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Sur les équations intégro-différentielles et la méthode des éléments finis

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Dedication

In loving memory of my dear mother, who now rests in heaven. This achievement is dedicated to you, Mom. Your love, sacrifices, and guidance have shaped me into who I am today.

Though you are not here to celebrate with me, I carry your love in my heart always.

This success is for you.

my brothers ***Said*** *and* ***Sofiane***

my sisters and all my family

my husband ***Zekari***

my best friend ***Lamia***

I dedicate this thesis.

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List of Symbols and Abbreviations

$\mathbb{R}$	$(-\infty, +\infty)$.
K, L	Integral operator.
$k(s,t)$	Kernel of integral operator.
$\langle ., . \rangle$	Scalar product
$H^m_w(\Omega)$	Sobolev space.
$L^2_w(\Omega)$	Space of measurable functions $u : \Omega \to \mathbb{R}$ such that $\|u\|_{w^{\alpha,\beta}} < \infty$.
$L^\infty(\Omega)$	Space of measurable functions $u : \Omega \to \mathbb{R}$ such that are bounded outside a set of measure zero.
$\|u\|_p$	Norm in $L^p(\Omega)$, $\|u\|_{\,p} = \left(\int_\Omega \lvert u(s)\rvert^P ds\right)^{\frac{1}{p}}$.
$\|u\|_{w^{\alpha,\beta}}$	Norm in $L^2_w(\Omega)$, $\|u\|_{w^{\alpha,\beta}} = \left(\int_\Omega \lvert u(s)\rvert^2 w^{\alpha,\beta}(s) ds\right)^{\frac{1}{2}}$.
$\|u\|_{L_\infty}$	Norm in $L^\infty(\Omega)$ defined by $ess \sup\limits_{s\in\Omega} \lvert u(s)\rvert$.
$P, \mathcal{P}_N$	Projection operators.
$\mathbb{R}[x]$	Set of all polynomials (of all degrees) in one variable x.
$\mu(\Omega)$	Lebesgue measure of Ω.
δ_{mn}	Kronecker symbol defined by $\delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$

Introduction

This thesis is part of the study and construction of numerical methods for some integro-differential equations

Integro-differential equations appear in various fields(see [1], [2], [3], [4]),These equations come from the mathematical modeling of very varied scientific phenomena, such as: the dynamics of fluids, solid physics, plasma physics, viscoelasticity of biology([5]),the economy([6]),heat transfer([7]),medicine, chemostat([8]),biotissues([9]),static analysis of towers and chimneys([10]), chemical kinetics([11]),ect.

Differential equations connect unknown functions, their derivatives and their independent variables. On the other hand, integral equations contain unknown functions under an integral. The term integro-differential equation is used in the case where the equation contains an unknown function with its derivatives and when either an unknown function or its derivatives or both appear under an integral([12])

Integral and integro-differential equations arise in many scientific and engineering applications. Volterra integral equations and Volterra integro-differential equations can be obtained from converting initial value problems with prescribed initial values. However, Fredholm integral equations and Fredholm integro-differential equations can be derived from boundary value problems with given boundary conditions. It is important to point out that converting initial value problems to Volterra integral equations, and converting Volterra integral equations to initial value problems are commonly used in the literature. This will be explained in detail in the coming section. However, converting boundary value problems to Fredholm integral equations, and converting Fredholm integral equations to equivalent boundary value problems are rarely used. The conversion techniques will be examined and illustrated examples will be presented. In what follows we will examine the steps that we will use to obtain these integral and integro-differential equations.

The concepts of integro-differential equations have motivated a large number of research works in recent years. These equations are difficult to solve analytically, so it is necessary to obtain an effective approximate solution. Several numerical methods have been applied for these equations such as:El Gendi([13]),Chebyshev polynomials([14],[15]), Chebyshev wavelets([16],[17]), Galerkin-Chebyshev([18]), Bernstein polynomials([19]), Taylor polynomials ([20]), Bernoulli matrix ([18]), Legendre polynomials ([21],[22],[23]), Legendre wavelet ([24]), Legendre-Galerkin ([25]), Legendre Collocation ([26]), Tau ([27]), Lagrange Interpolation ([28],[29]), Spline Interpolation [30], Bessel Polynomials ([31]), Negative Difference ([32]) and etc.

The finite element method (FEM) is a numerical technique for solving problems that are described by partial differential equations or can be formulated as a minimization function. A domain of interest is represented as an assembly of finite elements. Finite element approximation functions are determined in terms of point values of a physical field that is sought. A continuous physical problem is transformed into a discretized finite element problem with unknown point values. For a linear problem, a system of linear algebraic equations must be solved. The values inside the finite elements can be recovered using nodal values. Therefore, the basic idea of the finite element method is to replace continuous functions with piecewise approximations, usually polynomials.

The finite element method is a result of two areas of research: mathematics and engineering sciences. Mathematics: Tools that go back to the weighted residuals of Gauss (1775), Galerkin (1915) and Biezenokoch (1923), as well as the variational methods of Rayleigh (1870) and Ritz (1909). Engineering sciences: Whose contribution began in the 1940s with Hrenikoff (1941), Henry (1943) and Newmark (1949) who touched continuous structures for the first time, making an approximation on small portions in a continuous problem of a long bar. Hence the basic idea of finite elements. Argyris (1955), Turner, Glough and Martin (1956) made a direct analogy by adopting simplified behavior for small portions. The term "finite element" was first used by Glough (1960) and from then on there was rapid development of the method.

Today, the finite element method (FEM) is considered one of the well-established and practical techniques for the numerical solution of complex problems in different fields of engineering: civil engineering, mechanical engineering, nuclear engineering, biomedical engineering, hydrodynamics, conduction mechanical thermal, etc. On the other hand, FEM can be examined as a powerful tool for the approximate solution of differential equations describing different physical phenomena.

In this work, we presented the finite element method to solve linear integro-differential equations with Dirichlet boundary conditions problems, whether homogeneous or not, present significant challenges when it comes to solving them analytically. Our problem, which is first-order and non-elliptic, raises additional difficulties in establishing the existence and uniqueness of the solution. In this case, it has been proposed that a solution exists, and we now seek it numerically. The origins of the study of integral and integro-differential equations date back to the work of Abel, Lotka, Fredholm, Malthus, Verhulst and Volterra on problems in mechanics, mathematical biology and economics. Volterra's work on the competing species problem is of fundamental importance to the development of mathematical modeling of real-world problems.

Content of each chapter is summarized as follows:

The first chapter consists of a reminder of the definition of some functional spaces, which we used in our work. Then,we defined the theory of operators and their types(bounded, closed and unbounded operator),we have explained how to reverse each type and also how to disrupt the sum of two operators of different types. finally, we give the Lax Milgram theorem to show the existence and uniqueness of numerical solutions and define Galerkin method which is considered as one of the finite element methods to give the approximate solution to the equations.

In chapter two, consists of recalling the definition of integro-differential equations and their classifications, which aims to familiarize the reader with the concept of these equations. Then, we establish the origin of the integro-differential equations and the relationship between them and the differential equations. Finally, we give three major fixed point theorems: Banach, Brouwer and Schauder. These theorems are important for showing the existence of numerical solutions for nonlinear equations.

In chapter three, we described different analytical solution methods on the integro-differential equations of Volterra and Fredholm, such as Direct computation method,Adomian decomposition method, Modified decomposition method and Series solution method.We also made an approximate solution of the integral equations by Gauss Legendre quadrature and an approximate solution of the differential equations (Poisson equation) using the finite deference method.

The subject of the last chapter, is our work and we contribute as a publication of application of finite element methods to solve some linear Fredholm integro-differential equations, in which we will discuss our convergence analysis, and illustrate the efficiency of the present methods by instructive examples.

Chapter 1

Basic Concepts and Preliminaries

In this chapter we recall some notions and theories fondamentals, we use in next chapter

1.1 Functional spaces

We consider E and F linear spaces

1.1.1 Normed space

Definition 1.1 ([33]). *1. A norm on a linear space E is a real-valued function, whose value at x is denoted by $\parallel x \parallel$ and has the properties:*

- $\parallel x \parallel \geq 0$,
- $\parallel x \parallel = 0$ *iff* $x = 0$,
- $\parallel \alpha x \parallel = |\alpha| \parallel x \parallel$,$\alpha$ *is scalar,*
- $\parallel x + y \parallel \leqslant \parallel x \parallel + \parallel y \parallel$.

2. Normed linear space: A linear space with a defined norm is called a normed linear space.

3. The normed space E is said to be complete if every Cauchy sequence in E converges to an element of E.

1.1.2 Banach spaces

Definition 1.2 (Banach space [33]). *If a normed linear space E is complete, it is called a Banach space.*

Theorem 1.1 ([33]). *Let E be a normed linear space. Then there exists a normed linear space F such that F is complete and E is a dense subset of F. Up to isometry, the space F is unique.*

1.1.3 Hilbert space

Definition 1.3. *(Inner-product space)*[34]
A complex linear space E is called an inner-product space if to each pair of elements x, y of E there is associated a complex number (x,y)(called the inner product of x and y)with the following properties:

1. $(x+y,z)=(x,z)+(y,z)$, *for* $x,y,z \in E$
2. $(x,y)=\overline{(y,x)}$ *[the bar denotes complex conjugate],*
3. $(\alpha x,y)=\alpha(x,y)$, *for all scalars* α,
4. $(x,x) \geq 0$ *[it must be real by 2], and* $(x,x) \neq 0$ *if* $x \neq 0$.

Inner-product spaces are special cases of normed linear spaces. This is expressed by the following lemma.

Lemma 1.1. [34] *Let E be a linear space with inner product$(\cdot,\cdot)$.Then the expression*

$$\| x \|= \sqrt{(x,x)}, \quad x \in E,$$

defines a norm on E.

Definition 1.4. *(Hilbert space)*[34]
A Hilbert space is an inner product space which(as a normed linear space)is complete.

1.1.4 Lebesgue space [35]

Let $p \in R$ with $1 < p < \infty$ and $\Omega \subset R^N$ a measurable whole in the sense of Lebesgue, we define.

$$L^p(\Omega)=\{f:\Omega \to R \; f is measurable and \int_\Omega |f|^p dx < \infty\}.$$

We define the norm of f in $L^p(\Omega)$ by

$$\| f \|_p = (\int_\Omega |f|^p dx)^{1/p}$$

if $p=\infty$ we define

$$L^\infty(\Omega)=\{f:\Omega \to R \; f \text{ is measurable and } \exists C \geq 0 / \mid f(x) \mid \leq C \;\; \mu{-}almost\;\; everywhere\;\; on\;\; \Omega\}.$$

$$\| f \|_\infty = \inf\{C \geq 0 : \mid f(x) \mid \leq C \quad \mu - almost \;\; everywhere\},$$

is the norm of f in $L^\infty(\Omega)$

Definition 1.5. *(Espace $L^2(\Omega)$) for $p=2$ we note $L^2(\Omega)$ the set of summable square functions i.e*

$$L^2(\Omega)=\left\{u:\Omega \to R | \int_\Omega |u(x)|^2 dx < \infty\right\}.$$

$L^2(\Omega)$ is a linear functional space

Remark 1.1. *The space $L^2(\Omega)$ is a Hilbert space for the inner product*

$$(f,g)=\int_\Omega f(x)g(x)dx. \tag{1.1}$$

Theorem 1.2. *(Cauchy-Schwartz inequality)*
Let u and v functions of $L^2(\Omega)$, So

$$\mid \int_\Omega u(x)v(x)dx \mid\leqslant \left(\int_\Omega \mid u(x)\mid^2 dx\right)^{1/2}\left(\int_\Omega \mid v(x)\mid^2 dx\right)^{1/2}.$$

Definition 1.6. *(Hilbert Space $L^2(\Omega)$)*
The linear space $L^2(\Omega)$ provided with the inner product

$$< u,v >_{0,\Omega}=\int_\Omega u(x)v(x)dx.$$

is a Hilbert Space

1.1.5 Sobolev space [35]

Sobolev spaces are functional spaces (i.e. spaces made up of functions) whose powers and derivatives (in the sense of transposition, or in the weak sense) are integrable. Just like Lebesgue spaces, these spaces are Banach spaces (complete normalized vector spaces). The fact that they are complete is very important for the study of partial differential equations

Definition 1.7 ($H^1(\Omega)$ Space)**.** *We note $H^1(\Omega)$ the linear functional space defined by*

$$H^1(\Omega)=\left\{u\in L^2(\Omega):u'\in L^2(\Omega)\right\}.$$

Which we provide to the inner product:

$$< u,v >_{1,\Omega}=\int_\Omega u(x)v(x)dx+\int_\Omega u'(x)v'(x)dx,$$

and we note the corresponding norm:

$$\parallel u\parallel_{H^1(\Omega)}=\parallel u\parallel_{1,\Omega}=\left[\int_\Omega |u(x)|^2\right]^{1/2}+\left[\int_\Omega |u'(x)|^2dx\right]^{1/2}.$$

Definition 1.8. *(Trace at the edge)*
The restriction at the edge Γ of a function $w\in H^1(\Omega)$ is called a trace at the edge of w and is denoted $w|_\Gamma$ or $\gamma_0(w)$, where $\gamma_0:H^1(\Omega)\to L^2(\Omega)$.

Theorem 1.3. *(Trace at the edge)*
All traces at the edge of functions $H^1(\Omega)$ forms a subspace of $L^2(\Gamma)$ note $H^{\frac{1}{2}}(\Gamma)$. More

succinctly, we have:

$$\gamma_0(H^1(\Omega)) = H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma).$$

Theorem 1.4. *(Continuity of the trace at the edge)*
if $w \in H^1(\Omega)$, exists a constant C such that:

$$\|\gamma_0(w)\|_{L^2(\Gamma)} \leqslant C \parallel w \parallel_{H^1(\Omega)},$$

where

$$\|\gamma_0(w)\|_{L^2(\Gamma)} = \left[\int_\Gamma |\gamma_0(w)|^2 ds\right]^{1/2}.$$

The result is also true if we replace Γ by a non-zero part Γ_0 of Γ.

Definition 1.9. *We define the space $H_0^1(\Omega)$ the closure of $D(\Omega)$ for the norm $\parallel . \parallel$, thus, for each $v \in H_0^1(\Omega)$, there exists a suite $\varphi_n \in D(\Omega)$ such as*

$$lim_{n\to\infty} \parallel \varphi_n - v \parallel_{H^1(\Omega)} \to 0.$$

The functions of $H_0^1(\Omega)$ therefore cancel at the edge and we can write:

$$H_0^1(\Omega) = \{w \in H^1(\Omega) \quad w = 0 \quad on \quad \Gamma\} = H_\Gamma^1(\Omega) = ker \quad \gamma_0,$$

where

$$ker \quad \gamma_0 = \{w \in H^1(\Omega)\gamma_0(w) = 0\}.$$

We can also define

$$H_{\gamma_0}^1(\Omega) = \{w \in H^1(\Omega) \quad w = 0 \quad on \quad \Gamma_0\},$$

where Γ_0 is part of the border Γ of the domain Ω.

1.1.6 Poincaré's inequality

Theorem 1.5. *(Poincaré's inequality)*
Let Ω a regular bounded open of R,There then exists a constant C which only depends on Ω, and such that:

$$\|u\|_{L^2(\Omega)} \leqslant C(\Omega) \parallel \nabla u \parallel_{L^2(\Omega)}, \forall u \in H_0^1(\Omega).$$

1.2 Operator theory

1.2.1 Bounded Linear Operator

Let E and F be linear spaces over $\Bbbk$ The operator $L : E \to F$ is called linear if for every $u, v \in E$ and $\alpha, \beta \in \Bbbk$, we have

$$L(\alpha u + \beta v) = \alpha Lu + \beta Lv.$$

Definition 1.10. *Let E and F be normed linear spaces. A linear operator $L : E \to F$ is called a bounded linear operator if there exists a positive constant $c > 0$ such that*

$$\| Lx \|_F \leq c \| x \|_E \qquad for \;\; all \;\; x \in E.$$

1.2.2 Closed Operators

Definition 1.11. *Let E and F be normed spaces and $D \subset E$ a subspace. A linear operator $T : D \to E$ is called closed, if for each sequence $(x_n) \subset D$ we have that $(x_n) \to x$ $T(x_n) \to y$ imply $x \in D$ and $Tx = y$.*

1.2.3 Closable operators

Definition 1.12. *An operator T from E to F is said to be closable if T has closed extension. In other words, T is closable if and only if*

$$u_n \in D(T), u_n \to 0 \quad and \quad Tu_n \to v \quad imply \quad v = 0.$$

1.2.4 Unbounded Operators

Definition 1.13. *Let E and F be normed spaces we say that an linear operator A defined on $D(A) \subset E$ in F is an unbounded operator if $D(A) \neq E$.*

Remark 1.2. *Unbounded operator can only be in infinite spaces because in infinite spaces all linear operators are bounded.*

1.3 Operator inversion

Definition 1.14. *The set of linear applications of E in F is denoted $L(E, F)$. When $E = F$ the space $L(E, F)$ becomes by notation $L(E)$, The set of continuous linear applications of E in F is denoted $\ell(E, F)$. When $E = F$ the space $\ell(E, F)$ becomes by notation $\ell(E)$.*

Definition 1.15. *We denote by $R(A)$ the image of the operator $A : E \to F.R(A)$ is the vector subspace of F*

$$R(A) = \{Au : u \in E\}.$$

Definition 1.16. *We denote by $N(A)$ the kernel of the operator $A : E \to F$. $N(A)$ is the vector subspace of E*

$$N(A) = \{u \in E : Au = 0\}.$$

Definition 1.17. *Let E be a Banach space and $A \in \ell(H)$*
The resolvent set of A is the set

$$\rho(A) = \{\lambda \in C : (\lambda I - A)^{-1} \quad is \;\; bounded\}.$$

$$\rho(A) = \{\lambda \in C : (\lambda I - A)^{-1} \quad is \ \ bijective\}.$$

Definition 1.18. *Let A be a closed operator, then A is said to be regular if $R(A)$ closed and $\forall n \geq 0$, $N(A^n) \subseteq R(A)$.*

1.3.1 Inversibility of Operators

1.3.1.1 Inversibility of Bounded Operator

Let H Hilbert space and $A \in L(H)$ an operator, if $dimH < +\infty$. The following properties are equivalent:

(1) A is invertible.

(2) A is injective.

(3) A is surjective.

(4) A admits an inverse on the right (i.e. it exists $U \in L(H)$ such as $A \circ U = I_H$.

(5) A admits an inverse on the left (i.e. it exists $V \in L(H)$ such as $V \circ A = I_H$.

1.3.1.2 Inversibility of Closed Operator

Let $A \in L(H)$ regular, then the following two conditions are equivalent:

(a) $\exists B \in L(H)$generalized resolver from A to zero commuting with A.

(b) A is invertible.

1.3.1.3 Inversibility of Unbounded operators

Let H be a Hilbert space and $A : D(A) \subset H \to H$ a bijective linear operator.We define the operator A^{-1} defined from H in H such that:

$$AA^{-1}v = v, \forall v \in H_1 \ \ et \ \ A^{-1}Au = u, \forall u \in D(A).$$

The operator A^{-1} is said to be the inverse of A.

1.4 Perturbation of operators

Problem(1.1)

$$T + A \quad is \quad closed, \quad if \quad T \quad is \quad closed \quad and \quad A \quad is \quad bounded \quad with \quad D(A) \supset D(T). \tag{1.2}$$

Stability of closedness under relatively bounded perturbation

Let $T \in C(X,Y)$ where X, Y are Banach spaces [$C(X,Y)$ is the set of all closed operators from X to Y]. We have already noted (Problem 1.2) that $T+A$ is also closed if $A \in B(X,Y)$ [such that $B(X,Y)$ is a normed algebra]. This expresses the fact that closedness is stable under a bounded perturbation A. We now try to extend this stability theorem to a not necessarily bounded perturbation. An immediate extension of this kind can be made to the case of a relatively bounded perturbation. Let T and A be operators with the same domain space X (but not necessarily with the same range space) such that $D(T) \subset D(A)$ and

$$\| Au \| \leqslant a \| u \| + b \| Tu \|, \qquad u \in D(T), \tag{1.3}$$

where a, b are positive constants. Then we shall say that A is relatively bounded with respect to T or simply $T-$*bounded*. The greatest lower bound b_0 of all possible constants b in (1.3) will be called the relative bound of A with respect to T or simply the $T-$*bound* of A. If b is chosen very close to b_0, the other constant a will in general have to be chosen very large, thus it is in general impossible to set $b = b_0$ in (1.3).

Obviously a bounded operator A is $T-$*bounded* for any T with $D(T) \subset D(A), A)$, with $T-$*bound* equal to zero.

The extension of the stability theorem for closedness mentioned above is now given in the following.

Theorem 1.6. *Let T and A be operators from X to Y, and let A be $T-$bounded with $T-$bound smaller than 1. Then $S = T + A$ is closable if and only if T is closable, in this case the closures of T and S have the same domain. In particular S is closed if and only if T is closable.*

Proof. [42] page 189. □

1.5 Lax-Milgram theorem [35]

Definition 1.19. *(Continuous linear forms).*

A linear form L on Hilbert space V equipped with the norm $\| \, . \, \|_V$, is said to continue if there exists a constant $C > 0$ such that

$$| L(v) | \leq C \| v \|_V, \qquad \forall \; v \in V.$$

Definition 1.20. *(Dual space).*

The set of all continuous linear forms on a Hilbert space V is called the dual space of V and is denoted V'.

Definition 1.21. *(Continuous bilinear forms).*

A bilinear form a is said to be continuous on $V \times V$ if there exists a constant C such that:

$$\mid a(u,v) \mid \leq C \parallel u \parallel_V \parallel v \parallel_V, \qquad \forall u, v \in V.$$

Definition 1.22. *(Coercive bilinear forms).*
A bilinear form a is said to be continuous on $V \times V$ if there exists α constant C such that

$$a(u,v) \geq \alpha \parallel u \parallel_V^2, \qquad \forall u \in V.$$

Theorem 1.7. *(Lax-Milgram)[35].*
Let V be a Hilbert space and let L and a be continuous linear forms and continuous bilinear on V and $V \times V$ respectively, $(L \in V')$ If moreover a is coercive, then there exists a unique solution $u \in V$, such that

$$a(u,v) = L(v), \qquad \forall v \in V. \tag{1.4}$$

Theorem 1.8. *(Stampcchia)[35].*
Under the same hypotheses as those of the Lax-Milgram theorem and if in addition the bilinear form $a(.,.)$ is symmetrical, the variational problem 1.4 is equivalent to the following minimization problem:

$$J(u) = \inf_{v \in V} J(v) = \inf_{v \in V} \frac{1}{2} \left\{ a(v,v) - L(v) \right\}.$$

The functional J is often called an energy functional in applications.

1.6 Lagrange Interpolation

One-dimensional case

Let ξ_i,$\phi(\xi_i) = \phi_i$, $i = 1, \ldots, N$, we define

$$\phi(\xi) = \sum_{k=0}^{N} L_k(\xi)\phi_k, \qquad k = 0, \ldots, N.$$

such as

$$L_i(\xi_j) = \begin{cases} 1 & j = i, \\ 0 & j \neq i. \end{cases},$$

and

$$\sum_{k=0}^{N} L_i(\xi) = 1,$$

and $L_k(\xi)$ given by

$$L_i(\xi) = \prod_{m=0, \quad m \neq k}^{N} \left(\frac{\xi - \xi_m}{\xi_k - \xi_m} \right), \qquad k = 1, \ldots, N.$$

Two-dimensional case

Let $\phi(\xi_i, \eta)$,$i = 1, ..., N$ the local approximation of ϕ, and we write

$$\phi(\xi_i, \eta) = \sum_{k=0}^{N} L_i(\xi, \eta)\delta_i^e,$$

such that $L_i(\xi, \eta)$ are functions of local approximation of nodes.

1.7 Galerkin method

1.7.1 General principle

The Galerkin method makes it possible to approach the solution of model problems whose abstract formulation is as follows:

$$\begin{cases} Find \ \ u \in V \text{ such that:} \\ a(u, v) = L(v), \qquad \forall v \in V. \end{cases} \tag{1.5}$$

We assume that this problem checks the hypotheses of the Lax-Milgram theorem.
The idea of Galerkin's method is to replace the space V by a subspace $V_h \subset V$ of finite dimension (or $dimV_h = N$) and calculate u_h solution of problem:

$$\begin{cases} Find \ \ u_h \in V_h \ \ such \ \ that: \\ a(u_h, v) = L(v), \qquad \forall v \in V_h, \end{cases} \tag{1.6}$$

As $dimV_h = N$, so there is a basis $\varphi_1, ..., \varphi_N$ of V_h, by developing u_h on the basis $\varphi_1, ..., \varphi_N$ that's to say $u_h(x) = \sum\limits_{i=1}^{N} u_i\varphi_i(x)$ and $u_h(x_i) = u_i$ for all $i = 1, \dots, N$, and we take in (1.6) $v = \varphi_i$ we obtain

$$\begin{aligned} a(u_h, \varphi_i) &= L(\varphi_i), \qquad \forall i = 1, ..., N. \\ \Leftrightarrow \sum_{i=1}^{N} u_j a(\varphi_j, \varphi_i) &= L(\varphi_i), \qquad \forall i = 1, ..., N. \end{aligned}$$

Problem 1.6 is equivalent to

$$\begin{cases} Find \ \ U = (u_1, ..., u_N) \in R^N \ \ such \ \ that: \\ \sum\limits_{i=1}^{N} u_j a(\varphi_j, \varphi_i) = L(\varphi_i), \qquad \forall i = 1, ..., N. \end{cases}$$

we assume

$$A_h = (A_{ij})_{1 \leq i,j \leq N} \in R^{N \times N}, \quad A_{ij}) = a(\varphi_j, \varphi_i),$$

and

$$b_h = (b_i)_{1 \leq i \leq N} \in R^N, \qquad b_i = L(\varphi_i),$$

we obtain the linear system $A_h U = b_h$. The matrix A is called $stiffness\ matrix$ of the system.

1.7.2 Summary of the Galerkin technique

1. We give ourselves $V_h \subset V$,
2. We find a base of V_h,
3. We calculate A_h and the vector b_h,
4. We solve $A_h U = b_h$,
5. $u_h(x) = \sum\limits_{i=1}^{N} u_i \varphi_i(x)$.

Remark 1.3. *The matrix A is not symmetric in general.*
If the bilinear form $a(.,.)$ is symmetric, Galerkin's method is called Ritz's method, the only difference is that step 4 did not come from a minimization problem.

Chapter 2

Introductory to Integral and Integro-differential equations

In this chapter we introduce the basic concept of important equations in the applications of ours thesis and in ours contribute.

2.1 Integral Equations and Their Classifications

An integral equation is the equation in which the unknown function $u(x)$ appears inside an integral sign.
The form of the most common type of integral equation in $u(x)$ is as follows (see for example [2], [4] and [46]).

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t,u(t))dt, \tag{2.1}$$

where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $K(x,t)$ is a known function of two variables x and t, called the kernel of the integral equation. The unknown function $u(x)$ that will be determined appears inside the integral sign. In many other cases, the unknown function $u(x)$ appears inside and outside the integral sign. The functions $f(x)$ and $K(x,t)$ are given. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants, or mixed. Integral equations appear in many forms. Two distinct ways that depend on the limits of integration are used to characterize integral equations, namely

1. If the limits of integration are fixed, the integral equation is called a Fredholm integral equation given in the form:

$$u(x) = f(x) + \lambda \int_{a}^{b} K(x,t,u(t))dt. \tag{2.2}$$

2. If at least one limit is a variable, the equation is called a Volterra integral equation given in the form:

$$u(x) = f(x) + \lambda \int_{a}^{x} K(x,t,u(t))dt. \tag{2.3}$$

Moreover, two other distinct kinds, that depend on the appearance of the unknown function u(x), are defined as follows:

1. If the unknown function $u(x)$ appears only under the integral sign of Fredholm or Volterra equation, the integral equation is called a first kind Fredholm or Volterra integral equation respectively.
2. If the unknown function $u(x)$ appears both inside and outside the integral sign of Fredholm or Volterra equation, the integral equation is called a second kind Fredholm or Volterra equation integral equation respectively.

In all Fredholm or Volterra integral equations presented above, if $f(x)$ is identically zero, the resulting equation:

$$u(x) = \lambda \int_a^b K(x,t,u(t))dt. \tag{2.4}$$

or

$$u(x) = \lambda \int_a^x K(x,t,u(t))dt. \tag{2.5}$$

is called homogeneous Fredholm or homogeneous Volterra integral equation respectively.

2.1.1 Fredholm Integral Equations

Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x)$ may appear only inside integral equation in the form

$$f(x) = \lambda \int_a^b K(x,t,u(t))dt. \tag{2.6}$$

This is called Fredholm integral equation of the first kind. However, for Fredholm integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t,u(t))dt. \tag{2.7}$$

2.1.2 Volterra Integral Equations

Volterra integral equations, at least one of the limits of integration is a variable. For the first kind Volterra integral equations, the unknown function u(x) appears only inside integral sign in the form:

$$f(x) = \lambda \int_0^x K(x,t,u(t))dt. \tag{2.8}$$

However, Volterra integral equations of the second kind, the unknown function u(x) appears inside and outside the integral sign. The second kind is represented by the form

$$u(x) = f(x) + \int_0^x K(x,t,u(t))dt. \tag{2.9}$$

2.2 Integro-Differential Equations and their classifications

An integro-differential equation is an equation in which the unknown function $u(x)$ appears under an integral sign and contains an ordinary derivative $u^n(x)$ as well. A standard integro-differential equation is of the form:

$$\sum_{n=1}^{N} u^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t,u(t))dt. \tag{2.10}$$

where $g(x)$, $h(x)$, $f(x)$, λ and the kernel $K(x,t)$ are as prescribed before.

2.2.1 Fredholm Integro-Differential Equations

Fredholm integro-differential equations appear when we convert differential equations to integral equations. The Fredholm integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x)$, $n \geq 1$, inside and outside the integral sign respectively. The limits of integration in this case are fixed as in the Fredholm integral equations. The equation is labeled as integro-differential because it contains differential and integral operators in the same equation. It is important to note that initial conditions should be given for Fredholm integro-differential equations to obtain the particular solutions. The Fredholm integro-differential equation given by the form,

$$\sum_{n=1}^{N} u^{(n)}(x) = f(x) + \lambda \int_{a}^{b} K(x,t,u(t))dt, \tag{2.11}$$

where $u^{(n)}$ indicates the nth derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ at the left side. For example,

$$\begin{cases} u^{'}(x) = 1 - \dfrac{1}{3}x + \displaystyle\int_0^1 xu(t)dt, \\ u(0) = 0, \end{cases}$$

and

$$\begin{cases} u^{''}(x) + u^{'}(x) = x - sin(x) - \displaystyle\int_0^{\frac{\Pi}{2}} xtu(t)dt, \\ u(0) = 0, u^{'}(0) = 1. \end{cases}$$

2.2.2 Volterra Integro-Differential Equations

Volterra integro-differential equations appear when we convert initial value problems to integral equations. The Volterra integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x)$, $n \geq 1$, inside and outside the integral sign. At least one of the limits of integration in this case is a variable as in the Volterra integral equations. The equation is called integro-differential because differential and integral operators are involved in the same equation. It is important to note that initial conditions should be given for Volterra integro-differential equations to determine the particular solutions. The Volterra integro-differential equation appears in the form

$$\sum_{n=1}^{N} u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t,u(t))dt, \tag{2.12}$$

where $u^{(n)}$ indicates the nth derivative of $u(x)$. Other derivatives of less order may appear with $u(n)$ at the left side. For example,

$$\begin{cases} u^{'}(x) = \frac{1}{2}x^2 - xe^x - \int_0^x tu(t)dt, \\ u(0) = 0, \end{cases}$$

and

$$\begin{cases} u^{''}(x) + u^{'}(x) = 1 - x(sin(x) + cos(x)) - \int_0^x tu(t)dt, \\ u(0) = -1, u^{'}(0) = 1. \end{cases}$$

2.2.3 Volterra-Fredholm Integro-Differential Equations

The Volterra-Fredholm integro-differential equations arise in the same manner as Volterra-Fredholm integral equations with one or more of ordinary derivatives in addition to the integral operators. The Volterra-Fredholm integro-differential equations appear in the literature in two forms, namely

$$\sum_{n=1}^{N} u^{(n)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)dt, \tag{2.13}$$

and

$$\sum_{n=1}^{N} u^{(n)}(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau))d\xi d\tau, \quad (x,t) \in \Omega \times [0,T], \tag{2.14}$$

where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Omega \times [0,T]$, and Ω is a closed subset of $\Re$, $n = 1,2,3$. It is interesting to note that (2.13) contains disjoint Volterra and Fredholm integral equations, whereas (2.14) contains mixed integrals. Other derivatives of less order may appear as well. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind. Initial conditions should be given to determine the particular solution. Examples of the two types are given by

$$\begin{cases} u^{'}(x) = 24x + x^4 + 3 - \displaystyle\int_0^x (x-t)u(t)dt - \int_0^1 tu(t)dt, \\ u(0) = 0. \end{cases}$$

and

$$\begin{cases} u^{'}(x,t) = 1 + t^3 + \dfrac{1}{2}t^2 - \dfrac{1}{2}t - \displaystyle\int_0^t \int_0^1 \xi - \tau d\xi d\tau, \\ u(0,t) = t^3. \end{cases}$$

2.3 Linearity and Homogeneity

Integral equations and integro-differential equations fall into two other types of classifications according to linearity and homogeneity concepts. These two concepts play a major role in the structure of the solutions. In what follows we highlight the definitions of these concepts.

2.3.1 Linearity Concept

If the exponent of the unknown function $u(x)$ inside the integral sign is one, the integral equation or the integro-differential equation is called linear . If the unknown function $u(x)$ has exponent other than one, or if the equation contains nonlinear functions of $u(x)$, such as $\exp(u), \sinh u, \cos u, \ln(1+u)$, the integral equation or the integro-differential equation is called nonlinear.

It is important to point out that linear equations, except Fredholm integral equations of the first kind, give a unique solution if such a solution exists. However, solution of nonlinear equation may not be unique. Nonlinear equations usually give more than one solution and it is not usually easy to handle. Both linear and nonlinear integral equations of any kind will be investigated in this text by using traditional and new methods.

2.3.2 Homogeneity Concept

Integral equations and integro-differential equations of the second kind are classified as homogeneous or inhomogeneous, if the function f(x) in the second kind of Volterra or Fredholm integral equations or integro-differential equations is identically zero, the equation is called homogeneous. Otherwise it is called inhomogeneous. Notice that this property holds for equations of the second kind only

2.4 Origins of Integral and integro-differential Equations

Integral and integro-differential equations arise in many scientific and engineering applications. Volterra integral equations and Volterra integro-differential equations can be obtained from converting initial value problems with prescribed initial values. However, Fredholm integral equations and Fredholm integro-differential equations can be derived from boundary value problems with given boundary conditions. It is important to point out that converting initial value problems to Volterra integral equations, and converting Volterra integral equations to initial value problems are commonly used in the literature. This will be explained in detail in the coming section. However, converting boundary value problems to Fredholm integral equations, and converting Fredholm integral equations to equivalent boundary value problems are rarely used. The conversion techniques will be examined and illustrated examples will be presented. In what follows we will examine the steps that we will use to obtain these integral and integro-differential equations.

2.5 Fixed-point theorems and their applications to N.I.Es

In various branches of mathematical analysis we can solved many problems defined by nonlinear functional equations by converting them to an equivalent problem of a fixed point problems, in fact, an operator equation $Gu = u$ maybe expressed as a fixed point equation $Au = u$, where A is an operator. Here we discuss some different types of fixed point theorems that prove the existence of solutions for nonlinear integral equations. Starting from the basics of Banach's contraction theorem, one of the most main results and techniques that have been developed (see [39], [37], [41], [38], [40]), this theorem establishes a general criterion guarantees that the iteration procedure of a function results a fixed point. Then, we discuss the Brouwer fixed-point theorem which the finite dimensional version of Schauder's theorem, this latter stated that if Ω is a convex and compact subset of a Banach space X and A is a self-continuous mapping, then A has at least one fixed point, nevertheless, one of the defects of Schauder's fixed point theorem is the invariance condition $A(\Omega) \subset \Omega$ this must be insured for a bounded closed convex subset Ω in a Banach space. The principle of Leray-Schauder allows for the avoidance of such a condition by requiring the fulfilment of a boundary condition.

2.5.1 Banach's fixed-point theorem

Definition 2.1. *A bounded operator A on a Banach space X is a contraction, if there exists a constant L with $0 < L < 1$, such that*

$$\| A(u_1) - A(u_2) \| \leq L \| u_1 - u_2 \|, \qquad \forall\ u_1, u_2 \in X.$$

Theorem 2.1 (Banach's fixed-point theorem (1922) [37]). *Let A be a contraction on a Banach space X. Then the equation*

$$A(u) = u,$$

has a unique solution on X, this solution is the fixed point of operator A.

Proof. Let $u_0 \in X$ be arbitrary, make a sequence in X by establishing $u_1 = A(u_0)$ and $u_{n+1} = A(u_n)$ for $n > 0$.
Let we first demonstrate that $\{u_n\}$ is a Cauchy sequence. From the contractivity of A, we have

$$\| u_{n+1} - u_n \| \leq L \| u_n - u_{n-1} \| \leq \cdots \leq L^n \| u_1 - u_0 \|,$$

for $m \geq n \geq 1$,

$$\begin{aligned} \| u_m - u_n \| &\leq \sum_{j=0}^{m-n-1} \| u_{n+j+1} - u_{n+j} \| \\ &\leq \sum_{j=0}^{m-n-1} L^{n+j} \| u_1 - u_0 \| \\ &\leq \frac{L^n}{1-L} \| u_1 - u_0 \| . \end{aligned}$$

Since $L \in (0;1)$; $\| u_m - u_n \| \to 0$ as $m; n \to \infty$. Thus $\{u_n\}$ is a Cauchy sequence in the Banach space X, therefore $\{u_n\}$ converge to a limit $u \in X$, by the continuity of A we have

$$A(u) = A(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} A(u_n) = \lim_{n\to\infty} u_{n+1} = u,$$

then u is a fixed point of A.
Now by contradiction we prove that u is unique.
Assume that A possesses two or more distinct fixed points, say u_1 and u_2, Then

$$\| u_1 - u_2 \| = \| A(u_1) - A(u_2) \| \leq L \| u_1 - u_2 \|,$$

this implies that $\| u_1 - u_2 \| = 0$ since $0 < L < 1$, so a contractive mapping's fixed point is unique. □

Theorem 2.2 ([37]). *Let A be an operator on a Banach space X with A^n is a contraction. Then the equation*

$$A(u) = u,$$

has a unique solution on X.

Proof. As a result of the theorem, A^n has a fixed point noted by u_0, then

$$\begin{aligned} \| A(u_0) - u_0 \| &= \| A^n(A(u_0)) - A^n(u_0) \| \\ &\leq L \| A(u_0) - u_0 \|, \end{aligned}$$

implies that $Au_0 = u_0$ since $0 < L < 1$. The uniqueness is clear since a fixed point for A is also a fixed point for A^n. $\square$

2.5.2 Existence and uniqueness of Solution for N.I.E via Banach's Fixed Point Theorem

2.5.2.1 Case1: Nonlinear Volterra Integral Equation

Theorem 2.3. [37]
Assume that $K(s,t,u)$ is defined and continuous on the square $a \leq s;t \leq b$ and that it satisfies a Lipschitz condition of the form

$$\| K(s,t,u_1) - K(s,t,u_2) \| \leq L \| u_1 - u_2 \| .$$

Assume further that $f \in C[a,b]$. Then the nonlinear Volterra integral equation

$$u(s) = f(s) + \lambda \int_a^s K(s;t;u(t))dt, \tag{2.15}$$

has a unique solution on the interval $[a;b]$ for every value of λ, where $a \leq s \leq b$.

Proof. Let A an operator, such that

$$A : C[a,b] \to C[a,b],$$

defined by

$$A(u)(s) = f(s) + \lambda \int_a^s K(s,t,u(t))dt, \tag{2.16}$$

is a contraction for every valued of λ, then it will be obvious as an application of Theorem 2.1 that A has a unique fixed point, implying that the equation (2.15) has a unique solution. Let$u_1, u_2 \in C[a,b]$ and $s \in [a,b]$, we will show that for any $n \geq 1$

$$\| A^n(u_1) - A^n(u_2) \| \leq \frac{\lambda^n L^n (b-a)^n}{n!} \| u_1 - u_2 \|_\infty . \tag{2.17}$$

For $n = 1$, we have

$$\begin{aligned} |A(u_1(s)) - A(u_2(s))| &= |\lambda \int_a^s [K(s,t,u_1(t)) - K(s,t,u_2(t))] \, dt| \\ &\leq |\lambda L| \int_a^s |u_1(t) - u_2(t)| dt \\ &\leq |\lambda L(b-a) \| u_1 - u_2 \|_\infty, \end{aligned}$$

this implies that$\| A(u_1) - A(u_2) \| \leq |L(b-a) \| u_1 - u_2 \|_\infty$.
Assume that the property (2.17) is verified for $n = m$, and we will show that (2.17) is verified

for $n = m+1$.

$$\begin{aligned}
|A^{m+1}(u_1(s)) - A^{m+1}(u_2(s))| &= |A(A^m(u_1))(s) - A(A^m(u_2))(s)| \\
&= |\lambda \int_a^s [K(s,t,A^m u_1(t)) - K(s,t,A^m u_2(t))]dt| \\
&\leq |\lambda| \int_a^s L|A^m(u_1(t) - A^m(u_2(t)|dt \\
&\leq |\lambda| \int_a^s L\frac{|\lambda|^m L^m (b-a)^m}{m!} \parallel u_1 - u_2 \parallel_\infty dt \\
&\leq \frac{|\lambda|^{m+1} L^{m+1}(b-a)^{m+1}}{(m+1)!} \parallel u_1 - u_2 \parallel_\infty,
\end{aligned}$$

hence

$$\parallel A^{m+1}(u_1(s)) - A^{m+1}(u_2(s)) \parallel \leq \frac{|\lambda|^{m+1} L^{m+1}(b-a)^{m+1}}{(m+1)!} \parallel u_1 - u_2 \parallel_\infty,$$

then the property 2.17is valid for all $n > 0$.
Since the sequence $\frac{|\lambda|^n L^n (b-a)^n}{(n)!} \to 0$ there exist a power n_0 such that $\frac{|\lambda|^{n_0} L^{n_0}(b-a)^{n_0}}{(n_0)!} <$ 1,this implies that A^{n_0} is a contraction. □

2.5.2.2 Case2: Nonlinear Fredholm Integral Equation

Theorem 2.4. [37]
Assume that $K(s,t,u)$ is defined and continuous on the square $a \leq s; t \leq b$ and that it satisfies a Lipschitz condition of the form

$$\parallel K(s,t,u_1) - K(s,t,u_2) \parallel \leq L \parallel u_1 - u_2 \parallel .$$

Assume further that $f \in C[a,b]$. Then the nonlinear Fredholm integral equation

$$u(s) = f(s) + \lambda \int_a^b K(s,t,u(t))dt, \tag{2.18}$$

has a unique solution on the interval $[a,b]$ whenever $\lambda < 1/(L(b-a))$.

Proof. If it can be proved that an adequate power of the continuous operator

$$A : C[a,b] \to C[a,b],$$

defined by

$$A(u)(s) = f(s) + \lambda \int_a^b K(s,t,u(t))dt,$$

is a contraction for the constrained values of λ specified in the statements of the theorem, then it will be obvious as an application of Banach's fixed point theorem (Theorem 2.1) that A has an unique fixed point, implying that the integral equation 2.18 has a unique solution.

Let $u_1, u_2 \in C[a,b]$ and $s \in [a,b]$, we have

$$\begin{aligned} |A(u_1(s)) - A(u_2(s))| &= |\lambda \int_a^b (K(s,t,u_1(t)) - K(s,t,u_2(t)))dt|, \\ \leq |\lambda| L(b-a) \parallel u_1 - u_2 \parallel_\infty, \end{aligned}$$

this implies that

$$|\lambda| L(b-a) < 1,$$

(we can select $\lambda < \dfrac{1}{L(b-a)}$ then A is a contraction operator. □

2.5.3 Brouwer's fixed-point theorem

Theorem 2.5. *(Brouwer's fixed-point theorem (1912))* [37]
Let $\Omega \subset \Re^n$ be a nonempty,convex and compact set and let $A : \Omega \to \Omega$ be a continuous mapping. Then A has at least one fixed point.

See [42]. . □

2.5.4 Schauder's fixed-point theorem

Theorem 2.6 (Schauder's fixed-point theorem (1930) [37])**.** .
Let X be a Banach space, $\Omega \subset X$ a nonempty convex compact set and let $A : \Omega \to \Omega$ be a continuous operator. Then A has at least one fixed point.

Proof. Since A is a continuous mapping and is a compact, A is uniformly continuous, hence, for all $\epsilon > 0$, there exists a constant $\delta > 0$ such that for every $x, y \in \Omega$ satisfying $\parallel x - y \parallel \leq \delta$ we have that $\parallel A(x) - A(y) \parallel \leq \epsilon$ furthermore, there exists a finite set of points $\{x_1, ..., x_p\} \subset \Omega$ such that the open balls with radius δ centered at x_j cover Ω, i.e. $\Omega \subset \cup_{1 \leqslant j \leqslant p} B(x_j, \delta)$. Let $G = vect(A(x_j))_{1 \leqslant j \leqslant p}$, then G is of finite dimension, and $\Omega^* = \Omega \bigcap G$ is a compact convex and of finite dimension.
For $1 \leqslant j \leqslant p$, we define the continuous mappings $\psi_j : X \to \Re$ by

$$\psi_j = \begin{cases} 0 & if \qquad \parallel x - x_j \parallel \geq \delta, \\ 1 - \dfrac{\parallel x - x_j \parallel}{\delta}, & \qquad otherwise. \end{cases}$$

Obviously ψ_j is strictly positive on $B(x_j, \delta)$ and is equal to zero outside, hence

$$\forall x \in \Omega, \sum_{j=1}^{p} \psi_j(x) > 0,$$

this implies that the continuous mapping φ_j defined as follows

$$\varphi_j(x) = \frac{\psi_j(x)}{\sum\limits_{k=1}^{p} \psi_k(x)},$$

satisfying

$$\sum_{j=1}^{p} \psi_j(x) = 1,$$

for every $x \in \Omega$
Let us now define for all $x \in \Omega$, the mapping g by

$$g(x) = \sum_{j=1}^{p} \psi_j(x) A(x_j),$$

g is continuous mapping defined on Ω into Ω^*, and if we take the restriction g/Ω^*, the Brouwer fixed-point theorem implies that g has a fixed point $y \in \Omega^*$, furthermore,

$$\begin{aligned} A(y) - y &= A(y) - g(y) \\ &= \sum_{j=1}^{p} \varphi_j(y) A(y) - \sum_{j=1}^{p} \varphi_j(y) A(x_j) \\ &= \sum_{j=1}^{p} \varphi_j(y)(A(y) - A(x_j)), \end{aligned}$$

if $\varphi_j(y) \neq 0$ then $\| y - x_j \| \leq \delta$, this yields,$\| A(y) - A(x_i) \| \leq \epsilon$, and for all j

$$\| \varphi_j(y)(A(y) - A(x_j)) \| \leq \epsilon \varphi_j(y)$$

then

$$\| A(y) - y \| \leq \sum_{j=1}^{p} \| \varphi_j(y)(A(y) - A(x_j)) \|$$

$$\leq \sum_{j=1}^{p} \epsilon \varphi_j(y) = \epsilon$$

this means that, for each $\epsilon > 0$, there exists $y = y(\epsilon) \in \Omega$, hence

$$\forall m \in N \exists y_m \in \Omega : \| A(y_m) - y_m \| < 2^{-m}$$

Since Ω is a compact $y_{m_{m \in Z}}$ has a subsequence $y_{mk_{k \in Z}}$ convergent to some element $y^* \in \Omega$, and since A is continuous, the sequence $A(y_{mk})$ converge to $A(y^*)$, thus $A(y^*) = y^*$ i.e. y^* is at least one fixed point of A on Ω.

□

The following variation of Schauder's theorem is more commonly utilised in applications.

Theorem 2.7 ([37]). .
Let X be a Banach space, $\Omega \subset X$ a non empty convex bounded closed set and let $A : \Omega \to \Omega$ be a completely continuous operator. Then A has a fixed point.

2.5.5 Existence and uniqueness of solution for N.I.D.E via Banach's fixed point theorem

2.5.5.1 Local and Global Existence

This section is devoted to study the initial value problem (IVP) for integro-differential equations of the type

$$u^{'}(x) = f(x, u(x)) + \int_{x_0}^{x} k(x, t, u(t))dt, \qquad u(x_0) = u_0, \qquad dx(t)/dt = x^{'}(t), \tag{2.19}$$

where $f \in C[J \times R^n, R^n]$,$K \in C[J \times J \times R^n, R^n]$ and,$J = [x_0, x_0 + a]$ It is easy to show that the IVP (2.19) is equivalent to the integral equation

$$u(x) = u_0 + \int_{x_0}^{x} [f(t, u(t)) + \int_{t}^{x} k(\delta, t, u(t))d\delta]dt, \tag{2.20}$$

which can be seen by integrating (2.19) from x_0 to x and changing the order of integration. Since f and k are continuous, on differentiating (2.20), we obtain (2.19). Let us begin by proving the following local existence result by applying Schauder's fixed point theorem, which we state here in a suitable form.

Theorem 2.8 (Schauder [34]). *If E is a closed, bounded, convex subset of a Banach space B and $T : E \to E$ is completely continuous then T has a fixed point.*

Theorem 2.9 ([34]). .
Assume that $f \in C[J \times R^n, R^n]$,$K \in C[J \times J \times R^n, R^n]$ and $\int_t^x \mid k(\delta, t, u(t)) \mid d\delta \leqslant N$, for $x_0 \leqslant t \leqslant x \leqslant x_0 + a$, $u \in \Omega = \{\phi \in C[J, R^n] : \phi(x_0) = u_0$ and $|\phi(x) - u_0| \leqslant b\}$.Then the IVP (2.19) possesses at least one solution $u(t)$ on $x_0 \leq x \leq x_0 + \alpha$, for some $0 < \alpha \leq a$.

Proof. Consider the set $D = \{[(x, u) : x \in J \ |u - u_0| \leqslant b\}$,and let $|f(x, u)| \leqslant M$ on D. Choose
$\alpha = min \left[a, \dfrac{b}{M + N}\right]$ and let $\Omega_0 = \{\phi, C[j_0, R^n] : \phi(x_0 = u_0$ and $|\phi - u_0|_0 \leqslant b\}$, where $|\phi|_0 = max_{x_0 \leqslant x \leqslant x_0 + \alpha}|\phi(x)|$ and $J_0 = [x_0, x_0 + \alpha]$.Clearly the set Ω_0 is closed, convex and bounded. For any $\phi \in \Omega_0$ define the function $T\phi$ by

$$T\phi(x) = u_0 + \int_{x_0}^{x} [f(t, \phi(t)) + \int_{t}^{x} k(\delta, t, \phi(t)d\delta]dt, \quad x \in [x_0, x_0 + \alpha].$$

We may apply Schauder's fixed point theorem to prove the existence of a fixed point of T in Ω, which is equivalent to solving the IVP (2.19). Clearly $T\phi(x_0) = u_0$, and for $x \in J_0$

$$\begin{aligned} |T\phi(x) - u_0| &\leqslant \int_{x_0}^{x} [|f(t, \phi(t))| + \int_{t}^{x} |k(\delta, t, \phi(t)|d\delta]dt, \quad x \in [x_0, x_0 + \alpha] \\ &\leqslant (M + N)\alpha \leqslant b, \end{aligned}$$

which implies that $T\Omega_0 \subset \Omega_0$, Furthermore, for any $x_1, x_2 \in J_0$, such that $x_2 > x_1$, by changing the order of integration, we obtain

$$\begin{aligned} |T\phi(x_2) - \phi(x_2)| &\leqslant \int_{x_1}^{x_2} [|f(t, \phi(t))| + \int_{x_0}^{t} |k(t, \delta, u(\phi(t))| d\delta] dt, \\ &\leqslant (M + N)|x_2 - x_1|. \end{aligned}$$

This shows that the set $T(\Omega_0)$ is an equicontinuous family, and consequently the closure of $T(\Omega_0)$ is compact. For any ϕ, ψ,in Ω_0, it follows, using uniform continuity of f and K, that for any $\epsilon > 0$, there exist $\delta > 0$ such that

$$\begin{aligned} |T\phi(x) - T\psi(x)| &\leqslant \int_{x_0}^{x} [|f(t, \phi(t)) - f(t, \psi(t))| + \int_{t}^{x} |k(\delta, t, \phi(t) - k(\delta, t, \psi(t)| d\delta] dt \\ &\leqslant \epsilon(\alpha + \frac{\alpha^2}{\frac{1}{2}}) \quad forall x \in J_0, \end{aligned}$$

provided that $|\phi(t) - \psi(t)| < \delta$ for all $t \in J_0$ This implies that T is a continuous mapping. By Schauder's fixed point theorem, there is a fixed point of $T \in \Omega_0$, which completes the proof. Sometime, one needs to consider IVPs of integro-differential equations of the form

$$\begin{cases} u' = f(x, u, ku) \\ u(x_0) = u_0, \end{cases} \tag{2.21}$$

where

$$ku(x) = \int_{x_0}^{x} k(x, t) u(t) dt, \quad k(x, t),$$

is an $n \times n$ continuous matrix on $J \times J$, and
$f \in C[J \times R^n \times R^n, R^n]$. A local existence result for the IVP (2.21) can be proved using arguments similar to Theorem 2.9. We merely state such a result. □

Theorem 2.10 ([34]). .
Assume that $|k(x, t)| \leq k, \quad (x, t) \in J \times J$ *and* $|f(x, u, y)| \leq M$ *for* $x \in J$, $u, y \in \Omega = \{u \in R^n : |u - u_0| \leqslant b\}$, *Then there exists a solution* $u(t)$ *of* (2.21) *on* $[x_0, x_0 + a]$ *for some* $a > 0$.

We shall next discuss a global existence result for IVP (2.19) using Tychonoff's fixed point theorem, which we state in the following form.

Theorem 2.11 (Tychonoff [34]). *Let B be a complete, locally convex, linear space and B_0 a closed convex subset of B. Let the mapping $T : B \to B$be continuous and $T(B_0) \subset B_0$. If the closure of $T(B_0)$ is compact then T has a fixed point in B_0.*

Theorem 2.12 ([34]). *Assume that*

(i) $f \in C[R_+ \times R^n, R^n]$,$g \in C[R_+^2 \times R_+]$, $g(t, u)$ *is monotone non decreasing in* u *for each* $T \in J$ *and*

$$|f(t, x)| \leqslant g(t, |x|), \quad (t, x) \in R_+ \times R^n;$$

(ii) $K \in C[R_+^2 \times R^n, R^n]$, $G \in C[R_+^3 \times R_+]$, $G(t,s,u)$ *is monotone non decreasing in* u *for each* $(t,s) \in R_+^2$ *and*

$$|K(t,s,x)| \leqslant G(t,s,|x|), \quad (t,s,x) \in R_+^2 \times R^n;$$

(iii) *for every* $u_0 > 0$, *the scalar integro-differential equation*

$$\begin{cases} u'(x) = g(x,u(x)) + \int_{x_0}^{x} G(x,t,u(t))dt, \\ u(x_0) = u_0, \end{cases} \tag{2.22}$$

has a solution $u(t)$ *for* $t \geq t_0$,

(iv) $\int_t^x |K(\sigma,t,w(t))|d\sigma \leqslant N$ *for* $x,t \in R_+$, $w \in C[R_+,R^n]$.

Then for every $u_0 \in ER^n$, *such that* $|u_0| \leqslant u_0$, *there exists a solution* $u(t)$ *of* (2.19) *for* $x \geq x_0$ *satisfying* $|u(x)| \leqslant w(x)$,$x \geq x_0$.

Proof. Let us consider the real vector space B of all continuous functions from $[x_0,\infty)$ into R^n, the topology on B being that induced by the family of pseudo-norms $\{V_n(x)\}_{n=1}^{\infty}$ where for $x \in B$, $V_n(x) = sup_{x_0 \leqslant x \leqslant n}|u(x)|$.
A fundamental system of neighborhoods is then given by $\{S_n\}_{n=1}^{\infty}$, where $S_n = \{u \in B : V_n(u) \leq 1\}$ Under this topology, B is a complete, locally convex linear space.
Now define a subset B_0 of B as follows:

$$B_0 = \{u \in B : |u(x)| \leqslant w(x), x \geq x_0\},$$

where $w(x)$ is a solution of (2.22) existing for $x \geq x_0$. It is clear that in the topology of B, B_0 is closed, convex and bounded. Consider the integral operator defined by

$$Tu(x) = u_0 + \int_{x_0}^{x} \left[f(t,u(t)) + \int_t^x K(\sigma,t,u(t))d\sigma \right] dt$$

whose fixed point corresponds to a solution of (2.19). Evidently, the operator T is compact in the topology of B, and hence the closure of $T(B_0)$ is compact in view of the boundedness of B_0. Now to prove $T(B_0) \subset B_0$, observe that for any $u \in B_0$,

$$\begin{aligned} |Tu(x)| &\leq |u_0| + \int_{x_0}^{x} |f(t,u(t))|dt + \int_{x_0}^{x}\int_t^x |K(\sigma,t,u(t))|d\sigma dt, \\ &\leq |u_0| + \int_{x_0}^{x} g(t,|u(t)|)dt + \int_{x_0}^{x}\int_t^x G(\sigma,t,|u(t)|)d\sigma dt, \\ &\leq w_0 + \int_{x_0}^{x} g(t,w(t))dt + \int_{x_0}^{x}\int_t^x G(\sigma,t,w(t))d\sigma dt = w(x), \end{aligned}$$

using the monotonicity of g and G, the definition of B_0 , and the fact $w(t)$ is a solution of (2.22), Therefore $|Tu(x)| \leqslant w(x)$, which implies $T(B_0) \subset B_0$. Hence by Tychonoff's fixed point theorem, T has a fixed point in B0 , which completes the proof of the theorem. If, instead of the form (2.19), an integro-differential equation is of the type

$$u^{'}(x) = f(x, u(x)) + \int_0^x K(x, t, u(t))dt, \quad x \in R_+, \tag{2.23}$$

then it is sometimes convenient to specify an initial function $\phi_0(x)$on the interval $0 \leq x \leq x_0, x_0 \geq 0$, namely

$$u(x) = \phi_0(x), \quad 0 \leq x \leq x_0,$$

and look for solutions of

$$u^{'}(x) = f(x, u(x)) + \int_0^{x_0} K(x, t, \phi_0(t))dt + \int_{x_0}^x K(x, t, u(t))dt, \tag{2.24}$$

for $x \geq x_0$, that depend on (x_0, ϕ_0). Naturally, (2.24) can also be expressed in the form (2.23). To see this, set $y(x) = u(x + x_0)$, so that (2.24) is transformed into

$$y^{'}(x) = F(x, y(x)) + \int_0^x G(x, t, y(t))dt, \tag{2.25}$$

where $F(x, y(x)) = f(x + x_0, y(x)) + \int_0^{x_0} K(x + x_0, t, \phi_0(t))dt$ and
$G(x, t, y) = K(x + x_0, t + x_0, y)$. Here the initial function $(\phi_0(t)$ is absorbed into the source function F.
We shall also sometimes consider integro-differential equations of the form

$$u^{'}(x) = f(x, u(x)) + \int_{-\infty}^x K(x, t, u(t))dt,$$

and

$$u^{'}(x) = f(x, u(x)) + \int_{x-\tau}^x K(x, t, u(t))dt, \quad \tau > 0,$$

to which the methods developed here can also be applied. □

Chapter 3

Method of solutions and approximations

In this chapter we present some methods for solving a different type of integro-differential equations analytically using direct method and decomposition Adomian method and we use the finite difference method for the differential problem, also we discus for the convergence of this methods after calculate n terms and we extend how to approximate the problem of integro-differential type by using finite elements method in next chapter.

3.1 Approximations Method for IDE

3.1.1 Fredholm Integro-Differential Equations

In this section we will discuss the reliable methods used to solve Fredholm integro-differential equations. This type of equations was termed as Fredholm integro-differential equations, given in the form

$$u^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \tag{3.1}$$

we remark here that we will focus our concern on the equations that involve separable kernels where the kernel $K(x,t)$ can be expressed as a finite sum of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{3.2}$$

Without loss of generality, we will make our analysis on a one term kernel $K(x,t)$ of the form

$$K(x,t) = g(x)h(t), \tag{3.3}$$

and this can be generalized for other cases. The non-separable kernel can be reduced to separable kernel by using the Taylor expansion for the kernel involved. The Fredholm integro-differential equations are usually solved by a variety of methods, some are numerical methods, where as others are analytic methods. It is worth noting that in this chapter we will introduce the most recent and practical schemes that handle this type of equations, where we may obtain an exact or an approximation solutions with the highest desirable accuracy. We point out here that the methods to be discussed are introduced before, but we will focus our discussion on how these methods can be implemented in this type of equations. We first start with the most practical traditional method.

We point out that the direct computation method, that will be used first, requires integrating or multiple integrating of the derivatives that are involved in the equation. In what follows, we list some of the facts used in calculus courses:

$$\begin{aligned}\int_0^x u^{'}(t)dt &= u(x)-u(0),\\ \int_0^x\int_0^t u^{"}(s)dsdt &= u^{'}(x)-u(0)-xu^{'}(0),\\ \int_0^x\int_0^t\int_0^s u^{'''}(r)drdsdt &= u^{'}(x)-u(0)-xu^{'}(0)-\frac{1}{2}x^2u^{"}(0),\end{aligned}$$

and so on for other derivatives.

3.1.1.1 Direct Computation Method

The direct computation method has been extensively used in (see [43]) to handle Fredholm integral equations. Without loss of generality, we may assume a standard form to the Fredholm integro-differential equation given by

$$\begin{cases} u^{(n)}(x)=f(x)+\lambda\int_0^1 K(x,t)u(t)dt,\\ u^{(k)}(0)=b_k, \qquad 0\le k\le (n-1),\end{cases} \tag{3.4}$$

where $u^{(n)}(x)$ indicates the nth derivative of $u(x)$ with respect to x and b_k are constants that define the initial conditions. Substituting (3.3) into (3.4) yields

$$\begin{cases} u^{(n)}(x)=f(x)+g(x)\lambda\int_0^1 h(t)u(t)dt,\\ u^{(k)}(0)=b_k, \qquad 0\le k\le (n-1).\end{cases} \tag{3.5}$$

We can easily observe that the definite integral in the integro-differential equation 3.5 involves an integrand that completely depends on the variable t, and therefore, it seems reasonable to set that definite integral in the right side of (3.5) to a constant α, i.e. we set

$$\alpha=\int_0^1 h(t)u(t)dt. \tag{3.6}$$

With α defined in (3.6), the equation (3.5) can be written by

$$u^{(n)}(x)=f(x)+\alpha g(x). \tag{3.7}$$

It remains to determine the constant α to evaluate the exact solution $u(x)$. To find α, we should derive a form for $u(x)$ by using (3.7), followed by substituting this form in (3.6).

To achieve this we integrate both sides of (3.7) n times from 0 to x, and by using the given initial conditions $u^{(k)}(0) = b_k, \quad 0 \leq k \leq n-1$ we obtain an expression for $u(x)$, given by

$$u(x) = p(x, \alpha), \tag{3.8}$$

where $p(x, \alpha)$ is the result derived from integrating (3.7) and by using the given initial conditions. Substituting (3.8) into the right hand side of (3.6), integrating and solving the resulting equation lead to a complete determination of α. The exact solution of (3.4) follows immediately upon substituting the resulting value of α into (3.8). To give a clear view of the technique, we illustrate the method by solving the following examples.

Example 3.1. *Solve the first-order Fredholm integro-differential equation by using the direct computation method:*

$$\begin{cases} u^{'}(x) = 1 - \frac{1}{3}x + x\int_0^1 tu(t)dt, \\ u(0) = 0, \end{cases} \tag{3.9}$$

The equation (3.9) may be written in the form

$$\begin{cases} u^{'}(x) = 1 - \frac{1}{3}x + \alpha x, \\ u(0) = 0, \end{cases} \tag{3.10}$$

where the constant α is defined by

$$\alpha x = \int_0^1 tu(t)dt. \tag{3.11}$$

To determine α, we first need an expression for $u(x)$ to be used in (3.11). This can be easily done by integrating both sides of (3.10) from 0 to x, using (3.4), and by using the given initial condition we obtain

$$u(x) = x + (\frac{\alpha}{2} - \frac{1}{6})x^2. \tag{3.12}$$

Substituting (3.12) into (3.11) and evaluating the integral we find

$$\alpha = \frac{1}{3}$$

so that the exact solution

$$u(x) = x.$$

3.1.1.2 Adomian Decomposition Method

Adomian developed the so-called Adomian decomposition method or simply the decomposition method (ADM). The method was well introduced by Adomian in his recent book [43]. The method proved to be reliable and effective for a wide class of equations, differential, integral and integro-differential equations, linear and nonlinear models. The method provides the solution in a series. The method was applied mostly to ordinary and partial differential equations, and was rarely used for integral and integro-differential equations in [43]. The concept of convergence of the solution obtained by this method was addressed extensively in the literature. However, the decomposition method can be successfully applied towards linear and nonlinear integral and integro-differential equations.

In this section we will study how this powerful method can be implemented to determine a series solution to the Fredholm integro-differential equations. we may assume a standard form to the Fredholm integro-differential equation given by

$$\begin{cases} u^{(n)}(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \\ u^{(k)}(0) = b_k, \qquad 0 \leq k \leq (n-1), \end{cases} \tag{3.13}$$

where $u^{(n)}(x)$ indicates the nth derivative of $u(x)$ with respect to x and b_k are constants that give the initial conditions. Substituting (3.3) into (3.13) yields

$$u^{(n)}(x) = f(x) + g(x)\int_0^1 h(t)u(t)dt. \tag{3.14}$$

We can easily observe that the definite integral in the integro-differential equation (3.14) involves an integrand that completely depends on the variable t as discussed in the preceding section, the equation (3.14) can be written as

$$Lu(x) = f(x) + g(x)\int_0^1 h(t)u(t)dt, \tag{3.15}$$

where the differential operator L is given by

$$L = \frac{d^n}{dx^n}, \tag{3.16}$$

It is clear that L is an invertible operator, therefore the integral operator L^{-1} is an $n-fold$ integration operator and may be considered as definite integrals from 0 to x for each integral. Applying L^{-1} to both sides of (3.15) yields

$$u(x) = b_0 + b_1 x + \frac{1}{2!}b_2x^2 + \cdots + \frac{1}{(n-1)!}b_{(n-1)}x^{n-1} + L^{-1}(f(x)) + \left(\int_0^1 h(t)u(t)dt)L^{-1}(g(x)\right). \tag{3.17}$$

In other words we integrated (3.14) n times from 0 to x and we used the initial conditions at every step of integration. It is important to note that the equation obtained in (3.17) is a standard Fredholm integral equation. In the decomposition method we usually define the solution $u(x)$ of (3.13) in a series form given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{3.18}$$

Substituting (3.18) into both sides of (3.17) we get

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x) &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) \\ &\quad + L^{-1}(g(x)) \left(\int_0^1 h(t) (\sum_{n=0}^{\infty} u_n(t)) dt \right),\end{aligned}$$

or equivalently

$$\begin{aligned}u_0(x) + u_1(x) + u_2(x) + \cdots &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) + L^{-1}(g(x)) \int_0^1 h(t) u_0(t) dt \\ &\quad + L^{-1}(g(x)) \int_0^1 h(t) u_1(t) dt + L^{-1}(g(x)) \int_0^1 h(t) u_2(t) dt + \cdots .\end{aligned}$$

The components $u_i(x)$, $i \geq 0$ of the unknown function $u(x)$ are determined in a recurrent manner, in a similar fashion as discussed before, where we set

$$\begin{aligned}u_0(x) &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)), \\ u_1(x) &= L^{-1}(g(x)) \left(\int_0^1 h(t) u_0(t) dt \right), \\ u_2(x) &= L^{-1}(g(x)) \left(\int_0^1 h(t) u_1(t) dt \right), \\ u_3(x) &= L^{-1}(g(x)) \left(\int_0^1 h(t) u_2(t) dt \right).\end{aligned}$$

The above scheme for the determination of the components $u_0(x)$, $u_1(x)$,$u_2(x)$, $u_3(x)$, $\cdots$ of the solution $u(x)$ of the equation 3.13 can be written in a recursive relationship by

$$\begin{cases} u_0(x) &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)), \\ u_{n+1}(x) &= \left(\int_0^1 h(t) u_n(t) dt \right) L^{-1}(g(x)), \quad n \geq 0. \end{cases} \tag{3.19}$$

Before of (3.19), the components $u_0(x), u_1(x), u_2(x), u_3(x), u_4(x), \ldots$ of $u(x)$ are immediately determined. With these components established, the solution $u(x)$ of (3.13) is readily determined in a series form using (3.18). Consequently, the series obtained for $u(x)$ frequently provides the exact solution .

However, for some problems, where a closed form is not easy to find, we use the series form obtained to approximate the solution. We point out here that few terms of the series derived by the decomposition method usually provide the higher accuracy level of the approximate solution.

The decomposition method avoids massive computational work and difficulties that arise from other methods. The computational work can be minimized, sometimes, by using the modified decomposition method.

3.1.1.3 Modified Decomposition Method

The modified decomposition method is a powerful technique that minimizes the size of calculations. An essential requirement for the use of this method is that the data function $f(x)$ in (3.15) should consist of more than one term. Consequently, the data function $f(x)$ can be decomposed to two parts, as introduced in [47], of the form

$$f(x) = f_0(x) + f_1(x). \tag{3.20}$$

As stated before, the selection of these two parts depends mainly on a trial basis. Moreover, using this selection, we set the modified recurrence relation as

$$\begin{cases} u_0(x) & = \sum\limits_{k=0}^{n-1} \dfrac{1}{k!} b_k x^k + L^{-1}(f_0(x)), \\ u_1(x) & = L^{-1}(f_1(x) + \left(\int_0^1 h(t)u_0(t)dt\right) L^{-1}(g(x)), \\ u_{n+1}(x) & = \left(\int_0^1 h(t)u_n(t)dt\right) L^{-1}(g(x)), \quad n \geq 1. \end{cases} \tag{3.21}$$

Although the change between the Adomian decomposition method is slight, but it was proved from using the proposed modification that this techniques minimizes the calculations size, and mostly exact solution can be derived in using two iterations only for $u_0(x)$ and $u_1(x)$.

3.1.1.4 Noise Terms Phenomenon

The noise terms phenomenon, that handles the self-cancelling noise terms. It was proved by Adomian and Rach [44] and others that the exact solution of any integral or integro-differential equation, for some cases, may be obtained by considering the first two components u_0 and u_1 only. Instead of evaluating several components, it is useful to examine the first two components u_0 and u_1. The conclusion made in [44] suggests that if we observe the appearance of like terms in both components with opposite signs, then by cancelling these terms, the remaining non-cancelled terms of u_0 may in some cases provide the exact solution. This can be justified through substitution. The self-cancelling terms, the identical terms with opposite terms, between the components u_0 and u_1 are called the noise terms. It was formally proved that other terms in other components will vanish in the limit if the noise terms occurred in $u_0(x)$ and $u_1(x)$. However, if the exact solution was not attainable by using this phenomena, then we should continue determining other components of $u(x)$ to get a closed form solution or an approximate solution as discussed earlier.

Moreover, it is important to note that, even though this is a remarkable achievement that speeds the convergence of the solution and minimizes the size of calculations work, but unfortunately the self cancelling noise terms do not appear always, but a necessary condition should hold for the possibility that these terms may appear. The condition of the appearance of the noise terms, as proved in [45], requires the existence of the exact solution as one term of the zeroth component $u_0(x)$.

In the following we discuss an example which illustrate the above outlined decomposition scheme, the modified decomposition method, and the phenomenon of the self cancelling noise terms as well.

Example 3.2. *Consider the Fredholm integro-differential equation*

$$\begin{cases} u'(x) = cosx + \frac{1}{4}x - \frac{1}{4}\int_0^{\frac{\pi}{2}} xtu(t)dt, \\ u(0) = 0. \end{cases} \tag{3.22}$$

Integrating both sides of the (3.22) *from* 0 *to* x *gives*

$$\begin{cases} u(x) - u(0) = sinx + \frac{1}{8}x^2 - \frac{1}{8}x^2\int_0^{\frac{\pi}{2}} tu(t)dt, \\ u(0) = 0, \end{cases}$$

using the initial condition

$$\begin{cases} u(x) = sinx + \frac{1}{8}x^2 - \frac{1}{8}x^2\int_0^{\frac{\pi}{2}} tu(t)dt, \\ u(0) = 0, \end{cases} \tag{3.23}$$

(i) Using the Adomian decomposition method,

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{3.24}$$

Substituting (3.24) *into both sides of* (3.23) *yields*

$$\sum_{n=0}^{\infty} u_n(x) = sinx + \frac{1}{8}x^2 - \frac{1}{8}x^2\int_0^{\frac{\pi}{2}} t\left(\sum_{n=0}^{\infty} u_n(t)\right)dt. \tag{3.25}$$

The Adomian decomposition method admits by the recurrence relation

$$\begin{cases} u_0(x) &=& sinx + \frac{1}{8}x^2, \\ u_1(x) &=& -\frac{1}{8}x^2\left(\int_0^{\frac{\pi}{2}} tu_0(t)dt\right) = -\frac{1}{8}x^2 - \frac{\pi^4}{16^3}x^2, \\ u_2(x) &=& -\frac{1}{8}x^2\left(\int_0^{\frac{\pi}{2}} tu_1(t)dt\right) = \frac{\pi^4}{16^3}x^2 + \frac{\pi^8}{2\times 16^5}x^2, \\ u_3(x) &=& -\frac{1}{8}x^2\left(\int_0^{\frac{\pi}{2}} tu_2(t)dt\right) = -\frac{\pi^8}{2\times 16^5}x^2 + other\ terms, \\ \vdots \end{cases} \tag{3.26}$$

We obtain the exact solution

$$u(x) = \sin x.$$

(ii) Using the noise terms phenomenon: Using the first two components $u_0(x)$ *and* $u_1(x)$*, we observe that the two identical terms* $\frac{1}{8}x^2$ *appear in these two components with opposite signs. Cancelling this noise term from* $u_0(x)$*, the remaining non-cancelled term in* $u_0(x)$

gives the exact solution

$$u(x) = \sin x. \tag{3.27}$$

(iii) Using the modified decomposition method: We first decompose the data function $f(x) = \sin x + \frac{1}{8}x^2$ *as follows*

$$\begin{aligned} f_0(x) &= \sin x, \\ f_1(x) &= \frac{1}{8}x^2, \end{aligned}$$

using the modified recurrence relation (3.21) *gives*

$$\begin{aligned} u_0(x) &= \sin x, \\ u_1(x) &= \frac{1}{8}x^2 - \frac{1}{8}x^2\left(\int_0^{\frac{\pi}{2}} tu_0(t)dt\right) = 0, \end{aligned}$$

that gives the exact solution (3.27).

3.1.2 Volterra Integro-Differential Equations

In this section we will present the reliable methods that will be used to handle Volterra integro-differential equations. This new type of equations was termed as Volterra integro-differential equations, given in the form

$$u^{(n)}(x) = f(x) + \lambda\int_0^x K(x,t)u(t)dt. \tag{3.28}$$

We will focus our study on equations that involve separable kernels of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \tag{3.29}$$

Without loss of generality, we will consider the cases where the kernel K(x, t) consists of one product of the functions g(x) and h(t) given by

$$K(x,t) = g(x)h(t). \tag{3.30}$$

Where other cases can be generalized in the same manner. The nonseparable kernel can be reduced to separable kernel by using the Taylor expansion for the kernel involved. We first start with the most practical method.

3.1.2.1 Series Solution Method

We may consider a standard form to the Volterra integro-differential equation given by

$$\begin{cases} u^{(n)}(x) = f(x) + \int_0^x K(x,t)u(t)dt, \\ u^{(k)}(0) = b_k, \qquad 0 \leq k \leq (n-1). \end{cases} \tag{3.31}$$

where $u^{(n)}(x)$ indicates the nth derivative of $u(x)$ with respect to x, and b_k are constants that define the initial conditions. Substituting (3.30) into (3.31) yields

$$\begin{cases} u^{(n)}(x) = f(x) + g(x)\int_0^x h(t)u(t)dt, \\ u^{(k)}(0) = b_k, \qquad 0 \leq k \leq (n-1). \end{cases} \tag{3.32}$$

We will follow a manner parallel to the approach of the series solution method that usually used in solving ordinary differential equations around an ordinary point. To achieve this goal, we first assume that the solution $u(x)$ of (3.32) is an analytic function and hence can be represented by a series expansion given by

$$u(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{3.33}$$

where the coefficients a_k are constants that will be determined. It is to be noted that the first few coefficients a_k can be determined by using the initial conditions so that

$$\begin{aligned} a_0 &= u(0), \\ a_1 &= u'(0), \\ a_2 &= \frac{1}{2!}u''(0), \end{aligned}$$

and so on depending on the number of the initial conditions given, whereas the remaining coefficients a_k will be determined from applying the technique. Substituting (3.33) into both sides of (3.32) yields

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{(n)} = f(x) + g(x)\int_0^x h(t)\left(\sum_{k=0}^{\infty} a_k t^k\right)dt. \tag{3.34}$$

The integral equation (3.32) will be reduced to several calculable integrals in the right hand side of (3.34) that can be easily evaluated where we have to integrate terms of the form t^n, $n \geq 0$ only.

The next step is to write the Taylor expansion for $f(x)$, evaluate the resulting traditional integrals in (3.34), and then equating the coefficients of like powers of x in both sides of the equation. Accordingly, this leads to a complete determination of the coefficients a_i, $i \geq 0$. Consequently, substituting the obtained coefficients a_k, $k \geq 0$ into (3.33) produces the solution in a series form. This may converge to a solution in a closed form, if an exact solution exists, or we may use the obtained series for numerical purposes.

Example 3.3. *Consider the Volterra integro-differential equation.*

$$\begin{cases} u''(x) = x\cosh x - \int_0^x tu(t)dt, \\ u(0) = 0, \quad u'(0) = 1. \end{cases} \tag{3.35}$$

Substituting $u(x)$ *by the series*

$$u(x) = \sum_{k=0}^{\infty} a_n x^n, \tag{3.36}$$

into both sides of the equation (3.35) *and using the Taylor expansion of coshx we obtain*

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = x\left(\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k!)}\right) - \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt.$$

Using the initial conditions yields

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1. \end{aligned}$$

Evaluating the traditional integrals that involve terms of the form t_n, $n \geq 0$, and using a number of terms from both sides yield

$$\begin{aligned} 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots &= x\left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) \\ &\quad - \left(\frac{1}{3}x^3 + \frac{1}{4}a_2x^4 + \ldots\right). \end{aligned}$$

Equating the coefficients of like powers of x in both sides we find

$$\begin{aligned} a_2 &= 0, \\ a_3 &= \frac{1}{3!}, \\ a_4 &= 0, \end{aligned}$$

and generally

$$a_{2n} = 0, \quad for \quad n \geq 0,$$

and

$$a_{2n+1} = \frac{1}{(2n+1)!}, \quad for \quad n \geq 0.$$

Using (3.36) we find the solution $u(x)$ in a series form

$$u(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots,$$

and in a closed form, the exact solution is given by

$$u(x) = \sinh x.$$

3.1.2.2 Adomian Decomposition Method

In this section we will introduce how the Adomian decomposition method (ADM) can be implemented to determine a series solution to the Volterra integro-differential equations.we may assume a standard form to the Volterra integro-differential equation defined by the standard form

$$u^{(n)}(x) = f(x) + \int_0^x K(x,t)u(t)dt, \quad u^{(k)}(0) = b_k, \quad 0 \leqslant k \leqslant (n-1), \tag{3.37}$$

where $u^(n)(x)$ indicates the nth derivative of $u(x)$ with respect to x and b_k are constants that define the initial conditions. It is natural to seek an expression for $u(x)$ that will be derived from (3.37). This can be done by integrating both sides of (3.37) from 0 to x as many times as the order of the derivative involved. Consequently, we obtain

$$u(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) + L^{-1}\left(\int_0^x k(x,t)u(t)dt\right), \tag{3.38}$$

where $\sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k$ is obtained by using the initial conditions, and L^{-1} is an n-fold integration operator. Now we apply the decomposition method by defining the solution $u(x)$ of (3.38) in a decomposition series given by

$$u(x) = \sum_{n=0}^{\infty} a^n u_n(x). \tag{3.39}$$

Substituting (3.39) into both sides of (3.38) we get

$$\sum_{n=0}^{\infty} u_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) + L^{-1}\left(\int_0^x k(x,t)(\sum_{n=0}^{\infty} u_n(t))\right) dt,$$

or equivalently

$$\begin{aligned} u_0(x) + u_1(x) + u_2(x) + \cdots \quad &= \quad \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)) \\ &+ \int_0^x k(x,t)u_0(t)dt + \int_0^x k(x,t)u_1(t)dt \\ &+ \int_0^x k(x,t)u_2(t)dt + \cdots . \end{aligned}$$

The components $u_i(x)$, $i \geq 0$ of the unknown function $u(x)$ are determined in a recursive manner, in a similar way as discussed before, where we set

$$\begin{aligned}
u_0(x) &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)), \\
u_1(x) &= L^{-1}\left(\int_0^x k(x,t)u_0(t)dt\right), \\
u_2(x) &= L^{-1}\left(\int_0^x k(x,t)u_1(t)dt\right), \\
u_3(x) &= L^{-1}\left(\int_0^x k(x,t)u_2(t)dt\right),
\end{aligned}$$

and so on. The decomposition method discussed above for the determination of the components $u_i(x)$, $i \geq 0$ of the solution $u(x)$ of the equation (3.37) can be written in a recursive manner by

$$\begin{aligned}
u_0(x) &= \sum_{k=0}^{n-1} \frac{1}{k!} b_k x^k + L^{-1}(f(x)), \\
u_{n+1}(x) &= L^{-1}\left(\int_0^x k(x,t)u_n(t)dt\right), \quad n \geq 0.
\end{aligned} \tag{3.40}$$

The components $u_i(x)$ with $i \geq 0$ are immediately evaluated. The solution $u(x)$ of (3.37) is then obtained in a series form using (3.39). Consequently, the series obtained for $u(x)$ mostly provides the exact solution in a closed form. However, for concrete problems, where (3.39) cannot be evaluated, a truncated series is usually used to approximate the solution $u(x)$. It is convenient to point out that the phenomenon of the self-cancelling noise terms that was introduced before may be applied here if the noise terms appear between $u_0(x)$ and $u_1(x)$. Moreover, the modified decomposition method can be used as well. The following examples will explain how we can use the Adomian decomposition technique.

Example 3.4. *Solve the following Volterra integro-differential equation by using the decomposition method.*

$$\begin{cases} u''(x) = x + \int_0^x (x-t)u(t)dt, \\ u(0) = 0, \quad u'(0) = 1. \end{cases} \tag{3.41}$$

Applying the two-fold integral operator L^{-1}

$$L^{-1}(.) = \int_0^x \int_0^t (.)dtdx$$

to both sides of (3.41)*, i.e. integrating both sides of* (3.41) *twice from* 0 *to* x*, and using the given initial conditions yield*

$$u(x) = x + \frac{1}{3!}x^3 + L^{-1}\left(\int_0^x (x-t)u(t)dt\right).$$

Following the decomposition scheme (3.41) we find

$$\begin{aligned}
u_0(x) &= x+\frac{1}{3!}x^3,\\
u_1(x) &= L^{-1}\left(\int_0^x (x-t)u_0(t)dt\right),\\
&= \frac{1}{5!}x^5+\frac{1}{7!}x^7,\\
u_2(x) &= L^{-1}\left(\int_0^x (x-t)u_1(t)dt\right),\\
&= \frac{1}{9!}x^9+\frac{1}{11!}x^{11},\\
u_3(x) &= L^{-1}\left(\int_0^x (x-t)u_2(t)dt\right),\\
&= \frac{1}{13!}x^{13}+\frac{1}{15!}x^{15},\\
&\vdots\quad .
\end{aligned}$$

Combining the last results yields the solution $u(x)$ in a series form given by

$$u(x)=x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\frac{1}{7!}x^7+\frac{1}{9!}x^9+\frac{1}{11!}x^{11}+\frac{1}{13!}x^{13}+\frac{1}{15!}x^{15}+\cdots,$$

and this leads to the exact solution in a closed form

$$u(x)=\sinh x.$$

3.2 Methods of Approximating an Integral

The integral equations have motivated a large amount of research works. Integral equations have been the best way to formulate physics, mechanics, fluid, elastisity, radiation science and other fields problems. Moreover the numerical integral gives smaller relative errors than the numerical differentiation. Currently different numerical methods for finiding an approximate solution of integral equations were proposed, such as, collocation method [33], Galerkin method. ÷n this section we proposed an approximate solution to an integral equation using Gauss Legendre quadrature.

Gauss Legendre quadrature

The aim is to give the approximate solution of Fredholm integral equations defined as

$$u(x)-\int_a^b k(x,t)u(t)dt=f(x), \tag{3.42}$$

where $f(x)$, $k(x,t)$ are known functions, and $u(x)$ is unknown function. The integrals have been approximated using the following integration formula

$$\int_a^b k(x,t)u(t)dt \approx \frac{b-a}{2}\sum_{j=0}^{N} w_j k(x, \frac{b+a}{2}+\frac{b-a}{2}y_j)u(\frac{b+a}{2}+\frac{b-a}{2}y_j), \tag{3.43}$$

where, y_j and w_j are shifted Legendre-Gauss nodes and weights. By replacing equation (3.43) in equation (3.42), and by collocating at points $x = x_k$, such that

$$x_i = \frac{(b-a)}{2}\left[1-\frac{1}{2}\cos\left(\frac{(i-1)\pi}{N}\right)\right]+\frac{b+a}{2}, i = 0,\dots,N,$$

we obtain

$$u(x_i) - \frac{b-a}{2}\sum_{j=0}^{N} w_j k(x_i, \frac{b+a}{2}+\frac{b-a}{2}y_j)u(\frac{b+a}{2}+\frac{b-a}{2}y_j) = f(x_i).$$

Example 3.5. *Consider the Fredholm equation*

$$u(x) - \int_{-1}^{1} xtu(t)dt = e^{-x^2},$$

where the exact solution is $u(x) = e^{-x^2}$. *The numerical results are presented in the table 3.1 for* $N = 5$, *and the Figure 3.1 .*

x	Exact solution	Approximate solution	Absolute error
-1.0	0.36787944	0.37070743	0.00282799
-0.8	0.52729242	0.52497527	0.00231715
-0.6	0.69767633	0.70016097	0.0248464
-0.4	0.85214379	0.85486484	0.00272105
-0.2	0.96078944	0.95951565	0.00127379
0.0	1.00000000	0.99637067	0.00362933
0.2	0.96078944	0.95951565	0.00127379
0.4	0.85214379	0.85486484	0.00272105
0.6	0.69767633	0.70016097	0.00248464
0.8	0.52729242	0.52497527	0.00231715
1.0	0.36787944	0.37070743	0.00282799

Table 3.1: Approximate and exact solutions for Example 3.5.

3.3 Approximating Method for Differential Equations

In mathematics, a differential equation is an equation whose unknown(s) are functions; it is presented in the form of a relationship between these unknown functions and their

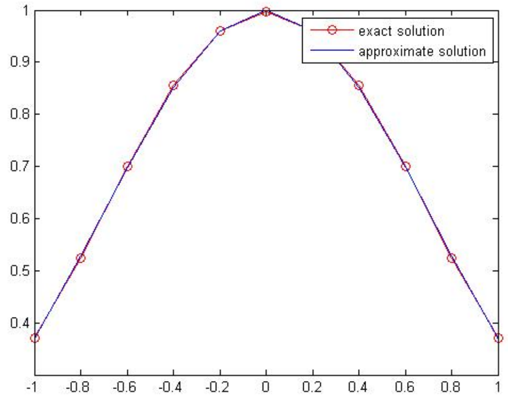

Figure 3.1: Exact and approximate solution for Example 3.5, at $N = 5$.

successive derivatives. This is a special case of functional equation.
There are generally two types of differential equations:

- Ordinary differential equations (ODE) where the unknown function(s) depend on only one variable.

- Partial differential equations (PDEs), where the unknown function(s) may depend on several independent variables.

The classical methods like Finite elements method (FEM), Finite difference method (FDM) and Finite Volume method (FVM) are still the most used methods for solving systems of partial differential equations in physical modeling problems.
Case 1: In this part, we used a FDM for solving the following famous partial differential equation called *Poisson's equation*,

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f, & (x,y) \in]0.1[\times]0,1[, \\ u(x,1) = u(0,y) = u(1,y) = 0, & \\ u(x,0) = sin(\pi x). & \end{cases}$$

We see that if $f = 0$, we obtain Laplace's equation, where $\Delta x = 1/n$ and $\Delta y = 1/m$, with $n,\ m$ are the numbers of subdivision. As this equation is only made up of second derivatives, we will approximate them by using the Taylor's formula of order 2 which gives us the centered approximation of the second partial derivatives, given as follow

$$\frac{\partial^2 u}{\partial x^2}(x,y) = \frac{u(x+\Delta x,y) - 2u(x,y) + u(x-\Delta x,y)}{(\Delta x)^2},$$

and

$$\frac{\partial^2 u}{\partial y^2}(x,y) = \frac{u(x,y+\Delta y) - 2u(x,y) + u(x,y-\Delta y)}{(\Delta y)^2},$$

Substituting by point (x_i, y_j), we have

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2},$$

$$\frac{\partial^2 u}{\partial y^2}(x_i, y_j) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2},$$

Replacing these approximations into Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f,$$

yields to

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = f_{i,j},$$

if $\Delta x = \Delta y = h$ the Poissson equation can be written as

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j}.$$

We notice that at each step, we need to know the points $u_{i+1,j}$, $u_{i-1,j}$, $u_{i,j+1}$ and $u_{i,j-1}$ to calculate the value of $u_{i,j}$ at the point (x_i, y_j)
The corresponding matrix system $Au = b$ is given by

$$k = (j-1)n + i, \quad \forall i = 2, \dots, n-1, \quad \forall j = 2, \dots, m-1.$$

$$\begin{cases} A(k,k) = -4,\ A(k,k-1) = 1,\ A(k,k+1) = 1, \\ A(k,k+n) = 1,\ A(k,k-n) = 1,\ b(k,1) = h^2 f_{i,j}. \end{cases}$$

$$\begin{cases} A(k,k) = 1,\ i = 1,\ \forall j = 1, \dots, m, \\ b(k,1) = 0, \end{cases}$$

$$\begin{cases} A(k,k) = 1,\ i = n,\ \forall j = 1, \dots, m, \\ b(k,1) = \sin(\pi x_i), \end{cases}$$

$$\begin{cases} A(k,k) = 1,\ j = 1, j, \dots, m,\ \forall i = 1, \dots, n, \\ b(k,1) = 0, \end{cases}$$

From the numerical results using Finite difference method, we obtain the approach values of u for $h = 0.1$ and $f = \frac{-5\pi^2}{4}\sin(\pi x)\cos(\frac{\pi y}{2})$ which are sitting in Table 3.2. The curves of approximate solution are given in Figure 3.2.

x/y	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0	0	0	0
0.1	0.309	0.306	0.295	0.276	0.251	0.220	0.183	0.141	0.096	0.049
0.2	0.588	0.582	0.561	0.562	0.478	0.418	0.347	0.268	0.183	0.093
0.3	0.809	0.801	0.772	0.742	0.658	0.575	0.478	0.369	0.251	0.127
0.4	0.951	0.941	0.907	0.851	0.773	0.676	0.562	0.434	0.296	0.150
0.5	1.000	0.990	0.954	0.895	0.813	0.711	0.591	0.457	0.311	0.157
0.6	0.951	0.941	0.907	0.851	0.773	0.676	0.562	0.434	0.296	0.150
0.7	0.809	0801	0.772	0.724	0.658	0.575	0.478	0.369	0.251	0.127
0.8	0.588	0.582	0.561	0.526	0.478	0.418	0.347	0.268	0.183	0.093
0.9	0.309	0.306	0.295	0.276	0.251	0.220	0.183	0.141	0.096	0.049
1	0	0	0	0	0	0	0	0	0	0

Table 3.2: Numerical results using FDM for solving Poisson's equation.

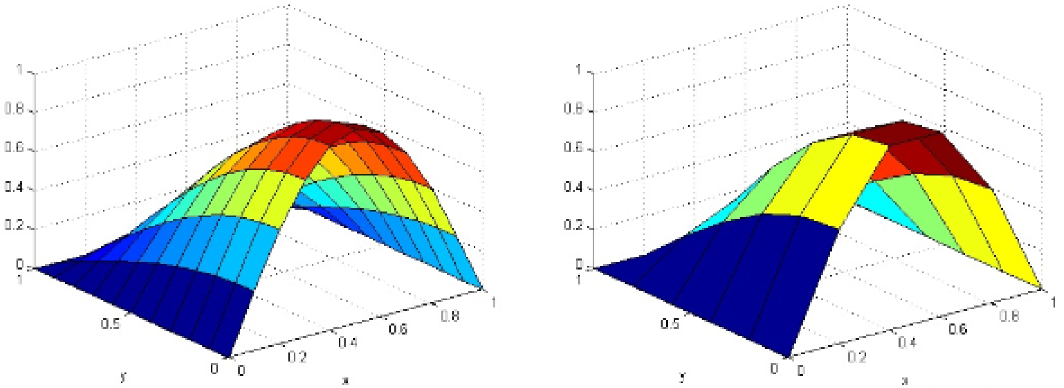

Figure 3.2: Numerical results using FDM for solving Poisson's equation : (Left) for $h = 0.1$ and (Right) for $h = 0.2$.

Case 2: In this case, we solve the following ordinary differential equation using FDM as well

$$\begin{cases} -u''(x) = f(x), & \forall x \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{3.44}$$

where $f(x)$ is a known function. We seek to obtain an approximation of the solution $u(x)$ in the interval $[0,1]$, to do this, the subdivisions of this interval into N length sub-intervals $h = 1/N$. We thus obtain $N+1$ points x_i checking $x_0 = 0$, $x_N = 0$ and for intermediate points

$$x_i = \frac{i}{N},\ x_{i+1} - x_i = h.$$

We notice u_i the approximation of $u(x_i)$ on point x_i. The boundary conditions impose that $u_0 = u_N = 0$. The finite difference method consists of directly discretizing the differential equation by replacing the derivatives of $u(x)$ by finite differences, at every point x_i, we can use for example a centered difference of order 2,

$$u''(x_i) = \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} + O(h^2) \simeq \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}.$$

The differential equation is written at each point x_i

$$-u''(x_i) = f(x_i).$$

So that by replacing by the centered difference, we obtain

$$-\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f(x_i),$$

and this for i ranging from 1 to $N-1$. In the previous equation, we have of course neglected the error term $O(h^2)$ and it results in an approximation of order two. We thus obtain a linear system of $(N-1)$ equations in $(N-1)$ unknowns of the form

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = h^2 \begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{N-2}) \\ f(x_{N-1}) \end{pmatrix}.$$

Solving this tridiagonal linear system is simple and provides an approximate solution to the initial differential equation at points x_i. we can see the numerical solution obtained with 10 intervals ($h = 0,1$) and the function $f(x) = -6x$. In this case, we easily verify that the analytical solution is $u(x) = x^3 - x$. We can therefore see that the numerical solution is a good approximation of the analytical solution.

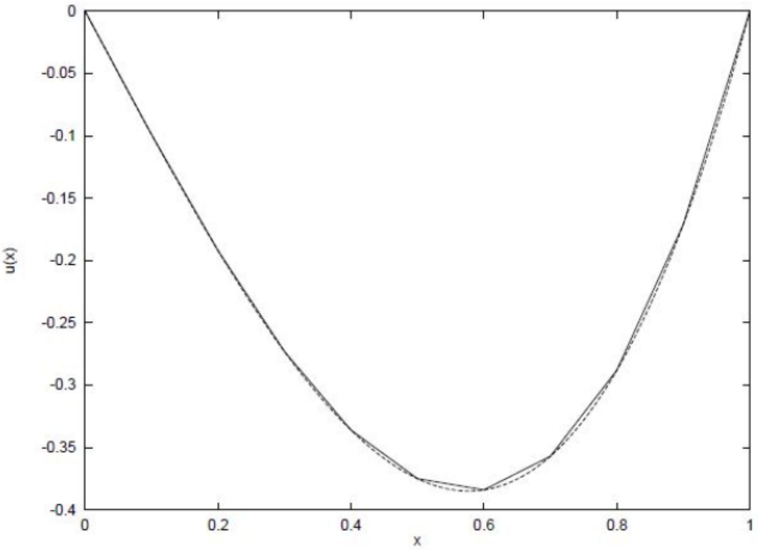

Figure 3.3: Numerical results using FDM for solving Poisson's equation (1D) for $h = 0.1$.

Chapter 4

Finite Elements Methods for Integro-Differential Equations

In this chapter we explain how to solve our interest problem in the form integro-differential equations using finite elements methods and we demonstrate the effectiveness and efficiency of current method described in the previous chapter, we solved several different examples using the Matlab code. The results are displayed in tables and figures, which include exact solutions, approximate solutions and absolute errors. Furthermore, we have compared these results with some existing methods.

4.1 Presentation of FEM

The general step of the finite element method is as follows.
We have an PDE or ODE to solve on a domain,

$$(PC)\begin{cases} Au = f \\ u(0) = \alpha, \quad u(1) = \beta. \end{cases}$$

We write the variational formulation of this PDE or ODE, and so we are back to a problem of type

$$(PV)\begin{cases} \text{Find a function } u \in V \text{ such that:} \\ a(u,v) = l(v), \qquad \forall v \in V. \end{cases}$$

We will look for an approximation of u by internal approximation. for this, we define a mesh of the domain, we will define an approximation space V_h, s.e.v de V of finite dimension N_h. The problem approached is then

$$\begin{cases} \text{Find a function } u \in V_h \text{ such that:} \\ a(u_h,v_h) = l(v_h), \qquad \forall v_h \in V_h. \end{cases} \tag{4.1}$$

Let $\varphi_1, \varphi_2, ..., \varphi_{N_h}$, a base of V_h, By decomposing u_h on this basis in the form

$$u_h = \sum_{i=1}^{N} u_i \varphi_i,$$

The problem 4.1 becomes with a bilinear and l linear

$$\begin{cases} \text{Find a function } u \in V_h \quad \text{such that:} \\ \sum\limits_{i=1}^{N} u_i a(\varphi_i, v_h) = l(v_h), \qquad \forall v_h \in V_h. \end{cases}$$

For $v_h = \varphi_j$,

$$\begin{cases} \text{Find a function } u \in V_h \quad \text{such that:} \\ \sum\limits_{i=1}^{N} u_i a(\varphi_i, \varphi_j) = l(\varphi_j), \qquad \forall j = 1, ..., N. \end{cases}$$

That is, solve the following linear system

$$\begin{pmatrix} a(\varphi_1,\varphi_1) & a(\varphi_2,\varphi_1) & \dots & a(\varphi_N,\varphi_1) \\ \vdots & \vdots & \ddots & \vdots \\ a(\varphi_1,\varphi_N) & a(\varphi_2,\varphi_N) & \dots & a(\varphi_N,\varphi_N) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} l(\varphi_1) \\ \vdots \\ l(\varphi_N) \end{pmatrix},$$

where simply

$$Au = b,$$

where the matrix A assumed to be full a priori. However, to limit the volume of calculations, we will define basic functions ϕ_i whose support will be small,that is, every function ϕ_i will be null everywhere except for a few miles, Thus the terms $a(i, j)$ will most often be zero,because corresponding a functions ϕ_i and ϕ_j of disjoint supports. The matrix A will therefore be a hollow matrix, and we will order the ϕ_i in such a way that A is a band structure, with the lowest possible bandwidth.

4.2 Finite Elements Method of Degree 1

Let's divide the interval $[0; 1]$ into $N + 1$ parts (N being a positive integer) and let's pose $h = \dfrac{1}{N+1}$, $x_i = ih$ with $i = 0, 1, ..., N + 1$.

We define, for $i = 0, 1, 2, ..., N + 1$, the following functions

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & if \quad x_{i-1} \leq x \leq x_i \\ \dfrac{x - x_{i+1}}{x_i - x_{i+1}}, & if \quad x_i \leq x \leq x_{i+1} \\ 0. & if \quad x \leq x_{i-1} \quad or \quad x \geq x_{i+1} \end{cases} \tag{4.2}$$

such that

$$\varphi_i(x_j) = \delta_{ij} = \begin{cases} 1, & if \;\; i = j, \\ 0. & if \;\; i \neq j, \end{cases} \quad 0 \leq j \leq N+1,$$

$\varphi_{i[x_{j-1},x_j]}$ is a polynomial of degree 1.
Thus the function φ_i belongs to V, the functions $\varphi_1, \varphi_2, ..., \varphi_N$ are linearly independent and we choose them to generate space V^N, we will thus say that
$x_1, x_2, .., x_{N+1}$ are the discretization nodes,
$[x_1, x_2], ..., [x_N, x_{N+1}]$ are the geometric elements,
$\varphi_1, \varphi_2, ..., \varphi_N$ are the basic functions of subspace V^N of the finite element type of degree 1 associated with the interior nodes $x_1, x_2, .., x_{N+1}$.

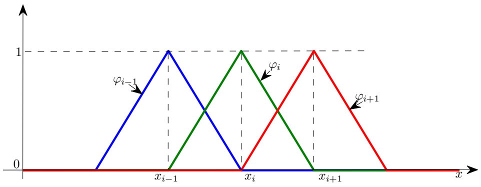

Figure 4.1: Graph of finite element basis functions of degree 1.

4.3 Finite Elements Method of Degree 2

Let's divide the interval $[0; 1]$ into $N + 1$ parts (N being a positive integer) and let's pose $h = \dfrac{1}{N+1}$, $x_i = ih$ with $i = 0, 1, 2, ..., N + 1$, $x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$, with $i = 0, 1, ..., N$
we define for $i = 0, 1, ..., N$ the following functions

$$\varphi_i(x) = \begin{cases} \dfrac{(x - x_{i-1})(x - x_{i-\frac{1}{2}})}{(x_i - x_{i-1})(x_i - x_{i-\frac{1}{2}})}, & if \;\; x_{i-1} \leq x \leq x_i, \\ \dfrac{(x - x_{i+1})(x - x_{i+\frac{1}{2}})}{(x_i - x_{i+1})(x_i - x_{i+\frac{1}{2}})}, & if \;\; x_i \leq x \leq x_{i+1}, \\ 0. & if \;\; x \leq x_{i-1} \;\; or \;\; x \geq x_{i+1}. \end{cases} \tag{4.3}$$

And for $i = 0, 1, ..., N$ the following functions

$$\varphi_{i+\frac{1}{2}}(x) = \begin{cases} \dfrac{(x - x_i)(x - x_{i+1})}{(x_{i+\frac{1}{2}} - x_i)(x_{i+\frac{1}{2}} - x_{i+1})}, & if \;\; x_i \leq x \leq x_{i+1}, \\ 0. & if \;\; x \leq x_{i-1} \;\; or \;\; x \geq x_{i+1}, \end{cases} \tag{4.4}$$

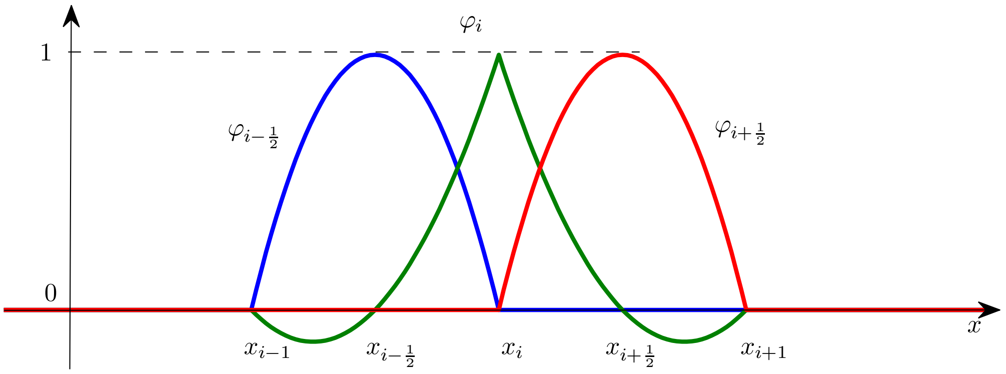

Figure 4.2: Graph of finite element basis functions of degree 2.

4.4 Applications

We consider the Fredholm integro-differential equation of the form

$$(CP)\begin{cases} -u'(x)+\displaystyle\int_0^1 k(x,t)u(t)dt = f(x), \quad 0<x<1, \\ u(0)=\alpha, \ u(1)=\beta, \end{cases} \tag{4.5}$$

where k and f are given continuous functions on $[0,1]\times[0,1]$ and $[0,1]$ respectively, u is unknown function to be determined. writing the abbreviation of Problem (4.5) by,

$$\begin{cases} -\mathcal{A}u+\mathcal{T}u = f, \\ u(0)=\alpha, \ u(1)=\beta, \end{cases} \tag{4.6}$$

where $\mathcal{A}$ is the differential operator and $\mathcal{T}$ is the integral one, given by

$$\mathcal{A}u(x)=u'(x), \quad \mathcal{T}u(x)=\int_0^1 k(x,t)u(t)dt, \qquad x,t\in[0,1]. \tag{4.7}$$

4.4.1 Variational Formulation

The conditions of Problem (4.5) are non-homogeneous. Firstly, we transform this problem into a problem with homogeneous conditions. Put $u(x)=\tilde{u}(x)+(\beta-\alpha)x+\alpha$, by replacing in (4.5), we obtain a homogeneous problem

$$(\widetilde{CP})\begin{cases} -\tilde{u}'(x)+\displaystyle\int_0^1 k(x,t)\tilde{u}(t)dt = g(x), \quad 0<x<1, \\ \tilde{u}(0)=\tilde{u}(1)=0, \end{cases} \tag{4.8}$$

where

$$g(x)=f(x)+(\beta-\alpha)-\int_0^1 k(x,t)\left[(\beta-\alpha)t+\alpha\right]dt.$$

Multiplying the equation in (4.8) by a test function $v \in H_0^1\left(]0,1[\right)$, vanish at the end-points of the interval $]0,1[$ and integrate, we get

$$-\int_0^1 \tilde{u}'(x)v(x)dx + \int_0^1 \left(\int_0^1 k(x,t)\tilde{u}(t)dt\right) v(x)dx = \int_0^1 g(x)v(x)dx, \tag{4.9}$$

this expression is called variational formulation of Problem (4.8), and due to the Dirichlet boundary conditions, we seek the unknown function $\tilde{u}$ in the space $H_0^1\left(]0,1[\right)$. Next, let $\mathbb{V} = H_0^1(]0,1[)$ be a Hilbert space, we note the bilinear form $a(\tilde{u},v)$ on $\mathbb{V}\times\mathbb{V}$ by

$$a(\tilde{u},v) = -\int_0^1 \tilde{u}'(x)v(x)dx + \int_0^1 \left(\int_0^1 k(x,t)\tilde{u}(t)dt\right) v(x)dx, \tag{4.10}$$

and the linear form $l(v)$ on $\mathbb{V}$ by

$$l(v) = \int_0^1 g(x)v(x)dx. \tag{4.11}$$

Hence, the variational formulation reads

$$(\widetilde{VP}) \begin{cases} \text{Find } \tilde{u} \in \mathbb{V}, \text{ such that} \\ a(\tilde{u},v) = l(v), \ \forall v \in \mathbb{V} \\ \tilde{u}(0) = \tilde{u}(1) = 0. \end{cases} \tag{4.12}$$

4.4.2 Lagrange Finite Element P_1 and P_2

The approximation of variational problem $(\widetilde{VP})$ using Lagrange finite element of degree one or two, i.e, approximation by Lagrange polynomial of degree equal to one (LFE1) or equal two (LFE2) respectively.

Let N be a positive integer and $i = 0,1,\ldots,N$. Define the points $x_0, x_1, \ldots, x_N$ on the interval $[0,1]$ such that $0 = x_0 < x_1 < \cdots < x_N = 1$. The points x_i are called vertices. We also introduce N sub-intervals $I_i = [x_{i-1}, x_i]\,,\ i = 1,2,\ldots,N$, where I_i are called elements. The size of the elements is given by $h_i = x_i - x_{i-1}$ and we denote $h = \max\ h_i$.

$$\begin{cases} x_0 = 0, \ x_N = 1, \\ h = \frac{1}{N+1}, \\ x_{i+1} = x_i + h, \ i = 0,1,\ldots,N-1, \\ x_{i+\frac{1}{2}} = x_i + \frac{h}{2}, \ i = 0,1,\ldots,N-1. \end{cases}$$

Let $\mathbb{V}_h^d$ denote the vector space of all piecewise linear $(d = 1)$ or quadratic $(d = 2)$ continuous functions v_h defined on $[0,1]$,

$$\mathbb{V}_h^d = \left\{ v_h : \ v_h \in H^1 \text{ and } v_h|_{I_i} \in P_d(I_i) \right\}, \qquad \text{for } d = 1 \text{ or } 2, \tag{4.13}$$

and $\mathbb{V}_{h,0}^d$ denote the vector space of $\mathbb{V}_h^d$ with $v_h(0) = v_h(1) = 0$,

$$\mathbb{V}_{h,0}^d = \left\{v_h \in \mathbb{V}_h^d : \ v_h(0) = v_h(1) = 0\right\}, \qquad \text{for } d = 1 \text{ or } 2, \tag{4.14}$$

where the dimension of the space $\mathbb{V}_h^d$ is finite (i.e. equal to $dN + 1$).
Let $\varphi_{i/d}$ be basis of the space $\mathbb{V}_h^d$, satisfying,

$$\varphi_{i/d}^d(x_{j/d}) = \delta_{i/d,j/d}, \qquad i, j = 0, 1, \ldots, N.d,$$

where $\delta_{..}$ is the Kronecker delta.

The basis functions $\left\{\varphi_{i/d}^d\right\}_{i=0}^{dN}$ characterizing the nodes of the mesh on the interval $[0, 1]$, are defined by the formula

- For $d = 1$

$$\varphi_i^1(x) := \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{if } x \in I_i, \\ \frac{x-x_{i+1}}{x_i-x_{i+1}} & \text{if } x \in I_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

- For $d = 2$

$$\varphi_i^2(x) := \begin{cases} \frac{(x-x_{i-1})(x-x_{i-1/2})}{(x_i-x_{i-1})(x_i-x_{i-1/2})} & \text{if } x \in I_i, \\ \frac{(x-x_{i+1})(x-x_{i+1/2})}{(x_i-x_{i+1})(x_i-x_{i+1/2})} & \text{if } x \in I_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\varphi_{i+1/2}^2(x) := \begin{cases} \frac{(x-x_i)(x-x_{i+1})}{(x_{i+1/2}-x_i)(x_{i+1/2}-x_{i+1})} & \text{if } x \in I_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that the support of φ_i^d is $I_i \cup I_{i+1}$. The exception is the two basis functions φ_0^d and φ_N^d at the leftmost and rightmost nodes a $x_0 = 0$ and $x_N = 1$ with support only on one interval (i.e, I_1 and I_N respectively).

In terms of basis functions this means that a basis for $\mathbb{V}_{h,0}$ is obtained by deleting the two basis φ_0^d and φ_N^d from the usual set $\left\{\varphi_{i/d}\right\}_{i=0}^{dN}$ of basis functions spanning $\mathbb{V}_h$.

4.4.3 Approximation Variational Formulation

The idea is to approximate in the finite dimensional space $\mathbb{V}_{h,0}^d$. Let v be in the space of approximation $\mathbb{V}_{h,0}^d$, we multiplied the integro-differential equations of continuous Problem (4.8) by the function v and integrated between 0 and 1, we get

$$-\int_0^1 \tilde{u}'(x)v(x)dx + \int_0^1 \left(\int_0^1 k(x,t)\tilde{u}(t)dt\right) v(x)dx = \int_0^1 g(x)v(x)dx.$$

Moreover, an integration by parts (Green's formula) and using the homogeneous conditions, then the approximate problem is given by

$$\begin{cases} \text{Find } \tilde{u} \in \mathbb{V}^d_{h,0}, \text{ such that} \\ a(\tilde{u}, v) = l(v), \ \forall v \in \mathbb{V}^d_h, \\ \tilde{u}(0) = \tilde{u}(1) = 0, \\ \text{where } a(\tilde{u}, v) = \displaystyle\int_0^1 \left[\tilde{u}(x)v'(x) + \left(\int_0^1 k(x,t)\tilde{u}(t)dt\right) v(x)\right] dx, \\ l(v) = \displaystyle\int_0^1 g(x)v(x)dx. \end{cases}$$

Now, the L^2-projection $\mathcal{P}^d_h\tilde{u} \in \mathbb{V}^d_{h,0}$ of $\tilde{u}$ is defined by

$$\int_I (\tilde{u} - \mathcal{P}^d_h\tilde{u})v dx = 0, \quad \forall v \in \mathbb{V}^d_{h,0}. \tag{4.15}$$

In order to actually compute the L^2-projection $\mathcal{P}^d_h\tilde{u}$, we first note that the formula (4.15) is equivalent to

$$\int_I (\tilde{u} - \mathcal{P}^d_h\tilde{u})\varphi^d_{i/d} dx = 0, \quad i = 1, \dots, dN-1. \tag{4.16}$$

Then, since $\mathcal{P}^d_h\tilde{u}$ belongs to $\mathbb{V}^d_{h,0}$ it can be written as the linear combination

$$\tilde{u}^d_N := \mathcal{P}^d_h\tilde{u} = \sum_{j=1}^{dN-1} u_j\varphi^d_{j/d}, \tag{4.17}$$

and choosing $v(x) = \varphi^d_{i/d}(x), \ i = 1, \dots, dN-1$, i.e $d = 1$ or $d = 2$.
Next, the approximate variational formulation is obtained by substituting approximate functions $\tilde{u}$, then the expression given by

$$\begin{cases} \text{Find } \tilde{u} \in \mathbb{V}^d_{h,0}([0,1]), \text{ such that} \\ \displaystyle\sum_{j=1}^{dN-1} \int_0^1 u_j\varphi^d_{j/d}(x)\varphi^{d\,\prime}_{i/d}(x)dx + \sum_{j=1}^{dN-1} \int_0^1\int_0^1 u_j k(x,t)\varphi^d_{j/d}(t)\varphi^d_{i/d}(x)dtdx \\ \qquad\qquad = \displaystyle\int_0^1 g(x)\varphi^d_{i/d}(x)dx, \quad i = 1, \dots, dN-1. \end{cases}$$

4.4.4 Matrix Representations of Method

Now, we approximate the functions k and g in the spaces $\mathbb{V}^d_h \times \mathbb{V}^d_h$ and $\mathbb{V}^d_h$ respectively, as

$$k(x,t) \approx \sum_{p=0}^{dN}\sum_{q=0}^{dN} k_{pq}\varphi^d_{p/d}(x)\varphi^d_{q/d}(t), \qquad k_{pq} = k(x_{p/d}, x_{q/d}),$$

and

$$g(x) \approx \sum_{j=0}^{dN} g_j\varphi^d_{j/d}(x), \qquad g_j = g(x_{j/d}).$$

This implies,

$$\begin{aligned}\sum_{j=1}^{dN-1} u_j \left[\int_0^1 \varphi^d_{j/d}(x){\varphi^d_{i/d}}'(x)dx + \sum_{p=0}^{dN}\sum_{q=0}^{dN} k_{pq}\left(\int_0^1 \varphi^d_{p/d}(x)\varphi^d_{i/d}(x)dx\right)\left(\int_0^1 \varphi^d_{q/d}(t)\varphi^d_{j/d}(t)dt\right)\right] \\ = \sum_{j=0}^{dN} g_j \int_0^1 \varphi^d_{j/d}(x)\varphi^d_{i/d}(x)dx, \quad i = 1, \ldots, dN-1, \qquad (4.18)\end{aligned}$$

we can write the system as follows

$$\sum_{j=1}^{dN-1}\left[a_{ji} + \sum_{p=0}^{dN}\sum_{q=0}^{dN} b_{ip}k_{pq}b_{qj}\right]u_j = \sum_{j=0}^{dN} b_{ij}g_j, \qquad i = 1, \ldots, dN-1, \tag{4.19}$$

where, $a_{ji} = \int_0^1 \varphi^d_{i/d}(x){\varphi^d_{j/d}}'(x)dx$ and $b_{ji} = \int_0^1 \varphi^d_{i/d}(x)\varphi^d_{j/d}(x)dx$.
Then, (4.19) leads to the matrix system

$$(\mathbf{A} + \mathbf{BKB}^t)U = \mathbf{B}G, \tag{4.20}$$

where

$U = [u_1, \ldots, u_{dN-1}]^t$, $G = [g(x_0), g(x_{1/d}), \ldots, g(x_{dN-1/d}), g(x_{dN})]^t$, $\mathbf{K} = [k(x_{i/d}, x_{j/d})]_{0 \le i,j \le dN}$,

if $d = 1$,

$$\mathbf{A} = [a_{ji}]_{1\le j,i\le N-1} = \begin{pmatrix} 0 & \frac{-1}{2} & & & 0 \\ \frac{1}{2} & 0 & \frac{-1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & 0 & \frac{-1}{2} \\ 0 & & & \frac{1}{2} & 0 \end{pmatrix}, \quad \mathbf{B} = [b_{ji}]_{\substack{1\le j\le N-1 \\ 0\le i\le N}} = \frac{h}{6}\begin{pmatrix} 1 & 4 & 1 & & 0 & \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & 4 & 1 \end{pmatrix},$$

and if $d = 2$,

$$\mathbf{A} = [a_{ji}]_{1\le j,i\le 2N-1} = \frac{1}{6}\begin{pmatrix} 0 & -4 & & & & & & & \\ 4 & 0 & -4 & 1 & & & 0 & & \\ & 4 & 0 & -4 & & & & & \\ & -1 & 4 & 0 & -4 & & & & \\ & & & 4 & 0 & -4 & 1 & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & 4 & 0 & -4 & 0 \\ & 0 & & & & -1 & 4 & 0 & -4 \\ & & & & & & & 4 & 0 \end{pmatrix},$$

$$\mathbf{B} = [b_{ji}]_{\substack{1\leq j\leq 2N-1\\ 0\leq i\leq 2N}} = \frac{h}{30}\begin{pmatrix} 2 & 16 & 2 & & & & 0 & & \\ -1 & 2 & 8 & 2 & -1 & & & & \\ & & 2 & 16 & 2 & & & & \\ & & -1 & 2 & 8 & 2 & -1 & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & -1 & 2 & 8 & 2 & -1 \\ & 0 & & & & & 2 & 16 & 2 \end{pmatrix}.$$

Which is a linear system of $(dN-1)$ algebraic equations for $(dN-1)$ unknown coefficients $u_1, u_2, \ldots, u_{dN-1}$. System (4.20) can be summarized as

$$\mathbf{M}U = \mathbf{G} \text{ or } \left[\begin{array}{c|c} \mathbf{M} & \mathbf{G} \end{array}\right], \tag{4.21}$$

where

$$\mathbf{M} = \mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{B}^t \text{ and } \mathbf{G} = \mathbf{B}G,$$

and M is invertible matrix in general, because it is positive definite symmetric matrix so is one as the sum of the symmetric band matrices, then

$$U = \mathbf{M}^{-1}\mathbf{G}. \tag{4.22}$$

Finally, after finding the unknown coefficients, we substitute them into formula (4.17) to obtain the general solution for Problem (4.8). Then can be write the solution of Problem (4.5) as

$$u_N^d(x) = \sum_{i=1}^{dN-1} u_i\varphi_{i/d}^d(x) + (\beta-\alpha)x + \alpha. \tag{4.23}$$

4.5 Error Analysis

In this section, we perform error analysis in the space $\mathbb{V}_h^d$, because $\mathbb{V}_{h,0}^d \subset \mathbb{V}_h^d$ and we have,

$$u(x) - u_N^d(x) = \tilde{u}(x) - \tilde{u}_N^d(x).$$

Firstly, we start with some propositions for interpolation.

Property 4.1. *The interpolant $\mathcal{I}_N^d u$ in the space $\mathbb{V}_h^d$ satisfies the estimates*

$$\|u - \mathcal{I}_N^d u\|_{L^2(I_1)} \leq Ch^{d+1}\|u^{(d+1)}\|_{L^2(I_1)}, \tag{4.24}$$

$$\|(u - \mathcal{I}_N^d u)'\|_{L^2(I_1)} \leq Ch^{d}\|u^{(d+1)}\|_{L^2(I_1)}, \tag{4.25}$$

where C is a constant positive, and $h = x_1 - x_0$.

Proof. (see [63] page 6). Indication, for $d = 2$ we have $(\mathcal{I}_N^2 u)^{(3)} = 0$. □

Property 4.2. *The interpolant $\mathcal{I}_N^d u$ satisfies the estimates*

$$\|u-\mathcal{I}_N^d u\|_{L^2(I)}^2 \leq C\sum_{i=1}^{N} h_i^{2d+2}\|u^{(d+1)}\|_{L^2(I_i)}^2, \tag{4.26}$$

$$\|(u-\mathcal{I}_N^d u)'\|_{L^2(I)}^2 \leq C\sum_{i=1}^{N} h_i^{2d}\|u^{(d+1)}\|_{L^2(I_i)}^2, \tag{4.27}$$

where C is a constant positive, and $h_i = x_i - x_{i-1}$.

Proof. Using the Triangle inequality and Proposition 4.1, we have

$$\begin{aligned} \|u-\mathcal{I}_N^d u\|_{L^2(I)}^2 &= \int_I (u(x)-\mathcal{I}_N^d u(x))^2 dx \\ &= \sum_{i=1}^{N}\int_{I_i}(u(x)-\mathcal{I}_N^d u(x))^2 dx \\ &= \sum_{i=1}^{N}\|u-\mathcal{I}_N^d u\|_{L^2(I_i)}^2 \\ &\leq C\sum_{i=1}^{N} h_i^{2d+2}\|u^{(d+1)}\|_{L^2(I_i)}^2. \end{aligned}$$

This confirms the first estimate. We follow the same steps to prove the second estimate. □

Now, in flowing the error estimate for projection.

Lemma 4.1 ([63]). *The L_2-projection $\mathcal{P}_N^d u$, defined by (4.15), satisfies the best approximation result*

$$\|u-\mathcal{P}_N^d u\|_{L^2(I)}^2 \leq \|u-v\|_{L^2(I)}^2, \qquad \forall v \in \mathbb{V}_h^d, \tag{4.28}$$

$$\|(u-\mathcal{P}_N^d u)'\|_{L^2(I)}^2 \leq \|(u-v)'\|_{L^2(I)}^2, \qquad \forall v \in \mathbb{V}_h^d. \tag{4.29}$$

Proof. (see [63], page 12). □

Lemma 4.2. *The L^2-projection $\mathcal{P}_N^d u$ satisfies the estimate*

$$\|u-\mathcal{P}_N^d u\|_{L^2(I)}^2 \leq C\sum_{i=1}^{N} h_i^{2d+2}\|u^{(d+1)}\|_{L^2(I_i)}^2, \tag{4.30}$$

$$\|(u-\mathcal{P}_N^d u)'\|_{L^2(I)}^2 \leq C\sum_{i=1}^{N} h_i^{2d}\|u^{(d+1)}\|_{L^2(I_i)}^2. \tag{4.31}$$

Proof. Commencing with the optimal approximation result, selecting $v = \mathcal{I}_N^d u$ as the interpolate for u, and applying the interpolation error estimate inequality (4.24) from Proposition

4.1, we obtain

$$\begin{aligned}\|u-\mathcal{P}_N^d u\|^2_{L^2(I)} &\leq \|u-\mathcal{I}_N^d u\|^2_{L^2(I)} && (4.32)\\ &\leq \sum_{i=1}^{N} \|u-\mathcal{I}_N^d u\|^2_{L^2(I_i)} && (4.33)\\ &\leq C\sum_{i=1}^{N} h_i^{2d+2}\|u^{(d+1)}\|^2_{L^2(I_i)}, && (4.34)\end{aligned}$$

which proves the estimate. Using the same steps, we obtain the second inequality by applying inequality (4.25). □

Corollary 4.1. *Recalling the definition* $h = \max h_i$ *we conclude that,*

$$\|u-\mathcal{P}_N^d u\|_{L^2(I)} \leq Ch^{d+1}\|u^{(d+1)}\|_{L^2(I)}, \qquad (4.35)$$

$$\|(u-\mathcal{P}_N^d u)'\|_{L^2(I)} \leq Ch^{d}\|u^{(d+1)}\|_{L^2(I)}. \qquad (4.36)$$

Theorem 4.1. *Let* u *the exact solution of to Problem* (4.5) *and the approximated solution* u_N *be obtained by using the finite element method* (4.23). *If* $u \in H_0^1(]0,1[)$, *then for*

$$\|(u-u_N^d)'\|_{L_2(I)} \leq Ch^{d+1}\left(\|g^{(d+1)}\|_{L^2(I)} + \lambda_{d,N}\|\tilde{u}^{(d+1)}\|_{L^2(I)} + \theta_{d,N}\|\tilde{u}\|_{L_2(I)}\right),$$

where

$$\lambda_{d,N} = \max_{0\leq i,j\leq dN} |k(x_i,x_j)|,$$

$$\theta_{d,N} = \max_{x,t\in I} |\partial_t^{d+1} k(x,t)|.$$

Proof. Let the Fredholm integro-differential equations in (4.5),

$$-u'(x) + \int_I k(x,t)u(t)dt = f(x), \qquad (4.37)$$

while using the transformation we apply (4.8),

$$-\tilde{u}'(x) = g(x) - \int_I k(x,t)\tilde{u}(t)dt. \qquad (4.38)$$

Now, we have the approximation solution of this equation as,

$$-(\tilde{u}_N^d)'(x) = \mathcal{P}_N^d g(x) - \int_I \mathcal{P}_{N,N}^{d,d} k(x,t)\tilde{u}_N^d(t)dt. \qquad (4.39)$$

Subtracting (4.39) from (4.38), we get the error equation

$$\tilde{u}'(x) - (\tilde{u}_N^d)'(x) = \mathcal{P}_N^d g(x) - g(x) + \int_I k(x,t)\tilde{u}(t)dt - \int_I \mathcal{P}_{N,N}^{d,d} k(x,t)\mathcal{P}_N\tilde{u}(t)dt, \qquad (4.40)$$

which can be rewritten as,

$$\begin{aligned}(\tilde{u}-\tilde{u}_N^d)'(x) &= \mathcal{P}_N^d g(x) - g(x) + \int_I \mathcal{P}_{N,N}^{d,d} k(x,t)(\tilde{u}(t) - \mathcal{P}_N^d \tilde{u}(t))dt \\ &\quad + \int_I \Big(k(x,t) - \mathcal{P}_{N,N}^{d,d} k(x,t)\Big)\,\tilde{u}(t)dt \\ &= E_1(x) + E_2(x) + E_3(x),\end{aligned}$$

where

$$E_1(x) = \mathcal{P}_N^d g(x) - g(x) \tag{4.41}$$

$$E_2(x) = \int_I \mathcal{P}_{N,N}^{d,d} k(x,t)(\tilde{u}(t) - \tilde{u}_N^d(t))dt \tag{4.42}$$

$$E_3(x) = \int_I \Big(k(x,t) - \mathcal{P}_{N,N}^{d,d} k(x,t)\Big)\,\tilde{u}(t)dt. \tag{4.43}$$

We have by the triangle inequality

$$\|(\tilde{u}-\tilde{u}_N^d)'\|_{L_2(I)} \leq \|E_1\|_{L_2(I)} + \|E_2\|_{L_2(I)} + \|E_3\|_{L_2(I)}. \tag{4.44}$$

Corollary 4.1 implies directly that

$$\|E_1\|_{L_2(I)} = \|g(x) - \mathcal{P}_N g(x)\|_{L_2(I)} \leq Ch^{d+1}\|g^{(d+1)}\|_{L^2(I)}. \tag{4.45}$$

By using Cauchy-Schwarz together with Corollary 4.1, we have

$$\begin{aligned}|E_2(x)| &= \left|\int_I \mathcal{P}_{N,N}^{d,d} k(x,t)(\tilde{u}(t) - \tilde{u}_N^d(t))dt\right| \\ &\leq \left(\int_I |\mathcal{P}_{N,N}^{d,d} k(x,t)|^2 dt\right)^{1/2} \left(\int_I |\tilde{u}(t) - \tilde{u}_N^d(t)|^2 dt\right)^{1/2} \\ &\leq Ch^{d+1}\|\tilde{u}^{(d+1)}\|_{L^2(I)} \left(\int_I |\mathcal{P}_{N,N}^{d,d} k(x,t)|^2 dt\right)^{1/2},\end{aligned}$$

then

$$\begin{aligned}\|E_2(x)\|_{L^2(I)} &= \left(\int_I |E_2(x)|^2 dx\right)^{1/2} \\ &\leq Ch^{d+1}\|\tilde{u}^{(d+1)}\|_{L^2(I)} \left(\int_I\int_I |\mathcal{P}_{N,N}^{d,d} k(x,t)|^2 dtdx\right)^{1/2} \\ &\leq Ch^{d+1} \max_{0\leq i,j\leq dN} |k(x_i,x_j)|.\|\tilde{u}^{(d+1)}\|_{L^2(I)}.\end{aligned}$$

Additionally, employing Cauchy-Schwarz once more

$$\begin{aligned}
|E_3(x)| &= \left|\int_I \left(k(x,t) - \mathcal{P}_{N,N}^{d,d}k(x,t)\right)\tilde{u}(t)dt\right| \\
&\leq \left(\int_I \left|k(x,t) - \mathcal{P}_{N,N}^{d,d}k(x,t)\right|^2 dt\right)^{1/2} \left(\int_I |\tilde{u}(t)|^2 dt\right)^{1/2} \\
&\leq Ch^{d+1}\|\tilde{u}\|_{L_2(I)} \left(\int_I |\partial_t^{d+1}k(x,t)|^2 dt\right)^{1/2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|E_3\|_{L^2(I)} &= \left(\int_I |E_3(x)|^2 dx\right)^{1/2} \\
&\leq Ch^{d+1}\|\tilde{u}\|_{L_2(I)} \left(\int_I \int_I |\partial_t^{d+1}k(x,t)|^2 dt dx\right)^{1/2} \\
&\leq Ch^{d+1} \max_{x,t\in I} |\partial_t^{d+1}k(x,t)|.\|\tilde{u}\|_{L_2(I)}.
\end{aligned}$$

In conclusion, the theorem's assertion is a consequence of the triangle inequality. □

4.6 Illustrative Examples

In this section, we show the numerical results and absolute errors for some examples are given. All computations were carried out by software MATLAB. Now we define the maximum absolute error for $u_N^d(x)$ as

$$E_N := \|u(x) - u_N^d(x)\|_\infty = \max_{i=0,100} \left|u(x_i) - u_N^d(x_i)\right|, \qquad 0 \leq x_i \leq 1.$$

Example 4.1. *Let us first consider the integro-differential equation*

$$\begin{cases} -u'(x) + \displaystyle\int_0^1 xtu(t)dt = \frac{-7}{4}x, \ 0 < x < 1, \\ u(0) = 0, \ u(1) = 1, \end{cases} \tag{4.46}$$

whose exact solution is $u(x) = x^2$. *We apply the methods that was explained in previous Section for* $N = 3$.

Firstly, for LFE1 method, the vrertices is $\{x_i\}_{i=0}^3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. *Then the augmented matrix is*

$$\left[\begin{array}{c|c} \mathbf{M} & \mathbf{G} \end{array}\right] = \left[\begin{array}{cc|c} 1/81 & -77/162 & 11/108 \\ 85/162 & 4/81 & -7/54 \end{array}\right].$$

By solving this system,

$$U = [-55/243 \quad -107/486]^T,$$

then the approximate solution of Problem (4.46) *is given by*

$$\begin{aligned} u_3^1(x) &= \sum_{i=1}^{2} u_N \varphi_i^1(x) + x \\ &= -55/243.\varphi_1^1(x) - 107/486.\varphi_2^1(x) + x \\ &= \begin{cases} 26x/81 & \text{if } 0 \leq x \leq \frac{1}{3}, \\ 55x/54 - 113/486 & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3}, \\ 269x/162 - 107/162 & \text{if } \frac{2}{3} \leq x \leq 1. \end{cases} \end{aligned}$$

Now, for LFE2 method, the vrertices is $\{x_i\}_{i=0}^{6} = \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\}$*. Then the augmented matrix is,*

$$\left[\ \mathbf{M} \,\middle|\, \mathbf{G}\ \right] = \left[\begin{array}{ccccc|c} \frac{1}{729} & \frac{-485}{729} & \frac{1}{243} & \frac{2}{729} & \frac{5}{729} & \frac{47}{324} \\ \frac{487}{729} & \frac{1}{729} & \frac{-161}{243} & \frac{247}{1458} & \frac{5}{729} & \frac{11}{324} \\ \frac{1}{243} & \frac{163}{243} & \frac{1}{81} & \frac{-160}{243} & \frac{5}{243} & \frac{-1}{108} \\ \frac{2}{729} & \frac{-239}{1458} & \frac{164}{243} & \frac{4}{729} & \frac{-476}{729} & \frac{-7}{162} \\ \frac{5}{729} & \frac{5}{729} & \frac{5}{243} & \frac{496}{729} & \frac{25}{729} & \frac{-53}{324} \end{array}\right].$$

By solving this system,

$$U = [-5/36 \quad -2/9 \quad -1/4 \quad -2/9 \quad -5/36]^T,$$

then the approximate solution of Problem (4.46) *is given by*

$$\begin{aligned} u_3^2(x) &= -5/36\varphi_{1/2}^2(x) - 2/9\varphi_1^2(x) - 1/4\varphi_{3/2}^2(x) - 2/9\varphi_2^2(x) - 5/36\varphi_{5/2}^2(x) + x \\ &= x^2, \end{aligned}$$

it is also the exact solution to the problem. The results in Table 4.1 shows the numerical solutions are in a very good agreement with the exact solution.

x	Exact Solution	LFE1($N=3$)	LFE2($N=3$)
0.0	0.00000000	0.00000000	0.00000000
0.1	0.01000000	0.03209877	0.01000000
0.2	0.04000000	0.06419753	0.04000000
0.3	0.09000000	0.09629630	0.09000000
0.4	0.16000000	0.17489712	0.16000000
0.5	0.25000000	0.27674897	0.25000000
0.6	0.36000000	0.37860082	0.36000000
0.7	0.49000000	0.50185185	0.49000000
0.8	0.64000000	0.66790123	0.64000000
0.9	0.81000000	0.83395062	0.81000000
1.0	1.00000000	1.00000000	1.00000000

Table 4.1: Approximate and exact solutions of Example 4.1.

Example 4.2. *[57] Consider the integro-differential*

$$-u'(x) + \int_0^1 xu(t)dt = x - (x+1)e^x, \ 0 < x < 1, \tag{4.47}$$

with $u(0) = 0$, $u(1) = e^1$. The exact solution of this problem is $u(x) = xe^x$. In Table 4.2, absolute errors in solutions obtained by Lagrange finite element P_1 and P_2 for $N = 64$ are compared with differential transfer method [51], Hybrid function method [54] and Improved homotopy perturbation method [57]. In Fig.4.3 ***a****, we plot the exact solution against the numerical solution obtained by LFE1 and LFE2 with $N = 4$, and in Fig.4.3* ***b****, we depict the rate of convergence.*

				Present Methods	
x	DTM [51]	HFM [54]	IHPM [57]	LFE1	LFE2
0.0	$0.00000e+00$	$1.11022e-16$	$0.00000e+00$	$0.00000e+00$	$0.00000e+00$
0.1	$1.00118e-02$	$9.07270e-03$	$2.31481e-05$	$4.11506e-06$	$6.09093e-05$
0.2	$2.78651e-02$	$1.13773e-02$	$9.25926e-05$	$9.21840e-05$	$4.10075e-05$
0.3	$5.08730e-02$	$9.84041e-03$	$2.08333e-04$	$8.35173e-05$	$4.20896e-05$
0.4	$7.55356e-02$	$6.87421e-03$	$3.70370e-04$	$3.25261e-05$	$6.69726e-05$
0.5	$9.71888e-02$	$4.30919e-03$	$5.78704e-04$	$3.38208e-10$	$2.37256e-06$
0.6	$1.09551e-01$	$3.31778e-03$	$8.33333e-04$	$6.67817e-05$	$7.39818e-05$
0.7	$1.04133e-01$	$4.32872e-03$	$1.13426e-03$	$3.86874e-05$	$5.18724e-05$
0.8	$6.94512e-02$	$6.93055e-03$	$1.48148e-03$	$2.22010e-05$	$5.53054e-05$
0.9	$1.00034e-02$	$9.76311e-03$	$1.87500e-03$	$1.36465e-04$	$8.96383e-05$
1.0	$1.55147e-01$	$1.03954e-02$	$2.31481e-03$	$0.00000e+00$	$0.00000e+00$

Table 4.2: A comparison of absolute errors between DTM, HFM, IHPM and present methods ($N = 64$) of Example 4.2.

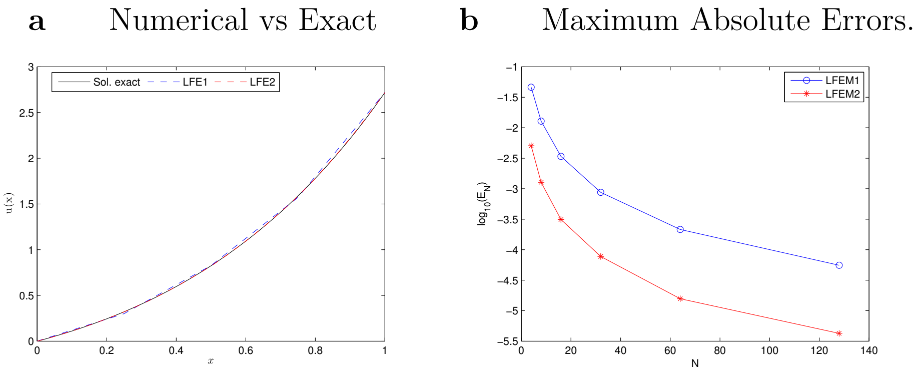

Figure 4.3: (**a**) Exact and numerical solution of Example 4.2 with $N = 4$. (**b**) Maximum absolute errors E_N of Example 4.2 for different values of N.

Example 4.3. *Consider the integro-differential*

$$\begin{cases} -u'(x) + \int_0^1 (x^2+t)u(t)dt = f(x), \ 0 < x < 1, \\ u(0) = 1, \ u(1) = \ln(2), \end{cases}$$

where $f(x)$ is chosen so that the exact solution is $u(x) = \ln(x+1)$. Table 4.3 we show the maximum absolute errors obtained by using present methods LFE1 and LFE2 for different values of N. The absolute errors are presented in Fig.4.4 for $N = 8$.

n	LFE1	LFE2
4	$4.36448e-03$	$1.78173e-03$
8	$1.24007e-03$	$4.72214e-04$
16	$3.32892e-04$	$1.21201e-04$
32	$8.63913e-05$	$3.06506e-05$
64	$2.19744e-05$	$7.71055e-06$
128	$5.50337e-06$	$1.93355e-06$
256	$1.37258e-06$	$4.84120e-07$

Table 4.3: Maximum errors E_N for different values of N for Example 4.3.

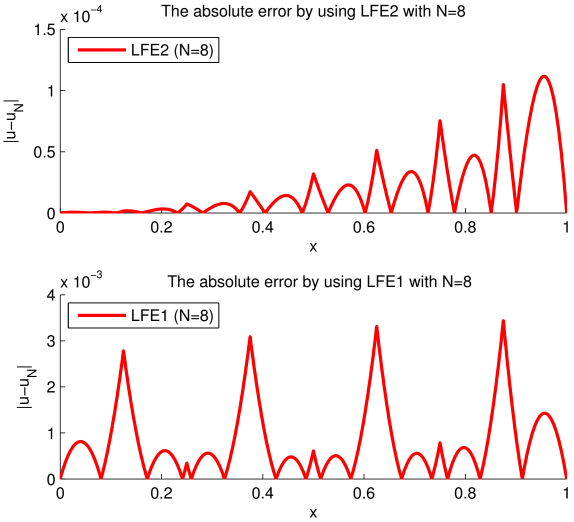

Figure 4.4: Absolute error of Example 4.3 for $N = 8$.

Conclusion

This part concludes the thesis by giving an assessment of the work carried out and the possible perspectives. The main goal of this work was to construct numerical methods for some integral, integro-differential and differential equations. We applied numerical methods such as: the finite deference method, the finite element method and Gauss Legendre quadrature. These methods consist of looking for solutions in the form of a linear combination of the elements of the base.

A considerable contribution has been made in this work for the numerical solution of integro-differential equations, using the finite element method. Considering the different tables given, we can affirm the effectiveness of this approach. This work can be extended to other types of integral and integro-differential equations.

Bibliography

[1] Delves, L.M. and Mohamed, J.L, *Computational methods for integral equations*, Cambridge University Press, 1985.

[2] Kress, R., *Linear Integral Equations*, Springer-Verlag, New York, 1999.

[3] Kythe, P.K. and Puri, P, *Computational methods for linear integral equations*, Birkhauser, Boston, 2002.

[4] Wazwaz, A.M, *Linear and Nonlinear Integral Equations Methods and Applications*, Higher edication press Bijing and Springer-Verlag Berlin Heidelderg, 2011.

[5] Sergeev, V. *On stability of viscoelastic plate equilibrium*, Automat. Remote Control 68 (2007) 1544-1550.

[6] Makroglou, A. *Numerical solution of some second order integrodifferential equations arising in ruin theory,*,Third Conference in Actuarial Science and Finance Held in Samos, Greece, pp. 2-5, September 2004.

[7] Boland, J. *The analytic solution of the differential equations describing heat ow in houses,*,Build. Environ. 37 (2002) 1027-1035.

[8] Wolkowicz, G., Xia, H. and Ruan, S. *Competition in the chemostat : a distributed delay model and its global asymptotic behavior*,SIAM J. Appl. Math. 57 (1997) 1281-1310.

[9] Fung, B.Y.C. *Mechanical Properties of Living Tissues*,Springer Verlag, New York,1993.

[10] Tomasiello, S. *A note on three numerical procedures to solve Volterra integro-differential equations in structural analysis*,Comput. Math. Appl. 62 (2011) 3183-3193.

[11] Thieme, H.R. *A model for the spatial spread of an epidemic*J. Math. Biol. 4 (1977) 337-351.

[12] Volterra. V. *Theory of functionals and of integral and integro-differential equations,*Dover Publ. Inc. 1959.

[13] El-Gendi, S.E. *Chebyshev solution of differential, integral and integro-differential equations,*Comput. J. 12 (1969) 282-287.

[14] Akyuz Dascioglu, A. *A Chebyshev polynomials approach for linear Fredholm-Volterra integro-differential equations in the most general form,*Appl. Math. Comput. 181 (2006) 103-112.

[15] Akyuz Dascioglu, A. and Sezer, M. *Chebyshev polynomial solutions of systems of higher-order linear Fredholm-Volterra integro-differential equations*J. Franklin Inst. 342 (2005) 688-701.

[16] Aminikhah, H., Hosseini, S. and Alavi, J. *Approximare analytical solution for high-order integro-differential equation by Chebyshev Wavelets*Inf. Sci. Lett. 4 (2015) 31-39.

[17] Barzkar, A., Oshagh, M., Assari, P. and Mehrpouya, M. *Numerical solution of the nonlinear Fredholm integral equation and the Fredholm integro-differential equation of second kind using Chebyshev Wavelets,*World Appl. Sci. J. 18 (2012) 1774-1782.

[18] Bhrawy, A.H., Tohidi, E. and Soleymani, F. *A new Bernoulli matrix method for solving high-order linear and nonlinear fredholm integro-differential equations with piecewise intervals,*Appl. Math. Comput. 219 (2012) 482-497.

[19] Behiry, S.H. *Solution of nonlinear Fredholm integro-differential equations using a hybrid of block pulse functions and normalized Bernstein polynomials,*Comput. Appl. Math. 260 (2014) 258-265.

[20] Maleknejad, K. and Mahmoudi, Y. *Taylor polynomial solution of high-order nonlinear Volterra-Fredholm integro-differential equations*Appl. Math. Comput. 145 (2003) 641-653.

[21] Maleknejad, K., Basirat, B. and Hashemizadeh, E *Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations.*Comput. Math. Appl. 61 (2011) 2821-2828.

[22] Venkatesh, S.G., Ayyaswamy, S.K. and Raja Balachandar, S. *Legendre approximation solution for a class of higher-order Volterra integro-differential equations,*Ain Shams Eng. J. 3 (2012) 417-422.

[23] Yalcinbas, S., Sezer, M. and Sorkun, H.H. *Legendre polynomial solutions of high-order linear Fredholm integro-differential equations,*Appl. Math. Comput. 210 (2009) 334-349.

[24] Chandel, R.S., Singh, A. and Chouhan, D. *Solution of higher order Volterra integro-differential equations by Legendre Wavelets,*I. J. App. Math. 28 (2015) 377-390.

[25] Fathy, M., El-Gamel, M. and El-Azab, M.S. *Legendre-Galerkin method for the linear Fredholm integro-differential equations,*Appl. Math. and Comput. 243 (2014) 789-800.

[26] Nemati, S. *Numerical solution of Volterra-Fredholm integral equations using Legendre collocation method,*J. Comput. Appl. Math. 278 (2015) 29-36.

[27] Ortiz, E.L. and Samara, H. *An operational approach to the Tau method for the numerical solution of non-linear differential equations*Computing. 27 (1981) 15-25.

[28] Rashed, M.T. *Lagrange interpolation to compute the numerical solutions of differential, integral and integro-differential equations,*Appl. Math. Comput. 151 (2004) 869-878.

[29] Shahsavaran, A. and Shahsavaran, A. *Application of Lagrange interpolation for nonlinear integro-differential equations,*App. Math. Sci. 6 (2012) 887-892.

[30] Franke, R. *Smooth interpolation of scattered data by local thin plate splines,*Comput. Math. Appl. 8 (1982) 273-281.

[31] Yuzbasi, S., Sahin, N. and Sezer, M. *Numerical solutions of systems of linear Fredholm integro-differential equations with Bessel polynomials bases,*Comput. Math. Appl. 61(2011) 3079-3096.

[32] Zhao, J. *Compact finite difference methods for high order integro-differential Appl. Math.* Comput. 221 (2013) 66-78.

[33] JL Lions and E Magenes., "Problemes aux limites non homogenes et applications volume1,dunod.", Mathematical work and research,1968.

[34] Jacques-Louis Lions and Walter A Strauss., "Some non-linear evolution equations.",Bulletin of the Mathmatical society of France, 93:43 96, 1965.

[35] Brezis, Haim and Brézis, Haim, "Functional analysis, Sobolev spaces and partial differential equations.",volume 2, Springer, 2011

[36] T, kato., "Perturbation Theory for Linear Operators",University of California Berkeley, CA 94720-3840 USA.

[37] R. P. Agarwal, M. Meehan, and D. O'regan, *Fixed point theory and applications,* volume 141. Cambridge university press, 2001.

[38] J. Appell, E. De Pascale, and P. Zabrejko., *On the application of the newtonkantorovich method to nonlinear integral equations of uryson type.*,Functional Analysis and Optimization, 12(3-4):271–283, 1991.

[39] C. Bardaro, J. Musielak, and G. Vinti., *Nonlinear integral operators and applications.*, de Gruyter, 2008.

[40] A. RAHMOUNE., *Équations intégrales linéaires et non linéaires.*, Analyse et techniques de résolution, 2018.

[41] S. M. ZEMYAN., *The classical theory of integral equations: a concise treatment.*, Springer Science and Business Media, 2012.

[42] E. ZEIDLER AND P. R. WADSACK, *Nonlinear Functional Analysis and Its Applications: Fixed-point Theorems/Transl*, by Peter R. Wadsack. Springer-Verlag, 1993.

[43] A.M WAZWAZ, *Nonlinear A First course in integral equations*,Saint Xavier University, USA.

[44] A. ALPINAH AND M. DEHGHAN, *Numerical solution of the nonlinear Fredholm integral equations by positive definite functions*, Appl. Math. Comput, (2007) 1754–1761.

[45] A. M. WAZWAZ, *Necessary conditions for the appearance of noise terms in decomposition solution series*, Applied Mathematics and Computation, 81, (1997), 265–274.

[46] BRUNNER, H., "COLLOCATION METHODS FOR VOLTERRA INTEGRAL AND RELATED FUNCTIONAL EQUATIONS",Cambridge University Press, 2004.

[47] A. M. WAZWAZ, *A reliable modification of the Adomian decomposition method*, Applied Mathematics and Computation, 102 (1999), 77–86.

[48] J. Biazar and M. G. Porshokouhi, *Application of variational iteration method for linear and nonlinear integro-differential-difference equations*, International Mathematical Forum, **5(67)**, pp. 3335–3341, (2010).

[49] S. M. Hosseini and S. Shahmorad, *Numerical piecewise approximate solution of Fredholm integro-differential equations by the Tau method*, Applied Mathematical Modelling, **29**, 1005–1021, (2005).

[50] H. Danfu and S. Xufeng, *Numerical solution of integro-differential equations by using CAS wavelet operational matrix of integration*, Applied Mathematics and Computation, **194(2)**, pp. 460–466, (2007).

[51] P. Darania and A. Ebadian, *A method for the numerical solution of the integro-differential equations*, Applied Mathematics and Computation, **188 (1)**, pp. 657–668, (2007).

[52] M. T. Kajani, M. Ghasemi and E. Babolian, *Comparison between the homotopy perturbation method and the sine-cosine wavelet method for solving linear integro-differential equations*, Computers and Mathematics with Applications, **54**, 1162–1168, (2007).

[53] Z. P. Atabakan, A. K. Nasab, and A. Kilikman, *On Solution of Fredholm Integro-differential Equations Using Composite Chebyshev Finite Difference Method*, Abstract and Applied Analysis, Abstr Appl Anal, 1–12, (2013).

[54] R. Jafri, R.Ezzati and K. Maleknejad, *Numerical Solution of Fredholm Integro-differential Equations By Using Hybrid Function Operational Matrix of Differentiation*, International Journal of Industrial Mathematics, **9**, 349–358, (2017).

[55] F. Ghomanjani, *Numerical solution for nonlinear* $(V.I.D.E)s$. Palest. J. Math. **9**, 164–169, (2020).

[56] P. Linz, *Convergence of a Discretization Method for Integro-differential Equations*, Numer. Math., Academic Press, (1975), 103–107.

[57] E. Yusufoglu (Agadjanov), *Improved homotopy perturbation method for solving Fredholm type integro-differential equations*, Chaos, Solitons and Fractals, **41(1)**, 28–37, 2009.

[58] K. P. Mredula and D. C. Vakaskar. *Collocation methods general approach for ordinary differential equations.* Palest. J. Math. **7(2)**, 650-662, (2018).

[59] S. Benyoucef and A. Rahmoune, *Efficient spectral-collocation methods for a class of linear Fredholm integro-differential equations on the half-line*, in J. Comput. Appl. Math., Academic Press, **377** 112–894, (2020).

[60] K. Atkinson and A. Bogomolny, *The Discrete Galerkin Method for Integral Equations*, in Mathematics of Computation, Academic Press, 595–616, (1987).

[61] K.E. Atkinson and W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, 3$^{\text{rd}}$ edition, Springer-Verlag, New York, 2009.

[62] K. E. Atkinson and F. A. Potra, *On the Discrete Galerkin Method for Fredholm Integral Equations of the Second Kind*, in IMA Journal of Numerical Analysis, Academic Press, 385–403, (1989).

[63] M. G. Larson and F. Bengzon, *The finite element method: Theory, Implementation, and applications*, Springer-Verlag Berlin Heidelberg 2013.

ملخص

يهدف هذا البحث إلى تقديم دراسة نظرية وعددية حول المعادلات التكاملية-التفاضلية الخطية. استخدمنا مبرهنة لاكس-ميلغرام لإثبات وجود وحيدة الحل لهذه المعادلات. بالإضافة إلى ذلك، قمنا بتطبيقٍ طريقة العناصر المحددة لحل بعض المعادلات التكاملية-التفاضلية الخطية تقريبياً. أخيراً، تم تقديم العديد من الأمثلة العددية لإظهار فعالية منهجياتنا.

كلمات مفتاحية : المعادلات التكاملية-التفاضلية، مبرهنة لاكس-ميلغرام، طريقة العناصر المحددة، لاغرانج، الاستيفاء.

Abstract

The main objective of this thesis is to propose a theoretical and numerical study on linear integro-differential equations. we used Lax Milgram theorems to provide existence and uniqueness results for linear integro-differentials. In addition, we applied the finite element method for the approximate solution of some linear integro-differential equations. Finally, several numerical examples are given to show the effectiveness of our approaches.

Keywords: Integro-differential equations, Lax-Milgram theorem, Finite element method, Lagrange, Interpolation.

Résumé

L'objectif principal de cette thèse est de proposer une étude théorique et numérique sur les équations intégro-differentielles linéaires. nous avons utilisé le théorèmes de Lax Milgram pour fournir des résultats d'existence et l'unicité pour les intégro-differentielles linéaires. En outre, nous avons appliqué la méthode des éléments finis pour la résolution approchée de quelques équations intégro-differentielles linéaires, Enfin, plusieurs exemples numériques sont donnés pour montrer l'efficacité de notre approches.

Mots-clés : Equations intégro-differentielles, Théorèms de Lax-Milgram, Méthode des éléments finis, Lagrange, Interpolation.