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Global phase portraits of quadratic differential systems exhibiting an invariant algebraic curve or an algebraic cubic first integral.

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Introduction

Quadratic polynomial differential systems appear frequently in many areas of applied mathematics, electrical circuits, astrophysics, in population dynamics, chemistry, neural networks, laser physics, hydrodynamics, etc. Although these differential systems are the simplest nonlinear polynomial systems, they are also important as a basic testing ground for the general theory of the nonlinear differential systems.

There are more than one thousand papers published on quadratic polynomial differential systems (simply QS) that are the differential systems of the form

$$\begin{aligned} \dot{x} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \dot{y} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2. \end{aligned} \tag{1}$$

Here the dot denotes the derivative with respect to an independent variable t, usually called time.

The difficulty of studying these differential systems is due to the fact that they depend on twelve parameters. The authors of these published papers studied many subclasses of quadratic systems. some of them :Lupan and Valu studied QS with a center [17], A.Gasull, S.Li-Ren and J. Llibre studied QS with no finite real singularities [11], Nikolaev and Vulpe studied QS with a unique finite singularity [19], Artés and Llibre studied QS with a focus and one anti-saddle [1], Artés, Llibre and Vulpe studied QS with an integrable saddle [5], Llibre and Schlomiuk studied QS with a third order weak focus [16], Gasull and Prohens studied QS with all points at infinity as singularities [12], Kalin and Vulpe studied Hamiltonian QS [14]), Llibre and Medrado studied Darboux integrable systems [15], Vulpe and Sibirskii studied homogeneous QS [23], Schlomiuk and Vulpe studied QS with invariant lines of total multiplicity greater than or equal to four [22], Cairó and Llibre studied QS with rational first integrals [7], García, Llibre, and Pérez del Río studied QS with polynomial first integrals [10], Coll Ferragut and Llibre studied QS with a polynomial inverse integrating factor [8]. More recently, the classification of some families of quadratic systems has been made using more modern methods such as the algebraic and geometric invariants ; see, for instance, the classification of the quadratic systems with a weak focus of second order [3], the classification of the quadratic systems with a weak focus and an invariant straight line [13], and the classification of the geometric configurations of singularities for quadratic systems [2] [4] [24] [25].

Recently, in 2019 Benterki and Llibre [6] classified the global phase portraits of quadratic polynomial differential systems having some relevant classic quadratic algebraic curves as invariant algebraic curves, i.e. these curves are formed by orbits of the quadratic polynomial differential system. More precisely, they realized 14 different well-known algebraic curves of degree 4 as invariant curves inside the quadratic polynomial differential systems. These realizations produced 28 topologically different phase portraits in the Poincaré disc for such quadratic polynomial differential systems.

Some results that we are need in our on differential systems and their qualitative theory are introduced in chapter one. There we analyze the local behavior of the orbits near singular points. The study of the singular points is the main objective of chapter 1. We mainly study the Non-hyperbolic singular points, and we provide the basic tool for studying all singularities of polynomial differential system in the plane. We end with a technique for constructing the global Phase Portrait of a differential system. These systems can be extended to infinity, compactifying $\mathbb{R}^2$ by adding a circle, and extending analytically the flow to this boundary. This is done by the so-called " Poincaré compactification, " and also by the more general "Poincaré Lyapunov compactification." In this way, we can study the behavior of the orbits near infinity.

In the second chapter, we construct a new class of quadratic polynomial differential system which exhibits an invariant algebraic curve of degree six. The system obtained has three parameters. We classify all its phase portraits. More precisely, we characterize the class of this quadratic system in the plane and we provide all the different topological phase portraits that this class exhibits in the Poincaré disc. This is made by using the techniques described in the first chapter.

In the third chapter, we provide a new class of quadratic polynomial differential system wich exhibits the whell- known cubic curve called **cubic egg** curve as the first integral, we study the global phase portraits of this class of integrable QS with two parameters, at all their finite and infinite singular points in the Poincaré disc.

Chapitre 1

Preliminary concepts of differential systems

In this chapter we introduce some basic results on the qualitative theory of differential equations with special emphasis on planar differential equations. First we recall the basic notions of singular points and their local phase portrait.
We end the chapter by a technique for constructing the global phase portrait of differential systems.

These systems can be extended to infinity, compactifying $\mathbb{R}^2$ by adding a circle, and extending analytically the flow to this boundary. This is done by the so-called " Poincaré compactification ".

1.1 Singular points

Definition 1.1 *A point $x_0 \in R^n$ is called an equilibrium point or critical point of*

$$\dot{x} = f(x), \tag{1.1}$$

if $f(x_0) = 0$.

1.1.1 Local structure of singular points

Let p be a singular point of a planar C^r vector field $\chi = (P, Q)$. In general the study of the local behavior of the flow near p is quite complicated. Already the linear systems show different classes, even for local topological equivalence.
We say that :

$$D\chi(p) = \begin{pmatrix} \dfrac{\partial P}{\partial x}(p) & \dfrac{\partial P}{\partial y}(p) \\ \dfrac{\partial Q}{\partial x}(p) & \dfrac{\partial Q}{\partial y}(p) \end{pmatrix}$$

is the linear part of the vector field χ at the singular point p.

The singular point p is called **non–degenerate** if 0 is not an eigenvalue.

The singular point p is called **hyperbolic** if the two eigenvalues of $D\chi(p)$ have real part different from 0.

The singular point p is called **semi-hyperbolic** if exactly one eigenvalue of $D\chi(p)$ is equal to 0.

Hyperbolic and **semi-hyperbolic** singularities are also said to be elementary singular points.

The singular point p is called **nilpotent** if both eigenvalues of $D\chi(p)$ are equal to 0 but $D\chi(p) \not\equiv 0$.

The singular point p is called **linearly zero** if $D\chi(p) \equiv 0$.

The singular point p is called a center if there is an open neighborhood consisting, besides the singularity, of periodic orbits.
The singularity is said to be a center if the eigenvalues of $D\chi(p)$ are purely imaginary without being zero.
The vector field χ can have either a center or a focus at p. To distinguish between a center and a focus, it is a difficult problem in the qualitative theory of planar differential equations.
We note that a center-focus problem also exists for nilpotent or linearly zero singular points.

In order to study the local Phase Portrait at the singular point p we define the determinant, the trace and the discriminant at p as :

$$det(p) = \begin{vmatrix} \frac{\partial P}{\partial x}(p) & \frac{\partial P}{\partial y}(p) \\ \frac{\partial Q}{\partial x}(p) & \frac{\partial Q}{\partial y}(p) \end{vmatrix},$$

$$tr(p) = \frac{\partial P}{\partial x}(p) + \frac{\partial Q}{\partial y}(p),$$

$$\Delta(p) = tr(p)^2 - 4det(p),$$

respectively. It is easy to check that

(1) if $det(p) \neq 0$, then the singular point is **non–degenerate** and it is either hyperbolic, or **linearly** a center ;
(2) if $det(p) = 0$ but $tr(p) \neq 0$, then the singular point is **semi-hyperbolic** ;
(3) if $det(p) = 0$ and $tr(p) = 0$, then the singular point is **linearly zero** or **nilpotent** depending on whether $D\chi(p)$ is the $zero$ matrix or not ;
(4) if $det(p) < 0$, λ_1 and λ_2 have different sign, the origin is a saddle ;
(5) if $det(p) > 0$, $tr(p) > 0$ and $\lambda_1, \lambda_2 > 0$, the origin is an unstable node ;
(6) if $det(p) > 0$, $tr(p) < 0$ and $\lambda_1, \lambda_2 < 0$, the origin is a stable node.

It is obvious that if $p = (x_0, y_0)$ is a singular point of the differential system :

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y). \end{aligned} \tag{1.2}$$

Then the point $(0, 0)$ is a singular point of the system

$$\begin{aligned} \dot{\bar{x}} &= P(\bar{x}, \bar{y}), \\ \dot{\bar{y}} &= Q(\bar{x}, \bar{y}). \end{aligned} \tag{1.3}$$

Where $x = \bar{x} + x_0$ and $y = \bar{y} + y_0$ and now the functions $P(\bar{x}, \bar{y})$ and $Q(\bar{x}, \bar{y})$ start with terms of order 1 in $\bar{x}$ and $\bar{y}$. In other words, we can always move a singular point to the origin of coordinates in which case system (1.2) becomes (dropping the bars over x and y).

$$\begin{aligned} \dot{x} &= ax + by + F(x, y), \\ \dot{y} &= cx + dy + G(x, y). \end{aligned} \tag{1.4}$$

Where F and G vanish together with their first partial derivatives at $(0,0)$. By a linear change of coordinates the linearization $D\chi(0,0)$ regarded as the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Can be placed in real Jordan canonical form. If the singularity is hyperbolic, the Jordan form is :

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} or \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix} or \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

With $\lambda_1, \lambda_2 \neq 0$, $\alpha \neq 0$ and $\beta > 0$.
In the semi-hyperbolic case and the linearly center case, we obtain, respectively

$$\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} and \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

With $\lambda \neq 0$ and $\beta > 0$, while we obtain

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} and \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In the nilpotent case and the linearly zero case, respectively
If moreover we allow a time rescaling, introducing a new time $\mu = \gamma t$ for some $\gamma > 0$ as is usual when working with equivalences, then we can also suppose that in the hyperbolic case one of the numbers λ_1 or λ_2 is equal to ± 1 and either $\alpha = \pm 1$ or $\beta = 1$, while in the semi-hyperbolic case $\lambda = \pm 1$ and in the linearly center case $\beta = 1$.

1.2 Phase portraits

Although it is often impossible (or very difficult) to determine explicitly the solutions of an ordinary differential system, it is still important to obtain information about these solutions, at least of qualitative nature. To a considerable extent, this can be done describing the Phase Portrait of the differential system. We note that in this section we consider only autonomous systems.

1.2.1 Orbits

Let $f : D \to \mathbb{R}^n$ be a continuous function in an open set $D \subset \mathbb{R}^n$ and consider the autonomous system :

$$\dot{x} = f(x). \tag{1.5}$$

The set D is called the Phase space of the system.

Definition 1.2 *If $x = x(t)$ is a solution of system (1.5) with maximal interval I, then the set $x(t) : t \in I \subset D$ is called an orbit of system (1.5).*

Definition 1.3 *The Phase Portrait of an autonomous ordinary differential system is obtained representing the orbits in the set D, also indicating the direction of motion. It is common not to indicate the directions of the axes, since these could be confused with the direction of motion.*

Definition 1.4 *A sector which is topologically equivalent to the sector shown in Figure $1.1(a)$ is called a hyperbolic sector.*
A sector which is topologically equivalent to the sector shown in Figure $1.1(b)$ is called a parabolic sector.
And a sector which is topologically equivalent to the sector shown in Figure $1.1(c)$ is called an elliptic sector. For more detail see [20].

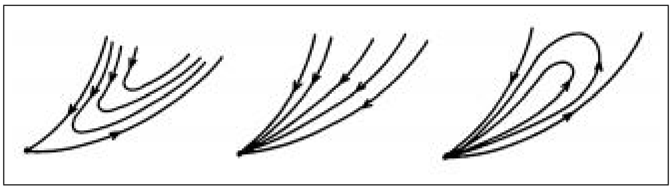


FIGURE 1.1 – (a) A hyperbolic sector. (b) A parabolic sector. (c) An elliptic sector

1.3 Hartman-Grobman Theorem

The Hartman-Grobman Theorem is another very important result in the local qualitative theory of ordinary differential systems. The theorem shows that near a hyperbolic equilibrium point x_0, the nonlinear system

$$\dot{x} = f(x), \tag{1.6}$$

has the same qualitative structure as the linear system :

$$\dot{x} = Ax, \tag{1.7}$$

with $A = Df(x_0)$. Throughout this section we shall assume that the singular point x_0 has been translated to the origin.

Definition 1.5 *Two autonomous systems of differential equations such as (1.6) and (1.7) are said to be topologically equivalent in a neighborhood of the origin or to have the same qualitative structure near the origin if there is a homeomorphism H mapping an open set U containing the origin onto an open set V containing the origin which maps trajectories of (1.6) in U onto trajectories of (1.7) in V and preserves their orientation by time in the sense that if a trajectory is directed from x_1 to x_2 in U, then its image is directed from $H(x_1)$ to $H(x_2)$ in V. If the homeomorphism H preserves the parameterization by time, then the systems (1.6) and (1.7) are said to be topologically conjugate in a neighborhood of the origin.*

Theorem 1.1 *(The Hartman-Grobman Theorem) Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let $\phi(t)$ be the flow of the nonlinear system (1.6). Suppose that $f(0) = 0$ and that the matrix $A = Df(0)$ has no eigenvalue with zero real part. Then there exists* *a homeomorphism* H *of an open set* U *containing the origin onto an open set* V *containing the origin such that for each* $x_0 \in U$*, there is an open interval* $I_0 \subset \mathbb{R}$ *containing zero such that for all* $x_0 \in U$ *and* $t \in I_0$

$$H \circ \phi_t(x_0) = exp(At) \circ H(x_0),$$

(i.e ; H *maps trajectories of* (1.6) *near the origin onto trajectories of* (1.7) *near the origin and preserves the parameterization by time).*

1.4 Semi-hyperbolic and Non-hyperbolic singular points in $\mathbb{R}^2$

In this part we present some results on non-hyperbolic singular points of planer analytic systems. This work originated with Poincaré and was extended by Bendixson and more recently by Andronov. We assume that the origin is an isolated singular point of the planar system

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1.8}$$

where P and Q are analytic in some neighborhood of the origin. We have already presented some results for the case when the matrix of the linear part $A = Df(0)$ has pure imaginary eigenvalues, i.e ; when the origin is a center for the linearized system. In this part we give some results for the case when the matrix A has one or two zero eigenvalues, but $A \neq 0$.
We first consider the case when the matrix A has one zero eigenvalue and $detA = 0$, but $trA \neq 0$. In this case, the system (1.8) can be put into the form :

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= y + Q(x,y), \end{aligned} \tag{1.9}$$

where P and Q are analytic in a neighborhood of the origin and have expansions that begin with second-degree terms in x and y.

Semi hyperbolic singular points

Theorem 1.2 *Let the origin be an isolated singular point for the analytic system* (1.9)*. Let* $y = \phi(x)$ *be the solution of the equation* $y + Q(x,y) = 0$ *in a neighborhood of the point* $(0,0)$*, and suppose that the function* $\psi(x) = P(x,\phi(x))$ *in a neighborhood of* $x = 0$ *have the form* $\psi(x) = a_m x^m + \ldots..$ *where* $m \geq 2$ *and* $a_m \neq 0$*. Then :*

1. *For* m *odd and* $a_m > 0$*, the origin is an unstable node ;*
2. *for* m *odd and* $a_m < 0$*, the origin is a (topological) saddle ;*
3. *for* m *even, the origin is a saddle-node.*

Example 1.1 *Let the system*

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= y. \end{aligned} \tag{1.10}$$

Then we can use Theorem 1.2 $y = \phi(x)$ *the solution of the equation* $y = 0$ *let the function* $\psi(x) = x^2$ *in a neighborhood of* $x = 0$ *have the form* $\psi(x) = a_m x^m$ *with* $a = 1$ *and* $m = 2$*. Since* m *is even then according of Theorem* 1.2 *the origin is a saddle-node.*

Even without Theorem 1.2*, this system is easy to discuss since it can be solved explicitly for* $x(t) = (\frac{1}{x_0} - t)^{-1}$ *and* $y(t) = y_0 e^t$*. The phase portrait for this system is shown in Figure* (1.3)*.*

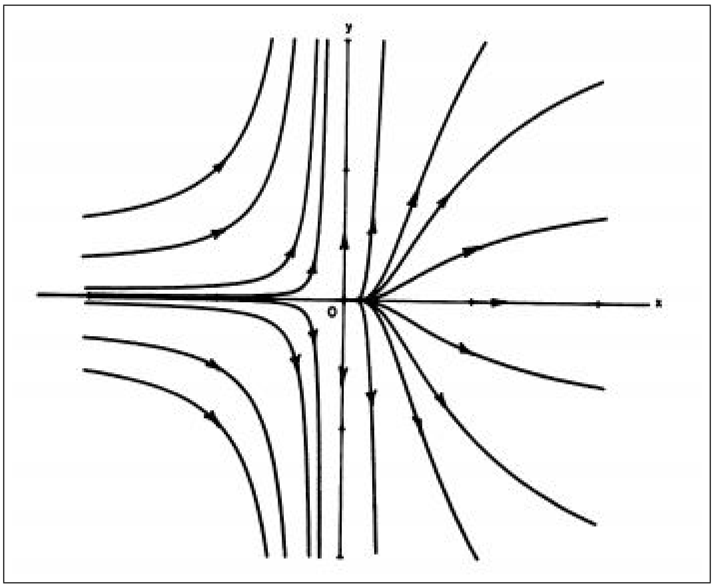


FIGURE 1.2 – A saddle-node at the origin.

1.4.1 Non-hyperbolic (Nilpotent singular points)

Theorem 1.3 *Let* $(0,0)$ *be an isolated singular point of the vector field* χ *given by :*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= a_k x^k[1 + h(x)] + b_n x^n y[1 + g(x)] + y^2 R(x,y), \end{aligned} \tag{1.11}$$

where $h(x)$*,* $g(x)$ *and* $R(x,y)$ *are analytic in a neighborhood of the origin,* $h(0) = g(0) = 0$*,* $k > 2$*,* $a_k \neq 0$ *and* $n > 1$*.*
Let $k = 2m + 1$ *with* $m > 1$ *in* (1.11) *and let* $\lambda = b_n^2 + 4(m+1)a_k$*. Then*

— *If* $a_k > 0$*, the origin is a (topological) saddle ;*
— *If* $a_k < 0$*, the origin is :*

1. *a focus or a center if* $b_n = 0$*, and also if* $b_n \neq 0$ *and* $n > m$*, or if* $n = m$ *and* $\lambda < 0$*;*
2. *a node if* $b_n \neq 0$*,* n *is an even number and* $n < m$ *and also if* $b_n \neq 0$*,* n *is an even number* $n = m$ *and* $\lambda \geq 0$*;*
3. *a critical point with an elliptic domain if* $b_n \neq 0$*,* n *is an odd number and* $n < m$ *and also if* $b_n \neq 0$*,* n *is an odd number* $n = m$ *and* $\lambda \geq 0$*.*

Let $k = 2m$ *with* $m \geq 1$ *in* (1.11)*. Then the origin is :*

1. *a cusp if* $b_n = 0$ *and also if* $b_n = 0$ *and* $n \geq m$*;*

2. *a saddle-node if* $b_n \neq 0$ *and* $n < m$.

Example 1.2 *Let the system :*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x^2 + 2x^2y + xy^2. \end{aligned} \tag{1.12}$$

Then we use the theorem 1.3. $a_K = 1$, $b_n = 2$, $n = 2$. $k = 2m$, $k > 0 \Rightarrow m = 1$, $k = 2m$. *The phase portrait for this system is shown in Figure* (1.3).

We see that a deleted neighborhood of the origin consists of two hyperbolic sectors and two separatrices. This type of critical point is called a cusp.

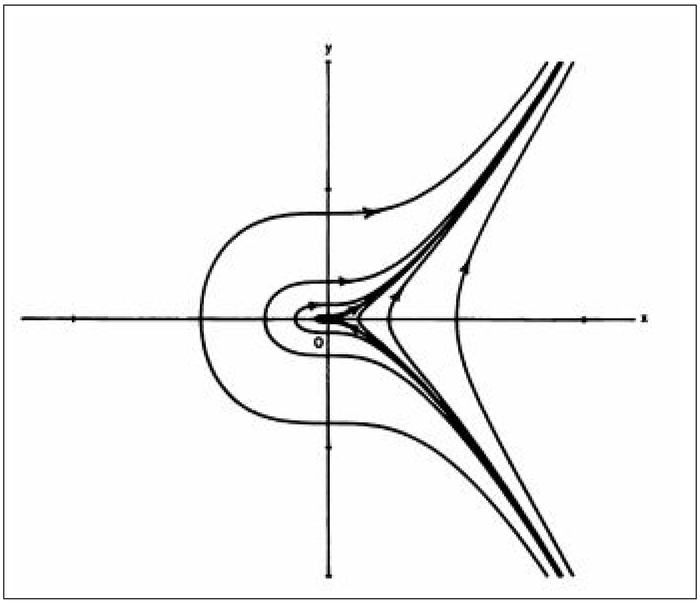


FIGURE 1.3 – A cusp at the origin

1.5 local charts

Let X be the planar polynomial vector field. We define the *Poincaré compactified vector field* $p(X)$ associated to X as follows (see all the details for instance in Chapter 5 of [9]).

The *Poincaré sphere* is defined as $s^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ and its tangent space at the point $y \in s^2$ is denoted by $T_y s^2$. We identify the plane $\mathbb{R}^2$ where we have our vector field X with the plane $T_{(0,0,1)}\mathbb{S}^2$.

We define the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ as follows : to each point $q \in T_{(0,0,1)}\mathbb{S}^2$ the central projection associates the two intersection points of the straight line which connects the points q and $(0,0,0)$ with the sphere $\mathbb{S}^2$.

This central projection gives two copies of X in $\mathbb{S}^2$, one in each hemisphere. Let X' be the vector field $Df \circ \mathcal{X}$, which is defined in $\mathbb{S}^2$ minus its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. The equator $\mathbb{S}^1$ can be identified with the infinity of $\mathbb{R}^2$.

We extend the vector field X' on $\mathbb{S}^2 \setminus \mathbb{S}^1$ to a vector field $p(X)$ on $\mathbb{S}^2$ as follows : $p(X)$ is the unique analytic extension of $y_3^7 \mathcal{X}'$ to $\mathbb{S}^2$.

In summary, we have two symmetric copies of X on $\mathbb{S}^2 \setminus \mathbb{S}^1$, and studying the dynamics of $p(X)$ near $\mathbb{S}^1$, we have the dynamics of X at infinity. The *Poincaré disc*, denoted by $\mathbb{D}^2$, is the closed northern hemisphere of $\{y \in \mathbb{S}^2 : y_3 \geq 0\}$ projected on $y_3 = 0$ under the projection $(y_1, y_2, y_3) \mapsto (y_1, y_2)$.

The infinity $\mathbb{S}^1$ is invariant under the flow of the Poincaré compactifcation $p(X)$.
Here two polynomial vector fields X and Y associated to systems (1.7) are *topologically equivalent* if there is a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow of $p(X)$ into orbits of the flow of $p(Y)$, either reversing or preserving the sense of all orbits.

For computing the analytic expression of $p(X)$ we use the fact that $\mathbb{S}^2$ is a differentiable manifold.
Thus we take the six local charts $U_i = \{y_2 \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y_2 \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1, 2, 3$; and the associated diffeomorphisms $F_i : U_i \longrightarrow \mathbb{R}^2$ and $G_i : V_i \longrightarrow \mathbb{R}^2$ for $i = 1, 2, 3$ are respectively the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$; $(-1, 0, 0)$; $(0, 1, 0)$; $(0, -1, 0)$; $(0, 0, 1)$ and $(0, 0, -1)$. The value of $F_i(y)$ or $G_i(y)$ for some
$i = 1, 2, 3$ is denoted by $z = (z_1, z_2)$, consequently according to the local charts under consideration the same letter z represents different coordinates.

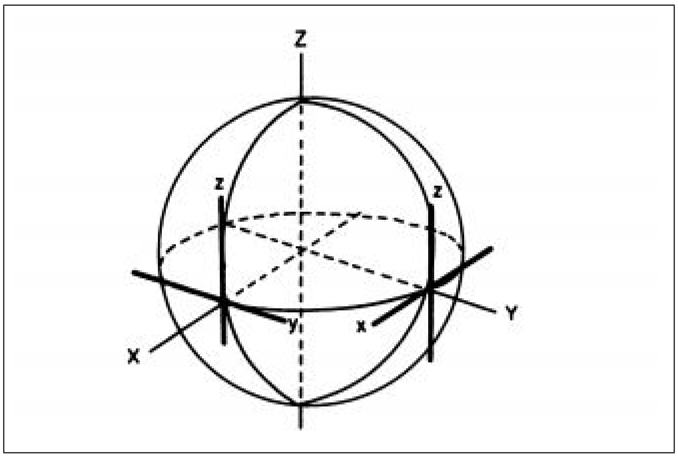


FIGURE 1.4 – The local charts in the Poincaré sphere

After a rescaling in the independent variable in the local chart (U_1, F_1) the expression for $p(X)$ is

$$\dot{u} = v^n \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \quad \dot{v} = -v^{n+1} P\left(\frac{1}{v}, \frac{u}{v}\right);$$

in the local chart (U_2, F_2) the expression for $p(X)$ is

$$\dot{u} = v^n \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right] \quad \dot{v} = -v^{n+1} Q\left(\frac{u}{v}, \frac{1}{v}\right);$$

and for the local chart (U_3, F_3) the expression for $p(X)$ is

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v).$$

In the chart (V_i, G_i) the expression for $p(X)$ is the same than in the chart (U_i, F_i) multiplied by $(-1)^{n+1}$ for $i = 1, 2, 3$. We note that the points at the infinity $\mathbb{S}^1$ in any chart have coordinates $(u, v) = (u, 0)$.

The equilibrium points of $p(X)$ which come from the equilibrium points of X are called *finite equilibrium points* of X, and the equilibrium points of $p(X)$ which are in $\mathbb{S}^1$ are called *infinite equilibrium points* of X.

We observe that the unique infinite equilibrium points which cannot be contained in the charts $U_1 \cup V_1$ are the origins of the local charts U_2 and V_2.
Therefore when we study the infinite equilibrium points on the charts $U_2 \cup V_2$, we only need to verify if the origin of these charts are equilibrium points.

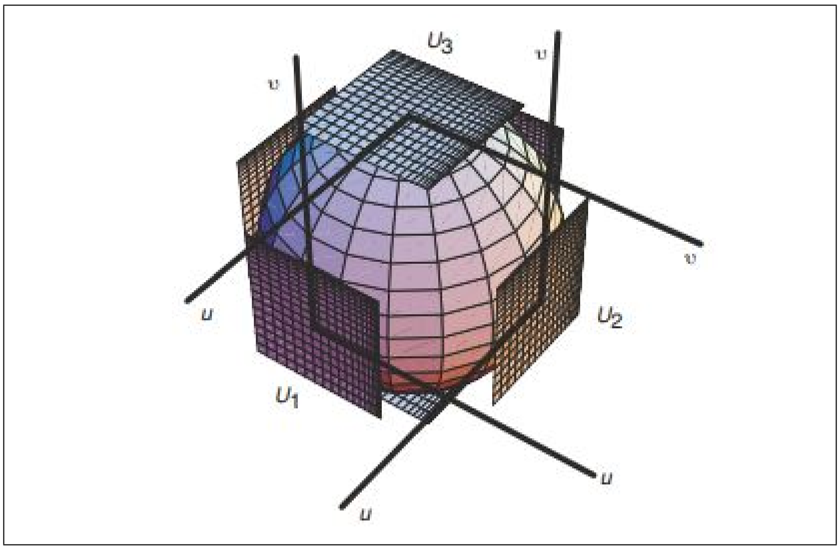


FIGURE 1.5 – The local charts in the Poincaré sphere

1.6 Topological equivalence

Two polynomial vector fields X and Y on $\mathbb{R}^2$ are *topologically equivalent* if there is a homeomorphism on the Poincaré sphere $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying trajectories of the flow of $p(X)$ into trajectories of the flow of $p(Y)$, either preserving or reversing the sense of all trajectories.

Here a *separatrix* of the Poincaré compactification $p(X)$ is a trajectory which is either an equilibrium point, or a limit cycle, or a trajectory which belongs to the boundary of a hyperbolic sector at an equilibrium point, finite or infinity, or any trajectory contained at the infinity $\mathbb{S}^1$. We denote by $S(p(X))$ the set formed by all separatrices of $p(X)$. It is known that the set $S(p(X))$ is closed, see for instance Neumann [18].

1.7 Invariant algebraic Curves

Definition 1.6 *The differential system of the form*

$$\begin{aligned} \dot{x} &= p(x, y), \\ \dot{y} &= q(x, y). \end{aligned} \tag{1.13}$$

where the dependent variables x *and* y *are real or complex, the independent one (the time)* t *is real, and* p *and* q *are polynomials in the variables* x *and* y*. We denote by* $m = max\{deg(p), deg(q)\}$ *the degree of the polynomial system.*

$$\mathcal{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

be the planar polynomial vector field of degree m *associated to system* (1.13)*. Suppose that system* (1.13) *has a trajectory (not a singular point) whose path is described by an algebraic curve. That is, it lies within the zero set of a polynomial,* $f(x, y) = 0$*. It is clear that the derivative of* f *with respect to time will not change along the curve* $f = 0$*. Since this derivative can be expressed as a polynomial in* x *and* y *which vanishes on* $f = 0$*, we are lead directly to the equation*

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = kf. \tag{1.14}$$

where k *must be a polynomial in* x *and* y *of degree at most* $m - 1$*, called the cofactor of* $f = 0$*. We shall also write this as* $\mathcal{X}f = kf$*. Conversely, given a polynomial f which satisfies* (1.14)*, it is easy to see that its zero set must be composed of trajectories of* (1.13)*. We call a polynomial solution of* (1.14) *an invariant algebraic curve of* (1.13)*.*

Chapitre 2

Global phase portraits of quadratic systems exhibiting an invariant algebric curve of degree six

In this chapter we study the global phase portraits of QS with three parameters at all their finite and infinite singular points in the Poincaré disc, this made by using the technics described in chapter one.

The polynomial quadratic system which we are going to study have an invariant algebraic curve of degree six.

2.1 Statement and the main results.

Our first result is the following

Theorem 2.1 *The algebraic curve of degree six given by :* $H(x,y) = 0$ *with* $H(x,y) = x^4(y^2+x^2) - (bx^2 - ax^3)^2$*, for* $ab \neq 0$ *is an invariant algebraic curve with associated cofactor* $k(x,y) = cy$ *of the quadratic system :*

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= \frac{1}{2}b^2(4-c) + ab(c-5)x + \frac{1}{2}(a^2-1)(6-c)x^2 + \frac{1}{2}(c-4)y^2. \end{aligned} \tag{2.1}$$

Proof 1 *To prove this theorem we can easely verify that we have the partially differential equation :*

$$P(x,y)\frac{\partial H}{\partial x} + Q(x,y)\frac{\partial H}{\partial y} = KH.$$

Our main result in this chapter is given by the following theorem.

Theorem 2.2 *The global phase portraits of QS* (2.1) *given in Theorem* 2.1 *are topologically equivalent to*

(1) for $a \in (1,+\infty)$ *and* $c \in (-\infty,4)$*;*

(2) for $a \in (1,+\infty)$ *and* $c \in (4,6)$*;*

(3) *for* $a \in (1, +\infty)$ *and* $c \in (6, +\infty)$*;*

(4) *for* $a \in (1, +\infty)$ *and* $c = 4$*;*

(5) *for* $a \in (1, +\infty)$ *and* $c = 6$ *or* $a \in (0, 1)$ *and* $c = 6$*;*

(6) *for* $a = 1$ *and* $c \in (-\infty, 4)$*;*

(7) *for* $a = 1$ *and* $c = 4$*;*

(8) *for* $a = 1$ *and* $c \in (4, 5)$*;*

(9) *for* $a = 1$ *and* $c = 5$*;*

(10) *for* $a = 1$ *and* $c \in (5, +\infty)$*;*

(11) *for* $a \in (0, 1)$ *and* $c \in (-\infty, 4)$*;*

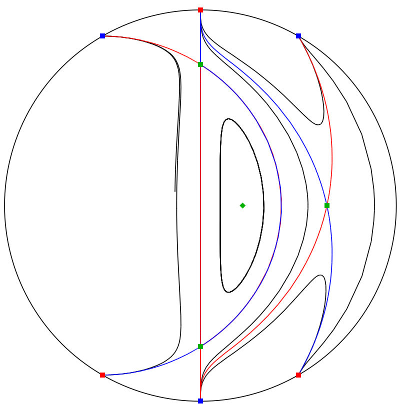
(12) *for* $a \in (0, 1)$ *and* $c = 4$*;*

(13) *for* $a \in (0, 1)$ *and* $c \in (4, c_1)$ *or* $c \in (c_2, 6)$ *where* $c_1 = 5 - \sqrt{1 - a^2}$ *and* $c_2 = 5 + \sqrt{1 - a^2}$*;*

(14) *for* $a \in (0, 1)$ *and* $c \in (c_1, c_2)$*;*

(15) *for* $a \in (0, 1)$ *and* $c \in (6, +\infty)$*;*

(16) *for* $a \in (0, 1)$ *and* $c = c_1$ *or* $c = c_2$*.*



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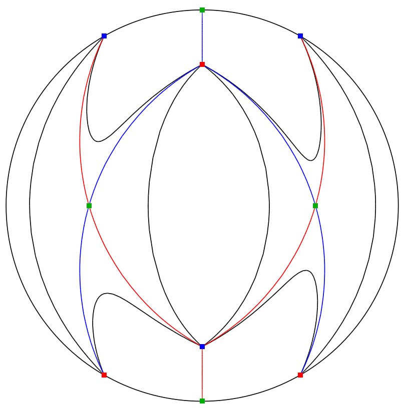

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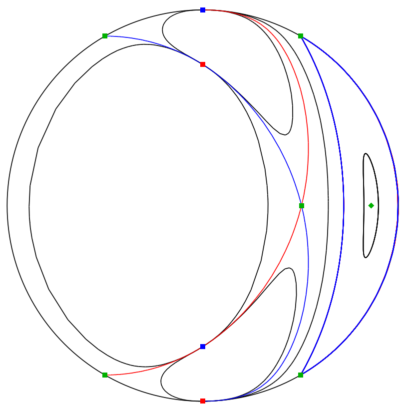

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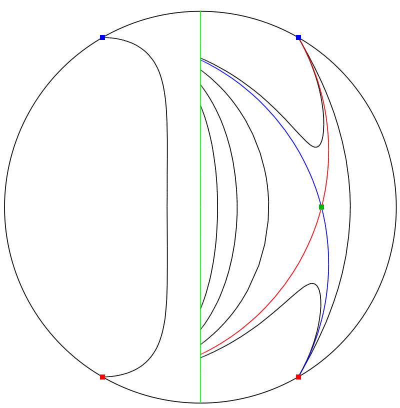

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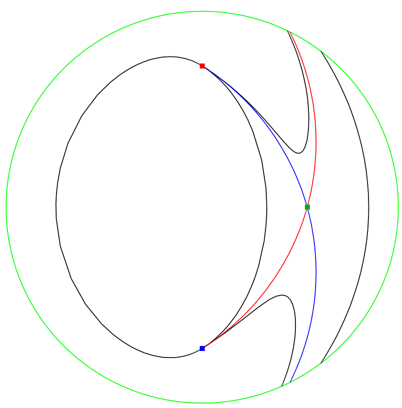

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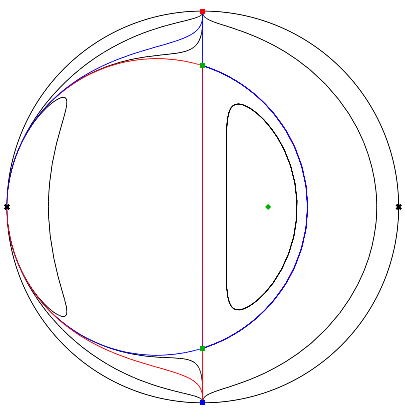

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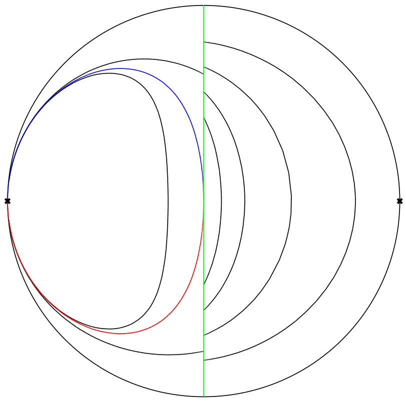

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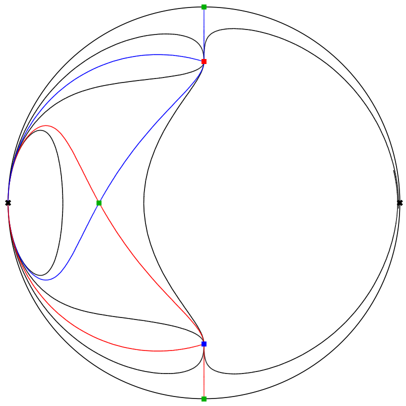

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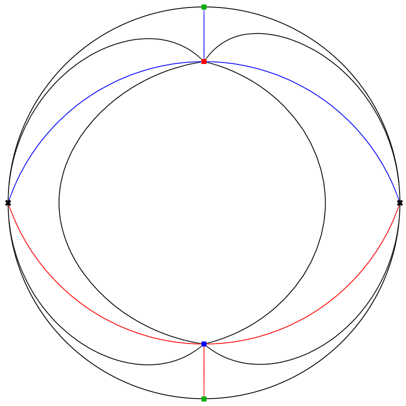

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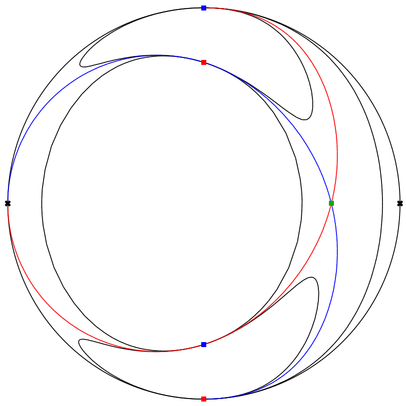

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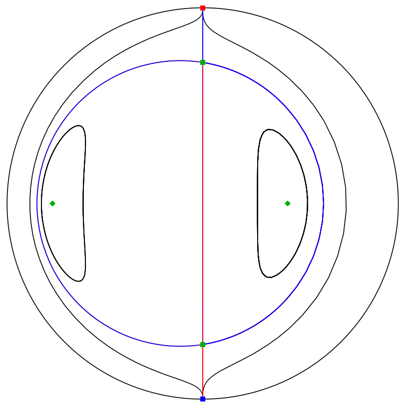

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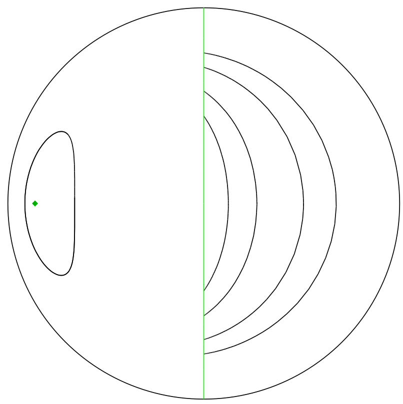

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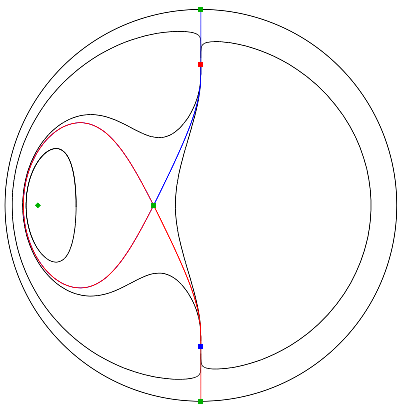

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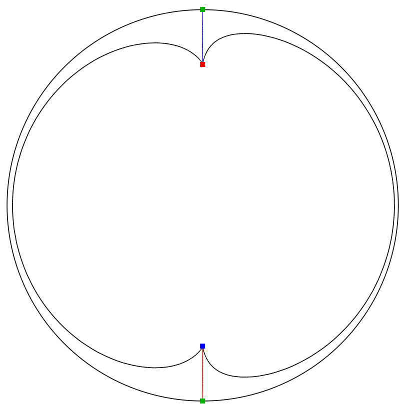

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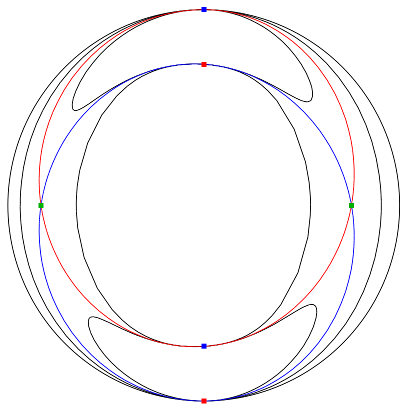

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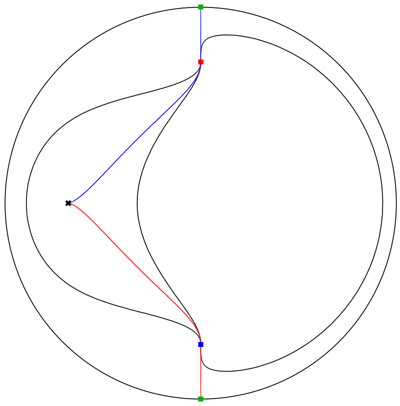

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2.2 Global phase portraits in the Poincaré disc

In this part we will study the global phase portraits in the Poincaré disc of the quadratic polynomail differential system (2.1).

Before start the study of system (2.1) we mention the following remark.

Remark 2.1 *System (2.1) is invariant under the changes*
$(x, y, t, a, b, c) \longrightarrow (-x, -y, -t, -a, b, c)$ *and* $(x, y, t, a, b, c) \longrightarrow (-x, -y, -t, a, -b, c)$
then we only need to study the system for $a > 0$ *and* $b > 0$.

2.3 Finite singular points

Taking into account the symmetry given in Remark 2.1, we can reduce the study of singular points of system (2.1) for $a > 0$ and $b > 0$.

Then the finite singular points of system (2.1) are given in the following proposition :

Proposition 2.1 *The following statements hold.*

For all the cases except $c = 4$ *system (2.1) has two hyperbolic singular points* $P_1 = (0, -b)$ *and* $P_2 = (0, b)$, *such that* P_1 *and* P_2 *are saddles if* $c \in (-\infty, 4)$, *and* P_1 *is a stable node and* P_2 *is an unstable node if* $c \in (4, +\infty)$.

(i) ***Assume*** $a > 1$

- *If* $c \in (-\infty, 4)$ *the system has four hyperbolic singularities : three saddles at* P_1, P_2 *and* P_3 *and a center at* P_4, *such that* $P_3 = \Big((ab(c-5)+A)/((a^2-1)(c-6)), 0\Big)$ *and* $P_4 = \Big((ab(c-5)-A)/((a^2-1)(c-6)), 0\Big)$. *Where* $A = -b\sqrt{a^2 + (c-6)(c-4)}$.
- *If* $c \in (4, 6)$ *it has four hyperbolic singularities : a stable node at* P_1, *an unstable node at* P_2 *and two saddles at* P_3 *and* P_4.
- *If* $c \in (6, +\infty)$ *the system has four hyperbolic singularities : a stable node at* P_1, *an unstable node at* P_2 , *a center at* P_3 *and a saddle at* P_4.
- *If* $c = 4$ *it has one saddle at* $(ab/(a^2-1), 0)$ *and* $x = 0$ *as a line of singulariy.*
- *If* $c = 6$ *it has three hyperbolic singularities : a stable node at* P_1, *an unstable node at* P_2 *and a saddle at* $(b/a, 0)$.

(ii) ***Assume*** $\boldsymbol{a = 1}$

- *If* $c \in (-\infty, 4)$ *the system has three hyperbolic singularities : two saddles at* P_1 *and* P_2 *and a center at* $R = (b(c-4)/2(c+5), 0)$.
- *If* $c = 4$ *it has a line of singularity* $x = 0$.
- *If* $c \in (4,5) \cup (5,+\infty)$ *it has three hyperbolic singularities : a stable node at* P_1*, an unstable nodes at* P_2 *and a saddle at* R.
- *If* $c = 5$ *it has two hyperbolic singularities : a stable node at* P_1 *and an unstable node at* P_2.

(iii) ***Assume*** $\boldsymbol{0 < a < 1}$

- *If* $c \in (-\infty, 4)$ *the system has four hyperbolic singularities : two saddles at* P_1 *and* P_2 *and two centers at* P_3 *and* P_4*, with* $P_3 = \Big((ab(c-5)+A)/((a^2-1)(c-6)), 0\Big)$*, and* $P_4 = \Big((ab(c-5)-A)/((a^2-1)(c-6)), 0\Big)$ *where* $A = -b\sqrt{a^2+(c-6)(c-4)}$.
- *If* $c \in (4, c_1) \bigcup (c_2, 6)$ *where* $c_1 = 5 - \sqrt{1-a^2}$ *and* $c_2 = 5 + \sqrt{1-a^2}$ *, it has four hyperbolic singularities : a stable node at* P_1*, an unstable node at* P_2*, a center at* P_3 *and a sadlle at* P_4.
- *If* $c \in (6, +\infty)$ *it has four hyperbolic singularities : a stable node at* P_1*, an unstable node at* P_2 *and two saddles at* P_3 *and* P_4.
- *If* $c = 6$ *it has three hyperbolic singularities :a stable node at* P_1*, an unstable node at* P_2 *and a saddle at* $(b/a, 0)$.
- *If* $c = c_1$ *or* $c = c_2$ *it has two hyperbolic nodes, a stable one at* P_1 *and an unstable at* P_2*, the third singularity is a nilpotent cusp at* P_5 *such that for* $c = c_1$ $P_5 = \Big((b(a^2-1+\sqrt{1-a^2}))/(a-a^3), 0\Big)$*, and for* $c = c_2$ $P_5 = \Big((b(a^2-1+\sqrt{1-a^2}))/(a^3-a), 0\Big)$.
- *If* $c \in (c_1, c_2)$ *it has two hyperbolic singularities : a stable node at* P_1 *and an unstable node at* P_2.
- *If* $c = 4$ *it has a center at* $(ab/(a^2-1), 0)$ *and* $x = 0$ *as a line of singulariy .*

Proof We start the proof by studing the nature of local phase portraits at $P_1 = (0, -b)$ and $P_2 = (0, b)$, when we look for the singularities of system (2.1) for all the cases of the parameters except when $c = 4$.

If $c \neq 4$ we can prove easely that P_1 and P_2 are singular points for system (2.1).

The Jacobian matrix of the vector field defined in (2.1) at (x_0, y_0) is given by :

$$M = D\chi(x_0, y_0) = \begin{pmatrix} y_0 & x_0 \\ ab(c-5) + (a^2-1)(6-c)x_0 & (c-4)y_0 \end{pmatrix}.$$

- At P_1 the matrix M becomes

$$M = \begin{pmatrix} -b & 0 \\ ab(c-5) & (4-c)b \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = -b$ and $\lambda_2 = (4-c)b$. Hence, P_1 is a hyberbolic saddle if $c \in (-\infty, 4)$ and a stable node if $c \in (4, +\infty)$.

- At P_2 the matrix M becomes

$$M = \begin{pmatrix} b & 0 \\ ab(c-5) & (c-4)b \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = b$ and $\lambda_2 = (c-4)b$. Hence, P_2 is a hyperbolic saddle if $c \in (-\infty, 4)$ and an unstable node if $c \in (4, +\infty)$.

▶ **Proof of statement** (i)

If $a \in (1+\infty)$ we distinguish many cases.

For $c \neq 4$ and $c \neq 6$, system (2.1) has four singular points : P_1, P_2(we have already studied them), $P_3 = \Big((ab(c-5)+A)/((a^2-1)(c-6)), 0\Big)$ and $P_4 = \Big((ab(c-5)-A)((a^2-1)(c-6)), 0\Big)$, such that $A = -bS$ where $S = \sqrt{a^2+(c-6)(c-4)}$.

• At P_3 the eigenvalues of linear part of system (2.1) are $\lambda_{1,2} = \pm\dfrac{bB}{(a^2-1)(c-6)}$, where $B = \sqrt{D}$ and $D = -S(a^2-1)(c-6)(S-a(c-5))$.

There fore, we have $D > 0$ if $c \in (-\infty, 4) \cup (4, 6)$ then P_3 is a saddle and we have $D < 0$ if $c \in (6, +\infty)$ then the eigenvalues of the matrix of vector field in (2.1) are imaginary purely such this equilibrum point is either a focus or a center, but due to the fact that system (2.1) is symetric with respect to (xx') axes, P_3 is a center.

• At P_4 the eigenvalues of the matrix M are $\lambda_{3,4} = \dfrac{\pm bV}{(a^2-1)(c-6)}$, such that $V = \sqrt{Q}$ and $Q = -S(a^2-1)(c-6)(S+a(c-5))$.

As a result, we have $Q < 0$ if $c \in (-\infty, 4)$ then the eigenvalues of the matrix of vector field in (2.1) are imaginary purely such this equilibrum point is either a focus or a center, but due to the fact that system (2.1) is symetric with respect to (xx') axes, P_4 is a center and we have $Q > 0$ if $c \in (4, 6) \cup (6, +\infty)$ then P_4 is a saddle.

For $c = 4$ the differential system (2.1) becomes

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= -abx + (a^2-1)x^2. \end{aligned} \tag{2.2}$$

This system has $x = 0$ as a line of singularity.

Doing a rescling of the time $xdt = ds$, the system (2.2) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ab + (a^2-1)x. \end{aligned} \tag{2.3}$$

This system has one singular point $q = (ab/(a^2-1), 0)$ and the Jacobian matrix of this system at q is

$$D\chi(\frac{ab}{a^2-1}, 0) = \begin{pmatrix} 0 & 1 \\ (a^2-1) & 0 \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = \sqrt{a^2-1}$ and $\lambda_2 = -\sqrt{a^2-1}$. Hence, q is a hyperbolic saddle.

For $c = 6$ the differential system (2.1) becomes

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= -b^2 + abx + y^2 \end{aligned} \tag{2.4}$$

This system has three singular points P_1, P_2 and $(b/a, 0)$

At $(b/a, 0)$ the eigenvalues of the linear part of system (2.4) are $\lambda_{1,2} = \pm b$. So this point is a hyperbolic saddle .

▶ **Proof of statement** (ii)

For $a = 1$ and $c \neq 5$ the differential system (2.1) becomes

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= \frac{1}{2}b^2(4-c) + b(c-5)x + \frac{1}{2}(c-4)y^2. \end{aligned} \tag{2.5}$$

In addition to P_1 and P_2, this system has an other singular points $R = (b(c-4)/2(c-5), 0)$.

- At R the eigenvalues of the matrix of vector field in (2.5) are $\lambda_{1,2} = \pm\dfrac{b\sqrt{c-4}}{\sqrt{2}}$.

If $c \in (-\infty, 4)$ the eigenvalues of the matrix of vector field in (2.5) at R are imaginary purely such this equilibrum point is either a focus or a center, but due to the fact that system (2.5) is symetric with respect to (xx') axes, R is center .

For $c = 4$ the differential system (2.5) becomes

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= -bx. \end{aligned} \tag{2.6}$$

The differential system (2.6) has a line of singularity $x = 0$.

If $c \in (4, 5) \cup (5, +\infty)$ R is a saddle.

For $c = 5$, the differential system (2.5) becomes

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= \frac{-b^2}{2} + \frac{y^2}{2}. \end{aligned} \tag{2.7}$$

It has only two singularities P_1 and P_2.

▶ **Proof of statement** (iii)

For $a \in (0, 1)$ we distinguish many cases.

If $c \in (-\infty, 4)$ system (2.1) has four singular points : P_1, P_2, P_3, and P_4 such that $P_3 = \left(\dfrac{ab(c-5)+A}{(a^2-1)(c-6)}, 0\right)$ and $P_4 = \left(\dfrac{ab(c-5)-A}{(a^2-1)(c-6)}, 0\right)$, where $A = -bS$ and $S = \sqrt{a^2 + (c-6)(c-4)}$.

- The jacobien matrix at P_3 is

$$M = \begin{pmatrix} 0 & \dfrac{A + ab(c-5)}{(a^2-1)(c-6)} \\ -A & 0 \end{pmatrix}.$$

Its eigenvalues are $\lambda_{1,2} = \pm\dfrac{ibB}{(a^2-1)(c-6)}$, such that $B = \sqrt{S(1-a^2)(c-6)(S-a(c-5))}$.

This eigenvalues are imaginary purely such this equilibrum point is either a focus or a center, but due to the fact that system (2.1) is symetric with respect to (xx') axes , P_3 is center.

- The jacobien matrix at P_4 is

$$N = \begin{pmatrix} 0 & \dfrac{-A + ab(c-5)}{(a^2-1)(c-6)} \\ A & 0 \end{pmatrix}.$$

Its eigenvalues are $\lambda_{1,2} = \pm \dfrac{ibV}{(a^2-1)(c-6)}$, such that $V = \sqrt{S(1-a^2)(c-6)(S + a(c-5))}$. This eigenvalues are imaginary purely such this equilibrum point is either a focus or a center, but due to the fact that system (2.1) is symetric with respect to (xx') axes , P_4 is center.

If $c \in (4, c_1) \cup (c_2, 6)$ the system has in addition to P_1 and P_2 two other singularities P_3 and P_4 wich we have already mentioned then.

The eigenvalues of the jacobien matrix at P_3 are $\lambda_{1,2} = \pm \dfrac{ibB}{(a^2-1)(c-6)}$. Hence, P_3 is a center.

The eigenvalues of the jacobien matrix at P_4 are $\lambda_{1,2} = \pm \dfrac{bV}{(a^2-1)(c-6)}$. Hence, P_4 is a saddle.

If $c \in (6, +\infty)$ it has four singularities P_1, P_2, P_3 and P_4.

The eigenvalues of the jacobien matrix at the point P_3 are $\lambda_{1,2} = \pm \dfrac{bB}{(a^2-1)(c-6)}$. Hence, P_3 is a saddle.

The eigenvalues of the jacobien matrix at the point P_4 are $\lambda_{1,2} = \pm \dfrac{bV}{(a^2-1)(c-6)}$. Hence, P_4 is a saddle.

If $c = 6$ the differential system (2.1) becomes (2.3).

In this case the differentail system (2.3) has in addition to P_1 and P_2 an other singular point $(b/a, 0)$.

The eigenvalues of the linear part of differential system (2.3) at $(b/a, 0)$ are $\lambda_{1,2} = \pm b$. So this point is a hyberbolic saddle.

If $c = c_1$ where $c_1 = 5 - \sqrt{1-a^2}$, the differential system (2.1) becomes

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= \frac{1}{2}\Big((-1 + \sqrt{1-a^2})b^2 - 2a\sqrt{1-a^2}bx\Big) \\ &+ \frac{1}{2}\Big((a^2-1)(1+\sqrt{1-a^2})x^2 - (\sqrt{1-a^2}-1)y^2\Big). \end{aligned} \tag{2.8}$$

System (2.8) has in addition to P_1 and P_2 an other singular point P_5 at $\left(\dfrac{(a^2-1+\sqrt{1-a^2})b}{a(a^2-1)}, 0\right)$.

The associated jacobien matrix of system (2.8) at P_5 is non zero with eigenvalues $\lambda_1 = \lambda_2 = 0$, then we conclude that it is a nilpotent singular point. In order to know the nature of this singular point.

First, we put this singular point at the origin of the coordinates by performing the translation

$(x, y) = (x_1 + \dfrac{(a^2 - 1 + \sqrt{1 - a^2})b}{a(a^2 - 1)}, y_1)$, and we get the folowing system :

$$\begin{aligned} \dot{x_1} &= \frac{(a^2 - 1 + \sqrt{1 - a^2})b}{a(a^2 - 1)} y_1 + x_1 y_1, \\ \dot{y_1} &= \frac{1 + \sqrt{1 - a^2}}{2} x_1^2 - \frac{\sqrt{1 - a^2} - 1)}{2} y_1^2. \end{aligned} \tag{2.9}$$

Second, we transforme this system by doing the following change $ds = \dfrac{(a^2 - 1 + \sqrt{1 - a^2})b}{a(a^2 - 1)} dt$ into its normal forme, then we get

$$\begin{aligned} \dot{x_1} &= y_1 + A(x_1, y_1) \\ \dot{y_1} &= B(x_1, y_1). \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} A(x_1, y_1) &= \frac{a(a^2 - 1)}{(a^2 - 1 + \sqrt{1 - a^2})b} x_1 y_1, \\ B(x_1, y_1) &= \frac{ab\sqrt{1 - a^2}}{2} x_1^2 - \frac{b(\sqrt{1 - a^2} - 1)(a^2 - 1 + \sqrt{1 - a^2})}{2a(a^2 - 1)} y_1^2. \end{aligned}$$

By solving the equation $y_1 + A(x_1, y_1) = 0$ for the variable y_1 we obtain $y_1 = f(x_1)$, where $f(x_1) = 0$.

We substituate the value of $y_1 = f(x_1)$ in the expresion $B(x_1, y_1)$, we get

$$F(x_1) = B(x_1, f(x_1)) = \frac{ab\sqrt{1 - a^2}}{2} x_1^2.$$

Now we have to concluate the function $G(x_1) = \left(\dfrac{\partial A}{\partial x_1} + \dfrac{\partial B}{\partial y_1}\right)(x_1, f(x_1)) = 0$.

By applying theorem (3.5) of [9] we obtain that the origin is a cusp.

The local phase portrait of P_5 consists of two hyperbolic sectors .

If $c = c_2$ where $c_2 = 5 + \sqrt{1 - a^2}$ the differential system (2.1) becomes

$$\begin{aligned} &\dot{x} = xy, \\ &\dot{y} = \frac{1}{2}(-1 - \sqrt{1 - a^2})b^2 + a\sqrt{1 - a^2} bx + \frac{1}{2}(a^2 - 1)(1 - \sqrt{1 - a^2})x^2 \\ &+ \frac{1}{2}(1 + \sqrt{1 - a^2})y^2. \end{aligned} \tag{2.11}$$

this system has three singular points P_1, P_2 and P_5, such that $P_5 = \left(\dfrac{(1 - a^2 + \sqrt{1 - a^2})b}{a(1 - a^2)}, 0\right)$

The associated jacobien matrix of system (2.8) at q_1 is non zero with eigenvalues $\lambda_1 = \lambda_2 = 0$, then we conclude that it is a nilpotent singular point. In order to know the nature of this singular point.

First, we put this singular point at the origin of the coordinates by performing the translation $(x, y) = (x_1 + \dfrac{(1 - a^2 + \sqrt{1 - a^2})b}{a(1 - a^2)}, y_1)$, and we get the folowing system :

$$\begin{aligned} \dot{x_1} &= \frac{(1-a^2+\sqrt{1-a^2})b}{a(a^2-1)}y_1 + x_1 y_1, \\ \dot{y_1} &= \frac{(a^2-1)(\sqrt{1-a^2}-1)}{2}x_1^2 + \frac{1+\sqrt{1-a^2}}{2}y_1^2. \end{aligned} \tag{2.12}$$

Second, we transforme this system by doing the following change $ds = \dfrac{(1-a^2+\sqrt{1-a^2})b}{a(1-a^2)}dt$ into its normal forme, then we get

$$\begin{aligned} \dot{x_1} &= y_1 + A(x_1, y_1) \\ \dot{y_1} &= B(x_1, y_1). \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} A(x_1, y_1) &= \frac{a(1-a^2)}{(1-a^2+\sqrt{1-a^2})b}x_1 y_1, \\ B(x_1, y_1) &= \frac{(a^2-1)(\sqrt{1-a^2}-1)}{2}x_1^2 + \frac{1+\sqrt{1-a^2}}{2}y_1^2. \end{aligned}$$

By solving the equation $y_1 + A(x_1, y_1) = 0$ for the variable y_1 we obtain $y_1 = f(x_1)$, where $f(x_1) = 0$.

We substituate the value of $y_1 = f(x_1)$ in the expresion $B(x_1, y_1)$, we get

$$F(x_1) = B(x_1, f(x_1)) = \frac{(a^2-1)(\sqrt{1-a^2}-1)}{2}x_1^2.$$

Now we have to concluate the function $G(x_1) = \left(\dfrac{\partial A}{\partial x_1} + \dfrac{\partial B}{\partial y_1}\right)(x_1, f(x_1)) = 0$ By applying theorem (3.5) of [9] we obtain that the origin is a cusp.

The local phase portrait of P_5 consists of two hyperbolic sectors.

If $c \in (c_1, c_2)$ the differential system (2.1) has two singular points P_1 and P_2.

If $c = 4$ the differential system (2.2) has one singular point $\left(\dfrac{ab}{a^2-1}, 0\right)$.

The Jacobian matrix of the vector field defined in (2.2) at $\left(\dfrac{ab}{a^2-1}, 0\right)$ is :

$$D\chi\left(\frac{ab}{a^2-1}, 0\right) = \begin{pmatrix} 0 & 1 \\ 1-a^2 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_{1,2} = \pm\sqrt{a^2-1}$. This eigenvalues are pure imaginary such this equilibrum point is either a focus or a center, but due to the fact that system (2.2) is symetric with respect to (xx') axes, So this point is center.

2.4 Infinite singular points

The main gool of this part is to give the local phase portrait of systems (2.1) at infinite singular points. To give a full study about the infinite singular points in the Poincaré disc we present the analysis of the vector field at infinity.

Proposition 2.2 *The local phase portraits at the infinite singular points of system (2.1) in the local chart U_1 are :*

1. *One nilpotent singularity at the origin of coordinate, where its local phase portrait consits of one hyperbolic, one elliptic and two parabolic sectors, for $a = 1$ and $c \neq 5$;*
2. *a linearly singular zero singular point at the origin and its local phase portrait consits of two parabolic and one hyperbolic sectors, for $a = 1$and $c = 5$;*
3. *no singularity if $a \in (0, 1)$;*
4. *the infinity is a line of singularily if $a \in (1, +\infty)$ and $c = 6$;*
5. *two singular points $q_1 = (\sqrt{a^2 - 1}, 0)$ and $q_2 = (-\sqrt{a^2 - 1}, 0)$, for $a \in (1, +\infty)$ and $c \neq 6$, such that :*
   (a) q_1 and q_2 are hyperbolic saddles if $c \in (6, +\infty)$;
   (b) q_1 is an unstable node and q_2 is a stable node if $c \in (-\infty, 6)$.

Proposition 2.3 *The local phase portrait of the origin of the local chart U_2 is :*

1. *A saddle if $c \in (4, 6)$ and a node if $c \in (-\infty, 4) \cup (6, +\infty)$;*
2. *not a singularity if $c = 4$ or $c = 6$.*

Proof of Proposition (2.2) The expression of system (2.1) in the local chart U_1 is given by

$$\begin{aligned} \dot{u} &= 1/2(-6 - a^2(c-6) - 6u^2 + 2ab(c-5)v + 4b^2v^2 + c(1 + u^2 - b^2v^2)), \\ \dot{v} &= -uv. \end{aligned} \tag{2.14}$$

Any arbitrary infinite sigular point of system (2.14) take the form $(u_0, 0)$.

▶ **Proof of statement** (1)

If $a = 1$ and $c \neq 5$ system (2.14) becomes

$$\begin{aligned} \dot{u} &= (c-5)v + \frac{1}{2}(c-6)u^2 + \frac{1}{2}(4b^2 - b^2c)v^2, \\ \dot{v} &= -uv. \end{aligned} \tag{2.15}$$

The origin is the only singular point of differential systems (2.15) which is a nilpotent singular point with eigenvalus $\lambda_1 = \lambda_2 = 0$.

In order to obtain the local phase portrait at this singular point we use the theorem (3.5) of [9] .

We transform this system into its normal form by doing the change $ds = b(c-5)dt$, then we get

$$\begin{aligned} \dot{u} &= v + A(u, v), \\ \dot{v} &= B(u, v). \end{aligned} \tag{2.16}$$

Where

$$\begin{aligned} A(u, v) &= \frac{c-6}{2b(c-5)}u^2 + \frac{b(4-c)}{2(c-5)}v^2, \\ B(u, v) &= -\frac{1}{b(c-5)}uv. \end{aligned}$$

By solving the equation $v + A(u,v) = 0$ for the variable v we obtain $v = f(u)$, where $f(u) = \dfrac{6-c}{2b(c-5)}u^2$.

We substituate the value of $v = f(u)$ in the expression $B(u,v)$ we get $F(u) = B(u, f(u)) = \dfrac{c-6}{2b^2(c-5)^2}u^3 = au^3$, such that $a = \dfrac{6-c}{2b(c-5)}$, then we have $m = 3$ is odd integer.

We need to calculate the function $G(u) = \dfrac{c-7}{b(c-5)}u = ku$, such that $k = \dfrac{c-7}{b(c-5)}$, then we have $n = 1$. By applying Theorem (3.5) of [9] we know that

If $c - 6 > 0$ the origin is a saddle.

If $c - 6 < 0$ we shoud calculate $\alpha_n = k^2 + 4a(n+1)$.

After calculation, we obtain that $\alpha_n = \dfrac{1}{b^2} > 0$, then the local phase portrait of $(0,0)$ consists of one hyperbolic, one elliptic and two parabolic sectors.

▶ **Proof of statement** (2)

If $c = 5$ the origin is a linearly zero singular point.

We need to do the blow-up to describe the local behavior at this point. We perform the directional blow-up $(u,v) \longrightarrow (u,w)$, with $w = \dfrac{v}{u}$ and we get the following system :

$$\begin{aligned} \dot{u} &= -\frac{1}{2}u^2 - \frac{b^2}{2}u^2w^2, \\ \dot{w} &= -\frac{1}{2}uw + \frac{b^2}{2}uw^3. \end{aligned} \tag{2.17}$$

We eliminate the common factor u between $\dot{u}$ and $\dot{w}$ by doing a rescaling of the independent variable $udt = ds$, then we get

$$\begin{aligned} \dot{u} &= -\frac{1}{2}u - \frac{b^2}{2}uw^2, \\ \dot{w} &= -\frac{1}{2}w + \frac{b^2}{2}w^3. \end{aligned} \tag{2.18}$$

This system has three singularities for $u = 0$: a stable node at $(0,0)$ and two saddle at $\left(0, \pm\dfrac{1}{b}\right)$.

Going back through the change of variables $w = \dfrac{v}{u}$ and $udt = ds$, and taking into acount the signe of the vector field at the axes $\dot{u}/u = 0$ and $\dot{w}/w = 0$, we conclud that the local phase portrait of the origin consists of two parabolic and one hyparbolic sector.

▶ **Proof of statement** (3)

If $a \in (0,1)$ system (2.14) has non sigularity.

▶ **Proof of statement** (4)

If $a > 1$ and $c = 6$ the expression for system (2.14) becomes

$$\begin{aligned} \dot{u} &= abv - bv^2, \\ \dot{v} &= -uv. \end{aligned} \tag{2.19}$$

For all u, $v = 0$ is a singularity for this system.Then the infinity if the singular for the system.

▶ **Proof of statement** (5)

If $a > 1$ and $c \neq 6$ the diferential systems (2.14) has two singularities $(\pm\sqrt{a^2-1}, 0)$.

The Jacobian matrix of the vector field defined in (2.14) at (u_0, v_0) is given by :

$$N = D\chi(u_0, v_0) = \begin{pmatrix} (c-6)u_0 & ab(c-5) + 4b^2(4-c)v_0 \\ -v_0 & -u_0 \end{pmatrix}$$

- At $(-\sqrt{a^2-1}, 0)$ the matrix N is given by :

$$N = \begin{pmatrix} (6-c)\sqrt{a^2-1} & ab(c-5) \\ 0 & \sqrt{a^2-1} \end{pmatrix}$$

The eigenvalues of the matrix N are $\lambda_1 = (6-c)\sqrt{a^2-1}$ and $\lambda_2 = \sqrt{a^2-1}$. So the singular point $(-\sqrt{a^2-1}, 0)$ is a saddle if $c > 6$ and an unstable node if $c < 6$.

- At $(\sqrt{a^2-1}, 0)$ the matrix N is given by :

$$N = \begin{pmatrix} (c-6)\sqrt{a^2-1} & ab(c-5) \\ 0 & -\sqrt{a^2-1} \end{pmatrix}$$

The eigenvalues of the matrix N are $\lambda_1 = (c-6)\sqrt{a^2-1}$ and $\lambda_2 = -\sqrt{a^2-1}$. So the singular point $(\sqrt{a^2-1}, 0)$ is a saddle if $c > 6$ and a stable node $c < 6$.

Proof of Proposition (2.3)

The diferential systems (2.1) in the local chart U_2 given by :

$$\begin{aligned} \dot{u} &= \tfrac{1}{2}u\left(\left(a^2-1\right)(c-6)u^2 - 2ab(c-5)uv + b^2(c-4)v^2 - c + 6\right), \\ \dot{v} &= \tfrac{1}{2}v\left(c\left(\left(a^2-1\right)u^2 - 1\right) - 6\left(a^2-1\right)u^2 - 2ab(c-5)uv + b^2(c-4)v^2 + 4\right). \end{aligned} \tag{2.20}$$

The Jacobian matrix of its vector field at $(0, 0)$ is given by :

$$D\chi(0,0) = \begin{pmatrix} 3 - c/2 & 0 \\ 0 & 2 - c/2 \end{pmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = 3 - \dfrac{c}{2}$ and $\lambda_2 = 2 - \dfrac{c}{2}$.

So, the origin is a saddle if $c \in (4, 6)$ and an unstable node if $c \in (-\infty, 4)$ and a stable node if $c \in (6, +\infty)$.

For $c = 4$ the expression of systems (2.20) becomes

$$\begin{aligned} \dot{u} &= u(1 + buv), \\ \dot{v} &= buv^2, \end{aligned} \tag{2.21}$$

$u = 0$ is singular of the system by doing a change of variable $ds = udt$, we have the system :

$$\begin{aligned} \dot{u} &= 1 + buv, \\ \dot{v} &= bv^2, \end{aligned} \tag{2.22}$$

it is clear the origin is not a singularty of the system.

For $c = 6$ the expression of systems (2.20) becomes

$$\begin{aligned} \dot{u} &= buv(-au + bv), \\ \dot{v} &= v(-1 - abuv + b^2v^2). \end{aligned} \tag{2.23}$$

$v = 0$ is singular of the system by doing a change of variable $ds = vdt$, we have the system :

$$\begin{aligned} \dot{u} &= bu(-au + bv), \\ \dot{v} &= b^2v^2 - 1 - abuv. \end{aligned} \tag{2.24}$$

it is clear the origin is not a singularty of the system.

Chapitre 3

Global phase portraits of an integrable quadratic polynomial differential system with two parameters

In this chapter we apply the techniques and the theorems that we saw in the first chapter to study a class of integrable quadratic polynomial system with two parameters and classify all its the global phase portraits. More precisely, we characterize locale phase portraits of all its finite and infinite singular points in the Poincaré disc.

3.1 Statement and the main results.

Firstly, we consider the well-known algebraic curve of degree three which is called ***Cubic egg*** and we find the differential quadratic polynomial system of second degree which accept this algebraic curve as a first integral.

Secondly, we analyze the local behavior of the finite and infinite singular points of the following system.

Theorem 3.1 *The algebraic curve* ***Cubic egg*** *of degree three given by* $H(x, y) = x^2 + (1 + x)y^2 - 1 = 0$ *is a first integral of the quadratic system,*

$$\begin{aligned} \dot{x} &= -1 + x^2 + ay + axy, \\ \dot{y} &= -b - \frac{a}{2} + (b - \frac{a}{2})x + \frac{y}{2} + \frac{xy}{2} + by^2, \end{aligned} \tag{3.1}$$

so, it's associated cofactor $k(x, y) = 0$.

Proof To prove this theorem we can easily verify that we have the following partial differential equation :

$$\frac{\partial H}{\partial t} = P(x, y)\frac{\partial H}{\partial x} + Q(x, y)\frac{\partial H}{\partial y} = 0.$$

we will solve geometrically the differential system of second degree (3.1) and we will give all its subsolution according to the two parameters a and b.

Our second main result in this chapter is given in the following Theorem.

Theorem 3.2 *The global phase portraits of QS (3.1) given in Theorem 3.1 are topologically equivalent to*

(1) *for* $a = 2b$ *and* $b \in \left(0, \frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, +\infty\right)$;

(2) *for* $a = 0$ *and* $b = 0$;

(3) *for* $a = \sqrt{2}$ *and* $b = \frac{\sqrt{2}}{2}$;

(4) *for* $a = b$ *and* $b \neq 0$;

(5) *for* $a = \sqrt{2}$ *and* $b = 0$;

(6) *for* $a \in \left(0, \sqrt{2}\right) \cup \left(\sqrt{2}, +\infty\right)$ *and* $b = 0$;

(7) *for* $a \in \left(0, \sqrt{2}\right)$ *and* $b \in (0, b_2)$; $a \in \left(\sqrt{2}, \frac{9}{4}\right)$ *and* $b \in (b_1, b_2)$ *or* $a \in (0, b_0)$ *and* $b \in (-\infty, b_1)$ *with* $b_0 = b + \sqrt{b^2 + 2}$, $b_1 = \dfrac{-9 + 8a^2 - \sqrt{81 - 16a^2}}{16a}$ *and* $b_2 = \dfrac{-9 + 8a^2 + \sqrt{81 - 16a^2}}{16a}$;

(8) *for* $a \in \left(0, \sqrt{2}\right)$ *and* $b \in \left(b_2, \frac{\sqrt{2}}{2}\right)$;

(9) *for* $a \in \left(\sqrt{2}, \frac{9}{4}\right)$ *and* $b \in \left(0, \frac{a^2 - 2}{2a}\right) \cup \left(\frac{a^2 - 2}{2a}, b_1\right) \cup \left(b_2, \frac{a}{2}\right)$; $a = \frac{9}{4}$ *and* $b \in \left(0, \frac{49}{72}\right) \cup \left(\frac{49}{72}, \frac{9}{8}\right)$; $a \in \left(\frac{9}{4}, b_0\right)$ *and* $b \in \left(\frac{a^2 - 2}{2a}, \frac{a}{2}\right)$ *or* $a \in (b_0, +\infty)$ *and* $b \in \left(0, \frac{a^2 - 2}{2a}\right)$;

(10) *for* $a = \sqrt{2}$ *and* $b \in \left(0, \frac{7}{8\sqrt{2}}\right) \cup \left(\frac{7}{8\sqrt{2}}, \frac{\sqrt{2}}{2}\right)$;

(11) *for* $a = 0$ *and* $b \in (-\infty, 0)$ *or* $a = \frac{9}{4}$ *and* $b \in (\left(\frac{9}{8}, \frac{9}{4}\right) \cup \left(\frac{9}{4}, +\infty\right)$;

(12) *for* $a = \sqrt{2}$ *and* $b \in \left(\frac{\sqrt{2}}{2}, \sqrt{2}\right) \cup \left(\sqrt{2}, +\infty\right)$;

(13) *for* $a = \sqrt{2}$ *and* $b \in (-\infty, 0)$;

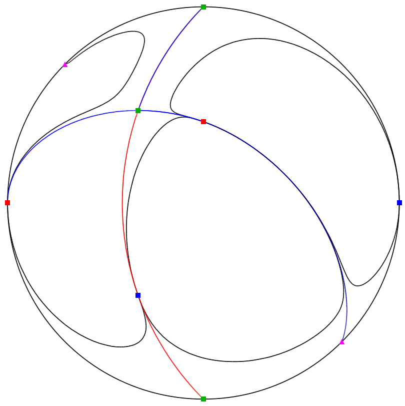
(14) *for* $a \in \left(\sqrt{2}, +\infty\right)$ *and* $b \in (-\infty, 0)$;

(15) *for* $a \in \left(b_0, \sqrt{2}\right)$ *and* $b \in (-\infty, b_1) \cup (b_1, 0)$;

(16) *for* $a \in (0, b_0)$ *and* $b \in (b_1, 0)$ *or* $a \in (b_0, +\infty)$ *and* $b \in \left(\frac{a}{2}, a\right) \cup (a, +\infty)$;

(17) *for* $a = b_0$ *and* $b \in (-\infty, 0)$;

(18) *for* $a = b_0$ *and* $b \in \left(0, \frac{b_0}{2}\right) \cup \left(\frac{b_0}{2}, b_0\right) \cup (b_0, +\infty)$.



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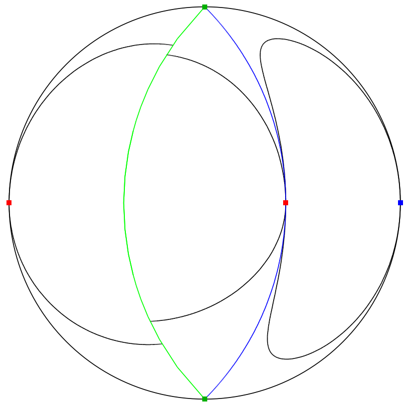

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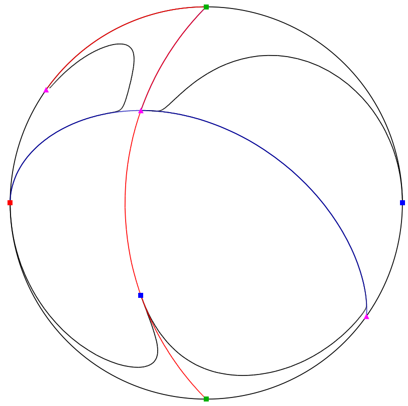

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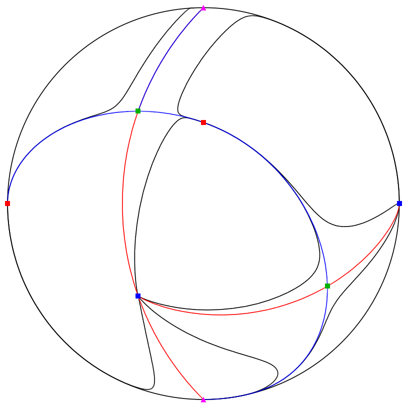

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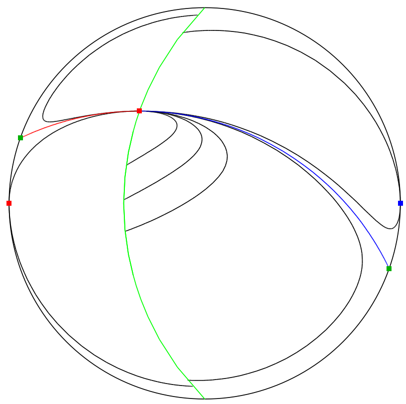

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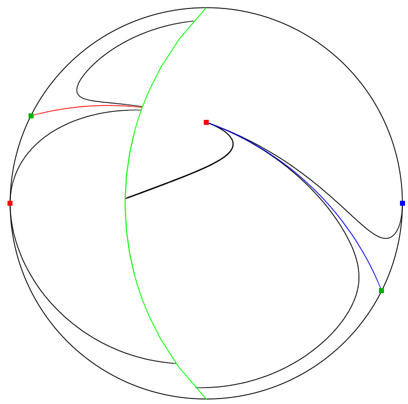

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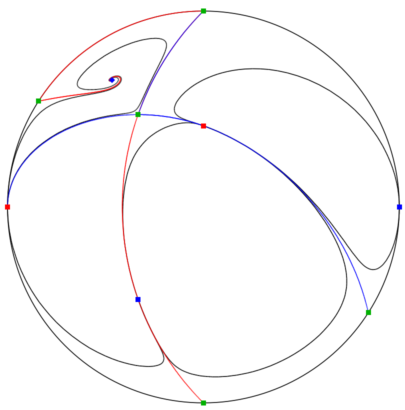

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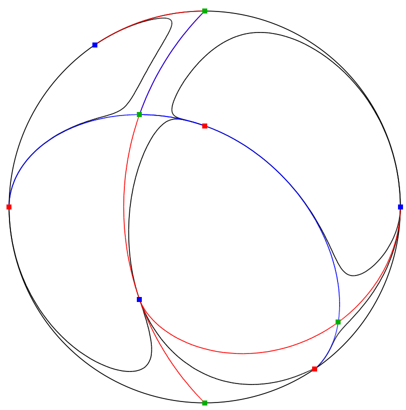

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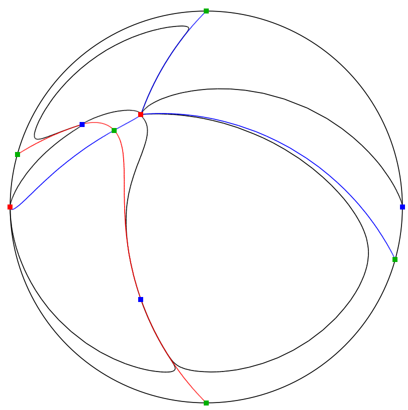

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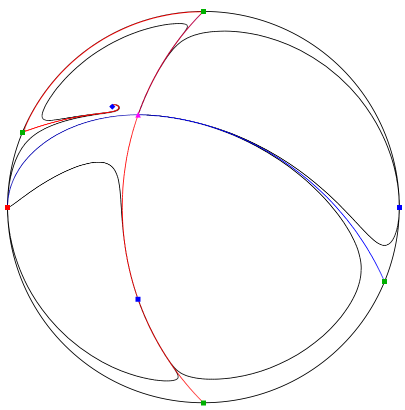

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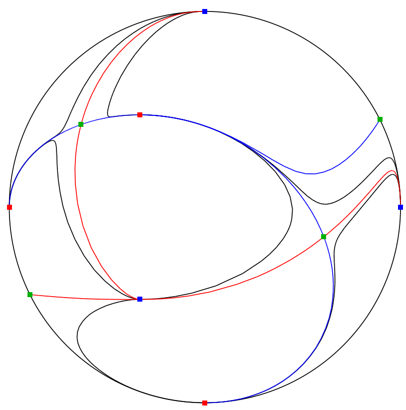

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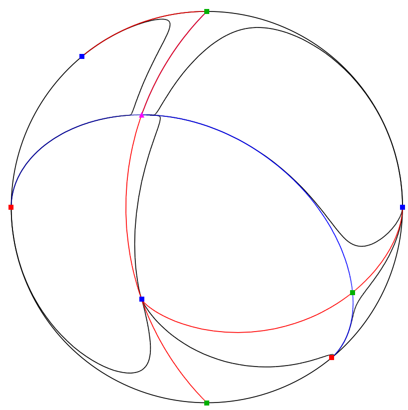

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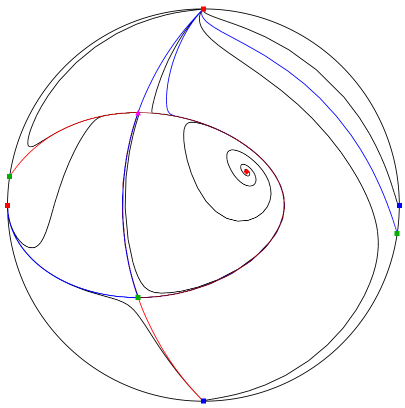

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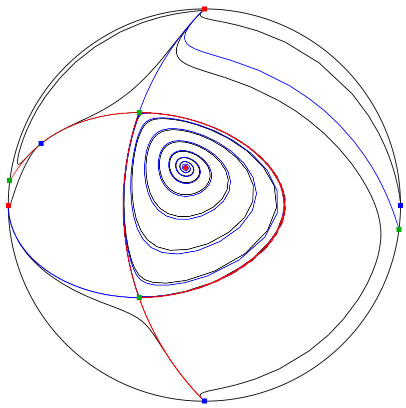

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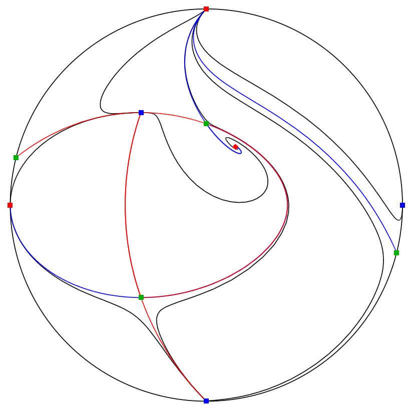

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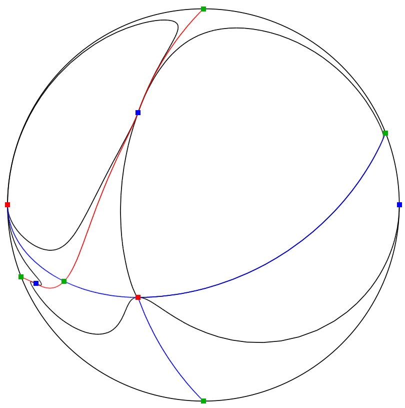

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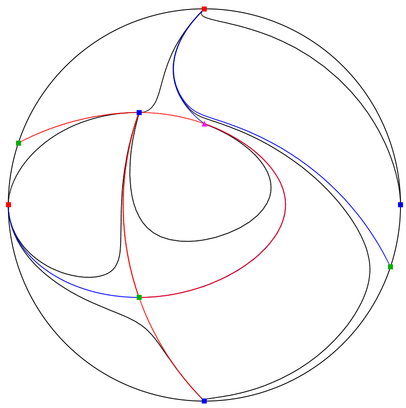

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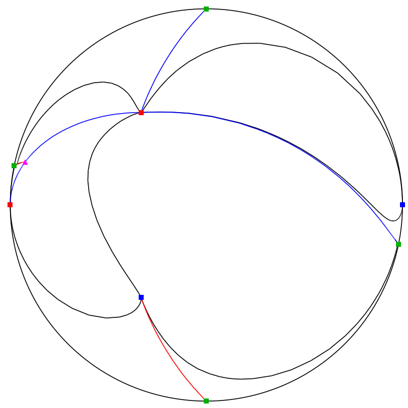

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3.2 Global phase portraits in the Poincaré disc

In this part we are going to study the global phase portraits in the Poincaré disc of the quadratic polynomail differential system (3.1).

We use the following remark to reduce the study of system (3.1)

Remark 3.1 *System (3.1) is invariant under the change* $(x, y, t, a, b) \longrightarrow (x, -y, t, -a, -b)$, *then we only need to study them for* $a > 0$ *or* $a = 0$ *and* $b \geq 0$.

3.3 Finite singular points

Taking into account the symmetry geving in Remark 3.1, then the finite singular points of system (3.1) are geving in the following proposition :

Proposition 3.1 *The following statements hold.*

I. Assume $a = 2b$

(i) *If* $b \in \left(0, \dfrac{\sqrt{2}}{2}\right)$ *the system has three hyperbolic singularities : a stable node at* p_1, *a saddle at* p_2 *and an unstable node at* q_1, *such that* $p_1 = \left(-1, -\sqrt{2}\right)$, $p_2 = \left(-1, \sqrt{2}\right)$ *and* $q_1 = (1 - 4b^2, 2b)$.

(ii) *If* $b \in \left(\dfrac{\sqrt{2}}{2}, +\infty\right)$ *it has three hyperbolic singularities : a stable node at* p_1, *an unstable node at* p_2 *and a saddle at* q_1.

(iii) *If* $b = 0$ *and* $a = 0$ *the system has a line of singularity* $x + 1 = 0$, *and a hyperbolic unstable node at* q_2, *with* $q_2 = (1, 0)$.

(iv) *If* $b = \dfrac{\sqrt{2}}{2}$ *and* $a = \sqrt{2}$ *it has two singularities : a stable node at* p_1 *and saddle-node at* p_2.

II. ***Assume*** $a \neq 2b$

(i) *For* $a \neq \sqrt{2}$ *and* $b \neq 0$ *the system has four hyperbolic singularities* p_1, p_2, p_3 *and* p_4, *such that* $p_1 = \left(-1, -\sqrt{2}\right)$, $p_2 = \left(-1, \sqrt{2}\right)$, $p_3 = (1 - a^2, a)$ *and* $p_4 = \left(1 - \dfrac{2a}{a - 2b}, \dfrac{2}{a - 2b}\right)$, *and we have the following subcases :*

(a) *If* $a \in (0, b_0)$ *and* $b \in (-\infty, 0)$ *system* (3.1) *has a saddle at* p_1, *a stable node at* p_2 *and an unstable node at* p_3, *with* $b_0 = b + \sqrt{b^2 + 2}$.

(b) *If* $a \in \left(b_0, \sqrt{2}\right)$ *and* $b \in (-\infty, 0)$ *it has two saddles at* p_1 *and* p_3, *and a stable node at* p_2.

(c) *If* $a \in \left(\sqrt{2}, +\infty\right)$ *and* $b \in (-\infty, 0)$ *it has two saddles at* p_1 *and* p_2 , *and a stable node at* p_3.

(d) *If* $a \in \left[0, \sqrt{2}\right)$ *and* $b \in (0, +\infty)$ *system* (3.1) *has a stable node at* p_1, *a saddle at* p_2 *and an unstable node at* p_3.

(e) *If* $a \in \left(\sqrt{2}, b_0\right)$ *and* $b \in (0, +\infty)$ *it has a stable node at* p_1, *an unstable node at* p_2 *and a saddle at* p_3.

(f) *If* $a \in (b_0, +\infty)$ *and* $b \in (0, +\infty)$ *it has two stable nodes at* p_1 *and* p_3, *and an unstable node at* p_2.

— *the nature of the fourth singuler point* p_4 *for all the six previous cases is given in the table (3.1)*

(ii) *If* $a \neq \sqrt{2}$ *and* $b = 0$ *it has a hyperbolic node at* p_3 *and a line of singularity* $x + 1 = 0$.

(iii) *If* $a = \sqrt{2}$ *and* $b \in (-\infty, 0)$ *it has two hyperbolic singularities a saddle at* p_1 *and an unstable focus at* q_3 *and a semi-hyperbolic saddle-node at* p_2,

where $q_3 = \left(1 + \dfrac{2}{\sqrt{2}b - 1}, \dfrac{\sqrt{2}}{1 - \sqrt{2}b}\right)$.

(iv) *If* $a = \sqrt{2}$ *and* $b \in \left(0, \dfrac{7}{8\sqrt{2}}\right)$ *it has two hyperbolic singularities a stable node at* p_1 *and a stable focus at* q_3 *and a semi-hyperbolic saddle-node at* p_2.

(v) *If* $a = \sqrt{2}$ *and* $b \in \left(\dfrac{7}{8\sqrt{2}}, \dfrac{\sqrt{2}}{2}\right)$ *it has two hyperbolic stable nodes at* p_1 *and* q_3 *and a semi-hyperbolic saddle-node at* p_2.

(vi) *If* $a = \sqrt{2}$ *and* $b \in \left(\dfrac{\sqrt{2}}{2}, +\infty\right)$ *it has two hyperbolic singularities a stable node at* p_1 *and a saddle at* q_3 *and a semi-hyperbolic saddle-node at* p_2.

(vii) *If* $a = \sqrt{2}$ *and* $b = 0$ *it has a hyperbolic unstable node at* p_2 *and a line of singularity* $x + 1 = 0$.

$0 < a < \sqrt{2}$	$b \in (-\infty, b_1)$	Unstable Focus
	$b \in (0, b_2)$	Stable Focus
	$b \in (b_1, 0)$	Unstable Node
	$b \in (b_2, +\infty)$	Saddle
$\sqrt{2} < a < \frac{9}{4}$	$b \in (-\infty, 0)$	Unstable Focus
	$b \in (0, (a^2-2)/2a)$	Saddle
	$b \in ((a^2-2)/2a, b_1)$	Stable Node
	$b \in (b_1, b_2)$	Stable Focus
	$b \in (b_2, a/2)$	Stable Node
	$b \in (a/2, +\infty)$	Saddle
$a > \dfrac{9}{4}$	$b \in (-\infty, 0)$	Unstable Focus
	$b \in ((a^2-2)/2a, a/2)$	Stable Node
	$b \in (0, (a^2-2)/2a) \cup (a/2, +\infty)$	Saddle
$a = 0$	Saddle	
$a = \dfrac{9}{4}$	$b \in (-\infty, 0)$	Unstable Focus
	$b \in (0, 49/72)$ or $b \in (9/8, +\infty)$	Saddle
	$b \in (49/72, 9/8)$	Stable Node
$b_1 = \dfrac{-9+8a^2-\sqrt{81-16a^2}}{16a}$,	$b_2 = \dfrac{-9+8a^2+\sqrt{81-16a^2}}{16a}$	

TABLE 3.1 – The nature of the singuler point p_4.

$(viii)$ *If* $a = b_0$ *and* $b \in (-\infty, 0)$ *it has two hyperbolic singularities a saddle at* p_1 *and a stable node at* p_2 *and a semi-hyperbolic saddle-node at* q_4*,*
where $q_4 = \left(-1 - 2b(b + \sqrt{2+b^2}, b + \sqrt{2+b^2}\right)$.

(ix) *If* $a = b_0$ *and* $b \in (0, +\infty)$ *it has two hyperbolic singularities a stable node at* p_1 *and an unstable node at* p_2 *and a semi-hyperbolic saddle-node at* q_4.

Proof

- **Proof of statement (I) of Proposition 3.1**

If $a = 2b$ the differential system (3.1) becomes

$$\begin{aligned} \dot{x} &= -1 + x^2 + 2by + 2bxy, \\ \dot{y} &= -2b + \frac{y}{2} + \frac{xy}{2} + by^2. \end{aligned} \tag{3.2}$$

This system is invariant under the change $(x, y, t, b) \longrightarrow (x, -y, t, -b)$, then we only need to study systems (3.2) for $b \geq 0$.

The differential system (3.2) has three singularities p_1, p_2 and q_1, such that $p_1 = \left(-1, -\sqrt{2}\right)$, $p_2 = \left(-1, \sqrt{2}\right)$ and $q_1 = (1 - 4b^2, 2b)$.

$\hookrightarrow$ At p_1 the eigenvalues of the matrix of vector field in (3.2) are $\lambda_1 = -2\sqrt{2}b$ and $\lambda_2 = -2(1 + \sqrt{2}b)$.

$\hookrightarrow$ At p_2 the eigenvalues of the matrix of vector field in (3.2) are $\lambda_1 = 2\sqrt{2}b$ and $\lambda_2 = -2 + \sqrt{2}b$.

$\hookrightarrow$ At q_1 the eigenvalues of the matrix of vector field in (3.2) are $\lambda_1 = 1 - 2b^2$ and $\lambda_2 = 2$. Then we have the following result :

- **If** $b \in \left(0, \frac{\sqrt{2}}{2}\right)$, p_1 is a hyperbolic stable node, p_2 is a hyperbolic saddle and q_1 is a hyperbolic unstable node. So the statement (i) holds for $b \in \left(0, \frac{\sqrt{2}}{2}\right)$.
- **If** $b \in \left(\frac{\sqrt{2}}{2}, +\infty\right)$, p_1 has the same nature as the previous case, p_2 is a hyperbolic unstable node and q_1 is a hyperbolic saddle. So the statement (ii) holds for $b \in \left(\frac{\sqrt{2}}{2}, +\infty\right)$.
- **If** $b = 0$ and $a = 0$ the diferential system (3.2) take the from

$$\begin{aligned} \dot{x} &= x^2 - 1, \\ \dot{y} &= \frac{y}{2} + \frac{xy}{2}. \end{aligned} \tag{3.3}$$

This system has a line of singularity $x + 1 = 0$, then we take the change of variable $ds = (x+1)dt$, we get the following system :

$$\begin{aligned} \dot{x} &= x - 1, \\ \dot{y} &= \frac{y}{2}. \end{aligned} \tag{3.4}$$

System (3.4) has a singular point $q_2 = (1, 0)$ and the eigenvalues of its associated matrix are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$, which means that q_2 is a hyperbolic unstable node. So the statement (iii) holds for $b = 0$ and $a = 0$.

- **If** $b = \frac{\sqrt{2}}{2}$ and $a = \sqrt{2}$ the differential system (3.2) becomes

$$\begin{aligned} \dot{x} &= -1 + x^2 + \sqrt{2}y + \sqrt{2}xy, \\ \dot{y} &= -\sqrt{2} + \frac{y}{2} + \frac{xy}{2} + \frac{y^2}{\sqrt{2}}. \end{aligned} \tag{3.5}$$

This system has the singularities p_1 and p_2.

$\hookrightarrow$ At p_1 the eigenvalues of the matrix of vector field in (3.5) are $\lambda_1 = -4$ and $\lambda_2 = -2$. Then p_1 is a hyperbolic stable node.

$\hookrightarrow$ At p_2 the eigenvalues of the matrix of vector field in (3.5) are $\lambda_1 = 0$ and $\lambda_2 = 2$. Then we conclude that p_2 is a semi-hyperbolic singularity. In order to know the local phase portrait of p_2, first must translate p_2 at the origin of coordinates by doing the change $(x, y) = (x_1 - 1, y_1 + \sqrt{2})$, and we get the following system

$$\begin{aligned} \dot{x_1} &= x_1^2 + \sqrt{2}x_1y_1, \\ \dot{y_1} &= \frac{\sqrt{2}}{2}x_1 + 2y_1 + \frac{x_1y_1}{2} + \frac{y_1^2}{\sqrt{2}}, \end{aligned} \tag{3.6}$$

second, we put the system (3.6) into the normal form

$$\begin{aligned} \dot{x} &= p(x, y), \\ \dot{y} &= y + q(x, y). \end{aligned} \tag{3.7}$$

In order to get the normal form of the system (3.6) we need to procedure the following steps. Firstly, we consider the following change of the variable

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.8}$$

where $P = \begin{pmatrix} -4 & 0 \\ \sqrt{2} & 1 \end{pmatrix}$ presents the passage matrix of the linear part of the system (3.6).
By doing the change of the variables $1/2dt = ds$ and by considering the change of the variables (3.8), system (3.6) becomes

$$\begin{aligned} \dot{x} &= \frac{1}{2}(\sqrt{2}xy + 2x^2), \\ \dot{y} &= y + \frac{y^2}{2\sqrt{2}} - xy + \frac{x^2}{\sqrt{2}}. \end{aligned} \tag{3.9}$$

Which is the norml form of the differential system (3.6) where $p(x, y) = \frac{1}{2}(\sqrt{2}xy - 2x^2)$ and $q(x, y) = -xy + \frac{x^2}{\sqrt{2}} + \frac{y^2}{2\sqrt{2}}$.
By solving the equation $y + q(x, y) = 0$ for the variable y we obtain that $y = \psi(x)$, where $\psi(x) = -\frac{x^2}{\sqrt{2}}$.
We substitute the value of $y = \psi(x)$ in the expresion of $p(x, y)$, we get
$\varphi(x) = p(x, \psi(x)) = -2x^2 - x^3 = a_m x^m + \ldots$.
By applying Theorem (2.19) of [20] we know that $m = 2$ and $a_m = -2 < 0$, then the origin is a semi-hyperbolic saddle-node. It result immediately that p_2 is semi-hyperbolic saddle-node. Then the statement (iv) holds for $b = \frac{\sqrt{2}}{2}$ and $a = \sqrt{2}$.

▶ **Proof of statement (II) of Proposition 3.1**

- **If** $a \neq \sqrt{2}$ and $b \neq 0$ the differential system (3.1) has in addition to p_1 and p_2 (mentioned in the first statement) two other singularities p_3 and p_4, with $p_3 = (1 - a^2, a)$ and $p_4 = \left(1 - \frac{2a}{a - 2b}, \frac{2}{a - 2b}\right)$.

↪ At p_1 the eigenvalues of the matrix of vector field in (3.1) are $\lambda_1 = -2\sqrt{2}b$ and $\lambda_2 = -2 - \sqrt{2}a$.

↪ At p_2 the eigenvalues of the matrix of vector field in (3.1) are $\lambda_1 = 2\sqrt{2}b$ and $\lambda_2 = -2 + \sqrt{2}a$.

↪ At p_3 the eigenvalues of the matrix of vector field in (3.1) are $\lambda_1 = -a^2 + 2ab + 2$ and $\lambda_2 = \frac{-a^2}{2} + 1$.

∗ **If** $a \in (0, b_0)$ and $b \in (-\infty, 0)$, p_1, p_2 and p_3 are a hyperbolic saddle, a stable node and an unstable node respectively, where $b_0 = b + \sqrt{b^2 + 2}$. Then the statement (a) holds.

∗ **If** $a \in \left(b_0, \sqrt{2}\right)$ and $b \in (-\infty, 0)$, p_1 and p_3 are two hyperbolic saddles and p_2 is a hyperbolic stable node. Then the statement (b) holds.

∗ **If** $a \in \left(\sqrt{2}, +\infty\right)$ and $b \in (-\infty, 0)$, p_1 and p_2 are two hyperbolic saddles, p_3 is a hyperbolic stable node. Then the statement (c) holds.

* **If** $a \in \left[0, \sqrt{2}\right)$ and $b \in (0, +\infty)$, then p_1, p_2 and p_3 are a hyperbolic stable node, a saddle and an unstable node respectively. Then the statement (d) holds.
* **If** $a \in \left(\sqrt{2}, b_0\right)$ and $b \in (0, +\infty)$, p_1, p_2 and p_3 are a hyperbolic stable node, an unstable node and a saddle respectively. Then the statement (e) holds.
* **If** $a \in (b_0, +\infty)$ and $b \in (0, +\infty)$, p_1 and p_3 are two hyperbolic stable nodes and p_2 is a hyperbolic unstable node. Then the statement (f) holds.

Now we are going to study the local phase portraits of the singuler point p_4.
At p_4 the jacobien matrix of the vector field defined in (3.1) is given by :

$$A = D\chi\Big(\frac{-a-2b}{a-2b}, \frac{2}{a-2b}\Big) = \begin{pmatrix} -\dfrac{4b}{a-2b} & \dfrac{4ab}{a-2b} \\ -\dfrac{a}{2} + \dfrac{1}{a-2b} + b & \dfrac{2b}{a-2b} \end{pmatrix}.$$

The eigenvalues of the matrix are $\lambda_{1,2} = \dfrac{b \pm \sqrt{b(2a(-2+(a-2b)^2)+9b)}}{2b-a}$.
We distinguish the following cases :

* **If** $a \in \left(0, \sqrt{2}\right)$ and $b \in (-\infty, b_1)$, $a \in \left(\sqrt{2}, \dfrac{9}{4}\right)$ and $b \in (-\infty, 0)$ or $a \in \left[\dfrac{9}{4}, +\infty\right)$ and $b \in (-\infty, 0)$, the eingenvalues λ_1 and λ_2 are complex and the real part of them is positive. Then p_4 is an unstable focus, such that $b_1 = \dfrac{-9+8a^2-\sqrt{81-16a^2}}{16a}$ and $b_2 = \dfrac{-9+8a^2+\sqrt{81-16a^2}}{16a}$.
* **If** $a \in \left(0, \sqrt{2}\right)$ and $b \in (0, b_2)$ or $a \in \left(\sqrt{2}, \dfrac{9}{4}\right)$ and $b \in (b_1, b_2)$, the eingenvalues λ_1 and λ_2 are complex and the real part of them is negative. Then p_4 is a stable focus.

For the following cases we have λ_1 and λ_2 are reals and we have $\lambda_1 + \lambda_2 = -\dfrac{2b}{a-2b}$ and $\lambda_1.\lambda_2 = -\dfrac{2b(-2+a^2-2ab)}{a-2b}$.

* **If** $a \in \left(\sqrt{2}, \dfrac{9}{4}\right)$ and $b \in \left(0, \dfrac{a^2-2}{2a}\right) \cup \left(\dfrac{a}{2}, +\infty\right)$, $a = \dfrac{9}{4}$ and $b \in \left(0, \dfrac{49}{72}\right) \cup \left(\dfrac{9}{8}, +\infty\right)$, $a \in \left(0, \sqrt{2}\right)$ and $b \in (b_2, +\infty)$, or $a = 0$, then $\lambda_1.\lambda_2 < 0$. So p_4 is a saddle.
* **If** $a \in \left(0, \sqrt{2}\right)$ and $b \in (b_1, 0)$, then $\lambda_1.\lambda_2 > 0$ and $\lambda_1 + \lambda_2 > 0$. So p_4 is an unstable node.
* **If** $a \in \left(\sqrt{2}, \dfrac{9}{4}\right)$ and $b \in \left(\dfrac{a^2-2}{2a}, b_1\right) \cup \left(b_2, \dfrac{a}{2}\right)$, $a \in \left(\dfrac{9}{4}, +\infty\right)$ and $b \in \left(\dfrac{a^2-2}{2a}, \dfrac{a}{2}\right)$ or $a = \dfrac{9}{4}$ and $b \in \left(\dfrac{49}{72}, \dfrac{9}{8}\right)$, then $\lambda_1.\lambda_2 > 0$ and $\lambda_1 + \lambda_2 < 0$. So p_4 is a stable node. Then the statement (i) holds.

- **If** $a \neq \sqrt{2}$ and $b = 0$ the differential system (3.1) becomes

$$\begin{aligned} \dot{x} &= -1 + x^2 + ay + axy, \\ \dot{y} &= \frac{1}{2}(-a - ax + y + xy). \end{aligned} \tag{3.10}$$

The differential system (3.10) has a line of singularity $x+1=0$.
Then, we take the change of variable $ds=(x+1)dt$, we get

$$\begin{aligned} \dot{x} &= (1-x)+ay, \\ \dot{y} &= \frac{y-a}{2}. \end{aligned} \tag{3.11}$$

This system has a singular point p_3 with the eigenvalues of its associated matrix are $\lambda_1 = 2-a^2$ and $\lambda_2 = \frac{1}{2}(2-a^2)$. Then p_3 is a hyperbolic unstable node if $a \in \left[0, \sqrt{2}\right)$ and a hyperbolic stable node if $a \in \left(\sqrt{2}, +\infty\right)$. So the statement (ii) holds for $a \neq \sqrt{2}$ and $b=0$.

- **If** $a=\sqrt{2}$ the differential system (3.1) becomes

$$\begin{aligned} \dot{x} &= -1+x^2+\sqrt{2}y+\sqrt{2}xy, \\ \dot{y} &= -\tfrac{1}{\sqrt{2}}-b+(b-\tfrac{1}{\sqrt{2}})x+\frac{y}{2}+\frac{xy}{2}+by^2. \end{aligned} \tag{3.12}$$

This system has three singularities p_1, p_2 and q_3, with $q_3 = \left(1+\dfrac{2}{\sqrt{2}b-1}, \dfrac{\sqrt{2}}{1-\sqrt{2}b}\right)$.

$\hookrightarrow$ At p_1 the eigenvalues of the matrix of vector field in (3.12) are $\lambda_1 = -4$ and $\lambda_2 = -2\sqrt{2}b$. So it's a hyperbolic saddle if $b \in (-\infty, 0)$ and a hyperbolic stable node if $b \in \mathbb{R}_*^+ \setminus \{\frac{7}{8\sqrt{2}}, \frac{\sqrt{2}}{2}\}$.

$\hookrightarrow$ At p_2 the eigenvalues of the matrix of vector field in (3.12) are $\lambda_1 = 2\sqrt{2}b$ and $\lambda_2 = 0$. So it's a semi-hyperbolic if $b \in \mathbb{R} \setminus \{0, \frac{7}{8\sqrt{2}}, \frac{\sqrt{2}}{2}\}$, now we are going to give the nature of semi-hyperbolic singuler point p_2 in the following steps. In order to know the local phase portrait of p_2, first we must translate it at the origin of coordinates by doing the change $(x,y)=(x_1-1, y_1+\sqrt{2})$, and we get the following system

$$\begin{aligned} \dot{x_1} &= x_1^2+\sqrt{2}x_1y_1, \\ \dot{y_1} &= bx+2\sqrt{2}by+by^2+\frac{xy}{2}, \end{aligned} \tag{3.13}$$

second,we put the system (3.13) into the normal form

$$\begin{aligned} \dot{x} &= p(x,y), \\ \dot{y} &= y+q(x,y). \end{aligned} \tag{3.14}$$

In order to get the normal form of the system (3.13) we need to procedure the following steps.
First, we consider the following change of the variable

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.15}$$

where $P = \begin{pmatrix} -4 & 0 \\ \sqrt{2} & 1 \end{pmatrix}$ presents the passage matrix of the linear part of the system (3.13).
By doing the change of the variable $1/2dt = ds$ and by considering the change of the variable (3.15), system (3.13) becomes

$$\begin{aligned}\dot{x} &= \frac{xy}{2b} - \frac{x^2}{\sqrt{2}b},\\ \dot{y} &= y + \frac{1}{2\sqrt{2}b}(by^2 - 4xy + 2\sqrt{2}bxy + 2bx^2).\end{aligned} \tag{3.16}$$

Which is the norml form of the differential system (3.13), where $q(x,y) = \dfrac{xy}{2b} - \dfrac{x^2}{\sqrt{2}b}$ and $q(x,y) = \dfrac{1}{2\sqrt{2}b}(by^2 - 4xy + 2\sqrt{2}bxy + 2bx^2)$.

By solving the equation $y + q(x,y) = 0$ for the variable y we obtain that $y = \psi(x)$, where $\psi(x) = \dfrac{-x^2}{\sqrt{2}}$.

We substitute the value of $y = \psi(x)$ in the expression $p(x,y)$, we get

$\varphi(x) = p(x,\psi(x)) = -\dfrac{x^2}{\sqrt{2}b} - \dfrac{x^3}{2\sqrt{2}b} = a_m x^m + \ldots$

By applying Theorem (2.19) of [20] we know that $m = 2$ and $a_m = -\dfrac{1}{\sqrt{2}b}$, then origin is semi-hyperbolic saddle-node.

It result immediately that p_2 is semi-hyperbolic saddle-node.

↪ At q_3 the eigenvalues of the matrix of vector field in (3.12) are $\lambda_{1,2} = \dfrac{-b \pm \sqrt{b^2(8\sqrt{2}b-7)}}{\sqrt{2}-2b}$.

we have $\lambda_1.\lambda_2 = \dfrac{b^2(8-8\sqrt{2}b)}{(\sqrt{2}-2b^2-2b)^2}$, and $\lambda_1+\lambda_2 = \dfrac{-2b}{\sqrt{2}-2b}$.

* **If** $b \in (-\infty, 0)$, the real part of λ_1, λ_2 negative. Then q_3 is a hyperbolic unstable focus.
* **If** $b \in \left(0, \dfrac{7}{8\sqrt{2}}\right)$, the real part of λ_1, λ_2 positive. Then q_3 is a hyperbolic stable focus.

For the following cases we have λ_1 and λ_2 are reals.

* **If** $b \in \left(\dfrac{7}{8\sqrt{2}}, \dfrac{\sqrt{2}}{2}\right)$, we have $\lambda_1.\lambda_2 > 0$ and $\lambda_1+\lambda_2 < 0$. Then q_3 is a hyperbolic stable node.
* **If** $b \in \left(\dfrac{\sqrt{2}}{2}, +\infty\right)$, we have $\lambda_1.\lambda_2 < 0$. Then q_3 is a hyperbolic saddle.

Abstract in summary . So the statements (iv), (v) and (vi) holds.

- **If** $a = \sqrt{2}$ and $b = 0$ the differential system (3.1) becomes

$$\begin{aligned}\dot{x} &= (1+x)(-1+x+\sqrt{2}y),\\ \dot{y} &= -\frac{1}{2}(1+x)(\sqrt{2}-y).\end{aligned} \tag{3.17}$$

This system has a line of singularity $x+1=0$.

Then, we take the change of variable $ds = (x+1)dt$, we get the following system

$$\begin{aligned}\dot{x} &= -1+x+\sqrt{2}y,\\ \dot{y} &= -\frac{1}{2}(\sqrt{2}-y).\end{aligned} \tag{3.18}$$

which has the singular point p_2 with its corresponded eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \dfrac{1}{2}$. So p_2 is a hyperbolic unstable node. Then the statement (vii) holds.

- If $a = b_0$ with $b_0 = b + \sqrt{b^2+2}$ the differential system (3.1) becomes

$$\begin{aligned}\dot{x} &= -1 + x^2 + (b+\sqrt{2+b^2})y + (b+\sqrt{2+b^2})xy,\\ \dot{y} &= -b - \frac{1}{2}(b+\sqrt{2+b^2}) + (b - \frac{1}{2}(b+\sqrt{2+b^2}))x + \frac{y}{2} + \frac{xy}{2} + by^2.\end{aligned} \tag{3.19}$$

This system has three singularities p_1, p_2 and q_4 where
$q_4 = \left(-1-2b(b+\sqrt{2+b^2}, b+\sqrt{2+b^2}\right)$.

$\hookrightarrow$ At p_1 the eigenvalues of the matrix of vector field in (3.19) are $\lambda_1 = -2\sqrt{2}b$ and $\lambda_2 = -2-\sqrt{2}(b+\sqrt{2+b^2})$. So it's a hyperbolic saddle if $b \in (-\infty, 0)$ and a hyperbolic stable node if $b \in (0, +\infty)$.

$\hookrightarrow$ At p_2 the eigenvalues of the matrix of vector field in (3.19) are $\lambda_1 = 2\sqrt{2}b$ and $\lambda_2 = -2+\sqrt{2}(b+\sqrt{2+b^2})$. So it's a hyperbolic stable node if $b \in (-\infty, 0)$ and a hyperbolic unstable node if $b \in (0, +\infty)$.

$\hookrightarrow$ At q_4 the eigenvalues of the matrix of vector field in (3.19) are $\lambda_1 = 0$ and $\lambda_2 = -b(b+\sqrt{2+b^2})$. So it's a semi-hyperbolic singular point.
By applying Theorem (2.19) of [20] we know that $m = 2$ is even, then the origin is semi-hyperbolic saddle-node. It result immediately that q_4 is semi-hyperbolic saddle-node. Then the statements (viii) and (ix) holds

3.4 Infinite singular points

The main goal of this part is to give the local phase portraits of system (3.1) at its infinite singular points. To give a full study about the infinite singular points in the Poincaré disc we present the analysis of the vector field at infinity.

Proposition 3.2 *The local phase portraits at the infinite singular points of system (3.1) in the local chart U_1 consists of*

(i) Two singular points : first singularity is a hyperbolic stable node at $q_1 = (0,0)$ and the second singularity is $q_2 = \left(\frac{1}{2(b-a)}, 0\right)$, such that $a \neq b$, which is a hyperbolic unstable node if $b \in (a/2, a)$, a hyperbolic saddle if $b \in (-\infty, a/2) \cup (a, +\infty)$, and a semi-hyperbolic saddle-node if $a = 2b$.

(ii) One singular point at $q_1 = (0,0)$, which is a stable node, for $a = b$.

Now, we give the local phase portaits of the origin of the local chart U_2.

Proposition 3.3 *The origin of the local chart U_2 is :*

(i) An unstable node if $b \in (-\infty, 0)$, a saddle if $b \in (0, a)$ and a stable node if $b \in (a, +\infty)$.

(ii) Not a singularity if $b = 0$ and $a \neq 0$.

(iii) A semi-hyperbolic saddle-node if $a = b$ and $b \neq 0$.

(iv) *linear zero singularity where its local phase portait consists of four hyperbolic sectors if* $a = 0$ *and* $b = 0$.

Proof of Proposition (3.2)

The expression of system (3.1) in the local chart U_1 is given by

$$\begin{aligned}\dot{u} &= (1/2)(-2u^2(a-b+av)+u(-1+v+2v^2)-\\ &\quad v(2b(-1+v)+a(1+v)))\\ \dot{v} &= -v(1+au-v)(1+v).\end{aligned} \tag{3.20}$$

Any arbitrary infinite singular points of differential system (3.20) take the forme $(u_0, 0)$.

► **Proof of statement** (i) **of Proposition** (3.2)

- **If** $a \neq b$ the differential system (3.20) has two singular points q_1 and q_2, such that $q_1 = (0,0)$ and $q_2 = \left(\dfrac{1}{2(b-a)}, 0\right)$.

  The eigenvalues of q_1 for the matrix of vector field in (3.20) are $\lambda_1 = -1$ and $\lambda_2 = -\dfrac{1}{2}$. So the origin is a stable node.

  The Jacobian matrix of vector field in (3.20) at q_2 is given by :

$$D\chi(\frac{1}{2(b-a)}, 0) = \begin{pmatrix} \frac{1}{2} & a(-\frac{1}{2} - \dfrac{1}{(4(a-b)^2}) - \dfrac{1}{(4(a-b)} + b \\ 0 & -1 + \dfrac{a}{(2a-2b)} \end{pmatrix}.$$

  The eigenvalues of this matrix are $\lambda_1 = \dfrac{1}{2}$ and $\lambda_2 = -\dfrac{(a-2b)}{2(a-b)}$. So q_2 is an unstable node if $b \in (a/2, a)$, a saddle if $b \in (-\infty, a/2) \cup (a, +\infty)$ and a semi-hyperbolic singularity if $a = 2b$. In order to know the local phase portrait at q_2, first must translate it at the origin of coordinates by doing the change $(u,v) = (u_1 - \dfrac{1}{2b}, v_1)$, we get the following system

$$\begin{aligned}\dot{u_1} &= \frac{1}{2}(u_1 - \frac{3v_1}{2b} - 2bu_1^2(1-2v_1) + 5bu_1v_1 + (2u_1 - 4b - \frac{1}{b})v_1^2),\\ \dot{v_1} &= -2bu_1v_1 + v_1^2 - 2bu_1v_1^2 + v_1^3,\end{aligned} \tag{3.21}$$

  second,we put the system (3.21) into the normal form

$$\begin{aligned}\dot{u} &= p(u,v),\\ \dot{v} &= v + q(u,v).\end{aligned} \tag{3.22}$$

  In order to get the normal form of the system (3.21) we need to procedure the following steps.

  First, we consider the following change of the variable

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.23}$$

  where $P = \begin{pmatrix} 1 & 3 \\ 0 & 2b \end{pmatrix}$ presents the passage matrix of the linear part of the system (3.21).

By doing the change of the variable $\frac{1}{2}dt = ds$ and by considering the change of the variable (3.23), system (3.21) becomes

$$\begin{aligned}\dot{u} &= -8bu^2 - 16b^2u^3 - 4buv - 8b^2u^2v,\\ \dot{v} &= v + 32bu^2 - 16b^3u^2 + 10buv - 16b^2u^2v - 2bv^2 - 8b^2uv^2.\end{aligned} \tag{3.24}$$

Which is the normal form of the differential system (3.21), where $p(u,v) = -8bu^2 - 16b^2u^3 - 4buv - 8b^2u^2v$ and $q(u,v) = 32bu^2 - 16b^3u^2 + 10buv - 16b^2u^2v - 2bv^2 - 8b^2uv^2$.

By solving the equation $v + q(u,v) = 0$ for the variable v we obtain that $v = \psi(u)$, where $\psi(u) = 32bu^2 - 16b^3u^2$.

We substitute the value of $v = \psi(u)$ in the expresion $p(u,v)$, we get $\varphi(u) = p(u,\psi(u)) = -8bu^2 - 144b^2u^3 + 64b^4u^3 - 256b^3u^4 + 128b^5u^4 = a_m x^m + \ldots$

By applying Theorem (2.19) of [20] we know that $m = 2$and $a_m = -8b$, then origin is semi-hyperbolic saddle-node. It result immediately that q_2 is semi-hyperbolic saddle-node. Then the statement (i) holds.

► **Proof of statement (ii) of Proposition (3.2)**

- **If** $a = b$ the differential system (3.20) becomes

$$\begin{aligned}\dot{u} &= (1/2)(-2au^2v + u(-1 + v + 2v^2) - av(3v - 1)),\\ \dot{v} &= -v(1 + au - v)(1 + v).\end{aligned} \tag{3.25}$$

This system has one singular point at the origin of coordinates which is a hyperbolic stable node. Then the statement (ii) holds.

Proof of Proposition (3.3)

The differential system (3.1) in the local chart U_2 takes the forme :

$$\begin{aligned}\dot{u} &= \tfrac{1}{2}((u - 2v)(u + v) + a(u + v)(2 + uv) + 2bu(-1 - uv + v^2)),\\ \dot{v} &= \tfrac{1}{2}v((u + v)(-1 + av) - 2b(1 + uv - v^2)).\end{aligned} \tag{3.26}$$

The origin is a singular point of this system.

The eigenvalues of $(0,0)$ for the matrix of vector field in (3.26) are $\lambda_1 = -b$ and $\lambda_2 = a - b$. So the origin is an unstable node if $b \in (-\infty, 0)$, a saddle if $b \in (0, a)$, and a stable node if $b \in (a, +\infty)$. Then the statement (i) holds.

- **If** $a \neq 0$ and $b = 0$ the differential system (3.26) becomes

$$\begin{aligned}\dot{u} &= (u + v)(u - 2v + a(2 + uv)),\\ \dot{v} &= \frac{1}{2}v(u + v)(-1 + av).\end{aligned} \tag{3.27}$$

This system has $u + v = 0$ as a line of singularity.

Then, we take the change of variable $(u + v)dt = ds$, we get

$$\begin{aligned}\dot{u} &= u - 2v + a(2 + uv),\\ \dot{v} &= \frac{1}{2}v(-1 + av).\end{aligned} \tag{3.28}$$

It is clear that the origin is not a singularity for this system. Then the statement (ii) holds.

- If $a = b$ and $b \neq 0$ the differential system (3.26) becomes

$$\dot{u} = \frac{1}{2}(u^2 - uv - 2v^2 - 2bu(1 + uv - v^2) + b(2v + u^2v + u(2 + v^2))),$$
$$\dot{v} = \frac{1}{2}v((u + v)(bv - 1) - 2b(1 + uv - v^2)). \tag{3.29}$$

The origin is a singular point of this system.
The eigenvalues of its corresponded matrix are $\lambda_1 = 0$ and $\lambda_2 = -b$, which means that the origin is a semi-hyperbolic singularity.
We should drive the system (3.29) into its normal form

$$\begin{aligned} \dot{u} &= p(u, v), \\ \dot{v} &= v + q(u, v). \end{aligned} \tag{3.30}$$

To do that we need to use the following change of variable

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.31}$$

where $P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ presents the passage matrix of the linear part of the system (3.29).
Now we use the change of variable $-bdt = ds$ and the change of variable (3.31) we get the following system

$$\begin{aligned} \dot{u} &= \frac{2}{b}uv - u^2v - \frac{u^2}{2b} + \frac{u^2v}{2}, \\ \dot{v} &= v - 2u^3 + \frac{uv}{2b} + \frac{vu^2}{2}, \end{aligned} \tag{3.32}$$

which it's the normal form where $p(u, v) = \frac{2}{b}uv - u^2v - \frac{u^2}{2b} + \frac{u^2v}{2}$ and $q(u, v) = -2u^3 + \frac{uv}{2b} + \frac{vu^2}{2}$.
By solving $v + q(u, v) = 0$ for the variable v we obtain $v = \psi(u)$, where $\psi(u) = 0$.
We substitute the value of $v = \psi(u)$ in the expresion $p(u, v)$, we get
$\varphi(u) = p(u, \psi(u)) = \frac{-u^2}{2b} = a_m x^m + \ldots$
By applying Theorem (2.19) of [20] we obtain $m = 2$ and $a_m = \frac{-1}{2b}$, so origin is a semi-hyperbolic saddle-node. Then the statement (iii) holds.

- If $a = 0$ and $b = 0$, the differential system (3.26) becomes

$$\begin{aligned} \dot{u} &= \tfrac{1}{2}(u^2 - uv - 2v^2), \\ \dot{v} &= \tfrac{1}{2}(-u - v)v. \end{aligned} \tag{3.33}$$

The origin is a linearly zero singular point.
We need to do the blow-up to describe the local behavior at this point. We perform the directional blow-up $(u, v) \longrightarrow (u, w)$, with $w = \frac{v}{u}$ and we get the following system :

$$\begin{aligned} \dot{u} &= -\frac{1}{2}u^2(-1 - w + 2w^2), \\ \dot{w} &= uw(w^2 - 1). \end{aligned} \tag{3.34}$$

We eliminate the common factor u between $\dot{u}$ and $\dot{w}$ by doing a rescaling of the independent variable $udt = ds$ then we get

$$\begin{aligned} \dot{u} &= -\frac{1}{2}u(-1 - w + 2w^2), \\ \dot{w} &= w(w^2 - 1). \end{aligned} \tag{3.35}$$

Going back through the change of variables $w = \dfrac{v}{u}$ and $udt = ds$, we conclud that the local phase portrait of the origin consists of four hyperbolic sectors. Than the statement (iv) holds.

Conclusion

In our work we ag to study the global phase portraits in the Poincaré disc of two quadratic polynomial differential systems .

We construct a new class of quadratic polynomial differential systems with three parametres a, b and c. These differential systems exhibit an invariant algebraic curve of the degree six, we classify all the phase portraits of these systems and we obtain **sixteen** phase portraits topologically non equivalent.

In the other hand we give another new class of integrable quadratic differential systems with two parameters a and b. Such systems have a cubic algebraic curve ***"Cubic egg"*** as a first integral, and we realize that these differential systems have **eighteen** phase portraits topologically non equivalent. re interestin

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