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THEME

Stabilité et convergence des méthodes spectrales
Application aux équations intégrales

By: Radjai abir

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Dedication

To my parents ***Rabah*** *and* ***Naima****, my husband* ***Walid***
my children ***Rafif-Iline*** *and* ***Mohamed-Amine***
. *my grand-mother* ***Zohra*** *and aunty* ***Hassina***
my brothers, my sisters and all my family
I dedicate this thesis.

Acknowledgment

I thank first and foremost Allah for giving me the courage, the patience, and the will to achieve this thesis. I wish to express my dearest gratitude to my supervisor Prof. Azedine RAHMOUNE for his continuous support, ideas, and patience in guiding me during the preparation of my thesis.

Secondly, I would like to thank members of my thesis committee: Pr.MAROUANI Abdelbaki , Pr.BENSERIDI Hamid and M.C.A. BOUKAROURA Ilyas from the Mathematics Department, University of Ferhat Abbas 1. M.C.A. ZEGHDANE Rebiha from the Mathematics Department, University Mohamed El Bachir El Ibrahimi of Bordj Bou Arréridj, and M.C.A. GAGUI Bachir from the Mathematics Department, University of Mohamed Boudiaf of M'sila.

Finally, I would also like to thank my comrades and my friends who supported me morally and to anyone who participated directly or indirectly in the realization of this work. Beyond all of them, a special thought goes to my family who has supported me during all these years to them, I dedicate this work.

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I : $[-1,1)$ or $(-1,1)$.

$\mathbb{R}$: $(-\infty,+\infty)$.

$J_n^{\alpha,\beta}$: Jacobi polynomial of degree n with $\alpha,\beta > -1$.

G_n^{α} : Gegenbauer polynomial of degree n with $\alpha > -1$.

L_n : Legendre polynomial of degree n .

K, L : Integral operator.

$k(s,t)$: Kernel of integral operator.

R_N : Residual functions.

$\langle .,. \rangle$: Scalar product

$H_w^m(\Omega)$: Sobolev space.

$I_N^{\alpha,\beta}$: Lagrange interpolation operator associated with the Gauss-Jacobi points.

$\{L_j^{\alpha,\beta}\}_{j=0}^N$: Lagrange interpolation basic function associated with Gauss-Jacobi points.

$L_w^2(\Omega)$: Space of measurable functions $u : \Omega \to \mathbb{R}$ such that $\|u\|_{w^{\alpha,\beta}} < \infty$.

$L^\infty(\Omega)$: Space of measurable functions $u : \Omega \to \mathbb{R}$ such that are bounded outside a set of measure zero.

$\|u\|_p$: Norm in $L^p(\Omega)$, $\|u\|_{\ p} = \left(\int_\Omega |u(s)|^P ds\right)^{\frac{1}{p}}$.

$\|u\|_{w^{\alpha,\beta}}$: Norm in $L_w^2(\Omega)$, $\|u\|_{w^{\alpha,\beta}} = \left(\int_\Omega |u(s)|^2 w^{\alpha,\beta}(s) ds\right)^{\frac{1}{2}}$.

$\|u\|_{L_\infty}$: Norm in $L^\infty(\Omega)$ defined by $ess \sup\limits_{s \in \Omega} |u(s)|$.

$P, \mathcal{P}_N$: Projection operators.

$\{\sigma_{N,j}^{\alpha,\beta}\}_{j=0}^N$: Points of the Gauss-Jacobi quadrature formula.

$\mathbb{R}[x]$: Set of all polynomials (of all degrees) in one variable x.

$\mu(\Omega)$: Lebesgue measure of Ω.

$\mathcal{O}(.)$: Order of convergence.

$\mathsf{B}(X,Y)$: bounded linear mapping from X into Y

$I.Es$: Integral equations.

LCM : Legendre collocation method.

$RLCM$: Rational Legendre collocation method.

Introduction

Many problems arising in applied mathematics or mathematical physics can be formulated in two distinct but connected ways, namely as differential equations or integral equations. The importance of these equations in all branches of science and engineering prompted several researchers to study certain integral equations analytically and numerically. A small number of these integral equations can be solved analytically and hence numerical methods are routinely used to solve such problems. Therefore, numerous methods have been developed and applied accordingly such as the spectral methods. The main idea of spectral method is to write the solution of a problem by a sum of any basis set of orthogonal functions. For instance, trigonometric polynomials for periodic problems, Legendre polynomials, and Chebyshev polynomials for non - periodic problems, Laguerre polynomials for problems on the half line, and Hermite polynomials for problems on the whole line. In order to solve linear and non-linear integral equations in bounded and unbounded domains, spectral approaches are becoming increasingly popular as higher order techniques. Note that analytically solving nonlinear integral equations on unbounded intervals is not a trivial task in general, thus numerical methods are required. Therefore, some efficient numerical algorithms have been developed by a few authors to solve such problems. For example, the Nyström method for convolution and non-convolution kernels was explored in [1], the finite-section approximation method was presented in [2], projection and multi-projection methods were discussed in [3, 4], Galerkin and multi-Galerkin methods based on Laguerre polynomials were investigated in [5], the Sinc-Nyström method based on single and double exponential transformations was proposed in [6], modified Legendre rational and exponential collocation methods were proposed in [7], and superconvergence results for non-linear Hammerstein integral equations on unbounded domain was discused in [8]. However, to the best of our knowledge, no numerical methods have yet been applied to quadratic integral equations on unbounded intervals. This is what we will address in the last chapter of our work.

The topic of spectral methods is very large, and various methods have been proposed and are actively used. The following description aims at giving the fundamental ideas, focusing on the collocation and Galerkin methods and analysing their basic theoretical properties (stability, accuracy, convergence). The concepts of convergence and stability are given very general definitions that apply to any numerical computation. The main feature of these definitions is the treatment of truncation error and rounding error as special cases of the unifying concept of a perturbation. Thus stability, which has to do with rounding errors. The fascinating merit of spectral methods is the high accuracy, the so-called convergence of "infinite order". It means that the error with increasing resolution (the number of grid points N) is decreasing exponentially. Indeed, increasing the number, N, of discretization points, a spectral method presents a converge rate of order $\mathcal{O}(N^{-m})$ for every natural number m, provided that the solution that we want to approximate is smooth (infinitely differentiable).

In this work, we study the convergence and stability of spectral methods. This thesis consists of an introduction, three chapters, and references. The first chapter is an introduction to the terminology and classification of integral equations, which aims to familiarize the reader of this thesis with the concept of integral equations. Thus, we provide a general introduction to spectral methods, in particular the aspect of a spectral method. We discuss the advantages and the general principle of these methods. Also, we try to expose both the intrinsic formulation of collocation and Galerking methods and their practical application using examples.

The second chapter establishes the theoretical framework of our study. We begin by recalling briefly a few basic concepts from general theory of stability (condition number, stability of linear system, stability of numerical algorithm) that we need in studying the stability of spectral methods wich presented in the second section of this chapter. Finally, we close by some numerical examples, they are presented to illustrate the convergence and stability of spectral method which we had been spoke about it.

The subject of the last chapter is the application of spectral method to solve non linear Fredholm integral equations in the first section and the rational Legendre collocation method to solve the quadratic Urysohn integral equations on half-line in the second section, in which we will discuss our convergence analysis, and illustrate the efficiency and the stability of the present methods by numerical examples.

Finally, we summarise the work that has been carried out in this thesis and consider some possible further work.

Chapter 1

Generalities on the theory of integral equations

Integral equations are among the most important branches of mathematics. They are used in various fields of science and engineering to model a wide range of phenomena. The name integral equation for any equation involving the unknown function under the integral sign. Abel is the initiator of the mention of integral equations, in 1823 he proposed a generalization of the Tautochrone problem whose solution referred to the solution of an integral equation, recently has been dubbed an integral equation of the first kind, and in 1837 Liouville proved that determining a particular solution of a linear differential equation of the second order might be obtained by solving an integral equation.

The terme integral equation was introduced by Du Bois-Reymond in 1888. Afterward, in 1896, Vito Volterra built up a theory of integral equations, viewing their solutions as a problem of finding the inverses of certain integral operators, without forgetting to mention the famous paper of Fredholm which he published in 1903 and it presented the fundamentals of the Fredholm integral equation theory. Poincaré, Fréchet, Hilbert, Schmidt, Hardy, and Riesz have also participated in the development of this area of research (For more information see [9–14]).

We recall in this chapter the forme and classifications of integral and integro-differential equations, applications of these equations and some solution methods. Also we discuss the general theory of spectral method and presenting the properties of Collocation, and Galerkin methods for integral equations

1.1 Introductory concept of integral equations

1.1.1 Classification of integral equations

An integral equation is defined as an equation in which the unknown function $u(x)$ to be determined appears under the integral sign. It is defined by the form

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt \tag{1.1.1}$$

where $k(x,t)$ is called the kernel of the integral equation (1.1.1), and $\alpha(x)$, $\beta(x)$ are the limits of integration. the kernel $k(x,t)$ and the function $f(x)$ are given functions , and λ is a constant parameter. The two main kinds of integral equations that are most commonly utilized are Fredholm and Volterra integral equations. Obviously, we have to categorize them as linear or nonlinear, as well as homogeneous or nonhomogeneous.

- ***Fredholm integral equations***

  The most standard form of Fredholm linear integral equations is given by the form:

  $$\mu(x)u(x) = f(x) + \lambda \int_{a}^{b} k(x,t)u(t)dt \tag{1.1.2}$$

  where the limits of integration a and b are constants. If the function $\mu(x) = 1$, then the equation (1.1.2) is called Fredholm integral equation of second kind; whereas if $\mu(x) = 0$, then the previous equation is called Fredholm integral equation of the first kind.

- ***Volterra integral equations***

  The most standard form of Volterra linear integral equations is given by the form:

  $$\mu(x)u(x) = f(x) + \lambda \int_{a}^{x} k(x,t)u(t)dt \tag{1.1.3}$$

  where the limits of integration are function of x. If the function $\mu(x) = 1$, then the equation (1.1.3) is called Volterra integral equation of second kind; whereas if $\mu(x) = 0$, then the previous equation is called Volterra integral equation of the first kind.

- ***Singular integral equations***

  When one or both limits of integration become infinite or when the kernel becomes infinite at a certain point in the interval, the integral equation is called singular. For examples, the integral equations

$$u(x) = f(x) + \lambda \int_0^{\infty} k(x,t)u(t)dt \tag{1.1.4}$$

and

$$u(x) = f(x) + \lambda \int_0^{x} \frac{k(x,t)}{x-t}u(t)dt \tag{1.1.5}$$

are classified as singular integral equations.

- ***Integro-differential equations***

  In the early 1900, VitoVolterra studied a new types of equations have been developed and termed as the integro-differential equations. In this type of equations, the unknown function $u(x)$ appears as the combination of the ordinary derivative and under the integral sign. Several phenomena in physics and biology [14, 15] and [16] give rise to this type of integro-differential equations.

  The most standard form of integro - differential equations is given by the form:

$$u^{(n)}(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)u(t)dt \ , u^{(j)}(0) = b_j \ , \ 0 \leq j \leq n-1; \tag{1.1.6}$$

  where $u^{(n)}(x) = \dfrac{d^n u}{dx^n}$ indicates the n^{th} derivative of $u(x)$. Because the resulted equation in (1.1.6) combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0)$, $u^{'}(0), ...u^{(n-1)}(0)$ for the determination of the particular solution $u(x)$.

◆ Remark

- If the unknown function $u(x)$ appearing under the integral sign is given in the functional form $\psi(u(x))$ e.g $sin(u(x)); u^2(x)$ etc. then the Volterra and Fredholm integral equations are classified as nonlinear integral equations.
- If we set $f(x) = 0$, in Volterra or Fredholm integral equations, then the resulting equation is called a homogeneous integral equation, otherwise it is called nonhomogeneous integral equation.

◆ One of the non-linear integral equations we have previously known is quadratic integral equations in which we distinguish two types, Urysohn and Hammerstien integral equations defined respectively as follows

$$u(x) = a(x) + f(x,u(x)) \int_a^b k(x,t,u(t))dt, \quad x \in [a,b], \tag{1.1.7}$$

$$u(x) = a(x) + f(x,u(x)) \int_a^b k(x,t)\psi(t,u(t))dt, \quad x \in [a,b]. \tag{1.1.8}$$

1.1.2 Applications

The subject of integral equations has held an eminent place in the attention of mathematicians, such equations arise naturally in applications in diverse areas of applied mathematics and physical sciences, engineering, biology, and many other fields, they also provide an effective technique for solving a wide range of practical problems.

For example, there are many problems that can be modeled with integral equations like the radiative transfer [17, 18], acoustic resonance scattering [19, 20], population dynamics [21, 22], electromagnetic and elastic waves [23], fluid dynamics [24], aerodynamics [25].

Integral or integro-differential equations are used to characterize a wide range of further applications in science and engineering. The Volterra's population growth model [26], propagation of stocked fish in a new lake [27]. The heat transfer and the heat radiation are among may areas that are described by integral equations [28]. Also the quadratic integral equations, which holds great importance due to their numerous significant applications in various fields such as radiative transfer [29, 30] kinetic theory of gases [31], the theory of neutron transport [32], and traffic theory. (see, e.g [33–36] and reference therein).

1.1.3 Theory of integral equations and some solution methods

The importance of integral equations in all branches of science and engineering prompted several researchers to study the existence and uniquness of solution of these equations. Fredholm alternative theorem and the geometric series of Neumann applied to prove the existence and uniqueness of the solution Fredholm integral equations (1.1.2) (See [16, 37]). To prove the existence and uniqueness of the solution of the Volterra equation (1.1.3) we use the contraction principle or the fix point theorem of Banach [38–40]. For the non linear integral equations Fredholm or Volterra, Some fixed point theorems have been used and the most famous of them is the theorem of Banach and Schauder fixed point theorems [41–45] . The technique of measures of noncompactness with the classical Schauder fixed point principle and the Darbo fixed point theorem are using to prove that the equations (1.1.7) has solution on the real half axis and having limits at infinity [46–53].

A variety of analytical and numerical methods are used in the literature to solve all kinds of equations we talked about earlier for example degenerate kernel method, the Adomian decomposition method we refer [12, 54, 55], successive approximation method (Picard iteration) [55–57] Nyström method and numerical quadrature methods [16, 57].

One of the things to keep in mind about (IE) is that most of these equations are not explicitly soluble analytically so mathematicians have resorted to solving them by numerical methods. With the advantage of numeric computing machines, especially computers, these methods have now become an essential tool for the investigation of the various fundamental scientific problems that are difficult, namely impossible to solve in the past. One of these methods is the spectral method that we will show in the next section.

1.2 Basic spectral methods for integral equations

The spectral methods possess high accuracy, and so play an important role in numerical solutions of differential and integral equations, the main idea is to write the spectral solution of the problem as a finite sum of various orthogonal systems of infinitely differentiable global functions and then to choose the coefficients in the sum to satisfy such problem as well as possible. Different functions lead to different spectral approximations. For instance in bounded domains, trigonometric polynomials for periodic problems and Jacobi polynomials for non-periodic problems. While, Laguerre, Hermite, and Mapped Jacobi functions for problems in unbounded domains.(for more details see [58–61]). The great importance of these methods makes us ask the question, what are the characteristics of the spectral methods that make them the best methods to solve integral equations?

1.2.1 Why spectral methods?

There are major benefits of spectral methods compared to other approaches. Mainly we can say that:

- Firstly, spectral discretizations of integral equations, based for example on Fourier bases or orthogonal polynomials of (Chebychev, Legendre, Hermite, . .) provided very low approximations errors. In many cases, these approaches may converge exponentially for a spectral development of order N, the difference between the analytical solution (exact) and the numerical solution tends towards zeros rapidly with the increase of the spectral development order.

- Since the numerical accuracy of the spectral methods is so high, the number of grid points needed to reach the desired precision is very small, therefore a spectral method requires less memory than other methods. This minimization is crucial, especially for the execution of algorithms.
- There is a high performance of implementations of algorithms necessary for basic transformation for most spectral methods, and the developer of a spectral method does not need to apply these codes.

1.2.2 Basic principle

Spectral methods are used extensively for the discretization of integral equations. The main idea is to approximate the solution $u(x)$ of integral equation on some interval Ω not necessarily bounded by a sum of some trial(or basis) functions (for example, Fourier bases, orthogonal polynomials . . .), then to choose the coefficients of this combination. In this method we have simplified the kernel in order to obtain a finite-dimensional problem. Now, we shall replace the (infinite-dimensional) space $\mathcal{X}$ $(\mathcal{X} = C(\Omega)\ or\ \mathcal{X} = L^2(\Omega))$ by a finite-dimensional subspace $\mathcal{X}_N$. These subspaces will be the ranges of projections, from which this method derives its name.
$-$ Our goal is to find an approximate solution the equation

$$u - Ku = f, \text{ for all } \quad f \in \mathcal{X}, \tag{1.2.1}$$

where K is assumed to be compact operator on Banach space $\mathcal{X}$ to $\mathcal{X}$. To do this, we choose a sequence of finite dimensional subspaces $\mathcal{X}_N \subset \mathcal{X}$ with

$$X_N = span\{\phi_0 .., \phi_N\} \Longrightarrow dim_{\mathcal{X}_N} = N + 1$$

and sequence of projectors $\mathcal{P}_N : \mathcal{X} \longrightarrow \mathcal{X}_N$, then the projection method assumes the form

$$u_N - \mathcal{P}_N K u_N = \mathcal{P}_N f, \tag{1.2.2}$$

whith

$$u_N(x) = \sum_{j=0}^{N} c_j \phi_j(x) \in \mathcal{X}_N,\ N \in \mathbb{N}^*$$

where the coefficients $(c_1, c_2, ..., c_N)$ are determined by forcing the equation to be almost exact in some sense (we see this later).

1.2.3 Choice of basic functions

The choice of basic functions is one of the features which distinguish spectral methods from other methods. The most commonly used trial/test functions are trigonometric functions or orthogonal polynomials.

It is obvious that we will like that our base is characterized by a number of properties, an easy calculation, a fast convergence, and that it is complete. This means that any solution can be represented with arbitrarily great precision by taking the N truncation sufficiently large.

The following table summarizes the Choice of basic functions:

Periodic Fourier $\theta \in [0, 2\pi]$	Halfe line Laguerre $x \in [0, \infty[$
No Periodic Chebychev or Legendre $x \in [-1, 1]$	Real line Hermite $x \in]-\infty, \infty[$

1.2.4 Polynomial basic functions and Quadratures

A basis set of linearly independent functions is used to expand the solution in the spectral method of solving partial differential and integral equations. One must consider the interval of interest and the expected behavior of the solutions while selecting a basis set for a given problem.

The most usable spectral methods are the orthogonal polynomials. A sequence of orthogonal polynomials is an infinite sequence of polynomials $P_0(x), P_1(x), P_2(x)$. with real coefficients, in which each $P_n(x)$ is of degree n, and tells that the polynomials of the sequence are orthogonal two to two for a given scalar product. The scalar product of the function is the integral product of these functions, over a limited interval

$$\langle f,g\rangle = \int_a^b f(x)g(x)dx$$

more generally, we can introduce "a weight function" $w(x)$ in the integral (on an integration interval $]a,b[$, we must be at finite and strictly positive values and the integral product of the weight function by a polynomial must be finished, the bounds a,b can be infinite).

$$\langle f,g\rangle = \int_a^b f(x)g(x)w(x)dx$$

with this scalar product definition, two functions are orthogonal to each other if their scalar product is equal to zero. Here are some examples of basic orthogonal functions. (for more details, see, e.g.,[62–68])

Jacobi Polynomials

Jacobi polynomials (occasionally called hypergeometric polynomials) $J_n^{\alpha,\beta}(x)$ are a class of classical orthogonal polynomials. They are orthogonal with respect to the weight $(1-x)^{\alpha}(1+x)^{\beta}$ on the interval $[-1,1]$. The Gegenbauer polynomials, and thus also the Legendre, Zernike and Chebyshev polynomials, are special cases of the Jacobi polynomials

- They are defined by the formulas

$$J_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}]$$
$$= 2^{-n}\Sigma_{m=0}^{n} C_{n+\alpha}^{m} C_{n+\beta}^{n-m}(x-1)^{n-m}(x+1)^m$$

where the C_n^m are binomial coefficients.

- They are generated by the three-term recurrence relation

$$J_{n+1}^{\alpha,\beta}(x) = (a_n^{\alpha,\beta}x - b_n^{\alpha,\beta})J_n^{\alpha,\beta}(x) - c_n^{\alpha,\beta}J_{n-1}^{\alpha,\beta}(x)\; n \geq 1,$$

$$J_0^{\alpha,\beta}(x) = 1,\;\; J_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta)$$

Where

$$a_n^{\alpha,\beta} = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}$$

$$b_n^{\alpha,\beta} = \frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}$$

$$c_n^{\alpha,\beta} = \frac{(2+\alpha)(n+\beta+1)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}$$

- The set of Jakobi polynomials forms an orthogonal system, namely,

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}J_n^{\alpha,\beta}(x)J_m^{\alpha,\beta}(x)dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\delta_{nm},\ \alpha,\beta>-1.$$

- Rodrigue's formula

$$J_n^{\alpha,\beta}(x)J^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\{(1-x)^{\alpha}(1+x)^{\beta}(1-x^2)^n\}$$

- Jakobi-Gauss quadrature formulas :the Gauss-Jacobi rule is defined by

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}f(x)dx \approx \Sigma_{i=0}^{n}w_i f(x_i)$$

where the nodes and weight given by
$-$ For Jacobi-Gauss : $\{x_i\}_{i=0}^{n}$ are the zeros of $J_n^{\alpha,\beta}(x)$ and

$$w_i = \frac{G_n^{\alpha,\beta}(x_i)}{J_n^{\alpha,\beta}(x_i)\partial J_n^{\alpha,\beta}(x_i)}.$$

$-$ For Jacobi-Gauss-Radau : $x_0=-1, \{x_i\}_{i=1}^{n}$are the zeros of $J_n^{\alpha,\beta+1}(x)$ and

$$w_0 = \frac{2^{\alpha+\beta+1}(\beta+2)\Gamma^2(\beta+1)n!\Gamma(n+\alpha+\beta)}{\Gamma(n+\beta+2)\Gamma(n+\alpha+\beta+2)}$$

$$w_i = \frac{1}{1+x_i}\frac{G_{n-1}^{\alpha,\beta+1}(x_i)}{J_{n-1}^{\alpha,\beta+1}(x_i)\partial J_n^{\alpha,\beta+1}(x_i)}.$$

− For Jacobi-Gauss-Lobatto : $x_0 = -1, x_n = 1, \{x_i\}_{i=1}^{n-1}$are the zeros of $\partial J_n^{\alpha,\beta}(x)$ and

$$w_0 = \frac{2^{\beta+1}(\beta+2)\Gamma^2(\beta+1)\Gamma(n)\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+2)}$$

$$w_n = \frac{2^{\alpha+1}(\beta+2)\Gamma^2(\alpha+1)\Gamma(n)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+2)}$$

$$w_i = \frac{1}{1+x_i^2}\frac{G_{n-2}^{\alpha+1,\beta+1}(x_i)}{J_{n-2}^{\alpha+1,\beta+1}(x_i)\partial J_{n-1}^{\alpha+1,\beta+1}(x_i)}.$$

Where

$$G_n^{\alpha,\beta} = \frac{2^{\alpha+\beta}(2n+\alpha+\beta+2)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(n+1)\Gamma(n+\alpha+\beta+2)}$$

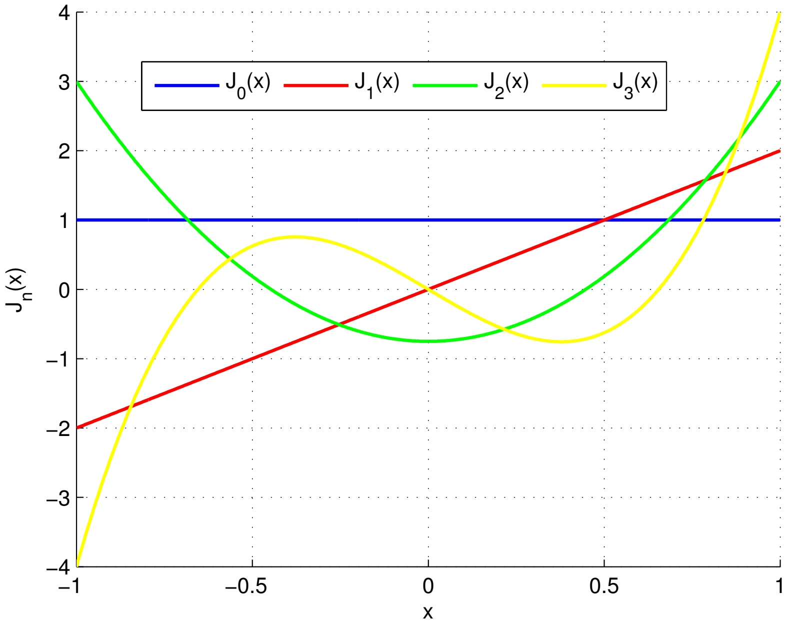


Figure 1.1: Jacobi polynomials $J_n^{1,1}(x)$ with $n = 0, 1, ..., 5$.

Gegenbauer Polynomials

Gegenbauer polynomials or ultraspherical polynomials $C_n^{\lambda}(x)$ $(\lambda > -1/2)$ are orthogonal polynomials on the interval $[-1,1]$ with respect to the weight function $(1-x^2)^{\lambda-1/2}$. They are a subset of the Jacobi polynomials with $\alpha = \beta = \lambda - 1/2$ and have applications to potential theory and harmonic analysis. They reduce to Legendre polynomials for $\lambda = 1/2$ and to Chebyshev polynomials for $\lambda = 0$.

- The polynomials can be defined in terms of their generating function ($Stein\, and\, Weiss$ 1971):

$$\frac{1}{(1-2xt+t^2)^{\lambda}} = \sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n$$

- They can be defined by the recurrence relation ($Suetin$ 2001):

$$C_n^{\lambda}(x) = \frac{1}{n}[2x(n+\lambda-1)C_{n-1}^{\lambda}(x) - (n+2\lambda-2)C_{n-2}^{\lambda}(x)].$$

- The first of these polynomials are

$$C_0^{\lambda}(x) = 1$$

$$C_1^{\lambda}(x) = 2\lambda x$$

$$C_2^{\lambda}(x) = 2\lambda(\lambda+1)x^2 - \lambda$$

$$C_3^{\lambda}(x) = \frac{4\lambda}{3}(\lambda+1)(\lambda+2)x^3 - 2\lambda(\lambda+1)\lambda x$$

- The set of Gegenbaeur polynomials forms an orthogonal system, namely,

$$\int_{-1}^{1} C_n^{\lambda}(x)C_m^{\lambda}(x)(1-x^2)^{\lambda-1/2}dx = \gamma_n^{\lambda}\delta_{n,m}$$

where $\delta_{n,m}$ is the Kronecker function and

$$\gamma_n^{\lambda} = \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)[\Gamma(\lambda)]^2}.$$

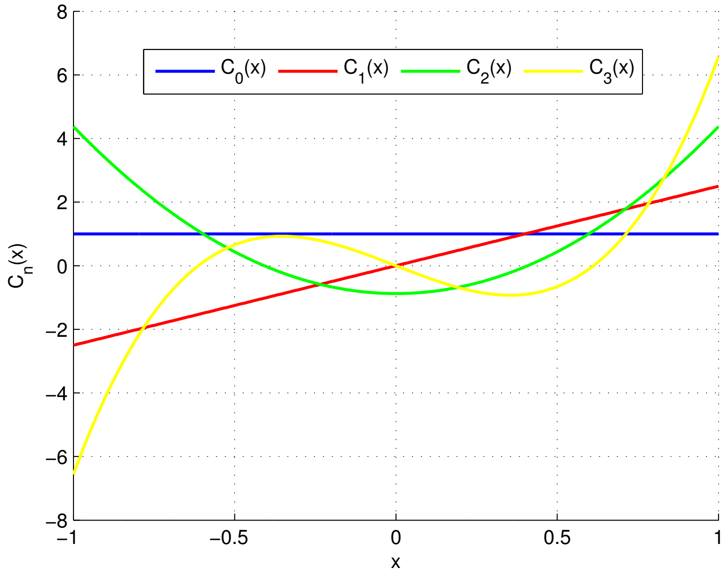


Figure 1.2: Gegenbauer polynomials with $\lambda = 2$ and $n = 0, 1, ..., 5$.

Legendre Polynomials

The well-known Legendre polynomials are orthogonal in the interval $I = [-1, 1]$ with respect to the uniform weight function $\omega(x) = 1$. They can be determined with the help of the following recurrence formula :

$$L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x) \;\; n \geq 1,$$

$$(2n+1)L_n(x) = L'_{n+1}(x) - L'_{n-1}(x) \; n \geq 1.$$

- The first of these polynomials are

$$L_0(x) = 1$$

$$L_1(x) = x$$

$$L_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$L_3(x) = \frac{1}{2}(5x^3 - 3x)$$

- The set of Legendre polynomials forms an orthogonal system, namely,

$$\int_{-1}^{1} L_n(x) L_m(x) dx = \frac{2}{2n+1} \delta_{n,m},$$

where $\delta_{n,m}$,is the Kronecker delta function.

- Sturm-Liouville Problem

$$(1-x^2)L_n''(x) - 2xL_n'(x) + n(n+1)L_n(x) = 0$$

- Rodrigue's formula

$$L_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right],\ n \geq 0.$$

- Legendre-Gauss quadrature formulas :

$$\int_{-1}^{1} f(x)dx \approx \Sigma_{j=0}^{n} w_j f(x_j)$$

$-$ For legendre- Gauss $(LG) : \{x_j\}_{j=0}^{N}$ are the zeros of $L_{N+1}(x)$ and

$$w_j(x) = \frac{2}{(1-x_j^2)[L_{N+1}'(x_j)]^2},\ 0 \leq j \leq N$$

$-$For legendre- Gauss-Radau $(LGR) : \{x_i\}_{j=0}^{N}$ are the zeros of $L_{N+1}(x) + L_N(x)$ and

$$w_0 = \frac{2}{(N+1)^2},\ w_j(x) = \frac{1}{(N+1)^2}\frac{1-x_j}{[L_N(x_j)]^2},\ 1 \leq j \leq N$$

$-$For legendre- Gauss-Lobatto $(LGL) : x_0 = -1, x_N = 1, \{x_i\}_{j=1}^{N-1}$ are the zeros of $L_N'(x)$ and

$$w_j = \frac{2}{N(N+1)}\frac{1}{[L_N(x_j)]^2},\ 0 \leq j \leq N$$

- Rodrigue's formula

$$C_n^{\lambda}(x) = \frac{(-1)^n}{2^n n!}\frac{\Gamma(\lambda+1/2)\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+1/2)}(1-x^2)^{-\lambda+1/2}\frac{d^n}{dx^n}[(1-x^2)^{n+\lambda-1/2}].$$

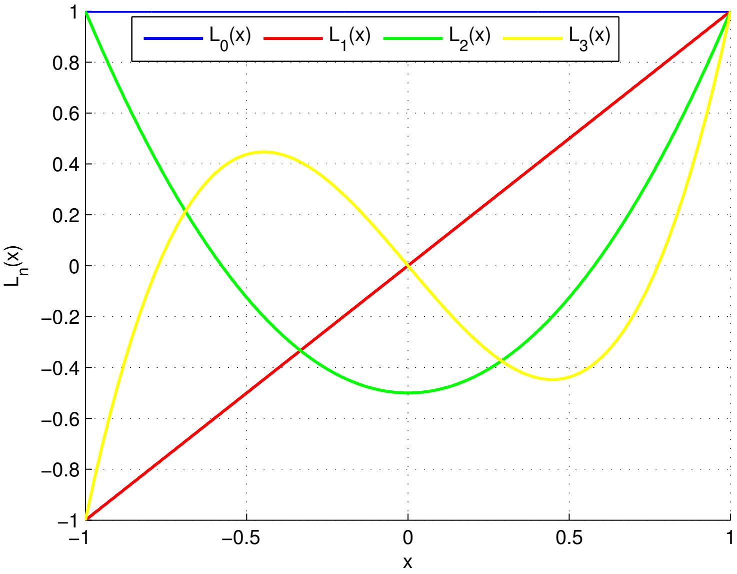


Figure 1.3: Legendre Polynomials.

1.3 Examples of spectral method

Spectral methods not only distinguish by the particular choice of test functions but also by the type of the method. The most often used are collocation and Galerkin methods.

1.3.1 Collocation method

We consider the linear problem

$$u - K(u) = f, \text{ for all } f \in \mathcal{X}, \tag{1.3.1}$$

where K is integral operator and $\mathcal{X}$ is a Banach space. To solve approximately the latter problem we choose a sequence of finite-dimensional subspaces $\mathcal{X}_N \subset \mathcal{X}$ with

$$X_N = span\{\phi_0 .., \phi_N\} \Longrightarrow dim_{\mathcal{X}_N} = N + 1$$

We expand the approximate solution as

$$u_N(x) = \sum_{j=0}^{N} c_j \phi_j(x) \in \mathcal{X}_N, \; N \in \mathbb{N}^* \tag{1.3.2}$$

The collocation method forces the residual to vanish point wisely at a set of preassigned points. More precisely, let $\{x_0, x_1, ..., x_N\}$ be a set of distinct node points in the interval $I = [a, b]$. The collocation method for Eq (1.3.1) amounts to finding $u_N \in \mathcal{X}_N$ such that the residual

$$R_N(x) = \sum_{j=0}^{N} c_j \phi_j(x) - K(\sum_{j=0}^{N} c_j \phi_j)(x) - f(x), \tag{1.3.3}$$

equal to zero at the collocation points, namely,

$$R_N(x_i) = 0 \quad i = 0, 1, \ldots, N. \tag{1.3.4}$$

This leads to determining $\{c_j\}_{j=0}^{N}$ as the solution of the linear system

$$\sum_{j=0}^{N} c_j \{\phi_j(x_i) - K(\phi_j)(x_i)\} = f(x_i), \quad i = 0, 1, \ldots, N. \tag{1.3.5}$$

This linear system has a unique solution if

$$\det[\phi_j(x_i) - K(\phi_j)(x_i)] \neq 0.$$

1.3.2 Galerkin's method

Galerkin's method is similar to the previous one, where $\mathcal{X} = L^2(I)$ be a Hilbert space provided with a scalar product $\langle \cdot, \cdot \rangle$ we consider a sequence of subsequence $\mathcal{X}_N \subset \mathcal{X}$ of finite dimension, let $\{\phi_0 .., \phi_N\}$ an orthonormal basis of $\mathcal{X}_N$, so that the unknown coefficients c_j are computed by requiring the residual $R_N((1.3.3)$ to satisfy

$$\langle R_N, \phi_i \rangle = \langle u_N - K(u_N) - f, \phi_i \rangle = 0, \quad i = 0, \ldots, N. \tag{1.3.6}$$

K is linear operator this yields the linear system

$$\sum_{j=0}^{N} c_j \{\langle \phi_j, \phi_i \rangle - \langle K(\phi_j), \phi_i \rangle\} = \langle f, \phi_i \rangle, \quad i = 0, \ldots, N, \tag{1.3.7}$$

where

$$\langle \phi_j, \phi_i \rangle = \delta_{i,j}, \quad i, j = 0, \ldots, N.$$

Solving this system leads to determine the coefficients $\{c_j\}_{j=0}^{N}$, therefore we find the resulting function $u_N(x)$ which we hope will be very close to the exact solution $u(x)$.

Example 1 *Consider the Fredholm integral equations of the forme*

$$u(x) = 1 - \frac{4}{3}x + \int_{-1}^{1} (xt^2 - 1)u(t)dt \tag{1.3.8}$$

We solve this equation by the Galerkin method 1.3.2 *with*

$$u_2(x) = \sum_{j=0}^{2} c_j L_j(x)$$

where $L_0(x), L_1(x), L_2(x)$ are the three first terms of Legendre polynomials 1.2.4 then

$$u_2(x) = c_0 + c_1 x + c_2 \frac{3x^2 - 1}{2}.$$

The residue is equal to

$$R_2(x) = u_2(x) - 1 + \frac{4}{3}x + \int_{-1}^{1} (xt^2 - 1)(c_0 + c_1 t + c_2 \frac{3t^2 - 1}{2})dt.$$

By calculating the integral one obtains

$$R_2(x) = \frac{4}{3}x + 3c_0 - \frac{2}{3}xc_0 + xc_1 - \frac{4}{15}xc_2 + c_2\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - 1$$

according to the orthogonality of residue R_2 to the system $\left\{1, x, \frac{3x^2-1}{2}\right\}$ we obtain

$$c_0 = \frac{1}{3}, c_1 = -\frac{10}{9}, c_2 = 0$$

so the solution is

$$u_2(x) = \frac{1}{3} - \frac{10}{9}x.$$

Chapter 2

Stability theory

Stability is a fundamental concept in numerical analysis and computational mathematics. It is an ambiguous term because it is defined differently in each fields of numerical mathematics. It refers to the ability of an algorithm or numerical method to provide accurate and reliable results even in the presence of small errors or uncertainties in the input data or in the calculation process. Despite the general concept of stability, we distinguish three types of it. The stability of a physical problem, of a mathematical problem and the stability of a numerical method [69–71].

We will focus in this work on studying the stability of spectral methods. Almost all numerical methods for solving integral equations leads to the solution of systems of equations. For this, in this chapter, we begin by recalling briefly a few basic concepts from general theory of stability (condition number, stability of linear system, stability of numerical algorithm) that we need in studying the numerical stability of spectral methods wich presented in the second section. We close this chapter by some numerical examples, They are presented to illustrate the convergence and stability of spectral method which we had been spoke about it.

2.1 Stability of spectral methods for integral equations

The stability of spectral methods for integral equations depends on several factors:

- ***Properties of the integral equation:*** The nature of the integral equation itself can influence its numerical stability. Before we start solving the equation, we need to know that the problem is a well or ill posed problem. For example, Voltera and Fredholm of the first kind becomes an ill posed problem, thus some kind of regularization method must be used to obtain the stable resolution of the equations.

- ***Choice of basic functions:*** As previously explained, spectral methods often use basic functions such as Chebyshev and Legendre polynomials in bounded domain, Hermite and Laguerre polinomials for unbounded domain. Different types of basis functions have distinct properties that can affect numerical stability and convergence in various ways. Let's explore this with numerical examples using three commonly used types of basis functions Jakobi, Legendre and Chebyshev polynomials for the numerical solution of Fredholm integral equation in the bounded domain $[-1, 1]$.

Example 2 *Consider the following equation*

$$u(x) = 2 - \exp(x) + \int_{-1}^{1} \exp(-t) u(t) dt,$$

where the exact solution is $u(x) = -\exp(x)$.

N	Legendre polynomials	Chebyshev polynomials
5	1.15e-04	2.40e-01
10	8.06e-11	2.40e-01
15	3.55e-15	2.40e-01
20	3.55e-15	2.40e-01

Table 2.1: The errors of Example 2

Example 3 *Consider the following equation*

$$u(x) = x^3 - 4x + (1/15) + \int_{-1}^{1} (x - t) u(t) dt,$$

whose exact solution is $u(x) = x^3 - 2x + 10$.

N	Legendre polynomials	Jakobi polynomials($\alpha = 1/2, \beta = 0$)
5	7.63e-16	2.93e-04
10	3.02e-15	2.65e-05
20	3.11e-15	2.12e-06

Table 2.2: The errors of Example 3.

Remark 1 *We observed that the Legendre collocation method achieves a fast convergence for the both examples while the Jakobi and Chebyshev collocation method obtain slow convergence rates. That's why a proper selection of basic functions is crucial to ensure the convergence and numerical stability of the spectral methods.*

- ***Smoothness of the solution :*** The performance of spectral methods can strongly depend on the properties of the solution, such as its regularity and asymptotic decay. Spectral methods tend to perform well for problems with smooth solutions. However, these methods may not be as effective in situations such as rapid, steep variations gradients, or steep front and for problems with discontinuous or highly oscillatory solutions. For example, for problems with rapidly (exponentially) decaying solutions spectral methods based on Laguerre or Hermite polynomials are not recommended because of their wild behaviors at infinity. Alternatively, in this case, Laguerre or Hermite functions are strongly suggested in this context. Nevertheless, when applied to problems with slow (algebraically) decaying solutions, Laguerre or Hermite polynomials and functions are ineffective. So, solutions with atypical behaviors can pose additional challenges to the convergence and thus affect the numerical stability of the spectral method. To address this limitation, addaptive methods are proposed for some types of integral equations [72, 73].

To maintain numerical stability of the spectral method, it's often necessary to conduct a thorough analysis of these factors.

2.2 Basic concepts of numerical stability

2.2.1 Well and Ill-conditioned problem

Every problem that we try to solve is based on an expression of some form or another. To have confidence in our solution we need to know that the expression is Well $-$ or Ill $-$ Conditioned Problems.

A problem is called well-conditioned if a small perturbation of the input data , leads to small variations of the results , i.e. of the same magnitude order. On the other hand, If small changes in the input lead to large changes in the output, then we call the problem ill-conditio-50ned.[74, 75]

2.2.2 Condition number of a problem

For many types of problems we can compute a condition number that indicates the magnification of the changes. Denoting by P the problem under consideration, if d represents the input data and r the output results, it is possible to define the condition number of the problem through the following inequality:

$$\frac{\|\delta r\|}{\|r\|} \leq \mathcal{K}(P)\frac{\|\delta d\|}{\|d\|} \tag{2.2.1}$$

where $\|.\|$ is a given norm which is able to measure the involved quantities. The condition number bounds the propagation of the input relative error in the output results, it is closely related to the maximum accuracy that can be attained in the solution. and we define the relative condition number to be

$$\mathcal{K}(P) = sup\left\{\frac{\|\delta r\|/\|r\|}{\|\delta d\|/\|d\|}\ ,\ \delta d \neq 0\right\}$$

2.2.3 Stability of linear systems

◆ Our problem is a system of equations:

$$Ax = b \tag{2.2.2}$$

Input data is $A \in \mathbb{R}^{(n,n)}$ and $b \in \mathbb{R}^{(n)}$, result is $x \in \mathbb{R}^{(n)}$. The condition number $\mathcal{K}(A)$ is involved in the answer to the question: how much can a change in the right hand side of a system of linear equations affect the solution?

The following system obtained by altering the right-hand side:

$$A(x + \delta_x) = b + \delta_b \tag{2.2.3}$$

Think of δ_b as being the error in b and δ_x as being the resulting error in x, although we need not make any assumptions that the errors are small. Consequently,

$$\frac{\|\delta_x\|}{\|x\|} \leq \mathcal{K}(A)\frac{\|\delta_b\|}{\|b\|} \tag{2.2.4}$$

The quantity $\frac{\|\delta_b\|}{\|b\|}$ is the relative change in the right-hand side, and the quantity $\frac{\|\delta_x\|}{\|x\|}$ is the resulting relative change in the solution.

More generally $\mathcal{K}_p(A)$ will denote the condition number of A in the $p - norm$ where $p = 1$, $p =$

2 *and* $p=\infty$ by:

$$\mathcal{K}_p(A)=\|A\|_p\,\|A^{-1}\|_p \tag{2.2.5}$$

The following system obtained by altering the left-hand side:

$$(A+\delta_A)(x+\delta_x)=b \tag{2.2.6}$$

then we have

$$\frac{\|\delta_x\|}{\|x+\delta_x\|}\leq\mathcal{K}(A)\frac{\|\delta_A\|}{\|A\|}. \tag{2.2.7}$$

◆ Relation between condition number and stability

With the concept of condition we are now able to characterize problems. Now we will have a look at the characterization of stability and its relation with condition number.

If $\mathcal{K}(P)$ is small, the problem will be $well-conditioned$. Otherwise, the problem is called $ill-conditioned$. (For more details from stability of linear system see [69, 76])

Remark 2 *If the system (2.2.2) is ill-conditioned, it can be replaced by the equivalent system $PAx=Pb$ where P is an invertible matrix and the new matrix $\tilde{A}=PA$ is well- conditioned. The new system is called a preconditioned system .*

Example 4 *The linear models of physics, astronomy ,.., often lead to the resolution of large linear systems which are represented by $AX=B$. It sometimes happens that a small variation on B leads to a large variation on X, In this case, we say that the matrix or the problem is ill-conditioned.*
$-$ *We consider $AX=B$ with*

$$A=\begin{pmatrix}10&7&8&8\\7&5&6&5\\8&6&10&9\\7&5&9&10\end{pmatrix},B=\begin{pmatrix}32\\23\\33\\31\end{pmatrix}$$

and exact solution $X=[1,1,1,1]^T$.

If we change the right-hand side of the problem to

$$B = \begin{pmatrix} 32.1 \\ 22.9 \\ 33.1 \\ 30.9 \end{pmatrix}$$

so that the exact solution becomes $X = [9.2, -12.6, 4.5, -11]^T$.
We remark that very small variations on B lead to large variations on X. On the other hand $\mathcal{K}_p(A) = \|A\|_p \, \|A^{-1}\|_p = 4488$, *where p is the matrix norm associated with the infinite norm on* $\mathbb{R}^4$ *is big then the problem is* $ill - conditioned$.

2.2.4 Stability of numerical algorithm

An algorithm or numerical process is called ”stable” if small errors in the inputs and at each step lead to small errors in the solution. Hence, even when a problem is well-conditioned, if we try to solve it with an unstable algorithm, the obtained results will be meaningless. The following example refer to a comparison between stable and unstable algorithms for the given problem.

Example 5 *Example of integral computation*

– In the next two steps, we compare two algorithms solving the following integral:

$$I_n = \frac{1}{e}\int_0^1 x^n e^x dx, \quad n \geq 0 \tag{2.2.8}$$

Both algorithms are based on the following theoretical considerations:

- $0 < I_n < \int_0^1 x^n dx = \dfrac{1}{1+n}$,
- $I_{n+1} < I_n$,
- $\lim_{n\to 0} I_n = 0$.

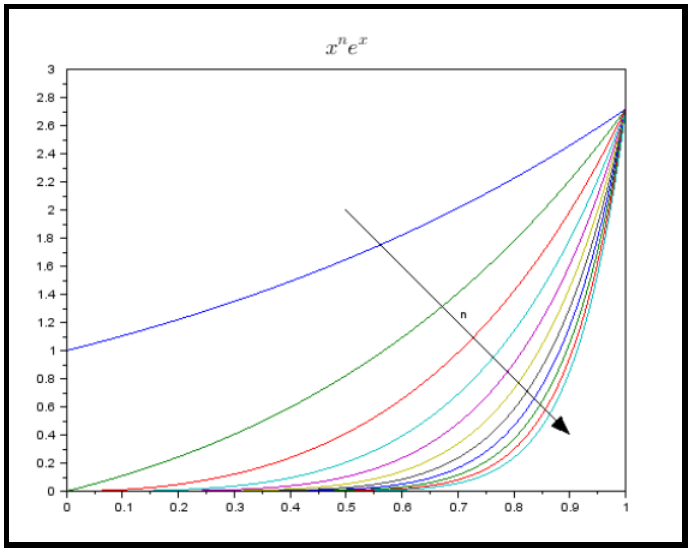


Figure 2.1: The integrand

• The first strategy (unstable formulation) is to develop an algorithm based on the following recursive formula:

- For $n = 0$ wa have:

$$I_0 = \frac{1}{e}\int_0^1 e^x dx = \frac{1}{e}(e-1) = 1 - e^{-1}.$$

- For $n > 0$ we can used integration by parts having

$$I_n = \frac{1}{e}\left(\int_0^1 x^n e^x dx = [x^n e^x]_0^1 - n\int_0^1 x^{n-1} e^x dx\right) = 1 - nI_{n-1}$$

The developed program starts from $n = 1$,where $I_1 = e^{-1}$.

To perform the error analysis we denote by $I'_n = I_n + \varepsilon_n$ the approximate value of the integral at step n with respect to the exact value I_n and making an error ε_n . Hence, it is possible to write the following recursive formula for the error

$$\varepsilon_n = I'_n - I_n = (1 - nI'_{n-1}) - (1 - nI_{n-1}) = -n(I'_{n-1} - I_{n-1}) = -n\varepsilon_{n-1}$$

and ε_n , with respect to the first error ε_1 , is

$$\varepsilon_n = (-1)^{n-1} n! \varepsilon_1$$

As a consequence, even if ε_1 is small, the error ε_n grows up to infinity as a factorial.

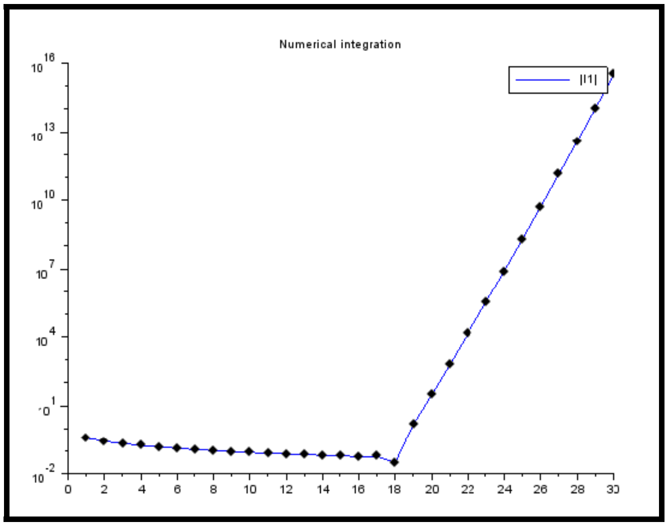


Figure 2.2: Absolute value of I_n

• The second strategy (stable formulation) is to develop an algorithm based on the following recursive formula:

- For $n = N$ we set: $I_N = 0$
- For $n > 0$ we rewrite the previous recursive formula $I_n = (1 - nI_{n-1})$ in terms of $n-1$ as follows:

$$I_{n-1} = \frac{1}{n}(1 - I_n).$$

The developed program starts from $N = 100$ and computes I_1 as the last integral.
To perform the error analysis we denote by $I'_n = I_n + \varepsilon_n$ the approximate value of the integral at step n with respect to the exact value I_n and making an error ε_n . Hence, it is possible to write the following recursive formula for the error

$$\varepsilon_n = I'_n - I_n = (1 - nI'_{n-1}) - (1 - nI_{n-1}) = -n(I'_{n-1} - I_{n-1}) = -n\varepsilon_{n-1}$$

giving $\varepsilon_{n-1} = -\frac{1}{n}\varepsilon_n$ Hence ε_1 can be expressed in terms of ε_N as

$$\varepsilon_1 = \frac{(-1)^{N-1}}{N!}\varepsilon_N$$

As a consequence, even if ε_N is “big”, the error ε_1 decreases to zero, since we have a factorial as denominator.

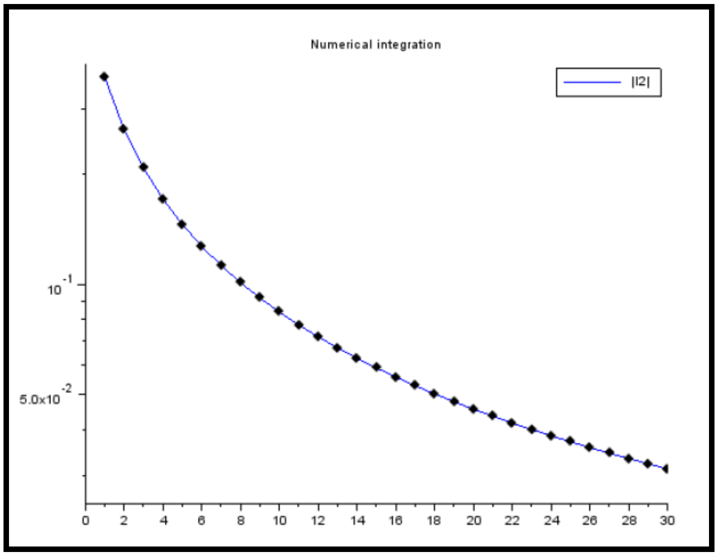


Figure 2.3: Value of I_n

◆ The relative error for I_1 starting from for $I_N = 0$ different values of N The relative error on ε_1 is computed as $err_{rel} = \dfrac{|I_1^{exact} - I_1|}{|I_1^{exact}|}$ where I_1 depends on N and $I_1^{exact} = e^{-1}$.
− Note that the values are visible only until $N = 17$, after that limit the values are less then ε and are not visible in a logarithmic scale.

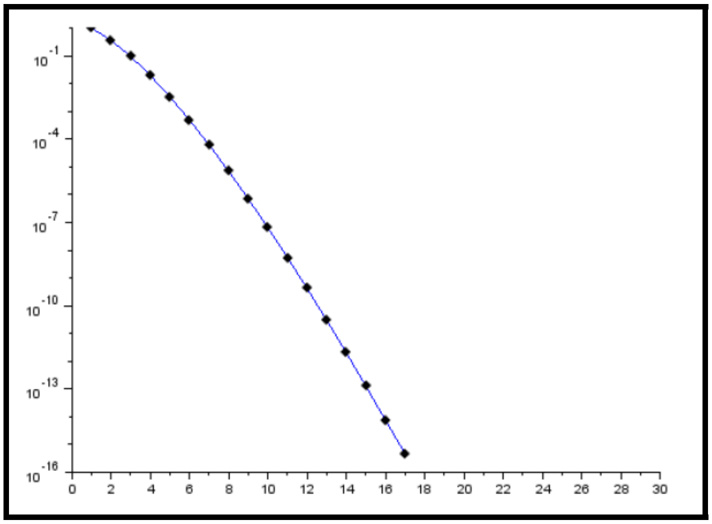


Figure 2.4: Relative error of the approximation of I_1 depending on N

2.3 Numerical stability and convergence of spectral methods

Convergence, consistency and stability are crucial in the analysis of spectral methods in differential and integral equations. As previously explained, all spectral methods for solving integral equations lead to the solution of linear systems. In this section we look at the conditioning of these linear systems, and we relate it to the conditioning of the original integral equation being solved. It's critical to understand the propagation of truncation and roundoff errors since approximation problems must be solved at every level. As a result, it is essential to research the stability of the equation and its approximation method. So what is consistency, convergence and stability? And how these are tested and defined?

2.3.1 Convergence Analysis

There are always two fundamental questions we should consider when solving the problem (1.3.1), How to find conditions under which u_N converges to u as $N \to \infty$ and how to bounded the error $\|u - u_N\|$? You will find the answer in the following section.
The spectral method is said to be convergent, if there is a rank N_0 from which the approximate equation (1.2.2) admits a unique solution $u_N \in \mathcal{X}_\mathcal{N}$ and that this solution converges to the solution (1.3.1).

Theorem 2.1 *Let $K : \mathcal{X} \longrightarrow \mathcal{X}$ be a compact linear operator in a Banach space $\mathcal{X}$ and $(I - K)$ be injective. Assume that the projection operators $\mathcal{P}_N : \mathcal{X} \longrightarrow \mathcal{X}_N$ then $\|\mathcal{P}_N K - K\| \longrightarrow 0, N \longrightarrow \infty$. Then, for sufficiently large N, the approximate equation (1.2.2) is uniquely solvable for all $f \in \mathcal{X}$ (i.e.$(I - \mathcal{P}_N K)^{-1}$) exist) and we have the error estimate*

$$\|u_N - u\| \leq M\|\mathcal{P}_N u - u\| \tag{2.3.1}$$

for some positive constant M depending on K.

Proof 2.1 – *See [16]*

2.3.2 Consistency

Consistency essentially requires that the discrete equations defining the approximate solution are at least approximately satisfied by the true solution.

Theorem 2.2 *$\mathcal{P}_N K$ is called consistent if some $K \in \mathsf{B}(\mathcal{X}, \mathcal{X})$ exists satisfying*

$$\lim_{N \longrightarrow \infty} \|\mathcal{P}_N K u - K u\| \longrightarrow 0, \;\; for\ all\ u \in \mathcal{X}$$

2.3.3 Stability and conditioning

For an integral equation of the second kind, there are two types of stability questions that one wishes to answer. First, is the original equation stable? Second, is the solution method stable? (see [12])

Almost all numerical methods for solving integral equations lead to the solution of linear systems $Ax = b$, which is described in (2.2.2).

In discussing stability one is concerned with either the absolute error determined by $\|A^{-1}\|$ and the relative error by the condition number

$$\mathcal{K}(A) = \|A\| \, \|A^{-1}\| \tag{2.3.2}$$

In both cases the analysis generally rests on the estimation of $\|A^{-1}\|$.

- For the first, the following problem

  $$u - K(u) = f, \text{ for all } f \in \mathcal{X}, \tag{2.3.3}$$

  can be tranformed into the linear system $Au = f$ where $A = I - K$. We have observed that $A = I - K$ has a bounded inverse provided that K is compact. Thus $\| A^{-1} \| = \| (I - K)^{-1} \|$, and the equation is stable if this quantity is not too large.

- For the second, we observe that the problem (1.2.2) can be defined by

  $$A_N u_N = f_N \tag{2.3.4}$$

  Where $A_N = I - \mathcal{P}_N K$ and $f_N = \mathcal{P}_N f$. The stability of the numerical algorithm is then determined by a knowledge of $\|A_N^{-1}\|$. If the approximation method is such that A_N^{-1} converges in norm to A^{-1} then $\|A_N^{-1}\|$ converges to $\|A^{-1}\|$, so that one has for some N_0

$$\sup_{N \geq N_0} \|A_N^{-1}\| \leq M$$

Where M is a constant and of course $\|A_N^{-1}\| \approx \|A^{-1}\|$ for large enough N.

Condition number

The condition number of equation (1.3.1) is usually defined as

$$\mathcal{K}(I-K) = \|(I-K)\|\|(I-K)^{-1}\| \tag{2.3.5}$$

For the projection methods considered earlier in the chapter, we have both $\mathcal{P}_N K \to K$ and $(I-\mathcal{P}_N K)^{-1} \to (I-K)^{-1}$ and consequently

$$\|(I-\mathcal{P}_N K)\| \to \|(I-K)\|$$

$$\|(I-\mathcal{P}_N K)^{-1}\| \to \|(I-K)^{-1}\|$$

For the condition number of $(I-\mathcal{P}_N K) = \mathcal{P}_N f$, we then have

$$\mathcal{K}(I-\mathcal{P}_N K) \to \mathcal{K}(I-K) \; as \; N \longrightarrow \infty$$

This means that $(I-K)u = f$ and its approximating equation $(I-\mathcal{P}_N K)u_N = \mathcal{P}_N f$ will be approximately the same in their conditioning.

◆ If the condition number (2.3.5) of the projection method (1.2.2) is not significantly large then the method is stable.

2.3.4 Relation between consistensy, convergence and Stability

Theorem 2.3 *Let the projection operators $\mathcal{P}_N$ be convergente and $K : \mathcal{X} \longrightarrow \mathcal{X}$ be compact. Then the operators $\mathcal{P}_N K$ converge with respect to the operator norm to K:*

$$\|\mathcal{P}_N K - K\| \longrightarrow 0, N \longrightarrow \infty.$$

If $(I - \mathcal{P}_N K)^{-1} \in \mathsf{B}(\mathcal{X}, \mathcal{X})$, then the projection method (1.2.2) is convergent, stable and consistent. The solutions u_N satisfy

$$u_N \longrightarrow u = (I - K)^{-1} f \text{ in } \mathcal{X}$$

Proof 2.2 – *See [77]*

Theorem 2.4 *(stability theorem).*

(a) Convergence implies stability.

(b) Convergence and consistency imply stability and $(I - K)^{-1}$ in $\mathsf{B}(\mathcal{X}, \mathcal{X})$

Theorem 2.5 *(convergence theorem). Suppose consistency, stability, and either*

(i) $(I - K)$ surjevtive or (ii) K compact, then

(a) $\mathcal{P}_N K$ is convergent and

$$u_N = (I - \mathcal{P}_N K)^{-1} f_N \to u = (I - K)^{-1} f \quad \text{for all } u \in \mathcal{X} \tag{2.3.6}$$

(b) if $u_N \to u$ then $u_N = (I - \mathcal{P}_N K)^{-1} f_N \to u = (I - K)^{-1} f$.

Theorem 2.6 *(equivalence theorem). Suppose consistency and one of the conditions (i) or (ii) from Theorem 2.5 . Then stability and convergence are equivalent. Furthermore, $u_N \to u$ implies*

$$u_N = (I - \mathcal{P}_N K)^{-1} f_N \to u = (I - K)^{-1} f \tag{2.3.7}$$

Proof 2.3 – *See [76, 77]*

2.4 Integral equations and ill-posed problems

There exist a number of problems arising in different scientific and technical fields belong to a class of ill-posed problems, As prominent example we find integral equation of the first kind. Integral equations of the first kind are characterized by having the unknown function only under the integral sign, whereas in equations of the second kind the unknown function appears both under and outside the integral sign. This small structural difference changes both the theory and the numerical analysis drastically. Generally a discretization of this equations lead to ill-conditioned linear systems. Moreover a slight perturbation in right hand side lead to enormous change of the solution. Thus some kind of regularization method must be utilized to obtain the stable resolution of the problems [78, 79].

2.4.1 Description of problem

Problems of solving integral equations of the first kind

$$\int_I k(x,t)u(t)dt = f(x),\ x \in I. \tag{2.4.1}$$

by their nature, are ill-posed in the sense of Hadamard [80, 81]. This means that the required solution is extremely sensitive to the perturbations in the observation data.

Definition 2.1 ***Well-posedness***
A problem is said to be well-posed (in the sense of Hadamard) if the following conditions hold:

- *existence: there exists at least one solution.*
- *uniqueness: the exists at most one solution.*
- *stability : the solution depends continuously on the given data.*

Otherwise, the problem is called ***ill-posed****.*

Definition 2.2 *Let $A : U \to V$ be an operator from a subset U of a normed space $\mathcal{X}$ into a subset V of a normed space $\mathcal{Y}$. The equation*

$$Au = f \tag{2.4.2}$$

is called well-posed if A is bijective and the inverse operator A^{-1} is continuous. Otherwise, the equation is called ill-posed.

According to this definition we may distinguish three types of ill-posedness. If A is not surjective, then equation (2.4.2) is not solvable for all $f \in V$ (nonexistence). If A is not injective, then the problem may have more than one solution (nonuniqueness). Finally, if A^{-1} exists but is not continuous, then the solution u of (2.4.2) does not depend continuously on the data f (instability). The latter case of instability is the one of primary interest in the study of ill-posed problems.

2.4.2 Projection methods for ill-posed equations

Let X and Y be Banach spaces and let $K : \mathcal{X} \longrightarrow \mathcal{X}$ be an injective bounded linear operator. Let $\mathcal{X}_N \subset \mathcal{X}$ and $\mathcal{Y}_N \subset \mathcal{Y}$ be two sequences of subspaces with $dim\mathcal{X}_N = dim\mathcal{Y}_N = N$ and let $\mathcal{P}_N : \mathcal{Y} \longrightarrow \mathcal{Y}_N$ be projection operators. Given $f \in \mathcal{Y}$ the projection method approximates the equation 2.4.2 for $u \in \mathcal{X}$ by the projected equation

$$\mathcal{P}_N K u_N = \mathcal{P}_N f \tag{2.4.3}$$

This projection method is said to be convergent, if there is a rank N_0 such that for each $f \in K(\mathcal{X})$ the approximate equation (2.4.3) has a unique solution $u_N \in \mathcal{X}_N$ and this solution converges to the solution (2.4.2). This convergence condition can be expressed simply according to the operator $\mathcal{P}_N K : \mathcal{X}_N \longrightarrow \mathcal{Y}_N$ are invertible and we have pointwise convergence $(\mathcal{P}_N K)^{-1}\mathcal{P}_N K u \to u, N \to \infty$ for all $u \in \mathcal{X}$. In general, we can expect convergence only if the subspaces $\mathcal{X}_N$ possess the denseness property

$$\inf_{v \in \mathcal{X}_N} = \|v - u\|,\ N \to \infty$$

Theorem 2.7 *A projection method for an injective linear operator $K : \mathcal{X} \longrightarrow \mathcal{Y}$ from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ converge if and only if there exist $N_0 \in \mathbb{N}$ from which the finite-dimensional operators $\mathcal{P}_N : \mathcal{X}_N \longrightarrow \mathcal{Y}_N$ are invertible and the operators $(\mathcal{P}_N K)^{-1}\mathcal{P}_N K : \mathcal{X} \to \mathcal{X}_N$ are uniformly bounded, i.e.*

$$\exists M > 0, N \geq N_0, \|(\mathcal{P}_N K)^{-1}\mathcal{P}_N K\| < M \tag{2.4.4}$$

In the case of convergence, we have the error estimate

$$\|u_N - u\| \leq (1 + M) \inf_{v \in \mathcal{X}_N} = \|v - u\|.$$

Proof 2.4 – *See [16]*

From theorem 2.3 and condition 2.4.4, we conclure that the projection method is stable.

2.5 Application

To illustrate the ideas of stability and conditioning explained in the previous sections we shall develop the collocation method of [82]. The authors using mapped Gegenbauer rational collocation method to solve a class of linear Fredholm integral equations on the real line defined by:

$$u(x) = f(x) + \int_{\infty}^{\infty} k(x,t)u(t)dt, \quad x \in \mathbb{R}, \tag{2.5.1}$$

We will now follow the same method with using the following variable change

$$x = \varphi_s(y) = \frac{sy}{\sqrt{1-y^2}}, \quad y = \theta_s(x) = \frac{sx}{\sqrt{1+x^2}}.$$

It is clear that

$$\frac{dx}{dy} = \frac{s}{(1-y^2)^{3/2}} \quad x \in \mathbb{R}, y \in I,$$

where s is a positive scaling factor. Then, the solution method define in section 3 in [82] leads to the matrix system

$$(P - sKDP)u = f$$

which can be easily solved by using a standard linear algebra routine.

2.5.1 Numerical experiments

Consider two exapmles of Fredholm integral equations (2.5.1) with diffrente kernel then we give maximum absolute errors at 1000 selected equally spaced points on the interval $[-50, 50]$ using algebraic map against various N and different scaling factor s.

Example 6 *Consider the following equation*

$$f(x) = e^{-x^2} - erf(1)\pi e/(x^2+1);\ k(x,t) = 1/(1+x^2)(1+t^2)\ where\ u(x) = e^{-x^2}.$$

	$\alpha = 0$		$\alpha = 1$	
N	$s = 2$	$s = 4$	$s = 1.5$	$s = 3$
30	1.12e-06	3.96e-06	3.97e-06	1.48e-07
60	3.38e-10	1.03e-12	3.46e-09	2.66e-12
90	2.49e-13	1.92e-13	8.85e-12	3.40e-14

Table 2.3: A comparison of the errors of Example 6.

Example 7 *[82] Consider*

$$f(x) = e^{-x^2} - \sqrt{\pi}\cos(x)e(-1/4);\; k(x,t) = \cos(x-t)\; where\; u(x) = e^{-x^2}.$$

	$\alpha = 1/2$		
N	$s = 2$	$s = 4$	[82]
20	3.03e-05	7.63e-05	1.15e-03
60	2.80e-10	4.31e-13	1.35e-05
80	9.85e-13	5.19e-15	2.72e-08

Table 2.4: A comparison of the errors of Example 7.

2.5.2 Numerical stability

The convergence and stability of MGRFs-collocation method is studied in [82]. For the stability, the authors camputed the relative condition number (2.2.2) defined in [83] which they found very good. Now we examine the stability of the above numerical examples by calculing the $\|A_N^{-1}\|_\infty = \|(P - sKDP)_N^{-1}\|_\infty$. From these examples, we can see that the norme of A_N^{-1} is bounded, such that $\|A_N^{-1}\|_\infty \simeq 54$ for example 6 (for $\alpha = 0$, $N = 20$ and $s = 4$) and $\|A_N^{-1}\|_\infty \simeq 70$ for example 7 (for $N = 20$ and $s = 2$).

Also, we observe that the convergence of the proposed method ($Theorem$ 4 [82]) is found under the following condition

$$\sup_{N \geq N_0} \|A_N^{-1}\| \leq \infty$$

wich found that the numerical results are in good accordance with the theoretical analysis ($Theorem$2.3). This illustrates that the proposed method is stable.

Chapter 3

Spectral solution methods for some integral equations

We discusse in this chapter the numerical solution of some integral equations by spectral collocation method that we saw earlier. In the first section, we presented a Legendre spectral collocation method to solve non linear Fredholm integral equations, the main idea in this approach is to convert the original problem into an equivalent one through an appropriate variable transformations, so that the resulting equation can be accurately solved by using spectral collocation at the Jacobi-Lagrange points. The convergence of the proposed method is validated by typical numerical examples.

In the second section, we have tried to apply the so-called rational Legendre functions that can be obtained by combining the classical Legendre polynomials with algebraic mapping, and then we apply the RLC method to solve the quadratic Urysohn integral equations in the half line in which the integral part is replaced by their operational matrix representations with collocation points. In addition, we discuss the convergence of the approximate solution to the exact solution for both L^∞ and weighted L^2 norms and address the stability of the proposed method. In order to show the efficiency and demonstrate the accuracy and stability of the proposed method, some numerical examples are presented

3.1 LCM for non linear Fredholm integral equations

The Fredholm Integral Equations of the second kind are capable of forming a variety of phenomena in different fields of science and technology. While many existing numerical algorithms have been extensively studied in many papers, many of them have suggested significant solutions to the non linear fredholm integral equations [84–89].

The development of an efficient numerical method is covered in this section. To solve the nonlinear Fredholm integral equation of the form

$$u(x) = f(x) + \int_a^b k(x,t)\psi(u(t))dt, \quad x \in [a,b], \tag{3.1.1}$$

where $f(x)$, $K(x,t)$ and $\psi(x)$ are given continuous functions and $u(x)$ is unknown function. Evidently, at first we must transform the problem set in the given interval to $[-1;1]$ by means of an appropriate variable transformation. Then, we apply the standard spectral Legendre collocation method to the resulting equation in which Lagrange interpolation polynomials together with the Gauss-Legendre quadrature rule are used to transform the source problem to a system of nonlinear algebraic equations. Afterwards, the resulting system can be solved by the iterative method. Finally, several selected numerical examples are presented and discussed to illustrate the application and effectiveness of the proposed approach.

3.1.1 Gauss-Legendre quadrature formula and Lagrange interpolation

For a given positive integer N, the Gauss-Legendre quadrature formula of order $N+1$ is given by

$$\int_{-1}^{1} f(x)dx = \sum_{i=0}^{N} w_i f(\zeta_i) \tag{3.1.2}$$

for some set of nodes $\{\zeta_i\}_{i=1}^{N}$ which is the set of $(N+1)$ legendre Gauss points, and weights $\{w_i\}_{i=1}^{N}$. It should be noted that when using the extensively tabulated roots of the Legendre polynomials and their corresponding weights to solve an equation whose integration interval is $[a;b]$; we need to first define a family of one-to-one mappings between $y \in I = [-1,1]$ and $x \in [a,b]$ of the form

$$x = \theta(y) = \frac{b-a}{2}y + \frac{b+a}{2} \, , \, y \in I. \tag{3.1.3}$$

Such that

$$\frac{dx}{dy} = \theta'(y) > 0, \ \ with \ \theta(a) = -1, \, \theta(b) = 1.$$

For a given mapping $y = \theta^{-1}(x), x \in [a,b]$ we define the so-called mapped Legendre functions by

$$L_n(x) = P_n(\theta^{-1}(x)), \ \ n = 0,1,2... \tag{3.1.4}$$

They are orthogonal in the interval $[a,b]$ with respect to the weight function

$$w_{a,b}(x) = \frac{dy}{dx} = \frac{2}{b-a}$$

which gives

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f(y)dy \approx \frac{b-a}{2}\sum_{i=0}^{N} w_i f(y_i). \tag{3.1.5}$$

Let $\mathbb{P}_N$ denote the space of all algebraic polynomials of degree not exceeding N. For any $U \in \mathcal{C}(I)$, we can define the Lagrange interpolating polynomial $\mathcal{P}_N U \in \mathbb{P}_N$ satisfying

$$\mathcal{P}_N U(x_i) = U(x_i), \ \ 0 \le i \le N.$$

It can be written as an expression of the form

$$\mathcal{P}_N U(\zeta_i) = \sum_{i=0}^{N} U(\zeta_i) L_i(y), \ \ y \in I$$

where $L_i(y)$ is the Lagrange interpolation basis function associated with the legendre collocation $\{\zeta_i\}_{i=1}^{N}$ namely,

$$L_i(y) := \frac{\pi(y)}{\pi'(\zeta_i)(y-\zeta_i)}, \ \ \pi(y) := (y-\zeta_0)(y-\zeta_1)...(y-\zeta_N).$$

3.1.2 Legendre collocation method

This section describes some basic properties of the Lagrange Interpolation method associate with the Legendre collocation points for solving the nonlinear Integral Equation on $[-1,1]$ interval, which can be obtained by substitute the mapping (3.1.3) in equation (3.1.1) as follow:

$$U(y) = F(y) + \int_{-1}^{1} K(y,z)\psi(U(z))dz \tag{3.1.6}$$

where

$$U(y) = u(\theta(y)), F(y) = f(\theta(y))$$

$$K(y,z)\psi(U(z)) = k(\theta(y),\theta(z))\psi(U(\theta(z)))\theta^{'}(z).$$

To obtain such a solution, we need to substitute $\{\zeta_i\}_{i=0}^{N}$ the Legendre points into the integral equation (3.1.6) so that the latter becomes

$$U(\zeta_i) = F(\zeta_i) + \int_{-1}^{1} K(\zeta_i, z), \psi(U(z))dz \quad i = 0\cdots N \tag{3.1.7}$$

The main difficulty in obtaining high order accuracy is to compute the integral term. To overcome this difficulty, we need to use the point Gauss quadrature formula relative to the Legendre weights $\{w_j\}_{j=0}^{N}$.Then the equation (3.1.7) can be written as follows:

$$U(\zeta_i) = F(\zeta_i) + \frac{b-a}{2}\sum_{j=0}^{N} K(\zeta_i,\zeta_j)\psi(U(\zeta_j))w_j, \quad i = 0\cdots N \tag{3.1.8}$$

We now need to represent $U(\zeta_i^N)$ using u_i^N, $0 \leq i \leq N$, i.e., the values at all the grid points. To this end, we expand u using Lagrange interpolation polynomials, i.e.,

$$U(y) = \sum_{i=0}^{N} u_i^N L_i(y), \quad U(\zeta_i^N) \backsim u_i^N, \tag{3.1.9}$$

where $L_i(y)$ is the i-th Lagrange polynomial. Combining (3.1.9) and (3.1.8) yields

$$u_i^N = F(\zeta_i^N) + \frac{b-a}{2}\sum_{j=0}^{N} K(\zeta_i^N,\zeta_j^N)\psi(u_j^N)w_j, \quad i = 0\cdots N \tag{3.1.10}$$

Then, the discrete spectral rational Legendre-collocation method for solving Eq. (3.1.1) leads to the following nonlinear system:

$$\mathbf{u} = \mathbf{H}(\mathbf{u}), \quad \mathbf{H}(\mathbf{u}) = \mathbf{F} + \mathbf{MPW}, \tag{3.1.11}$$

where **F**,**M**,**P** and **W** are given by:

$$\begin{aligned} \mathbf{F} = F(\zeta_i^N) \quad &, \quad \mathbf{M} = K(\zeta_i^N,\zeta_j^N), \\ \mathbf{P} = \psi(u_j^N) \quad &, \quad \mathbf{W} = diag(\theta'(x)w_j). \end{aligned} \tag{3.1.12}$$

To achieve a highly accurate numerical solution of (3.1.11), we would need to apply the following iterative process

$$\mathbf{u}^{(m)} = \mathbf{H}(\mathbf{u}^{(m-1)}), \tag{3.1.13}$$

with the initial value $\mathbf{u}^{(0)} = F$.

Finally, the recurrence relation (3.1.13) gives the approximate solution of (3.1.1) as follow

$$u_N(x) = U(y) = \sum_{i=0}^{N} u_i^{N,m} L_i(\theta^{-1}(x)), \tag{3.1.14}$$

where m represent the number of iteration to obtain the approximate solution by using the iterative method.

3.1.3 Convergence analysis

Before presenting error estimates for the above scheme, we need to introduce the following weighted Hilbert space with some useful lemmas about Lagrange interpolation based on the Legendre Gauss points. For a nonnegative integer m, define

$$H^m(I) = \{U : \partial_y^r U \in L^2(I),\ 0 \leq r \leq m\}$$

Where

$\partial_y^r U(y) = \partial^r U(y)/\partial y^r$ related to the following semi-norm and the norm:

$$|U|^m = \|\partial_y^m U\|\ ,\ \|U\|^m = \left(\sum_{r=0}^{m} \|\partial_y^r U\|^2\right)^{1/2}$$

Also, it is convenient to introduce the semi-norms

$$|U|^{m,N} := |U|_{H^m} = \left(\sum_{r=min(m,N+1)}^{m} \|U^r\|^2\right)^{1/2}$$

The following estimates follows from [90].

Lemma 3.1 *Assume that $U \in H^m(I)$ and denote by $\mathcal{P}_N U$ the interpolation operator associated* *with the legendre-Gauss points. Then the following estimates hold:*

$$\|U - \mathcal{P}_N U\|_\infty \leq C N^{1/2-m} |U|^{m,N}$$

where C is a positive constant independent of N and U. For simplicity, from now we denote all constants by the symbol C. Also we quote the following result from [91].

Lemma 3.2 [92] *Let $\{L_i(y)\}_{i=1}^N$ be the $N-th$ Lagrange interpolation polynomials associated with the legendre collocation points. Then*

$$\|\mathcal{P}_N\|_\infty := \max_{y \in I} \sum_{i=1}^{N} |L_i(y)| = \mathcal{O}(N^{1/2}).$$

Theorem 3.1 *Let U be the exact solution to Eq. (3.1.1) and U_N be the approximate solution obtained by using the spectral-collocation scheme (3.1.10). For $m \geq 1$, If $U \in H^m(I)$, then*

$$\|U - U_N\|_\infty = \mathcal{O}(N^{\frac{1}{2}-m}). \tag{3.1.15}$$

Proof. Let the integral equation

$$U(x) = F(x) + \int_{-1}^{1} K(x,t)\psi(U(t))dt, \tag{3.1.16}$$

while using the approximate solution, we have

$$U_N(x) = \mathcal{P}_N F(x) + \int_{-1}^{1} \mathcal{P}_{N,N}(K(x,t)\psi(U(t)))dt. \tag{3.1.17}$$

Subtracting (3.1.17) from (3.1.16), we get the error equation

$$\begin{aligned} U(x) - U_N(x) &= F(x) - \mathcal{P}_N F(x) \\ &+ \int_{-1}^{1} K(x,t)\psi(U(t))dt - \int_{-1}^{1} \mathcal{P}_{N,N}(K(x,t)\psi(U(t))) \\ &= J_1(x) + J_2(x), \end{aligned} \tag{3.1.18}$$

where

$$\begin{aligned} J_1(x) &= f(x) - \mathcal{P}_N F(x), \\ J_2(x) &= \int_{-1}^{1} K(x,t)\psi(U(t))dt - \int_{-1}^{1} \mathcal{P}_{N,N}(K(x,t)\psi(U(t))), \end{aligned}$$

It follows immediately from lemma (3.1) that

$$\|J_1\|_\infty \leq C_0 N^{1/2-m} |F|^{m;N}. \tag{3.1.19}$$

We suppose that $\Psi(x,t) = K(x,t)\psi(U(t))$, then

$$\begin{aligned} |J_2(x)| &= |\int_{-1}^{1} K(x,t)\psi(U(t))dt - \int_{-1}^{1} \mathcal{P}_{N,N}(K(x,t)\psi(U(t)))dt| \\ &= |\int_{-1}^{1} \Psi(x,t)dt - \int_{-1}^{1} \mathcal{P}_{N,N}\Psi(x,t)|dt \\ &\leq C\int_{-1}^{1} \lim_{N\longrightarrow\infty} |\Psi(x,t) - \mathcal{P}^{s}_{N,N}\Psi(x,t)|dt \end{aligned}$$

By lemma 3.1, we have

$$\lim_{N\to\infty} |\Psi(x,t) - \mathcal{P}_{N,N}\Psi(x,t)| = 0.$$

Hence, using Proposition 4.2.3 in [93], we get

$$\int_{-1}^{1} \lim_{N\to\infty} |\Psi(x,t) - \mathcal{P}_{N,N}\Psi(x,t)|dt = 0.$$

This implies for every $\varepsilon_N > 0$, there exists a positive integer N (depending on ε_N) such that

$$\int_{-1}^{1} |\Psi(x,t) - \mathcal{P}_{N,N}\Psi(x,t)|dt \leq \varepsilon_N.$$

By taking $\varepsilon_N = N^{1/2-m}|\Psi|^{m;N}$, we get

$$\|J_2\|_\infty \leq 2C_3 N^{1/2-m}|\Psi|^{m;N}.$$

3.1.4 Numerical examples

In this section, in order to illustrate the performance of the presented methods in solving non linear Fredholm integral equations and justify the accuracy and stability of the methods, we consider the following examples.

Example 8 *[86, 87] For the first example, consider the following integral equation*

$$u(x) = \sin(\pi x) + \frac{1}{5}\int_{0}^{1} \cos(\pi x)\sin(\pi t)u^3(t)dt$$

where the exact solution is $u(x) = \sin(\pi x) + \frac{1}{3}(20 - \sqrt{391})\cos(\pi x)$. *The absolute error* $E_N = \|U - U_N\|$ *for* $N = 4, 10, 18$ *and* $x \in [0, 1]$ *is shown in the following table.*

x	N=4	N=10	N=18	[87]N=16	[86] N=25
0	2.15e-03	5.56e-10	8.88e-16	1.13e-03	———-
0.2	2.18e-04	1.02e-11	1.11e-16	21.5e-03	3.79e-10
0.4	6.33e-05	8.51e-11	6.66e-16	1.26e-03	1.45e-10
0.6	2.46e-04	1.77e-10	1.11e-16	1.23e-03	1.45e-10
0.8	6.72e-04	3.64e-10	4.44e-16	1.39e-03	3.79e-10
1	2.67e-03	2.05e-09	3.47e-16	6.27e-02	———

Table 3.1: Comparison of errors for Example 8.

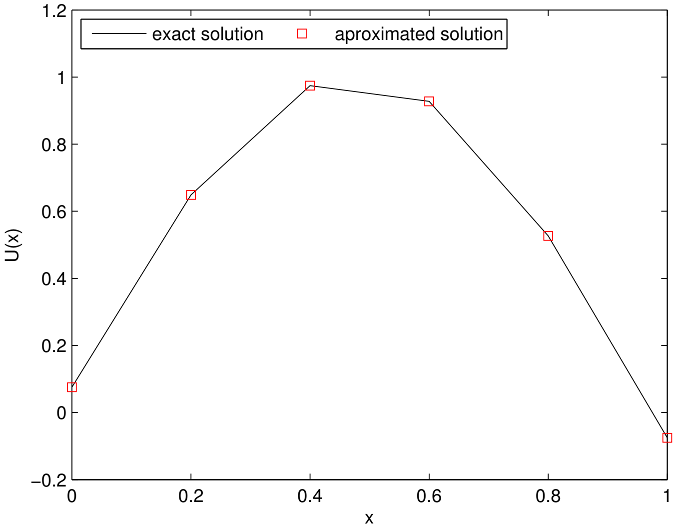


Figure 3.1: Exact and approximated solutions of Example 8 for $N = 10$.

Example 9 *[84, 86] For the second example, consider the following integral equation*

$$u(x) = 2 - \frac{1}{3}(2\sqrt{2}-1)x - x^2) + \int_0^1 xt\sqrt{u(t)}dt$$

where the exact solution is $u(x) = 2 - x^2$. *The absolute error for* $N = 4, 7, 20$ *and* $x \in [0, 1]$ *is shown in the following table.*

x	N=4	N=7	N=20	[86]N=20	[84]N=7
0.1	1.94e-08	6.31e-12	4.44e-16	1.48e-12	1.4e-10
0.3	5.83e-08	1.89e-11	2.22e-16	4.45e-12	4.0e-10
0.5	9.72e-08	3.16e-11	8.88e-16	7.42e-12	4.0e-10
0.7	1.36e-07	4.42e-11	8.88e-16	1.03e-11	0.0
0.9	1.75e-07	5.68e-11	4.44e-16	1.33e-11	8.0e-10

Table 3.2: Comparison of errors for Example 9.

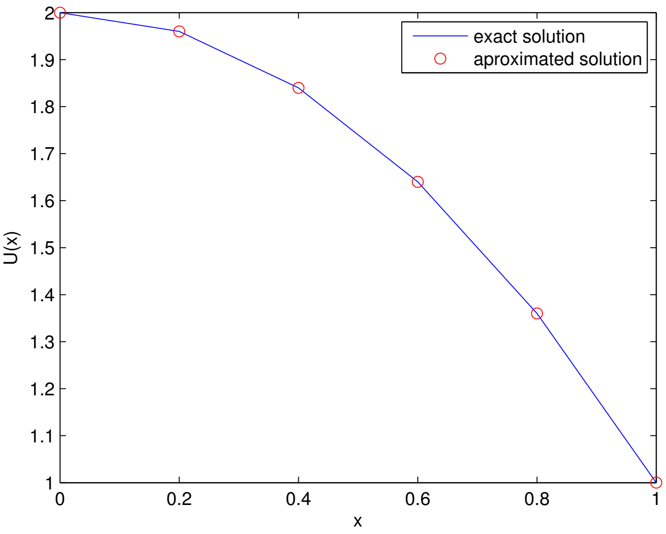


Figure 3.2: Exact and approximated solutions of Example 9 for $N = 8$.

Example 10 *[85, 89] For the third example, consider the following integral equation*

$$u(x) = e^{2x} - x + \int_0^1 xe^{-2t}u(t)dt$$

where the exact solution is $u(x) = e^{2x}$. *The absolute error* E_N *for* $N = 4, 7, 10$ *and* $x \in [0, 1]$ *is shown in the following table*

x	N=4	N=7	N=10	[85, 89]N=10
0.1	1.04e-03	2.09e-08	5.43e-11	2.68e-06
0.3	7.49e-04	3.50e-07	4.78e-11	8.04e-06
0.5	4.44e-16	3.74e-07	8.88e-16	1.33e-05
0.7	8.55e-04	3.82e-07	5.11e-11	1.76e-05
0.9	1.36e-03	2.50e-08	6.20e-11	5.23e-06

Table 3.3: Comparison of errors for Example 10.

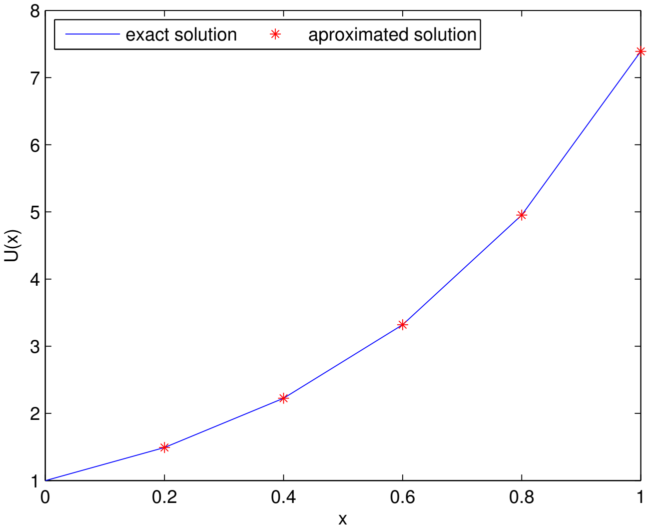


Figure 3.3: Exact and approximated solutions of Example 10 for $N = 10$.

3.2 RLCM for the solution of quadratic Urysohn integral equations on the half-line

The importance of quadratic integral equations in all branches of science and engineering prompted several researchers in recent years to examine the existence and uniqueness of solutions for different types of these equations, limited attention has been given to those defined on unbounded intervals. For instance, Banaś and al. [46] studied the solvability of nonlinear quadratic integral equation of Hammerstein type on an unbounded interval in some Banach space, consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+$. The authors in [47–53], investigated the existence of solutions for the Urysohn integral equation on an unbounded interval. In a related context, Karaoui et al. [94] examined the existence of solutions to nonlinear quadratic integral equations in the Banach space $L^p(\mathbb{R}_+)$. Furthermore, in the second part of their work, they provided a numerical method for solving nonlinear quadratic Volterra integral equations, but over a bounded interval only.

This section is the subject of our work which have been published [95] where we presented an efficient numerical method for the solution of nonlinear quadratic integral equations of Urysohn type, defined on the half-line, namely

$$u(x) = a(x) + f(x,u(x)) \int_0^\infty k(x,t,u(t))dt, \quad x \in [0,\infty), \tag{3.2.1}$$

where $k(x,t,.)$, $a(x)$ and $f(x,u(x))$ are given continuous functions and $u(x)$ is the unknown function.

We first derive the so-called rational Legendre functions that can be obtained by combining the classical Legendre polynomials with an appropriate mapping [7, 96], and then we apply the rational Legendre collocation method to solve the given equation where the problem is converted into a nonlinear system of equations. We provided an error analysis and the stability of the proposed schemes. Finally, several numerical examples are given to show the effectiveness of the present methods.

3.2.1 Orthogonal rational Legendre functions for the semi-infinite interval

In this section, we introduce rational Legendre functions and recall some basic properties. Even more, we present function approximations in some weighted L^2-space.

The well-known Legendre polynomials are orthogonal in the interval $I = [-1,1]$ with respect to the uniform weight function. They can be determined with the help of the following recurrence formula [90]:

$$(n+1)P_{n+1}(y) = (2n+1)yP_n(y) - nP_{n-1}(y) \quad n \geq 1. \tag{3.2.2}$$

Besides

$$P_0(y) = 1, \quad P_1(y) = y, \quad P_n(1) = 1, \quad P_n(-1) = (-1)^n. \tag{3.2.3}$$

The set of Legendre polynomials forms an orthogonal system, namely,

$$\int_{-1}^{1} P_n(y)P_m(y)dy = \frac{2}{2n+1}\delta_{n,m}, \tag{3.2.4}$$

where $\delta_{n,m}$,is the Kronecker delta function. Furthermore, for any function $U \in L^2(I)$, we write

$$U(y) = \sum_{j=0}^{\infty} c_j P_j(y) \text{ with } c_j = \frac{2j+1}{2} \int_{-1}^{1} U(y) P_j(y) dy. \tag{3.2.5}$$

For a given positive integer N, let $\mathcal{P}_N$ denote the space of all algebraic polynomials of degree not exceeding N. we denote the collocation points by $\{\sigma_i^N\}_{i=0}^N$ which is the set of $(N+1)$ Legendre-Gauss points, and by $\{\omega_i^N\}_{i=0}^N$ the corresponding weights. The associated Gauss-Legendre quadrature formula is defined by :

$$\int_{-1}^{1} \phi(y) dy = \sum_{i=0}^{N} \phi(\sigma_i^N) \omega_i^N, \quad \forall \phi \in \mathcal{P}_{2N+1}. \tag{3.2.6}$$

Let us consider the following one to one invertible mapping between $x \in \mathbb{R}_+ = [0, \infty)$ and $y \in I$, with $s > 0$ of the form:

$$y = \eta_s(x) = \frac{x-s}{x+s}, \quad x = \varphi_s(y) = \frac{s(1+y)}{1-y}. \tag{3.2.7}$$

It is clear that

$$\frac{dy}{dx} = \frac{2s}{(x+s)^2}, \quad \frac{dx}{dy} = \frac{2s}{(1-y)^2}, \tag{3.2.8}$$

where s is a positive scaling factor. The *rational Legendre functions* can be defined by

$$R_{s,n}(x) := P_n(\eta_s(x)), \quad n = 0, 1, 2, \ldots, . \tag{3.2.9}$$

They are orthogonal on the interval $\mathbb{R}_+$ with respect to the weight function

$$\rho_s(x) = \frac{dy}{dx} = \frac{2s}{(x+s)^2}, \tag{3.2.10}$$

equivalently

$$\int_0^{\infty} R_{s,n}(x) R_{s,m}(x) \rho_s(x) dx = \frac{2}{2n+1} \delta_{n,m}. \tag{3.2.11}$$

Let us define

$$L^2_{\rho_s}(\mathbb{R}_+) = \{u : \mathbb{R}_+ \to \mathbb{R} \mid u \text{ is measurable and } \|u\|_{\rho_s} < \infty\},$$

where

$$\|u\|_{\rho_s} = \int_0^{\infty} |u(x)|^2 \rho_s(x) dx,$$

is the norm induced by the inner product of the space $L^2_{\rho_s}(\mathbb{R}_+)$,

$$\langle u,v\rangle_{\rho_s} = \int_0^\infty u(x)v(x)\rho_s(x)dx. \tag{3.2.12}$$

It is not hard to show that $\{R_{s,j}\}_{j=0}^\infty$ forms a complete basis in $L^2_{\rho_s}(\mathbb{R}_+)$. For any function $u \in L^2_{\rho_s}(\mathbb{R}_+)$, the following expansion holds

$$u(x) = \sum_{j=0}^\infty \hat{u}_{s,j}R_{s,j}(x) \text{ with } \hat{u}_{s,j} = \frac{2j+1}{2}\int_0^\infty u(x)R_{s,j}(x)\rho_s(x)dx. \tag{3.2.13}$$

In the sequel, $k(x,t,u(t))$ in Eq. (3.2.1) will be assumed, so that $k_s : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_s(x,t,u(t)) = \frac{k(x,t,u(t))}{\rho_s(t)},$$

is a bounded function. This allows us to rearrange Eq. (3.2.1) into the following form:

$$u(x) = a(x) + f(x,u(x))\int_0^\infty k_s(x,t,u(t))\rho_s(t)dt, \quad x \in [0,\infty). \tag{3.2.14}$$

3.2.2 Rational Lagrange interpolation

First, we define $\mathbb{X}^s_N$ the finite dimensional approximation subspace spanned for a given positive integer N by the set of rational Legendre functions as

$$\mathbb{X}^s_N := \{v \mid v(x) = \phi(\eta_s(x)), \forall \phi \in \mathcal{P}_N\}. \tag{3.2.15}$$

The set of rational Legendre-Gauss $\{\zeta^s_{N,i}\}_{i=0}^N$, which is defined as

$$\zeta^s_{N,i} = \varphi_s(\sigma^N_i), \quad 0 \leqslant i \leqslant N. \tag{3.2.16}$$

Applying a mapping (3.2.7) to the quadrature formula (3.2.6) leads to the rational Legendre-Gauss quadrature:

$$\int_0^\infty v(x)\rho_s(x)dx = \sum_{i=0}^N v(\zeta^s_{N,i})\omega^N_i, \quad \forall v \in \mathbb{X}^s_{2N+1}. \tag{3.2.17}$$

The rational Lagrange basis functions are defined by the following formula:

$$L_{i,s}^{N}(x) = \prod_{j=0,j\neq i}^{N} \frac{\eta_s(x) - \eta_s(\zeta_{N,j}^{s})}{\eta_s(\zeta_{N,i}^{s}) - \eta_s(\zeta_{N,j}^{s})}, \quad 0 \leq i \leq N, \tag{3.2.18}$$

then it is clear that the functions $L_{i,s}^{N}(x)$ satisfy

$$L_{i,s}^{N}(\zeta_{N,j}^{s}) = \delta_{i,j}. \tag{3.2.19}$$

For any $u \in \mathcal{C}(\mathbb{R}^+)$, we can define the Lagrange interpolating polynomial $\mathcal{I}_N^s u \in \mathbb{X}_N^s$, satisfying:

$$\mathcal{I}_N^s u \in \mathbb{X}_N^s \text{ such that } \mathcal{I}_N^s u(\zeta_{N,j}^s) = u(\zeta_{N,j}^s), \quad 0 \leqslant j \leqslant N, \tag{3.2.20}$$

which can be expanded as

$$\mathcal{I}_N^s u(x) = \sum_{i=0}^{N} u(\zeta_{N,i}^s) L_{i,s}^{N}(x). \tag{3.2.21}$$

The following estimate quoted from lemma 5.5 of [92].

Lemma 3.3 *Let $\{L_{i,s}^{N}(x)\}_{i=1}^{N}$ be the $N-$th rational Lagrange interpolation functions associated with the rational Legendre collocation points. Then*

$$\|\mathcal{I}_N^s\|_\infty := \sup_{x\in\mathbb{R}^+} \sum_{i=0}^{N} |L_{i,s}^{N}(x)| = \mathcal{O}(N^{1/2}). \tag{3.2.22}$$

In order to describe the approximation errors, we introduce new differential operators as follows:

$$D_x u = g_s(x)\frac{du}{dx}, \qquad g_s(x) := \frac{dx}{dy}, \tag{3.2.23}$$

and an induction argument leads to

$$D_x^m u = g_s(x)\frac{d}{dx}\left(g_s(x)\frac{d}{dx}\left(\cdots\left(g_s(x)\frac{du}{dx}\right)\cdots\right)\right) = \partial_y^m U_s, \quad m = 0,1,\ldots, \tag{3.2.24}$$

where

$$u(x) = u(\varphi_s(y)) := U_s(y). \tag{3.2.25}$$

To prove error estimates for the above scheme, we begin by defining the following weighted Hilbert space with some useful lemmas about rational Lagrange interpolation based on the rational

Legendre-Gauss points. For a nonnegative integer m, define

$$H^m_{\rho_s}(\mathbb{R}_+) = \{u \mid D^r_x u \in L^2_{\rho_s}(\mathbb{R}_+) \;\; 0 \le r \le m\},$$

related to the following semi-norm and the norm:

$$|u|^s_m = \|D^m_x u\|_{L^2_{\rho_s}(\mathbb{R}_+)}, \;\; \|u\|^s_m = \left(\sum_{r=0}^m \|D^r_x u\|^2_{L^2_{\rho_s}(\mathbb{R}_+)}\right)^{1/2}.$$

Also, it is convenient to introduce the semi-norms

$$|u|^{m;N}_{\rho_s} := |u|_{H^{m;N}_{\rho_s}(\mathbb{R}_+)} = \left(\sum_{r=\min(m,N+1)}^{m} \|D^r_x u\|^2_{L^2_{\rho_s}(\mathbb{R}_+)}\right)^{1/2}.$$

In the following, we prove the below lemma, which estimates the error between the approximate and exact solutions.

Lemma 3.4 *Assume that $u \in H^m_{\rho_s}(\mathbb{R}_+)$ we have*

$$\|u - \mathcal{I}^s_N u\|_{L^2_{\rho_s}(\mathbb{R}_+)} \le cN^{-m}|u|^{m;N}_{\rho_s}, \tag{3.2.26}$$

$$\|u - \mathcal{I}^s_N u\|_\infty \le cN^{1/2-m}|u|^{m;N}_{\rho_s}, \tag{3.2.27}$$

where c is a positive constant independent of N and u.

Proof. Let I_N be the Lagrange interpolation operator associated with the Legendre collocation points, we have for the weighted L^2-norm

$$\|u - \mathcal{I}^s_N u\|^2_{L^2_{\rho_s}(\mathbb{R}_+)} = \int_0^\infty |u(x) - \mathcal{I}^s_N u(x)|^2 \rho_s(x) dx = \int_{-1}^1 |U_s(y) - I_N U_s(y)|^2 dy = \|U_s - I_N U_s\|^2_{L^2(I)}. \tag{3.2.28}$$

Next for the infinity norm, we consider

$$\|u - \mathcal{I}^s_N u\|_\infty = \sup_{x \in \mathbb{R}_+} |u(x) - \mathcal{I}^s_N u(x)| = \sup_{y \in I} |U_s(y) - I_N U_s(y)| = \|U_s - I_N U_s\|_\infty. \tag{3.2.29}$$

According to Lemma 1 of [91], it is mentioned that for any $U_s \in H^m(I)$ and $m \ge 0$,

$$\|U_s - I_N U_s\|_{L^2(I)} \leqslant cN^{-m}|U_s|^{m;N}, \tag{3.2.30}$$

$$\|U_s - I_N U_s\|_\infty \leqslant cN^{1/2-m}|U_s|^{m;N}. \tag{3.2.31}$$

From (3.2.23), (3.2.24) and (3.2.25), we have

$$\|\partial_y^m U_s\|_{L^2(I)}^2 = \int_{-1}^{1} |\partial_y^m U_s(y)|^2 dy = \int_0^\infty |D_x^m u(x)|^2 \rho_s(x) dx = \|D_x^m u\|_{L^2_{\rho_s}(\mathbb{R}_+)}^2. \tag{3.2.32}$$

This implies $|U_s|^{m;N} = |u|_{\rho_s}^{m;N}$. Hence, we obtain the desired estimate, i.e.,

$$\|u - \mathcal{I}_N^s u\|_{L^2_{\rho_s}(\mathbb{R}_+)} \le cN^{-m} |u|_{\rho_s}^{m;N}, \tag{3.2.33}$$

$$\|u - \mathcal{I}_N^s u\|_\infty \le cN^{1/2-m} |u|_{\rho_s}^{m;N}. \tag{3.2.34}$$

This completes the proof.

3.2.3 Existence and uniqueness

In order to consider the Eq. (3.2.1) in the weighted space $L^2_{\rho_s}(\mathbb{R}_+)$, we will adopt the following assumptions:

C1. There exist a continuous and bounded function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|k(x,t,y) - k(x,t,z)| \le g(x,t)|y - z|, \text{ for all } x,t \in \mathbb{R}_+.$$

C2. For every $x \ge 0$,

$$\int_0^\infty |g(x,t)|^2 (t+s)^2 dt \le M_1 < \infty \text{ for all } s > 0.$$

C3. For every $x \ge 0$,

$$\int_0^\infty |k(x,t,u(t))| dt \le M_2 < \infty.$$

From now onwards, we make the following assumptions on the nonlinear function $f(.,u(.))$:

C4. $f(x,z)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}$ and bounded:

$$\sup_{x \in \mathbb{R}_+} |f(x,u(x))| \le M_3.$$

C5. The functions $f(x,u(x))$ is Lipschitz continuous in u i.e., for any $u, v \in L^2_{\rho_s}(\mathbb{R}_+)$, there exist constants L such that

$$|f(x,u(x)) - f(x,v(x))| \le L|u(x) - v(x)|.$$

Next, we define the operator $\mathcal{T}$ on $L^2_{\rho_s}(\mathbb{R}_+)$ by

$$\mathcal{T}(u) := F(u)\mathcal{K}(u) + a, \text{ where } \mathcal{K}(u)(x) = \int_0^\infty k(x,t,u(t))dt \text{ and } F(u)(x) = f(x,u(x)),$$

so that Eq. (3.2.1) can be written as

$$\mathcal{T}(u) = u. \tag{3.2.35}$$

Theorem 3.2 *Assume $\mathcal{K} : L^2_{\rho_s}(\mathbb{R}_+) \to L^2_{\rho_s}(\mathbb{R}_+)$ is bounded, if the following condition holds:*

$$\left(LM_2 + M_3\sqrt{\frac{M_1}{s}}\right) < 1. \tag{3.2.36}$$

Then Eq. (3.2.1) has a unique solution in $L^2_{\rho_s}(\mathbb{R}_+)$ for all $a \in L^2_{\rho_s}(\mathbb{R}_+)$.

Proof. For all $u, v \in L^2_{\rho_s}(\mathbb{R}_+)$, we have

$$\begin{aligned}|\mathcal{T}(u)(x) - \mathcal{T}(v)(x)| &= |F(u)(x)\mathcal{K}(u)(x) - F(v)(x)\mathcal{K}(v)(x)| \\ &= |F(u)(x)\mathcal{K}(u)(x) - F(v)(x)\mathcal{K}(u)(x) + F(v)(x)\mathcal{K}(u)(x) - F(v)(x)\mathcal{K}(v)(x)| \\ &\leq |F(u)(x) - F(v)(x)||\mathcal{K}(u)(x)| + |F(v)(x)||\mathcal{K}(u)(x) - \mathcal{K}(v)(x)|.\end{aligned} \tag{3.2.37}$$

From Lipschitz continuity of f and the assumptions C1, C3 and C4, we get

$$\begin{aligned}|\mathcal{T}(u)(x) - \mathcal{T}(v)(x)| &\leq L|u(x) - v(x)||\mathcal{K}(u)(x)| + M_3|\mathcal{K}(u)(x) - \mathcal{K}(v)(x)| \\ &\leq LM_2|u(x) - v(x)| + M_3\int_0^\infty |g(x,t)||u(t) - v(t)|dt.\end{aligned} \tag{3.2.38}$$

By applying Cauchy-Schwarz inequality and using assumption C2, we obtain

$$\int_0^\infty |g(x,t)||u(t) - v(t)|dt \leq \left(\int_0^\infty |g(x,t)|^2 \frac{1}{\rho_s(t)} dt\right)^{1/2} \left(\int_0^\infty |u(t) - v(t)|^2 \rho_s(t) dt\right)^{1/2} \leq \sqrt{\frac{M_1}{2s}} \|u - v\|_{L^2_{\rho_s}(\mathbb{R}_+)}. \tag{3.2.39}$$

This implies

$$|\mathcal{T}(u)(x) - \mathcal{T}(v)(x)| \leq LM_2|u(x) - v(x)| + M_3\sqrt{\frac{M_1}{2s}}\|u - v\|_{L^2_{\rho_s}(\mathbb{R}_+)}.$$

Hence

$$\|\mathcal{T}(u)-\mathcal{T}(v)\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq \left(LM_2+M_3\sqrt{\frac{M_1}{s}}\right)\|u-v\|_{L^2_{\rho_s}(\mathbb{R}_+)}. \tag{3.2.40}$$

It follows that for $\left(LM_2+M_3\sqrt{\frac{M_1}{s}}\right)<1$, $\mathcal{T}$ is a contraction operator, so that it has a unique fixed point and that fixed point is the solution of of Eq. (3.2.1).

3.2.4 Rational Legendre collocation method

An approximate solution of Eq. (3.2.1) may be obtained by simply collocating, that is forcing Eq. (3.2.14) to be exact at the rational Legendre-Gauss points $\{\zeta^s_{N,j}\}^N_{j=0}$, namely

$$u(\zeta^s_{N,j}) = a(\zeta^s_{N,j}) + f(\zeta^s_{N,j}, u(\zeta^s_{N,j}))\int_0^\infty k_s(\zeta^s_{N,j}, t, u(t))\rho_s(t)dt, \quad j=0\cdots N. \tag{3.2.41}$$

By applying the rational Legendre-Gauss quadrature formula (3.2.17) to the above equation, we obtain

$$u(\zeta^s_{N,j}) = a(\zeta^s_{N,j}) + f(\zeta^s_{N,j}, u(\zeta^s_{N,j}))\sum_{i=0}^N k_s(\zeta^s_{N,j}, \zeta^s_{N,i}, u(\zeta^s_{N,i}))\omega_i^N, \quad j=0\cdots N, \tag{3.2.42}$$

Using $u^N_{s,j}, 0\leq j\leq N$, to approximate the function value $u(\zeta^s_{N,j})$, and use

$$u^N_s(x) = \sum_{j=0}^N u^N_{s,j}L^N_{j,s}(x), \tag{3.2.43}$$

to approximate the function $u(x)$, namely

$$u(\zeta^s_{N,j}) \sim u^N_{s,j}, \quad u(x)\sim u^N_s(x). \tag{3.2.44}$$

Then, the discrete spectral Legendre-collocation method for solving Eq. (3.2.1) leads to the following fully discrete problem

$$u^N_{s,j} = a(\zeta^s_{N,j}) + f(\zeta^s_{N,j}, u^N_{s,j})\sum_{i=0}^N k_s(\zeta^s_{N,j}, \zeta^s_{N,i}, u^N_{s,i})\omega_i^N, \quad j=0\cdots N, \tag{3.2.45}$$

which is a nonlinear system of the form

$$\mathrm{u} = \mathbf{H}(\mathrm{u}), \quad \mathbf{H}(\mathrm{u}) = \mathbf{A} + \mathbf{F}(\mathrm{u})\mathbf{M}(\mathrm{u})\mathbf{W}, \tag{3.2.46}$$

where $\mathbf{M}, \mathbf{W}, \mathbf{A}$ and $\mathbf{F}$ are given by:

$$\mathbf{M}(\mathrm{u}) = (k_s(\zeta^s_{N,j}, \zeta^s_{N,i}, u^N_{s,j}))_{0\le i,j\le N}, \ \mathbf{W} = \operatorname{diag}((\omega^N_i)_{0\le i\le N}), \ \mathbf{A} = (a(\zeta^s_{N,j}))_{0\le j\le N},$$

$$\mathbf{F}(\mathrm{u}) = \operatorname{diag}((f(\zeta^s_{N,j}, u^N_{s,j}))_{0\le j\le N}),$$

and the unknown is the vector $\mathrm{u} \equiv [u^N_{s,0}, u^N_{s,1}, \ldots, u^N_{s,N}]^T$.

To achieve a highly accurate numerical solution of (3.2.46), we would need to apply the following iterative process

$$\mathrm{u}^{(k)} = \mathbf{H}(\mathrm{u}^{(k-1)}), \tag{3.2.47}$$

with the initial value $\mathrm{u}^{(0)} = \mathbf{A}$.

3.2.5 Convergence analysis

In this section we provided error analysis for the proposed method to indicate its exponential rate of convergence, provided that a and f are bounded sufficiently smooth functions. In order to do that, the above assumptions are taken into account.

Theorem 3.3 *Let u be the exact solution to Eq. (3.2.1) and u^N_s be the approximate solution obtained by using the spectral-collocation scheme (3.2.45). For $m \ge 1$, assume that $D^r_t k_s(x, t, I^s_N u(t)) \in L^2_{\rho_s}(\mathbb{R}_+)$ for $\min(m, N+1) \le r \le m$, If $u \in H^m_{\rho_s}(\mathbb{R}_+)$, then*

$$\|u - u^N_s\|_{L^2_{\rho_s}(\mathbb{R}_+)} = \mathcal{O}(N^{-m}), \quad \|u - u^N_s\|_\infty = \mathcal{O}(N^{\frac{1}{2}-m}). \tag{3.2.48}$$

Proof. Let the quadratic Urysohn integral equation

$$u(x) = a(x) + f(x, u(x)) \int_0^\infty k_s(x, t, u(t))\rho_s(t)dt, \tag{3.2.49}$$

while using the approximate solution, we have

$$u^N_s(x) = \mathcal{I}^s_N a(x) + \mathcal{I}^s_N f(x, \mathcal{I}^s_N u(x)) \int_0^\infty \mathcal{I}^s_{N,N} k_s(x, t, \mathcal{I}^s_N u(t))\rho_s(t)dt. \tag{3.2.50}$$

Subtracting (3.2.50) from (3.2.49), we get the error equation

$$\begin{aligned} u(x)-u_s^N(x) &= a(x)-\mathcal{I}_N^s a(x)+f(x,u(x))\int_0^\infty (k_s(x,t,u(t))-k_s(x,t,\mathcal{I}_N^s u(t)))\rho_s(t)dt \\ &\quad +(f(x,u(x))-f(x,\mathcal{I}_N^s u(x)))\int_0^\infty k_s(x,t,\mathcal{I}_N^s u(t))\rho_s(t)dt \\ &\quad +f(x,\mathcal{I}_N^s u(x))\int_0^\infty (k_s(x,t,\mathcal{I}_N^s u(t))-\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t)))\rho_s(t)dt \\ &\quad +(f(x,\mathcal{I}_N^s u(x))-\mathcal{I}_N^s f(x,\mathcal{I}_N^s u(x)))\int_0^\infty \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t))\rho_s(t)dt \\ &= J_0(x)+J_1(x)+J_2(x)+J_3(x)+J_4(x), \end{aligned} \tag{3.2.51}$$

where

$$J_0(x)=a(x)-\mathcal{I}_N^s a(x), \tag{3.2.52}$$

$$J_1(x)=f(x,u(x))\int_0^\infty (k_s(x,t,u(t))-k_s(x,t,\mathcal{I}_N^s u(t)))\rho_s(t)dt, \tag{3.2.53}$$

$$J_2(x)=(f(x,u(x))-f(x,\mathcal{I}_N^s u(x)))\int_0^\infty k_s(x,t,\mathcal{I}_N^s u(t))\rho_s(t)dt, \tag{3.2.54}$$

$$J_3(x)=f(x,\mathcal{I}_N^s u(x))\int_0^\infty (k_s(x,t,\mathcal{I}_N^s u(t))-\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t)))\rho_s(t)dt, \tag{3.2.55}$$

$$J_4(x)=(f(x,\mathcal{I}_N^s u(x))-\mathcal{I}_N^s f(x,\mathcal{I}_N^s u(x)))\int_0^\infty \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t))\rho_s(t)dt. \tag{3.2.56}$$

By the triangle inequality, we have

$$\|u-u_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)} \le \sum_{k=0}^{4}\|J_k\|_{L^2_{\rho_s}(\mathbb{R}_+)}, \quad \|u-u_s^N\|_\infty \le \sum_{k=0}^{4}\|J_k\|_\infty. \tag{3.2.57}$$

It follows immediately from Lemma 3.4 that

$$\|J_0\|_{L^2_{\rho_s}(\mathbb{R}_+)} \le cN^{-m}|a|_{\rho_s}^{m;N}, \quad \|J_0\|_\infty \le cN^{1/2-m}|a|_{\rho_s}^{m;N}. \tag{3.2.58}$$

On the other hand, by assumptions C1 and C4, we have

$$\begin{aligned} |J_1(x)| &= \left| f(x,u(x))\int_0^\infty (k_s(x,t,u(t))-k_s(x,t,\mathcal{I}_N^s u(t)))\rho_s(t)dt \right| \\ &\le M_3\int_0^\infty |g(x,t)||u(t)-\mathcal{I}_N^s u(t)|dt. \end{aligned}$$

Using Cauchy Schwarz inequality as in (3.2.39), we get

$$\|J_1\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq M_3\sqrt{\frac{M_1}{2s}}\|u-\mathcal{I}^s_N u\|_{L^2_{\rho_s}(\mathbb{R}_+)}\left(\int_0^\infty \rho_s(x)dx\right)^{1/2} = M_3\sqrt{\frac{M_1}{s}}\|u-\mathcal{I}^s_N u\|_{L^2_{\rho_s}(\mathbb{R}_+)}. \tag{3.2.59}$$

Next for the infinity norm, by using assumption C2 we obtain

$$\begin{aligned} |J_1(x)| &\leq M_3\|u-\mathcal{I}^s_N u\|_\infty \int_0^\infty |g(x,t)|dt \\ &\leq M_3\|u-\mathcal{I}^s_N u\|_\infty \left(\int_0^\infty |g(x,t)|^2\frac{1}{\rho_s(t)}dt\right)^{1/2}\left(\int_0^\infty \rho_s(t)dt\right)^{1/2} \\ &\leq M_3\sqrt{\frac{M_1}{s}}\|u-\mathcal{I}^s_N u\|_\infty. \end{aligned} \tag{3.2.60}$$

Then, according to Lemma 3.4, it follows that:

$$\|J_1\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq C_1 N^{-m}|u|^{m;N}_{\rho_s}, \quad \|J_1\|_\infty \leq C_1 N^{1/2-m}|u|^{m;N}_{\rho_s}. \tag{3.2.61}$$

Using assumptions C1, C3 and invoking (3.2.39), we can write

$$\begin{aligned} \int_0^\infty |k(x,t,\mathcal{I}^s_N u(t))|dt &\leq \int_0^\infty |k(x,t,\mathcal{I}^s_N u(t)) - k(x,t,u(t)|dt + \int_0^\infty |k(x,t,u(t)|dt \\ &\leq \int_0^\infty |g(x,t)||u(t)-\mathcal{I}^s_N u(t)|dt + M_2 \leq \sqrt{\frac{M_1}{2s}}\|u-\mathcal{I}^s_N u\|_{L^2_{\rho_s}(\mathbb{R}_+)} + M_2, \end{aligned}$$

so that from C5, we get

$$\begin{aligned} |J_2(x)| &\leq |f(x,u(x) - f(x,\mathcal{I}^s_N u(x))|\int_0^\infty |k_s(x,t,\mathcal{I}^s_N u(t))|\rho_s(t)dt \\ &\leq L|u(x)-\mathcal{I}^s_N u(x)|\int_0^\infty |k(x,t,\mathcal{I}^s_N u(t))|dt \\ &\leq L\left(M_2 + \sqrt{\frac{M_1}{2s}}\|u-\mathcal{I}^s_N u\|_{L^2_{\rho_s}(\mathbb{R}_+)}\right)|u(x)-\mathcal{I}^s_N u(x)|. \end{aligned}$$

Hence by Lemma 3.4, we have

$$\|J_2\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq C_2 N^{-m}|u|^{m;N}_{\rho_s}, \quad \|J_2\|_\infty \leq C_2 N^{1/2-m}|u|^{m;N}_{\rho_s}. \tag{3.2.62}$$

Also, we have

$$|J_3(x)| \leq |f(x,\mathcal{I}^s_N u(x))|\int_0^\infty |e^s_{N,N}(x,t)|\rho_s(t)dt.$$

where $e^s_{N,N}(x,t) = k_s(x,t,\mathcal{I}^s_N u(t)) - \mathcal{I}^s_{N,N} k_s(x,t,\mathcal{I}^s_N u(t))$. By using assumptions C4 and C5 we get

$$|f(x,\mathcal{I}^s_N u(x))| \leq |f(x,u(x))| + |f(x,\mathcal{I}^s_N u(x)) - f(x,u(x))| \leq M_3 + L\|u - \mathcal{I}^s_N u\|_\infty.$$

Thus

$$|J_3(x)| \leq (M_3 + L\|u - \mathcal{I}^s_N u\|_\infty) \int_0^\infty |e^s_{N,N}(x,t)|\rho_s(t)dt.$$

Finally, by using Cauchy-Schwarz inequality we write

$$|J_3(x)| \leq \sqrt{2}(M_3 + L\|u - \mathcal{I}^s_N u\|_\infty)\|e^s_{N,N}(x,.)\|_{L^2_{\rho_s}(\mathbb{R}_+)}.$$

Now, from (3.2.26) we have

$$\|e^s_{N,N}(x,.)\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq cN^{-m}|k_s(x,.,\mathcal{I}^s_N u(.))|^{m;N}_{\rho_s}.$$

Hence, we get

$$\begin{aligned}\|J_3\|_{L^2_{\rho_s}(\mathbb{R}_+)} &\leq \sqrt{2}(M_3 + L\|u - \mathcal{I}^s_N u\|_\infty)cN^{-m}|k_s(x,.,\mathcal{I}^s_N u(.))|^{m;N}_{\rho_s}\left(\int_0^\infty \rho_s(x)dx\right)^{1/2} \\ &\leq 2(M_3 + L\|u - \mathcal{I}^s_N u\|_\infty)cN^{-m}|k_s(x,.,\mathcal{I}^s_N u(.))|^{m;N}_{\rho_s}.\end{aligned}$$

For the infinity norm, we use (3.2.27) to get

$$\begin{aligned}\|J_3\|_\infty &\leq (M_3 + L\|u - \mathcal{I}^s_N u\|_\infty)\|e^s_{N,N}(x,.)\|_\infty \int_0^\infty \rho_s(t)dt \\ &\leq 2(M_3 + L\|u - \mathcal{I}^s_N u\|_\infty)cN^{1/2-m}|k_s(x,.,\mathcal{I}^s_N u(.))|^{m;N}_{\rho_s}.\end{aligned}$$

Hence

$$\|J_3\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq C_3 N^{-m}|k_s(x,.,\mathcal{I}^s_N u(.))|^{m;N}_{\rho_s}, \tag{3.2.63}$$

$$\|J_3\|_\infty \leq C_3 N^{1/2-m}|k_s(x,.,\mathcal{I}^s_N u(.))|^{m;N}_{\rho_s}. \tag{3.2.64}$$

To estimate J_4 using Lemma 3.3, we obtain the following expression:

$$\begin{aligned}|J_4(x)| &\leq |f(x,\mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x,\mathcal{I}_N^s u(x))| \int_0^\infty |\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t))| \rho_s(t)dt \\ &\leq |f(x,\mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x,\mathcal{I}_N^s u(x))| \sup_{x,t\in\mathbb{R}_+} |\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t))| \int_0^\infty \rho_s(t)dt \\ &\leq 2|f(x,\mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x,\mathcal{I}_N^s u(x))| \|\mathcal{I}_N^s\|_\infty^2 \sup_{0\leq i,j\leq N} |k_s(\zeta_{N,j}^s,\zeta_{N,i}^s,\mathcal{I}_N^s u(\zeta_{N,i}^s))|. \quad (3.2.65)\end{aligned}$$

Therefore, by using Lemma 3.4, we can derive the following result:

$$\|J_4\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq C_4 N^{-m} \|\mathcal{I}_N^s\|_\infty^2 \sup_{0\leq i,j\leq N} |k_s(\zeta_{N,j}^s,\zeta_{N,i}^s,\mathcal{I}_N^s u(\zeta_{N,i}^s))| |f|_{\rho_s}^{m;N}, \quad (3.2.66)$$

$$\|J_4\|_\infty \leq C_4 N^{1/2-m} \|\mathcal{I}_N^s\|_\infty^2 \sup_{0\leq i,j\leq N} |k_s(\zeta_{N,j}^s,\zeta_{N,i}^s,\mathcal{I}_N^s u(\zeta_{N,i}^s))| |f|_{\rho_s}^{m;N}. \quad (3.2.67)$$

Finally, the statement of the theorem follows from the triangle inequality.

3.2.6 Stability of the RLCM

In this subsection, we focus on discussing the stability of the RLC method for solving integral equation (3.2.1). To analyze the stability, we introduce a function $a_\varepsilon(x) = a(x) + \varepsilon$. This addition allows us to use the impulse from theorem 3.2, which guarantees the existence of a solution u_ε. The solution u_ε is obtained by satisfying the following equation:

$$u_\varepsilon(x) = a_\varepsilon(x) + f(x,u_\varepsilon(x)) \int_0^\infty k(x,t,u_\varepsilon(t))dt, \quad x \in [0,\infty). \quad (3.2.68)$$

Applying the RLC method to the perturbed problem (3.2.68) we can obtain the corresponding scheme

$$u_{s,\varepsilon}^N(x) = \mathcal{I}_N^s a_\varepsilon(x) + \mathcal{I}_N^s f(x,\mathcal{I}_N^s u_\varepsilon(x)) \int_0^\infty \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t)) \rho_s(t)dt. \quad (3.2.69)$$

Definition 3.1 *The numerical method is said to be stable if*

$$\|u_s^N - u_{s,\varepsilon}^N\| \leq C\|a - a_\varepsilon\|, \quad (3.2.70)$$

where $u_{s,\varepsilon}^N$ *is the numerical solution of the perturbed problem.*

In the following theorem, we focus solely on demonstrating the maximum norm stability of the RLC method because the same argument can be applied in the $L^2_{\rho_s}-$norm. For notational convenience, let us denote

$$\gamma_s = LM_2 + M_3\sqrt{\frac{M_1}{s}}.$$

Theorem 3.4 *Let* $\{u_s^N\}$ *and* $\{u_{s,\varepsilon}^N\}$ *be two sequences of numerical solutions obtained by the RLC schemes* (3.2.50) *and* (3.2.69), *respectively. Further assume*

$$\sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2 < \infty. \tag{3.2.71}$$

If $\gamma_s < 1$, *then we have*

$$\|u_s^N - u_{s,\varepsilon}^N\|_\infty \le \|\mathcal{I}_N^s\|_\infty \left(1 + \frac{\tau_s}{1-\gamma_s}\|\mathcal{I}_N^s\|_\infty^3\right)\varepsilon, \tag{3.2.72}$$

where

$$\tau_s = 2L\sup_{0\le i,j\le N}|k_s(\zeta_{N,j}^s,\zeta_{N,i}^s,\mathcal{I}_N^s u_\varepsilon(\zeta_{N,i}^s))| + \frac{1}{s}\sup_{0\le j\le N}|f(\zeta_{N,j}^s,\mathcal{I}_N^s u_\varepsilon(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2.$$

Proof. Note that $\|a-a_\varepsilon\|_\infty = \varepsilon$. By (3.2.50) and (3.2.69), we have

$$\begin{aligned} u_s^N(x) - u_{s,\varepsilon}^N(x) &= \mathcal{I}_N^s a(x) - \mathcal{I}_N^s a_\varepsilon(x) \\ &\quad + (\mathcal{I}_N^s f(x,\mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x,\mathcal{I}_N^s u_\varepsilon(x)))\int_0^\infty \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t))\rho_s(t)dt \\ &\quad + \mathcal{I}_N^s f(x,\mathcal{I}_N^s u_\varepsilon(x))\int_0^\infty (\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t)) - \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t)))\rho_s(t)dt.. \end{aligned}$$

Let us denote

$$J_0^\varepsilon(x) = \mathcal{I}_N^s a(x) - \mathcal{I}_N^s a_\varepsilon(x), \tag{3.2.73}$$

$$J_1^\varepsilon(x) = (\mathcal{I}_N^s f(x,\mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x,\mathcal{I}_N^s u_\varepsilon(x)))\int_0^\infty \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t))\rho_s(t)dt, \tag{3.2.74}$$

$$J_2^\varepsilon(x) = \mathcal{I}_N^s f(x,\mathcal{I}_N^s u_\varepsilon(x))\int_0^\infty (\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t)) - \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t)))\rho_s(t)dt. \tag{3.2.75}$$

where

$$u_s^N(x) - u_{s,\varepsilon}^N(x) = J_0^\varepsilon(x) + J_1^\varepsilon(x) + J_2^\varepsilon(x).$$

For J_0^ε, it is easy to get

$$\|J_0^\varepsilon\|_\infty \leq \|\mathcal{I}_N^s\|_\infty \|a - a_\varepsilon\|_\infty \leq \|\mathcal{I}_N^s\|_\infty \varepsilon. \tag{3.2.76}$$

Before starting the estimation of J_1^ε and J_2^ε, we need to estimate $u - u_\varepsilon$ in the infinity norm, where u is the exact solution of equation (3.2.1) and u_ε is the exact solution of the perturbed problem (3.2.69), under the above assumptions as follows.

$$|u(x) - u_\varepsilon(x)| \leq LM_2|u(x) - u_\varepsilon(x)| + M_3 \int_0^\infty |g(x,t)||u(t) - u_\varepsilon(t)|dt + |a(x) - a_\varepsilon(x)|.$$

By applying Cauchy-Schwarz inequality, we get

$$\|u - u_\varepsilon\|_\infty \leq \gamma_s \|u - u_\varepsilon\|_\infty + \|a - a_\varepsilon\|_\infty. \tag{3.2.77}$$

This implies

$$\|u - u_\varepsilon\|_\infty \leq \frac{\|a - a_\varepsilon\|_\infty}{(1 - \gamma_s)} \leq \frac{\varepsilon}{(1 - \gamma_s)}. \tag{3.2.78}$$

To estimate J_1^ε, we can write

$$\begin{aligned} |J_1^\varepsilon(x)| &\leq |\mathcal{I}_N^s f(x, \mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x, \mathcal{I}_N^s u_\varepsilon(x))| \int_0^\infty |\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t))| \rho_s(t) dt \\ &\leq |\mathcal{I}_N^s f(x, \mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x, \mathcal{I}_N^s u_\varepsilon(x))| \sup_{x,t \in \mathbb{R}_+} |\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t))| \int_0^\infty \rho_s(t) dt \\ &\leq 2\|\mathcal{I}_N^s\|_\infty^2 \sup_{0 \leq i,j \leq N} |k_s(\zeta_{N,j}^s, \zeta_{N,i}^s, \mathcal{I}_N^s u_\varepsilon(\zeta_{N,i}^s))| |\mathcal{I}_N^s (f(x, \mathcal{I}_N^s u(x)) - f(x, \mathcal{I}_N^s u_\varepsilon(x)))|. \end{aligned}$$

Using the assumption C5, we obtain

$$\|J_1^\varepsilon\|_\infty \leq 2L\|\mathcal{I}_N^s\|_\infty^4 \sup_{0 \leq i,j \leq N} |k_s(\zeta_{N,j}^s, \zeta_{N,i}^s, \mathcal{I}_N^s u_\varepsilon(\zeta_{N,i}^s))| \|u - u_\varepsilon\|_\infty. \tag{3.2.79}$$

Hence, by (3.2.78), we get

$$\|J_1^\varepsilon\|_\infty \leq 2L\|\mathcal{I}_N^s\|_\infty^4 \sup_{0 \leq i,j \leq N} |k_s(\zeta_{N,j}^s, \zeta_{N,i}^s, \mathcal{I}_N^s u_\varepsilon(\zeta_{N,i}^s))| \frac{\varepsilon}{(1 - \gamma_s)}. \tag{3.2.80}$$

In order to estimate J_2^ε, we use assumption C1 and C4, which allows us to derive that

$$\begin{aligned}
|J_2^\varepsilon(x)| &\leq |\mathcal{I}_N^s f(x,\mathcal{I}_N^s u_\varepsilon(x))| \int_0^\infty |\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t)) - \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t))|\rho_s(t)dt \\
&\leq \|\mathcal{I}_N^s\|_\infty \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s,\mathcal{I}_N^s u_\varepsilon(\zeta_{N,j}^s))| \int_0^\infty |\mathcal{I}_{N,N}^s(k_s(x,t,\mathcal{I}_N^s u(t)) - k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t))|\rho_s(t)dt \\
&\leq \|\mathcal{I}_N^s\|_\infty^3 \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s,\mathcal{I}_N^s u_\varepsilon(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} |k_s(x,t,\mathcal{I}_N^s u(t)) - k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t))| \int_0^\infty \rho_s(t)dt \\
&\leq \frac{1}{s}\|\mathcal{I}_N^s\|_\infty^3 \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s,\mathcal{I}_N^s u_\varepsilon(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2 |\mathcal{I}_N^s(u(t)-u_\varepsilon(t))| \\
&\leq \frac{1}{s}\|\mathcal{I}_N^s\|_\infty^4 \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s,\mathcal{I}_N^s u_\varepsilon(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2 \|u-u_\varepsilon\|_\infty.
\end{aligned} \tag{3.2.81}$$

Then, we have

$$\|J_2^\varepsilon\|_\infty \leq \frac{1}{s}\|\mathcal{I}_N^s\|_\infty^4 \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s,\mathcal{I}_N^s u_\varepsilon(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2 \|u-u_\varepsilon\|_\infty, \tag{3.2.82}$$

Hence, using (3.2.78), we get

$$\|J_2^\varepsilon\|_\infty \leq \frac{1}{s}\|\mathcal{I}_N^s\|_\infty^4 \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s,\mathcal{I}_N^s u_\varepsilon(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2 \frac{\varepsilon}{(1-\gamma_s)}. \tag{3.2.83}$$

Finally, by using the triangle inequality, we obtain the desired result.

3.2.7 Illustrative examples

In this section, we provide numerical examples to illustrate the practical application of our theoretical results. All computations were carried out using Matlab. For the subsequent part of our analysis, we investigate the stability of the system at specific points by considering various values of the perturbation parameter ε. This examination allows us to assess the system's robustness and sensitivity to perturbations. Furthermore, we introduce the notation $e_s^N = u - u_s^N$ to represent the error between the exact solution u and the numerical solution u_s^N obtained using our proposed method.

Example 11 Let us consider the quadratic Urysohn integral equation

$$u(x) = \frac{x+2}{x+3}e^{-x} + u(x)\int_0^{+\infty} e^{-t(x+1)}u^2(t)dt, \quad x \in [0,\infty). \tag{3.2.84}$$

In order to confirm the effectiveness of our method, we present this example with the following known smooth exact solution: $u(x) = e^{-x}$. This choice of exact solution allows us to assess the accuracy and reliability of our approach in solving the quadratic Urysohn integral equation. To evaluate the performance of our approach, we conducted a series of experiments using the RLC method described above. The obtained results are summarized in Table 3.4. We can observe that they are in good accordance with the theoretical analysis provided by Theorem 3.3.

Table 3.4: Comparison of errors for Example 11.

	$s=1$		$s=2$		$s=3$	
N	$\|e_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|e_s^N\|_\infty$	$\|e_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|e_s^N\|_\infty$	$\|e_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|e_s^N\|_\infty$
4	1.01e−02	3.35e−01	5.06e−03	6.33e−02	5.81e−03	1.54e−01
8	9.99e−04	3.95e−02	3.60e−04	1.19e−02	1.40e−04	4.05e−03
16	2.13e−05	7.90e−04	2.87e−06	8.66e−05	9.64e−07	1.40e−05
32	6.00e−08	2.24e−06	2.26e−09	7.44e−08	2.02e−10	1.13e−09
64	5.04e−12	8.56e−11	2.32e−14	7.16e−13	1.60e−14	2.13e−13

Example 12 [47] Let us consider the following quadratic Urysohn integral equation:

$$u(x) = xe^{-4x^2} + \arctan(x + u(x)) \int_0^{+\infty} e^{-t(x+1)} u^2(t)dt, \quad x \in [0, \infty). \tag{3.2.85}$$

In Table 3.5, we present the numerical errors obtained by computing the $L^2_{\rho_s}$ norm and the infinity norm of the difference between u_s^{256} and u_s^N using the RLC scheme for various $s-$parameter. These results indicate that the spectral accuracy is obtained for this problem. Furthermore, Table 3.6 evaluates u_s^N at various points using the RLC method with $s = 3/2$. Additionally, we plot the numerical solution for $n = 128$ and $s = 3/2$ (refer to Figure 3.4).

Example 13 [97] Let us consider the following quadratic Urysohn integral equation:

$$u(x) = \frac{x}{x^2+4} + u(x)^4 \int_0^{+\infty} \frac{u(t)}{(1+x+t)^2(1+u^2(t))} dt, \quad x \in [0, \infty). \tag{3.2.86}$$

The numerical errors for the $L^2_{\rho_s}$ norm and the infinity norm are presented in Table 3.7. The analysis of the obtained results also demonstrates that the RLC method yields excellent results with

Table 3.5: Comparison of errors for Example 12 with $E_s^N = u_s^{256} - u_s^N$.

	$s = 3/2$		$s = 2$		$s = 3$	
N	$\|E_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|E_s^N\|_\infty$	$\|E_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|E_s^N\|_\infty$	$\|E_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|E_s^N\|_\infty$
4	3.80e−02	6.63e−02	4.40e−02	6.71e−02	6.18e−02	1.80e−01
8	8.84e−03	3.07e−02	8.76e−03	2.52e−02	1.57e−02	5.97e−02
16	3.28e−04	5.52e−04	3.73e−04	6.17e−04	6.51e−04	1.17e−03
32	4.10e−07	2.20e−06	5.46e−07	1.29e−06	1.13e−06	7.09e−06
64	1.72e−12	9.21e−12	8.08e−12	5.89e−12	3.18e−11	6.92e−12
128	8.11e−15	2.22e−15	6.81e−14	6.22e−15	2.71e−13	7.70e−15

Table 3.6: Some values of $u_s^N(x)$ at selected points for Example 12

	x			
N	0.5	5	10	15
4	0.1	0.00	0.0	0.0
8	0.1	0.00	0.00	0.00
16	0.191	0.00	0.001	0.000
32	0.19115	0.0040215	0.001356	0.00057
64	0.19115470283	0.0040215733	0.00135659880	0.000574038487
128	0.1911547028388	0.004021573357	0.0013565988032	0.00057403848735

all choices of s. To provide additional clarity, we further examine the values of u_s^N at various points using the RLC method with $s = 2$ in Table 3.8. Moreover, Figure 3.5 represents the corresponding numerical solution plotted over the interval $[0, 30]$.

Table 3.7: Comparison of errors for Example 13 with $E_s^N = u_s^{128} - u_s^N$.

	$s = 1$		$s = 2$		$s = 3$	
N	$\|E_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|E_s^N\|_\infty$	$\|E_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|E_s^N\|_\infty$	$\|E_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|E_s^N\|_\infty$
4	1.13e−02	3.02e−02	2.35e−02	7.13e−02	2.06e−02	9.21e−02
8	5.58e−04	1.55e−03	1.03e−03	1.31e−03	2.03e−03	1.10e−02
16	3.87e−06	6.29e−06	9.40e−06	1.33e−05	2.46e−05	6.53e−05
32	4.02e−10	1.75e−09	1.96e−09	1.14e−08	1.04e−08	7.67e−08
64	7.40e−16	1.44e−15	6.80e−16	1.22e−15	3.59e−15	3.80e−14

Example 14 [49] Let us consider the following quadratic Urysohn integral equation:

$$u(x) = xe^{-x} + \frac{\sqrt{u^2(x)+1}}{x+1}\int_0^{+\infty} e^{-(x+t+1)}\sqrt{1+|u(t)|}dt, \quad x \in [0, \infty). \tag{3.2.87}$$

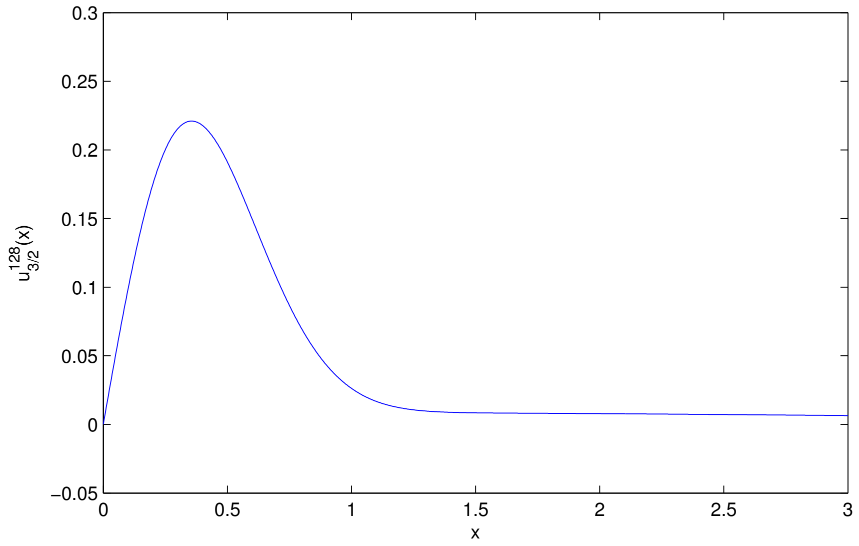


Figure 3.4: Numerical solution of Example 12 with $N = 128$ and $s = 3/2$

Table 3.8: Some values of $u_s^N(x)$ at selected points for Example 13

	x			
N	0.5	5	10	15
4	0.4	0.19	0.09	0.06
8	0.40	0.19	0.09	0.06
16	0.4043	0.1923	0.099	0.06637
32	0.40436119	0.19234757	0.099011074518	0.066371815
64	0.404361192947411	0.19234757172204	0.09901107471898	0.066371815933000

Table 3.9 presents the $L^2_{\rho_s}$ and infinity errors, indicating the discrepancy between u_s^{128} and u_s^N obtained using the RLC scheme with various s. The results reveal that the numerical errors for $s = 3$ and $s = 4$ exhibit remarkable rapprochement, outperforming those obtained for $s = 2$. Consequently, we perform an evaluation of u_s^N at specific points using the RLC method with $s = 3$, as outlined in Table 3.10. Furthermore, we provide the absolute values of the RLC coefficients and illustrate the numerical solution in Figure 3.6.

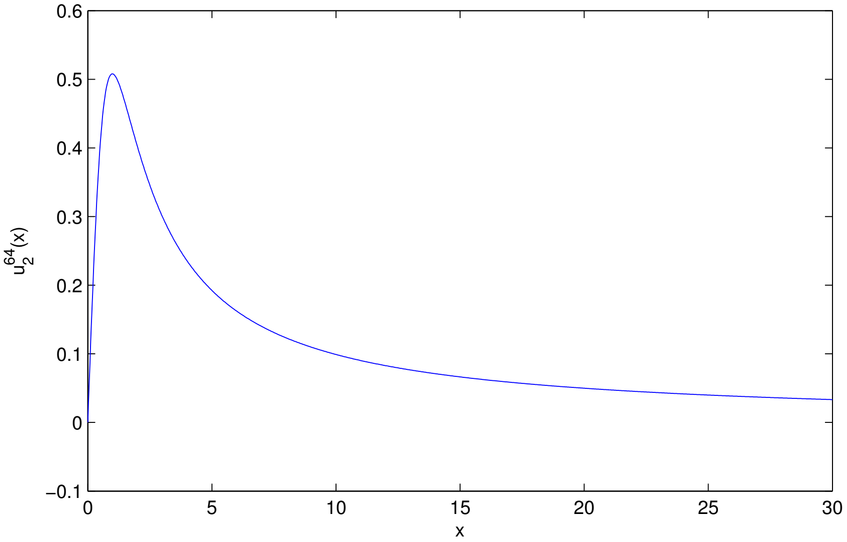


Figure 3.5: Numerical solution of Example 13 with $N = 64$ and $s = 2$

Table 3.9: Comparison of errors for Example 14 with $E_s^N = u_s^{128} - u_s^N$.

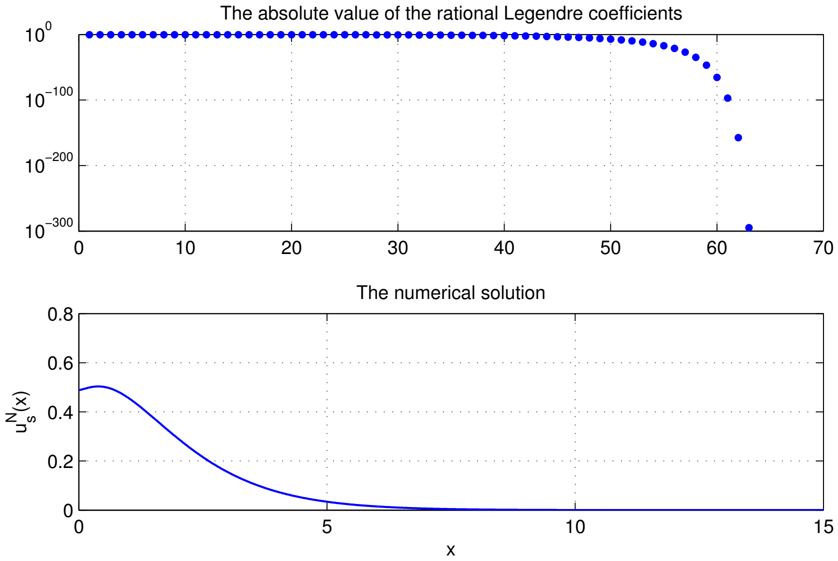
	$s=2$		$s=3$		$s=4$	
N	$\|E_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|E_s^N\|_\infty$	$\|E_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|E_s^N\|_\infty$	$\|E_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|E_s^N\|_\infty$
4	2.26e−02	3.11e−02	1.99e−02	5.62e−02	1.64e−02	5.71e−02
8	1.27e−03	3.17e−03	1.31e−03	1.97e−03	7.88e−04	3.15e−03
16	2.68e−05	6.15e−05	9.47e−06	2.00e−05	3.53e−06	7.65e−06
32	2.42e−08	7.78e−08	3.17e−09	7.16e−09	5.81e−10	2.11e−09
64	4.47e−13	1.56e−12	1.65e−14	3.14e−14	1.01e−15	3.42e−14

Table 3.10: Some values of $u_s^N(x)$ at selected points for Example 14

	x			
N	0.5	5	10	15
8	0.501	0.034	0.000	0.000
16	0.5018	0.03418	0.00045	0.00000
32	0.501879827	0.034183019	0.0004558	0.00000459
64	0.5018798271510	0.034183019457366	0.00045581118021	0.000004596928060
128	0.50187982715105	0.0341830194573660	0.000455811180218	0.0000045969280602

3.2.8 Stability results

In order to demonstrate the stability of the examples, we investigate the effect of perturbation ε on the non-linear system of algebraic equations (3.2.41). We specifically focus on the input perturbation $(\mathbf{A} + \varepsilon)$ and observe that the output of the system undergoes minimal changes. The


The absolute value of the rational Legendre coefficients

The numerical solution

Figure 3.6: Numerical results of RLC-scheme for Example 14 with $N = 64$ and $s = 3$

stability of Example 12 is shown in Table 3.11 for different values of $\varepsilon = 10^{-2}, 10^{-3}$, and 10^{-4}. The same principle is applied to the third example, as presented in Table 3.13. Remarkably, we observe that the approximate solutions exhibit negligible variation across various values of perturbation ε. To provide further insight, we have included the relative errors for Examples 2 and 3 obtained through the RLC method in Tables 3.12 and 3.14 for $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$, and 10^{-4}, respectively. Also we denote

$$R^{\varepsilon}_{\rho_s} = \frac{\|u^N_s - u^N_{s,\varepsilon}\|_{L^2_{\rho_s}(\mathbb{R}_+)}}{\|u^N_s\|_{L^2_{\rho_s}(\mathbb{R}_+)}}, \quad R^{\varepsilon}_{\infty} = \frac{\|u^N_s - u^N_{s,\varepsilon}\|_{\infty}}{\|u^N_s\|_{\infty}}.$$

Table 3.11: Stability results of Example 12 with $s = 3/2$

x	u^N_s	$u^N_s(\varepsilon = 10^{-2})$	$u^N_s(\varepsilon = 10^{-3})$	$u^N_s(\varepsilon = 10^{-4})$
0.5	0.1911547	0.2022581	0.1922591	0.1912651
5	0.0040216	0.0146030	0.0050771	0.0041271
10	0.0013566	0.0115974	0.0023793	0.0014588
20	0.0002872	0.0103658	0.0012944	0.0003880
40	0.0000449	0.0100682	0.0010469	0.0001451
80	0.0000061	0.0100131	0.0010066	0.0001061

Table 3.12: The relative errors for Example 12 with $s = 3/2$ and $N = 128$.

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$R^{\varepsilon}_{\rho_s}$	1.19e−00	1.14e−01	1.13e−02	1.13e−03
R^{ε}_{∞}	5.44e−01	5.13e−02	5.10e−03	5.10e−04

Table 3.13: Stability results of Example 13 with $s = 2$ and $N = 64$.

x	u_s^N	$u_s^N(\varepsilon = 10^{-2})$	$u_s^N(\varepsilon = 10^{-3})$	$u_s^N(\varepsilon = 10^{-4})$
10	0.0990111	0.0990013	0.0990101	0.0990110
20	0.0498753	0.0498741	0.0498752	0.0498753
30	0.0332963	0.0332960	0.0332963	0.0332963
40	0.0249844	0.0249842	0.0249844	0.0249844
50	0.0199920	0.0199919	0.0199920	0.0199920
60	0.0166620	0.0166620	0.0166620	0.0166620
70	0.0142828	0.0142828	0.0142828	0.0142828
80	0.0124980	0.0124980	0.0124980	0.0124980

Table 3.14: The relative errors for Example 13 with $s = 2$ and $N = 64$.

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$R^{\varepsilon}_{\rho_s}$	3.30e−01	3.24e−02	3.24e−03	3.24e−04
R^{ε}_{∞}	2.29e−01	2.17e−02	2.16e−03	2.16e−04

Conclusion and prospects

Spectral methods have developed rapidly in the past two decades. They have been applied successfully to numerical simulations in many fields, such as heat conduction, fluid dynamics, quantum mechanics, and so on. Nowadays, they are some of the most powerful tools for numerical solutions of integral equations. The principal aim is to present the fundamental principles of spectral methods to the solution of integral equations, and to demonstrate the improved convergence and stability obtained with classical basis functions for certain problems.

Firstly, we present an introduction to the terminology and classification of integral equations through certain characteristics and criteria. We saw the aspect of a spectral method, general principle and advantages. We also present some spectral methods namely the collocation and Galerkin method.

Secondly, we recall basic consepts of numerical stability with some examples then we analyze the basic theoretical properties of spectral method (stability, convergence and consistency). We illustrate also the relation between the convergence and stability of the previous methods. Then, we close with some numerical examples they are presented to illustrate the convergence and stability of collocation method for the resolution of Fredholm integral equations on the real line.

Finally, we have tried to apply the rational Legendre functions which are created by combining the classical Legendre polynomials with algebraic mapping for the numerical solution of quadratic Urysohn integral equations in the half line.

We have stated the theorems on the convergence and error estimates of the method, and we have proved them for both L^∞ and weighted L^2 norms. The obtained results from the numerical examples have shown that the present method is efficient and stable.

This approach can be extended to solve a broad class of unbounded interval problems as quadratic Hammerstein integral equations on the half-line by the use of the rational collocation method where the idea of a future work will be.

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Abstract

In recent years, there has been a growing interest in the formulation of many problems in terms of integral equations, and this has fostered a parallel rapid growth of the literature on their numerical solution. In this sense, our focus will be on spectral methods for solving integral equations. One of the purposes of this research is to provide the mathematical foundations of spectral methods and to analyze their basic theoretical properties (stability, accuracy, computational complexity, and convergence). Furthermore, we have applied the spectral collocation method to find numerical solutions to quadratic Urysohn integral equations. This method reduces the nonlinear integral equation to a system of nonlinear algebraic equations and that algebraic system has been solved by the iterative method. We have derived an error analysis for the current method, which proves that it has exponential convergence order. Finally, several numerical examples are given to show the effectiveness and stability of our approach.

Keywords: Nonlinear integral equations, Collocation method, rational approximation, Legendre polynomials, convergence analysis, stability, half line.

Résumé

Au cours des dernières années, on s'est intéressé de plus en plus à la formulation de nombreux problèmes en termes d'équations intégrales, ce qui a favorisé une croissance rapide parallèle de la littérature sur leur solution numérique. C'est la raison pour laquelle nous allons nous focaliser sur les méthodes spectrales pour résoudre les équations intégrales. L'un des objectifs de cette recherche est de fournir les bases mathématiques des méthodes spectrales, pour analyser leurs propriétés théoriques de base (stabilité, précision et convergence). Nous avons également présenté une méthode efficace pour résoudre les équations intégrales non linéaires avec une étude approfondie sur la convergence. En outre, nous avons appliqué la méthode spectrale de collocation pour trouver la solution numérique de l'équation intégrale quadratique du type Urysohn, cette méthode transforme l'équation intégrale non linéaire en un système d'équations algébriques non linéaires ce système algébrique a été résolu par la méthode itérative. Enfin, plusieurs exemples numériques sont donnés pour montrer l'efficacité et la stabilité de nos approches.

Mots clés: Equations intégrales non lineaires, la méthode de collocation, approximation rationnelle, les polynômes de Legendre analyse de la convergence, la stabilité, demi-droite.

ملخص

في السنوات الاخيرة كان هناك اهتمام متزايد بصياغة العديد من المشاكل باستعمال المعادلات التكاملية، وقد ادى ذلك الى تطور الخوارزميات المتعلقة بحلها. لهذا سنهتم في هذا العمل بالطرق الطيفية لحل المعادلات التكاملية. احد اهداف هدا البحث هو تقديم مفهوم الطرق الطيفية ، مبدأ عملها و وخصائصها النظرية (تقارب، دقة، استقرار) . بالاضافة الى ذلك طبقنا طريقة التجميع الطيفي من اجل ايجاد حلول تقريبية للمعادلات التكاملية غير الخطية من نوع ايريسون. هاته الطريقة تحول المعادلة التكاملية غير الخطية الى نظام المعادلات الجبرية غير الخطية وهذا النظام الجيري تم حله بالطريقة التكرارية واستخلصنا تحليلا للخطأ فيما يتعلق بالأساليب الحالية والتي تثبت أن لها ترتيبا أسيا للتقارب. وأخيرا، قدمنا عدة أمثلة رقمية لاثبات فعالية واستقرار طريقتنا التقريبية .

الكلمات المفتاحية: معادلات تكاملية غير خطية، نصف المستقيم العددي، طريقة كولولكاسيو، التقريب الجذري، كثرات الحدود ليجوندر ، تحليل التقارب ، الاستقرار .