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List of symbols

$\langle \cdot \rangle$: Inner product.

H : Hilbert space.

$B(H)$: The set of all bounded linear operator defined on a Hilbert space H .

T : A linear operator defined on a Hilbert space H .

$D(T)$: The domain of T .

\bar{T} : The closure of the closable operator T .

I : The identity operator .

$\|T\|$: The norm of T .

$\ker(T)$: The kernel of T .

$Im(T)$: The image of T .

$G(T)$: The graph of T .

T^* : Adjoint operator of T .

$\sigma(T)$: The spectrum of T .

$\rho(T)$: The resolvent set of T .

$\sigma_p(T)$: The point spectrum of T .

T^{-1} : The inverse of T .

U : A partial isometry defined on a Hilbert space H .

$r(T)$: The spectral radius of T .

$|T|$: The modulus of T .

$\Re(T)$: The real part of T .

$\Im(T)$: The imaginary part of T .

Introduction

The class of closed linear operators is an important class of bounded and unbounded linear operators which is large enough to cover almost all interesting operators occurring in applications. Some of the standard textbooks on closed linear operators theory are [11] and [15].

The absolute value plays a prominent role in operator theory. For example, it appears in the polar decomposition of any closed linear operator.

The purpose of our works is to investigate when relations of the types $|ST| = |S||T|$, $|S \pm T| \leq |S| + |T|$ and $||S| - |T|| \leq |S \pm T|$ hold in an unbounded operator setting. In fact, our work is inspired by the paper "On the absolute value of unbounded operators" (see [1]).

Our work is divided into three chapters:

In the first chapter, we give a basic definitions and results about general closed operators on Hilbert space. We define and study closed linear operators, closable linear operators, the spectrum of closed operators, and adjoint operators. We also introduce some fundamental classes of unbounded operators as symmetric operators, self-adjoint operators, and normal operators.

In the second chapter, we present a generalization of the famous Reid inequality related to linear operators (see [13, 4]). Also, We gave a necessary conditions to make sure that

the relation $|ST| = |S||T|$ hold for closed operators S and T .

In the last chapter, we prove the triangle inequalities for closed linear operators. As consequences, we obtain a characterization of invertibility for the class of unbounded normal operators.

Essential background

In this chapter, we present some basic definitions and fundamental properties of closed linear operators, that we need in the coming chapters. Note that, most of the notions and the definitions are taken from [15].

1.1 Hilbert Spaces

Let E be a linear space over $\mathbb{K} = \mathbb{C}$.

Definition 1.1.1. *An inner product on E is a function $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{C}$, such that*

1. $\langle \lambda_1 x_1 + \lambda_2 x_2, y \rangle = \lambda_1 \langle x_1, y \rangle + \lambda_2 \langle x_2, y \rangle$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$.
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in E$ (Hermitian property).
3. $\langle x, x \rangle \geq 0$ for all $x \in E$, and $\langle x, x \rangle = 0$ implies $x = 0$.

Definition 1.1.2. *A complete normed inner product space is said to be a Hilbert space.*

Example 1.1.3. 1. Let $C([a, b])$ denote the space of all complex-valued continuous functions defined on the interval $[a, b]$. We define an inner product on $C([a, b])$

by

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx$$

where $f, g : [a, b] \rightarrow \mathbb{C}$ are continuous function. This space is not complete, so it is not a Hilbert space.

2. We can define an inner product on \mathbb{C} over itself by

$$\langle x, y \rangle = x\bar{y}$$

where x and y are complex numbers. Since \mathbb{C} is a complete space, then \mathbb{C} is a Hilbert space.

1.2 Closed linear operators

Definition 1.2.1. A linear operator from a Hilbert space H_1 into a Hilbert space H_2 is a linear mapping T of a linear subspace of H_1 , called the domain of T and denoted by $D(T)$.

Remark 1.2.2. It is not assumed that T is bounded. Of course, if T is bounded then T has a continuous extension to H_1 . In that case T is the restriction of some bounded linear operator.

Definition 1.2.3. Let T be a linear operator for all $x, y \in D(T)$ and $\alpha, \beta \in \mathbb{C}$. The linear subspace:

$$Im(T) = \{T(x) : x \in D(T)\} = T(D(T))$$

is called the image or the range of T , and the linear subspace:

$$\ker(T) = \{x \in D(T) : T(x) = 0\}$$

is called the the kernel of T .

We shall say that T is an extension of linear operator S or that S is a restriction of T and write $S \subseteq T$ or $S = T \upharpoonright D(S)$, when $D(S) \subseteq D(T)$ and $S(x) = T(x)$ for all $x \in D(S)$. That is, we have $S \subseteq T$ if and only if $S = T \upharpoonright D(S)$.

Proposition 1.2.4. *Let R, S and T be a linear operators from H into H , and let λ be a complex number. Then:*

- $D(\lambda T) = D(T)$.
- $D(S + T) = D(S) \cap D(T)$.
- $D(ST) = \{x \in D(T) : Tx \in D(S)\}$.
- $(R + S)T = RT + ST$ and $T(R + S) \supseteq TR + TS$.

Definition 1.2.5. *The graph of a linear operator T from H_1 into H_2 is the set*

$$G(T) = \{(x, Tx) : x \in D(T)\}.$$

The graph $G(T)$ is a linear subspace of the Hilbert space $H_1 \times H_2$ which contains the full information about the operator T . Obviously, the relation $S \subseteq T$ is equivalent to the inclusion $G(S) \subseteq G(T)$.

Definition 1.2.6. *Let T be a linear operator. T is called closed if its graph $G(T)$ is a closed subset of the Hilbert space $H_1 \times H_2$, and T is called closable if there exists a closed linear operator S from H_1 into H_2 such that $T \subseteq S$.*

Proposition 1.2.7 ([15]). *The following statements are equivalent:*

1. T is closed.
2. If $(x_n)_{n \in \mathbb{N}}$ is a sequence of vectors $x_n \in D(T)$ such that $\lim x_n = x$ in H_1 and $\lim T(x_n) = y$ in H_2 , then $x \in D(T)$ and $Tx = y$.
3. $(D(T), \|\cdot\|_T)$ is complete, or equivalently, $(D(T), \langle \cdot, \cdot \rangle_T)$ is a Hilbert space, where

$$\|x\|_T = \|x\|_1 + \|Tx\|_2.$$

Remark 1.2.8. *Often we are dealing with a linear operator that maps a Hilbert space H into itself. We usually express this by saying that we have an operator on H . That is, whenever we speak about an operator T on a Hilbert space H , we mean a linear mapping T of a linear subspace $D(T)$ of H into H .*

Proposition 1.2.9. *Let T be a linear operator. The following statements are equivalent:*

1. T is closable.
2. If $(x_n)_{n \in \mathbb{N}}$ is a sequence of vectors $x_n \in D(T)$ such that $\lim x_n = 0$ in H_1 and $\lim T(x_n) = y$ in H_2 , then $y = 0$.
3. The closure of the graph $G(T)$ is the graph of a linear operator.

Definition 1.2.10. *Let T be a closable operator. The operator \bar{T} is called the closure of the closable operator T , where $D(\bar{T})$ is the set of vectors $x \in H_1$ for which there exists a sequence $(x_n)_{n \in \mathbb{N}}$ from $D(T)$ such that $x = \lim x_n$ in H_1 and $(T(x_n))_{n \in \mathbb{N}}$ converges in H_2 .*

Recall that a linear operator T is continuous (that is, if $\lim x_n = x$ in H_1 for $x_n, x \in D(T), n \in \mathbb{N}$, then $\lim T(x_n) = T(x)$ in H_2) if and only if T is bounded (that is, there is a constant $c > 0$ such that $\|T(x)\|_2 \leq c\|x\|_1$ for all $x \in D(T)$).

If an operator T is continuous, then it is clearly closable by Proposition 1.2.9. If T is bounded, the graph norm $\|\cdot\|_T$ on $D(T)$ is obviously equivalent to the Hilbert space norm $\|\cdot\|_1$. Therefore, a bounded linear operator T is closed if and only if its domain $D(T)$ is closed in H_1 . Conversely, if T is a closed linear operator whose domain $D(T)$ is closed in H_1 , then the closed graph theorem implies that T is bounded. In particular, it follows that a closed linear operator which is not bounded cannot be defined on the whole Hilbert space.

Definition 1.2.11. *A linear subspace D of $D(T)$ is called a core for T if D is dense in $(D(T), \|\cdot\|_T)$, that is, for each $x \in D(T)$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of vectors $x_n \in D$ such that $x = \lim x_n$ in H_1 and $Tx = \lim T(x_n)$ in H_2 .*

If T is closed, a linear subspace D of $D(T)$ is a core for T if and only if T is the closure of its restriction $T \upharpoonright D$.

1.2.1 Adjoint operators

We wish to associate a Hilbert space adjoint T^* to T . Its domain $D(T^*)$ is to consist of all $y \in H$ for which the linear functional

$$x \rightarrow (Tx, y) \tag{1.1}$$

is continuous on $D(T)$. If $y \in D(T^*)$, then the Hahn-Banach theorem extends the functional (1.1) to a continuous linear functional on H , and therefore there exist an element $T^*y \in H$ that satisfies

$$(Tx, y) = (x, T^*y). \tag{1.2}$$

Obviously, T^*y will be uniquely determined by(1.2) if and only if $D(T)$ is dense in H , that is, if only if T is densely defined. the only operators T that will be given an adjoint T^* are therefore the densely defined ones.

Theorem 1.2.12 ([15]). *Let S and T be linear operators from H_1 into H_2 such that $D(T)$ is dense in H_1 . Then*

1. T^* is a closed linear operator from H_2 into H_1 .
2. $\text{Im}(T)^\perp = \ker(T^*)$.
3. If $D(T^*)$ is dense in H_2 , then $T \subseteq T^{**}$, where $T^{**} = (T^*)^*$.
4. If $T \subseteq S$, then $S^* \subseteq T^*$.
5. $(\lambda T)^* = \bar{\lambda}T^*$ for $\lambda \in \mathbb{C}$.
6. If $D(T + S)$ is dense in H_1 , then $(T + S)^* \supseteq T^* + S^*$.
7. If S is bounded and $D(S) = H_1$, then $(T + S)^* = T^* + S^*$.

Proposition 1.2.13 ([14]). *Let $T : H_1 \rightarrow H_2$ and $S : H_2 \rightarrow H_3$ be linear operators such that $D(ST)$ is dense in H_1 .*

1. If $D(S)$ is dense in H_2 , then $(ST)^* \supseteq T^*S^*$.

2. If S is bounded and $D(S) = H_2$, then $(ST)^* = T^*S^*$.

Definition 1.2.14. Let T be a closed densely defined linear operator on a Hilbert space H . The inverse operator T^{-1} is the linear operator defined on H , such that

$$TT^{-1} = I \quad \text{and} \quad T^{-1}T \subseteq I$$

where I is the identity operator. In this case we say that T is invertible.

Note that, T is invertible if and only if $\ker(T) = \{0\}$ and $\text{Im}(T) = H$.

Theorem 1.2.15 ([15]). Let T be a densely defined linear operator on a Hilbert space H .

1. T is closable if and only if $D(T^*)$ is dense in H_2 .

2. If T is closable, then $(\overline{T})^* = T^*$, and setting $T^{**} := (T^*)^*$, we have

$$\overline{T} = T^{**}.$$

3. T is closed if and only if $T = T^{**}$.

4. Suppose that $\ker(T) = \{0\}$ and $\text{Im}(T)$ is dense in H_2 . Then T^* is invertible and

$$(T^*)^{-1} = (T^{-1})^*.$$

5. If T is invertible, then T is closed if and only if T^{-1} is closed.

Example 1.2.16. Let D be a linear dense subspace of a Hilbert space H , and let $e \neq 0$ be a vector of H . Let F be a linear functional on D which is not continuous in the Hilbert space norm. Define the operator T by

$$D(T) = D \quad \text{and} \quad T(x) = F(x)e$$

where $x \in D$.

T is not closable. In fact, Since F is not continuous, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ from D such that $\lim x_n = 0$ in H and $(F(x_n))$ does not converge to zero. We can assume that there is a constant $c > 0$ such that $|F(x_n)| \geq c$ for all $n \in \mathbb{N}$. Putting $x'_n = F^{-1}(x_n)x_n$,

we have $\lim x'_n = 0$ and $T(x'_n) = F(x'_n)e = e \neq 0$. Hence, T is not closable by Proposition 1.2.9.

Since the functional F is discontinuous, the map $x \rightarrow \langle Tx, y \rangle = F(x) \langle e, y \rangle$ is continuous if and only if $\langle e, y \rangle = 0$. Hence, $D(T^*) = e^\perp$ in H and $T^*y = 0$ for $y \in D(T^*)$.

Example 1.2.17. Let J be an interval. For a continuous function φ on J , we define the multiplication operator M_φ on the Hilbert space $L^2(J)$ by

$$(M_\varphi f)(x) = \varphi(x)f(x) \quad \text{for } f \in D(M_\varphi) := \{f \in L^2(J) : \varphi f \in L^2(J)\}.$$

Since $D(M_\varphi)$ contains all continuous functions with compact support, M_φ is dense, so the adjoint operator $(M_\varphi)^*$ exists and $(M_\varphi)^* = M_{M_\varphi}$.

1.3 Spectrum of closed linear operators

In this section, we defined the spectrum of closed linear operators.

Definition 1.3.1. A complex number λ belongs to the resolvent set $\rho(T)$ of T if the operator $T - \lambda I$ has a bounded everywhere on H defined inverse $(T - \lambda I)^{-1}$, called the resolvent of T at λ and denoted by $R_{\lambda(T)}$.

The set $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the spectrum of the operator T .

Remark 1.3.2. A complex number λ is in $\rho(T)$ if and only if there is an operator $B \in B(H)$ such that

$$B(T - \lambda I) \subseteq I \text{ and } (T - \lambda I)B = I.$$

The operator B is then uniquely determined and equal to the resolvent $R_\lambda(T)$.

Proposition 1.3.3 ([15]). $\rho(T)$ is an open subset, and $\sigma(T)$ is a closed subset of \mathbb{C} .

If T is bounded everywhere defined operator then $\sigma(T)$ is a compact set of \mathbb{C} .

Proposition 1.3.4. Let T be a closed densely defined operator on H .

1. $\rho(T)$ is the set of all numbers $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is a bijective mapping of $D(T)$ on H (or equivalently, $\ker(T - \lambda I) = \{0\}$ and $\text{Im}(T - \lambda I) = H$).

2. $\lambda \in \sigma(T)$ if and only if $\bar{\lambda} \in \sigma(T^*)$.

Definition 1.3.5. The set $\sigma_p(T) := \{\lambda \in \mathbb{C} : \ker(T - \lambda I) \neq \{0\}\}$ is called the point spectrum of T .

We call $\lambda \in \sigma_p(T)$ an eigenvalue of T , the dimension of $\ker(T - \lambda I)$ its multiplicity, and any nonzero element of $\ker(T - \lambda I)$ an eigenvector of T at λ .

1.4 Some classes of closed unbounded linear operators

1.4.1 Symmetric operators

Definition 1.4.1. The operator T is called symmetric (or Hermitian) if

$$\langle Tx, y \rangle = \langle x, Ty \rangle \quad \text{for all } x, y \in D(T).$$

If T is a densely defined operator then T is symmetric if and only if $T \subseteq T^*$. If $\langle Tx, x \rangle \geq 0$ for all $x \in D(T)$, then T is called positive.

Proposition 1.4.2 ([14]). Let T be a symmetric operator

1. Any eigenvalue of T is real.
2. Eigenvectors belonging to different eigenvalues of T are mutually orthogonal.
3. Suppose that T is densely defined and $\langle Tx, x \rangle \geq 0$ for all $x \in D(T)$. If $\langle Tx, x \rangle = 0$ for some $x \in D(T)$, then $Tx = 0$.

Proof. 1. Let x be a nonzero vector from $N(T - \lambda I)$. Since $\langle Tx, x \rangle = \lambda \|x\|^2$ and $\langle Tx, x \rangle$ is real, then λ is real.

2. Let $x \in N(T - \lambda I)$ and $y \in N(T - \mu I)$, where $\lambda \neq \mu$. Since μ is real and T is symmetric, $\mu \langle x, y \rangle = \langle x, Ty \rangle = \langle Tx, y \rangle = \lambda \langle x, y \rangle$, and hence $\langle x, y \rangle = 0$.

3. Since $T \geq 0$, then the Cauchy–Schwartz inequality implies that

$$|\langle Tx, y \rangle|^2 \leq \langle Tx, x \rangle \langle Ty, y \rangle = 0$$

for all $y \in D(T)$. Since $D(T)$ is dense, then $Tx = 0$.

□

1.4.2 Self-adjoint operators

Definition 1.4.3. Let T be a closed densely defined operator. We say that T is self-adjoint if and only if $T = T^*$.

Remark 1.4.4. If T is symmetric and its adjoint is symmetric, then T is self-adjoint.

Example 1.4.5. Let T be a closed densely defined operator. We define the operator S by :

$$S = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$$

then, S is a self-adjoint operator.

Theorem 1.4.6 ([6]). If T is a symmetric operator where $\text{Im}(T) = H$, then T is self-adjoint, and its inverse T^{-1} is a bounded self-adjoint operator on H .

Proposition 1.4.7. A closed symmetric linear operator T on H is self-adjoint if and only if $\sigma(T) \subseteq \mathbb{R}$.

Proposition 1.4.8 ([15]). Let T be a densely defined symmetric operator on a Hilbert space H . Then there exists a self adjoint operator A on a Hilbert space G which contains H as a subspace such that $T \subseteq A$ and $D(\overline{T}) = D(A) \cap H$.

Proposition 1.4.9. Let T be a densely defined closed linear operator on a Hilbert space H . Then:

1. $I + T^*T$ is a bijective mapping and its inverse $C = (I + T^*T)^{-1}$ is a bounded self-adjoint operator, such that $0 \leq C \leq I$.
2. T^*T is a positive self-adjoint operator on H , and $D(T^*T)$ is a core for T .

1.5 Normal operators

Definition 1.5.1. Let T be a densely defined operator. We say that T is formally normal if $D(T) \subseteq D(T^*)$ and

$$\|Tx\| = \|T^*x\| \quad \text{for all } x \in D(T)$$

and that T is normal if T is formally normal and $D(T) = D(T^*)$.

Remark 1.5.2. 1. If T is normal, then T^* is normal.

2. All self-adjoint operators are normal, but normal operators are not necessarily self-adjoint operators.

Theorem 1.5.3 ([15]). T is normal if and only if T is closed and $T^*T = TT^*$.

Proof. First suppose that T is normal. Obviously, The graph norms $\|\cdot\|_T$ and $\|\cdot\|_{T^*}$ coincide on $D(T) = D(T^*)$. Since T^* is closed, $(D(T^*), \|\cdot\|_{T^*})$ is complete, and so is $(D(T), \|\cdot\|_T)$. therefore, T is closed.

For $x, y \in D(T)$, we have $\|T(x + \tau y)\| = \|T^*(x + \tau y)\|$ for $\tau = 1, -1, i, -i$, and $\langle Tx, Ty \rangle = \langle T^*x, T^*y \rangle$.

Let $y \in D(T^*T)$. Then $y \in D(T)$, and so

$$\langle x, T^*Ty \rangle = \langle Tx, Ty \rangle = \langle T^*x, T^*y \rangle$$

for all $x \in D(T)$. Therefore, $T^*y \in D(T^{**}) = D(T)$ and $TT^*y = T^*Ty$. This proves that $T^*T \subseteq TT^*$. Since $T = T^{**}$ and T^* is also normal, so we can interchange T and T^* , which yields $TT^* \subseteq T^*T$. Thus, $T^*T = TT^*$.

Conversely, suppose that T is closed and $T^*T = TT^*$. For $x \in D(T^*T) = D(TT^*)$, we have

$$\|Tx\|^2 = \langle T^*Tx, x \rangle = \langle TT^*x, x \rangle = \|T^*x\|^2,$$

and hence $\|x\|_T = \|x\|_{T^*}$. By Proposition 1.4.9, $D(T^*T)$ is a core for T , and $D(TT^*)$ is a core for T^* . That is, both $D(T)$ and $D(T^*)$ are completions of $(D, \|\cdot\|_T) = (D, \|\cdot\|_{T^*})$. This implies $\|Tx\| = \|T^*x\|$ for $x \in D(T) = D(T^*)$.

□

Theorem 1.5.4 ([15]). *Let T be a normal operator on H . Then we have:*

1. *For any $\lambda \in \mathbb{C}$, the operator $T - \lambda I$ is normal, and $\ker(T - \lambda I) = \ker(T^* - \bar{\lambda}I)$.
In particular, λ is an eigenvalue of T and $x \in H$ is an eigenvector of T for λ if and only if $\bar{\lambda}$ is an eigenvalue of T^* and x is an eigenvector of T^* for $\bar{\lambda}$.*
2. *Eigenvectors of T belonging to different eigenvalues are orthogonal.*
3. $H = \overline{\text{Im}(T - \lambda I)} \oplus \ker(T - \lambda I)$
4. *If S is an operator such that $T \subseteq S$ and $D(S) \subseteq D(S^*)$, then $T = S$.
In particular, each normal operator is maximal formally normal, that is, if T is normal, S is formally normal and $T \subseteq S$, then $T = S$.*
5. *If T is injective, then its inverse T^{-1} is also normal. In particular, the resolvents $R_\lambda(T), \lambda \in \rho(T)$, of a normal operator T are normal.*

The next Theorem is known as Fuglede-Putnam theorem.

Theorem 1.5.5 ([6]). *If T is a bounded operator and N and M are not necessarily bounded normal operators, then*

$$TN \subset MT \Rightarrow TN^* \subset M^*T.$$

Theorem 1.5.6 ([6]). *Let T be a normal operator on a Hilbert space H . Then there exists a unique spectral measure E such that*

$$T = \int_{\mathbb{C}} \lambda dE_T(\lambda).$$

Remark 1.5.7. *If T is a self-adjoint operator, then:*

$$T = \int_{\mathbb{R}} \lambda dE_T(\lambda).$$

Theorem 1.5.6, is known as the spectral Theorem, and one of its consequences is the next proposition.

Proposition 1.5.8. *If T is a positive self-adjoint operator, then there exists unique positive self-adjoint operator S on H such that $T = S^2$.*

S is called the square root of T and denoted by $T^{\frac{1}{2}}$ or \sqrt{T} .

Definition 1.5.9. *A densely defined operator T with domain $D(T)$ is called hyponormal if*

$$D(T) \subset D(T^*) \text{ and } \|T^*x\| \leq \|Tx\|, \quad \forall x \in D(T).$$

Remark 1.5.10. *Clearly, normal operators are hyponormal operators.*

The absolute value of the product of linear operators

In this chapter, we present a generalization of the famous Reid inequality related to linear operators. We also gave a sufficient conditions to ensure that the relation $|ST| = |S||T|$ hold for closed operators S and T .

2.1 Polar decomposition of a closed linear operators

Let T be a closed densely defined linear operator on a Hilbert space H . Since T^*T is a positive self adjoint operators, then according to Proposition 1.5.8, T^*T has a unique positive square root denoted $|T|$ where $|T| = (T^*T)^{\frac{1}{2}}$.

In the rest of our work all the operators are densely defined linear operators on a Hilbert space H .

Proposition 2.1.1. *Let T be a closed linear operator on a Hilbert space H . Then*

$$D(T) = D(|T|) \quad \text{and} \quad \|Tx\| = \||T|x\|$$

for all $x \in D(T)$.

Proof. If $x \in D(T^*T) = D(|T|^2)$, then

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle = \langle |T|^2x, x \rangle = \||T|x\|^2.$$

By Proposition 1.4.9, $D(T^*T)$ is a core for the operator T . $D(|T|^2)$ is a core for $|T|$. Hence, the assertion follows by taking the closure of $D(T^*T) = D(|T|^2)$ in the graph norm $\|T\cdot\| + \|\cdot\| = \||T|\cdot\| + \|\cdot\|$. \square

Definition 2.1.2. A bounded linear operator U on a Hilbert space H is said to be a partial isometry operator if there exists a closed subspace M such that $\|Ux\| = \|x\|$ for all $x \in M$, and $Ux = 0$ for any $x \in M^\perp$.

Theorem 2.1.3 ([15]). Let T be a closed linear operator on a Hilbert space H . Then there is a partial isometry U defined on H , such that

$$T = UT \tag{2.1}$$

We call the operator $|T|$ the modulus of T , the operator U_T the phase of T , and the decomposition (2.1) the polar decomposition of T .

Remark 2.1.4. Let T be a closed densely defined linear operator then

1. $|T| = U^*T = T^*U$.
2. $U^*U|T| = |T|$
3. $|T^*| = UT^* = TU^*$.
4. $T = |T^*|U$.
5. $T^* = U^*|T^*| = |T|U^*$.

2.2 Reid's inequality

Definition 2.2.1. Let T and S be unbounded positive self-adjoint operators. We say that $S \geq T$ if $D(S^{\frac{1}{2}}) \subseteq D(T^{\frac{1}{2}})$ and $\|S^{\frac{1}{2}}x\| \geq \|T^{\frac{1}{2}}x\|$ for all $x \in D(S^{\frac{1}{2}})$.

The following assertion is usually called the Heinz inequality.

Theorem 2.2.2. Let S and T be a positive self-adjoint operators. If $S \geq T \geq 0$, then $S^\alpha \geq T^\alpha$, where, $0 \leq \alpha \leq 1$.

The inequality of Reid which first appeared in [13] is recalled next:

Theorem 2.2.3. Let $S, T \in B(H)$ such that T is positive and TS is self-adjoint. Then

$$|\langle TSx, x \rangle| \leq \|S\| \langle Tx, x \rangle \quad \text{for all } x \in H.$$

The next Theorem shows that we can replace the hypothesis TS is self-adjoint in Reid's inequality by the hypothesis TS is normal.

Theorem 2.2.4. Let $S, T \in B(H)$ such that T is positive and TS is a normal. then

$$|\langle TSx, x \rangle| \leq \|S\| \langle Tx, x \rangle \quad \text{for all } x \in H.$$

Remark 2.2.5. Reid's inequality is not true even if we assume that the operator TS is hyponormal (see [4]).

Theorem 2.2.6. Let T be a closed hyponormal operator. then

$$T^*T \geq TT^*.$$

Proof. First, since T is closed, both T^*T and TT^* are self-adjoint and positive (cf. [15]).

Then it is known that

$$D(|T|) = D(T) \subseteq D(T^*) = D(|T^*|).$$

Finally, for all $x \in D(T)$, we have

$$\| |T^*| x \| = \| T^* x \| \leq \| T x \| = \| |T| x \|.$$

Therefore, according to Definition 2.2.1, we have $T^*T \geq TT^*$, as required. □

Theorem 2.2.7. *Let T be a closed hyponormal operator. Then*

$$|\langle Tx, x \rangle| \leq \langle |T|x, x \rangle \quad \text{for all } x \in D(T).$$

Proof. Let $T = U|T|$ be the polar decomposition of T where U is partial isometry. Remember that (see e.g. [15])

$$|T^*| = U|T|U^*$$

and

$$U^*U|T| = |T|.$$

By Proposition 2.2.6, $TT^* \leq T^*T$. Hence by Heinz inequality, we infer that $|T^*| \leq |T|$.

Now, let $x \in D(T)$. Then, we have

$$\begin{aligned} |\langle Tx, x \rangle|^2 &= |\langle U|T|x, x \rangle|^2 = |\langle |T|x, U^*x \rangle|^2 \\ &\leq \langle |T|x, x \rangle \langle |T|U^*x, U^*x \rangle \\ &= \langle |T|x, x \rangle \langle U|T|U^*x, x \rangle = \langle |T|x, x \rangle \langle |T^*|x, x \rangle \\ &\leq \langle |T|x, x \rangle \langle |T|x, x \rangle \quad (\text{as } T \text{ is hyponormal}). \end{aligned}$$

Accordingly,

$$|\langle Tx, x \rangle| \leq \langle |T|x, x \rangle,$$

as required. □

Proposition 2.2.8. *Let $S \in B(H)$ and let T be a positive self-adjoint not necessarily bounded operator on H . Then*

$$|(TS)^*| \leq \|S\|T.$$

Proof. First observe that TS is closed since T is closed and S is bounded and observe that

$$D(T) = D(S^*T) \subseteq D((TS)^*) = D(|(TS)^*|).$$

We have

$$\| |(TS)^*|x \|^2 = \| (TS)^*x \|^2 = \| S^*Tx \|^2 \leq \|S\|^2 \|Tx\|^2,$$

for each x in $D(T)$. It follows that

$$|(TS)^*|^2 \leq \|S\|^2 T^2$$

By the Heinz inequality, it follows that

$$|(TS)^*| \leq \|S\|T.$$

□

The next result is a natural generalization of Theorem 2.2.3.

Theorem 2.2.9 ([4]). *Let $S \in B(H)$ and let T be a positive self-adjoint not necessarily bounded operator such that $(TS)^*$ is hyponormal. Then*

$$|\langle TSx, x \rangle| \leq \|S\| \langle Tx, x \rangle \quad \text{for all } x \in D(T).$$

Proof. First, since $(TS)^*$ is hyponormal, we clearly have

$$D(T) \subseteq D((TS)^*) \subset D((TS)^{**}) = D(TS)$$

because TS is also closed. Proposition 2.3.15 then yields

$$|(TS)^*| \leq \|S\|T.$$

Therefore, for all $x \in D(T)$

$$\begin{aligned} |\langle TSx, x \rangle| &= |\langle x, (TS)^*x \rangle| = \overline{|\langle (TS)^*x, x \rangle|} \\ &= |\langle (TS)^*x, x \rangle| \\ &\leq \langle |(TS)^*| x, x \rangle \quad ((TS)^* \text{ is hyponormal }) \\ &\leq \|S\| \langle Tx, x \rangle \end{aligned}$$

and this marks the end of the proof. □

Now, we give the generalization of Halmos-Reid inequality (cf. [7]). The proof is more technical in the unbounded case.

Theorem 2.2.10 ([4]). *Let S be a bounded operator, and let T be an unbounded self-adjoint positive operator such that $S^*T \subseteq TS$, then*

$$|\langle TSx, x \rangle| \leq r(S) \langle Tx, x \rangle \quad \text{for all } x \in D(T),$$

where $r(S) = \lim_{n \rightarrow +\infty} \|S^n\|^{\frac{1}{n}}$ denotes the spectral radius of S .

Proof. Since $S^*T \subseteq TS$, we can get by induction that

$$S^{*n}T \subseteq TS^n \text{ or merely } (TS^n)^* \subseteq TS^n$$

for all n . Hence $(TS^n)^*$ is hyponormal. We also observe that as $T^{\frac{1}{2}}T^{\frac{1}{2}} = T$, then $D(T) \subset D\left(T^{\frac{1}{2}}\right)$. Now, let us prove the following result:

$$|\langle TSx, x \rangle| \leq \langle TS^{2n}x, x \rangle^{\frac{1}{2n}} \langle Tx, x \rangle^{\frac{2^n-1}{2^n}}$$

We use a proof by induction. For $n = 1$, we have for all $x \in D(T)$:

$$\begin{aligned} |\langle TSx, x \rangle| &\leq \left| \left\langle T^{\frac{1}{2}}T^{\frac{1}{2}}Sx, x \right\rangle \right| \\ &= \left| \left\langle T^{\frac{1}{2}}Sx, T^{\frac{1}{2}}x \right\rangle \right| \quad \text{because } T^{\frac{1}{2}}Sx \in D\left(T^{\frac{1}{2}}\right). \\ &\leq \left\| T^{\frac{1}{2}}Sx \right\| \left\| T^{\frac{1}{2}}x \right\| \\ &= \left\langle T^{\frac{1}{2}}Sx, T^{\frac{1}{2}}Sx \right\rangle^{\frac{1}{2}} \langle Tx, x \rangle^{\frac{1}{2}} \\ &= \left\langle T^{\frac{1}{2}}T^{\frac{1}{2}}Sx, Sx \right\rangle^{\frac{1}{2}} \langle Tx, x \rangle^{\frac{1}{2}} \\ &= \langle TSx, Sx \rangle^{\frac{1}{2}} \langle Tx, x \rangle^{\frac{1}{2}} \\ &= \langle S^*TSx, x \rangle^{\frac{1}{2}} \langle Tx, x \rangle^{\frac{1}{2}} \\ &= \langle TS^2x, x \rangle^{\frac{1}{2}} \langle Tx, x \rangle^{\frac{1}{2}}. \end{aligned}$$

Using a similar argument we can prove it for $n + 1$ if it holds for n . Therefore, for all n (and all $x \in D(T)$)

$$|\langle TSx, x \rangle| \leq \langle TS^{2n}x, x \rangle^{\frac{1}{2n}} \langle Tx, x \rangle^{\frac{2^n-1}{2^n}}.$$

Since $(TS^{2n})^*$ is hyponormal, we might apply Theorem 2.2.9 to get

$$\begin{aligned} |\langle TSx, x \rangle| &\leq \langle TS^{2n}x, x \rangle^{\frac{1}{2n}} \langle Tx, x \rangle^{\frac{2^n}{2^n}} \\ &\leq \|S^{2n}\|^{\frac{1}{2^n}} \langle Tx, x \rangle^{\frac{1}{2^n}} \langle Tx, x \rangle^{\frac{2^n}{2^n}} = \|S^{2n}\|^{\frac{1}{2^n}} \langle Tx, x \rangle \end{aligned}$$

whichever n . Passing to the limit finally yields

$$|\langle TSx, x \rangle| \leq r(S) \langle Tx, x \rangle \quad \text{for all } x \in D(T).$$

□

Corollary 2.2.11. *Let S be a bounded operator, and let T be an unbounded self-adjoint positive operator such that TS , then*

$$|\langle TSx, x \rangle| \leq r(S) \langle Tx, x \rangle \quad \text{for all } x \in D(T),$$

2.3 The absolute value and products

In this section we are mainly interested in investigating when the equalities $|ST| = |S||T| = |T||S|$ are valid in an unbounded setting.

In general, there is no reason why we should expect $|S||T| = |T||S|$ to hold. Even when $ST = TS$, the equality $|S||T| = |T||S|$ need not hold. For example, let

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Then

$$ST = TS,$$

but $|S||T| \neq |T||S|$.

Note that most of this section is taken from [1]. We start with the definition of strongly commuting normal operators.

Definition 2.3.1. *Two unbounded normal operators are said to strongly commute if their corresponding spectral measures commute.*

Remark 2.3.2. *If $S \in B(H)$ and T is unbounded, then S commutes with T if $ST \subseteq TS$. If also S and T are normal, then the previous amounts to the commutativity of their spectral measures*

In the rest of our work we will write $ST \subset TS$ instead of $ST \subseteq TS$, which means $D(ST) \subset D(TS)$ and ST and TS coincide on $D(ST)$.

Proposition 2.3.3. *Let T and S be a self-adjoint positive operators where only S is a bounded operator. If $ST \subset TS$, then TS (and \overline{ST}) is self-adjoint, positive and $\overline{ST} = TS$. Moreover,*

$$\sqrt{TS} = \sqrt{T}\sqrt{S}.$$

Proof. The self-adjointness and the positiveness of TS follow by invoking the spectral theorem. The equality $\sqrt{TS} = \sqrt{T}\sqrt{S}$ may also be obtained via a similar argument. Actually, we have

$$ST \subset TS \Rightarrow \sqrt{ST} \subset T\sqrt{S} \Rightarrow \sqrt{S}\sqrt{T} \subset \sqrt{T}\sqrt{S}.$$

Since, $\sqrt{T}\sqrt{S}$ is self-adjoint (and positive) and hence so is $(\sqrt{T}\sqrt{S})^2$. Then

$$(\sqrt{T}\sqrt{S})^2 = \sqrt{T}\sqrt{S}\sqrt{T}\sqrt{S} \subset \sqrt{T}\sqrt{T}\sqrt{S}\sqrt{S} = TS.$$

As both $(\sqrt{T}\sqrt{S})^2$ and TS are self-adjoint, then by the maximality of self-adjoint operators, we obtain

$$(\sqrt{T}\sqrt{S})^2 = TS.$$

Accordingly, by the uniqueness of the positive square root, we infer that

$$\sqrt{TS} = \sqrt{T}\sqrt{S},$$

as needed. □

Proposition 2.3.4 ([1]). *Let T be normal and let $S \in B(H)$. Then*

$$ST \subset TS \Rightarrow |S||T| \subset |T||S|,$$

$|S||T|x = |T||S|x$ whenever $x \in D(|S||T|) = D(|T||S|)$.

Theorem 2.3.5 ([1]). *Let T be a normal operator and let $S \in B(H)$. If $ST \subset TS$, then*

$$\overline{|ST|} = |TS| = |T||S| = \overline{|S||T|}.$$

Proof. The equality $|T||S| = \overline{|S||T|}$ is a consequence from Proposition 2.3.4, we have $|S||T| \subset |T||S|$. Since $|S|$ and $|T|$ are self-adjoint, it follows by Proposition 2.3.3 that

$$\overline{|S||T|} = |T||S|.$$

Now, $ST \subset TS$ gives $S^*T^* \subset T^*S^*$. Also and as above, the normality of T yields

$$ST \subset TS \Rightarrow ST^* \subset T^*S.$$

Hence by passing to adjoints, $S^*T \subset TS^*$. Therefore,

$$(TS)^*TS \supset S^*T^*TS = S^*TT^*S \supset S^*TST^* \supset S^*STT^*.$$

The closedness of TS gives the self-adjointness of $(TS)^*TS$. The previous “inclusion” becomes after taking adjoints

$$(TS)^*TS \subset TT^*S^*S = T^*TS^*S.$$

Since $(TS)^*TS$ and T^*TS^*S are self-adjoint, by the maximality of self-adjoint operators, we get

$$(TS)^*TS = T^*TS^*S.$$

By Proposition 2.3.3, we have

$$|TS| = \sqrt{(TS)^*TS} = \sqrt{T^*TS^*S} = \sqrt{T^*T}\sqrt{S^*S} = |T||S|.$$

To prove the last equality, we proceed as before. We have

$$(ST)^*ST = T^*S^*ST \supset S^*T^*ST \supset S^*ST^*T.$$

Since ST is closable and $S^*S \in B(H)$, it follows by passing to adjoints that

$$(ST)^*\overline{ST} \subset [(ST)^*ST]^* \subset T^*TS^*S.$$

As above, we obtain

$$(ST)^*\overline{ST} = T^*TS^*S.$$

Accordingly,

$$|\overline{ST}| = \sqrt{(ST)^*\overline{ST}} = \sqrt{(ST)^*ST} = \sqrt{T^*TS^*S} = |T||S|.$$

□

Remark 2.3.6. *We may obtain The condition $ST \subset TS$ if TS is normal, T and S are self-adjoint (one of them is positive) and $S \in B(H)$ (see [6], [8], [10]).*

Corollary 2.3.7. *Let T be a normal operator and let $S \in B(H)$. If $ST \subset TS$, then*

$$|TS| = |T^*S| = |\overline{ST}| = |\overline{ST^*}|.$$

Proof. Since T is normal then $|T| = |T^*|$, hence

$$|\overline{ST}| = |TS| = |T||S| = |T^*||S| = |T^*S| = |\overline{ST^*}|.$$

□

Corollary 2.3.8. *Let T be a boundedly invertible normal operator. Then*

$$|T^{-1}| = |T|^{-1}.$$

Proof. Since T is invertible, then

$$T^{-1}T \subset TT^{-1} = I$$

where $T^{-1} \in B(H)$. Therefore

$$|T||T^{-1}| = |TT^{-1}| = I.$$

Hence, $|T^{-1}| = |T|^{-1}$.

□

By using the spectral theorem for (unbounded) normal operators, it is easy to see that $|T^n| = |T|^n$ for any $n \in \mathbb{N}$ and T an unbounded normal operator. In [8], Jabłoński–Jung–Stochel extended the above to quasinormal operators (a class for which there is no spectral theorem). We recall that a closed densely defined operator T is said to be quasinormal if $U|T| \subset |T|U$ where $T = U|T|$ is the polar decomposition of T .

Proposition 2.3.9 ([8]). *If T is unbounded and quasinormal, then for all $n \in \mathbb{N}$*

$$|T^n| = |T|^n.$$

The previous Proposition is not true for the class of bounded hyponormal operators. Indeed, we have the following example.

Example 2.3.10. *Let $S \in B(l^2)$ be the usual shift. Remember that $SS^* \neq I$ and $S^*S = I$ where I is the identity operator on l^2 . Now, take $T = S + I$ so that T is hyponormal. Then*

$$|T|^2 = |S + I|^2 = (S^* + I)(S + I) = 2I + S + S^*.$$

On the other hand,

$$|T^2| = \sqrt{(S^* + I)^2(S + I)^2} = \sqrt{6I + 4S^* + 4S + S^{*2} + S^2}.$$

If $|T^2| = |T|^2$ held, then we would obtain $|T^2|^2 = |T|^4$. Which is impossible.

Remark 2.3.11. Proposition 2.3.9 does not hold for the class of symmetric and closed operators either. In fact, Let T be a closed, symmetric and semi-bounded operator as in [2]. Then $D(T^2) = \{0\}$ and so $|T^2|$ does not make sense whereas $|T|^2$ is self-adjoint.

Proposition 2.3.12 ([1]). Let T and S be two strongly commuting normal operators. Then $\overline{TS} = \overline{ST}$ are normal and

$$|\overline{ST}| = |\overline{TS}| = \overline{|T||S|} = \overline{|S||T|}.$$

Corollary 2.3.13. Let T and S be two strongly commuting normal operators.

Then

$$\overline{ST} = \overline{TS} = U\overline{|T||S|} = U\overline{|S||T|} = \overline{|S||T|}U = \overline{|T||S|}U$$

for some unitary operator $U \in B(H)$.

Proof. The proof follows from Proposition and from $|T| = |T^*|$ and $|S| = |S^*|$ □

Proposition 2.3.14 ([1]). Let T and S be two strongly commuting normal operators.

Then

$$(TS)^* = (\overline{TS})^* = \overline{S^*T^*} = \overline{T^*S^*} = (ST)^*.$$

Lemma 2.3.15. Let $S \in B(H)$ and let T be a densely defined operator such that T is normal. If $ST \subset TS$, then

$$ST^* \subset T^*S, \quad \overline{ST} \subset \overline{TS} \quad \text{and} \quad S^*\overline{T} \subset \overline{TS}^*.$$

Proof. Since \overline{T} is normal, $\overline{T^*} = T^*$ stays normal. By using the Fuglede theorem and taking adjoints, we have

$$ST^* \subset T^*S, \quad \overline{ST} \subset \overline{TS} \quad \text{and} \quad S^*\overline{T} \subset \overline{TS}^*.$$

□

Theorem 2.3.16. *Let T and K be strongly commuting normal operators and let $S \in B(H)$. If $ST \subset TS, SK \subset KS$ and TK is densely defined, then*

$$|\overline{TSK}| = |\overline{TK}||S| = |\overline{T}||\overline{K}||S|.$$

Proof. By assumptions,

$$S(TK) \subset TSK \subset (TK)S.$$

Since TK is densely defined and $D(TK) = D(STK) \subset D(TSK)$, then TSK is densely defined. Therefore

$$(TSK)^* \subset [S(TK)]^* = (TK)^*S^*.$$

Since \overline{TK} is normal, then Lemma 2.3.15 gives

$$S(TK)^* \subset (TK)^*S \text{ or } S^*\overline{TK} \subset \overline{TK}S^*$$

and

$$TSK \subset TKS \subset \overline{TK}S$$

and so

$$\overline{TSK} \subset \overline{TK}S$$

for $\overline{TK}S$ is closed. Hence

$$(TSK)^*\overline{TSK} \subset (TK)^*S^*\overline{TSK} \subset (TK)^*S^*\overline{TK}S \subset (TK)^*\overline{TK}S^*S.$$

Since $(TK)^*\overline{TK}$ and S^*S are self-adjoint operators and because \overline{TSK} is closed, we then get

$$(TSK)^*\overline{TSK} = (TK)^*\overline{TK}S^*S.$$

Therefore, Proposition 2.11.4.9 implies

$$|\overline{TSK}| = |\overline{TK}||S| = |\overline{T}||\overline{K}||S|$$

establishing the result. □

Corollary 2.3.17. *Let $(T_i)_{i=1,\dots,n}$ be pairwise strongly commuting normal operators that $T_1T_2\cdots T_n$ is densely defined. Let $S \in B(H)$ be such that $ST_i \subset T_iS$ for $i = 1, 2, \dots, n$. Then*

$$|\overline{ST_1T_2\cdots T_n}| = |\overline{T_1T_2\cdots T_nS}| = |\overline{T_1T_2\cdots T_iST_{i+1}\cdots T_n}| = |T_1||T_2|\cdots|T_n||S|.$$

Proof. These equalities are consequences of the results obtained above. For example, to prove the last equality, observe that $T_1T_2\cdots T_i$ and $T_{i+1}\cdots T_n$ are normal. Then

$$|\overline{T_1T_2\cdots T_iST_{i+1}\cdots T_n}| = |(\overline{T_1T_2\cdots T_i})(\overline{T_{i+1}\cdots T_n})||S| = |T_1||T_2|\cdots|T_n||S|,$$

as required. □

The absolute value of the sum of linear operators

In this chapter, we prove the triangle inequalities for closed linear operators.

3.1 Triangle inequalities for closed linear operators

We start with the next Theorem.

Theorem 3.1.1. *Let T be a normal operator and let $S \in B(H)$ be hyponormal. If $ST \subset TS$, then T^*S is hyponormal, which means,*

$$\|(T^*S)^*x\| \leq \|T^*Sx\|$$

for all $x \in D(T^*S)$.

Proof. Since $ST \subset TS$, by the Fuglede theorem $ST^* \subset T^*S$ because T is normal. By Corollary 2.3.7 we have

$$|T^*S| = |\overline{ST^*}|.$$

Since T^*S is closed, we obtain

$$D(T^*S) = D(|T^*S|) = D(\overline{|ST^*|}) = D(\overline{|ST^*|}).$$

Since T^*S is densely defined, we have

$$S^*T^{**} = S^*T \subset (T^*S)^*.$$

Hence

$$\overline{S^*T} \subset (T^*S)^*$$

and $D(T^*S) = D(\overline{S^*T}) \subset D[(T^*S)^*]$.

Now, for all $x \in D(T) = D(T^*) = D(ST^*) \subset D(T^*S) \subset D[(T^*S)^*]$, we have (using the hyponormality of S)

$$\|(T^*S^*)x\| = \|S^*Tx\| \leq \|STx\| = \|TSx\| = \|T^*Sx\|,$$

proving the hyponormality of T^*S , as wished. \square

Theorem 3.1.2 ([1]). *Let T be a normal operator and let $S \in B(H)$ be hyponormal. If $ST \subset TS$, then*

$$|T + S| \leq |T| + |S|.$$

Proof. We have $ST \subset TS$ and $ST^* \subset T^*S$. Whence, if $x \in D(T) = D(T^*)$, then $Sx \in D(T) = D(T^*)$. Moreover, the self-adjointness of $|T|$ and $|S|$ implies the self-adjointness (and positiveness) of $|T| + |S|$ given that $|S| \in B(H)$. Since T is closed and $S \in B(H)$, $T + S$ is closed. Hence $|T + S|$ makes sense and besides

$$D(|T| + |S|) = D(|T|) = D(T + S) = D(|T + S|).$$

By using the Heinz inequality (Theorem 2.2.2), it suffices to show that

$$|T + S|^2 \leq (|T| + |S|)^2.$$

Let $x \in D(T)$. Then

$$\begin{aligned}
\| |T + S|x \|^2 &= \|(T + S)x\|^2 = \langle (T + S)x, (T + S)x \rangle \\
&= \|Tx\|^2 + \|Sx\|^2 + \langle Bx, Ax \rangle + \langle Tx, Bx \rangle \\
&= \|Tx\|^2 + \|Sx\|^2 + \langle Sx, Tx \rangle + \overline{\langle Sx, Tx \rangle} \\
&= \|Tx\|^2 + \|Sx\|^2 + 2\Re\langle Sx, Tx \rangle \\
&= \|Tx\|^2 + \|Sx\|^2 + 2\Re\langle T^*Sx, x \rangle \\
&\leq \|Tx\|^2 + \|Sx\|^2 + 2|\langle T^*Sx, x \rangle|.
\end{aligned}$$

According to Theorem 3.1.1 and Theorem 2.2.7, we obtain

$$|\langle T^*Sx, x \rangle| \leq \langle |T^*S|x, x \rangle$$

for each x . By Proposition 2.3.4 and Theorem 2.3.5, we have for each $x \in D(T)$:

$$|T^*S|x = |T^*||S|x = |T||S|x = |S||T|x.$$

Therefore

$$2|\langle T^*Sx, x \rangle| \leq 2\langle |T^*S|x, x \rangle = \langle |T||S|x, x \rangle + \langle |S||T|x, x \rangle.$$

Accordingly,

$$\| |T + S|x \|^2 \leq \|Tx\|^2 + \|Sx\|^2 + \langle |T||S|x, x \rangle + \langle |S||T|x, x \rangle = \|(|T| + |S|)x\|^2$$

or merely

$$\| |T + S|x \| \leq \|(|T| + |S|)x\|$$

for all $x \in D(T)$. Hence

$$|T + S|^2 \leq (|T| + |S|)^2,$$

as needed above. □

Next, we give several examples.

Example 3.1.3. 1. Let F be the Fourier transform on $L^2(\mathbb{R})$. Then F is unitary and

so

$$F = \int_0^{2\pi} e^{i\lambda} dE_\lambda$$

in terms of a spectral measure E_λ . Now, set

$$T = \int_{\mathbb{R}} \lambda dE_\lambda$$

which is unbounded and self-adjoint and, by construction, it strongly commutes with F . Therefore, $T + F$ satisfies

$$|T + F| \leq |T| + I.$$

2. Let $a \in \mathbb{R}$ and U be the unitary operator defined on $L^2(\mathbb{R})$ by

$$(Uf)(x) = f(x + a), \quad x \in \mathbb{R}.$$

Let $Tf(x) = xf(x)$ defined on

$$D(T) = \{f \in L^2(\mathbb{R}) : xf \in L^2(\mathbb{R})\}.$$

Then

$$|T|f(x) = |x|f(x) \text{ and } |U| = I.$$

Also, U commutes with T . By Theorem 3.1.2,

$$|T + U| \leq M$$

where $Mf(x) = (|x| + 1)f(x)$ defined on $D(T)$.

3. Let T be an invertible normal operator and let T^{-1} be its everywhere bounded inverse.

Then

$$|T + T^{-1}| \leq |T| + |T^{-1}| = |T| + |T|^{-1}.$$

Corollary 3.1.4. Let T be a normal operator and let $S \in B(H)$ be hyponormal. If $ST \subset TS$, then

$$|T - S| \leq |T| + |S|.$$

The real and imaginary parts of a densely defined operator T are defined respectively, by

$$\Re(T) = \frac{T + T^*}{2} \quad \text{and} \quad \Im(T) = \frac{T - T^*}{2i}$$

Proposition 3.1.5 ([1]). *Let T be a closed hyponormal operator. Then*

$$|\overline{\Re(T)}| \leq |T|$$

where $\overline{\Re(T)}$ denotes the closure of $\Re(T)$.

Proof. Since T is hyponormal, we have $D(T) \subset D(T^*)$ and so $T + T^*$ is densely defined. Since T is closed, $T + T^*$ is symmetric. Therefore, $\Re(T) \subset \overline{\Re(T)}$. Let $x \in D(T)$. Then

$$(\Re(T))x = \overline{(\Re(T))x}.$$

It suffices to show that

$$|\overline{\Re(T)}|^2 \leq |T|^2.$$

By the closedness of T , we have

$$D(|T|) = D(T) = D(\Re(T)) \subset D(\overline{\Re(T)}) = D(|\overline{\Re(T)}|).$$

Let $x \in D(|T|)$. The use of the hyponormality of T implies that

$$\|\overline{\Re(T)}x\| = \|\overline{(\Re(T))x}\| = \|(\Re(T))x\| \leq \frac{1}{2}(\|Tx\| + \|T^*x\|) \leq \|Tx\| = \||T|x\|,$$

which means that $|\overline{\Re(T)}|^2 \leq |T|^2$. By using the Heinz inequality, we infer that

$$|\overline{\Re(T)}| \leq |T|.$$

□

Similarly, we may prove the following proposition.

Proposition 3.1.6. *Let T be a closed hyponormal operator. Then*

$$|\overline{\Im(T)}| \leq |T|$$

where $\overline{\Im(T)}$ denotes the closure of $\Im(T)$.

The following result gives an important application to self-adjointness. It is also a partial answer in an unbounded setting to a famous conjecture posed by Fong–Tsui in [5].

Theorem 3.1.7. *Let T be a normal operator such that*

$$|T| \leq |\overline{\Re(T)}|.$$

Then T is self-adjoint.

Proof. Since T is normal, then by Proposition 3.1.5, $|\overline{\Re(T)}| \leq |T|$. A glance at the hypothesis of the theorem yields

$$|T| = |\overline{\Re(T)}|.$$

By the normality of T , we have $D(T) = D(T^*)$. Since $\overline{\Re(T)}$ is self-adjoint, we may write

$$D[(\Re(T))^*] = D[(\overline{\Re(T)})^*] = D(\overline{\Re(T)}) = D(|T|) = D(T) \text{ and}$$

$$\|T^*x\| = \|Tx\| = \||T|x\| = \||\overline{\Re(T)}|x\| = \|\overline{\Re(T)}x\| = \|(\Re(T))^*x\|.$$

Therefore, the conditions of Theorem 3 in [9] are fulfilled and so $T = T^*$, as needed. \square

In Theorem 3.1.2 we assume that one of the operators is hyponormal and so there is no room for the (two parameter) spectral theorem. If we work with strongly commuting normal operators (even both unbounded), then the proof becomes a slightly simpler. Besides, this will be used to generalize some of the results above.

Theorem 3.1.8 ([1]). *Let T and S be two strongly commuting unbounded normal operators. Then,*

$$|\overline{T+S}| \leq |T| + |S|.$$

The next result is an immediate consequence of the above theorem.

Corollary 3.1.9. *Let T be a normal operator. Then*

$$|T| \leq |\overline{\Re(T)}| + |\overline{\Im(T)}|.$$

As an interesting consequence, we obtain a characterization of invertibility for the class of unbounded normal operators.

Theorem 3.1.10. *Let T be a normal operator. Then*

$$T \text{ is invertible} \Leftrightarrow |\overline{\Re}(T)| + |\overline{\Im}(T)| \text{ is invertible} .$$

Particularly, if $\lambda = \alpha + i\beta$, then

$$\lambda \in \sigma(T) \Leftrightarrow |\overline{\Re}(T) - \alpha I| + |\overline{\Im}(T) - \beta I| \text{ is not invertible} .$$

Proof. Since T is invertible, so is $|T|$. Hence by Corollary 3.1.9, $|\overline{\Re}(T)| + |\overline{\Im}(T)|$ too is invertible. Conversely, by Propositions 3.1.5 and 3.1.6

$$|\overline{\Re}(T)| \leq |T| \text{ and } |\overline{\Im}(T)| \leq |T|.$$

Hence, by the same arguments as in the previous proofs, we may easily establish

$$|\overline{\Re}(T)| + |\overline{\Im}(T)| \leq 2|T|.$$

Thus, the invertibility of $|\overline{\Re}(T)| + |\overline{\Im}(T)|$ implies the invertibility of $|T|$. Since the latter means that the normal operator T is left invertible, by [3] we get that T is invertible. \square

Another consequence is the following.

Corollary 3.1.11. *Let T be a normal operator. Then*

$$\sigma(T) \subset \sigma(|\overline{\Re}(T)|) + i\sigma(|\overline{\Im}(T)|).$$

Proof. Let $\lambda = \alpha + i\beta \in \sigma(T)$. Theorem 3.1.10 implies that $|\overline{\Re}(T) - \alpha I| + |\overline{\Im}(T) - \beta I|$ is not invertible. If either $|\overline{\Re}(T) - \alpha I|$ or $|\overline{\Im}(T) - \beta I|$ is invertible, then $|\overline{\Re}(T) - \alpha I| + |\overline{\Im}(T) - \beta I|$ would be invertible! Therefore, both $|\overline{\Re}(T) - \alpha I|$ are not invertible, so, $\overline{\Re}(T) - \alpha I$ and $\overline{\Im}(T) - \beta I$ are not invertible.

In other words, $\alpha \in \sigma(\overline{\Re}(T))$ and $\beta \in \sigma(\overline{\Im}(T))$. Consequently, $\lambda \in \sigma(\overline{\Re}(A) + i\sigma(\overline{\Im}(T)))$, as needed. \square

Next, We obtain a very simple proof of a famous result, which is the “realness” of the spectrum of unbounded self-adjoint operators.

Corollary 3.1.12. *Let T be a self-adjoint operator. Then $\sigma(T) \subset \mathbb{R}$*

Proof. Let $\lambda \in \mathbb{C}$, where $\lambda = \alpha + i\beta$ and $\beta \neq 0$. Since $T - \alpha I - i\beta I$ is normal, then by the invertibility of $|\beta|I$ it follows that $|T - \alpha I| + |\beta I|$ is also invertible. By Theorem 3.1.10, this just means that $T - \lambda I$ is invertible, that is, $\lambda \notin \sigma(T)$. \square

Theorem 3.1.13 ([1]). *Let $(T_i)_{i=1,\dots,n}$ be pairwise strongly commuting normal operators. Then*

$$\overline{|T_1 + T_2 + \dots + T_n|} \leq |T_1| + |T_2| + \dots + |T_n|.$$

Next, we give another triangle inequality.

Theorem 3.1.14. *Let T be a normal operator and let $S \in B(H)$ be hyponormal. If $ST \subset TS$, then*

$$\| |T| - |S| \| \leq \| T + S \|.$$

Proof. The proof is in essence fairly similar to that of Theorem 3.1.2. Clearly, $|T| - |S|$ and $T + S$ are closed and

$$D(|T + S|) = D(T + S) = D(T) + D(|T|) = D(|T| - |S|) = D(\| |T| - |S| \|).$$

Let $x \in D(T)$. Then

$$\begin{aligned} \| \| |T| - |S| \| x \|^2 &= \| (|T| - |S|)x \|^2 = \langle (|T| - |S|)x, (|T| - |S|)x \rangle \\ &= \| |T|x \|^2 + \| |S|x \|^2 - \langle |T|x, |S|x \rangle - \langle |S|x, |T|x \rangle \\ &= \| |T|x \|^2 + \| |S|x \|^2 - \langle |S||T|x, x \rangle - \langle |T||S|x, x \rangle \\ &= \| |T|x \|^2 + \| |S|x \|^2 - 2\langle |T||S|x, x \rangle \\ &= \| |T|x \|^2 + \| |S|x \|^2 - 2\langle |T^*||S|x, x \rangle \\ &= \| |T|x \|^2 + \| |S|x \|^2 - 2\langle |T^*S|x, x \rangle. \end{aligned}$$

Since T^*S is closed and hyponormal, we have for all x :

$$|\langle |T^*S|x, x \rangle| \leq \langle |T^*S|x, x \rangle$$

and so

$$-2\langle |T^*S|x, x \rangle \leq -2|\langle |T^*S|x, x \rangle| \leq 2\Re\langle |T^*S|x, x \rangle.$$

Hence

$$\begin{aligned}
\| |T| - |S| \|x\|^2 &\leq \| |T|x \|^2 + \| |S|x \|^2 + 2\Re(\langle T^*Sx, x \rangle) \\
&= \| |T|x \|^2 + \| |S|x \|^2 + \langle T^*Sx, x \rangle + \overline{\langle T^*Sx, x \rangle} \\
&= \| |T|x \|^2 + \| |S|x \|^2 + \langle T^*Sx, x \rangle + \langle x, T^*Sx \rangle \\
&= \| |T|x \|^2 + \| |S|x \|^2 + \langle Sx, Tx \rangle + \langle Tx, Sx \rangle \\
&= \|Tx\|^2 + \|Sx\|^2 + \langle Sx, Tx \rangle + \langle Tx, Sx \rangle \\
&= \|(T + S)x\|^2 = \| |T + S|x \|^2.
\end{aligned}$$

Therefore

$$\| |T| - |S| \|x\| \leq \| |T + S|x \|,$$

we have shown that

$$\| |T| - |S| \|^2 \leq \| |T + S| \|^2$$

or merely

$$\| |T| - |S| \| \leq \| |T + S| \|.$$

□

Corollary 3.1.15. *Let T be a normal operator and let $S \in B(H)$ be hyponormal. If $ST \subset TS$, then*

$$\| |T| - |S| \| \leq \| |T - S| \|.$$

Conclusion

In this work, we have studied the relations of the types $|ST| = |S||T|$, $|S \pm T| \leq |S| + |T|$ and $||S| - |T|| \leq |S \pm T|$ where S and T are closed linear operators. For example, we have presented a generalization of the famous Reid inequality related to linear operators. Also, we have proved that if T is a normal operator, $S \in B(H)$, and $ST \subset TS$, then

$$|\overline{ST}| = |TS| = |T||S| = \overline{|S||T|}.$$

We have showed that if T is a normal operator, $S \in B(H)$ is hyponormal, and $ST \subset TS$, then

$$|S \pm T| \leq |S| + |T| \quad \text{and} \quad ||S| - |T|| \leq |S \pm T|.$$

As consequences, we obtain a characterization of (unbounded) self-adjointness as well as a characterization of invertibility for the class of unbounded normal operators. Some examples accompany our results.

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