3.1. *Let $T \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ such that $\mathrm{supp}(T) \cap \mathrm{supp}(\varphi) = \emptyset$ then*

$$\langle T, \varphi\rangle = 0.$$

Proof. We have $\mathrm{supp}(\varphi) \subset (\mathrm{supp}(T))^c$, so for every $x \in \mathrm{supp}(\varphi)$, there exists an open neighborhood $\mathcal{V}_x$ such that

$$\langle T, \varphi\rangle = 0, \quad \forall\varphi \in \mathcal{D}(\mathcal{V}_x).$$

Therefore,

$$\mathrm{supp}(\varphi) \subset \bigcup_{x\in\mathrm{supp}(\varphi)} \mathcal{V}_x,$$

since $\mathrm{supp}(\varphi)$ is compact, the Borel-Lebesgue property shows that there exist points $x_1, x_2, \ldots, x_N$ in $\mathrm{supp}(\varphi)$ such that

$$\mathrm{supp}(\varphi) \subset \bigcup_{i=1}^{N} \mathcal{V}_{x_i}.$$

Let (χ_i) be a partition of unity associated with the cover $(\mathcal{V}_{x_i})_{i=1,\ldots,N}$, i.e., $\chi_i \in \mathcal{D}(\mathcal{V}_{x_i})$ and $\sum_{i=1}^{N} \chi_i(x) = 1$ for $x \in \mathrm{supp}(\varphi)$. Then

$$\varphi = \sum_{i=1}^{N} \chi_i\varphi \quad \text{and hence} \quad \langle T, \varphi\rangle = \sum_{i=1}^{N} \langle T, \chi_i\varphi\rangle = 0, \quad \text{because } \chi_i\varphi \in \mathcal{D}(\mathcal{V}_{x_i}).$$

□

Proposition 3.3. *For all function $\psi \in C^\infty(\Omega)$, we have*

$$\operatorname{supp}(\psi T) \subset \operatorname{supp}(\psi) \cap \operatorname{supp}(T).$$

Proof. Let $x_0 \in (\operatorname{supp}(\psi))^c \cup (\operatorname{supp}(T))^c$. If $x_0 \in (\operatorname{supp}(\psi))^c$, there exists a neighborhood $\mathcal{V}_{x_0}$ of x_0 such that $\psi(x) = 0$ for all $x \in \mathcal{V}_{x_0}$. Then if $\varphi \in \mathcal{D}(\mathcal{V}_{x_0})$, we have, for all $x \in \Omega$, $\psi(x)\varphi(x) = 0$. Hence

$$\langle \psi T, \varphi \rangle = \langle T, \psi\varphi \rangle = 0 \quad \text{and} \quad x_0 \notin \operatorname{supp}(\psi T).$$

If $x_0 \in (\operatorname{supp}(T))^c$, there exists $\mathcal{V}_{x_0}$ such that $\langle T, \psi \rangle = 0$ for $\psi \in \mathcal{D}(\mathcal{V}_{x_0})$.
Let $\varphi \in \mathcal{D}(\mathcal{V}_{x_0}))$. Then $\psi\varphi \in \mathcal{D}(\mathcal{V}_{x_0})$ so $\langle \psi T, \varphi \rangle = \langle T, \psi\varphi \rangle = 0$ and $x_0 \notin \operatorname{supp}\,(\psi T)$. □

3.1.1 Distributions with compact support

Definition 3.3. *The linear map $T : \mathcal{E}(\Omega) \to \mathbb{K}$ belong to $\mathcal{E}'(\Omega)$ if*

$$\lim_{j\to\infty} T(\varphi_j) = T(\varphi),$$

for any sequence $(\varphi_j)_j \subset \mathcal{E}(\Omega)$ which converges to φ in $\mathcal{E}(\Omega)$.

Note that if $T \in \mathcal{E}'(\Omega)$, then since $\mathcal{D}(\Omega)$ is a subspace of $\mathcal{E}(\Omega)$, $T : \mathcal{D}(\Omega) \to \mathbb{K}$ is a linear map. Moreover, we know that if $(\varphi_j)_j$ is a sequence converges to φ in $\mathcal{D}(\Omega)$, then it also tends to φ in $\mathcal{E}(\Omega)$. Thus $T \in \mathcal{E}'(\Omega)$ is naturally an element of $\mathcal{D}'(\Omega)$, we have

$$\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega).$$

We are justified then in referring to elements of $\mathcal{E}'(\Omega)$ as distributions. Moreover, the inclusion map $i : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$ is continuous ($\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$) and $\mathcal{E}'(\Omega)$ is dense in $\mathcal{D}'(\Omega)$.
We can give a useful characterisation of continuity for linear maps from $\mathcal{E}(\Omega)$ to $\mathbb{K}$ as follows

Theorem 3.2. *Let $T : \mathcal{E}(\Omega) \to \mathbb{K}$ be a linear map. The following are equivalent*

1. $T \in \mathcal{E}'(\Omega)$.
2. *There exists a compact $K \subset \Omega$, there exists a constant $C > 0$ and $m \in \mathbb{N}$ such that*

$$|\langle T, \varphi \rangle| \leq C \sum_{|\alpha|\leq m} \sup_{x\in K} |\partial^\alpha \varphi(x)|, \quad \text{for all } \varphi \in \mathcal{E}(\Omega). \tag{3.1}$$

Proof. The proof of this theorem is very similar to the proof of Theorem 2.1. □

Theorem 3.3. *The elements of the space $\mathcal{E}'(\Omega)$ are distributions with compact support.*

Proof. Let $K \subset \Omega$ be compact set as in (3.1). For any $\varphi \in \mathcal{D}(\Omega)$ we have

$$\text{supp}(\varphi) \cap K = \emptyset,$$

and by using (3.1) we obtain

$$\langle T, \varphi \rangle = 0.$$

Therfore $\text{supp}(T) \subset K$. □

3.2 Convolution of distributions

In this section, we come to the main objective : the convolution of distributions. The general notion of convolution of distributions is defined in terms of the tensor product. We therefore begin by defining this product.

3.2.1 Tensor product

Tensor product of two functions

Let Ω_1, Ω_2 be two open subsets of $\mathbb{R}^{n_1}, \mathbb{R}^{n_2}$ respectively.

Definition 3.4. *For $f_i \in C^0(\Omega_i), i = 1, 2$, define the function $f = f_1 \otimes f_2$ (read as "f_1 tensor f_2") on $\Omega_1 \times \Omega_2$ by*

$$(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1) f_2(x_2), \quad (x_1, x_2) \in \Omega_1 \times \Omega_2. \tag{3.2}$$

Then $f_1 \otimes f_2 \in C^0(\Omega_1 \times \Omega_2)$.

Example 3.2. *The tensor product of the Heaviside function by*

$$\Pi(x) = \begin{cases} 1 & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}], \\ 0 & \text{otherwise.} \end{cases}$$

is the function

$$(H \otimes \Pi)(x, y) = H(x)\Pi(y) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and } y \in [-\frac{1}{2}, \frac{1}{2}], \\ 0 & \text{otherwise.} \end{cases}$$

Tensor product of two distributions

Lemma 3.2. (Derivation under the bracket) *Let $\phi \in \mathcal{D}(\mathbb{R}^q_x \times \mathbb{R}^n_y)$ and let Ω be an open subset of $\mathbb{R}^n$. Assume that for every $x \in \mathbb{R}^q$, $\operatorname{supp} \phi(x,\cdot) \subset \Omega$. Let $T \in \mathcal{D}'(\Omega)$. Define, for $x \in \mathbb{R}^q$, $\theta(x) = \langle T, \phi(x,\cdot)\rangle$. Then, $\theta \in \mathcal{D}(\mathbb{R}^q)$ and we have*

$$\forall \alpha \in \mathbb{N}^q,\ \forall x \in \mathbb{R}^q,\ \partial^\alpha \theta(x) = \langle T, \partial_x^\alpha \phi(x,\cdot)\rangle\ ,$$

where ∂_x^α denotes a derivative of order α with respect to x.

Let us now define the tensor product of two distributions. Let $f_i \in C^0(\Omega_i), i = 1,2$, for $\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$, by Fubini's theorem, we have

$$\begin{aligned}\langle f_1 \otimes f_2, \varphi\rangle &= \iint_{\Omega_1\times\Omega_2} f_1(x) f_2(y)\varphi(x,y)\,dx\,dy \\ &= \int_{\Omega_1} f_1(x)\left(\int_{\Omega_2} f_2(y)\varphi(x,y)\,dy\right)dx \\ &= \langle f_1, \langle f_2, \varphi(x,\cdot)\rangle\rangle.\end{aligned}$$

We write also

$$\langle T_{f_1} \otimes T_{f_2}, \varphi\rangle = \langle (T_{f_1})_x, \langle (T_{f_2})_y, \varphi(x,y)\rangle\rangle, \quad \forall \varphi \in \mathcal{D}(\Omega_1 \times \Omega_2).$$

In particular, if $\varphi(x,y) = \varphi_1(x)\,\varphi_2(y)$ where $\varphi_i \in \mathcal{D}(\Omega_i)$, then

$$\langle f_1 \otimes f_2, \varphi_1 \otimes \varphi_2\rangle = \langle f_1, \varphi_1\rangle\langle f_2, \varphi_2\rangle.$$

ie.

$$\langle T_{f_1} \otimes T_{f_2}, \varphi_1 \otimes \varphi_2\rangle = \langle T_{f_1}, \varphi_1\rangle\langle T_{f_2}, \varphi_2\rangle.$$

We can therefore generalize this operation to distributions.

Theorem 3.4. (and definition) *Let $T_i \in \mathcal{D}'(\Omega_i)$, $i = 1,2$. There exists a unique distribution $T \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ such that, for all functions $\varphi_i \in \mathcal{D}(\Omega_i)$, $i = 1,2$, we have*

$$\langle T, \varphi_1 \otimes \varphi_2\rangle = \langle T_1, \varphi_1\rangle\langle T_2, \varphi_2\rangle. \tag{3.3}$$

Moreover, for $\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ we have

$$\langle T, \varphi\rangle = \langle T_1, \langle T_2, \varphi(x,.)\rangle\rangle. \tag{3.4}$$

and

$$\langle T, \varphi\rangle = \langle T_2, \langle T_1, \varphi(\cdot,y)\rangle\rangle. \tag{3.5}$$

We write also

$$\langle T, \varphi \rangle = \langle (T_1)_x, \langle (T_2)_y, \varphi(x,y) \rangle \rangle = \langle (T_2)_y, \langle (T_1)_x, \varphi(x,y) \rangle \rangle.$$

If $T_i \in \mathcal{E}'(\Omega_i), i = 1, 2$, the same formulas hold for $\varphi \in C^\infty(\Omega_1 \times \Omega_2)$. We denote thus $T = T_1 \otimes T_2 = T_2 \otimes T_1$ and T is called the tensor product of the distributions T_1 and T_2.

Proof. Let $\theta : x \mapsto \langle T_2, \varphi(x, \cdot) \rangle$. Then by the Lemma 3.2, θ is a function in $\mathcal{D}(\Omega_1)$.
We define $\langle T, \varphi \rangle = \langle T_1, \theta \rangle$ and prove that it is a distribution in $\mathcal{D}'(\Omega_1 \times \Omega_2)$. Indeed, by successively applying the characterization of a distribution to T_1 then T_2, we obtain for every compact $K = K_1 \times K_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, the existence of constants $C_1 > 0$ and $C_2 > 0$ and integers m_1 and m_2 such that, for every $\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ with $\mathrm{supp}(\varphi) \subset K$,

$$\begin{aligned} |\langle T, \varphi \rangle| &= |\langle T_1, \theta \rangle| \\ &\leq C_1 \sum_{|k| \leq m_1} \| \langle T_2, \partial_x^\alpha \varphi(x, \cdot) \rangle \|_{\infty, x} \\ &\leq C_1 C_2 \sum_{|\alpha| \leq m_1, |\beta| \leq m_2} \| \partial_y^\beta \partial_x^\alpha \varphi(x,y) \|_\infty. \end{aligned}$$

Therefore, T is a distribution in $\mathcal{D}'(\Omega_1 \times \Omega_2)$. If $\varphi(x,y) = \varphi_1(x)\varphi_2(y)$ with $\varphi_1 \in \mathcal{D}(\Omega_1)$ and $\varphi_2 \in \mathcal{D}(\Omega_2)$, then

$$\begin{aligned} \langle T, \varphi_1 \otimes \varphi_2 \rangle &= \langle (T_1)_x, \langle (T_2)_y, \varphi_1(x)\varphi_2(y) \rangle \rangle \\ &= \langle (T_1)_x, \varphi_1(x) \langle (T_2)_y, \varphi_2(y) \rangle \rangle \\ &= \langle T_1, \varphi_1 \rangle \langle T_2, \varphi_2 \rangle. \end{aligned}$$

Now, we use the density of $\mathcal{D}(\Omega_1) \otimes \mathcal{D}(\Omega_2)$ in $\mathcal{D}(\Omega_1 \times \Omega_2)$ to prove uniqueness (see Chapter 1 Proposition 1.6). Indeed, let $\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$, then

$$\varphi = \lim_{k \to \infty} (\psi_k \otimes \phi_k),$$

where $\psi_k \in \mathcal{D}(\Omega_1)$, $\phi_k \in \mathcal{D}(\Omega_2)$. Let T, S be two distributions satisfying (3.3), then $U = T - S$ satisfies,

$$\langle U, \varphi \rangle = \lim_{k \to \infty} \langle U, \psi_k \otimes \phi_k \rangle = \lim_{k \to +\infty} \left(\langle T_1, \psi_k \rangle \langle T_2, \phi_k \rangle - \langle T_1, \psi_k \rangle \langle T_2, \phi_k \rangle \right) = 0\,.$$

Finally, the distribution $\tilde{T}$, defined by $\langle \tilde{T}, \varphi \rangle = \langle T_2, \langle T_1, \varphi(\cdot, y) \rangle \rangle$, also satisfies (3.3). Uniqueness shows that $T = \tilde{T}$, hence (3.5). □

Example 3.3. *The tensor product of distributions δ_0 and H for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$,*

$$\begin{aligned}\langle \delta_0 \otimes H, \varphi \rangle &= \langle \delta_0, \langle H, \varphi(x, \cdot) \rangle \rangle \\ &= \langle \delta_0, \int_0^{+\infty} \varphi(x, y)\, dy \rangle \\ &= \int_0^{+\infty} \varphi(0, y)\, dy.\end{aligned}$$

The tensor product of distributions δ_a and δ_b is

$$\begin{aligned}\langle \delta_a \otimes \delta_b, \varphi \rangle &= \langle (\delta_a)_x, \langle (\delta_b)_y, \varphi(x, y) \rangle \rangle \\ &= \langle (\delta_a)_x, \varphi(x, b) \rangle \\ &= \varphi(a, b) = \langle \delta_{(a,b)}, \varphi \rangle.\end{aligned}$$

Remark 3.1. - *If $T_1 \in \mathcal{D}'(\Omega_1)$ and $T_2 \in \mathcal{D}'(\Omega_2)$ then we have*

$$\frac{\partial}{\partial x_k}(T_1 \otimes T_2) = \left(\frac{\partial T_1}{\partial x_k}\right) \otimes T_2, \quad \forall k, \quad k = 1, \cdots n_1,$$

and

$$\frac{\partial}{\partial y_m}(T_1 \otimes T_2) = T_1 \otimes \left(\frac{\partial T_2}{\partial y_m}\right), \quad \forall m, \quad m = 1, \cdots, n_2.$$

For example, if $H(t)$ denotes the Heaviside function on $\mathbb{R}$, then for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ we have

$$\frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} \left(H(x_1) \otimes H(x_2) \otimes \cdots \otimes H(x_n)\right) = \delta_{x_1=0} \otimes \delta_{x_2=0} \otimes \cdots \otimes \delta_{x_n=0} = \delta_{(0,\cdots,0)}.$$

Let $T_1 \in \mathcal{D}'(\Omega_1)$ and $T_2 \in \mathcal{E}'(\Omega_2)$. Let $\chi \in \mathcal{D}(\Omega_2)$, with $\chi = 1$ in a neighborhood of the support of T_2. Then for $\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ we have

$$\langle T_1 \otimes T_2, \varphi \rangle = \langle T_1 \otimes T_2, \chi\varphi \rangle.$$

Indeed,

$$\langle T_1 \otimes T_2, \varphi \rangle = \langle T_1, \langle T_2, \varphi(x, \cdot) \rangle \rangle = \langle T_1, \langle T_2, \chi(.)\varphi(x, \cdot) \rangle \rangle = \langle T_1 \otimes T_2, \chi\varphi \rangle.$$

This shows that we can extend $T_1 \otimes T_2$ to functions $\varphi \in C^\infty(\Omega_1 \times \Omega_2)$ such that $\chi\varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$.

Proposition 3.4. *The tensor product of two distributions is commutative, continuous and associative.*

The tensor product of distributions satisfies distributivity with respect to addition. In other

words, if T_1, T_2, and T_3 are distributions, then the following property holds

$$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3.$$

Moreover, we have

(i) *If $(T_j)_{j\geq 0}$ converges to T in $\mathcal{D}'(\Omega_1)$ and $S \in \mathcal{D}'(\Omega_2)$, then $(T_j \otimes S) \to T \otimes S$ as $j \to +\infty$ in $\mathcal{D}'(\Omega_1 \times \Omega_2)$,*

(ii) *if $(T_j)_{j\geq 0}$ converges to T in $\mathcal{D}'(\Omega_1)$ and $(S_j)_{j\geq 0}$ converges to S in $\mathcal{D}'(\Omega_2)$ $(j \to +\infty)$, then $(T_j \otimes S_j)_{j\geq 0}$ converges to $T \otimes S$ as $j \to +\infty$ in $\mathcal{D}'(\Omega_1 \times \Omega_2)$.*

Proposition 3.5. *For $(T_1, T_2) \in \mathcal{D}'(\Omega_1) \times \mathcal{D}'(\Omega_2)$, it holds that*

$$\operatorname{supp}(T_1 \otimes T_2) = \operatorname{supp}(T_1) \times \operatorname{supp}(T_2).$$

Proof. Let $y_0 \notin \operatorname{supp} T_2$. There exists a neighborhood $\mathcal{V}$ of y_0 such that for all $\psi \in \mathcal{D}(\mathcal{V})$, $\langle T_2, \psi\rangle = 0$. Thus, for $\varphi \in \mathcal{D}(\mathbb{R}^{n_1} \times \mathcal{V})$, we get

$$\langle T_1 \otimes T_2, \varphi\rangle = \langle T_1, \langle T_2, \varphi\rangle\rangle = \langle T_1, 0\rangle = 0,$$

which shows that $\operatorname{supp}(T_1 \otimes T_2) \subset \mathbb{R}^{n_1} \times \operatorname{supp}(T_2)$. The same way we obtain

$$\operatorname{supp}(T_1 \otimes T_2) \subset \operatorname{supp}(T_1) \times \mathbb{R}^{n_2},$$

and thus

$$\operatorname{supp}(T_1 \otimes T_2) \subset (\mathbb{R}^{n_1} \times \operatorname{supp}(T_2)) \cap (\operatorname{supp}(T_1) \times \mathbb{R}^{n_2}) = \operatorname{supp}(T_1) \times \operatorname{supp}(T_2).$$

Conversely, assume that $(x_0, y_0) \in \operatorname{supp}(T_1) \times \operatorname{supp}(T_2)$. Let $\mathcal{W}$ be a neighborhood of (x_0, y_0). There are neighborhoods $\mathcal{U}$ of x_0 and $\mathcal{V}$ of y_0 such that $\mathcal{W} \supset \mathcal{U} \times \mathcal{V}$. Since $x_0 \in \operatorname{supp}(T_1)$ and $y_0 \in \operatorname{supp}(T_2)$, there is $\varphi_1 \in \mathcal{D}(\mathcal{U})$ and $\varphi_2 \in \mathcal{D}(\mathcal{V})$ such that

$$\langle T_1 \otimes T_2, \varphi_1 \otimes \varphi_2\rangle = \langle T_1, \varphi_1\rangle\langle T_2, \varphi_2\rangle \neq 0.$$

Therefore $(x_0, y_0) \in \operatorname{supp}(T_1 \otimes T_2)$. □

3.2.2 Convolution

Let $f \in C_c(\mathbb{R}^n)$ and $g \in C(\mathbb{R}^n)$, the convolution $f * g \in C(\mathbb{R}^n)$ is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy, \quad x \in \mathbb{R}^n.$$

We may consider $f * g$ as a regular distribution on $\mathbb{R}^n$ and calculate its action on a test function as follows

$$\begin{aligned} \langle f * g, \varphi \rangle &= \int_{\mathbb{R}^n} f * g(z)\varphi(z)\,dz \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(z-y)g(y)\varphi(z)\,dy\,dz \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(z-y)g(y)\varphi(z)\,dz\,dy, \end{aligned}$$

we set $x = z - y$ we obtain

$$\langle f * g, \varphi \rangle = \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(x)g(y)\varphi(x+y)\,dx\,dy,$$

by Fubini's theorem we have

$$\begin{aligned} \langle f * g, \varphi \rangle &= \int_{\mathbb{R}^n} f(x) \left[\int_{\mathbb{R}^n} g(y)\varphi(x+y)\,dy \right] dx \\ &= \langle (f)_x \otimes (g)_y, \varphi(x+y) \rangle. \end{aligned}$$

This suggests to generalize the convolution to distributions T_1, T_2 by

$$\begin{aligned} \langle T_1 * T_2, \varphi \rangle &= \langle (T_1)_x \otimes (T_2)_y, \varphi(x+y) \rangle \\ &= \langle (T_1)_x, \langle (T_2)_y, \varphi(x+y) \rangle \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \end{aligned} \tag{3.6}$$

However, we immediately see that it will not be possible to define in general the convolution of two distributions, since the function $(x, y) \mapsto \varphi(x+y)$ has not compact support in $\mathbb{R}^n \times \mathbb{R}^n$ for a general $\varphi \in \mathcal{D}(\mathbb{R}^n)$. The support of $\varphi(x+y)$ is not bounded. Indeed, if the support of $\varphi(x)$ is contained in

$$\{x \in \mathbb{R}^n : \|x\| \leq a,\ a > 0\},$$

then the support of $\varphi(x+y)$ is contained in the infinite strip

$$\{(x, y) : \|x + y\| \leq a,\ a > 0\} \subset \{(x, y) : |x_i + y_i| \leq a, i = 1, 2, \ldots, n\}.$$

Thus, we can give meaning to the definition (3.6) if we assume that one of the two distributions has compact support. Indeed, since

$$\operatorname{supp}(T_1 \otimes T_2) = \operatorname{supp}(T_1) \times \operatorname{supp}(T_2),$$

then $\operatorname{supp}(T_1 \otimes T_2) \cap \operatorname{supp}(\varphi(x+y))$ is bounded for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$. For example, if

$$\operatorname{supp}(\varphi) \subset \{x \in \mathbb{R}^n : \|x\| \leq a,\ a > 0\} \quad \text{and} \quad \operatorname{supp}(T_1) \subset \{x \in \mathbb{R}^n : \|x\| \leq b,\ b > 0\},$$

then

$$\begin{aligned}
&[\operatorname{supp}(T_1) \times \operatorname{supp}(T_2)] \cap \operatorname{supp}(\varphi(x+y)) \\
&= \{(x,y) : \|x\| \leq b,\ \|x+y\| \leq a\} \\
&\subset \{(x,y) : |x_i| \leq b,\ |x_i + y_i| \leq a, i = 1, 2, \ldots, n\},
\end{aligned}$$

which is indeed bounded. Then, if $T_1 \in \mathcal{E}'(\mathbb{R}^n)$, $T_2 \in \mathcal{D}'(\mathbb{R}^n)$, and let $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\chi(x) = 1$ in a neighborhood of the support of T_1 and equal to zero outside a larger neighborhood. Then $\theta(x,y) = \chi(x)\varphi(x+y) \in \mathcal{D}(\mathbb{R}^{2n})$ for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and we have

$$\langle T_1 * T_2, \varphi \rangle = \langle T_1, \langle T_2, \chi(x)\varphi(x+y) \rangle \rangle = \langle T_1, \chi(x) \langle T_2, \varphi(x+y) \rangle \rangle, \tag{3.7}$$

since the value of the distribution T_1 depends on the value of the test function φ in a neighborhood of the support of the distribution and is not altered by a change of values of φ outside any neighborhood of the support of T_1. That is, the definition (3.7) is independent of the choice of $\chi(x)$, and therefore we can write

$$\langle T_1 * T_2, \varphi \rangle = \langle (T_1)_x \otimes (T_2)_y, \varphi(x+y) \rangle = \langle (T_1)_x, \langle (T_2)_y, \varphi(x+y) \rangle \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Thus, we have the following result

Theorem 3.5. *Let $T_1 \in \mathcal{E}'(\mathbb{R}^n)$ and $T_2 \in \mathcal{D}'(\mathbb{R}^n)$ (or the reverse). Then the convolution product $T_1 * T_2$ given by (3.6) is a distribution in $\mathcal{D}'(\mathbb{R}^n)$.*

Example 3.4. *We have*

$$\begin{aligned}
\langle \delta_a * \delta_b, \varphi \rangle &= \langle (\delta_a)_x \otimes (\delta_b)_y, \varphi(x+y) \rangle, \\
&= \langle (\delta_a)_x, \langle (\delta_b)_y, \varphi(x+y) \rangle \rangle, \\
&= \langle (\delta_a)_x, \varphi(x+b) \rangle, \\
&= \varphi(a+b), \\
&= \langle \delta_{a+b}, \varphi \rangle.
\end{aligned}$$

Hence $\delta_a * \delta_b = \delta_{a+b}$.

Proposition 3.6. 1. **Commutativity**. *Let* $T_1 \in \mathcal{E}'(\mathbb{R}^n)$ *and* $T_2 \in \mathcal{D}'(\mathbb{R}^n)$. *The convolution product* $T_1 * T_2$, *is commutative, i.e.,* $T_1 * T_2 = T_2 * T_1$.

2. **Associativity**. *For* T_1, T_2, *and* $T_3 \in \mathcal{D}'(\mathbb{R}^n)$ *such that at least two of them have compact support. Then* $(T_1 * T_2) * T_3 = T_1 * (T_2 * T_3)$.

3. **Differentiation**. $T_1 \in \mathcal{E}'(\mathbb{R}^n)$ *and* $T_2 \in \mathcal{D}'(\mathbb{R}^n)$, *then for every* $\alpha \in \mathbb{N}^n$ *we have*

$$\partial^\alpha(T_1 * T_2) = \partial^\alpha(T_1) * T_2 = T_1 * \partial^\alpha(T_2).$$

If $n = 1$, *we have*

$$(T_1 * T_2)' = (T_1)' * T_2 = T_1 * (T_2)',$$

and

$$(T_1 * T_2)^{(\alpha)} = (T_1)^{(\alpha)} * T_2 = T_1 * (T_2)^{(\alpha)},$$

when $\alpha \in \mathbb{N}$.

4. *Let* $T \in \mathcal{D}'(\mathbb{R}^n)$ *we have*

$$\delta_a * T = T * \delta_a = \tau_a T.$$

In particular $\delta_0 * T = T * \delta_0 = T$.

5. *The convolution product of distributions satisfies distributivity with respect to addition. In other words, if* $T_1 \in \mathcal{E}'(\mathbb{R}^n)$ *and* T_2, T_3 *are distributions, then the following property holds*

$$T_1 * (T_2 + T_3) = T_1 * T_2 + T_1 * T_3.$$

Proof. For all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have

1.

$$\begin{aligned} \langle T_1 * T_2, \varphi\rangle &= \langle (T_1)_x, \langle (T_2)_y, \varphi(x+y)\rangle\rangle \\ &= \langle (T_2)_y, \langle (T_1)_x, \varphi(y+x)\rangle\rangle \\ &= \langle T_2 * T_1, \varphi\rangle. \end{aligned}$$

2.

$$\begin{aligned} \langle (T_1 * T_2) * T_3, \varphi\rangle &= \langle (T_1 * T_2)_x, \langle (T_3)_z, \varphi(x+z)\rangle\rangle \\ &= \langle (T_1)_x, \langle (T_2)_y \langle (T_3)_z, \varphi(x+y+z)\rangle\rangle\rangle \\ &= \langle T_1 * (T_2 * T), \varphi\rangle. \end{aligned}$$

3.

$$\langle \partial^\alpha (T_1 * S), \varphi \rangle = (-1)^{|\alpha|} \langle T_1 * T_2, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle (T_1)_x, \langle (T_2)_y, (\partial^\alpha \varphi)(x+y) \rangle \rangle.$$

Now, $(\partial^\alpha \varphi)(x+y) = \partial_y^\alpha(\varphi(x+y))$ and $\langle (T_2)_y, \partial_y^\alpha(\varphi(x+y)) \rangle = (-1)^{|\alpha|} \langle \partial^\alpha T_2, \varphi(x+\cdot) \rangle$. Hence,

$$\langle \partial^\alpha (T_1 * S), \varphi \rangle = \langle (T_1)_x, \langle (\partial^\alpha T_2)_y, \varphi(x+y) \rangle \rangle = \langle T_1 * (\partial^\alpha T_2), \varphi \rangle.$$

The other equality is proved similarly using $(\partial^\alpha \varphi)(x+y) = \partial_x^\alpha(\varphi(x+y))$.

4. $\langle \tau_a T, \varphi \rangle = \langle T, \tau_{-a} \varphi \rangle$, and

$$\begin{aligned} \langle \delta_a * T, \varphi \rangle &= \langle (\delta_a)_x, \langle (T)_y, \varphi(x+y) \rangle \rangle \\ &= \langle (T)_y, \langle (\delta_a)_x, \varphi(x+y) \rangle \rangle \\ &= \langle (T)_y, \varphi(a+y) \rangle \\ &= \langle (T)_y, \tau_{-a} \varphi \rangle \\ &= \langle \tau_a T, \varphi \rangle. \end{aligned}$$

Then $\delta_a * T = \tau_a T$. We also show that $T * \delta_a = \tau_a T$. In particular, we have $\delta_0 * T = T * \delta_0 = T$.

5. It's enough to use the distributivity of the tensor product and the linearity of distributions.

□

Remark 3.2. 1. *In general* $(T_1 * T_2) * T_3 \neq T_1 * (T_2 * T_3)$. *It suffices to choose* $T_1 = 1$, $T_2 = \delta_0'$ *and* $T_3 = H$ *(Heaviside distribution). We have*

$$(T_1 * T_2) * T_3 = (1 * \delta_0') * H = 1' * H = 0 * H = 0.$$

Similarly, we have

$$T_1 * (T_2 * T_3) = 1 * (\delta_0' * H) = 1 * H' = 1 * \delta_0 = 1.$$

Therefore, $(T_1 * T_2) * T \neq T_1 * (T_2 * T_3)$.

2. *Let* $T_1 \in \mathcal{D}'(\mathbb{R}^n)$ *and* $T_2 \in \mathcal{E}'(\mathbb{R}^n)$. *Then*

$$\operatorname{supp}(T_1 * T_2) \subset \operatorname{supp}(T_1) + \operatorname{supp}(T_2). \tag{3.8}$$

But the reverse inclusion in (3.8) *is not true in general. It suffices to take two locally integrable functions f and g with*

$$f = 1 \quad \textit{and} \quad \int_{\mathbb{R}^n} g(x)\,dx = 0.$$

Then $f * g = 0$, hence $\operatorname{supp}(T_f * T_g) = \emptyset$, while $\operatorname{supp}(T_f) + \operatorname{supp}(T_g) = \mathbb{R}^n$ since $\operatorname{supp}(T_f) = \mathbb{R}^n$.

Proposition 3.7. 1. *If $((S_j)_{j\in\mathbb{N}}) \subset \mathcal{E}'(\mathbb{R}^n)$ and $(S_j)_{j\in\mathbb{N}}$ converges to S in $\mathcal{E}'(\mathbb{R}^n)$ and $T \in \mathcal{D}'(\mathbb{R}^n)$. Then $(T * S_j)_{j\in\mathbb{N}}$ converges to $T * S$ in $\mathcal{D}'(\mathbb{R}^n)$.*

2. *If $((T_j)_{j\in\mathbb{N}}) \subset \mathcal{D}'(\mathbb{R}^n)$ and $(T_j)_{j\in\mathbb{N}}$ converges to T in $\mathcal{D}'(\mathbb{R}^n)$ and $S \in \mathcal{E}'(\mathbb{R}^n)$. Then $(T_j * S)_{j\in\mathbb{N}}$ converges to $T * S$ in $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\langle T * S_j, \varphi\rangle = \langle S_j * T, \varphi\rangle = \langle S_j, \langle T, \varphi(y + \cdot)\rangle\rangle.$$

Since the function $y \mapsto (T, \varphi(y + \cdot))$ is $C^\infty(\mathbb{R}^n)$ and $S_j \to S$ in $\mathcal{E}'(\mathbb{R}^n)$ we have

$$\langle T * S_j, \varphi\rangle \to \langle S, \langle T, \varphi(y + \cdot)\rangle\rangle = \langle S * T, \varphi\rangle.$$

Similarly for the case $(T_j)_{j\in\mathbb{N}} \subset \mathcal{D}'(\mathbb{R}^n)$. □

Now, we consider the following density theorem

Theorem 3.6. *Let Ω be an open subset of $\mathbb{R}^n$. Then $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}'(\Omega)$.*

Proof. We know that Ω is the union of an exhaustive sequence of compact sets, $\Omega = \bigcup_{j=1}^{+\infty} K_j$ (see Theorem 1.4). Let

$$\chi_j \in \mathcal{D}(\overset{\circ}{K}_{j+1}), \quad \chi_j = 1 \text{ on } K_j.$$

Let

$$\theta \in \mathcal{D}(\mathbb{R}^n), \quad \operatorname{supp}(\theta) \subset \{x \in \mathbb{R}^n : |x| \le 1\} \text{ and } \int \theta(x)\,dx = 1.$$

Set $\theta_j = j^n\theta(jx)$. Then for j sufficiently large, $\operatorname{supp}(\chi_j) + \operatorname{supp}(\theta_j) \subset \Omega$.
On the other hand, $\chi_j T \in \mathcal{E}'(\Omega) \subset \mathcal{E}'(\mathbb{R}^n)$. Set

$$T_j = (\chi_j T) * \theta_j.$$

Then $T_j \in \mathcal{D}(\Omega)$ for j sufficiently large, because $\operatorname{supp}(T_j) \subset \operatorname{supp}(\chi_j) + \operatorname{supp}(\theta_j) \subset \Omega$. Let

us show that

$$T_j \to T \text{ in } \mathcal{D}'(\Omega).$$

Let $\varphi \in \mathcal{D}(\Omega)$. We have

$$\langle T_j, \varphi \rangle = \langle \chi_j T, \langle \theta_j, \varphi(x+y) \rangle \rangle.$$

But

$$\langle \theta_j, \varphi(x+y) \rangle = \int \theta_j(y)\varphi(x+y)dy = (\tilde{\theta_j} * \varphi)(x) \text{ where } \tilde{\theta}_j(y) = \theta_j(-y).$$

Therefore

$$\langle T_j, \varphi \rangle = \langle T, \chi_j(\tilde{\theta_j} * \varphi) \rangle.$$

The sequence $(\chi_j(\tilde{\theta_j} * \varphi))_j$ converges to φ in $\mathcal{D}(\Omega)$. Indeed, if j_0 is sufficiently large, we have, for $j \geq j_0$,

$$\text{supp}(\tilde{\theta_j} * \varphi) \subset \text{supp}(\varphi) + B(0, \frac{1}{j}) \subset \text{supp}(\varphi) + \overline{B(0, \frac{1}{j_0})} := K \subset\subset \Omega.$$

If j is sufficiently large, $\chi_j = 1$ on K (since the K_j are increasing) and therefore

$$\text{supp}(\chi_j(\tilde{\theta_j} * \varphi)) = \text{supp}(\tilde{\theta_j} * \varphi) \subset K.$$

Let us show that, for every $\alpha \in \mathbb{N}^n$, $(\partial^\alpha(\tilde{\theta_j} * \varphi))$ converges to $\partial^\alpha \varphi$ uniformly on K. We have

$$(\tilde{\theta_j} * (\partial^\alpha \varphi))(z) - \partial^\alpha \varphi(z) = \int \theta_j(-y)\partial^\alpha \varphi(z-y)dy - \left(\int \theta_j(-y)dy \right) \partial^\alpha \varphi(z).$$

Setting $-jy = t$ in the integrals, we get

$$(\tilde{\theta_j} * (\partial^\alpha \varphi))(z) - \partial^\alpha \varphi(z) = \int \theta(t) \left[\partial^\alpha \varphi \left(z + \frac{1}{j} t \right) - \partial^\alpha \varphi(z) \right] dt,$$

hence,

$$|(\tilde{\theta_j} * (\partial^\alpha \varphi))(z) - \partial^\alpha \varphi(z)| \leq \frac{1}{j} \sum_{k=1}^{n} \sup_{\mathbb{R}^n} \left| \frac{\partial}{\partial z_k} \partial^\alpha \varphi \right| \int |t_k||\theta(t)|\, dt \leq \frac{C}{j},$$

therefore,

$$\sup_K |(\tilde{\theta_j} * (\partial^\alpha \varphi))(z) - \partial^\alpha \varphi(z)| \leq \frac{C}{j} \to 0 \quad \text{as } j \to +\infty.$$

Then $\chi_j(\tilde{\theta_j} * \varphi) \to \varphi$ in $\mathcal{D}(\Omega)$. Since $T \in \mathcal{D}'(\Omega)$ we get

$$\langle T_j, \varphi \rangle = \langle T, \chi_j(\tilde{\theta_j} * \varphi) \rangle \to \langle T, \varphi \rangle \quad \text{as } j \to +\infty.$$

□

3.3 Application to differential equations

A linear differential operator with constant coefficients can conveniently be expressed in multi-index notation as

$$P = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \qquad (c_\alpha \in \mathbb{C}).$$

If for some multi-index α of length m we have $c_\alpha \neq 0$, then P is said to be of order m. If also the coefficients c_α satisfy $c_\alpha = 0$ for all $|\alpha| < m$, then we say that P is homogeneous of order m.
We can view P as a mapping from $\mathcal{D}'(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$.

Example 3.5. *In one dimension $(n = 1)$ a linear differential operator with constant coefficients has the form*

$$Pu = c_0 u + c_1 u' + c_2 u'' + \cdots + c_m u^{(m)}.$$

*The Laplace operator (or **Laplacian**) given by*

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2},$$

is a special case of a linear differential operator with constant coefficients, of order 2.

Definition 3.5. *A distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is called an elementary solution of the differential operator P if $PE = \delta_0$.*

The existence of an elementary solution is guaranteed by the following result.

Theorem 3.7. (Malgrange-Ehrenpreis) *Every differential operator with constant coefficients in $\mathbb{R}^n$ admits an elementary solution.*

The notion of elementary solution is used to find solutions to certain PDEs.

Theorem 3.8. *Let P be a differential operator with constant coefficients in $\mathbb{R}^n$ and let $E \in \mathcal{D}'(\mathbb{R}^n)$ be an elementary solution of P. Then for every $f \in \mathcal{E}'(\mathbb{R}^n)$, the equation $Pu = f$ has at least one solution $u \in \mathcal{D}'(\mathbb{R}^n)$ of the form $u = E * f$.*

Proof. We have f has compact support, then the expression for u makes sense and

$$\begin{aligned} Pu &= P(E * f) \\ &= \sum_{|\alpha|\leq m} c_\alpha \partial^\alpha (E * f) \\ &= \sum_{|\alpha|\leq m} c_\alpha (\partial^\alpha E) * f \\ &= \Big(\sum_{|\alpha|\leq m} c_\alpha \partial^\alpha E \Big) * f \\ &= PE * f. \end{aligned}$$

Since E is an elementary solution we have

$$Pu = P(E * f) = \delta_0 * f = f.$$

□

Example 3.6. 1. *It is easy to verify that H is an elementary solution of $Pu = u'$. Therefore $u = H * f$ is a solution of $u' = f$.*

2. *Consider the equation $u'' = f$ (i.e. $Pu = u''$). A solution is given by $u = E * f$ where E satisfies $E'' = \delta_0$, i.e. $E' = H + c_0$ and consequently*

$$E = x^+ + c_0 x + c_1, \quad c_0, c_1 \in \mathbb{R},$$

where $x^+ = \max(x, 0)$.

3. *Let $f \in L^1_{loc}(\mathbb{R})$ with compact support. We aim to find a solution $u \in \mathcal{D}'(\mathbb{R})$ of the ordinary differential equation*

$$u' + u = f. \tag{3.9}$$

Since $E(x) = e^{-x}H(x)$ is an elementary solution of the operator $Pu = u' + u$, a solution of (3.9) is given by $u = E * f$. As both E and f are functions, u is also a function given by

$$u(x) = \int_{\mathbb{R}} E(x-t) f(t)\, dt = \int_{-\infty}^{x} e^{-(x-t)} f(t)\, dt.$$

(because $H(x-t) = 0$ when $t > x$).

Exercises

Exercise 3.1. *Let $T^{(\alpha)}$ be the α-th derivative of an arbitrary distribution T. Show that $\mathrm{supp}(T^{(\alpha)}) \subset \mathrm{supp}(T)$. Is the converse true? Justify your answer.*

Exercise 3.2. *I) Let $u(x,t)$ be the function defined in $\mathbb{R}^2$ by*

$$u(x,t) = \begin{cases} \alpha & \text{if } t^2 - x^2 \geq 0, \quad t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

Show that $u(x,t)$ defines a distribution in $\mathbb{R}^2$ and determine its support.
II) Determine the support of $VP(\frac{1}{x})$.

Exercise 3.3. *Let T be a distribution on $\mathbb{R}^n$ and f a C^∞ function on $\mathbb{R}^n$.*

1. *Show that if $fT = 0$, then the support of T is included in the set*

$$Z(f) = \{x \in \mathbb{R}^n, \quad f(x) = 0\}.$$

2. *Determine the C^∞ functions f such that $f\delta_0' = 0$.*

Exercise 3.4. *For $\varphi \in \mathcal{D}(\mathbb{R})$, define*

$$\langle T, \varphi \rangle = \int_{\mathbb{R}} e^{-x^2} \varphi(\sin(x))\, dx.$$

1. *Show that T is a distribution.*
2. *Determine the support of T.*

Exercise 3.5.

1. *Compute $\delta_0' * \delta_0'$, $\quad H(x) * (x^2 H(x))$ and $\frac{d^2}{dx^2}(H * H)$.*
2. *Let $T_1 \in \mathcal{D}'(\mathbb{R})$ and $T_2 \in \mathcal{E}'(\mathbb{R})$. Prove that $e^{ax}(T_1 * T_2) = (e^{ax} T_1) * (e^{ax} T_2)$, $a \in \mathbb{R}$.*
3. *Let T and S be two distributions on $\mathbb{R}$, with S having compact support, x^n denotes the function $x \mapsto x^n$. Prove the following formula*

$$x^n (T \star S) = \sum_{k=0}^{n} \binom{n}{k} (x^k T) \star (x^{n-k} S).$$

Exercise 3.6. *In $\mathbb{R}$, consider the function*

$$G_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \quad \sigma > 0.$$

This function represents the normal (or Gaussian) density. The number $\sigma > 0$ is the standard deviation. Prove the relation

$$G_\sigma * G_\tau = G_{\sqrt{\sigma^2+\tau^2}}.$$

Exercise 3.7. *Let P be a constant coefficient differential operator defined by*

$$Pu = \frac{d^2u}{dx^2} - \frac{du}{dx} - 2u.$$

Let z be the unique solution of the homogeneous differential equation

$$\begin{cases} Pz = 0; \\ z(0) = 0, \quad \frac{dz}{dx}(0) = 1. \end{cases}$$

1. *Verify that $E = zH$ is an elementary solution of P.*
2. *Find a particular solution of the equation*

$$Pu = \chi_{[0,1]} \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Exercise 3.8. *We consider in $\mathbb{R}^2$ the function $E(x,t) = \frac{H(t)}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}$.*

1. *Show that we can associate to the function E a distribution of order 0 on $\mathbb{R}^2$.*
2. *The heat operator is defined by*

$$P = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}.$$

Prove that for all $t > 0$ and all $x \in \mathbb{R}$, we have

$$\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}\right).$$

3. *Let $\varepsilon > 0$ and let $\varphi \in \mathcal{D}'(\mathbb{R}^2)$, we set*

$$I_\varepsilon = -\int_{\mathbb{R}} \int_{\varepsilon}^{+\infty} E(x,t) \frac{\partial}{\partial t} \varphi(x,t)\, dt\, dx, \quad J_\varepsilon = -\int_{\mathbb{R}} \int_{\varepsilon}^{+\infty} E(x,t) \frac{\partial^2}{\partial x^2} \varphi(x,t)\, dt\, dx.$$

(i) Show that

$$I_\varepsilon = \int_{\mathbb{R}} E(x,\varepsilon)\varphi(x,\varepsilon)\,dx + \int_{\mathbb{R}}\int_{\varepsilon}^{+\infty} \frac{\partial}{\partial t}(E(x,t))\varphi(x,t)\,dt\,dx.$$

Deduce that

$$I_\varepsilon = \int_{\mathbb{R}} E(x,\varepsilon)\varphi(x,\varepsilon)\,dx + \int_{\mathbb{R}}\int_{\varepsilon}^{+\infty} \frac{\partial^2}{\partial x^2}(E(x,t))\varphi(x,t)\,dt\,dx.$$

(ii) Using Fubini's theorem and two integrations by parts, show that

$$I_\varepsilon = \int_{\mathbb{R}} E(x,\varepsilon)\varphi(x,\varepsilon)\,dx - J_\varepsilon.$$

(iii) By using a change of variables, show that

$$I_\varepsilon + J_\varepsilon = \frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} e^{\frac{-y^2}{4}}\varphi(\sqrt{\varepsilon}y,\varepsilon)\,dy.$$

Using the dominated convergence theorem, deduce that

$$\lim_{\varepsilon\to 0}(I_\varepsilon + J_\varepsilon) = \varphi(0,0).$$

Hint: $\dfrac{1}{\sqrt{4\pi}}\displaystyle\int_{\mathbb{R}} e^{\frac{-y^2}{4}}\,dy = 1.$

4. Show that in $\mathcal{D}'(\mathbb{R}^2)$

$$PE = \delta_{(0,0)}.$$

Solutions of the exercises

Exercise 3.1

Let Ω_1 be the largest open set on which $T = 0$, and Ω_2 the largest open set on which $T^{(\alpha)} = 0$. By definition, we have $\text{supp}(T) = \Omega_1^c$ and $\text{supp}(T^{(\alpha)}) = \Omega_2^c$. Since $T = 0$ on Ω_1, it follows that $T^{(\alpha)} = 0$ on Ω_1. Therefore, $\Omega_2 \supset \Omega_1$, and consequently $\Omega_2^c \subset \Omega_1^c$. The converse is false. Indeed, consider the distribution $T = \text{constant}$. Then $\text{supp}(T) = \mathbb{R}$, whereas $\text{supp}(T^{(\alpha)}) = \emptyset$.

Exercise 3.2

I) The function $u(x,t)$ defines a distribution on $\mathbb{R}^2$ because it is locally integrable. The support of $u(x,t)$ is the cone $t^2 - x^2 \geq 0, \quad t \geq 0$.

II) Let $F = \text{supp}\left(\text{PV}\left(\frac{1}{x}\right)\right)$ and denote by F^c the complement of F. By definition, F is the smallest closed set such that $\text{PV}\left(\frac{1}{x}\right)$ vanishes on F^c. This means that $\left\langle \text{PV}\left(\frac{1}{x}\right), \varphi \right\rangle = 0$ for every function $\varphi \in \mathcal{D}(\mathbb{R})$ with support contained in F^c. To show that $F = \mathbb{R}$, we proceed by contradiction. Suppose that $F \neq \mathbb{R}$, with $\left\langle \text{PV}\left(\frac{1}{x}\right), \varphi \right\rangle = 0$ for every $\varphi \in \mathcal{D}(\mathbb{R})$ with support in $F^c \neq \emptyset$. We will show that in this case one can construct a function $\psi \in \mathcal{D}(\mathbb{R})$ with support in F^c such that $\left\langle \text{PV}\left(\frac{1}{x}\right), \psi \right\rangle \neq 0$, which is a contradiction. Indeed, since $F^c \neq \emptyset$ and F^c is open in $\mathbb{R}$, every point $c \in F^c$ is interior to F^c, i.e, there exists $r > 0$ such that $[c-r, c+r] \subset F^c$. In other words, F^c is a neighborhood of c. Consider a function $\psi \in \mathcal{D}(\mathbb{R})$ defined by $\psi(x) = x\varphi(x)$, where

$$\varphi(x) = \begin{cases} e^{-\frac{1}{r^2-(x-c)^2}} & \text{if } |x-c| < r, \\ 0 & \text{if } |x-c| \geq r. \end{cases}$$

We have $\text{supp}(\psi) = [c-r, c+r]$ and

$$\left\langle \text{PV}\left(\frac{1}{x}\right), \psi \right\rangle = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\psi(x)}{x}\, dx = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \varphi(x)\, dx = \int_{c-r}^{c+r} \varphi(x)\, dx.$$

Since $\varphi(x) > 0$ for all $x \in [c-r, c+r]$, it follows that $\left\langle \text{PV}\left(\frac{1}{x}\right), \psi \right\rangle > 0$, which is a contradiction. Therefore, $F = \mathbb{R}$.

Exercise 3.3

1. Let $T \in \mathcal{D}'(\mathbb{R})$ such that $fT = 0$. Show that $\text{supp}(T) \subset Z(f)$. Since f is continuous, $Z(f)$ is closed. Let $x \notin Z(f)$, there exists an open interval I such that $x \in I \subset \mathbb{R} \backslash Z(f)$. For every $\varphi \in \mathcal{D}(I) \subset \mathcal{D}(\mathbb{R})$, the function f does not vanish on a neighborhood of

$\mathrm{supp}(\varphi)$, and therefore $\frac{1}{f}\varphi$ is well-defined and belongs to $\mathcal{D}(I)$. Then

$$\langle T, \varphi\rangle = \left\langle fT, \frac{1}{f}\varphi \right\rangle = 0.$$

Hence $T|_I = 0$ and $x \notin \mathrm{supp}(T)$. Therefore, $\mathrm{supp}(T) \subset Z(f)$.

2. We have
$$\langle \delta_0', f\varphi\rangle = -f(0)\varphi'(0) - f'(0)\varphi(0),$$
and this is identically zero if and only if $f(0) = f'(0) = 0$.

Exercise 3.4 It is clear that T is well-defined and linear. Let us show that T is continuous at 0. Let $(\varphi_j) \subset \mathcal{D}(\mathbb{R})$ be such that $\varphi_j \to 0$. As $j \to +\infty$. By definition of convergence in $\mathcal{D}(\mathbb{R})$, there exists a compact $K \subset \mathbb{R}$ such that

$$\begin{aligned} &\mathrm{supp}(\varphi_j) \subset K, \quad \forall j \in \mathbb{N}, \\ &\varphi_j^{(\alpha)} \to 0, \quad \text{uniformly on } K, \quad \forall \alpha \in \mathbb{N}. \end{aligned}$$

We have

$$\begin{aligned} |\langle T, \varphi_j\rangle| &\leq \int_{\mathbb{R}} e^{-x^2} |\varphi_j(\sin(x))|\, dx \\ &\leq \sqrt{\pi} \sup_{x\in K} |\varphi_j(x)| \to 0, \end{aligned}$$

so T is continuous at 0.
$\mathrm{supp}(T) = [-1, 1]$. Indeed, since the image of the sin function is $[-1, 1]$, only the values of φ on $[-1, 1]$ matter for T. Let us show that $\mathrm{supp}(T) \subset [-1, 1]$. Let $\varphi \in \mathcal{D}(\mathbb{R})$ such that $[-1, 1] \cap \mathrm{supp}(\varphi) = \emptyset$. Then

$$\langle T, \varphi\rangle = \int_{\mathbb{R}} e^{-x^2} \varphi(\sin(x))\, dx = 0.$$

Conversely, let us show that $[-1, 1] \subset \mathrm{supp}(T)$. Let $x_0 \in]-1, 1[$ and $\varepsilon > 0$. We show that there exists $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\mathrm{supp}(\varphi) \subset]x_0 - \varepsilon, x_0 + \varepsilon[$ and $\langle T, \varphi\rangle \neq 0$. Using the function φ defined by

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

and we set

$$\varphi_{x_0,\varepsilon}(x) = \frac{1}{\varphi(0)} \varphi\left(\frac{2(x - x_0)}{\varepsilon}\right),$$

we obtain a function $\varphi_{x_0,\varepsilon} \in \mathcal{D}(\mathbb{R})$ supported on $]x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}[$ such that $\varphi_{x_0,\varepsilon}(x_0) = 1$ and $\varphi_{x_0,\varepsilon} \geq 0$.

By continuity of $\varphi_{x_0,\varepsilon}$, there exists $n \in \mathbb{N}$ such that $\varphi_{x_0,\varepsilon}(x) \geq \frac{1}{2}$ on $I =]x_0 - \frac{\varepsilon}{n}, x_0 + \frac{\varepsilon}{n}[$. Then

$$\begin{aligned} \langle T, \varphi \rangle &= \int_{\mathbb{R}} e^{-x^2} \varphi_{x_0,\varepsilon}(x)(\sin(x))\, dx \\ &\geq \frac{1}{2} \int_{\arcsin(I)} e^{-x^2}\, dx > 0. \end{aligned}$$

We deduce that $\operatorname{supp}(T) = [-1, 1]$.

Exercise 3.5

1. We have

$$\begin{aligned} \langle \delta_0' * \delta_0', \varphi \rangle &= \langle \delta_0' \otimes \delta_0', \varphi(x+y) \rangle \\ &= \langle \delta_0', -\varphi'(x) \rangle \\ &= \varphi''(0). \end{aligned}$$

Let $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned} \langle H(x) * (x^2 H(x)), \varphi \rangle &= \langle H(x) \otimes (y^2 H(y)), \varphi(x+y) \rangle \\ &= \langle H(x), \langle y^2 H(y), \varphi(x+y) \rangle \rangle \\ &= \left\langle H(x), \int_0^\infty y^2 \varphi(x+y)\, dy \right\rangle, \\ &= \int_0^\infty \int_0^\infty y^2 \varphi(x+y)\, dy\, dx \\ &= \int_0^\infty \left(\int_0^\infty \varphi(x+y) dx \right) y^2\, dy. \end{aligned}$$

Let $\psi(y) = \displaystyle\int_0^\infty \varphi(x+y)\, dx$, hence

$$\begin{aligned} \langle H(x) * (x^2 H(x)), \varphi \rangle &= \int_0^\infty \psi(y) y^2 dy \\ &= \frac{y^3}{3}\psi(y)\Big|_0^\infty - \frac{1}{3}\int_0^\infty y^3 \psi'(y)\, dy \\ &= -\frac{1}{3}\int_0^\infty y^3 \psi'(y)\, dy. \end{aligned}$$

Now $\psi(y) = \displaystyle\int_y^\infty \varphi(u)\,du$, where $u = x + y$, so $\psi'(y) = -\varphi(y)$ and

$$\begin{aligned}\langle H(x) * (x^2 H(x)), \varphi\rangle &= \frac{1}{3}\int_0^\infty y^3\varphi(y)\,dy \\ &= \frac{1}{3}\int_0^\infty x^3\varphi(x)\,dx \\ &= \left\langle \frac{x^3}{3}H(x), \varphi \right\rangle.\end{aligned}$$

Therefore

$$H(x) * (x^2 H(x)) = \frac{x^3}{3}H(x).$$

We have

$$\frac{d}{dx}(H * H) = H * \frac{dH}{dx} = H * \delta_0 = H,$$

then

$$\frac{d^2}{dx^2}(H * H) = \frac{dH}{dx} = \delta_0.$$

2. Let $T_1 \in \mathcal{D}'(\mathbb{R})$, $T_2 \in \mathcal{E}'(\mathbb{R})$ and $a \in \mathbb{R}$. For every $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned}\langle e^{ax}(T_1 * T_2), \varphi\rangle &= \langle T_1 * T, e^{ax}\varphi\rangle \\ &= \langle (T_1)_x, \langle (T_2)_y, e^{a(x+y)}\varphi(x+y)\rangle\rangle \\ &= \langle (T_1)_x, \langle (T_2)_y, e^{ax}e^{ay}\phi(x+y)\rangle\rangle \\ &= \langle (T_1)_x, e^{ax}\langle (T_2)_y, e^{ay}\phi(x+y)\rangle\rangle \\ &= \langle e^{ax}(T_1)_x, \langle (T_2)_y, e^{ay}\varphi(x+y)\rangle\rangle \\ &= \langle (e^{ax}T_1) * (e^{ax}T_2), \varphi\rangle\end{aligned}$$

hence

$$e^{ax}(T_1 * T_2) = (e^{ax}T_1) * (e^{ax}T_2).$$

3. We have

$$\begin{aligned}\langle x^n(T * S), \theta\rangle &= \langle T * S, x^n\varphi\rangle \\ &= \langle T_x \otimes S_y, (x+y)^n\varphi(x+y)\rangle \\ &= \sum_{k=0}^n \binom{n}{k}\langle T_x \otimes S_y, x^k y^{n-k}\varphi(x+y)\rangle \\ &= \sum_{k=0}^n \binom{n}{k}\langle (x^k T_x) \otimes (y^{n-k}S_y), \varphi(x+y)\rangle \\ &= \left\langle \sum_{k=0}^n \binom{n}{k}(x^k T) * x^{n-k}S), \varphi \right\rangle.\end{aligned}$$

Exercise 3.6 Let $x \in \mathbb{R}$, we have

$$G_\sigma * G_\tau(x) = \int_{-\infty}^{+\infty} G_\sigma(t) G_\tau(x-t)\, dt = \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{t^2}{\sigma^2} + \frac{(x-t)^2}{\tau^2}\right)}\, dt.$$

Moreover, we have

$$\frac{t^2}{\sigma^2} + \frac{(x-t)^2}{\tau^2} = \frac{x^2}{\sigma^2+\tau^2} + \frac{\sigma^2+\tau^2}{\sigma^2\tau^2}\left(t - \frac{\sigma^2}{\sigma^2+\tau^2}x\right)^2,$$

then

$$G_\sigma * G_\tau(x) = \frac{1}{2\pi\sigma\tau} e^{-\frac{x^2}{2(\sigma^2+\tau^2)}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\frac{\sigma^2+\tau^2}{\sigma^2\tau^2}\left(t - \frac{\sigma^2}{\sigma^2+\tau^2}x\right)^2}\, dt.$$

Let $u = \frac{\sqrt{\sigma^2+\tau^2}}{\sigma\tau}\left(t - \frac{\sigma^2}{\sigma^2+\tau^2}x\right)$, hence

$$G_\sigma * G_\tau(x) = \frac{1}{2\pi\sqrt{\sigma^2+\tau^2}} e^{-\frac{x^2}{2(\sigma^2+\tau^2)}} \int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}}\, du.$$

Since $\int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}}\, du = \sqrt{2\pi}$ we obtain

$$G_\sigma * G_\tau(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma^2+\tau^2}} e^{-\frac{x^2}{2(\sigma^2+\tau^2)}} = G_{\sqrt{\sigma^2+\tau^2}}(x).$$

Exercise 3.7

1. $E = zH$ is an elementary solution of P if and only if $PE = \delta_0$. We have

$$\begin{aligned}
\frac{dE}{dx} &= \frac{d(zH)}{dx} \\
&= \frac{dz}{dx}H(x) + z\frac{dH}{dx} \\
&= \frac{dz}{dx}H(x) + z\delta_0 \\
&= \frac{dz}{dx}H(x) + z(0)\delta_0 \quad (z(0) = 0) \\
&= \frac{dz}{dx}H(x),
\end{aligned}$$

and

$$\begin{aligned}\frac{d^2E}{dx^2} &= \frac{d^2z}{dx^2}H(x) + \frac{dz}{dx}\delta_0 \\ &= \frac{d^2z}{dx^2}H(x) + \frac{dz(0)}{dx}\delta_0 \qquad \left(\frac{dz(0)}{dx} = 1\right) \\ &= \frac{d^2z}{dx^2}H(x) + \delta_0.\end{aligned}$$

Therefore

$$\begin{aligned}PE &= \frac{d^2E}{dx^2} - \frac{dE}{dx} - 2E \\ &= \frac{d^2z}{dx^2}H(x) + \delta_0 - \frac{dz}{dx}H(x) - 2zH(x) \\ &= (\frac{d^2z}{dx^2} - \frac{dz}{dx} - 2z)H(x) + \delta_0 \\ &= Pz\,H(x) + \delta_0,\end{aligned}$$

since $Pz = 0$ we get $PE = \delta_0$.

2. The solution to the equation $Pu = \chi_{[0,1]}$ in $\mathcal{D}'(\mathbb{R})$ is therefore

$$u = E * \chi_{[0,1]},$$

where E is an elementary solution of P.

Exercise 3.8

1. Let K be a compact subset of $\mathbb{R}^2$. There exists $a > 0$ such that $K \subset [-a, a]^2$. Then we have

$$\begin{aligned}\|E\|_{L^1(K)} &= \int_K E(x,t)dxdt \\ &\leq \int_{-a}^{a}\int_{-a}^{a} E(x,t)\,dx\,dt \\ &= \int_{0}^{a}\int_{-a}^{a} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\,dx\,dt \\ &\leq \int_{0}^{a}\int_{-a}^{a} \frac{1}{\sqrt{4\pi t}}\,dx\,dt \\ &= 2a\sqrt{\frac{a}{\pi}}.\end{aligned}$$

Then $E \in L^1_{\text{loc}}(\mathbb{R}^2)$ and defines a distribution of order 0 on $\mathbb{R}^2$.

2. Standard calculations show that for all $t > 0$ and all $x \in \mathbb{R}$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\right) &= \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4\pi t}}\right)e^{-\frac{x^2}{4t}} + \frac{1}{\sqrt{4\pi t}}\frac{\partial}{\partial t}\left(e^{-\frac{x^2}{4t}}\right)\\ &= \left(-\frac{1}{2\sqrt{4\pi}t^{\frac{3}{2}}} + \frac{x^2}{4\sqrt{4\pi}t^{\frac{5}{2}}}\right)e^{-\frac{x^2}{4t}}\\ &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\right).\end{aligned}$$

3. Let $\varepsilon > 0$ and let $\varphi \in \mathcal{D}(\mathbb{R}^2)$. Define

$$I_\varepsilon = -\int_{\mathbb{R}}\int_{\varepsilon}^{+\infty} E(x,t)\frac{\partial\varphi}{\partial t}(x,t)\,dt\,dx \quad \text{and} \quad J_\varepsilon = -\int_{\mathbb{R}}\int_{\varepsilon}^{+\infty} E(x,t)\frac{\partial^2\varphi}{\partial x^2}(x,t)\,dt\,dx.$$

i) A simple integration by parts, combined with the identity proved in part 2, shows that

$$\begin{aligned}I_\varepsilon &= -\int_{\mathbb{R}}[E(x,t)\varphi(x,t)]_{t=\varepsilon}^{t=+\infty}\,dx + \int_{\mathbb{R}}\int_{\varepsilon}^{+\infty}\frac{\partial E}{\partial t}(x,t)\varphi(x,t)\,dt\,dx\\ &= \int_{\mathbb{R}}E(x,\varepsilon)\varphi(x,\varepsilon)\,dx + \int_{\mathbb{R}}\int_{\varepsilon}^{+\infty}\frac{\partial E}{\partial t}(x,t)\varphi(x,t)\,dt\,dx\\ &= \int_{\mathbb{R}}E(x,\varepsilon)\varphi(x,\varepsilon)\,dx + \int_{\mathbb{R}}\int_{\varepsilon}^{+\infty}\frac{\partial^2 E}{\partial x^2}(x,t)\varphi(x,t)\,dt\,dx.\end{aligned}$$

ii) Using part i), Fubini's theorem and integrating by parts twice, we obtain

$$\begin{aligned}I_\varepsilon &= \int_{\mathbb{R}}E(x,\varepsilon)\varphi(x,\varepsilon)\,dx + \int_{\mathbb{R}}\int_{\varepsilon}^{+\infty}\frac{\partial^2 E}{\partial x^2}(x,t)\varphi(x,t)\,dt\,dx\\ &= \int_{\mathbb{R}}E(x,\varepsilon)\varphi(x,\varepsilon)\,dx + \int_{\varepsilon}^{+\infty}\int_{\mathbb{R}}\frac{\partial^2 E}{\partial x^2}(x,t)\varphi(x,t)\,dx\,dt\\ &= \int_{\mathbb{R}}E(x,\varepsilon)\phi(x,\varepsilon)\,dx + \int_{\varepsilon}^{+\infty}\underbrace{[E(x,t)\varphi(x,t)]_{x=-\infty}^{x=+\infty}}_{=0}\,dt\\ &\quad - \int_{\varepsilon}^{+\infty}\int_{\mathbb{R}}\frac{\partial E}{\partial x}(x,t)\frac{\partial\varphi}{\partial x}(x,t)\,dx\,dt\\ &= \int_{\mathbb{R}}E(x,\varepsilon)\varphi(x,\varepsilon)\,dx - \int_{\varepsilon}^{+\infty}\int_{\mathbb{R}}\frac{\partial E}{\partial x}(x,t)\frac{\partial\varphi}{\partial x}(x,t)\,dx\,dt\\ &= \int_{\mathbb{R}}E(x,\varepsilon)\varphi(x,\varepsilon)\,dx - \int_{\varepsilon}^{+\infty}\underbrace{[E(x,t)\frac{\partial\varphi}{\partial x}(x,t)]_{x=-\infty}^{x=+\infty}}_{=0}\,dt\\ &\quad + \int_{\varepsilon}^{+\infty}\int_{\mathbb{R}}E(x,t)\frac{\partial^2\varphi}{\partial x^2}(x,t)\,dx\,dt\\ &= \int_{\mathbb{R}}E(x,\varepsilon)\varphi(x,\varepsilon)\,dx - J_\varepsilon.\end{aligned}$$

iii) From part ii), we have

$$I_\varepsilon + J_\varepsilon = \int_{\mathbb{R}} E(x,\varepsilon)\varphi(x,\varepsilon)\,dx = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\varepsilon}} e^{-\frac{x^2}{4\varepsilon}} \varphi(x,\varepsilon)\,dx.$$

We set $y = \frac{x}{\sqrt{\varepsilon}}$ we get

$$I_\varepsilon + J_\varepsilon = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} e^{-\frac{y^2}{4}} \varphi(\sqrt{\varepsilon}y,\varepsilon)\,dy.$$

Using the dominated convergence theorem, we deduce that

$$\begin{aligned}
\lim_{\varepsilon\to 0}(I_\varepsilon + J_\varepsilon) &= \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \lim_{\varepsilon\to 0} e^{-\frac{y^2}{4}} \varphi(\sqrt{\varepsilon}y,\varepsilon)\,dy \\
&= \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \lim_{\varepsilon\to 0} e^{-\frac{y^2}{4}} \varphi(0,0)\,dy \\
&= \varphi(0,0)\frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \lim_{\varepsilon\to 0} e^{-\frac{y^2}{4}}\,dy \\
&= \varphi(0,0) \\
&= \langle \delta_{(0,0)}, \varphi \rangle.
\end{aligned}$$

4. Using distributional differentiation, we obtain

$$\begin{aligned}
\left\langle \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) E, \varphi \right\rangle &= \left\langle \frac{\partial E}{\partial t}, \varphi \right\rangle - \left\langle \frac{\partial^2 E}{\partial x^2}, \varphi \right\rangle \\
&= -\left\langle E, \frac{\partial \varphi}{\partial t} \right\rangle - (-1)^2 \left\langle E, \frac{\partial^2 \varphi}{\partial x^2} \right\rangle \\
&= -\left\langle E, \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} \right\rangle \\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}} E(x,t) \left(\frac{\partial \varphi}{\partial t}(x,t) + \frac{\partial^2 \varphi}{\partial x^2}(x,t)\right) dx\,dt \\
&= \lim_{\varepsilon\to 0}(I_\varepsilon + J_\varepsilon).
\end{aligned}$$

Then

$$\left\langle \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) E, \varphi \right\rangle = \langle \delta_{(0,0)}, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^2).$$

In other words,

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) E = \delta_{(0,0)} \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

(ie $PE = \delta_{(0,0)}$ and E is an elementary solution of P).

Additional exercises

Exercise 3.9. *Let h be a C^1-diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Let T be a linear map defined by*

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^2), \quad \langle T, \varphi \rangle = \int_{\mathbb{R}} \varphi(x, h(x))\, dx.$$

1. *Show that $T \in \mathcal{D}'(\mathbb{R}^2)$. What is its order?*
2. *Determine the support of T.*

Exercise 3.10. *Consider the linear form T on $\mathcal{D}(\mathbb{R})$ given by*

$$T : \varphi \mapsto \sum_{k=1}^{+\infty} \frac{1}{\sqrt{k}} \left(\varphi\left(\frac{1}{k}\right) - \varphi\left(-\frac{1}{k}\right) \right).$$

1. *Show that T is a distribution, of order 1.*
2. *Show that the support of T is*

$$\mathcal{A} = \{0\} \cup \{\frac{1}{k},\, k \in \mathbb{Z}^*\}.$$

Exercise 3.11. *Let T be defined by*

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^2), \quad \langle T, \varphi \rangle = \int_{\mathbb{R}} \varphi(x, -x),\, dx.$$

Determine the support of T.

Exercise 3.12. *I) Calculate $\delta'_a \otimes \delta'_b$.*
II) Let $\psi(x) \in C^\infty(\mathbb{R}^m)$. Show that

$$\psi(x)\, [T_1 \otimes T_2] = [\psi(x) T_1 \otimes T_2],$$

where $T_1 \in \mathcal{D}'(\mathbb{R}^m)$, $T_2 \in \mathcal{D}'(\mathbb{R}^n)$.

Exercise 3.13. *I) Calculate*

$$\delta_a^{(m)} * \delta_b^{(n)}, \quad \delta_a * H \quad \text{and} \quad (1_{[a,b]} * 1_{[c,d]})''.$$

II) Let $f_1, f_2 \in L^1_{loc}(\mathbb{R})$. Show that

$$T_{Hf_1} * T_{Hf_2} = T_{Hf_1 \star Hf_2} = T_F,$$

where H the Heaviside function and

$$F(x) = H(x) \int_0^x f_1(t) f_2(x-t) dt.$$

Exercise 3.14. *Consider the differential operator $P = \frac{d^2}{dx^2} + a\frac{d}{dx} + b$, $a, b \in \mathbb{R}$, acting on $\mathcal{D}'(\mathbb{R})$.*
Let f and g be two functions of class C^2 on $\mathbb{R}$ such that $Pf = Pg = 0$, $f(0) = g(0)$ and $f'(0) - g'(0) = 1$. Consider the function h defined by

$$\forall x \in \mathbb{R},\ h(x) = \begin{cases} f(x) & \text{if} \quad x \leq 0 \\ g(x) & \text{if} \quad x > 0. \end{cases}$$

Finally, let T be the distribution defined by,

$$\forall \varphi \in \mathcal{D}(\mathbb{R}),\ \langle T, \varphi \rangle = -\int_{\mathbb{R}} h(x)\varphi(x) dx.$$

Show that $PT = \delta_0$, in the sense of distributions.

Exercise 3.15. *Consider the distribution defined in the plane by the locally integrable function*

$$E(x,t) = \begin{cases} \frac{1}{2} & \text{if } t - |x| > 0, \\ 0 & \text{if } t - |x| < 0. \end{cases}$$

Let $\Box$ be a wave operator defined by

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}.$$

Solve $\Box E$ in the sense of distributions.

Bibliography

[1] **L. Ait Mahiout.** Théorie des Distributions et Analyse de Fourier-Résumés de cours et exercices corrigés. 2021.

[2] **F. Bayen.** Distributions, analyse de Fourier, transformation de laplace, Ellipses, 1988.

[3] **J. M. Bony.** Cours d'analyse, théorie des distributions et analyse de fourier, les éditions de l'école polytechnique, Ellipses, 2001.

[4] **H. Boumaza**. Théorie des distributions, U.P.N- Sup Galilée, Formation Ingénieurs MACS/M1 Mathématiques, 2015.

[5] **H. Brezis.** Analyse Fonctionnelle, Théorie et applications, Masson, 1987.

[6] **G.B. Folland.** Fourier Analysis and Its Applications, Pacific Grove, CA: Brooks/Cole, 1992.

[7] **G.B. Folland.** Real Analysis: Modern Techniques and Their Applications, 2nd Edition,2nd Edition, John Wiley & Sons, Inc., New York, 1999.

[8] **G. Friedlander, M. Joshi.** Tntroduction to the Theory of Distributions. 2nd Edition, Cambridge University Press, UK, 1998.

[9] **L. Hormander.** The analysis of linear partial differential operators I: Distribution theory and Fourier analysis. Springer, 2015.

[10] **G. Hormann, R. Steinbauer.** Theory of distributions, Facultat fur Mathematik, Universitat Wien, 2009.

[11] **A. Lesfari.** Distributions, analyse de Fourier et transformation de Laplace, France, 2012.

[12] R. S. Pathak. A Course in Distribution Theory and Applications. Alpha Science, 2001.

[13] **D. E. Pérez.** Distribution Theory and Fundamental Solutions of Differential Operators, , 2015.

[14] **L. Schwartz.** Théorie des Distributions. Hermann, Paris, 1966.

[15] **V. Vladimirov.** Distributions en Physique Mathématique. Edition Mir, Moscou, 1979.

[16] **C. Zuily.** Éléments de distributions et d'équations aux dériveés partielles, Sciences Sup, Dunod, 2002.

[17] **C. Zuily.** Problems in Distributions and Partial Differential Equations. Hermann, Paris, 1988.