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Stochastic viability under the flow of a stochastic equation (Itô - Stratonovich) and application "the kubo oscillator equation"

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Dedication

I dedicate this memoir to my family, whose unconditional support and love have always encouraged me to pursue my dreams. To my parents for their confidence in me and their sacrifices, and to my friends for their presence and motivation throughout this journey.

I also dedicate this work to all those who aspire to knowledge and research, in the hope that my efforts will inspire others to follow their passion.

ملخص

في هذا العمل، ندرس خاصية الحفاظ على حل معادلة مذبذب كوبو ضمن نموذج عشوائي، وذلك بعد مراجعة لحساب الاحتمالات العشوائي ومفاهيم الحفاظ والصلاحية في السياقات الحتمية والعشوائية.

الكلمات المفتاحية : الحركة البراونية، عملية فينر، الضوضاء البيضاء ، التكاملات العشوائية، معادلة إيتو التفاضلية العشوائية، معادلة ستراتونوفيتش التفاضلية العشوائية، القابلية العشوائية، الثبات العشوائي، مذبذب كوبو

Abstract

In this work, we study the invariance property of the solution of the Kubo oscillator equation under a stochastic model, following a review of stochastic calculus and the concepts of invariance and viability in both deterministic and stochastic settings.

Key words : Brownian motion, Wiener process, White noise, stochastic Integrals, Itô stochastic differential equation, Stratonovich stochastic differential equation, stochastic viability, stochastic invariance, kubo oscillator.

Résumé

Dans ce travail, nous étudions la propriété d'invariance de la solution de l'équation de l'oscillateur de Kubo sous un modèle stochastique, après une revue du calcul stochastique et des concepts d'invariance et de viabilité dans les cadres déterministe et stochastique.

Mots-clés : Mouvement brownien, processus de Wiener, Bruit blanc, intégrales stochastiques, équation différentielle stochastique d'Itô, équation différentielle stochastique de Stratonovich, viabilité stochastique, invariance stochastique, Oscillateur de Kubo.

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List of Symbols

Symbol	Description
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{N}$	Set of natural numbers
Ω	Probability sample space
$\mathscr{F}$	σ-algebra on Ω
$\mathbb{P}$	Probability measure
$.^T$	Transpose vector
$\mathscr{B}(I)$	Borel σ-algebra on the interval I, i.e., the smallest σ-algebra containing all open subsets of I.
$(\mathscr{F}_t)_{t\geq 0}$	Filtration (information available up to time t)
$\langle \cdot, \cdot \rangle$	Inner product in $\mathbb{R}^n$
$\|\cdot\|$	Euclidean norm
W_t	Standard Brownian motion (Wiener process)
B_t	Brownian motion
$\xi(t)$	White noise (formal derivative of W_t)
$\mathbb{E}[\cdot]$	Expectation with respect to $\mathbb{P}$
$\mathrm{Var}(X_t)$	Variance of the stochastic process X_t
$\mathrm{Cov}(X_s, X_t)$	Covariance between X_s and X_t
$\circ dB_t$	Stratonovich differential
$\lim_{m\to\infty} {}^{qm}$	Limit on a quadratic mean
$\int_0^t f(s)\, dB_s$	Itô stochastic integral
$\int_0^t f(s) \circ dB_s$	Stratonovich stochastic integral
$[B,B]_t$	Quadratic variation of Brownian motion B up to time t
$M_2[t_0;t)$	Space of measurable matrix-valued functions square-integrable in $[t_0, t)$ with respect to probability
$L^2[t_0,t]$	Space of square-integrable functions on the interval $[t_0, t]$
$V(s,t)$	Space of adapted, square-integrable processes on $[s,t]$, i.e., $\mathbb{E}[\int_s^t f^2(u)\, du] < \infty$
$H(Y_1, Y_2)$	Hamiltonian function (first integral) that remains constant along solutions of the equation

Introduction

The theory of stochastic processes is a dynamic branch of probability theory, focusing on the interdependence and long-term behavior of random variables [8, 19]. Stochastic processes are observed when phenomena envolve over time according to probabilistic laws. Examples include Brownian motion [19], population growth [1], radioactive decay, and fluctuating outputs in industrial processes. These processes are found in diverse fields such as medicine [1], biology [21, 12], physics [22], and economics [7].

Stochastic theory is crucial for modeling natural systems with inherent randomness, particularly in population growth [1, 19]. By incorporating random perturbations, stochastic differential equations transform deterministic models into probabilistic frameworks that capture the complexity of real-world biological and ecological processes, providing more nuanced and accurate predictions of system behavior. Incorporating randomness into deterministic models allows for a more faithful representation of real-world phenomena. For instance, Øksendal's model of stochastic population growth describes the population size N_t through the following stochastic differential equation [19]:

$$dN_t = rN_t dt + \sigma N_t dB_t$$

where r is the growth rate, σ represents the intensity of random fluctuations, and B_t is a Brownian motion process. This formulation accounts for environmental variability and offers a probabilistic perspective on population dynamics, incorporating the uncertainty and fluctuations that deterministic models cannot address [1, 19].

The Itô and Stratonovich integrals are two approaches in stochastic calculus [8, 19]. The Itô integral is preferred in finance and mathematics due to its compatibility with Itô's formula and martingale theory, while the Stratonovich integral aligns with classical calculus, making it ideal for physics and engineering [19, 22]. Their key difference lies in the correction term present in Itô's interpretation. This distinction is evident in models like stochastic population growth, where the Itô solution includes a drift adjustment absent in the Stratonovich framework [19].

Though the Itô and Stratonovich equations are mathematically equivalent, they offer distinct conceptual interpretations of SDEs. The Itô framework, which relies on non-anticipative processes, is particularly useful in financial mathematics and probability theory [7, 19]. It explicitly incorporates the drift correction term to capture the stochastic effects of Brownian motion. In contrast, the Stratonovich interpretation is often preferred in physics and engineering, as it maintains consistency with classical differential equations and better reflects the natural occurrence of noise in physical systems [19, 22].

When explicit solutions to SDEs are unavailable, theoretical results on existence, uniqueness, and regularity are crucial [8, 19]. The existence theorem ensures that a solution exists under conditions such as Lipschitz continuity and linear growth of the coefficients. Uniqueness guarantees that the solution is singular, while regularity addresses the smoothness of the solution, depending on the properties of the drift and diffusion terms [2, 19]. These theoretical foundations, as discussed by Ludwig Arnold [2] and Bernt Øksendal [19], enable rigorous analysis and facilitate the use of numerical methods, such as the Euler-Maruyama scheme, when analytical solutions are not feasible [8, 19, 12].

In this study, we focus on the invariance property applied to the Kubo oscillator equation. Our objective is to determine the necessary conditions on the diffusion term that ensure the preservation of this invariance property [12].

In the first chapter, we give a reminder of fundamental concepts in stochastic processes, covering Brownian motion [19], the Wiener process, and white noise [8]. We also introduce Itô and Stratonovich integrals, which are essential tools in stochastic calculus [19, 22].

The second chapter contains a reminder of fundamental concepts in stochastic differential equations. The Itô and Stratonovich equations, as well as the conversion formula between these two formulations [19, 2].

In the third chapter, we examine the problem of stochastic viability persistence applied to the Kubo oscillator. First, we explore deterministic invariance and viability [2]. Then, we move to stochastic invariance and viability [19, 12]. Finally, we apply these concepts to the Kubo oscillator equation and analyse the conditions for stochastic persistence [12].

Chapter 1

Stochastic Processes, Brownian Motion and Stochastic Integrals

1.1 Stochastic Process

A set of random variables that characterizes a system's evolution across time is called a stochastic process. It shows how a system's state varies over time or place, frequently in an unanticipated or random fashion. These procedures are used to model uncertain scenarios, such shifts in the population, weather conditions, or the financial markets. A probability distribution that controls the system's state transitions across time is the mathematical definition of a stochastic process.. Notable examples of stochastic processes include **Brownian motion** and **Markov chains**.

Definition 1.1.1 *A **stochastic process** is a family of random variables $(Z_t)_{t\in I}$ indexed by a set I, where I can be either $[0,T]$ for some $T>0$ or $[0,\infty)$. Each random variable Z_t is defined as a function*

$$Z_t:\Omega\to\mathbb{R}$$

where Ω denotes the underlying probability space.

A stochastic process can be defined more broadly by considering more general index sets I and allowing state spaces beyond $\mathbb{R}$. In our context, there are two distinct perspectives on the process Z:

- The collection $Z=(Z_t)_{t\in I}$ represents a family of random functions, where each outcome w corresponds to a function
$$f(w)=(Z_t(w))_{t\in I}.$$
The function $t\mapsto Z_t(w)$ is referred to as a **path** or **trajectory** of the process.

- Alternatively, the process $Z=(Z_t)_{t\in I}$ can be understood as an indexed collection of random variables, ordered with respect to the time parameter t, forming a mapping
$$t\mapsto Z_t.$$

These two perspectives differ in the interpretation of the roles played by w and t.

1.1.1 Some properties of stochastic process

Definition 1.1.2 *A process $Z(t,w)$ is stochastically continuous at a point $s \in [s_0, s_1] \subset I$ if for each $\epsilon > 0$,*

$$\lim_{t \to s} \mathbb{P}\{|Z(t,w) - Z(s,w)| > \varepsilon\} = 0. \quad (1.1)$$

The definitions of right and left stochastic continuity are analogous.
The process Z is said to be **continuous** if the mapping $t \mapsto Z_t(w)$ is continuous for all t.

Definition 1.1.3 *Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. A collection of σ-algebras $\{\mathscr{F}_t\}_{t\in I}$ is referred to as a **filtration** if it satisfies the inclusion property:*

$$\mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \mathscr{F}, \quad \text{for all } 0 \le s \le t \in I.$$

The tuple $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in I}, \mathbb{P})$ is referred to as a stochastic basis.

Definition 1.1.4 *Consider a stochastic process $Z = (Z_t)_{t\in I}$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, along with a filtration $(\mathscr{F}_t)_{t\in I}$.*

- The process Z is said to be **measurable** if the mapping $(w,t) \mapsto Z_t(w)$, interpreted as a function from $\Omega \times I$ to $\mathbb{R}$, is measurable with respect to $\mathscr{F} \otimes \mathscr{B}(I)$ and $\mathscr{B}(\mathbb{R})$.
- It is called **adapted** to the filtration $(\mathscr{F}_t)_{t\in I}$ if, for every $t \in I$, the random variable Z_t is $\mathscr{F}_t$-measurable.

Next, we introduce the concept of a martingale.

Definition 1.1.5 *Let $Z = (Z_t)_{t\in I}$ be a stochastic process that is adapted to the filtration $(\mathscr{F}_t)_{t\in I}$ and satisfies the integrability condition $\mathbb{E}[Z_t] < \infty$ for all $t \ge 0$. The process Z is called a **martingale** if, for all $0 \le s \le t \in I$, the following property holds:*

$$\mathbb{E}[Z_t | \mathscr{F}_s] = Z_s. \quad a.s$$

Esperance, Variance and Covariance of a stochastic process

The most important characteristics of a stochastic process are its expectation, variance, and covariance matrix. Let $Z = \{Z_t\}_{t\in I}$ be a stochastic process and let $Z \in \mathbb{R}^d$.

1. Expectation of Z_t

The expectation (or mean) of the stochastic process Z_t at time t is defined as the expected value of the random variable Z_t. It is a vector of expected values for each component of the process. where the expected value of a random variable is a real number which gives the mean value of this random variable.

Formula:

$$m_Z(t) = \mathbb{E}[Z_t] = \left(\mathbb{E}[Z_t^{(1)}], \mathbb{E}[Z_t^{(2)}], \ldots, \mathbb{E}[Z_t^{(d)}]\right)$$

where $Z_t = \left(Z_t^{(1)}, Z_t^{(2)}, \ldots, Z_t^{(d)}\right)$ is a d-dimensional random vector. Each $Z_t^{(i)}$ is a random variable representing the i-th component of Z_t, and $\mathbb{E}[Z_t^{(i)}]$ is the expectation of the i-th component.

For a continuous stochastic process, the expectation is given by the integral over the probability space Ω:

$$m_Z(t) = \mathbb{E}[Z_t] = \int_\Omega Z_t(w) P(dw)$$

This is the vector of expected values of each component at time t.

2. Variance of Z_t

The variance of Z_t at time t describes the spread or dispersion of the random vector Z_t around its mean. The variance of a vector-valued process is represented by the covariance matrix.

Formula:

$$\mathrm{var}(Z_t) = \mathbb{E}[Z_t Z_t^T] - \mathbb{E}[Z_t]\mathbb{E}[Z_t]^T$$

Here:

- $\mathbb{E}[Z_t Z_t^T]$ is the expected value of the outer product of the vector Z_t with itself, representing the second moment.
- $\mathbb{E}[Z_t]\mathbb{E}[Z_t]^T$ is the outer product of the mean vector of Z_t with itself.

The result is a covariance matrix $\Sigma_Z(t)$, which is a $d \times d$ matrix. The diagonal elements of this matrix represent the variances of each component of Z_t, and the off-diagonal elements represent the covariances between the components.

Thus, the covariance matrix is:

$$\Sigma_Z(t) = \mathbb{E}[Z_t Z_t^T] - m_Z(t) m_Z(t)^T$$

3. Covariance between Z_s and Z_t

The covariance between Z_s and Z_t measures the relationship between the values of the stochastic process at two different times, s and t.

Formula:

$$K(s,t) = \mathrm{cov}(Z_s, Z_t) = \mathbb{E}\left[(Z_s - \mathbb{E}[Z_s])(Z_t - \mathbb{E}[Z_t])^T\right]$$

This can also be written as:

$$K(s,t) = \mathbb{E}[Z_s Z_t^T] - \mathbb{E}[Z_s]\mathbb{E}[Z_t]^T$$

Where:

- $\mathbb{E}[Z_s Z_t^T]$ is the expected value of the outer product between Z_s and Z_t.
- $\mathbb{E}[Z_s]\mathbb{E}[Z_t]^T$ is the outer product of the mean vectors $m_Z(s)$ and $m_Z(t)$.

The result is a $d \times d$ covariance matrix for the pair (s,t). This matrix reflects how the components of Z_s and Z_t are related at different times. The diagonal elements give the variances, and the off-diagonal elements give the covariances between the components of Z_s and Z_t.

Definition 1.1.6 *Assume that $\{Z_t\}$ and $\{\tilde{Z}\}$ are stochastic processes on $(\Omega, \mathscr{F}, \mathbb{P})$. The process $\{Z_t\}$ is then considered a version of $\{\tilde{Z}\}$ if:*

$$\mathbb{P}\{w : Z_t(w) = \tilde{Z}(w)\} = 1, \quad \forall t \geq 0.$$

Definition 1.1.7 *A stochastic process is said to be **(strictly) stationary** if its distributions remain invariant under time shifts. In other words, for $t_i, t_i + t \in [t_0, T]$, we have:*

$$F_{t_1+t,\ldots,t_n+t}(Z_1,\ldots,Z_n) = F_{t_1,\ldots,t_n}(Z_1,\ldots,Z_n).$$

This means that the joint distribution of the process at times $t_1,\ldots,t_n$ is identical to that at shifted times $t_1 + t,\ldots,t_n + t$, for any t.

Proposition 1.1.1 *Two stochastic processes $Z = (Z_t)_{t\in I}$ and $Y = (Y_t)_{t\in I}$ are **independent** if for all $t, s \in I$, their joint distribution satisfies:*

$$P(Z_t \leq z, Y_s \leq y) = P(Z_t \leq z)P(Y_s \leq y).$$

*If Z and Y are independent, then their **covariance matrix is zero**, meaning:*

$$\mathrm{Cov}(Z_t, Y_s) = 0, \quad \forall\, t, s \in I.$$

Gaussian processes constitute an important class of stochastic processes, frequently applied in diverse fields of both pure and applied mathematics.

Definition 1.1.8 *A real-valued (one-dimensional) stochastic process $Z = (Z_t)_{t\geq 0}$ is called a* Gaussian process *if, for every integer $n \geq 1$ and every choice of time points $0 \leq t_1 < t_2 < \cdots < t_n$, the random vector $\Gamma = (Z_{t_1}, \ldots, Z_{t_n})$ follows a (possibly degenerate) multivariate normal distribution.*

(see the section (1.2.2))

1.2 Brownian motion and Wiener process

Brownian motion is named after the Scottish Botanist Robert Brown, who first observed that pollen grains move in random directions when placed in water. It is one of the most basic stochastic processes. Louis Bachelier first proposed it in 1900, and Einstein conducted his own independent research on it in his article from 1905. Wiener created the Wiener process, a contemporary mathematical expression for Brownian motion, in 1923. He proved that there is a continuous-path variant of this procedure. Being a Gaussian process, a Markov process, and a martingale, Brownian motion is a key concept in stochastic process theory. These characteristics make it an essential part of building more intricate stochastic models. For a detailed historical account of Brownian motion and related processes, see the works of Meyer [14], Klebaner [10], and Pitman [20].
Brownian motion $B(t)$ is usually noted by Wiener process W because a rigorous mathematical treatment of BM began with N.Wiener, who provided the first existence proof. Brownian motion is also called the Wiener-Lévy process.

1.2.1 One-Dimensional Wiener Process (Brownian motion)

We now introduce and examine Brownian motion, also referred to as the Wiener process.

Definition 1.2.1 *A stochastic process $\{B(t), t \geq 0\}$ is said to be a Brownian motion with variance parameter $\sigma^2 > 0$ if it satisfies the following conditions:*

(i) ***Initial Condition:*** $B(0) = 0, a.s.$

(ii) ***Independent Increments:*** *For any sequence of times $0 \leq t_1 < t_2 < \cdots < t_m$, the random variables*

$$B(t_1), B(t_2) - B(t_1), \ldots, B(t_m) - B(t_{m-1})$$

are independent.

(iii) ***Stationary Increments:*** *For any $0 \leq s < t$, the increment $B(t) - B(s)$ follows a normal distribution with mean zero and variance $\sigma^2(t-s)$:*

$$B(t) - B(s) \sim \mathcal{N}(0, \sigma^2(t-s)).$$

(iv) ***Continuous Paths:*** *The sample paths* $\{B(t)\}_{t\geq 0}$ *are continuous functions of* t.

Remark 1.2.1 :

- *The natural filtration associated with the Brownian motion is given by:* $\mathscr{F}_t^B = \sigma\{B(s) : s \leq t\}$.
- *When the variance parameter satisfies* $\sigma^2 = 1$, *we refer to the process* $\{B(t) : t \geq 0\}$ *as a* ***standard Brownian motion***.

Consider a standard BM (Wiener process) B_t, and define the process

$$Y_t = \sigma B_t + mt, \quad \sigma, m \in \mathbb{R}.$$

The process Y_t follows a Wiener process with drift m and variance σ^2, and it satisfies the two fundamental conditions outlined earlier. Therefore, demonstrating the existence of Wiener processes (or Brownian motion) reduces to constructing a standard one-dimensional Wiener process [11].

1.2.2 An n-dimensional Wiener process

$$W_t = (W_t^1, \dots, W_t^n) = B_t = (B_t^1, \dots, B_t^n)$$

is referred to as an n-dimensional Brownian motion originating from zero, with drift $M = (M_1, \dots, M_n) \in \mathbb{R}^n$ and an $n \times n$ covariance matrix Γ, relative to the filtration $\{\mathscr{F}_t\}$, if the following properties hold:

1- $W_0 = 0$ almost surely.

2- For $s < t$, the increment $W_t - W_s$ follows a multivariate normal distribution with mean $(t-s)M$ and covariance matrix $(t-s)\Gamma$.

3- For $s < t$, the random vector $W_t - W_s$ is independent of $\mathscr{F}_s$.

4- The sample paths of W_t are continuous almost surely.

Each component W_t^k of the process W_t behaves as a Brownian motion with drift M_k and variance Γ_{kk}, relative to the filtration $\mathscr{F}_t$. In the special case where $M = 0$ and $\Gamma = I$, the processes $W_t^1, \dots, W_t^n$ are independent standard Brownian motions, and the process W_t is referred to as an **n-dimensional standard Brownian motion.**

Additionally, recall that a process $X_t \in \mathbb{R}^n$ is said to be a Gaussian process (or jointly normal process) if, for any selection of time points $0 \leq t_1 \leq \cdots \leq t_k$, the random vector

$$Z = (X_{t_1}, \dots, X_{t_k}) \in \mathbb{R}^{nk}$$

follows a multivariate normal distribution. This implies the existence of a mean vector $M \in \mathbb{R}^{nk}$ and a non-negative definite covariance matrix

$$\Gamma = [c_{jm}] \in \mathbb{R}^{nk \times nk}$$

where Σ belongs to the set of all $nk \times nk$ real-valued matrices, such that
For a given random vector Z, its characteristic function is expressed as:

$$\mathbb{E}^x \left[\exp\left(i \sum_{j=1}^{nk} u_j Z_j \right) \right] = \exp\left[-\frac{1}{2} \sum_{j,m} u_j c_{jm} u_m + i \sum_j u_j M_j \right] \tag{1.2}$$

for any vector $u = (u_1, \dots, u_{nk}) \in \mathbb{R}^{nk}$, where $i = \sqrt{-1}$ represents the imaginary unit, and $\mathbb{E}^x$ denotes expectation with respect to P_x.

Furthermore, if the above equality holds, then the expectation of Z is given by:

$$M = \mathbb{E}^x[Z],$$

and its covariance matrix is given by:

$$\Gamma = [c_{jm}], \quad \text{where} \quad c_{jm} = \mathbb{E}^x\left[(Z_j - M_j)(Z_m - M_m)\right].$$

A key requirement for an Itô stochastic differential equation to be well-defined is that the matrix $\sigma\sigma^T$ must be non-negative definite. As a result, studying stochastic equations driven by a standard n-dimensional Wiener process is sufficient.

The following is an outline of some fundamental properties of Wiener processes:

Proposition 1.2.1 ***(A)*** *A process of this kind is known as a Wiener process originating from x, and it satisfies the condition:*

$$P_x\{W_0 = x\} = 1, \quad x \in \mathbb{R}^n.$$

(B) *Consider an n-dimensional Wiener process W_t that starts at zero. Let $U \in \mathbb{R}^{n\times n}$ be a constant orthogonal matrix such that $UU^T = I$. Then, the transformed process*

$$\tilde{W}_t := UW_t$$

also follows the properties of a Wiener process, meaning that Wiener processes remain invariant under rotations in $\mathbb{R}^n$.

(C) *A Wiener process has independent increments. Specifically, for any time sequence $0 \le t_1 < t_2 < \cdots < t_k$, the increments*

$$W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_k} - W_{t_{k-1}}$$

are mutually independent. This property can be established by recognizing that normal random variables are independent if and only if they are uncorrelated. Therefore, it is sufficient to verify that

$$E^x[(W_{t_i} - W_{t_{i-1}})(W_{t_j} - W_{t_{j-1}})] = 0, \quad \text{for } t_{i-1} < t_i \le t_{j-1} < t_j.$$

This follows from the structure of the covariance matrix:

$$E^x[W_{t_i}W_{t_j} - W_{t_{i-1}}W_{t_j} - W_{t_i}W_{t_{j-1}} + W_{t_{i-1}}W_{t_{j-1}}] = n(t_i - t_{i-1} - t_i + t_{i-1}) = 0.$$

Thus, it follows that $W_s - W_t$ is independent of $\mathscr{F}_t$ whenever $s > t$.

(D) *If W_t is a Wiener process, then the following transformations also yield Wiener processes:*

$$-W_t, \quad cW_{t/c^2} \quad (\text{for } c \neq 0), \quad tW_{1/t}, \quad \text{and} \quad W_{t+s} - W_s \quad (\text{for fixed } s \text{ and } t \ge 0).$$

1.2.3 Properties of Brownian motion (Wiener process)

1- Martingale property

A martingale represents a fundamental class of stochastic processes. It states that the conditional expectation of a process Z_s is simply the current value:

$$\mathbb{E}[Z_t \mid \mathscr{F}_s] = Z_s \quad t > s$$

In simple terms„ the martingale property ensures that in a " game", knowledge of the past will be of no use in predicting future winnings.

Lemma 1.2.1 *Let B_t be a Wiener process adapted to the filtration $\mathscr{F}_t$. Then, B_t is an $\mathscr{F}_t$-martingale.*

Proof: To verify this property, we must demonstrate that for any $t > s$,

$$\mathbb{E}[B_t \mid \mathscr{F}_s] = B_s.$$

Since B_s is $\mathscr{F}_s$-measurable (as the process is adapted), proving the above is equivalent to showing that

$$\mathbb{E}[B_t - B_s \mid \mathscr{F}_s] = 0.$$

According to the definition of the Wiener process, the increment $B_t - B_s$ is independent of $\mathscr{F}_s$ and has an expected value of zero.

Furthermore, Brownian motion is a square-integrable martingale with a quadratic variation given by

$$\langle B \rangle_t = t, \quad t \geq 0.$$

This property holds not only in the probabilistic sense of the Doob-Meyer decomposition but also almost surely for all $w \in \Omega$.

Lemma 1.2.2 *Consider $B(\cdot)$ as a standard one-dimensional Wiener process. The expected value of the product $W(t)W(s)$ is given by*

$$E[B(t)B(s)] = \min(t, s), \quad \textit{for } t \geq 0, s \geq 0.$$

Proof:
Suppose $t \geq s \geq 0$. Using the decomposition property of the Wiener process, we write

$$B(t) = B(s) + (B(t) - B(s)).$$

Taking the expectation, we obtain

$$E[B(t)B(s)] = E[(B(s) + (B(t) - B(s)))B(s)].$$

Expanding the terms, we get

$$E[B(t)B(s)] = E[B^2(s)] + E[(B(t) - B(s))B(s)].$$

Since $W(s)$ follows a normal distribution with mean zero and variance s, and the increment $B(t) - B(s)$ is independent of $B(s)$, it follows that

$$E[(B(t) - B(s))B(s)] = E[B(t) - B(s)] \cdot E[B(s)] = 0.$$

Thus, we conclude that

$$E[B(t)B(s)] = E[B^2(s)] = s = \min(t, s).$$

2- Markov property

The Markov property states that a stochastic process essentially has "no memory". This means that the conditional probability distribution of the future states of the process are independent of any previous state, with the exception of the current state.

Definition 1.2.2 (Markov process) *A Markov process is a stochastic process $\{Z_t,\ t \in [0, T]\}$ given an index set $[t_0, T] \subseteq [0, \infty]$ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. This means that the following equation holds with probability one:*

$$\mathbb{P}(Z_t \in B \mid \mathscr{F}([t_0, s])) = \mathbb{P}(Z_t \in B \mid Z_s)$$

for all $t_0 \leq s \leq t \leq T$ and for all $B \in \mathscr{B}(\mathbb{R}^d)$.

Definition 1.2.3 *A stochastic process Z_t is said to be Markov if, for any time t, the conditional distribution of $\{Z_s : s \geq t\}$ given $\{Z_r : r \leq t\}$ depends only on Z_t. In other words, the future evolution of the process is independent of its past, given the present state.*

Lemma 1.2.3 *An $\mathscr{F}_t$-Brownian motion B_t qualifies as an $\mathscr{F}_t$-Markov process.*

Proof: We have :

$$\mathbb{E}[f(B_{s+t}) \mid \mathscr{F}_s] = \mathbb{E}[f((B_{s+t} - B_s) + B_s) \mid \mathscr{F}_s].$$

Since the increment $B_{s+t} - B_s$ is independent of $\mathscr{F}_s$, and in particular of B_s, we obtain

$$\mathbb{E}[f(B_{s+t}) \mid \mathscr{F}_s] = \mathbb{E}[f((B_{s+t} - B_s) + x)]\big|_{x=B_s}.$$

As $B_{s+t} - B_s \sim \mathcal{N}(0, t)$, we get

$$\mathbb{E}[f(B_{s+t}) \mid \mathscr{F}_s] = \int_{\mathbb{R}} f(y + B_s) \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \, dy.$$

By the change of variables $z = y + B_s$, we obtain

$$= \int_{\mathbb{R}} f(z) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(z-B_s)^2}{2t}} \, dz = (P_t f)(B_s),$$

where $P_t f$ denotes the action of the heat semi-group at time t on the function f.

Thus,

$$\mathbb{E}[f(B_{s+t}) \mid \mathscr{F}_s] = (P_t f)(B_s) = \mathbb{E}[f(B_{s+t}) \mid B_s],$$

which proves the Markov property.

3- Non-differentiability:

Theoreme 1.2.1 *For any t, almost every path of Brownian motion is non-differentiable at t*

Proof: Let B_t be a standard Brownian motion and fix $t \geq 0$. Consider the difference quotient

$$\frac{B_{t+h} - B_t}{h}.$$

Since $B_{t+h} - B_t \sim \mathcal{N}(0, h)$, we compute:

$$\mathbb{E}\left[\left(\frac{B_{t+h} - B_t}{h}\right)^2\right] = \frac{1}{h^2}\mathbb{E}[(B_{t+h} - B_t)^2] = \frac{1}{h^2} \cdot h = \frac{1}{h}.$$

As $h \to 0$, we have

$$\lim_{h\to 0} \mathbb{E}\left[\left(\frac{B_{t+h} - B_t}{h}\right)^2\right] = \infty.$$

This implies that the limit defining the derivative cannot exist in L^2, and in fact, almost surely

$$\limsup_{h\to 0} \left|\frac{B_{t+h} - B_t}{h}\right| = \infty.$$

Therefore, B_t is almost surely not differentiable at t.

4- Quadratic variation:

Definition 1.2.4 *The quadratic variation of Brownian motion $B(t)$ is given by*

$$[B, B](t) = [B, B]([0, t]) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}[B_{t_i^n} - B_{t_{i-1}^n}]^2, a.s$$

where, for each n, the set $\{t_i^n, 0 \leq i \leq n\}$ forms a partition of $[0, t]$. The limit is taken over all partitions satisfying $\delta_n = \max_i (t_{i+1}^n - t_i^n) \to 0$ as $n \to \infty$, in the sense of convergence in probability.

The quadratic variation of a Brownian motion over $[0, t]$ is t. As stated in Klebaner [10].

Theoreme 1.2.2 (Lévy's Theorem) *A Brownian motion is characterized as a continuous martingale whose quadratic variation over any interval $[0, t]$ is equal to t.*

1.3 White noise

In many physical and engineering models, the stochastic perturbations are often represented using *white noise*, typically denoted by $\xi(t)$. Although $\xi(t)$ is not a classical function, it is formally defined as the time derivative of the Wiener process W_t, that is:

$$\xi(t) = \frac{dW_t}{dt}$$

This expression is symbolic, since W_t is almost surely nowhere differentiable. The concept of white noise is understood in the sense of generalized functions (distributions), where it satisfies the following properties:

- It has zero mean: $\mathbb{E}[\xi(t)] = 0$,
- It is delta-correlated: $\mathbb{E}[\xi(t)\xi(s)] = \delta(t-s)$, where δ is the Dirac delta function.

In practice, white noise is used to describe instantaneous random disturbances in stochastic differential equations. These are often written in the following informal form:

$$dX(t) = a(X(t))\,dt + b(X(t))\,\xi(t)\,dt$$

Such expressions are made rigorous by interpreting $\xi(t)\,dt$ as dW_t, leading to the Itô or Stratonovich formulation:

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW_t$$

White noise provides a useful physical intuition for systems affected by random fluctuations at every instant of time, even though it does not exist as an ordinary stochastic process.

1.4 Itô and Stratonovich integrals

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where B_t is a standard Brownian motion.
The set $V(s,t)$ consists of all real-valued **measurable** functions $f(t,w)$ defined on $[0,\infty)\times\Omega$ that satisfy the following properties:

a- $f(t,w)$ is adapted to the filtration $(\mathscr{F}_t)$.

b- $\mathbb{E}\left[\int_s^t f(t,\cdot)^2\,dt\right] < \infty$.

1.4.1 Itô integral

For simple functions

Our aim is to define the Itô integral of f over the interval $[s,t)$, given by

$$I(f)(w) = \int_s^t f(t,w)\,dB_t(w). \tag{1.3}$$

where $f \in V(s,t)$.

The approach is straightforward: We first introduce $I(f)$ for a specific class of simple functions f as Lobesgue integral. Then, we define $\int_s^t f(t,w)\,dB_t(w)$ as the limit of $\int_s^t \phi\,dB_t(w)$ when f is approximated by the simple function ϕ.

Let start by defining the integral for the follwing basic class of functions.

Definition 1.4.1 (Simple Functions) *A function $\phi \in V(s,t)$ is said to be a simple (or elementary) function if it can be represented as a finite sum of characteristic functions:*

$$\phi(t,w) = \sum_{k\geq 0} e_k(w)\mathbb{1}_{[t_k,t_{k+1})}(t). \tag{1.4}$$

Definition 1.4.2 *Suppose $\phi \in V(s,t)$ is a simple function as described in 1.4. The stochastic integral of ϕ over $[s,t]$ is defined by*

$$\int_s^t \phi(t,w)\,dB_t = \sum_{k\geq 0} e_k(w)(B_{t_{k+1}} - B_{t_k})(w). \tag{1.5}$$

Lemma 1.4.1 (Itô Isometry [19]) *Let $\phi \in V(s,t)$ be a simple function. Then, the following identity holds:*

$$\mathbb{E}\left[\left(\int_s^t \phi(t,\cdot)\,dB_t\right)^2\right] = \mathbb{E}\left[\int_s^t \phi^2(t,\cdot)\,dt\right]. \tag{1.6}$$

Proof: See [19, p.26] .

Notice that equation (1.6) indeed defines an isometry. More precisely, it can be expressed as an equality of norms in L^2 spaces:

$$\left\|\int_s^t \phi(t,\cdot)\,dB_t\right\|_{L^2(\Omega,\mathbb{P})} = \|\phi\|_{L^2([s,t]\times\Omega)}.$$

This leads to the following important proposition.

Proposition 1.4.1 *For any $f \in V$, there exists a sequence of simple functions $\phi_n \in V$, with $n \in \mathbb{N}$, that approximates f in the L^2-norm. More precisely,*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_s^t (f(t,\cdot) - \phi_n(t,\cdot))^2\,dt\right] = \lim_{n\to\infty} \|f - \phi_n\|^2_{L^2([s,t]\times\Omega)} = 0. \tag{1.7}$$

Definition 1.4.3 (Itô integral) *Let $f \in V(s,t)$. The Itô integral of f over $[s,t]$ is defined as the $L^2(\Omega,P)$ limit*

$$I(f) = \int_s^t f(t,w)\,dB_t(w) = \lim_{\substack{qm \\ n\to\infty}} \int_s^t \phi_n(t,w)\,dB_t(w), \tag{1.8}$$

where ϕ_n is a sequence of simple functions in V, with $n \in \mathbb{N}$, that converges to f in $L^2([s,t]\times\Omega)$.

Remark 1.4.1 *Note that, as indicated by (1.7), the above definition is independent of the particular sequence $\{\phi_n\}_{n\in\mathbb{N}}$ chosen.*

By definition, the Itô isometry property holds for Itô integrals.

Corollary 1.4.1 (Itô Isometry for Itô Integrals) *For any $f \in V(s,t)$, the following identity holds:*

$$\mathbb{E}\left(\int_s^t f(t,\cdot)\,dB_t\right)^2 = \mathbb{E}\left[\int_s^t f^2(t,\cdot)\,dt\right]. \tag{1.9}$$

Corollary 1.4.2 ([18]) *If a sequence of functions $f_n(t,w) \in V(s,t)$ converges to $f(t,w) \in V(s,t)$ in the $L^2([s,t]\times\Omega)$-norm as $n\to\infty$, then*

$$\int_s^t f_n(t,\cdot)\,dB_t \to \int_s^t f(t,\cdot)\,dB_t \tag{1.10}$$

in the $L^2(\Omega,P)$-norm.

Extensien of Itô integral

The definition of the Itô integral can be extended to a broader class of functions $f(t,w)$ that satisfy a weaker integrability condition. This extension is essential, as there exist functions that do not belong to V. To address this, we introduce the following class of functions.

Definition 1.4.4 *Let $\mathbb{W}(s,t)$ denote the set of real-valued measurable functions $f(t,w)$ defined on $[0,\infty)\times\Omega$ that satisfy the following conditions:*

a- $f(t,w)$ is adapted to the filtration $\mathscr{F}_t$.

b- With probability one, the integral $\int_s^t f(t,\cdot)^2 dt$ is finite, i.e.,

$$\mathbb{P}\left(\int_s^t f(t,\cdot)^2 dt < \infty\right) = 1.$$

For any $f \in \mathbb{W}$, it can be demonstrated that there exists a sequence of simple functions $\phi_n \in \mathbb{W}$ such that

$$\int_s^t \left|\phi_n(t,\cdot) - f(t,\cdot)\right|^2 dt \to 0 \tag{1.11}$$

in probability. Given such a sequence, the stochastic integrals $\left(\int_s^t \phi_n(t,w)dB_t(w), n\in\mathbb{N}\right)$ converge in probability to a certain random variable. Moreover, this limit is uniquely determined and does not depend on the chosen approximating sequence ϕ_n. This allows us to define.

Definition 1.4.5 (Itô Integral II) *Let $f \in \mathbb{W}(s,t)$. The Itô integral of f over the interval $[s,t]$ is defined as the limit in probability:*

$$\int_s^t f(t,w)dB_t(w) = \lim_{n\to\infty}\int_s^t \phi_n(t,w)dB_t(w) \tag{1.12}$$

where $\phi_n \in \mathbb{W}$, for $n \in \mathbb{N}$, is a sequence of simple functions that approximates f in probability.

Definition 1.4.6 ($M_2[t_0,t]$) *A $(n\times d)$ matrix-valued function $\sigma = \sigma(s,w)$ defined on $[t_0,t[\times\Omega$ and measurable in (s,w); $\sigma(s,\cdot)$ is $\mathscr{F}_s$-measurable for all $s\in[t_0,t[$; the sample functions $\sigma(\cdot,w)$ are with probability 1 in $L^2[t_0,t]$, that is, with probability 1*

$$\int_{t_0}^t \|\sigma(s,w)\|^2 ds < \infty.$$

For arbitrary integrand

For any partition $t_0 < t_1 < \cdots < t_m$ of the interval $[t_0,t]$ and any choice of intermediate points $\tau_i \in [t_{i-1},t_i)$, we define the step function

$$W_s^m = \begin{cases} \sum_{i=1}^m W_{\tau_i}\mathbf{1}_{[t_{i-1},t_i)}(s), & t_0 \le s < t,\\ W_t, & s = t.\end{cases} \tag{1.13}$$

which approximates W_s. Thanks to the continuity of W_s, we obtain

$$\lim_{\substack{qm\\ m\to\infty}} W_s^m = W_s$$

uniformly on $[t_0,t]$, independently of the choice of the τ_i.

The integral of the step function W_s^m with respect to W_s is then defined via the corresponding Riemann–Stieltjes sum:

$$\int_{t_0}^{t} W_s^m \, dW_s = S_m = \sum_{i=1}^{m} W_{\tau_i}(W_{t_i} - W_{t_{i-1}}).$$

However, the existence and the value of the limit of S_m depend on the choice of the intermediate points τ_i.

A key property of Itô's choice of intermediate points, namely $\tau_i = t_{i-1}$, is that the approximating step function W_s^m can be determined at any fixed time $s \in [t_0, t]$ using only the values of W from t_0 up to s.

According to Lemma (4.4.9) and Corollary (4.5.2) of [2], the step function defined in (1.13), implies that for any almost surely continuous function $f(\cdot, \mathbb{W}) \in M_2[t_0, t]$, the simplest non-anticipating step functions of the form

$$f_m(s) = \sum_{i=1}^{m} f(t_{i-1}, W_{t_{i-1}}) \mathbf{1}_{[t_{i-1}, t_i)}(s),$$

which are directly constructed from the function itself, can be used to approximate the stochastic integral. Consequently, we obtain

$$\int_{t_0}^{t} W_s \, dW_s = \underset{m \to \infty}{\operatorname{qm}\lim} \sum_{i=1}^{m} W_{i-1}(W_{t_i} - W_{t_{i-1}}) = \frac{W_t^2 - W_{t_0}^2}{2} - \frac{t - t_0}{2}. \tag{1.14}$$

For a one-dimensional integrand $f(s) = W_s$, the stochastic integral is defined as

$$\int_{t_0}^{t} f(s, W_s) dW_s := \underset{m \to \infty}{\operatorname{qm}\lim} \sum_{i=1}^{m} f(t_{i-1}, W_{t_{i-1}})(W(t_i) - W(t_{i-1})). \tag{1.15}$$

for any function f in $M_2[t_0, t]$ that is continuous in probability. When generalizing the integral from step functions to arbitrary functions in $M_2[t_0, t]$, the key properties of Theorem (1.4.2) naturally extend as well.

Some properties of the Itô integral

In the following section, we will highlight some fundamental properties of this integral.

Proposition 1.4.2 *[2]and[18]*
Let $f, g \in M_2[t_0, t]$ and let $0 \le s < u < t$. Additionally, let $a, b \in \mathbb{R}$. Then

1- $\int_s^t (af + bg) \, dB_t = a \int_s^t f \, dB_t + b \int_s^t g \, dB_t$.

2- $\int_s^t f \, dB_t = \int_s^u f \, dB_t + \int_u^t f \, dB_t$.

3- $\int_s^t f(s) \, dB_s = \begin{bmatrix} \sum_{k=1}^d \int_s^t f_{1k}(s) \, dB_k(s) \\ \vdots \\ \sum_{k=1}^d \int_s^t f_{nk}(s) \, dB_k(s) \end{bmatrix}, \quad B_t = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}.$

4- If $\mathbb{E}[|f(s)|] < \infty$ for all $s \in [s, t]$, then

$$\mathbb{E}\left[\int_s^t f(s) \, dB_s\right] = 0.$$

5- If $\mathbb{E}[|f(s)|^2] < \infty$ for all $s \in [s, t]$, then the covariance matrix of the stochastic integral satisfies

$$\mathbb{E}\left[\left(\int_s^t f(s) \, dB_s\right)\left(\int_s^t f(s) \, dB_s\right)^T\right] = \int_s^t \mathbb{E}[f(s) f^T(s)] \, ds.$$

In particular,

$$\mathbb{E}\left|\int_s^t f(s)\,dB_s\right|^2 = \int_s^t \mathbb{E}\left|f(s)\right|^2 ds.$$

6- *The process* $M_t(w) = \int_0^t f(s,w)\,dB_s(w)$, *where* $f \in V(0,t)$ *for any* $t > 0$, *is a martingale with respect to the filtration* $\mathscr{F}_t$.

1.4.2 Stratonovich integral

In this section, we introduce another type of stochastic integral, wich called the Stratonovich integral, and explore its relationship with the Itô integral. We previously studied the Itô integral in the form

$$I(t) = \int_0^t C_s\,dB_s, \quad t \in \mathbb{R}_+,$$

where $(B_t)_{t\geq 0}$ is a Brownian motion and $(C_t)_{t\geq 0}$ is a process adapted to the Brownian filtration $\mathscr{F}_t = \sigma(B_s, s \leq t)$, we restrict ourselves to the case where

$$C_t = f(B_t), \quad t \in \mathbb{R}_+,$$

where f is a differentiable function on $\mathbb{R}_+$.

The associated Riemann-Stieltjes sum is given by

$$S_n = \sum_{k=1}^{n} f(B_{y_k})\Delta B_k,$$

where the intermediate points are defined as

$$y_k = \frac{t_{k-1} + t_k}{2}, \quad k = 1, \ldots, n.$$

It can be shown that the mean-square limit of this sum exists as the partition step size tends to zero.

The unique mean-square limit $S_T(f(B))$ of the sum S_n exists if

$$\int_0^T \mathbb{E}[f^2(B_t)]\,dt < \infty.$$

This limit is called the Stratonovich stochastic integral of $f(B)$, and it is denoted by

$$S_T(f(B)) = \int_0^T f(B_t) \circ dB_t.$$

The integral process is defined as

$$S_t(f(B)) = \int_0^t f(B_s) \circ dB_s, \quad t \in \mathbb{R}_+.$$

Example 1.4.1 *Consider the Stratonovich integral of the Brownian motion itself:*

$$S_t(B) = \int_0^t B_u \circ dB_u.$$

It can be shown that

$$S_t(B) = \frac{1}{2}B_t^2.$$

Derivation:

Using the relationship between Stratonovich and Itô integrals:

$$\int_0^t B_u \circ dB_u = \int_0^t B_u \, dB_u + \frac{1}{2}[B,B]_t.$$

Since $[B,B]_t = t$*, we have:*

$$\int_0^t B_u \circ dB_u = \int_0^t B_u \, dB_u + \frac{1}{2}t.$$

From Itô's formula:

$$\int_0^t B_u \, dB_u = \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

Thus, the final result is:

$$S_t(B) = \frac{1}{2}B_t^2.$$

Theoreme 1.4.1 *([13], pp. 154-160)*
Suppose that f *satisfies the conditions*

$$\int_0^t \mathbb{E}[f^2(B_u)] \, du < \infty, \quad \textit{and} \quad \int_0^t \mathbb{E}[(f'(B_u))^2] \, du < \infty.$$

Then, the relationship between the Stratonovich and Itô integrals is given by

$$\int_0^t f(B_u) \circ dB_u = \int_0^t f(B_u) dB_u + \frac{1}{2}\int_0^t f'(B_u) du.$$

From this formula, it is clear that the process $S_t(f(B))$, $t \in \mathbb{R}_+$, is no longer a martingale, unlike the Itô integral.

Now, suppose that the integrand is of the form

$$C_t = f(t, B_t), \quad t \in \mathbb{R}_+,$$

where $f(t,x)$ is a function with continuous second-order partial derivatives. Consider a process X satisfying the Itô stochastic differential equation

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) \circ dB_s,$$

where the functions a and b are continuous and satisfy the conditions for the existence and uniqueness of the solution.

The associated Riemann-Stieltjes sum is defined by

$$\tilde{S}_n = \sum_{k=1}^{n} f(t_{k-1}, \frac{1}{2}(X_{k-1} + X_k))\Delta B_k,$$

The mean-square limit of $\tilde{S}_n$ exists if

$$\int_0^t \mathbb{E}[f^2(s, X_s)] ds < \infty.$$

In this case, the Stratonovich integral satisfies the relation

$$\int_0^t f(u, X_u) \circ dB_u = \int_0^t f(u, X_u) dB_u + \frac{1}{2}\int_0^t b(u, X_u)\frac{\partial f}{\partial x}(u, X_u) du. \tag{1.16}$$

A One-Dimensional Example

In the framework of Itô calculus for a one-dimensional Brownian motion, the Itô integral is defined as the limit of Riemann sum approximations:

$$\int_0^t B_s\,dB_s := \lim_{m\to^{qm}\infty}\sum_{k=1}^{m} B(t_{k-1})(B(t_k)-B(t_{k-1})) = \frac{B^2(t)-t}{2},$$

where $\Pi = \{0 = t_0 \le t_1 \le \cdots \le t_m = t\}$ represents a partition of $[0,t]$, $\Delta = \max_i(t_i - t_{i-1})$ and the limite in quadratic mean is equal to the limite in L^2 which is used in Itô case. In this formulation, the integrand is evaluated at the left endpoint of each subinterval $[t_{k-1}, t_k]$.

On the other hand, the Stratonovich integral is defined differently:

$$\int_0^t B_s \circ dB_s := \lim_{m\to\infty}\sum_{k=1}^{m}\left(\frac{B(t_k)+B(t_{k-1})}{2}\right)(B(t_k)-B(t_{k-1})) = \frac{B^2(t)}{2}.$$

Since the sum in this formulation depends on the values of the process at intermediate points within each subinterval $[t_{k-1}, t_k]$, we observe that:

$$\frac{S_1+S_0}{2} = \frac{S_1}{2},$$

which leads to the equivalent formulation:

$$\int_0^t B_s \circ dB_s := \lim_{n\to 0}\sum_{k=0}^{m-1} B\left(\frac{t_{k+1}+t_k}{2}\right)(B(t_{k+1})-B(t_k)).$$

Thus, in this particular case, the Stratonovich integral corresponds to a Riemann sum approximation where the integrand is evaluated at the midpoint of each subinterval $[t_{k-1}, t_k]$.[12]

For arbitrary integrand

More generally, we define the Stratonovich integral as follows:

$$\int_0^t H(s,W_s)\circ dW_s := \lim_{m\to^{qm}\infty}\sum_{k=1}^{m} H\left(t_{k-1}, \frac{W_{t_{k-1}}+W_{t_k}}{2}\right)(W(t_k)-W(t_{k-1})). \tag{1.17}$$

Here, W_t represents a d-dimensional Wiener process, and $H(t,x)$ is an $n\times d$ matrix-valued function that is continuous in t, possesses first-order partial derivatives H_{x_j} with respect to each component x_j of x, and satisfies the integrability condition:

$$\int_0^t \mathbb{E}[H(s,W_s)]^2 ds < \infty. \tag{1.18}$$

According to [2] (Theorem 10.2.5, p.169), the limit in the above definition exists. This formulation is closely related to Itô's integral, which is defined in this context as:

$$\int_0^t H(s,W_s)dW_s := \lim_{n\to^{qm}0}\sum_{k=1}^{m} H(t_{k-1}, W_{t_{k-1}})(W(t_k)-Wt_{k-1}). \tag{1.19}$$

The relationship between the Stratonovich and Itô integrals is given by the conversion formula:

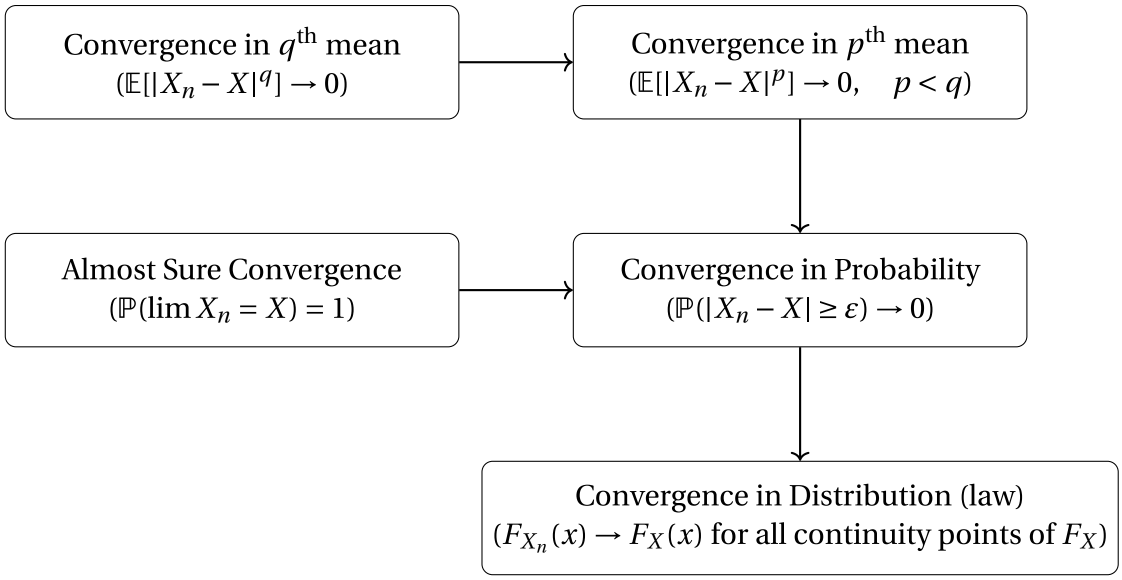
$$\int_0^t H(s,W_s)\circ dW_s = \int_0^t H(s,W_s)dW_s + \frac{1}{2}\int_0^t \bar{H}(s,W_s)ds, \tag{1.20}$$

where

$$\bar{H}_i(s, W_s) = \sum_{j=1}^{n} \sum_{k=1}^{d} \frac{\partial H_{i,j}}{\partial x_k}(s, W_s) H_{k,j}(s, W_s), \quad i = 1, \ldots, n. \tag{1.21}$$

Notably, when $H(t,x)$ is independent of x, meaning $H(t,x) = H(t)$, the Itô and Stratonovich integrals yield the same result.

Reminder: Convergence Concepts



Chapter 2

Stochastic Differential Equations

2.1 Stochastic Differential Equations

A stochastic differential equation (SDE) is an equation expressed as follows:

$$dX_t = a(t, X_t)\,dt + b(t, X_t)\,dB_t, \quad X_0 = x, \tag{2.1}$$

where $(B_t)_{t\geq 0}$ is a m-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. The initial condition x is $\mathscr{F}_0$-measurable. The functions $a : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ satisfy certain regularity conditions depending on the context. The solution $(X_t)_{t\geq 0}$ is a d-dimensional continuous adapted process. Equation 2.1 conveys the same meaning as the following integral equation

$$X_t = x + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s. \tag{2.2}$$

If a stochastic process X_t satisfies this equation, it is said to be a solution of the stochastic differential equation 2.1.

The primary objective of this section is to establish conditions on the coefficients a and b that ensure the existence and uniqueness of solution.

However, several subtle aspects need to be considered.

- First, to ensure the existence of the integrals in (2.2), a certain level of regularity is required for X_t and the functions a and b. Specifically, for all $t \geq 0$, it must hold with probability one that $\int_0^t |a(s, X_s)|\,ds < \infty$ and $\int_0^t b^2(s, X_s)\,ds < \infty$.
- Second, the process X_t must be defined on the same probability space as the Wiener process B_t and be adapted to the associated filtration. Notably, for some choices of the coefficient functions a and b, solutions to the stochastic integral equation (2.2) may exist for certain Wiener processes and filtrations but fail to do so for others.
  - A solution is considered a strong solution if it remains valid for any given Wiener process and initial value, meaning it is unique in a pathwise sense.
  - The solution is referred to as a weak solution if it exists for given coefficients but without specifying the Wiener process, meaning its probability distribution is uniquely determined.

More specifically.

Definition 2.1.1 *Consider a standard Brownian motion $\{B_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an admissible filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$. A process X_t is said to be a* ***strong solution*** *of the stochastic differential equation (2.2) with the initial condition $x \in \mathbb{R}$ if it is adapted, has continuous sample paths, and satisfies the equation*

$$X(t) = x + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dB(s), \quad \textit{for all } t \geq 0.$$

Definition 2.1.2 *A continuous stochastic process X_t is said to be a* ***weak solution*** *of the stochastic differential equation (2.2) with initial condition x if it is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and there exists a Wiener process B_t along with an admissible filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $X(t)$ is adapted and satisfies the corresponding stochastic integral equation (2.2).*

Now, let us discuss uniqueness. As with existence, there are two distinct notions.

Definition 2.1.3 (pathwise uniqueness) *Pathwise uniqueness for equation (2.2) is said to hold if, for any filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with a given Brownian motion $(B_t)_{t\geq 0}$ and a deterministic initial condition $X_0 = x$, any two continuous, $\mathcal{F}_t$-adapted processes $(X_t^{(1)})_{t\geq 0}$ and $(X_t^{(2)})_{t\geq 0}$ that satisfy equation (2.2) must be indistinguishable.*

Definition 2.1.4 (uniqueness in law) *Uniqueness in law for equation (2.1) holds if, for any two weak solutions defined on different probability spaces, their distributions are identical.*

Theoreme 2.1.1 *The functions a and b satisfy the following conditions:*

1. ***Lipschitz condition :*** *There exists a constant K such that for all $x, y \in \mathbb{R}^n$ and $t \geq 0$,*

$$\left\| a(t,x) - a(t,y) \right\| + \left\| b(t,x) - b(t,y) \right\| \leq K \left\| x - y \right\|.$$

2. ***Linear growth condition :*** *There exists a constant K such that for all $x \in \mathbb{R}^n$ and $t \geq 0$,*

$$\| a(t,x) \| + \| b(t,x) \| \leq K(1 + \|x\|).$$

Under these conditions, there exists a unique strong solution X to the stochastic differential equation (2.1) with continuous paths. Moreover, there is a constant C such that

$$\mathbb{E}\left[\|X_t\|^2\right] \leq Ce^{Ct}\left(1 + \|x\|^2\right).$$

Proof : See [19]

2.2 d-Dimensional Stochastic Differential Equations

The relationship between vector-valued and scalar stochastic differential equations is analogous to that between vector and scalar stochastic integrals. In the following, vectors are interpreted as column vectors, and their transposes as row vectors.

Consider an m-dimensional Brownian motion $B = \{B_t; t \geq 0\}$, where the components $B_t^1, B_t^2, \ldots, B_t^m$ are independent scalar Brownian motions, meaning that $\mathrm{Cov}(B^i, B^j) = 0$ for $i \neq j$. It is evident that each component B_t^i is $\mathcal{F}_t$-adapted.

Let us define a vector-valued function

$$a(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$$

and an $(d \cdot m)$-dimensional matrix-valued function

$$b(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}.$$

Then, we consider the following d-dimensional stochastic differential equation:

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t, \tag{2.3}$$

which can also be expressed in integral form as

$$X_t = x_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s. \tag{2.4}$$

The Lebesgue and Itô integrals are computed component-wise:

$$X_t^i = x_0^i + \int_0^t a^i(s, X_s)ds + \sum_{j=1}^{m} \int_0^t b^{i,j}(s, X_s)dB_s^j, \tag{2.5}$$

for $i = 1, \dots, d$. In matrix form, this equation can be rewritten as

$$d\begin{bmatrix} X_t^1 \\ X_t^2 \\ \vdots \\ X_t^d \end{bmatrix} = \begin{bmatrix} a^1(t, X_t) \\ a^2(t, X_t) \\ \vdots \\ a^d(t, X_t) \end{bmatrix} dt + \begin{bmatrix} b^{1,1}(t, X_t) & \cdots & b^{1,m}(t, X_t) \\ b^{2,1}(t, X_t) & \cdots & b^{2,m}(t, X_t) \\ \vdots & \ddots & \vdots \\ b^{d,1}(t, X_t) & \cdots & b^{d,m}(t, X_t) \end{bmatrix} d\begin{bmatrix} B_t^1 \\ B_t^2 \\ \vdots \\ B_t^m \end{bmatrix}. \tag{2.6}$$

All previously introduced definitions for weak and strong solutions remain valid in this vector setting. Moreover, the existence and uniqueness theorem for strong solutions applies without additional difficulty, provided that absolute values in previous assumptions are replaced by vector norms.

2.3 Itô equation

The formal differential form of an Itô stochastic differential equation defined on a probability space$(\Omega, (\mathscr{F}_t)_{t \geq t_0}, \mathbb{P})$ is as follows:

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t, \tag{2.7}$$

which the integral equation correlates to :

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s. \tag{2.8}$$

Here, the second integral represents an Itô integral, where $(B_t)_{t \geq 0}$ is a m-dimensional Wiener process, and $(\mathscr{F}_t)_{t \geq 0}$ is the associated filtration. The initial condition X_0, the drift function a and the diffusion function satisfy the following properties:

1. For all $t \geq t_0$, X_0 is an $\mathscr{F}_{t_0}$-measurable random variable that is independent of $B_t - B_{t_0}$.This is especially true when X_0 is not random.

2. The function a is an $\mathbb{R}^d$-valued function, measurable in (s, w), and non-anticipative, meaning that for all $t \in [t_0, T]$, $a(t, \cdot)$ is $\mathscr{F}_t$-measurable. Additionally, it satisfies the integrability condition
$$\mathbb{E}\left[\int_{t_0}^t |a(s, w)|^2\, ds\right] < \infty.$$

3. The function b is an $d \times m$ matrix-valued function belonging to $L^2([t_0, t])$, meaning that it satisfies the integrability condition
$$\mathbb{E}\left[\int_{t_0}^t \|b(s, w)\|^2 ds\right] < \infty, \quad \text{for almost all } w \in \Omega.$$

2.4 Itô's formula

Similar to ordinary differential equations, it is generally not possible to obtain an analytical form for the solutions of a stochastic differential equation. Nevertheless, a number of SDEs do admit an analytic solution (i.e., the differential of a stochastic process) which can often be derived using Itô's formula. This section is dedicated to presenting and applying Itô's formula through several examples.

Moreover, the definition of the Itô integral is not very practical; and, as with the Lebesgue integral, it is essential to rely on powerful results instead of attempting to approximate the functions of interest by elementary functions.

Definition 2.4.1 *A stochastic process $(X_t)_{t\geq 0}$ adapted to a filtration $\{\mathscr{F}_t\}$ is called an Itô process or a semi martingale if it can be expressed in the form*

$$X_t = X_0 + \int_0^t K(s)\, ds + \int_0^t H(s)\, dB_s, \tag{2.9}$$

where $H(t) \in H^2([0,\infty))$, $K(t)$ is adapted to the filtration $\{\mathscr{F}_t\}$, and

$$\mathbb{P}\left(\int_0^t |K(u)|\, du < \infty\right) = 1.$$

The process $(X_t)_{t\geq 0}$ is said to have a unique stochastic differential given by

$$dX_t = K_t\, dt + H_t\, dB_t. \tag{2.10}$$

It is clear that the differential operator "d" is linear. We now present the differentiation formulas for the product of two Itô processes and for a composed function.

Theoreme 2.4.1 (Integration by Parts Formula) *If the processes $X_1(t)$ and $X_2(t)$ admit stochastic differentials given by*

$$dX_1(t) = K_1(t)dt + H_1(t)dB_t,$$
$$dX_2(t) = K_2(t)dt + H_2(t)dB_t,$$

then the product process $X_1(t)X_2(t)$ also has a stochastic differential given by

$$d(X_1(t)X_2(t)) = X_1(t)dX_2(t) + X_2(t)dX_1(t) + d\langle X_1(t), X_2(t)\rangle \tag{2.11}$$

where

$$d\langle X_1(t), X_2(t)\rangle = d[X_1(t), X_2(t)] = H_1(t)H_2(t)dt.$$

In particular, for a continuously differentiable function f on $[0,T]$, we have the following relation:

$$\int_0^t f(u)\, dB_u = f(t)B_t - \int_0^t f'(u)B_u\, du.$$

Remark 2.4.1 *We have the following multiplication table:*

$$\begin{cases} (dt)^2 = 0, \\ dB_t^i dB_t^j = \delta_{ij} dt, \\ dB_t dt = dt dB_t = 0, \end{cases}$$

where δ_{ij} is the Kronecker delta.

Let us now state a theorem that defines the bracket process $\langle\cdot,\cdot\rangle$.

Theoreme 2.4.2 *Let $X_1(t)$ and $X_2(t)$ be two Itô processes satisfying*

$$dX_1(t) = K(t)dt + H(t)dB_t,$$

$$dX_2(t) = K'(t)dt + H'(t)dB_t.$$

Then, the bracket process $\langle X_1, X_2 \rangle$ is defined as follows:

1 $\langle X_1, X_2 \rangle_t$ *is a symmetric bilinear form.*

2 *If $X(t)$ is an Itô process, i.e., $dX(t) = K_t dt + H_t dB_t$, then*

$$\left\langle \int_0^t K_s ds, X_t \right\rangle_t = 0.$$

3 *If $i \neq l$, then*

$$\left\langle \int_0^t H_s dB_s^i, \int_0^t H_s dB_s^l \right\rangle_t = 0.$$

4

$$\left\langle \int_0^t H_s dB_s, \int_0^t H_s' dB_s \right\rangle_t = \int_0^t H_s H_s' ds.$$

2.4.1 One dimensional formula

Theoreme 2.4.3 (First Itô Formula) *Let $X(t)$, $t \geq 0$, be an Itô process satisfying*

$$dX(t) = K_t dt + H_t dB_t,$$

and let $f(x)$ be a function that is twice continuously differentiable. Then, the first Itô formula states that

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X, X\rangle_t,$$

where

$$d\langle X, X\rangle_t = H_t^2 dt.$$

Thus, we can write it in a more explicit form as

$$df(X_t) = \left(f'(X_t)K_t + \frac{1}{2}f''(X_t)H_t^2\right)dt + f'(X_t)H_t dB_t.$$

Theoreme 2.4.4 (Second Itô Formula) *If a process $X(t)$ has a stochastic differential given by*

$$dX(t) = a(t)dt + b(t)dB_t,$$

and if $f(t,x)$ is a real-valued function defined for $t \in [0,T]$ and $x \in \mathbb{R}$, which is continuous and has partial derivatives $f_t'(t,x)$, $f_x'(t,x)$, and $f_{xx}''(t,x)$, then the process $f(t,X(t))$ also has a stochastic differential given by

$$df(t,X(t)) = f_t'(t,X(t))dt + f_x'(t,X(t))dX_t + \frac{1}{2}f_{xx}''(t,X(t))d\langle X,X\rangle_t.$$

Equivalently, we can write

$$df(t,X(t)) = \left[f_t'(t,X(t)) + f_x'(t,X(t))a(t) + \frac{1}{2}f_{xx}''(t,X(t))b^2(t)\right]dt + f_x'(t,X(t))b(t)dB_t. \quad (2.12)$$

Remark 2.4.2 *Let $B(t)$ be a Brownian motion, and consider a function $f(x)$ that is twice differentiable. Then, applying Itô's formula, we obtain*

$$df(B(t)) = f'(B(t))dB(t) + \frac{1}{2}f''(B(t))d\langle B,B\rangle_t.$$

Since for a Brownian motion, we have $d\langle B,B\rangle_t = dt$, it follows that

$$df(B(t)) = f'(B(t))dB(t) + \frac{1}{2}f''(B(t))dt. \tag{2.13}$$

Theoreme 2.4.5 (Third Itô Formula) *Let X^1 and X^2 be two Itô processes starting from x_1 and x_2, respectively. These processes have drift coefficients a_1 and a_2, diffusion coefficients b_1 and b_2, and are driven by two correlated Brownian motions B_1 and B_2 with correlation coefficient ρ. Assume that a_i and b_i are adapted to the filtration $\mathscr{F}_{B_i}(t)$ for $i = 1,2$.*

Then, applying Itô's formula, we obtain:

$$\begin{aligned} f(X_t^1, X_t^2) &= f(x_1,x_2) + \int_0^t f_1'(X_s^1,X_s^2)\,dX_s^1 + \int_0^t f_2'(X_s^1,X_s^2)\,dX_s^2 \\ &+\frac{1}{2}\int_0^t \left[f_{11}''(X_s^1,X_s^2)(b^1(s))^2 + 2\rho f_{12}''(X_s^1,X_s^2)b^1(s)b^2(s) + f_{22}''(X_s^1,X_s^2)(b^2(s))^2\right]ds. \end{aligned} \tag{2.14}$$

Here, f_i' denotes the partial derivative of f with respect to x_i, and f_{ij}'' represents the second-order partial derivative with respect to x_i and x_j, for $i,j = 1,2$.

2.4.2 Multi dimensional formula

Suppose that the stochastic processes $X_1(t), X_2(t), \dots, X_n(t)$ satisfy the stochastic differential equations:

$$dX_i(t) = a_i(t)dt + b_i(t)dB_t, \quad (i = 1,\dots,n).$$

Let $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with partial derivatives

$$\frac{\partial f}{\partial t}, \quad \frac{\partial f}{\partial x_i}, \quad \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad \text{for } i,j = 1,\dots,n.$$

Then, the process $f(t, X_1(t), X_2(t), \dots, X_n(t))$ satisfies the following stochastic differential equation:

$$\begin{aligned} df(t,X_1(t),X_2(t),\dots,X_n(t)) = &\left[\frac{\partial f}{\partial t}(t,X_1,\dots,X_n) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t,X_1,\dots,X_n)a_i(t)\right. \\ &\left. + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i\partial x_j}(t,X_1,\dots,X_n)b_i(t)b_j(t)\right]dt \\ &+ \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t,X_1,\dots,X_n)b_i(t)dB_t. \end{aligned} \tag{2.15}$$

Remark 2.4.3 *An analogous formula to (2.15) is as follows: if the processes $X_1(t),\dots,X_n(t)$ satisfy the stochastic differential equations in $[0,T]$*

$$dX_k(t) = a_k(t)dt + \sum_{j=1}^m b_{kj}(t)dB_j(t), \quad (k = 1,\dots,n),$$

and if $U(t,x_1,\dots,x_n)$ is a continuous function with continuous partial derivatives

$$\frac{\partial U}{\partial t}, \quad \frac{\partial U}{\partial x_k}, \quad (k=1,\dots,n), \quad \frac{\partial^2 U}{\partial x_i\partial x_j}, \quad (i,j=1,\dots,n),$$

then the function $\phi(t) = U(t, X_1(t), \dots, X_n(t))$ also has a stochastic differential (see [9], page 456).

$$\begin{aligned} d\phi(t) = \Bigg[&\frac{\partial U}{\partial t}(t, X_1(t), \dots, X_n(t)) + \sum_{k=1}^{n} \frac{\partial U}{\partial x_k}(t, X_1(t), \dots, X_n(t)) a_k(t) \\ &+ \frac{1}{2} \sum_{i,k=1}^{n} \frac{\partial^2 U}{\partial x_i \partial x_k}(t, X_1(t), \dots, X_n(t)) \sum_{j=1}^{m} b_{ij}(t) b_{kj}(t) \Bigg] dt \\ &+ \sum_{j=1}^{m} \left[\sum_{k=1}^{n} \frac{\partial U}{\partial x_k}(t, X_1(t), \dots, X_n(t)) b_{jk}(t) \right] dB_j(t). \end{aligned} \tag{2.16}$$

Remark 2.4.4 *Itô's formula allows us to show that if B is a standard one-dimensional Brownian motion and X is an $\mathscr{F}_t^B$-adapted process, with $p \in [1, \infty)$ and $T > 0$ such that*

$$\mathbb{E}\left(\int_0^T |X_s|^{2p} \, ds \right) < \infty,$$

then

$$\mathbb{E}\left[\left| \int_0^T X_s dB_s \right|^{2p} \right] \le \left[p(2p-1) \right]^p T^{p-1} \mathbb{E}\left[\int_0^T |X_s|^{2p} \, ds \right].$$

2.4.3 Example: How to apply Itô's formula to an equation

Consider the stochastic differential equation (SDE)

$$dX(t) = \mu X(t) \, dt + \sigma X(t) \, dB_t, \quad X(0) = x_0, \tag{2.17}$$

where $\mu, \sigma \in \mathbb{R}$, and $(B_t)_{t \ge 0}$ is a standard Brownian motion.

We aim to find a function $f(t, B_t)$ such that $X(t) = f(t, B_t)$, and apply Itô's formula to derive the explicit solution of this SDE.

Guess for the solution. Suppose that $X(t)$ has the form

$$X(t) = f(t, B(t)) = \exp\left(\sigma B(t) + a g(t) \right), \tag{2.18}$$

where $a \in \mathbb{R}$ is a constant and $g(t)$ is a differentiable deterministic function to be determined.

Step 1: Compute partial derivatives. Let $f(t, x) = \exp(\sigma x + a g(t))$. Then:

$$\begin{aligned} \frac{\partial f}{\partial t} &= a g'(t) f(t, x), \\ \frac{\partial f}{\partial x} &= \sigma f(t, x), \\ \frac{\partial^2 f}{\partial x^2} &= \sigma^2 f(t, x). \end{aligned}$$

Step 2: Apply Itô's formula. Using Itô's formula for functions of t and $B(t)$, we obtain:

$$\begin{aligned} dX(t) &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} dt \\ &= f(t, B(t)) \left[a g'(t) \, dt + \sigma \, dB_t + \frac{1}{2} \sigma^2 \, dt \right] \\ &= X(t) \left[\left(a g'(t) + \frac{1}{2} \sigma^2 \right) dt + \sigma \, dB_t \right]. \end{aligned}$$

Step 3: Match with the original SDE. To match the drift and diffusion terms with the original SDE, we require:

$$ag'(t)+\frac{1}{2}\sigma^2=\mu \quad \Longrightarrow \quad g'(t)=\frac{1}{a}\left(\mu-\frac{1}{2}\sigma^2\right).$$

Integrating:

$$g(t)=\frac{1}{a}\left(\mu-\frac{1}{2}\sigma^2\right)t+C,$$

for some constant $C\in\mathbb{R}$.

Step 4: Determine the constant C. From the initial condition $X(0)=x_0$, we get:

$$X(0)=\exp(\sigma\cdot 0+ag(0))=\exp(aC)=x_0 \quad \Longrightarrow \quad C=\frac{1}{a}\ln x_0.$$

Conclusion. Replacing $g(t)$ in the guessed form yields the explicit solution:

$$X(t)=x_0\exp\left(\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma B(t)\right). \tag{2.19}$$

This confirms that the guessed form for $X(t)$ satisfies the SDE, and illustrates how Itô's formula can be used to derive solutions to stochastic differential equations.

Remark 2.4.5 *The explicit solution obtained via Itô's formula illustrates not only the method's effectiveness but also provides key properties of the process, such as continuity, positivity, and adaptability. These aspects will be fundamental in the study of stochastic viability presented later.*

2.5 Stratonovich equation

In certain applications, it is essential to express the stochastic differential equation (SDE) in a form where the standard Itô integral is replaced by the Stratonovich integral. Such an equation is referred to as a Stratonovich stochastic differential equation. It takes the form

$$dX_t=a(t,X_t)dt+b(t,X_t)\circ dB_t, \tag{2.20}$$

or, in its integral representation,

$$X_t=x_0+\int_0^t a(s,X_s)ds+\int_0^t b(s,X_s)\circ dB_s. \tag{2.21}$$

Here, the symbol $\circ$ indicates that the integration follows Stratonovich's interpretation.

The advantage of the Stratonovich integral is that it induces classical chain rule formulas under a change of variables:

$$dF(t,X_t)=\frac{\partial F}{\partial t}dt+(\nabla_X^\top F)dX_t. \tag{2.22}$$

Any solution X_t satisfying the Stratonovich equation is also a solution to the corresponding Itô equation, since the drift term is related by

$$a(t,x)=a(t,x)+\frac{1}{2}b(t,x)\frac{\partial b}{\partial x}(t,x). \tag{2.23}$$

it will be called conversion formula.

Example 2.5.1 *Consider the Stratonovich stochastic differential equation*

$$dX_t = 2X_t \circ dB_t,$$

and the corresponding Itô stochastic differential equation

$$dX_t = 2X_t dt + 2X_t dB_t.$$

Both equations share the same solution, given by

$$X_t = X_0 e^{B_t - B_0}.$$

2.6 Conversion Formula

The solution of a modified Itô equation is equivalent to the solution of a Stratonovich stochastic differential equation (2.20):

$$dX_t = a_{\text{cor}}(t, X_t)dt + b(t, X_t)dB_t, \tag{2.24}$$

where the corrected drift term is given by

$$a_{\text{cor}}(t,x) = a(t,x) + \frac{1}{2} b(t,x) \frac{\partial b}{\partial x}(t,x). \tag{2.25}$$

The additional term $\frac{1}{2} b(t,x) \frac{\partial b}{\partial x}(t,x)$ is known as the Wong-Zakai correction term.

For the multidimensional case, where

$$a : \mathbb{R}^{d+1} \to \mathbb{R}^d, \quad a(t,x) = (a^1(t,x), \ldots, a^n(t,x)),$$

and

$$b : \mathbb{R}^{d+1} \to \mathbb{R}^{d \times m}, \quad b(t,x) = (b^{i,j}(t,x))_{1 \le i \le n, 1 \le j \le d},$$

the correction term generalizes to

$$a_{\text{cor},i}(t,x) = a^i(t,x) + \frac{1}{2} \sum_{j=1}^{d} \sum_{k=1}^{n} \frac{\partial b^{i,j}}{\partial x_k} b^{k,j}, \quad 1 \le i \le n. \tag{2.26}$$

Due to the explicit connection (2.24) between the Stratonovich and Itô formulations, many theoretical results can be established for only one type of integral. The Stratonovich integral has the advantage of obeying the classical chain rule under changes of variables, which makes it particularly useful in geometric contexts, such as stochastic differential equations on manifolds (see Ikeda and Watanabe (1989)). However, unlike the Itô integral, the Stratonovich integral does not preserve the martingale property, which makes the Itô formulation more suitable for computational purposes.

Chapter 3

Stochastic viability and stochastic persistence problem on Kubo oscillator

3.1 introduction

A deterministic ordinary differential equation of the following type is examined:

$$\frac{dy}{dt} = f(t, y), \quad y \in \mathbb{R}^n,\ n \in \mathbb{N}^*. \tag{3.1}$$

By including a "noise" term in the conventional deterministic equation, a stochastic perturbation is considered as follows:

$$\frac{dy}{dt} = f(t, y) + \text{"noise"} \tag{3.2}$$

and to use a stochastic term in place of the "noise" term as

$$dy_t = f(t, y_t)\,dt + \sigma(t, y_t)\,dW_t \tag{3.3}$$

W_t is the normal Wiener process .

This work focuses on the selection of admissible stochastic models that preserve viability (e.g., positivity, conservation laws) rather than determining the exact form of the stochastic perturbation. In another way, assume that the invarance of a closed subset S satisfied by a classical ODE of the form (3.1). In what circumstances does a stochastic perturbation of type (3.3) also satisfy this properties ?

The literature on manifold invariance for stochastic differential equations is extensive but abstract, especially regarding the stochastic analog of the Nagumo-Brezis theorem. This part provides a simple derivation of necessary and sufficient conditions for preserving a submanifold under stochastic perturbations, with distinct conditions for Stratonovich and Itô frameworks.

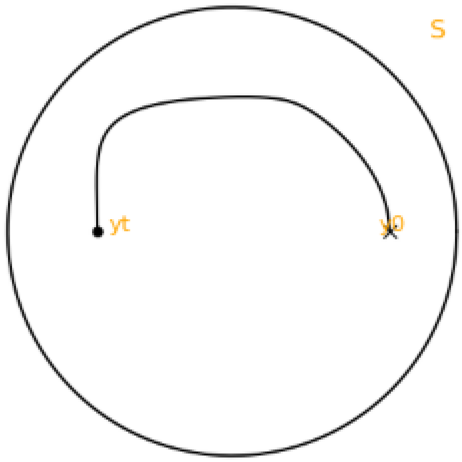
3.2 Deterministic invariance - viability

We examine an ordinary differential equation with the following form:

$$\begin{cases} \dot{y}_t &= f(t, y_t), \\ y(0) &= y_0 \end{cases} \tag{ODE}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function, and $y_0 \in \mathbb{R}^n$ is the initial condition.

Definition 3.2.1 *A closed set $S \subset \mathbb{R}^n$ is considered invariant with respect to the flow generated by the differential equation (ODE) if, for every initial condition $y_0 \in S$, the solution $y_t(y_0)$, which originates from y_0 at $t = 0$, remains entirely within S for all $t \in \mathbb{R}^+$.*



The distinction between viability and invariance becomes clear when multiple solutions to the (ODE) exist. Viability is preserved if at least one solution to the (ODE) corresponding to the initial condition y_0 remains within the subset S, whereas invariance is preserved if all solutions starting from y_0 lie within S. In other words, for any $t_0 \geq 0$ and any y_0, every solution to the (ODE) is viable within S.

The basic premise of viability is a tangency condition, which generally asserts that the vector $f(t, y)$ is either tangent to S or points into the interior of S at a boundary point $y \in \partial S$. In fact, whenever y lies in the interior of S, the tangent space at y is equal to $\mathbb{R}^n$.

Theoreme 3.2.1 (Nagumo's theorem) *Let $S \subset \mathbb{R}^n$ be a closed set, and suppose that the function f satisfies the linear growth condition . Then, S is viable under the dynamics induced by the differential equation if and only if the following criterion is fulfilled.*

$$f(t, y) \in \left\{ v \in \mathbb{R}^n : \lim_{h \to 0^+} \frac{d_S(y + hv)}{h} = 0 \right\}, \quad \forall (t, y) \in [t_0, \infty) \times S \tag{3.4}$$

where $d_S(y) = d(y, S)$ denotes the distance from the point y to the set S.

Specifically, a global solution satisfying $y(t) \in S$ for every $t \geq t_0$ exists if S is compact and if f is Lipschitz and continuous on $[t_0, \infty) \times S$.

If we represent the tangent plane of S at y by $T_y S$, that is,

$$T_y S = \left\{ v \in \mathbb{R}^n : \lim_{t \to 0^+} \frac{d_S(y + hv)}{h} = 0 \right\}$$

The invariance standard can thus be written as follows:

$$f(t, y) \in T_y S, \quad \forall (t, y) \in [t_0, \infty) \times \bar{S} \tag{Tc}$$

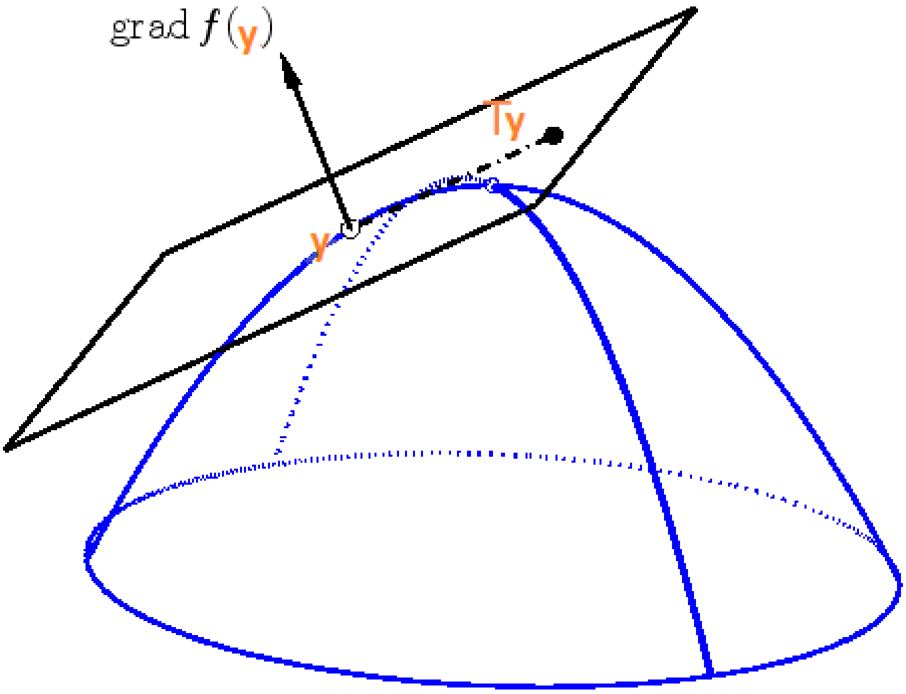
When S is a closed subset of $\mathbb{R}^n$, the final condition can be expressed as

$$N(y) \cdot f(t, y) \leq 0, \quad \forall (t, y) \in [t_0, \infty) \times \partial S \tag{Nc}$$

where the open ball with radius $|N(y)|$ and center at $y + N(y)$, which has no point in common with S, defines the outer normal $N(y)$, geometrically, the ball B touches S at y, with $y \in \partial S \cap B$.

The condition (Tc) implies (Nc), and if the map f is continuous in $\mathbb{R}^+ \times S$, the opposite is true. We find the proof in [3].

Theoreme 3.2.2 *Let f be a Lipschitz continuous bounded function and S be a closed set of $\mathbb{R}^n$. If and only if f meets the conditions (Tc) or (Nc), then a set S is invariant under the differential equation's flow.*



In 1942, Nagumo developed condition (Tc) and demonstrated the final theorem. The condition (Nc) dates back to 1969, when Bony demonstrated the equivalence assuming that $f = f(y)$ is locally Lipschitz continuous; Brezis demonstrated the same under condition (Tc) in 1970. This theorem clearly implies invariance in the following demonstration if f is continuous and if the condition about f ensures uniqueness (see [23], Chap.III).

Proof

Let $y_t(y_0) \in S$ for all t. Then,

$$y(t+h) = y(t) + h \cdot \dot{y} + h \cdot \epsilon(h),$$

where $\epsilon(h) \to 0$ as $h \to 0$. Therefore, $\dot{y} = f(t, y) \in T_{y(t)}S$ by the definition of the tangent space, where z is tangent to S if it can be expressed as the limit of a sequence z_n such that $y + h_n z_n$ lies within S for some sequence $h_n \to 0$.

For sufficiency, let y be a solution of the differential equation, with $y(t_0) \in S$ and $\rho(t) = \text{dist}(y(t), S)$. Suppose $y(s) \notin S$, i.e., $\rho(s) > 0$ for some $s > t_0$, and let $z \in \partial S$ such that $\rho(s) = |y(s) - z|$. The map $\sigma(t) = |y(t) - z|$ satisfies $\rho(t) \leq \sigma(t)$, with $\rho(s) \leq \sigma(s)$, and hence $D^+\rho(s) \leq \sigma'(s)$ (where D^+ is the Dini derivative). Note that σ is differentiable near s, and we have:

$$\frac{1}{2}\sigma^2 = \sigma\sigma' = (y(t) - z) \cdot y'(t) = (y(t) - z) \cdot f(t, y(t)).$$

Now, $N(z) = y(s) - z$ is a normal to S at z, and the invariance condition (Nc) implies $N(z) \cdot f(s, z) \leq 0$. Therefore, for $t = s$, we obtain:

$$\sigma\sigma' = N(z) \cdot f(s, y(s)) = N(z) \cdot (f(s, y) - f(s, z)) \leq L\rho^2,$$

where L is the Lipschitz constant. Thus, $D^+\rho(s) \leq L\rho$ whenever $\rho > 0$. Given our assumption, there exists an interval $[b, c]$ such that $\rho(b) = 0$ and $\rho > 0$ in $(b, c]$. By the comparison theorem (viability), it follows that $\rho = 0$ on $[b, c]$, which leads to a contradiction.

3.2.1 Geometric definition

The tangent plane of S at y is represented by $T_y S$, and the invariance requirement can be expressed as follows.

$$f(t,y) \in T_y S, \quad \forall (t,y) \in \mathbb{R}^+ \times S \tag{3.5}$$

We can define the normal vector $N(y)$ to the tangent hyperplane $T_y S$ in y since S has codimension 1 for any $y \in S$. This way:

$$T_y S = \{\, z \in \mathbb{R}^n \mid z \cdot N(y) = 0\}$$

Consequently, the invariance condition can be expressed as:

$$N(y) \cdot f(t,y) = 0, \quad \forall (t,y) \in \mathbb{R}^+ \times S \tag{3.6}$$

The normal vector to S at y equals $\nabla F(y)$ when S has the form (3.7): After that, the invariance requirement is as follows:

$$\nabla F(y) \cdot f(t,y) = 0, \quad \forall (t,y) \in \mathbb{R}^+ \times S \tag{IF}$$

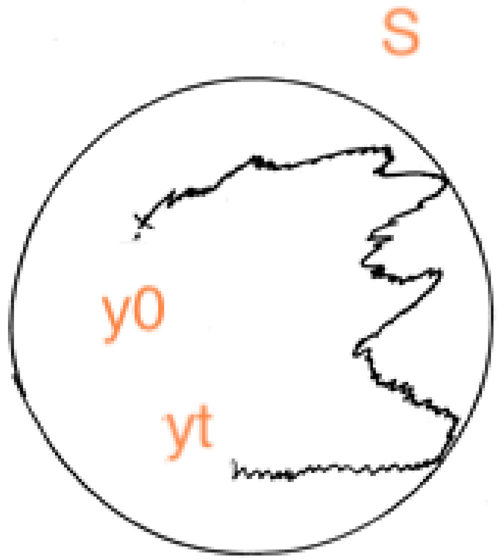
The trajectories are continuous but not differentiable in the stochastic situation. Therefore, it is not possible to use the preceding geometric condition. Two logical extensions of the concept of invariance in a stochastic context are covered in the sections that follow.

3.3 stochastic invariance - viability

The feasibility of a closed convex subset S of $\mathbb{R}^n$ in relation to the autonomous stochastic system in the Itô scenario has been studied by **J.P. Aubin** and **G.Da Prato**.

$$dy_t = f(y_t)\,dt + \sigma(y_t)\,dW_t \tag{aIE}$$

The requisite condition was provided in [4], and its sufficiency was demonstrated in [5] . **A. Milian** extended these findings in 1993 in [15] to any closed subset, even if it is not convex under the flow of a non autonomous equation.



Theoreme 3.3.1 (Stochastic Nagumo's viability theorem) *: Assuming that S is a closed subset of $\mathbb{R}^n$, we anticipate that either*

- *f and σ are Lipschitz maps.*

- *The f and σ maps are bounded and continuous.*

If and only if each $\mathscr{F}_t$-random variable y in S is within the flow of,

$$\begin{cases} dy_t = f(t, y_t)\,dt + \sigma(t, y_t)\,dW_t, \\ y_{t_0} = y_0 = c, \quad t_0 \le t \le T < 1 \end{cases} \tag{IE}$$

then S is viable;

$$(f(y), \sigma(y)) \in T_S(t, y) \tag{STC}$$

Assuming that the diffusion part σ is zero, the stochastic contingent set coincides with the tangent of S as provided by deterministic Nagumo's theorem. In contrast, the set $T_S(t, y)$ is the stochastic contingent set to S.

The following is the definition of the stochastic contingent set:

Definition 3.3.1 *The stochastic contingent set of a closed subset S is the collection of pairs (γ, v) of $\mathscr{F}_t$-random variables that meet the following criteria:*
Sequences of $\mathscr{F}_{t+h_n}$-measurable random variables a^n and b^n, and of $h_n > 0$, converge to 0 in such a way that

$$\begin{cases} i) & \mathbb{E}[\|a^n\|^2] \to 0 \\ ii) & \mathbb{E}[\|b^n\|^2] \to 0 \\ iii) & \mathbb{E}[b^n] = 0 \\ iv) & b^n \text{ is independent of } \mathscr{F}_t \end{cases}$$

and fulfilling for nearly every $w \in \Omega$;

$$\forall n \ge 0: \quad y_w + v_w\,(W_w(t + h_n) - W(t)) + h_n \gamma_w + h_n a^n + \sqrt{h_n b^n} \in S.$$

Using a function F as the unit sphere, we specialize this theorem in this chapter to sub-manifolds of $\mathbb{R}^n$ of codimension 1.

where, an m-dimensional topological sub-manifold (subset) of an n-dimensional topological manifold S is a subset $N \subset S$ which is an m-dimensional manifold in the induced topology. So, the number $n - m$ is called the co-dimension of the sub-manifold. as an example the sphere S^2, which is of co-dimension 1 define by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

3.3.1 Stochastic invariance for co-dimension 1 sub-manifolds

This section presents an invariance condition for a submanifold S of co-dimension 1 in $\mathbb{R}^n$, which corresponds to the zero set of a function $F : \mathbb{R}^n \to \mathbb{R}$ of class $\mathscr{C}^2$, i.e.,

$$S = \{y \in \mathbb{R}^n \mid F(y) = 0\}. \tag{3.7}$$

This section discusses an explicit stochastic Nagumo-Brezis Theorem, as proven by Milian [16] and Aubin-Da Prato [4], is often a good challenging for non-specialists. The key is that our criterion can be easily concluded using fundamental stochastic calculus results.

Definition 3.3.2 (Viability in probability 1) *For the stochastic system (IE), we refer to a subset S as invariant in probability 1 if, for any random variable y_0 in S, there is almost certainly a solution $y(t)$ to the equation (IE) beginning at y_0 that satisfies*

$$\mathbb{P}\left\{w \in \Omega : y(t; w) \in S, \textit{for all } t \in [t_0, +\infty)\right\} = 1. \tag{3.8}$$

One is led to a weaker concept of invariance in particular circumstances.

Definition 3.3.3 (Viability in mean) *When a solution $y(t)$ to the equation (IE) beginning at y_0 exists for any random variable y_0 such that $E(y_0) \in S$, we refer to this subset S as viable in mean for the stochastic system (IE) if it satisfies:*

$$\mathbb{E}[y(t)] \in S, \quad \textit{for all } t \in [t_0, +\infty). \tag{Vm}$$

These final two definitions are provided as follows in our instance:

Definition 3.3.4 (Strong persistence) *For the stochastic system (IE), a submanifold S is invariant in the strong sense if, for each initial data set $y_0 \in S$, the corresponding solution $y(t)$ almost certainly satisfies:*

$$\mathbb{P}\left\{F(y(t)) = 0 \quad \forall t \in [t_0, +\infty)\right\} = 1.$$

That is to say, the solution most likely achieves values inside the submanifold S.

Definition 3.3.5 (Weak persistence) *In the weak sense for the stochastic system (IE), we refer to a submanifold S as invariant if the corresponding solution $y(t)$, meets the following for each initial data $y_0 \in S$ in the mean:*

$$\mathbb{E}[F(y(t))] = 0, \quad \textit{for all } t \in [t_0, +\infty).$$

The weak persistence is obviously implied by the strong one. Yes, if

$$\mathbb{E}[F(y(t))] = \mathbb{E}\left[F(y(t))\mathbb{1}_A + F(y(t))\mathbb{1}_{A^c}\right] = 0.$$

so that

$$A = \left\{w \in \Omega : F(y(t; w)) = 0, \forall t \in [t_0, +\infty)\right\}.$$

The opposite isn't always the case.

3.3.2 Explicit conditions in Itô case

In this instance, the deterministic invariance condition (IF), when applied to the stochastic perturbation that is, $\sigma(t, y_t) \in T_{y_t}S$ ensures the strong persistence of the invariant submanifold S. However, for any stochastic perturbation satisfying $\mathbb{E}\left[\int_0^t \nabla F(y_s) \cdot \sigma(s, y_s) dW_s\right] = 0$, the specific form of σ has no effect on the weak persistence (W_s is a d standard Brownian motion).

Theoreme 3.3.2 (strong invariance) *Assume that function F invariant under the deterministic flow associated with $dy_t = f(t, y_t, \mu)\, dt$, $y \in \mathbb{R}^n$ defines S as a submanifold, that is:*

$$\nabla F(y) \cdot f(t, y) = 0, \quad \textit{for all } y \in S,\ t \geq 0.$$

Under the stochastic system's (IE) flow, the submanifold S is strongly invariant if and only if:

$$\nabla F(y) \cdot \sigma(t, y) = 0, \quad \textit{for all } y \in S,\ t \geq 0.$$

and

$$\sum_{i,j} \frac{\partial^2 F}{\partial y_i \partial y_j}(y_t) \sum_{l=1}^{d} \sigma_{i,l}(t, y_t)\sigma_{j,l}(t, y_t) = 0 \tag{3.9}$$

Proof: The Itô formula, which will assist us in establishing the invariance requirement, is the crucial instrument in this situation. Indeed, a process y_t leaves the submanifold S invariant if and only if the stochastic process associated with y_t satisfies $F(y_t) = 0$ for every t, almost surely, where it is defined for all initial conditions $y_0 \in S$ a.s.

The formula for multidimensional Itô is as follows:

$$d[F(y_t)] = \nabla F(y_t)\, dy_t + \frac{1}{2}\sum_{i,j} \frac{\partial^2 F}{\partial y_i \partial y_j}(y_t)\, dy_i(t)\, dy_j(t)$$

Consequently, we get:

$$\begin{aligned} d[F(y_t)] &= \nabla F(y_t) f(t, y_t)\, dt + \nabla F(y_t)\, \sigma(t, y_t)\, dW_t \\ &+ \frac{1}{2}\sum_{i,j} \frac{\partial^2 F}{\partial y_i \partial y_j}(y_t) \sum_{l=1}^{d} \sigma_{i,l}(t, y_t)\, \sigma_{j,l}(t, y_t)\, dt \end{aligned} \tag{3.10}$$

Since the manifold S is supposed to be invariant in the deterministic situation, we have $\nabla F(y_t) \cdot f(t, y_t) = 0$, where the gradient of F is always normal to the tangent space of S. It is still

$$\begin{aligned} d[F(y_t)] &= \nabla F(y_t)\, \sigma(t, y_t)\, dW_t \\ &+ \frac{1}{2}\sum_{i,j} \frac{\partial^2 F}{\partial y_i \partial y_j}(y_t) \sum_{l=1}^{d} \sigma_{i,l}(t, y_t)\, \sigma_{j,l}(t, y_t)\, dt. \end{aligned} \tag{3.11}$$

If and only if the perturbation σ satisfies the invariance condition (IF), the only contribution to the stochastic part, $\nabla F(y_t)\, \sigma(t, y_t)$, is equal to zero. The preceding expression then becomes:

$$d[F(y_t)] = \frac{1}{2}\sum_{i,j} \frac{\partial^2 F}{\partial y_i \partial y_j}(y_t) \sum_{l=1}^{d} \sigma_{i,l}(t, y_t)\, \sigma_{j,l}(t, y_t)\, dt \tag{3.12}$$

It provides us with the third conditional.

We obtain the following condition if we suppose that the stochastic perturbation takes the simplest form, where $\sigma_{i,j} = \delta^i_j$ and $d = n$:

$$\sum_{i=1}^{d} \left(\frac{\partial^2 F}{\partial y_i^2}(y_t)\, \sigma_{i,i}(t, y_t)^2 \right) = 0, \quad \forall (t, y) \in \mathbb{R}^+ \times S.$$

According to the preceding Theorem, in the Itô example, it is impossible to restore the invariance of a given manifold using a direct stochastic perturbation of a deterministic equation unless σ and F take very precise forms.

We can specialize the general finding to the case of the sphere, which is useful for exploring invariance properties connected to the Kubo oscillator.

Corollary 3.3.1 *If the stochastic perturbation on the sphere is null, that is, if and only if the sphere S is invariant under the flow of the stochastic system (IE), then*

$$\sigma_{i,i}(t, y) = 0, \quad i = 1, \dots, n, \quad \forall\, t \in \mathbb{R}^+ and y \in S^{n-1}.$$

The fact that $F(y) = \sum_{i=1}^{n} y_i^2$ provides the proof, which means that condition (3.9) reduces to:

$$\sum_{i=1}^{n} [\sigma_{i,i}(t, y_t)]^2 = 0, \quad \forall\, (t, y) \in \mathbb{R}^+ \times S. \tag{3.13}$$

3.3.3 Explicit conditions in Stratonovich case

The Stratonovich equation can be directly interpreted as follows:

Theoreme 3.3.3 (strong invariance) *Under the stochastic system's flow, the submanifold S is invariant in the Stratonovich sense if and only if:*

$$f(t,y) \in T_y S \quad and \quad \sigma(t,y) \in T_y S, \quad \forall (t,y) \in \mathbb{R}^+ \times S. \tag{3.14}$$

Proof: Since the Stratonovich stochastic calculus functions similarly to the standard differential calculus, we get

$$d[F(y_t)] = \nabla F(y_t) \cdot f(t,y_t)\, dt + \nabla F(y_t) \cdot \sigma(t,y_t) \circ dW_t$$

The proof is concluded when this quantity is zero if and only if $\nabla F(y_t) \cdot f(t,y_t) = 0$ and $\nabla F(y_t) \cdot \sigma(t,y_t)) = 0$.

The Stratonovich scenario is quite similar to the typical deterministic scenario.

The following "explicit" conditions for the sphere in the Stratonovich example are a simple result of theorem (3.3.3).

Corollary 3.3.2 *Under the stochastic system's (2.20) flow, the sphere S is invariant if and only if:*

$$y \cdot f(t,y) = y \cdot \sigma(t,y) = 0, \quad \forall\, (t,y) \in \mathbb{R}^+ \times S.$$

Proof: This is because, if and only if $N \wedge y = 0$, a vector N is normal to the tangent plane of S^2 at point y. Consequently, if and only if $y \cdot f(t,y) = 0$ and $y \cdot \sigma(t,y) = 0$, then the vectors $f(t,y)$ and $\sigma(t,y)$ belong to the tangent manifold $T_y S^2$ at point y.

3.4 Applications: the Kubo oscillator system

Kubo oscillator with strong invariance in Stratonovich interpretation: Let's look at the stochastic Kubo oscillator model that Milstein defined in [17]

$$\begin{cases} dY_1 = -aY_2\, dt - \sigma Y_2 \circ dW_t, & Y_1(0) = 1, \\ dY_2 = \ \ aY_1\, dt + \sigma Y_1 \circ dW_t, & Y_2(0) = 0, \end{cases} \tag{3.15}$$

where W_t is a one-dimensional Brownian motion, and $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in \mathbb{R}^2$, with $a, \sigma \in \mathbb{R}$.

A Hamiltonian system is defined by

$$\begin{cases} \dot{Y}_1 = -\nabla_{Y_2} H(Y_1, Y_2), \\ \dot{Y}_2 = \nabla_{Y_1} H(Y_1, Y_2), \end{cases} \tag{3.16}$$

where $H(Y_1, Y_2)$ is a function which called first integral that is constant along solutions to the equation (3.16),i.e.;

$$H(Y_{1,t}, Y_{2,t}) = H(Y_{1,0}, Y_{1,0}), \forall t.$$

We assume that the subset S is defined by the function F which is given by

$$S = \left\{ \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in \mathbb{R}^2 : F(Y_1, Y_2) = H(Y_{1,t}, Y_{2,t}) - H(Y_{1,0}, Y_{2,0}) \right\}.$$

The systems (3.15) possess particular characteristics. Specifically, the function H_0 is a first integral that corresponds to the energy of the underlying system when it is independent of time. This means that for any solution of the deterministic system $(Y_{1,t}, Y_{2,t}) \in \mathbb{R}^2$, the subset S is invariant under the flow of (3.16), in other ways; we have

$$\frac{d}{dt}\left[H_0(Y_{1,t}, Y_{2,t})\right] = 0. \tag{3.17}$$

Consequently, under the flow of the deterministic system, every level surface of the function H_0 is invariant. Generally, a stochastic Itô disturbance destroys this quality (see corollary(3.3.1)). But as we'll show, the Kubo oscillator maintains these certain deterministic system characteristics in Stratonovich interpretation.

The Hamiltonian family in the Kubo oscillator instance is provided by:

$$H = \left\{H_0(Y) = Y_1^2 + Y_2^2, \quad H_1 = \sigma H_0.\right\}$$

where $\sigma \in \mathbb{R}$. Under the stochastic perturbation, the earlier finding on the invariance of the level surfaces of the Hamiltonian H_0 is maintained. Specifically, we have:

Lemma 3.4.1 *A strong first integral for the Kubo oscillator is the function $H_0(Y) = Y_1^2 + Y_2^2$, i.e.*

$$d\left[H_0(Y_{1,t}, Y_{2,t})\right] = 0. \tag{3.18}$$

indicating that the level surfaces defined by H_0 are highly invariant under the Kubo oscillator's flow over the system's solution.

Proof: We possess

$$\begin{aligned} d\left[H_0(Y_{1,t}, Y_{2,t})\right] &= 2Y_{1,t}\,dY_1 + 2Y_{2,t}\,dY_2 \\ &= -2aY_1Y_2\,dt - 2\sigma Y_1Y_2 \circ dW_t + 2aY_1Y_2\,dt + 2\sigma Y_1Y_2 \circ dW_t \\ &= 0. \end{aligned} \tag{3.19}$$

This ends the proof.

This means that any Kubo oscillator solution that begins with the initial condition $(Y_{1,0}, Y_{2,0})$ will stay on the circle $Y_1^2 + Y_2^2 = r_0^2$ which noted by H_r, where $r_0^2 = Y_{1,0}^2 + Y_{2,0}^2$ in stratonovich case, never in Itô.

Conclusion

Scientists are increasingly required to include stochastic effects in problems that were initially modeled in a deterministic way. The persistence of the original deterministic system's features during the stochasticization process is one of the basic modeling concerns that greatly depends on the type of stochasticity. The aim of this work, is the persistence of invariance or viability of some subset under the stochastic equation, in Itô interpretation and stratonovich interpretation. In the application we study the persistence of invariance property of the Kubo oscillator under a stochastic equation, in Itô we always lost the property but in stratonovich we need to the below condition on the diffusion part to preserve the invariance:

$$\nabla F(y_t) \cdot \sigma(t, y_t) = \nabla H(y_1, y_2) \cdot \sigma(t, y_t) = 0, \forall t, \forall y_t,$$

as the example given by G.N.Milstein in [17].

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