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Thème

Birfucations of limit cycles from a multiple focus and bifurcations at non-hyperbolic periodic orbits

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Thank you.

Dédicace

Every challenging work needs self effort
guidance of elders speatly tho
our heart. My humble effort I dedicate my sweet and loving

Mother & Father

Debich Dalila and Mustapha Baymout, Who
effection, love, encouragement and pray
night make me able to gut such succe
forgat my dear Grandfather rahmat allah alayh, who hel

and Abd Arrafik.

❖ *Baymout Louiza* ❖

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INTRODUCTION

Ordinary differential equations appear in many interdisciplinary areas and are the favored language for the study of various natural phenomena that are employed extensively in natural sciences, engineering, and technology. At present, ordinary differential equations are integrated into any standard undergraduate science curriculum, while continuing to be the subject of intensive research.

In general, most of the nonlinear differential equations cannot be solved by terms of elementary functions. The qualitative or geometrical theory of differential equations is being used to analyze differential equations whose explicit solutions are hard to find. These tools are originated by Henri Poincaré in his work on differential equations at the end of the nineteenth century [1].

The main goal of this thesis is the global analysis of the behavior of solutions, under the point of view of the qualitative or geometrical theory of nonlinear planar differential systems, especially those depend on a parameter or several parameters then the problem is what's happened if our differential equation depends on a parameter and this parameter change?. In this work we address the question of how the qualitative behavior of a differential equation change as we change the function of vector field, here we are on the presence of bifurcation.

The qualitative theory offers two types of tools that permit the analysis of a differential equation. On the one hand, there are tools of local character, some of these tools such as the Hartman-Grobman Theorem, enable describing the singular points of the dynamical system; other techniques are used to the analysis of the flow in the neighborhood of singular points or periodic orbits. We should also mention the Poincaré–Bendixson Theorem, which allows the analysis of the α and ω-limit sets in planar dynamical systems, i.e., the values to which the orbits of the dynamical system tend, as the time approaches the extreme values in the interval of definition. Furthermore, the qualitative theory contains tools of the global portraits, such as the study of the invariant algebraic curves which are invariant by the flow of the differential system, which the calculation of a sufficient number of them enables the calculation of first integrals.

Now, we describe the structure of this thesis which is divided into three chapters, in the first one we present the necessary background information to perform our study as singular points and their nature, Hartman-Grobman theorem, Poincaré map, phase portraits, structural stability (see [2]).

Chapter 2 begins with our examination concern the Hopf bifurcations and bifurcations of limit cycles from a multiple focus and bifurcations at non-hyperbolic periodic orbit [7], we present theorems of creation of limit cycles from a multiple focus, bifurcations in the neighborhood of a multiple focus of multiplicity $m = 1$.

In chapter 3 we tackle the Hopf bifurcations and classifies all phase portraits of a family of rigid systems under the form

$$\dot{x} = -y + x(a + bx^2 + cy^2), \quad \dot{y} = x + y(a + bx^2 + cy^2),$$

where $b^2 + c^2$ is not zero. Moreover, it distinguish between center and focus for these systems.

CHAPTER I

INTRODUCTION TO BIFURCATION THEORY

In this chapter we adress the question of how the qualitative behavior of the ordinary differential equation change as we change parameters. If the qualitative behavior remains the same for all nearby vector fields,

$$\dot{x} = f(x). \tag{I.1}$$

then the system $(I.1)$ or the function f said to be structurally stable, if a vector field $f \in C^1(E)$ is not structurally stable, it belongs to the bifurcation set in $C^1(E)$.

Our aim is to give the basic results to study a what we called a bifurcation, firstly we mention in which case we have bifurcation after that we need to define what's we mean by structurally stable or unstable and we end the chapter by a technique for studying the stability and bifurcation of the periodic orbits. This is done by the so-called "Poincaré map".

I.1 Some concepts of differential equations

A good place to start analyzing the nonlinear system $(I.1)$ is to determine the equilibrium points of $(I.1)$ and describe the behavior of this system near its equilibrium points.

***Definition* I.1** (Equilibrium points). *A point $x_0 \in \mathbb{R}^n$ is called an equilibrium point or critical point of $(I.1)$ if $f(x_0) = 0$.*

***Definition* I.2.** *An equilibrium point x_0 of $(I.1)$ is called a sink if all of the eigenvalues of the matrix $Df(x_0)$ have a negative real part, it is called source if all of the eigenvalues of $Df(x_0)$ have a positive real part. And it is a saddle if there exists an eigenvalue positive, and there exists another one negative.*

***Definition* I.3** (Flow). *Let E be an open subset of $\mathbb{R}^n$ and $f \in C^1(E)$. For $x_0 \in E$, let $\phi_t(x_0)$ be the solution of the initial value problem $(I.1)$ with x_0 on its maximal interval of existence $I(x_0)$. Then* *the set*

$$\{t \in I(x_0) : \phi_t(x_0) = \phi(t, x_0)\},$$

is called the flow of the differential equation $(I.1)$*. Which satisfies the following basic properties for all* x *in* $\mathbb{R}^n$

- $\phi_0(x) = x$,
- $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$ *for all* $s,\ t \in \mathbb{R}$,
- $\phi_t(\phi_{-t}(x)) = \phi_{-t}(\phi_t(x)) = x$ *for all* $t \in \mathbb{R}$.

The same properties preserve for a linear system have the flow $\phi_t = e^{At}$ *defined from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$.

In general, the study of the local behavior of the flow near an equilibrium point x_0 is quite complicated. Already the linear systems show different classes, even for local topological equivalence. We say that $Df(x_0)$ is the linear part of the vector field f at x_0. There are many types of equilibrium points of a differential equation $(I.1)$ that classify from the eigenvalues of $Df(x_0)$.

I.1.1 Hyperbolic and non-hyperbolic equilibrium points

Hyperbolic equilibrium points

***Definition* I.4** (Hyperbolic equilibrium point)**.** *The equilibrium* x_0 *is said to be hyperbolic if all eigenvalues of the Jacobian matrix* $Df(x_0)$ *have non-zero real parts.*

Hartman-Grobman theorem

The Hartman-Grobman theorem is one of the very important results in the qualitative theory of ordinary differential equations. The theorem shows that near a hyperbolic equilibrium point x_0, the nonlinear system $(I.1)$ has the same qualitative structure as the linear system

$$\dot{x} = Ax. \tag{I.2}$$

with $A = Df(x_0)$, in what follow we shall assume that the equilibrium point x_0 has been translated to the origin.

***Definition* I.5** (Topologically equivalent)**.** *two autonomous systems of differential equations are said to be topologically equivalent in a neighboorhood of the origin* $N_\delta(0)$ *or have the same qualitative structure near the origin if there is a homeomorphism* H *mapping an open* U *containing the origin onto an open set* V *containing the origin which map trajectories of the first system in* U *to the second one in* V *and preserves their orientation by time. cf. Figure* $(I.1)$*, for more details see* [2].

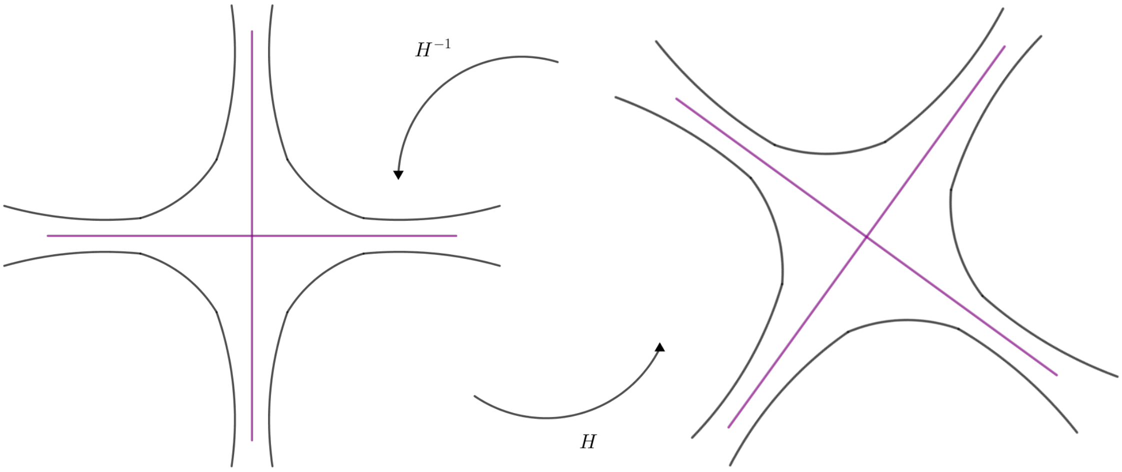


Figure I.1: Topologically equivalent.

***Theorem* I.1.** *Let E be an open sub set of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and ϕ_t be the flow of the nonlinear system (I.1). Suppose that the origin is an equilibrium point of (I.1) which mean $f(0) = 0$ and that the matrix $Df(0)$ has no eigenvalue with zero real part. Then there exists $H : U \longrightarrow V$ Homeomorphism such that for all $x_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x_0 \in U$ and $t \in I_0$*

$$H \circ \phi(x_0) = e^{At} H(x_0);$$

i.e., H maps trajectories of (I.1) near the origin onto trajectories of (I.2) near the origin and preserves the parametrization by time.

The proof consists of five steps, see [2] and [9].

By the Hartman-Grobman theorem the nature and stability of any hyperbolic equilibrium point x_0 of the nonlinear system (I.1) is determine by the signs of the real parts of the eigenvalues λ_j of the matrix $Df(x_0)$. The stability of non-hyperbolic equilibrium points is typically more difficult to determine.

Non-hyperbolic equilibrium points

***Definition* I.6** (Non-hyperbolic equilibrium point)**.** *If at least one eigenvalue of the Jacobian matrix is zero or has a zero real part, then the equilibrium is said to be non-hyperbolic.*

***Definition* I.7** (Center)**.** *The origin is called a center for the nonlinear system (I.1) if there exists a strictly positive ε such that every solution curve of (I.1) in the neighborhood $N_\varepsilon(0)$ containing the origin in the interior, is a closed curve.*

***Definition* I.8** (Focus)**.** *The origin is called a focus for the nonlinear system (I.1) if there exists a* *positive* $\varepsilon > 0$ *such that for* $0 < r_0 < \varepsilon$ *and* $\theta_0 \in \mathbb{R}$,

$$r(t, r_0, \theta_0) \to 0 \quad \textit{and} \quad \theta(t, r_0, \theta_0) \to \infty,$$

as $t \to \infty$ *for a stable focus, and* $t \to -\infty$ *for unstable focus.*

***Definition* I.9** (Center-focus). *The origin is called a center-focus for* $(I.1)$ *if there exists a decreases sequence of closed solution curves* Γ_n*; i.e.,* Γ_{n+1} *in the interior of* Γ_n *such that* $\Gamma_n \to 0$ *as* $n \to \infty$ *and such that every trajectory between* Γ_n *and* Γ_{n+1} *spirals toward* Γ_n *or* Γ_{n+1} *as* $t \to \pm$.

***Theorem* I.2.** *Let* E *be an open subset of* $\mathbb{R}^2$ *containing the origin and let* $f \in C^1(E)$ *with* $f(0) = 0$. *Suppose that the origin is a center for the linear system* $(I.2)$ *with* $A = Df(0)$. *Then the origin is either a center, a center-focus or a focus for the nonlinear system* $(I.1)$.

A center-focus cannot occur in an analytic system. This is a consequence of Dulac's theorem [2]. Liapunov's method is one tool that can be used to distinguiche a center from a focus for a nonlinear system. In our work, we interested in the second tool which is the so-called "Poincaré map".

I.1.2 Phase portraits

Although it is often impossible (or very difficult) to determine explicitly the solutions of a differential equation, it is still important to obtain information about these solutions, at least of qualitative nature. To a considerable extent, this can be done describing the phase portrait of the differential equation.

Let $f : D \to \mathbb{R}^n$ be a continuous function in an open set $D \subset \mathbb{R}^n$ and consider the autonomous equation $(I.1)$. The set D is called the phase space of the equation.

***Definition* I.10** (Orbits). *If* $x(t) = \Phi_t(x)$ *is a solution of equation* $(I.1)$ *with maximal interval* I, *then the set* $x(t) : t \in I \subset D$ *is called an orbit of the equation* $(I.1)$.

***Definition* I.11.** *The phase portrait of an autonomous ordinary differential equation is obtained by representing the orbits in the set* D*, also indicating the direction of motion. It is common not to indicate the directions of the axes, since these could be confused with the direction of motion.*

I.1.3 Global phace portais

In order to study the behavior of trajectories of a planar differential system near infinity, it is possible to use a compactification. One of the possible constructions relies on the stereographic projection of the sphere onto the plane (for more information see [6]). A better approach to studying the behavior of trajectories "at infinity" is to use the so-called Poincaré sphere. However, some of the singular points at infinity, on the Poincaré sphere my still be very complicated (see all the details for instance in chapter 5 of [5] and [2]).

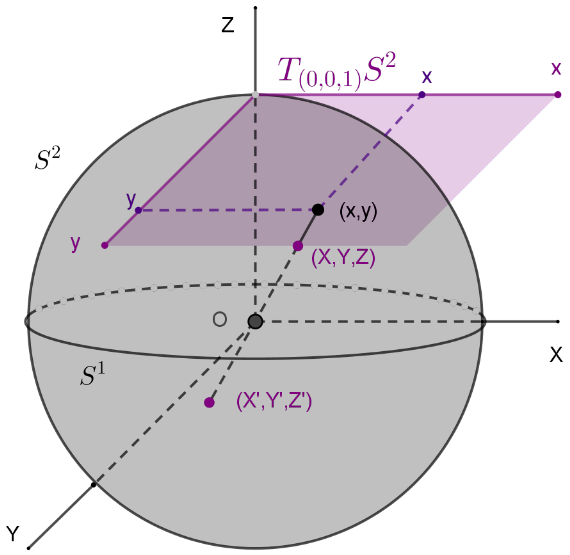
Local charts

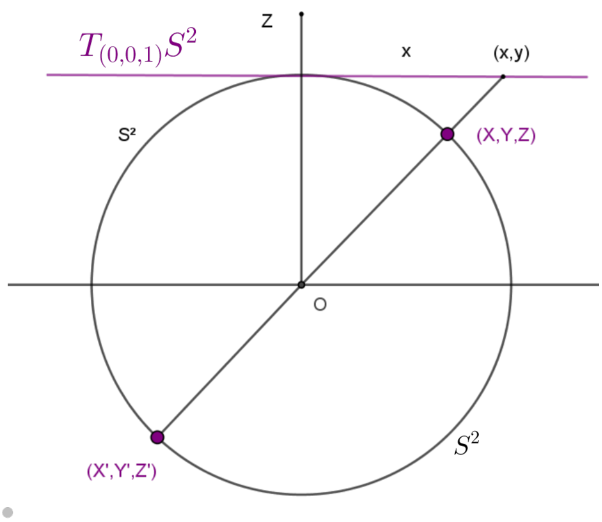
Let $\chi = \varphi\partial \setminus \partial x + \psi\partial \setminus \partial y$ be the planar polynomial vector field or in other words

$$\begin{cases} \dot{x} &= \varphi(x,y), \\ \dot{y} &= \psi(x,y), \end{cases} \tag{I.3}$$

We recall that n the degree of the system $(I.3)$, is the maximum between d_1, d_2 degrees of φ and ψ. The Poincaré sphere is defined as $\mathbb{S}^2 = \{y = (X,\ Y,\ Z) \in \mathbb{R}^3 :\ X^2 + Y^2 + Z^2 = 1\}$ and its tangent space at the point $Y \in \mathbb{S}^2$ is denoted by $T_Y\mathbb{S}^2$, which is tangent to $\mathbb{R}^2$ in $T_{(0,0,1)}\mathbb{S}^2 \ = \mathbb{R}^2$.

We define the central projection $f\ :\ T_{(0,0,1)}\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ as follows: to each point $(x,\ y) \in T_{(0,0,1)}\mathbb{S}^2$ the central projection associates the two intersection points $f^+(x,\ y)$, $f^-(x,\ y)$ of the straight line which connects the points $(x,\ y)$ and $(0,0,0)$ with the sphere $\mathbb{S}^2$. This central projection gives two copies of $(I.3)$ in $\mathbb{S}^2$, one in each hemisphere, $H_+ \ = \{Z \in \mathbb{S}^2 : Z < 0\}$ the northen hemisphere and $H_- \ = \{Z \in \mathbb{S}^2 : Z > 0\}$ the southern hemisphere; cf. Figure $(I.1.3)$.





$FIGURE\ \ I.1.3\ -\ central\ projection,$ $\quad Across - section\ of\ the\ central\ of\ the\ upper\ hemisphere.$

$$f^+(x,\ y) \ = \ (X,\ Y,\ Z) \ = \ \Big(\frac{x}{\Delta(x,y)}, \frac{y}{\Delta(x,y)}, \frac{1}{\Delta(x,y)}\Big),$$

$$f^-(x,\ y) \ = \ (X',\ Y',\ Z') \ = \ \Big(-\frac{x}{\Delta(x,y)}, -\frac{y}{\Delta(x,y)}, -\frac{1}{\Delta(x,y)}\Big).$$

Where

$$\Delta(x,y) \ = \ \sqrt{x^2 + y^2 + 1}.$$

In this way, we obtain induced vector fields in each hemisphere. Of course, all of them are analytically conjugate to χ. the induced vector field on H_+ is $\overline{\chi}(f^+(x, y)) = Df^+(x, y)\chi(x, y)$, and the one in H_- is $\overline{\chi}(f^-(x, y)) = Df^-(x, y)\chi(x, y)$. The equator $\mathbb{S}^1 = \{Z \in \mathbb{S}^2 : Z = 0\}$ can be identified with the infinity of $\mathbb{R}^2$.

***Remark* I.1.** *We remark that $\overline{\chi}$ is a vector field on $\mathbb{S}^2 \backslash \mathbb{S}^1$ that is everywhere tangent to $\mathbb{S}^2$.*

We extend the vector field $\overline{\chi}$ from $\mathbb{S}^2 \setminus \mathbb{S}^1$ to $\mathbb{S}^2$, then the extended vector field on $\mathbb{S}^2$ is called the Poincaré compactification of the vector field χ on $\mathbb{R}^2$, and is denoted by $P(\chi)$ (see all the details for instance in chapter 5 of [5]).

In summary, we have two symmetric copies of χ on $\mathbb{S}^2 \setminus \mathbb{S}^1$, and studying the dynamics of $P(\chi)$ near $\mathbb{S}^1$, we have the dynamics of χ at infinity. The Poincaré disc, denoted by D^2, is the closed northern hemisphere of $\{Z \in \mathbb{S}^2 : Z \geq 0\}$ projected on $Z = 0$ under the projection $(X, Y, Z) \mapsto (x, y)$. The infinity $\mathbb{S}^1$ is invariant under the flow of the Poincaré compactification $P(\chi)$.

Here two polynomial vector fields X and Y associated to system $(I.1)$ are topologically equivalent if there is a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow of $P(X)$ into orbits of the flow of $P(Y)$, either reversing or preserving the sense of all orbits. For computing the analytic expression of $P(\chi)$ we use the fact that $\mathbb{S}^2$ is a differentiable manifold. Thus we take the six local charts $U_i = \{y = (y_1, y_2, y_3) \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y = (y_1, y_2, y_3) \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1, 2, 3$; and the associated diffeomorphisms $F_i : U_i \longrightarrow \mathbb{R}^2$ and $G_i : V_i \longrightarrow \mathbb{R}^2$ for $i = 1, 2, 3$ are respectively the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$; $(-1, 0, 0)$; $(0, 1, 0)$; $(0, -1, 0)$; $(0, 0, 1)$ and $(0, 0, -1)$. The value of $F_i(y)$ or $G_i(y)$ for some $i = 1, 2, 3$ is denoted by $z = (z_1, z_2)$, consequently according to the local charts under consideration the same letter z represents different coordinates.

After a rescaling in the independent variable in the local chart (U_1, F_1) the expression for $P(\chi)$ is

$$\dot{u} = v^n \left[-u\varphi\left(\frac{1}{v}, \frac{u}{v}\right) + \psi\left(\frac{1}{v}, \frac{u}{v}\right) \right], \quad \dot{v} = -v^{n+1}\varphi\left(\frac{1}{v}, \frac{u}{v}\right);$$

in the local chart (U_2, F_2) the expression for $P(\chi)$ is

$$\dot{u} = v^n \left[\varphi\left(\frac{u}{v}, \frac{1}{v}\right) - u\psi\left(\frac{u}{v}, \frac{1}{v}\right) \right] \quad \dot{v} = -v^{n+1}\psi\left(\frac{u}{v}, \frac{1}{v}\right);$$

and for the local chart (U_3, F_3) the expression for $P(\chi)$ is

$$\dot{u} = \varphi(u, v), \quad \dot{v} = \psi(u, v).$$

In the chart (V_i, G_i) the expression for $P(\chi)$ is the same than in the chart (U_i, F_i) multiplied by $(-1)^{n+1}$ for $i = 1, 2, 3$. We note that the points at the infinity $\mathbb{S}^1$ in any chart have coordinates $(u, v) = (u, 0)$.

The equilibrium points of $P(\chi)$ which come from the equilibrium points of χ are called finite equilibrium points of χ, and the equilibrium points of $P(\chi)$ which are in $\mathbb{S}^1$ are called infinite equilibrium points of χ. We observe that the unique infinite equilibrium points which cannot be contained in the charts $U_1 \cup V_1$ are the origins of the local charts U_2 and V_2. Therefore when we study the infinite equilibrium points on the charts $U_2 \cup V_2$, we only need to verify if the origin of these charts is an equilibrium point.

I.2 Structurally stable

In this Section, we present the concept of structurally stable vector field or dynamical system and give necessary and sufficient conditions for a C^1-vector fieled f on a compact to be structurally stable.

The idea of structural stability originated with Andronov and Pontrygin in 1937; cf. [7], p.56. We say that f is structurally stable vector field if for any vector field g near f, the vectors f and g topologically equivalent, which means that those two vectors fields are close to each other.

***Definition* I.12** (The C^1-Norme)**.** *If $f \in C^1$ where E is an open subset of $\mathbb{R}^n$, then the C^1-Norme of f which defined from E into E,*

$$\|f\|_1 = \sup_{x\in E} |f(x)| + \sup_{x\in E} \|Df(x)\|.$$

where $|.|$ the euclidien norme, and $\|.\|$ the usual norme. So, we use the C^1-Norme to measure the distance between any two functions in C^1, and if E is a compact implie that $\|f\|_1 < +\infty$.

***Definition* I.13** (Structurally stable)**.** *Let E be an open subset of $\mathbb{R}^n$, we said that the vector field $f \in C^1$ is structurally Stable, if there exist $\varepsilon > 0$ such that for all $g \in C^1$ with*

$$\|f - g\|_1 < \varepsilon,$$

f and g are topologically equivalent on E, which means that there exit h from E onto E Homeomorphisme which map trajectories of $x' = f(x)$, onto trajectories of $x' = g(x)$ and preserve their orientation by time, then we said f structurally unstable if the vector field f is not structurally stable.

As we know, a periodic orbit or a cycle of a differential equation is any closed solution curve, that can be stable or unstable. In the next section we are going to define the Poincaré map which allow us to study the stability and bifurcation of periodic orbits.

I.3 The Poincaré map

Probably the most basic tool for studying the stability and bifurcation of the periodic orbits is the Poincaré map. The idea of the Poincaré map when Γ is a periodic orbit of the system $(I.1)$ through x_0, with Σ is a hyperplane perpendicular to Γ at x_0, then for any point $x \in \Sigma$ sufficiently near x_0 the solution of $(I.1)$ through x at $t = 0$, $\Phi_t(x)$, will cross Σ again at $P(x)$ near x_0 ; cf.Figure $(I.2)$, the mapping $x \to P(x)$ is called the Poincaré map. The Poincaré map can also be defined when Σ is a smooth surface.

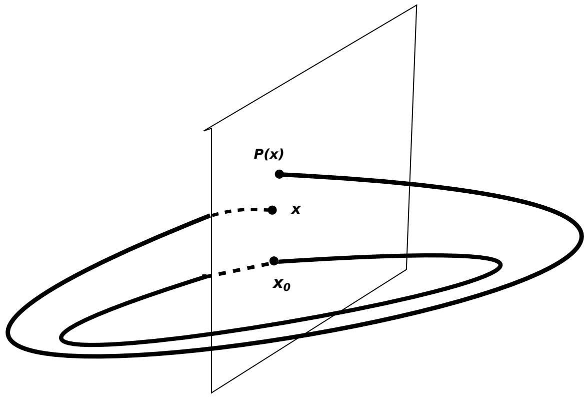


Figure I.2: The Poincaré map.

***Theorem* I.3** (The existence and continuity of the Poincaré map and its first derivative). *Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$. Suppose that $\Phi_t(x_0)$ is a periodic solution of $(I.1)$ of period T and that the cycle*

$$\Gamma = \{x \in \mathbb{R}^n \mid x = \Phi_t(x_0), \;\; 0 \le t \le T\},$$

is contained in E. Let $\sum$ be the hyperplane orthogonal to Γ at x_0; i.e., let

$$\Sigma = \{x \in \mathbb{R}^n \mid (x - x_0) \cdot f(x_0) = 0\},$$

then $\exists \delta > 0$ and $\exists!$ function $\tau(x)$ defined and continuously differentiable for $x \in N_\delta(x_0)$ such that

$$\begin{cases} \tau(x_0) = T \\ \Phi_{\tau(x)}(x) \in \Sigma \end{cases} \quad \textit{for all } x \in N_\delta(x_0).$$

Proof. The proof of this theorem is an immediate application of the implicit function theorem, by the supposition of

$$F(t, x) = (\Phi_t(x) - x_0) \cdot f(x_0), \quad \text{for a given } x_0 \in \Gamma \subset E.$$

for more details see [2]. ■

***Definition* I.14** (The Poincaré map). *Let* Γ, Σ, δ, *and* $\tau(x)$ *be defined as in theorem* $(I.3)$. *Then, for* $x \in N_\delta(x_0) \cap \Sigma$, *the function*

$$P(x) = \Phi_{\tau(x)},$$

is called the Poincaré map for Γ *at* x_0.

***Remark* I.2.** *It follows from theorem* $(I.3)$ *that* $P \in C^1(U)$ *where* $U = N_{\delta(x_0)} \cap \Sigma$.

- *If* f *analytic in* $E \Rightarrow P$ *analytic in* U ,
- *Fixed points of the Poincaré map, i.e.,(points* $x \in \Sigma : P(x) = x$*) are periodic orbits of* $(I.1)$,
- *By considering the system* $(I.1)$ *with* $t \to -t$, *we can show that the Poincaré map* P *has a* C^1*-inverse,* $P^{-1}(x) = \Phi_{-\tau(x)}(x)$. *Thus,* P *is a diffeomorphism; i.e., a smooth function with a smooth inverse.*

I.3.1 The Poincaré map of planar systems

Now, we are going to cite some specific results for the Poincaré map for planar systems. For planar systems, if we translate the origin to the point $x_0 \in \Sigma \cap \Gamma$. The point $0 \in \Gamma \cap \Sigma$ divide Σ on two open segments $\Sigma^+ \wedge \Sigma^-$; cf. Figure $(I.3)$ below. Let s be the signed distance along Σ with $s > 0$ for points in Σ^+ .

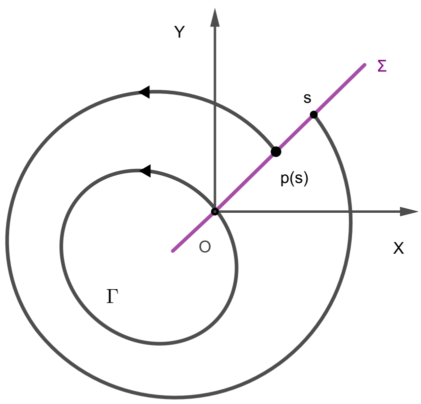


Figure I.3: The straight line normal Σ to Γ at 0

By theorem $(I.3)$, the Poincaré map $P(s)$ defined for $|s| < \delta$ and we have $P(0)$. In order to see how the stability of the cycle Γ is determined by $P'(0)$, let us introduce the displacement function, which defined for all $|s| < \delta$ by

$$d(s) = P(s) - s. \tag{I.4}$$

with $P(0) = 0$ and $d'(s) = P'(s) - 1$. From the mean value theorem, for $|s| < \delta$, $\exists\, \sigma \in [0, s]$ such that $: d(s) = d'(\sigma)s$. Since $d'(s)$ is continuous, the sign of $d'(s)$ will be the same as $d'(0)$ for $|s|$ sufficiently small as long as $d'(0) \neq 0$. Thus, if $d'(0) < 0$ implie that $d(s) < 0$ for $s > 0$ and $d(s) > 0$ for $s < 0$ and that $s < 0$ *in* Σ^-; i.e., the cycle Γ is a stable limit cycle Cf. Figure $(I.3)$. Similarly, if $d'(0) > 0$ then Γ is an unstable limit cycle. So, we have the corresponding results that if $P(0) = 0$ and $P'(0) < 1$, then Γ is stable limit cycle and if $P(0) = 0$ and $P'(0) > 1$, then Γ is an unstable limit cycle.

***Theorem* I.4.** *Let the differential equation $(I.3)$ in the plane, and let $\phi(x, y, t)$ be the flow of $(I.3)$, and $\nabla \cdot f(x, y)$ be the divergence of the vector field $f = (\varphi, \psi)$ at (x, y). Now, let us take y_* and $L = \{x, x_1 \leq x \leq x_2\}$ with $x_1, x_2 \in \mathbb{R}$, we chose these so that the horizontal line $\Sigma = L \times \{y_*\}$ is transversal; i.e., $\psi(x, y_*) \neq 0$ for x in L. Assume that $L' \subset L$ is an open subinterval such that for each $x \in L'$, the solution of $(I.3)$ starting from (x, y_*) returns to $L \times \{y_*\}$ for some $\tau(x) > 0$; i.e., $\psi(x, y_*) \in L \times \{y_*\}$, and $P(x)$ be the first coordinate of the first return map or the Poincaré map as indicated in $(I.4)$. Then, for any $x \in L'$*

$$P'(x) = \frac{\psi(x, y_*)}{\psi(P(x), y_*)} \cdot exp\{\int_0^{\tau(x)} \nabla \cdot f(\phi(x, y_*, t))\, dt\}.$$

In particular if $P(x_0) = x_0$, then

$$P'(x) = exp\{\int_0^{\tau(x)} \nabla \cdot f(\phi(x, y_*, t))\, dt\}.$$

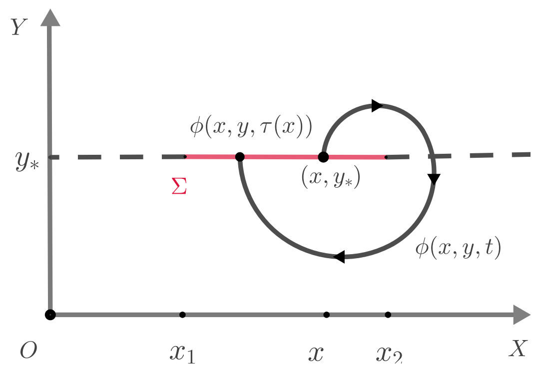


Figure I.4: The straight line normal $\Sigma = L \times \{y_*\}$ to $\phi(t, x, y)$ at (x, y_*).

Proof. The derivative of the flow $D\phi$ is known to be a fundamental matrix solution of the first variational equation $\phi(x, y, t)$, i.e.,

$$\frac{d}{dt} D\phi(x, y, t) = Df(\phi(x, y, t)) \cdot D\phi(x, y, t),$$

Since $det\, D\phi(x, y, x_0) = det(id) = 1$ because of $\phi(x, y, x_0) = \phi(x, y) = (x, y)$, the

Abel-Liouville formula gives that

$$det\, D\phi(x, y_*, \tau(x)) \; = \; exp\{\int_0^{\tau(x)} \nabla \cdot f(\phi(x, y_*, t))\, dt\}.$$

To complete the proof it is necessary to relate $P'(x)$ with $D\phi(x, y_*, \tau(x))$. Taking the partial derivative of $(P(x), y_*) \; = \; \phi(x, y_*, \tau(x))$ gives

$$(P'(x), 0) \; = \; \frac{\partial \phi}{\partial t}(x,\; y_*,\; \tau(x)) + \tau'(x) \cdot f(\phi(x, y_*, \tau(x)).$$

Using the fact at $t \; = \; 0$

$$\frac{d}{dt}\phi(\cdot, t) \circ \phi(x, y_*, \tau(x)) \; = \; \frac{d}{dt}\phi(\cdot, \tau(x)) \cdot \phi(x, y_*, t),$$

it follows that $f(P(x),\; y_*) \; = \; D\phi(x, y_*, \tau(x)) \cdot f(x,\; y_*)$. So,

$$\begin{aligned}
\psi(P(x),\; y_*) \cdot P'(x) \; &= \; det \begin{bmatrix} P'(x) & \varphi(P(x),\; y_*) \\ 0 & \psi(P(x),\; y_*) \end{bmatrix} \\
&= \; det \begin{bmatrix} \frac{\partial \phi}{\partial t}(x,\; y_*,\; \tau(x)) & D\phi(x, y_*, \tau(x)) \cdot f(x,\; y_*) \end{bmatrix} \\
&\quad + det \begin{bmatrix} \tau'(x) \cdot f(\phi(x, y_*, \tau(x))) & f(\phi(x, y_*, \tau(x))) \end{bmatrix} \\
&= \; det \begin{bmatrix} D\phi(\cdot\cdot) \begin{pmatrix} 1 \\ 0 \end{pmatrix} & D\phi(\cdot\cdot) f(x, y_*) \end{bmatrix} + 0 \\
&= \; det\, [D\phi(\cdot\cdot)] \cdot det \begin{bmatrix} 1 & \\ & f(\cdot\cdot) \\ 0 & \end{bmatrix} \\
&= \; \left(exp \int_0^{\tau(x)} \nabla \cdot f \left[\phi(x, y_*, t) \right] \right) \cdot \psi(x, y_*).
\end{aligned}$$

Dividing by $\psi(P(x),\; y_*)$ gives the desired formula.

■

Now, we are going to cite the most useful formula of the Poincaré map for studying the stability of limit cycles of the vector field f.

***Corollary* I.1.** *Let E be an open subset of $\mathbb{R}^2$ and suppose that $f = (\varphi, \psi) \in C^1(E)$, and $\gamma(t)$ be a periodic solution of (I.3) of period T. then*

$$P'(0) = \exp\{\int_0^T \nabla \cdot f(\gamma(t))dt\},$$

is the derivative of the Poincaré map $P(s)$ along Σ.

***Corollary* I.2.** *Under the hypotheses of corollary* $(I.1)$*, the periodic solution* $\gamma(t)$ *is*

- *a stable limit cycle if* $\int_0^T \nabla \cdot f(\gamma(t))dt < 0$,
- *an unstable limit cycle if* $\int_0^T \nabla \cdot f(\gamma(t))dt > 0$.

***Remark* I.3.** *It may be a stable, unstable, or semi-stable limit cycle or it may belong to a continuous band of cycles if this quantity is* $zero$.

I.3.2 A multiple limit cycle of multiplicity k

***Definition* I.15** (A multiple limit cycle of multiplicity k). *Let* $P(s)$ *be The Poincaré map for a cycle* Γ *of planar analytic system* $(I.1)$ *and let*

$$d(s) = P(s) - s,$$

be the displacement function. Then if

$$d(0) = d'(0) = \cdots = d^{(k-1)}(0) = 0 \text{ and } d^k(0) \neq 0,$$

Γ *is called a multiple limit cycle of multiplicity* k. *If* $k = 1$ *then* Γ *is called a simple limit cycle.*

***Remark* I.4.** *It can be shown the stability of the limit cycle* Γ *from "*k*".*

1. k *even* $\Rightarrow \Gamma$ *is semi-stable limit cycle,*
2. k *odd* $\Rightarrow \Gamma$ *is stable limit cycle if* $d^{(k)}(0) < 0$ *and unstable limit cycle if* $d^{(k)}(0) > 0$.

We shall see in the next chapter that if Γ *multiple limit cycle of multiplicity* k*, then "*k*" limit cycles can be made to bifurcate from* Γ *under a small periodic perturbation of the differential system* $(I.1)$*. Then it can be shown that in the analytic case,* $d^{(k)}(0) = 0$ *for* $k = 0, 1, 2, \ldots$ *iff* Γ *belongs to a continuous band of cycles.*

I.3.3 The Poincaré map for a focus

In this part, we discuss The Poincaré map in the neighborhood of a focus, of course for planar analytic systems, and to define what we mean by a multiple focus.

Suppose that the planar analytic system $(I.1)$ has a focus at the origin and that $Df(0) \neq 0$. Then $(I.1)$ is linearly equivalent to the system

$$\begin{cases} \dot{x} = ax - by + p(x, y), \\ \dot{y} = bx + ay + q(x, y), \end{cases} \tag{I.5}$$

with $b \neq 0$, and the power series expansions of p, q with second or higher degree terms. In polar coordinates $(I.5)$ equivalent to

$$\begin{cases} \dot{r} &= ar + O(r^2), \\ \dot{\theta} &= b + O(r^2), \end{cases} \tag{I.6}$$

Suppose that $r(t, r_0, \theta_0)$ and $\theta(t, r_0, \theta_0)$ are the solution of $(I.6)$ satisfying $r(0, r_0, \theta_0) = r_0$ and $\theta(0, r_0, \theta_0) = \theta_0$. Then for $r_0 > 0$ sufficiently small and $b > 0$, θ is strictly increasing function of t. Let $t(\theta, r_0, \theta_0)$ be the inverse of this strictly increasing function and for a fixed θ_0, we define the function

$$P(r_0) = r(t(\theta_0 + 2\pi, r_0, \theta_0), r_0, \theta_0).$$

$P(r_0)$ is called the Poincaré map for the focus at the origin of $(I.6)$; cf. Figure$(I.5)$.

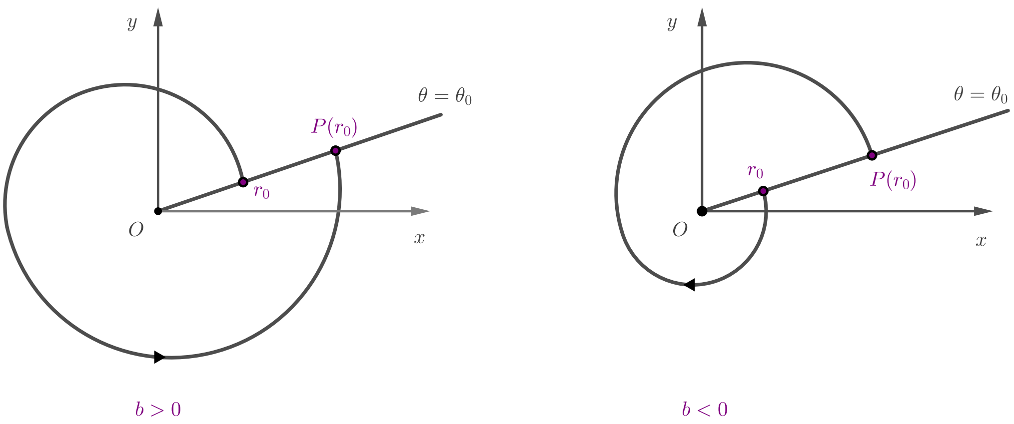


Figure I.5: The Poincaré map for a focus at the origin.

***Lemma* I.1.** *There exist $r > 0$, such that for all s, $0 < |s| \leqslant r$*

$$d(s) \cdot d(-s) < 0.$$

For the proof see ([7]).

The following theorem gives us the stability and the multiplicity of a multiple focus .

***Theorem* I.5.** *Let $P(s)$ be the Poincaré map for a focus at the origin of planar analytic system $(I.5)$, and $d(s) = P(s) - s$ the displacement function then by lemma $(I.1)$ and*

$$d(0) = d'(0) = \cdots = d^{(k-1)}(0) = 0 \text{ and } d^{(k)}(0) \neq 0,$$

k is an odd number; i.e., $k = 2m + 1$ this fact provide in the next chapter. The integer $m = (k - 1)/2$ is called the multiplicity of the focus.

- *If $m = 0$ the focus is called a simple focus, then the sign of $d'(0) = a \neq 0$ determine the stability of the focus; i.e., (stable if $a < 0$ else unstable). But if $d'(0) = 0$ which means $a = 0$; i.e., ((I.5) has a multiple focus or center at the origin), and the first nonzero derivative $\delta \equiv d^k(0) \neq 0$ is called the Laypunov number for the focus, and the stability of this focus depends on the sign of δ.*

- *If $k = 3$ then*

$$\begin{aligned}\delta_3 = d'''(0) = & \tfrac{3\pi}{2b}\{[3(a_{30}+b_{03})+(a_{12}+b_{21})]-\tfrac{1}{b}[2(a_{20}b_{20}-a_{02}b_{02}) \\ & -a_{11}(a_{02}+b_{20})+b_{11}(a_{02}+b_{20})]\},\end{aligned}$$

where

$$p(x,y) = \sum_{i+j\geq 2} a_{ij}x^i y^j \text{ and } q(x,y) = \sum_{i+j\geq 2} b_{ij}x^i y^j.$$

This information will be useful in the next chapter where we shall see that m limit cycles can be made to bifurcate from a multiple focus of multiplicity m under a suitable small perturbation of the system (I.5). Now we are going to prove the Laypunov number for $k = 3$.

Proof. Suppose that the planar analytic system (1.5) has a focus at the origin and that $Df(0) \neq 0$,

$$\begin{cases} x' = ax - by + a_{02}y^2 + a_{03}y^3 + a_{11}xy + a_{12}xy^2 + a_{20}x^2 + a_{21}x^2y + a_{30}x^3, \\ y' = ay + bx + b_{02}y^2 + b_{03}y^3 + b_{11}xy + b_{12}xy^2 + b_{20}x^2 + b_{21}x^2y + b_{30}x^3, \end{cases}$$

with $b \neq 0$. In polar coordinates this system has the form

$$\begin{cases} \dot{r} = & ar + r^2\big((a_{02}+b_{11})\sin^2(\theta)\cos(\theta) + a_{11} + b_{20})\sin(\theta)\cos^2(\theta) + a_{20}\cos^3(\theta) \\ & +b_{02}\sin^3(\theta)) + r^3((a_{03}+b_{12})\sin^3(\theta)\cos(\theta) + (a_{12}+b_{21})\sin^2(\theta)\cos^2(\theta) \\ & +(a_{21}+b_{30})\sin(\theta)\cos^3(\theta) + a_{30}\cos^4(\theta) + b_{03}\sin^4(\theta)). \\ \dot{\theta} = & r(-a_{02}\sin^3(\theta) - (a_{11}-b_{02})\sin^2(\theta)\cos(\theta) - (a_{20}-b_{11})\sin(\theta)\cos^2(\theta) \\ & +b_{20}\cos^3(\theta)) + r^2(-a_{03}\sin^4(\theta) - (a_{12}-b_{03})\sin^3(\theta)\cos(\theta) - (a_{21}-b_{12})\times \\ & \sin^2(\theta)\cos^2(\theta) - (a_{30}-b_{21})\sin(\theta)\cos^3(\theta) + b_{30}\cos^4(\theta)) + b. \end{cases}$$

then

$$\begin{aligned}
\frac{dr}{d\theta} &= (ar + r^2((a_{02}+b_{11})\sin^2(\theta)\cos(\theta) + (a_{11}+b_20)\sin(\theta)\cos^2(\theta) + a_{20}\cos^3(\theta) \\
&\quad + b_{02}\sin^3(\theta)) + r^3((a_{03}+b-12)\sin^3(\theta)\cos(\theta) + (a_{12}+b_{21})\sin^2(\theta)\cos^2(\theta) \\
&\quad + (a_{21}+b_{30})\sin(\theta)\cos^3(\theta) + a_{30}\cos^4(\theta) + b_{03}\sin^4(\theta)))/(r(-a_{02}\sin^3(\theta) \\
&\quad - (a_{11}-b_{02})\sin^2(\theta)\cos(\theta) - (a_{20}-b_{11})\sin(\theta)\cos^2(\theta) + b_{20}\cos^3(\theta)] \\
&\quad + r^2[-a_{03}\sin^4(\theta) - (a_{12}-b_{03})\sin^3(\theta)\cos(\theta) - (a_{21}-b_{12})\sin^2(\theta)\cos^2(\theta) \\
&\quad - (a_{30}-b_{21})\sin(\theta)\cos^3(\theta) + b_{30}\cos^4(\theta)) + b).
\end{aligned}$$

Now, by using the series of Taylor of the 5^{th} order with $a = 0$, then $F_2(r,\theta)$ the coefficients of r^2 which given by

$$\begin{aligned}
F_2(r,\theta) &= \Big[(a_{02}+b_{11})\sin^2(\theta)\cos(\theta) + a_{20}\cos^3(\theta) + b_{02}\sin^3(\theta) + (b_{20}+a_{11}) \\
&\quad \sin(\theta)\cos^2(\theta)\Big]/b,
\end{aligned}$$

by integration of $F_2(r,\theta)$ between 0 and 2π all over 2π we find 0, the next degree is 3 and $F_3(r,\theta)$ is the coefficients of r^3. By integration, we obtain δ_3. ∎

CHAPTER II

BIFURCATION THEORY

In this chapter, we consider two types of bifurcations, that can occur at a non-hyperbolic equilibrium point x_0 of a differential system which depends on a parameter μ,

$$\dot{x} = f(x, \mu). \tag{II.1}$$

with $\mu \in \mathbb{R}$, here for studying the stability we have two cases for the matrix $Df(x_0, \mu_0)$. The first one is if it has a simple zero eigenvalue, in the second case we see if the saddle-node bifurcations were generic.

II.1 Hopf bifurcations and bifurcations of limit cycles from a multiple focus

In this section, we are interested in the one which has only a simple pair of purely imaginary eigenvalues (i.e., no other eigenvalues with zero real part). Here the implicit function theorem guarantees that in the neighborhood of μ_0 there will be a unique equilibrium point x_μ near x_0.

We illustrate the idea and present a general theory for planar systems. For the more general theory of Hopf bifurcation in higher dimensional system see [3] or [4]. Let the planar analytic system

$$\begin{cases} \dot{x} &= \mu x - y + p(x, y), \\ \dot{y} &= x + \mu y + q(x, y), \end{cases} \tag{II.2}$$

where the analytic functions p, q defined as in chapter (I).

Changing over polar coordinates (r, θ) we first obtain the system

$$\begin{cases} \frac{dr}{dt} &= F(r,\theta) = \mu r + p(rcos\theta, rsin\theta)cos\theta + q(rcos\theta, rsin\theta)sin\theta, \\ \\ \frac{d\theta}{dt} &= 1 + \Theta(r,\theta) = 1 + \frac{q}{r}cos\theta - \frac{p}{r}sin\theta, \end{cases} \tag{II.3}$$

and then the equation

$$\frac{dr}{d\theta} = R(r,\theta) = \frac{F(r,\theta)}{1+\Theta(r,\theta)}. \tag{II.4}$$

***Definition* II.1** (Hopf bifurcation). *A Hopf bifurcation occurs, where a periodic orbit or limit cycle is created as the stability of the equilibrium point x_μ changes, arises or goes away as a parameter μ varies. When a stable limit cycle surrounds an unstable equilibrium point, the bifurcation is called a supercritical Hopf bifurcation. If the limit cycle is unstable and surrounds a stable equilibrium point, then the bifurcation is called a subcritical Hopf bifurcation.*

Let F_0 be a function of class k_0 or analytical function in an open region G of $\mathbb{R}^n$, δ is some positive number, r is a natural number such that $r \leq k_0$.

***Definition* II.2** (δ-Closeness to rank r). *A function F_1 of class k_1, $r \leq k_1$ or analytical in an open region G of $\mathbb{R}^n$ is said to be δ-close to rank r to the function F_0, if at any point of the region*

$$|F_1 - F_0| < \delta, \quad |F^{(l)}_{1x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}} - F^{(l)}_{0x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}}| < \delta,$$

where $l = 1, 2, \dots, r$, all α_i are non-negative numbers and $\alpha_1 + \alpha_2 + \dots + \alpha_n = l$.

***Remark* II.1.** *It is clear that*

- *If two functions are δ-close to rank r in some region G, they are δ-close to any rank; moreover, for any δ_1, $\delta_1 \geq \delta$ they are δ_1-close to rank r in any subregion of G.*
- *If we only have the one inequality everywhere in the region G,*

  $$|F_1 - F_0| < \delta,$$

  i.e., only the functions as such are δ-close, but not their derivatives, the functions F_1 and F_0 are said to be δ-close to rank 0.

***Definition* II.3** (Focal value). *The k-th focal value of the focus O is the value of the k-th derivative of the displacement function $(I.15)$ at the origin, i.e., $d^{(k)}(0)$.*

***Lemma* II.1.** *If there exists k such that*

$$d'(0) = d''(0) = \dots = d^{(k-1)}(0) = 0, \; and \; d^{(k)}(0) \neq 0. \tag{II.5}$$

and the origin is a focus, then k is odd number.

Proof. Let suppose that k the multiplicity of the focus is even number, by $(II.3)$ and $(II.4)$, $r = 0$ is a solution of equation $(II.4)$. Therefore

$$d(0) = 0. \tag{II.6}$$

By using the series of Taylor to the displacement function d and using relations $(II.5)$ and $(II.6)$ we find

$$d(s) = \frac{d^{(k)}(\eta s)}{k!} s^k,$$

where $0 < \eta < 1$. Therefore, if k is even, $d(r)$ has the same sign for all sufficiently small r both negative and positive. contraduction with $(I.1)$. ∎

***Definition* II.4.** *If the lemma $(II.1)$ satisfied, and $k = 2m+1,\ m \geqslant 0,$ we shall say that the focus O is a focus of multiplicity m.*

The next theorem shows that a dynamic system may only have a finite number of different bifurcations in the neighborhood of a focus of a finite multiplicity m.

***Theorem* II.1** (Theorem of creation of limit cycles from a multiple focus)**.** *If the origin O is a multiple focus of multiplicity $m \geq 1$ of a dynamic system (A), of class $N \geq 2m+1$ then*

1. *there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that any system $(\tilde{A})$ δ_0-close to rank $2m+1$ to system (A) has at most m closed paths in the neighborhood of the origin $N_{\varepsilon_0}(O)$;*
2. *for any $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$ there exists a system $(\tilde{A})$ of class N to rank $2m+1$ to (A) and has m closed paths in $N_{\varepsilon_0}(O)$.*

i.e., If the origin O is a multiple focus of multiplicity $m \geq 1$ of (A), systems $(\tilde{A})$ sufficiently close to (A) to rank $2m+1$ can have at most m closed paths in a sufficiently small neighborhood of the focus. Thus it may create m, but no more than m limit cycles.

For the details of the proof of this theorem, see [7].

Now lets consider one particular case, which is often encountered in applications, namely a system dependent on a single parameter and its bifurcations in the neighborhood of a multiple focus of multiplicity 1 when the parameter is varied. Let the planar analytic system

$$\begin{cases} \dot{x} &= a(\mu)\,x + b(\mu)\,y + \varphi(x,y,\mu), \\ \dot{y} &= c(\mu)\,x + d(\mu)\,y + \psi(x,y,\mu), \end{cases} \tag{A_μ}$$

***Theorem* II.2** (Bifurcations in the neighborhood of a multiple focus of multiplicity $m = 1$). *For a planar analytic system has a focus at the origin, if $\delta_3 \neq 0$ defined as in the first chapter theorem $(I.5)$ then a Hopf bifurcation occurs at the origin of the planar analytic system (A_μ) at the bifurcation value $\mu = 0$;*

1. *if $\delta_3 < 0$, a unique stable limit cycle bifurcates from the origin of (A_μ) in this case, we have what is called a supercritical Hopf bifurcation;*
2. *in the second case, if $\delta_3 > 0$, the critical point generates an unstable limit cycle as μ passes through the bifurcation value $\mu = 0$, we have what is called a subcritical Hopf bifurcation.*

***Proof*.** Let (A_μ) be a dynamic system which depends on the parameter μ, so we will consider the bifurcations of this system associated with the variation of the parameter μ , in the neighborhood of an equilibrium point $O(0, 0)$, when $O(0, 0)$ is a multiple focus of multiplicity 1. For simplicity, we assume that $\mu = 0$ is the bifurcation value. Let

$$\sigma(\mu) = a(\mu) + d(\mu), \tag{II.7}$$

$$\Delta(\mu) = \begin{vmatrix} a(\mu) & b(\mu) \\ c(\mu) & d(\mu) \end{vmatrix}. \tag{II.8}$$

Then

$$\sigma(0) = 0, \tag{II.9}$$

$$\Delta(\mu) > 0. \tag{II.10}$$

We apply the transformation

$$\xi = x, \quad \eta = -\frac{a(0)}{\sqrt{\Delta(0)}}x - \frac{b(0)}{\sqrt{\Delta(0)}}, \tag{II.11}$$

which reduces (A_0) ((A_μ) with $\mu = 0$) to the canonical form

$$\frac{d\xi}{dt} = -\sqrt{\Delta(0)}\eta + \varphi\,(\xi, \eta), \quad \frac{d\eta}{dt} = \sqrt{\Delta(0)}\xi + \psi\,(\xi, \eta). \tag{II.12}$$

Since $(II.11)$ is non-singular transformation, O remains a multiple focus of multiplicity 1 for $(II.12)$ also, and its stability does not change either, with the third derivative of the displacement function does not vanish. We have seen in the first chapter that if $\delta_3 < 0$, the origin is a stable focus, and if $\delta_3 > 0$ it is an unstable focus.

Let V_0 be a sufficiently small neighborhood of the point O bounded by a cycle without contact C of system (A_0) which contains no closed paths or equilibrium point other than O of this system, and let $\sigma_0 > 0$ be so small that any system (A_μ) for which has the following properties :

1. the curve C is a cycle without contact for this system;
2. system (A_μ) has no equilibrium point, other than O, in V_0;
3. the point O is a focus of (A_μ);
4. system (A_μ) has at most one closed path in V_0.

Suppose $\sigma(\mu)$ that reverses its sign as the system passes through the bifurcation value of the parameter $\mu = 0$, i.e., the focus O changes its stability. This condition is clearly satisfied if $\sigma'(0) \neq 0$. Let us now consider the different possible cases.

The case (i): If $\delta_3 < 0$ we assume that when we passing through the bifurcation value $\mu = 0$, $\sigma(\mu)$ changes its sign from minus to plus. If $\sigma'(\mu) \neq 0$ then this conditions are satisfied when μ increases, for $\sigma'(\mu) > 0$; when μ decreases, for $\sigma'(\mu) < 0$.

Since $\delta_3 < 0$, the focus O is a stable focus of (A_0) for $\mu = 0$. Therefore all the paths of (A_μ) enter into the cycle without contact C as t increases. For $\sigma(\mu) < 0$, O is a stable focus of (A_μ)). By theorem $(II.1)$, (A_μ) has at most one limit cycle in V_0, and if this cycle exists, it is a simple cycle, i.e., either stable or unstable. Clearly, for $\sigma(\mu) < 0$ no such cycle exists. Indeed, if this cycle existed, it would be stable from outside and unstable from the inside, i.e., it could not be simple. We have thus established that if $\delta_3 < 0$ and $\sigma(\mu) < 0$, (A_μ) has no limit cycles in V_0.

Conversely, if $\sigma(\mu) > 0$, O is an unstable focus of (A_μ). Then, reasoning as before, we conclude that there is a single limit cycle L_μ of (A_μ) inside V_0, and this is a simple stable cycle. It is ok seen that if μ is sufficiently small, the cycle L_μ is arbitrarily close to O.

We thus obtain the following results. If $\delta_3 < 0$ and $\sigma'(0) > 0$, system (A_μ) has no limit cycles in V_0 for small negative μ and $\mu = 0$, and O is the stable focus. As the system crosses the bifurcations value (i.e., for $\mu > 0$). The focus becomes unstable, and a stable limit cycle develops inside the neighborhood V_0. If μ varied in the opposite direction, i.e., we move from positive to negative μ, the stable limit cycle which originally existed in V_0 would contract to the focus O and vanish for $\mu = 0$, and the focus will change its stability accordingly.

As μ is further decreased, the focus remains stable and the topological structure of V_0 does not change.

For $\delta_3 < 0$ and $\sigma'(0) < 0$, the stable limit cycle is created on passing from positive to negative μ, and conversely it disappears when μ increases and reaches zero.

The case (ii): For $\delta_3 > 0$. The investigation proceeds along the same lines as before.

The above results can be summarized in the following table :

	$\mu < 0$	$\mu = 0$	$\mu > 0$
$\delta_3 < 0, \sigma'(0) > 0$	Unstable focus, no cycle	Stable focus, no cycle	Unstable focus, stable cycle
$\delta_3 < 0, \sigma'(0) < 0$	Unstable focus, stable cycle	Stable focus, no cycle	Stable focus, no cycle
$\delta_3 > 0, \sigma'(0) > 0$	Stable focus, unstable cycle	Unstable focus, no cycle	Unstable focus, no cycle
$\delta_3 > 0, \sigma'(0) < 0$	Unstable focus, no cycle	Unstable focus, no cycle	Stable focus, unstable cycle

The above analysis shows that the change in μ brings about a change in the stability of the focus if a limit cycle is created from the focus disappears contracting into the focus. A stable focus creates a stable cycle, and an unstable focus, an unstable cycle. Thus a focus creates a limit cycle of the same stability, and the stability of the focus changes in the process.
Conversely, when the cycle disappears (when it is absorbed by the focus), the focus acquires the same stability as that of the cycle before absorption. This state of things is not limited to the case of a focus of multiplicity 1 : it is observed whenever a focus creates or absorbs a cycle of definite stability(i.e., not semistable cycle). ■

***Remark* II.2.** *The same results are reserve for system $(II.2)$, because of $(II.2)$ is a special case of (A_μ).*

The next theorem proved the existance of the Hopf bifurcation in higher dimensional systems where the Jacobian matrix has a pair of pure imaginary eigenvalues and no other eigenvalues with zero real part; i.e., $\lambda_i = \pm\beta i$ for all i, with $\beta > 0$.

***Theorem* II.3** (Hopf)**.** *Suppose that the C^4-system $(II.1)$ with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$ which has (x_0, μ_0) as a critical point, with simple pair of purely imaginary eigenvalues and no other eigenvalues with zero real part. Then there is a smooth curve of equilibrium points $x(\mu)$ and the eigenvalue, $\lambda(\mu)$ and $\overline{\lambda}(\mu)$ of $Df(x(\mu), \mu)$ which are pure imaginary at $\mu = \mu_0$. Furthermore, if*

$$\frac{d}{dt}\left[Re\lambda(\mu)\right]_{\mu=\mu_0} \neq 0,$$

then there is a unique two-dimensional center manifold passing through the point (x_0, μ_0) and a smooth transformation of coordinates such that the system $(II.1)$ on the center manifold is transformed into the normal form

$$\begin{cases} \dot{x} &= -y + ax(x^2+y^2) - by(x^2+y^2) + O(|x|^4), \\ \dot{y} &= x + bx(x^2+y^2) + ay(x^2+y^2) + O(|x|^4), \end{cases} \tag{II.13}$$

in a neighborhood of the origin for $a \neq 0$, has a weak focus of multiplicity one at the origin and

$$\begin{cases} \dot{x} &= \mu x - y + ax(x^2+y^2) - by(x^2+y^2), \\ \dot{y} &= x + \mu y + bx(x^2+y^2) + ay(x^2+y^2), \end{cases} \tag{II.14}$$

is a universal unfolding of this normal form in a neighborhood of the origin on the center manifold.

This theorem can be proved by a direct application of the center manifold (cf.[10]).

The following theorem shows that at most m limit cycles can bifurcate from the origin which is a weak focus or a multiple focus of multiplicity $m > 1$ as the parameter μ varies through the bifurcations value and that there is an analytic perturbation of the vector field in $(II.15)$, which has exactly m limit cycles.

$$\begin{cases} \dot{x} &= -y + p(x,y), \\ \dot{y} &= x + q(x,y), \end{cases} \tag{II.15}$$

***Theorem* II.4** (The bifurcation of limit cycles from a multiple focus)**.** *If the origin is a multiple focus of multiplicity m of the analytic system $(II.15)$ then for $k \geq 2m+1$*

1. *there is a $\delta > 0$ and an $\varepsilon > 0$ such that any system ε-close to $(II.15)$ in the C^k-norm has at most m limit cycles in $N_\delta(0)$ and,*
2. *for any $\delta > 0$ and $\varepsilon > 0$ there is an analytic system which is ε-close to $(II.15)$ in the C^k-norm has exactly m simple limit cycles in $N_\delta(0)$.*

For the proof you can see [7].

In this section, we considered a multiple focus and showed that it may create closed paths. In next section we will elucidate the number of paths that may be created in the neighborhood of a multiple limit cycle.

II.2 Bifurcations at non-hyperbolic periodic orbit

Many interesting types of bifurcations can take place at a non-hyperbolic periodic orbit. This is the case when the derivative of the Poincaré map at $x_0 \in \Gamma$, has an eigenvalue equal to one.

***Definition* II.5** (Non-hyperbolic periodic orbit)**.** *A non-hyperbolic periodic orbit is a periodic orbit have two or more characteristic exponents with zero real part.*

The system $(II.1)$ is said to have a non-hyperbolic periodic orbit Γ_0 through x_0 at the bifurcation value μ_0 if $DP(x_0, \mu_0)$ has an eigenvalue of unit modulus.

***Definition* II.6.** *A closed path Γ_0 of a dynamic system (A) of class N is said to be limit cycle of multiplicity k if*

$$d'(0) = d''(0) = \dots = d^{(k-1)}(0) = 0, \text{ and } d^{(k)}(0) \neq 0,$$

(k is a natural number, $k \leq N$).

The three simplest types of bifurcations that can occurs at a non-hyperbolic periodic orbit in $\mathbb{R}^2$ are illustrated in the following theorem.

***Theorem* II.5.** *Suppose that $f \in C^2(E \times J)$ where E is an open subset of $\mathbb{R}^2$ and $J \subset \mathbb{R}$. Assume that the system $(II.1)$ has a periodic orbit Γ_0 at the bifurcation value $\mu = \mu_0$ and its Poincaré map is $P(s, \mu)$ defined in a neighborhood $N_\delta(0)$ of the origin which is a multiple limit cycle. Then if $P(0, \mu_0) = 0$, $DP(0, \mu_0) = 1$ we have three cases;*

- *if $D^2P(0, \mu_0) \neq 0$ and $P_\mu(0, \mu_0) \neq 0$, a saddle-node bifurcation occurs at the non-hyperbolic periodic orbit Γ_0 at the bifurcation value $\mu = \mu_0$, the periodic orbit Γ_0 is a multiple limit cycle of multiplicity 2 and exactly two limit cycles.*
- *if $P_\mu(0, \mu_0) = 0$, $DP_\mu(0, \mu_0) \neq 0$ and $D^2P(0, \mu_0) \neq 0$, then a transcritical bifurcation occurs at the non-hyperbolic periodic orbit Γ_0 at the bifurcation value $\mu = \mu_0$,*
- *then if $P_\mu(0, \mu_0) = 0$, $DP_\mu(0, \mu_0) \neq 0$, $D^2P(0, \mu_0) = 0$ and $D^3P(0, \mu_0) = 0$ a pitchfork bifurcation occurs at the non-hyperbolic periodic orbit Γ_0 at the bifurcation value $\mu = \mu_0$.*

***Definition* II.7.** *The root O of the equation*

$$g_0(x) = 0, \tag{II.16}$$

is called a root of multiplicity r of $(II.16)$ if g_0 is a function of class $k \geqslant r$ and we have the following condition are satisfied

1. *there exist $\varepsilon_0 > 0$, $\sigma_0 > 0$ such that any equation $g(x) = 0$ of class r which is σ_0-close to rank r to the function $g_0(x)$ has at most r roots for $|x| < \varepsilon_0$,*
2. *for any positive $\varepsilon < \varepsilon_0$ and σ there exists a function $g(x)$, σ-close to rank r to the function $g_0(x)$ such that the equation $(II.16)$ has precisely r roots for $|x| < \varepsilon$.*

A root x_0 of a function $g_0(x)$ is said to be of multiplicity $r \geq 1$ if functions $g(x)$ sufficiently close to $g_0(x)$ cannot have more than r roots in a sufficiently small neighborhood of the root x_0, but there is any number of functions sufficiently close to $g_0(x)$ which have exactly r roots in any arbitrarily small neighborhood of x_0.

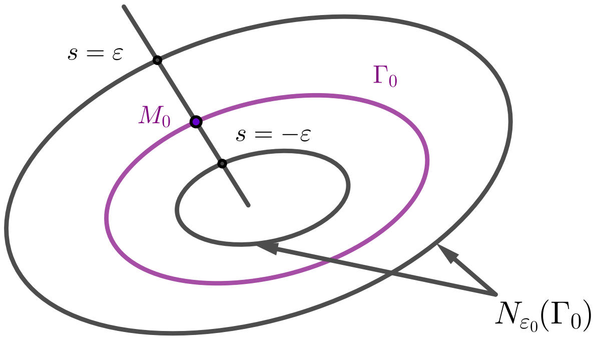


Figure II.1: The values of Γ_0 correspond to the values of the parameter s .

***Lemma* II.2.** *We called the number x_0 a root of multiplicity r of the function $g_0(x)$ if and only if*

$$g_0(x_0) = g_0'(x_0) = g_0''(x_0) = \dots = g_0^{(r-1)}(x_0) = 0, \ \ and \ \ g_0^{(r)}(x_0) \neq 0.$$

For more detail see [7].

***Theorem* II.6** (Theorem of the creation of limit cycles from a multiple limit cycle)**.** *If (A) is a dynamic system of class $N > 1$ or an analytical system, and Γ_0 is a multiple limit cycle of multiplicity k ($2 \leq k \leq N$), then*

1. *there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that any system $(\tilde{A})$ δ_0-close to rank k to system (A) has at most k closed paths in $N_{\varepsilon_0}(\Gamma_0)$;*
2. *for any $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$ there exists a system $(\tilde{A})$ of class N (of analytical class, respectively) which is δ_0-close to rank k to (A) and has k closed paths in $N_{\varepsilon_0}(\Gamma_0)$.*

Proof. Lets prove the first proposition,let the displacement function d defined for all s, $|s| \leq \eta_*$, where η_* some positive number. And let Γ_0 be a limit cycle of multiplicity k of system (A), so

$$d'(0) = d''(0) = \dots = d^{(k-1)}(0) = 0, \quad d^{(k)}(0) \neq 0.$$

we see that 0 is a root of multiplicity k of the displacement function d, then by lemma $(II.2)$ and definition $(II.7)$, there exist a positive number $\varepsilon < \eta_*$ and σ such that any function $\tilde{d}(s)$ defined for all s, $|s| \leq \varepsilon$, and σ-close to $d(s)$ to rank k may have at most k roots on the segment $[-\varepsilon, \varepsilon]$ (Figure $(II.1)$). By the second proposition of the definition $(II.7)$, a sufficiently small positive number is taken ε_0, and σ_0 is taken also so small that the following condition is satisfied: if system $(\tilde{A})$ σ_0-close to rank k to A the function $\tilde{d}(s)$ is defined for $(\tilde{A})$ on the arc Σ for all s, $|s| \leq \eta_*$, and for $|s| \leq \eta_*$ the function $\tilde{d}(s)$ is σ_0-close to $d(s)$ to rank k. ∎

CHAPTER III

RIGID SYSTEM

All rigid planar systems are given by the differential equation of the form

$$\begin{cases} \dot{x} &= -y + x\, q(x, y), \\ \dot{y} &= x + y\, q(x, y), \end{cases} \tag{III.1}$$

where $q : \mathbb{R}^2 \to \mathbb{R}$ is an analytic function. A differential planar system in which angular speed is constant is called a rigid system. It is simple to see this system has the origin as the only equilibrium point which is of center-focus type when $q(0, 0) = 0$. There are the following open questions:

- How to decide the stability of the equilibrium point at the origin?
- How to know whether the system presents or not a center at the origin?

These questions are also related to the second part of Hilbert's 16^{th} problem is still unsolved, even in the quadratic case. The objective of this chapter is to classify the phase portraits of system

$$\dot{x} = -y + x(a + bx^2 + cy^2), \quad \dot{y} = x + y(a + bx^2 + cy^2), \tag{III.2}$$

in the Poincaré disc. We assume that $b^2 + c^2$ is not zero.

***Theorem* III.1.** *The phase portraits in the Poincaré disc of systems* $(III.2)$ *with* $b^2 + c^2 \neq 0$ *are topologically equivalent to one of the four phase portraits given in Figure* $(III.1)$.

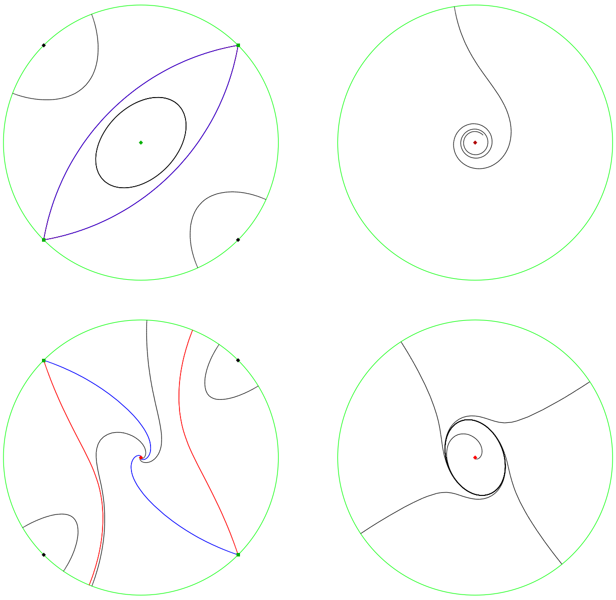


Figure III.1: All topologically different phase portraits in the Poincaré disc of systems (III.2).

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III.1 The local phase portraits of system (3.2)

Study of systems (3.2) in the finite region

Let us study the singular points and periodic orbits of systems (III.2) in the finite region. For that, the following lemma is needed.

***Lemma* III.1.** *For all $\theta \in [0, 2\pi]$ and $\lambda > 0$, the function*

$$\phi : \theta \to \phi(\theta) = -\int_0^\theta e^{2as}(b + c + (b - c)\cos 2s)ds + \lambda,$$

is strictly positive if one of the following conditions holds.

(i) $b > c,\ c > 0$ *and* $(b + 2a^2b + c)(1 - e^{4a\pi})/2a(1 + a^2) + \lambda > 0.$

(ii) $b > c$ *and* $b < 0.$

(iii) $b < c,\ b > 0$ *and* $(b + 2a^2b + c)(1 - e^{4a\pi})/2a(1 + a^2) + \lambda > 0.$

(iv) $b < c$ *and* $c < 0.$

(v) $b = c \neq 0$ *and* $(b/a)(1 - e^{4a\pi}) + \lambda > 0.$

Proof.

The case (i): We assume that $b > c,\ c > 0$ and

$$(b + 2a^2 b + c)(1 - e^{4a\pi})/2a(1 + a^2) + \lambda > 0.$$

Let the function

$$\phi : \theta \to \phi(\theta) = -\int_0^\theta e^{2as}(b + c + (b - c)\cos 2s)ds + \lambda.$$

The derivative of this function with respect to θ is

$$\frac{d\phi}{d\theta} = -(b - c)e^{2a\theta}(\frac{b + c}{b - c} + \cos 2\theta).$$

From the conditions $b > c$ and $c > 0$ it is easy to notice that $(b + c)/(b - c) > 1$ which gives $(b + c)/(b - c) + \cos 2\theta > 0$ for all $\theta \in [0, 2\pi]$, thus ϕ is a strictly decreasing function for all $\theta \in [0, 2\pi]$. Since $\phi(0) = \lambda > 0,\ \ \phi(2\pi) = (b + 2a^2 b + c)(1 - e^{4a\pi})/2a(1 + a^2) + \lambda > 0$ and ϕ is a strictly decreasing function we obtain that ϕ is strictly positive for all $\theta \in [0, 2\pi]$. We use the same argument to prove the case (iii).

The case (ii): We assume that $b > c$ and $b < 0$, this implies $(b + c)/(b - c) < -1$ what means $(b + c)/(b - c) + \cos 2s < 0$, we get

$$\phi(\theta) = (c - b)\int_0^\theta e^{2as}(\frac{b + c}{b - c} + \cos 2s)ds + \lambda > 0.$$

Then ϕ is strictly positive. We use the same argument to prove the case (iv).

The case (v): Under the condition $b = c \neq 0$ the function ϕ becomes

$$\phi(\theta) = -\int_0^\theta 2be^{2as}ds + \lambda.$$

The derivative of this function with respect to θ is

$$\frac{d\phi}{d\theta} = -2be^{2a\theta}.$$

So, ϕ is strictly increasing or decreasing between $\phi(2\pi) = (b/a)(1 - e^{4a\pi}) + \lambda > 0$ and $\phi(0) = \lambda > 0$. Then ϕ is strictly positive. ▪

The singular points of systems $(III.2)$ in finite are studied in the proposition below.

***Proposition* III.1.** *Polynomial differential systems $(III.2)$ have only one finite singular point which is the origin of coordinates. If $a > 0$ this singular point is an unstable focus or a stable focus if $a < 0$. When $a = 0$, the origin is a center if $b = -c$ and a focus if $b \neq -c$.*

***Proof*.** Since the eigenvalues of the linear part at the origin are $a \pm i$, it follows that the origin is a focus, which is unstable if $a > 0$ and stable if $a < 0$. In the case when $a = 0$, we use the Poincaré map to distinguish between center and focus. The use of the polar coordinates systems $(III.1)$ become equivalent to the following Bernoulli equation

$$\frac{dr}{d\theta} = ar + \frac{1}{2}(b + c + (b - c)\cos 2\theta)r^3.$$

By solving this last equation, we get

$$r(\theta, r_0) = e^{(-2a\theta)}\left(-\int_0^\theta e^{2as}(b + c + (b - c)\cos 2s)ds + r_0^{-2}\right)^{-\frac{1}{2}}.$$

From Lemma $(III.1)$ we see that $r(\theta, r_0)$ is well defined. For this, we can define the Poincaré map by

$$P : r_0 \to P(r_0) = r(2\pi, r_0),$$

where $r(2\pi, r_0) = e^{(-4a\pi)}\left(-\int_0^{2\pi} e^{2as}(b + c + (b - c)\cos 2s)ds + r_0^{-2}\right)^{-\frac{1}{2}}$.

For $a = 0$, we have

$$P(r_0) = r(2\pi, r_0) = (-2(b + c)\pi + r_0^{-2})^{-\frac{1}{2}}.$$

If $b = -c$ we obtain $r(2\pi, r_0) = r_0$ for all $r_0 \in \mathbb{R}^+$, then the origin is center. If $b \neq -c$ we get $r(2\pi, r_0) \neq r_0$ for all $r_0 \in \mathbb{R}^+$, hence the origin is a focus. ▪

In the following Proposition, limit cycles of systems (III.2) are studied.

***Proposition* III.2.** *The polynomial differential systems $(III.1)$ have a unique limit cycle if $(a > 0,\ b < 0,\ c < 0)$ or $(a < 0,\ b > 0,\ c > 0)$, and its expression in polar coordinates is*

$$r(\theta) = e^{(-2a\theta)}\left(-\int_0^\theta e^{2as}(b + c + (b - c)\cos 2s)ds + r_*^{-2}\right)^{-\frac{1}{2}},$$

where

$$r_* = \left(\frac{-(b + 2a^2b + c)}{2a(1 + a^2)}\right)^{-\frac{1}{2}}$$

***Proof*.** We have defined P in the proof of Proposition $(III.1)$ by

$$P(r_0) = r(2\pi, r_0) = e^{(-4a\pi)}\left(-\int_0^{2\pi} e^{2as}(b + c + (b - c)\cos 2s)ds + r_0^{-2}\right)^{-\frac{1}{2}}.$$

To get a periodic orbit we must verify the equality $r(2\pi, r_0) = r_0$. So the unique positive value to verify this equation is

$$r_* = \left(\frac{-(b + 2a^2b + c)}{2a(1 + a^2)} \right)^{-\frac{1}{2}}.$$

From Lemma $(III.1)$ and the value r_* to be positive, we must take $(a > 0,\ b < 0,\ c < 0)$ or $(a < 0,\ b > 0,\ c > 0)$. Then the proof of the proposition is completed. ∎

The study of systems (3.2) in the infinite region

Throughout this part, we will study the infinite singular points in the Poincaré disc. Therefore, we need to study all singular points in the chart U_1 and verify whether the origin of the chart U_2 is a singular point. For that, we use the notations given in chapter 1 section $(I.1.3)$, consequently, then following proposition is deduced.

***Proposition* III.3.** *In the local chart U_1, if $cb > 0$ the infinity of systems (III.2) is filled by singular points, and if $cb < 0$ the infinity of systems (III.2) is filled by singular points and when eliminating the common factor we get two others singular points $(\pm\sqrt{-b/c}, 0)$. If $b > 0$ the singular point $(\sqrt{-b/c}, 0)$ is saddle and $(-\sqrt{-b/c}, 0)$ is weak focus. If $b < 0$ the singular point $(\sqrt{-b/c}, 0)$ is weak focus and $(-\sqrt{-b/c}, 0)$ is saddle.*

***Proof*.** The systems (III.2) on the local chart U_1 is

$$\dot{u} = v^2 + u^2v^2, \quad \dot{v} = -bv - cu^2v - av^3 + uv^3. \tag{III.3}$$

If $bc > 0$, the line $\{v = 0,\ \forall u \in \mathbb{R}\}$ verify the algebraic system

$$\dot{u} = 0, \quad \dot{v} = 0,$$

thus the infinity of systems (III.2) is a line of singularities. when we eliminate the common factor v systems (III.2) on the local chart U_1 becomes

$$\dot{u} = v + u^2v, \quad \dot{v} = -b - cu^2 - av^2 + uv^2. \tag{III.4}$$

If $bc < 0$ these systems have two singular points $(\sqrt{-b/c}, 0)$ and $(-\sqrt{-b/c}, 0)$ with eigenvalues $\pm\sqrt{2(b-c)\sqrt{-b/c}}$ and $\pm\sqrt{-2(b-c)\sqrt{-b/c}}$, respectively. In the case $b > 0$ the value $-2(b-c)\sqrt{-b/c}$ is negative, then the point $(\sqrt{-b/c}, 0)$ is saddle and the eigenvalues of $(-\sqrt{-b/c}, 0)$ are $\pm i\sqrt{2(b-c)\sqrt{-b/c}}$. To distinguish if the singular point $(-\sqrt{-b/c}, 0)$ is center or focus, we need to move this singular point at the origin by the change of variable $u = w - \sqrt{-b/c}$, thus systems $(III.4)$ become

$$\dot{w} = -2\sqrt{\frac{-b}{c}}w + (\frac{-b}{c} + 1)v + w^2, \quad \dot{v} = 2\sqrt{\frac{-b}{c}}cw - (a + \sqrt{\frac{-b}{c}})v^2 - cw^2 + v^2w.$$

The eigenvalues of the origin are $-\sqrt{-b/c} \pm i\sqrt{-2\sqrt{-b/c}(c-b+\sqrt{-b/c})/c}$. Then, the singular point $(-\sqrt{-b/c}, 0)$ is focus. We use same argument to prove in the case $b < 0$ that the singular point $(\sqrt{-b/c}, 0)$ is focus. The systems (III.2) on the chart U_2 is

$$\dot{u} = -v^2 - u^2v^2, \quad \dot{v} = -bv - cu^2v - av^3 - uv^3. \tag{III.5}$$

The origin of the local chart U_2 is one point of the line of singularities $v = 0$. ■

III.2 The global phase portraits of systems (3.2)

This section includes all information from subsection $(III.1)$ to prove the global phase portraits of systems $(III.2)$.

III.2.1 The case: $a = 0$

In this case, the eigenvalues of the origin in finite are purely imaginary. We use the Proposition $(III.1)$ to distinguish if the origin is a center or a focus. And with the help of Proposition $(III.3)$, we prove all different cases in infinite.

The case: ($a = 0, b = -c$)

From Proposition $(III.1)$ the origin in finite is a center and there is no limit cycle. And from Proposition $(III.3)$ in infinite there is a line of singularities and two singular points $(\sqrt{-b/c}, 0)$ and $(-\sqrt{-b/c}, 0)$ in the chart U_1; saddle and focus respectively if $b > 0$, focus and saddle respectively if $b < 0$. Then there is a unique possible global phase portrait in the Poincaré disc given in Figure $(III.2)$.

The case: ($a = 0, bc > 0$)

From Proposition $(III.1)$, the origin in finite is a focus; additionally, there is no limit cycle. Also, from Proposition $(III.3)$ in infinite there is a line of singularities; along with a unique possible global phase portrait in the Poincaré disc given in Figure $(III.3)$.

The case: ($a = 0, b \neq -c, bc < 0$)

From Proposition $(III.1)$, the origin in finite is a focus and there is no limit cycle. Furthermore, from Proposition $(III.3)$ in infinite there is a line of singularities and two singular points $(\sqrt{-b/c}, 0)$ and $(-\sqrt{-b/c}, 0)$ in the chart U_1; saddle and focus respectively if $b > 0$, focus and

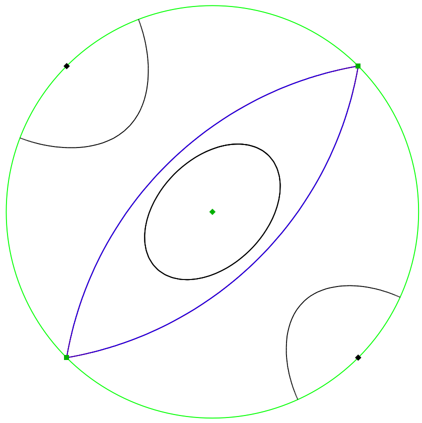

Figure III.2: The global phase portrait of systems (III.2) for $a = 0$ and $b = -c$, has eleven separatrices (S=11) and three canonical regions (R=3). The phase portrait can be obtained by taking $a = 0$, $b = 1$, $c = -1$.

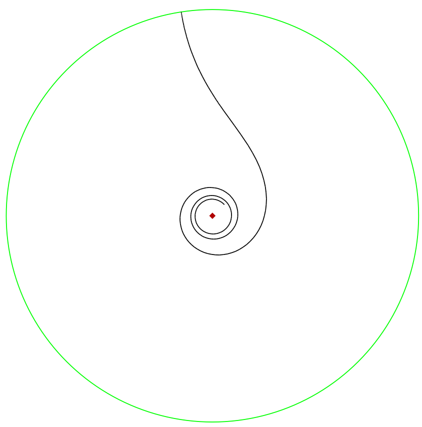

Figure III.3: The global phase portrait of systems (III.2) for $a = 0$, $b \neq -c$ and $bc > 0$, has tow separatrices (S=2) and one canonical region (R=1). The phase portrait can be obtained by taking $a = 0$, $b = 1$, $c = 1$.

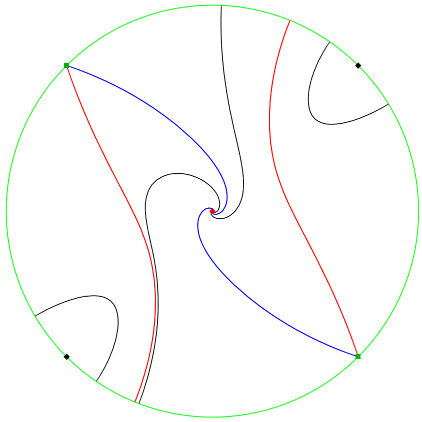

Figure III.4: The global phase portrait of systems (III.2) for $a = 0$, $b \neq -c$ and $bc < 0$, has thirteen separatrices (S=13) and four canonical regions (R=4). The phase portrait can be obtained by taking $a = 0$, $b = 1$, $c = -2$.

saddle respectively if $b < 0$. Then there is a unique possible global phase portrait in the Poincaré disc given in Figure $(III.4)$.

III.2.2 The case: $a \neq 0$

In this case, the eigenvalues of the origin are complex where the real part is not zero, thus the origin in finite is focus.

The case: when there is a limit cycle

From Proposition $(III.2)$, in the cases ($a > 0$, ,$b < 0$, $c < 0$) or ($a < 0$, ,$b > 0$, $c > 0$) there is a limit cycle. And from Proposition $(III.3)$ in infinite, there is a line of singularities. From there on, there is a unique possible global phase portrait in the Poincaré disc given in Figure $(III.5)$.

The case: when there is no limit cycle

From Proposition $(III.2)$, there is a limit cycle if only if one of these conditions ($a > 0$, $b < 0$, $c < 0$) or ($a < 0$, $b > 0$, $c > 0$) is verified. If these conditions are not verified and $a \neq 0$, the global phase portrait of systems (III.2) in the Poincaré disc is given in Figure $(III.4)$ if $bc < 0$ and is given in Figure $(III.3)$ if $bc > 0$.

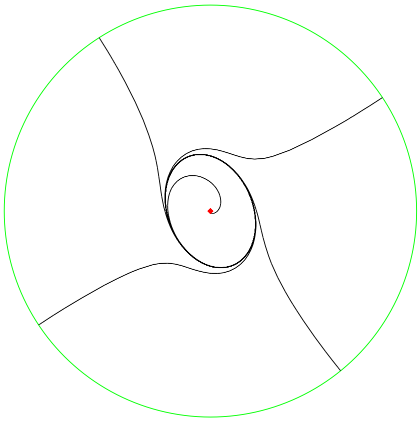

Figure III.5: The global phase portrait of systems (III.2) when there is limit cycle has three separatrices (S=3) and tow canonical regions (R=2). The phase portrait can be obtained by taking $\boldsymbol{a = 1}$, $\boldsymbol{b = -6}$, $\boldsymbol{c = -2}$.

CONCLUSION

This thesis has tried to show the global analysis of the behavior of solutions of a non-linear planar differential system, more precisely those depend on a parameter or several parameters. For that, we have first presented the necessary information as singular points and their nature, Hartman-Grobman theorem, Poincaré map, phase portraits and structural stability. Then next we have tried to understand in general theorems concern Hopf bifurcations and bifurcations of limit cycles from a multiple focus and bifurcations at non-hyperbolic periodic orbit where limit cycles can be made to bifurcate. Finlly, we have presented an example of a family of rigid systems that Hopf bifurcation and global phase portraits have been studied, and we get four diffrent global phase portraits where only one limit cycle created from the origin which is a multiple focus of multiplicity one.

BIBLIOGRAPHY

[1] H. Poincaré, *Mémoire sur les courbes définies par une équation differentielle I, II,* J. Math. Pures Appl. **7** (1881) 375-422; 8 (1882) 251-296; *Sur les courbes définies par les equations differentielles III, IV* 1 (1885) 167-244; 2 (1886) 155-217.

[2] L. Perko, *Differential equations and dynamical systems.* Third Edition. Texts in Applied Mathematics, **7**. Springer-Verlag, New York, Berlin Heidelberg, 2001.

[3] Guckenheimer and P.J. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields.* Springer-Verlag, New York, 1983.

[4] D. Ruelle, *Elements of differentiable dynamics and bifurcation theory,* Academic press, Inc, 1989.

[5] F. Dumortier, J. Llibre and J.C. Artés, *qualitative theory of planar differential systems,* Universitext, Spring-Verlag, 2006.

[6] I. Bendixson, *Sur les courbes définies par des equations différentielles,* Acta Math, 1901.

[7] A.A Andronov, E.A. LEONTOVICH, I.I. GORDON, and A.G. MAIER, *Theory of bifurcations of dynamical systems on a plane* , Israel program for scientific translations, Jerusalem, 1971.

[8] Shangjiang. Guo, Jianhong. Wu, *Bifurcation theory of functional differential equations,* Springer 2013.

[9] P. Hartman, *Ordinary differential equations*, The University of New South Wales, 2010.

[10] J.E.Marsden, and M.McCracken, *The Hopf bifurcation and its applications*, Applied mathematical sciences. p66, Volume 19, Springer-Verlag New York 1976.

[11] J.Reyn, *Phase portrait of planar quadratic systems*, Mathematics and its Applications, 2007.

[12] J.Llibre, and A.E. Teruel, *Introduction to the qualitative theory of differential systems*, Springer Basel 2014.

Abstract

In this work, we have focused on the planar polynomial systems and studied the Hopf bifurcations and bifurcations of limit cycles from a multiple focus and bifurcations at non-hyperbolic equilibrium point where limit cycles can be made to bifurcate. In particular, the limit cycles and global phase portrait of class of rigid systems under its parameters. The results of bifurcations of limit cycles are proved by the uses of the Poincaré map and drawing all diffrent phase portraits.

Key words : **Hopf bifurcation, Poincaré map, phase portrait, rigid system, limit cycle, non-hyperbolic.**

ملخص

فى هذا العمل ، ركزنا على الأنظمة المستوية متعددة الحدود ودرسنا تشعبات هوبف والتشعبات للحلول الدورية المعزولة من نقاط التركيز و التشعبات عند نقطة التوازن غير القطعى حيث يمكن نشاة الحلول الدورية المعزولة. على وجه الخصوص ، الحلول الدورية المعزولة والرسم الكلى لفئة الأنظمة الصلبة تحت تاثير متغيراتها. النتائج المستخرجة من دراسة الحلول الدورية المعزولة مبرهنة باستخدام ماب بوانكارى ورسم الحلول.

الكلمات المفتاحية: تشعب هوبف ،ماب بوانكاريه ، رسم الحلول ، الانظمة الصلبة ، دورة حد ، نقطة توازن غير زائدى.

Ré

Dans ce travail, nous sommes concentrés sur les systèmes polynomiaux planaires et nous sommes étudié les bifurcations de Hopf et les bifurcations des cycles limites à partir d'un foyer multiple, et des bifurcations au point d'équilibre non hyperbolique où les cycles limites peuvent cree. En particulier, les cycles limites et le portrait de phase global de la classe des systèmes rigides sous ses paramètres. Les résultats des bifurcations de cycles limites sont prouvés par les utilisations de la carte de Poincaré et par le dessin de tous différents les portraits de phase .

Mots clés : **Bifurcation de Hopf, carte de Poincaré , portrait de phase, system rigide, cycle limite, non hyperbolique.**