Université Mohamed El Bachir El Ibrahimi de Bordj Bou Arréridj
Faculté des Mathématiques et de l'Informatique
Département des Mathématiques

Mémoire

Présenté par

Tababouchet Ines et Saidani Sara

Pour l'obtention du diplôme de

Master

Filière : Mathématiques
Spécialité : Systémes Dynamiques

Thème

On the limit cycles of some classes of kolmogorov differential systems

---

Soutenu publiquement le 05 septembre 2020 devant le jury composé de

Salhi Tayeb	Président
Berbache Aziza	Encadrant
Ghermoul Bilal	Examinateur

Promotion 2019/2020

Acknowledgement

We thanks all who in one way or another contributed in completion of this work. Firstly our heartfelt thanks are addressed to Allah Almighty who giving us strength, desire and courage to accomplish this modest work and protected us during our research and my whole period of us life especial during us studies.

Special and warmest thanks to our supervisor, "Dr. A. Barbash" for her tireless encouragement, motivational criticism, guidelines and advice to make this work more meaningful.

We wish to show our gratitude to our respected teachers of mathematics department , BBA university for their precious pieces of advice that guided us during our studies and for their in valuable help in preparing this work.

Finally, we wish to express our deepest gratitude to to our families for their considerable either material or moral support to the completionof this work particularly and to the achievement of our studies at large and getting at this level throughout five years of studies at the university and our thankfulness to every one participated in the realization of this dissertation.

May God bless all of them.

Contents

List of Figures

Introduction

Differential equations were invented by ***Newton*** (1642-1727). It is the beginning of modern physics and the use of analysis to solve the law of universal gravitation leading to the ellipsity of the orbits of the planets in the solar system. ***Leibniz*** (1646-1716) erects analysis in autonomous discipline but it is necessary to await the work of ***Euler*** (1707-1783) and ***Lagrange*** (1736-1813) to see appear the methods allowing the resolution of the linear equations.

The significant development of the theory of dynamical systems during this century has helped to develop methods of studying the properties of their solutions.The direct solution to the differential system is usually difficult or impossible. Since Andronov (1932), three different approaches have traditionally been used to study dynamic systems: qualitative methods, analytical methods and numerical methods the most important of which are qualitative methods.

One of the main problems in **the qualitative theory** of differential equations is the study of **the integrability (the first integrals)** and **the limit cycles** of polynomial planar differential systems.

The notion of first integral appeared for the first time in the work of ***G. Darboux*** (1842-1917) [7] in 1878. He built so-called general integrals for ordinary first order differential equations, having on many invariant algebraic curves.

Concerning the limit cycles, ***Henri Poincare*** is the first who speaks about there in his second thesis "on curves definied by a differential equation" [14],[15],[16],[17].

On August 8, 1900, at the Second International Congress of Mathematicians meeting in Paris, ***David Hilbert*** profoundly changed the face of mathematics. And yet, that day, he did not announce any new theorem, no result. Nothing of the sort. On the contrary, that very day, ***Hilbert*** posed 23 problems for the mathematician community [11]. These problems have been the driving force of much research throughout the last century. In problem 16, he raises the question of the number and the arrangement of limit cycles noted $H(n)$ of a planar polynomial differential system of degree n. In 1923, ***Dulac*** [8] proposed a resolution makes it possible to prove that the polynomial vector field has a finite number of limit cycles n.

In biology, the first model with a limit cycle is that introduced in 1936 by the Russian mathematician ***Andrei Kolmogorov*** (1903-1987) who bears his name and has the form.

$$\begin{cases} \dot{x} = xf(x,y), \\ \dot{y} = yg(x,y), \end{cases}$$

where f and g are polynomials. There are works interested in studying the existence, the number and the stability of the cycles limite to the systems Kolmogorov.

In our work, we will use **the qualitative theory** of ordinary differential equations to treat some classes of **Kolmogorov**. This work has been structured in three chapters:

The first chapter is dedicated to reminders of some preliminary notions on the planar differential systems.

In the second chapter, in the first part we study the nonexistence of an algebraic limit cycle for kolmogorv system of degree j which has an invariant algebraic curve of degree $j-1$ in the quadrant $\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$. In the second part we give sufficient conditions for the existence of hyperbolic algebraic limit cycles for certain classes of kolmogorov systems.

In third chapter, we study the integrability and the existence of limit cycles of a class of 2-dimensional Kolmogorov systems and we determine sufficient conditions for a polynomial differential system to possess an explicit non-algebraic or algebraic limit cycle.

Chapter 1

Preliminaries

1.1 Introduction

This chapter contains some basic concepts for the qualitative study of dynamic systems. To understand this chapter, we start by definition of polynomial differential systems, we will discuss the notions of: vector field, phase portrait, solution and periodic solution, limit cycle, nature of the critical points. We also quote some theorems used as tools in our work. Most part of the results are given without proof, however references where they can found , are included.

1.2 Polynomial differential systems

Definition 1.2.1 *A planar polynomial differential system (or simply a polynomial system) is a differential system of the form*

$$\begin{cases} \dfrac{dx}{dt} = \dot{x} = P(x,y) \\ \dfrac{dy}{dt} = \dot{y} = Q(x,y) \end{cases} \tag{1.1}$$

or equivalent

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)},$$

where P and Q are polynomials in the variables x and y.

-The degree n of the polynomial system (1.1) *will be the maximum of the degrees of the polynomials P and Q.*

-A differential system (1.1) *is said autonomous if the function P and Q depends only on the vector variable (x,y) . Otherwise ,it is non-outonomous .*

-If P and Q are homogeneous,in this case the system (1.1) *is called homogeneous polynomial differential system.*

1.3 Vector Field

Definition 1.3.1 *The region of the plane in which there exists at any point a vector* $\overrightarrow{V}(M,t)$ *is called a vector field.*
We assume that we have given a vector field of the class C^1 *in an open* $\Omega \in \mathbb{R}$, *that is to say an application*

$$M=(x,y)\rightarrow \overrightarrow{V}(M)=\begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix},$$

where P *and* Q *are class* C^1 *on* Ω, *we can also write*

$$X=P(x,y)\frac{\partial}{\partial x}+Q(x,y)\frac{\partial}{\partial y}.$$

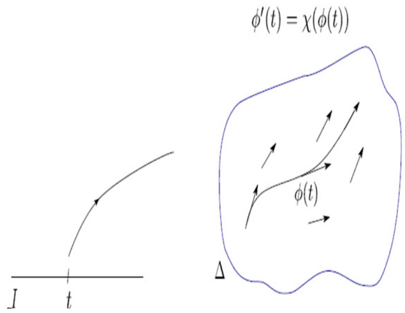


Figure 1.1: Vector field

1.4 Solution and periodic solution

Definition 1.4.1 *We say that* $(x(t),y(t))_{t\in\mathbb{R}}$ *is a solution of system* (1.1) *if the vector Field* $X=(P,Q)$ *is always tangent to the trajectory representing this solution in the phase plane.*

Definition 1.4.2 *Called periodic solution of system* (1.1) *all solution* $(x(t),y(t))$ *for which there exists a real* $T>0$ *such that:*

$$\forall t\in\mathbb{R},\;\; x(t+T)=x(t) \;\; and \;\; y(t+T)=y(t).$$

The smallest number $T>0$ *is called the periodic of this solution.*

1.5 Phase portrait

The plane $\mathbb{R}^2$ called phase plane and the solution of a vector field X, represent in the phase plane of the orbits or the trajectory, the phase portrait of a vector field X is the set of the solutions in the phase plane.

Definition 1.5.1 *A phase portrait is a geometric representation of the trajectory of a dynamic system in the phase space, at each set of initial conditions corresponds a curve or a point.*

Example 1.5.1 *we consider the system:*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 2y(1 - x^2) - x, \end{cases} \tag{1.2}$$

The phase portrait of this system :

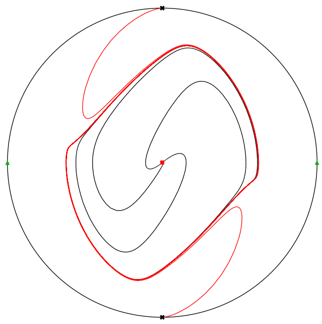


Figure 1.2: The phase portrait of system

1.6 Equilibrium point

The fixed points or equilibrium points play a vital role in the study of dynamic system, Henri Poincare $(1854 - 1912)$ showed that to characterize a dynamic system with multiple variables it is not necessary to calculate the detailed solution, it is enough to know equilibrium points and their stability.

Definition 1.6.1 *A point* $(x^*, y^*) \in \mathbb{R}^2$ *is called an equilibrium point (singular point, critical point) of system* (1.1) *if* $(P(x^*, y^*) = 0, Q(x^*, y^*) = 0)$.

1.6.1 Linearization of polynomial system

The most nature approach to study the behavior of the trajectories of a nonlinear autonomous differential system, in the neighborhood of a singular point, consist in to bring back to the associated linear system, then to make the link between the trajectories of the two system.

Remark 1.6.1 *At a critical point* (x^*, y^*) *of the system* (1.1) *we define the Jacobian by*

$$A = J(x^*, y^*) = \begin{pmatrix} \frac{dP}{dx}(x^*, y^*) & \frac{dP}{dy}(x^*, y^*) \\ \frac{dQ}{dx}(x^*, y^*) & \frac{dQ}{dy}(x^*, y^*) \end{pmatrix}.$$

Definition 1.6.2 *The system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dfrac{dP}{dx}(x^*, y^*) & \dfrac{dP}{dy}(x^*, y^*) \\ \dfrac{dQ}{dx}(x^*, y^*) & \dfrac{dQ}{dy}(x^*, y^*) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.3}$$

is called the linearization of system (1.1) *on* (x^*, y^*).

Example 1.6.1 *we consider the system (1.2):*
It is clear that $\dot{x} = 0, \dot{y} = 0$*; entails that* $(x^*, y^*) = (0,0)$ *is the only point of equilibrium of this system.*
We are looking for the linearized of this system in $(0,0)$

$$J(x^*, y^*) = \begin{pmatrix} 0 & 1 \\ -1 - 4x^*y^* & 2 - 2(x^*)^2 \end{pmatrix}$$

Thus

$$J(x^*, y^*) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

So the linearization of system (1.2) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x + 2y \end{pmatrix}$$

Definition 1.6.3 *An equilibrium point* (x^*, y^*) *is called hyperbolic equilibrium point of* (1.1) *if the matrix* $A = J(x^*, y^*)$ *has no eigenvalue with zero real part. Otherwise, the singular point is said to be non-hyperbolic.*

Example 1.6.2 *The equilibrium point* $(0,0)$ *of the system (1.2) is hyperbolic because the eigenvalues of the matrix* $J(0,0)$ *is* $\lambda_1 = \lambda_2 = 1$.

1.6.2 Classification of equilibrium points

Consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

$A \in \mathbb{M}_2(\mathbb{R})$ is an invertible matrix.
Let λ_1, λ_2 two eigenvalues of A and (x^*, y^*) is an equilibrium point of $x' = Ax$.

Definition 1.6.4

- *If the eigenvalues* λ_1, λ_2 *are real and of the same sign, the equilibrium point* (x^*, y^*) *is called node.*

- *If the eigenvalues* λ_1, λ_2 *are real, non-zero and of different sign, the point* (x^*, y^*) *is called a saddle.*
- *If the eigenvalues* λ_1, λ_2 *are complex with* $Im(\lambda_1, \lambda_2) \neq 0$*, and* $Re(\lambda_1, \lambda_2) \neq 0$*the equilibrium point* (x^*, y^*) *is called the focus.*
- *If the eigenvalues* λ_1, λ_2 *are complex with* $Im(\lambda_1, \lambda_2) \neq 0$ *and* $Re(\lambda_1, \lambda_2) = 0$*;the equilibrium point* (x^*, y^*) *is called the center.*

1.6.3 The Hartman-Grobman Theorem

This theorem allows us to reduce the study of a dynamic system (1.1) in the vicinity of a hyperbolic singular point, to the study of a linear system (1.3) topologically equivalent to (1.1), near the origin. It is very useful in practice and generally facilitates the study of dynamic systems defined on an open plane. It sets out below (see [15] for demonstration).

Theoreme 1.6.1 *Suppose that the Jacobian matrix at the point* (x^*, y^*) *has tow eigenvalues such as* $Re(\lambda_{1,2}) \neq 0$*; then the solutions of the system* (1.1) *are given approximately by the solution of the system linearized* (1.3) *near equilibrium point. In other words, the portrait phase system linearized* (1.3) *is near this equilibrium point, a good approximation of the system* (1.1).

Remark 1.6.2 *In the case where* $Re(\lambda_{1,2}) = 0$ *this method of linearization not working, for example if the equilibrium point* (x^*, y^*) *is a center for the system linearity* (1.3)*, the determination of its kind in the case of the system* (1.3) *requires further investigations, that's the problem from the center.*

1.6.4 Stability of equilibrium point

Any non-linear system can have several equilibrium positions which can be stable or unstable, in some situations the stability of equilibrium required which is defined as follows:
Let (x^*, y^*) be a point of equilibrium of system (1.1) denote by

$$X^* = (P(x^*, y^*), Q(x^*, y^*)).$$

Definition 1.6.5 *Let* $X(t) = (x(t), y(t))$ *the solution of the differential system* (1.1) *defined for all* $t \in \mathbb{R}$.

- *An equilibrium point* $X^* = (x^*, y^*)$ *of system* (1.1) *is stable if*
$$\forall\ \varepsilon > 0, \exists\ \sigma > 0, \| (x, y) - (x^*, y^*) \| < \sigma \Longrightarrow (\forall\ t > 0 : \| X(t) - X^* \| < \varepsilon).$$
- *An equilibrium point* $X^* = (x^*, y^*)$ *of system* (1.1) *is asymptotically stable if it is stable and there exist a* $\sigma > 0$ *such that if* $\| (x, y) - (x^*, y^*) \| < \sigma$*, then*
$$\lim_{t\to\infty} \| X(t) - X^* \| = 0.$$
- *An equilibrium point* $X^* = (x^*, y^*)$ *of system* (1.1) *is unstable if it is not stable.*

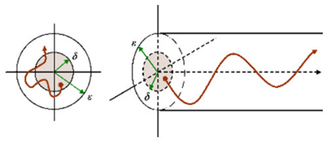


Figure 1.3: Stability of equilibrium point

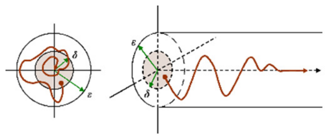


Figure 1.4: Asymptotic stability

1.7 Invariant curve

The invariant algebraic curve play an important role in the integrability of differential planar polynomial system, and are also used in the study of the existence and non-existence of periodic solution and consequently the existence and non-existence of limit cycle.

Definition 1.7.1 *Called invariant curve of the system* (1.1) *any curve of equation* $U(x, y) = 0$ *of the phase plane for which there exists a function* $K = K(x, y)$ *called the cofactor associated with the invariant curve, such that*

$$P(x,y)\frac{\partial U(x,y)}{\partial x} + Q(x,y)\frac{\partial U(x,y)}{\partial y} = K(x,y)U(x,y). \tag{1.4}$$

Example 1.7.1 *The curve defined by the equation* $x^2+y^2 = 2$ *is an invariant curve for the system*

$$\begin{cases} \dot{x} = x^2 + y^2 + 2y - 2, \\ \dot{y} = x^2 + y^2 - 2x - 2, \end{cases} \tag{1.5}$$

if we pose $U(x, y) = x^2 + y^2 - 2 = 0$, *then we have*

$$\begin{aligned} P\frac{\partial U}{\partial x} + Q\frac{\partial U}{\partial y} &= 2x(x^2 + y^2 + 2y - 2) + 2y(x^2 + y^2 - 2x - 2) \\ &= (2x + 2y)(x^2 + y^2 - 2), \end{aligned}$$

thus $K(x, y) = 2x + 2y$ *is the cofactor associated.*

Definition 1.7.2 *An invariant curve* $U(x, y) = 0$ *is said to be algebraic of degree* d *if* $U(x, y)$ *is a polynomial of degree* d. *If not it is said that is non-algebraic.*

Remark 1.7.1 *In the case where the system* (1.1) *is polynomial and has an algebraic invariant curve* $U(x, y) = 0$ *of degree* d, *the cofactor is also algebraic and its degree*

satisfies $deg(K) \leq d-1$.
We recall that the notion div is the divergence of the system (1.1)*, where*

$$div\,(P,Q) = \frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}.$$

Proposition 1.7.1 *Suppose* $U \in \mathbb{C}[x,y]$ *and let* $U = U_1^{n_1}...U_r^{n_r}$ *be its factorization into irreducible factors over* $\mathbb{C}[x,y]$.
Then for a polynomial system (1.1)*,* $U = 0$ *is an invariant algebraic curve with cofactor* k_u *if and only if* $U_i = 0$ *is an invariant algebraic curve for each* $i = 1, ..., r$ *with cofactor* k_{u_i}*. Moreover* $k_u = n_1 k_{u_1} + ... + n_r k_{u_r}$.

Theoreme 1.7.1 *Consider* $\Gamma(t)$ *a periodic orbit of system* (1.1) *of period* $T > 0$. *Assume that* $U : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ *is a invariant curve with*

$$\Gamma(t) = \{(x,y) \in \Omega : U(x,y) = 0\}$$

and $K(x,y) \in C^1$ *is the cofactor given in the equation* (1.4)*, of invariant curve* $U(x,y) = 0$*. Assume that* $p \in \Omega$ *such that* $U(p) = 0$ *and* $\nabla U(p) \neq 0$*, then* p *is a singular point of system* (1.1)*, and*

$$\int_0^T div(\Gamma(t))dt = \int_0^T K(\Gamma(t))dt.$$

1.8 Limit cycle

We have seen that the solution refers to a singular point, another possible behavior for a trajectory is to refers to a periodic movement in the case of a planar system, that means that the trajectories refers to what is called a limits cycles.

Definition 1.8.1 *A limit cycle* $\Gamma = \{(x(t), y(t)), t \in [0,T]\}$ *is an isolated periodic solution in the set of all periodic solution of system* (1.1).
If a limit cycle is contained in a algebraic curve of the plan, then we say that it is algebraic, otherwise it is called non algebraic.

Example 1.8.1 *The limit cycle of the system*

$$\begin{cases} \dot{x} = x(-2 + 2x - 2x^2 + x^3 + 2y - 2xy + xy^2), \\ \dot{y} = y(2 - 2x + 2y - 2xy + x^2y - 2y^2 + y^3), \end{cases} \tag{1.6}$$

is given in the following figure:

Remark 1.8.1 *The limit cycle appear only in non-linear differential systems.*

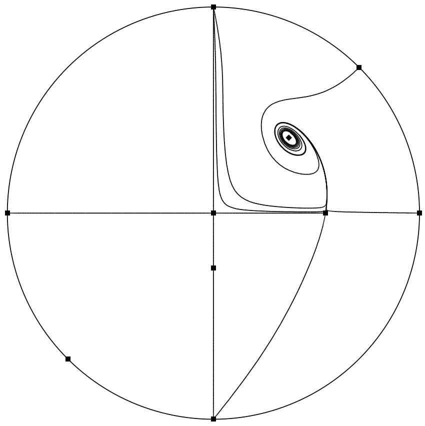


Figure 1.5: Cycle limit of system (1.6)

1.8.1 Stability of limit cycle

Theoreme 1.8.1 *[13] Consider* $\Gamma(t)$ *is a periodic orbit of system* (1.1) *of periodic* T.

- *If* $\int_0^T div(\Gamma(t))dt < 0$, *then* Γ *is a stable limit cycle.*
- *If* $\int_0^T div(\Gamma(t))dt > 0$, *then* Γ *is a unstable limit cycle.*

And if $\int_0^T div(\Gamma(t))dt = 0$, *then* Γ *may be a stable, unstable or semi-stable limit cycle or it may belong to a continuous band of cycles.*

Definition 1.8.2 *We say that the limit cycle* Γ *is hyperbolic if* $\int_0^T div(\Gamma(t))dt \neq 0$.

Example 1.8.2 *The system*

$$\begin{cases} \dot{x} = -y + \eta x(1 - x^2 - y^2), \\ \dot{y} = x + \eta y(1 - x^2 - y^2), \end{cases} \tag{1.7}$$

has a limit cycle Γ *represented by*

$$\Gamma(\theta) = (\cos\theta, \sin\theta),$$

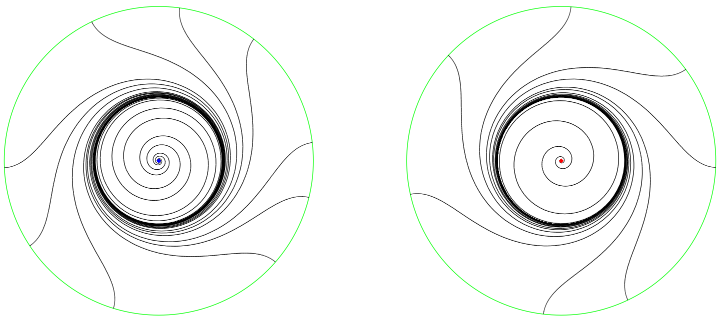
because we have

$$div(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \eta(2 - 4x^2 - 4y^2),$$

and

$$\begin{aligned}\int_0^T div(\Gamma(\theta))dt &= \int_0^T div(\cos\theta, \sin\theta)dt,\\ &= \eta\int_0^{2\pi}(4(\cos\theta)^2+4(\sin\theta)^2-2)d\theta,\\ &= \eta\int_0^{2\pi}2d\theta.\end{aligned}$$

So the cycle $\Gamma(\theta) = (\cos(\theta), \sin(\theta))$ *is a unstable limit cycle if* $\eta > 0$ *and is a stable limit cycle if* $\eta < 0$.



(a) limit cycle for $\eta = -0.2$. (b) limit cyle for $\eta = 0.2$.

Figure 1.6: limit cyle of system (1.7)

1.9 Integrability problem

1.9.1 First integrals

Definition 1.9.1 *The vector field* X *or equivalently the system* (1.1) *is integrable on an open subset* Ω *of* $\mathbb{R}^2$ *if there exists a non constant analytic function* $\Omega \to \mathbb{R}^2$ *, called a first integral of the system on* Ω *, which is constant on all solution curves* $(x(t), y(t))$ *of* X *contained in* Ω*; i.e., if*

$$\frac{dH(x,y)}{dt} = P(x,y)\frac{\partial H(x,y)}{\partial x} + Q(x,y)\frac{\partial H(x,y)}{\partial y} \equiv 0.$$

in the points of Ω*. Moreover* $H = h$ *is the general solution of this equation, where* h *is an arbitrary constant.*

Remark 1.9.1

- *We say that the differential system* (1.1) *is integrable on an open* Ω *if it admits a first integral on* Ω.
- *It is well known that for differential system defined on the plan* $\mathbb{R}^2$ *the existence of a first integral determines their phase portrait.*

1.9.2 Integrating factors

Definition 1.9.2 *Let Ω be an open subset of $\mathbb{R}^2$ and $R : \Omega \to \mathbb{R}$ be an analytic function which is not identically zero on Ω. The function R is an integrating factor of the differential system* (1.1)*, or of on Ω if one of the following three equivalent conditions on Ω.*

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}$$
$$div(RP, RQ) = 0 \tag{1.8}$$
$$\mathcal{X}R = -R div(\mathcal{X})$$

on Ω. As usual the divergence of the vector field X is defined by

$$div\,(X) = div\,(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

The first integral H associated to the integrating factor R is given by

$$H\,(x, y) = \int R\,(x, y)\,P\,(x, y)\;dy + h(x), \tag{1.9}$$

where h is chosen such that $\frac{\partial H}{\partial x} = -RQ$. Then

$$\begin{cases} \dot{x} = RP = \dfrac{\partial H}{\partial y}, \\ \dot{y} = RQ = -\dfrac{\partial H}{\partial x}. \end{cases} \tag{1.10}$$

In (1.9) *we suppose that the domain of integration Ω is well adapted to the specific expression.*
Conversely, given a first integral H of system (1.1) *we always can find an integrating factor R for which* (1.10) *holds.*

Proposition 1.9.1 *If a vector field X has two integrating factors R_1 and R_2 on the open subset Ω of $\mathbb{R}^2$, then in the open set $\Omega \setminus \{R_2 = 0\}$ the function R_1/R_2 is a first integral, provided R_1/R_2 is non-constant.*

1.9.3 Darbouxian theory of integrability

Before stating the main results of the Darboux theory of integrability we need some definitions. If $S(x, y) = \sum_{i+j=0}^{m-1} a_{ij}x^i y^j$ is a polynomial of degree $m - 1$ with $m(m + 1)/2$ coefficients in $\mathbb{C}$, then we write $S \in \mathbb{C}_{m-1}[x, y]$. We identify the linear vector space $\mathbb{C}_{m-1}[x, y]$ with $\mathbb{C}^{m(m+1)/2}$ through the isomorphism $S \to (a_{00}, a_{10}, a_{01}, ..., a_{m-1,0},$
$a_{m-2,1}, ..., a_{0,m-1})$.

We say that r points $(x_k, y_k) \in \mathbb{C}^2, k = 1, ..., r$, are independent with respect to $\mathbb{C}_{m-1}[x, y]$ if the intersection of the r hyperplanes

$$\left\{(a_{ij}) \in \mathbb{C}^{m(m+1)/2} : \sum_{i+j=0}^{m-1} a_{ij} x_k^i y_k^j = 0, k = 1, ..., r\right\},$$

is a linear subspace of $\mathbb{C}^{m(m+1)/2}$ of dimension $\frac{m(m+1)}{2} - r > 0$.
We recall that (x_0, y_0) is a singular point of system (1.1) if $P(x_0, y_0) = Q(x_0, y_0) = 0$. We remark that the maximum number of isolated singular points of the polynomial system (1.1) is m^2 (by Bezout theorem), that the maximum number of independent isolated singular points of the system is $\frac{m(m+1)}{2} - 1$, and that $\frac{m(m+1)}{2} < m^2$ for $m \geq 2$. A singular point (x_0, y_0) of system (1.1) is called weak if the divergence $div(P, Q)$ of system (1.1) at (x_0, y_0) is zero.

Theoreme 1.9.1 *[5] There exist $\lambda_i, \mu_i \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$, if and only if the (multi-valued) function*

$$f_1^{\lambda_1} ... f_p^{\lambda_p} \left(exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} ... \left(exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} \tag{1.11}$$

is a first integral of X.

Proposition 1.9.2 *[10] Consider the $\mathbf{C}^r$ differential system (1.1) and let $U_i = 0$ be a $\mathbf{C}^{r+1}$ invariant curve of (1.1) with cofactor k_i for $i = 1, ..., M$, Then $H(x, y) = \prod_{i=1}^{m} U_i^{\alpha_i}(x, y)$, is a first integral of (1.1) if and only if $\sum_{i=1}^{m} \alpha_i K_i(x, y) = 0$, for some convenient $\alpha_i \in \mathbb{C}$.*

1.10 The first return map

Probably the most basic tool for studying the stability of periodic orbits is the Poincare map or first return map, defined by Henri Poincare in 1881. The idea of Poincare map is quite simple : If Γ is a periodic orbit of system (1.1), through the point (x_0, y_0) and Σ is a hyperplane perpendicular to Γ at (x_0, y_0), then for any point $(x, y) \in \Sigma$ sufficiently near (x_0, y_0), the solution of (1.1) through (x, y) at $t = 0, \Phi_t(x, y)$ will cross Σ again at a point $\Pi(x, y)$ near (x_0, y_0), the mapping $(x, y) \to \Pi(x, y)$ is called the Poincare map.
The next theorem establishes the existence and continuity of the Poincare map $\Pi(x, y)$ and of its first derivative $D\Pi(x, y)$.

Theoreme 1.10.1 *[13] Let E be an open subset of $\mathbb{R}^2$ and let $(P, Q) \in \mathbb{C}^1(E)$. Suppose that $\Phi_t(x_0, y_0)$ is a periodic solution of (1.1) of period T and that the cycle*

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 | (x, y) = \Phi_t(x_0, y_0), 0 \leq t \leq T\right\},$$

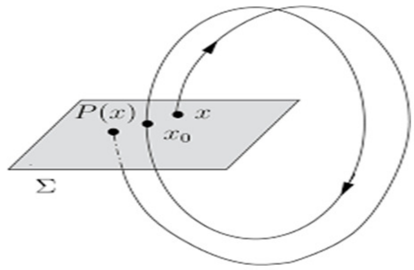


Figure 1.7: The poincare map

is contained in E. Let Σ *be the hyperplane orthogonal to* Γ *at*(x_0, y_0)*, i.e.; let*

$$\Sigma = \left\{(x, y) \in \mathbb{R}^2 | (x - x_0, y - y_0).(P(x_0, y_0), Q(x_0, y_0)) = 0\right\}.$$

Then there is a $\delta > 0$ *and a unique function* $\tau(x, y)$*, defined and continuously differentiable for* $(x, y) \in N_\delta(x_0, y_0)$*, such that*

$$\tau(x_0, y_0) = T,$$

and

$$\Phi_{\tau(x,y)}(x, y) \in \Sigma,$$

for all $(x, y) \in N_\delta(x_0, y_0)$.

Definition 1.10.1 *Let* Γ, Σ, δ *and* $\tau(x, y)$ *be defined as in Theorem* (2.5)*. Then for* $(x, y) \in N_\delta(x_0, y_0) \cap \Sigma$*, the function*

$$\Pi(x, y) = \Phi_{\tau(x,y)}(x, y),$$

is called the Poincare map for Γ *at* (x_0, y_0).
The following theorem gives the formula of $\Pi'(0, 0)$.

Example 1.10.1 *Let the following system of equations:*

$$\begin{cases} \dot{x} = -y + x(4 - x^2 - y^2), \\ \dot{y} = x + y(4 - x^2 - y^2). \end{cases} \tag{1.12}$$

Note that $(0, 0)$ *is the only critical point of (1.12). In polar coordinates, the system (1.12) becomes:*

$$\begin{cases} \dot{r} = r(4 - r^2), \\ \dot{\theta} = 1. \end{cases} \tag{1.13}$$

The differential system (1.13) is equivalent to the following Bernoulli differential equation

$$\frac{dr}{d\theta} = 4r - r^3.$$

thus general solution is

$$r(\theta) = \left[\frac{1}{4} + ce^{-8\theta}\right]^{-\frac{1}{2}}$$

a solution of (1.13) satisfying the initial condition $r(0) = r_0$ *is given by*

$$r(\theta, r_0) = \left[\frac{1}{4} + (\frac{1}{r_0^2} - \frac{1}{4})e^{-8\theta}\right]^{-\frac{1}{2}}.$$

For $\theta = 2\pi$*; It follows that the Poincare first return map is given by:*

$$\Pi(r_0) = r(2\pi, r_0) = \left[\frac{1}{4} + (\frac{1}{r_0^2} - \frac{1}{4})e^{-16\pi}\right]^{-\frac{1}{2}}.$$

Definition 1.10.2 *A fixed point of the application* Π *is a point* (x, y) *such that* $\Pi(x, y) = (x, y)$*. It corresponds to a periodic orbit of the system (1.1).*

1.10.1 Stability of poincare map

Theoreme 1.10.2 *[13] Let* $\Gamma(t)$ *be a periodic solution of* (1.1) *of period* T*. Then the derivative of the Poincare map* $\Pi(s)$ *along a straight line* Σ *normal to* $\Gamma = \{(x, y) \in \mathbb{R}^2 : (x, y) = \Phi_t(x_0, y_0), 0 \leq t \leq T\}$ *at* $(x, y) = (0, 0)$ *is given by*

$$\Pi'(0) = \exp \int_0^T \nabla.(P(\Gamma(t)), Q(\Gamma(t)))dt.$$

Corollary 1.10.1 *[13] Under the hypotheses of* (1.10.2)*, the periodic solution* $\Gamma(t)$ *is a stable limit cycle if*

$$\int_0^T \nabla.(P(\Gamma(t)), Q(\Gamma(t)))dt < 0$$

and it is an unstable limit cycle if

$$\int_0^T \nabla.(P(\Gamma(t)), Q(\Gamma(t)))dt > 0.$$

Example 1.10.2 *The system (1.12) has a limit cycle* Γ_1 *represented by*

$$\Gamma_1(\theta) = (2\cos\theta, 2\sin\theta),$$

we have $\nabla.(P(x, y), Q(x, y)) = 8 - 4x^2 - 4y^2$
and

$$\begin{aligned}\int_0^{2\pi} \nabla.(P(\Gamma_1(\theta)), Q(\Gamma_1(\theta)))d\theta &= \int_0^{2\pi} 8 - 4((2\cos(\theta))^2 + (2\sin(\theta))^2)d\theta. \\ &= -8\pi < 0.\end{aligned}$$

Then, the cycle Γ_1 *is a stable limit cycle.*

Chapter 2

Algebraic limit cycle for some classes of Kolmogorov systems

2.1 Introduction

The existence of limit cycles is interesting and very important for understanding the dynamics of polynomial differential systems. In this chapiter we are interested in studying the hyperbolic algebraic limit cycles of the some classes of 2-dimensional Kolmogorov systems of the form:

$$\begin{cases} \dot{x} = x(F(x,y)U(x,y) + yA_3(x,y)U_y(x,y)), \\ \dot{y} = y(G(x,y)U(x,y) - xA_3(x,y)U_x(x,y)), \end{cases}$$

where F, G and A_3 are polynomials of degree $\leqslant j-k-1$.

2.2 Construction of a class Kolmogorov system

Definition 2.2.1 *We call planar Kolmogorov polynomial differential system, a system of the form*

$$\begin{cases} \dot{x} = xf(x,y), \\ \dot{y} = yg(x,y), \end{cases} \tag{2.1}$$

where f and g are polynomials in the variable x and y with real coefficients.

- The order of system (2.1) is $m = 1 + \max(\deg(f), \deg(g))$.

Remark 2.2.1 *The system accepts at least three trajectories :*
i) The origin $(0,0)$, which is a equilibrium point.
ii) The positive y axis.
iii) The positive x-axis.

- The union of the three orbits forms the border of the positive quadrant

$$\mathbb{R}_+^2 = \left\{(x,y) \in \mathbb{R}^2 : x \geqslant 0, y \geqslant 0\right\}.$$

Lemma 2.2.1 *[6] Let $U_i = 0$ for $i = 1, ..., p$ be different irreducible invariant algebraic curves of system (2.1) with $\deg U_i = u_i$. Assume that U_i satisfy conditions:*
i) There are no points at which U_i and its first derivatives all vanish.
ii) If two curves intersect at a point in the finite plane, they are transversal at this point.
iii) There are no more than two curves $U_i = 0$ meeting at any point in the finite plane.
(a) If $(U_{ix}, U_{iy}) = 1$ for $i = 1, ..., p$, then system (2.1) has the normal form:

$$\begin{cases} \dot{x} = \left(B - \sum_{i=1}^{p} \dfrac{A_i(U_i)_y}{U_i}\right) \prod_{i=1}^{p} U_i, \\ \dot{y} = \left(D + \sum_{i=1}^{p} \dfrac{A_i(U_i)_x}{U_i}\right) \prod_{i=1}^{p} U_i, \end{cases} \tag{2.2}$$

where B, D and A_i are suitable polynomials.
(b) If U_i satisfy conditions:
i) The highest order terms of U_i have no repeated factors.
ii)There are no two curves having a common factor in the highest order terms.
then system (2.1) has the normal form (2.2) with degree B ,degree $D \leq j - \sum_{i=1}^{p} u_i$ and degree $A_i \leq j - \sum_{i=1}^{p} u_i + 1$.

Theoreme 2.2.1 *A Kolmogorov system of degree $\leqslant j$ has an irreducible invariant algebraic curve $U = 0$ of degree k if and only if this system can be written via an via an affine change of variables into the form*

$$\begin{cases} \dot{x} = x(F(x, y)U(x, y) + yA_3(x, y)U_y(x, y)), \\ \dot{y} = y(G(x, y)U(x, y) - xA_3(x, y)U_x(x, y)), \end{cases} \tag{2.3}$$

where F, G and A_3 are polynomials of degree $\leqslant j - k - 1$.

Proof: Assume that the degree system (2.1) is j. This system has three invariant curves: the y-axis ($U_1(x, y) = x = 0$), the x-axis ($U_2(x, y) = y = 0$) and $U(x, y) = 0$. then all the hypothesis of Lemma 2.2.1 are satisfied. So the system (2.1) has the general form (2.2):

$$\begin{cases} \dot{x} = \left(B(x, y) - \frac{A_2(x,y)}{y} + \frac{A_3(x,y)U_y(x,y)}{U(x,y)}\right) xyU(x, y), \\ \dot{y} = \left(D(x, y) + \frac{A_1(x,y)}{x} - \frac{A_3(x,y)U_x(x,y)}{U(x,y)}\right) xyU(x, y), \end{cases}$$

we can write this system in the form :

$$\begin{cases} \dot{x} = x\left((yB(x, y) - A_2(x, y))U(x, y) + yA_3(x, y)U_y(x, y)\right), \\ \dot{y} = y\left((xD(x, y) - A_1(x, y))U(x, y) - xA_3(x, y)U_x(x, y)\right), \end{cases}$$

with $\deg B$, $\deg D \leqslant j - (k+2)$, $\deg A_i \leqslant j - (k+1)$. If we pose $F(x,y) = yB(x,y) - A_2(x,y)$ and $G(x,y) = xD(x,y) - A_1(x,y)$, thus the system has normal form

$$\begin{cases} \dot{x} = x\left(F(x,y)U(x,y) + yA_3(x,y)U_y(x,y)\right), \\ \dot{y} = y\left(G(x,y)U(x,y) - xA_3(x,y)U_x(x,y)\right), \end{cases}$$

withe $\deg F, G, A_3 \leqslant j - k - 1$.

2.3 On the non-existence of limit cycles for a Kolmogorov systems of degree $j = k + 1$

Theoreme 2.3.1 *A Kolmogorov system of degree $j = k + 1$ has an irreducible invariant algebraic curve $U = 0$ of degree k has no limit cycles.*

Proof: According to Lemma 2.2.1, the Kolmogorov system (2.3) of degree $j = k+1$ has an irreducible invariant algebraic curve $U = 0$ of degree k, can be written as

$$\begin{cases} \dot{x} = x\left(nU(x,y) + ypU_y(x,y)\right), \\ \dot{y} = y\left(mU(x,y) - xpU_x(x,y)\right), \end{cases} \tag{2.4}$$

where, $n = F, m = G$ and $p = A_3$ are real constants. First, we prove that $U(x,y) = 0$ is invariant algebraic curve, we have

$$\begin{aligned} \frac{dU(x,y)}{dt} &= \dot{x}U_x(x,y) + \dot{y}U_y(x,y) \\ &= (x\left(nU(x,y) + ypU_y(x,y)\right))U_x(x,y) \\ &\quad + (y\left(mU(x,y) - xpU_x(x,y)\right))U_y(x,y) \\ &= (xnU_x(x,y) + ymU_y(x,y))U(x,y). \end{aligned}$$

Consequently, $U(x,y) = 0$ is invariant algebraic curve with cofactor

$$K(x,y) = xnU_x(x,y) + ymU_y(x,y).$$

Suppose that $U = 0$ contains an equilibrium point (x^*, y^*) of the system, therefore (x^*, y^*) is a solution of the system

$$\begin{cases} x\left(nU(x,y) + pyU_y(x,y)\right) = 0, \\ y\left(mU(x,y) - pxU_x(x,y)\right) = 0. \end{cases}$$

This implies that (x^*, y^*) is a solution of the system:

$$\begin{cases} U_y(x,y) = 0, \\ U_x(x,y) = 0. \end{cases}$$

which is impossible because the curve $U = 0$ is not singular, therefore $U = 0$ is a periodic solution on the other hand, the Kolmogorov system (2.3) admits three invariant algebraic curve.

1) The positive ordinate axis $x = 0$ and $y > 0$ withe the cofactor

$$K_1 = nU(x, y) + ypU_y(x, y).$$

2) The positive $x-$axis $y = 0$ and $x > 0$ withe the cofactor

$$K_2 = mU(x, y) - xpU_x(x, y).$$

3) The curve $U(x, y) = 0$ withe the cofactor

$$K(x, y) = xnU_x(x, y) + ymU_y(x, y).$$

We have

$$mK_1(x, y) - nK_2(x, y) - pK(x, y) = 0.$$

Then according to darboux's theorem [see chapter 1 Theorem 1.9.1] the system has a first integral of the form

$$H(x, y) = x^m y^{-n}(U(x, y))^{-p},$$

implies that

$$H(x, y) = \frac{x^m}{y^n(U(x, y))^p}.$$

Since the first integral of system (2.4) is a rational first integral, then this system has no limit cycles.

2.3.1 Application examples

Now we give some examples to show that the system (2.3) cannot have a limit cycle, although this system admits a family of periodic solutions.

Cubic system

Example 2.3.1 *We take* $U(x, y) = (x - 2)^2 + (y - 2)^2 - 1 = 0$, $m = 1, n = 1$ *and* $p = 1$, *system (2.3) become*

$$\begin{cases} \dot{x} = x(7 - 4x + x^2 - 12y + 5y^2), \\ \dot{y} = y(7 + 4x - 3x^2 - 4y + y^2). \end{cases} \tag{2.5}$$

The function

$$H(x, y) = \frac{x}{y(7 - 4x + x^2 - 4y + y^2)},$$

is a first integral of this system
we find that

$$H_x = \frac{7 - x^2 - 4y + y^2}{y(7 - 4x + x^2 - 4y + y^2)^2}$$

$$H_y = -\frac{x(7 - 4x + x^2 - 8y + 3y^2)}{y^2(7 - 4x + x^2 - 4y + y^2)^2}$$

Thus

$$\begin{aligned}
\frac{dH}{dt} &= \dot{x}H_x + \dot{y}H_y \\
&= x(x^2 + 3y^2 - 4x - 12y + 11)\frac{7 - x^2 - 4y + y^2}{y(7 - 4x + x^2 - 4y + y^2)^2} \\
&\quad -y(-x^2 + y^2 - 6y + 11)\frac{x(7 - 4x + x^2 - 8y + 3y^2)}{y^2(7 - 4x + x^2 - 4y + y^2)^2}) \\
&\equiv 0.
\end{aligned}$$

Therefore, this system (2.5) admits a rational first integral, then the system has no algebraic limit cycles and the system accepts a family of periodic solutions.

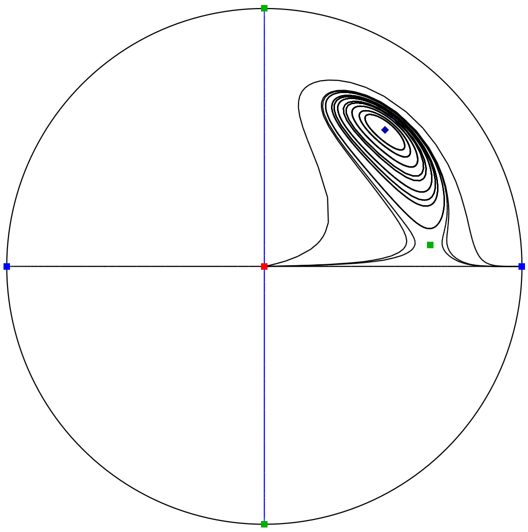


Figure 2.1: The phase portrait of System (2.5).

Quartic system

Example 2.3.2 *We pose* $U(x, y) = (x - 2)^2 + x(y - 3)^2 - 2 = 0$ $m = 2, n = 1$ *and* $p = 1$*, system (2.3) become*

$$\begin{cases} \dot{x} = x(2 + 5x + x^2 - 12xy + 3xy^2), \\ \dot{y} = y(4 + 5x - 6xy + xy^2), \end{cases} \tag{2.6}$$

the function

$$H(x, y) = \frac{x^2}{y(2 + 5x + x^2 - 6xy + xy^2)}$$

is a first integral of system (2.6), then the system (2.6) has no limit cycles.

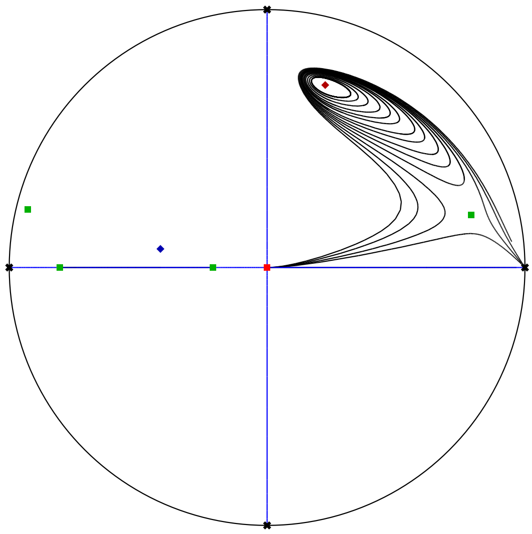


Figure 2.2: The phase portrait of System (2.6).

Quintic system

Example 2.3.3 *We pose* $U(x,y) = (x-2)^2 + x^2(y-2)^2 - 1 = 0$ $m = 1, n = 2$ *and* $p = 2$*, system (2.3) become*

$$\begin{cases} \dot{x} = x(6 - 8x + 10x^2 - 16x^2y + 6x^2y^2), \\ \dot{y} = y(3 + 4x - 15x^2 + 12x^2y - 3x^2y^2), \end{cases} \tag{2.7}$$

the function

$$H(x,y) = \frac{x}{y^2(2 - 4x + 10x^2 - 6x^2y + x^2y^2)^2}$$

is a first integral of system (2.7), thus the system has no limit cycles.

2.4 On the existence of algebraic limit cycles of some classes of Kolmogorov systems

2.4.1 Class 1

In this subsection we consider the polynomial differential system (2.3) where $F = axy + f(y)$, $G = bxy + g(x)$ and $A_3 \in \mathbb{R}^*$ with $a + b \neq 0$, then System (2.3) can be written in the form

$$\begin{cases} \dot{x} = x((axy + f(y))U(x,y) + yA_3U_y(x,y)), \\ \dot{y} = y((bxy + g(x))U(x,y) - xA_3U_x(x,y)) \end{cases} \tag{2.8}$$

As a main result, we have

Theoreme 2.4.1 *The polynomial differential system (2.8), admits the ovals of* $U = 0$ *located in* $\Omega = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ *as a limit cycles.*

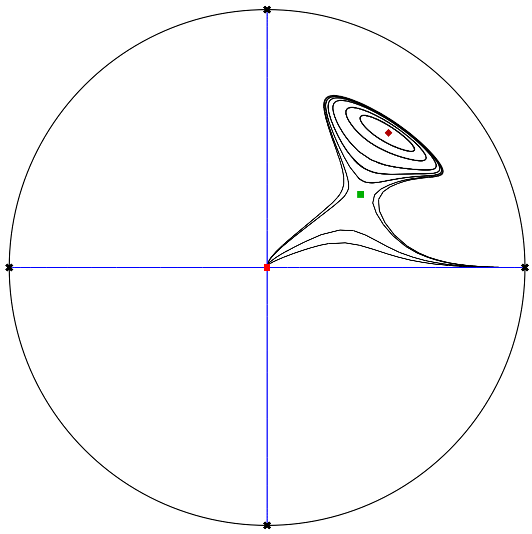


Figure 2.3: The phase portrait of System (2.7)

Proof: We have

$$\begin{aligned}\frac{dU(x,y)}{dt} &= \dot{x}U_x(x,y)+\dot{y}U_y(x,y)\\ &= x((axy+f(y))U(x,y)+yA_3U_y(x,y))U_x(x,y)\\ &\quad +y((bxy+g(x))U(x,y)-xA_3U_x(x,y))U_y(x,y)\\ &= (x(axy+f(y))U_x+y(bxy+g(x))U_y)U(x,y).\end{aligned}$$

Therefor the curve defined by $U(x,y)=0$ is invariant curve for system (2.8) withe associated cofactor

$$K(x,y)=x(axy+f(y))U_x+y(bxy+g(x))U_y.$$

It is clear that $\Gamma:\{(x,y):U(x,y)=0\}$ is a trajectory of (2.8).
In the other hand, according to [Theorem 1.7.1 chapter 1] we have

$$\int_0^T div(\Gamma)dt=\int_0^T K(x,t)dt$$

and

$$\begin{aligned}\int_0^T div(\Gamma)dt &= \int_0^T (x(axy+f(y))U_x+y(bxy+g(x))U_y)dt,\\ &= \int_0^T x(axy+f(y))U_xdt+\int_0^T y(bxy+g(x))U_ydt,\end{aligned}$$

Since

$$dt=\frac{dx}{x((axy+f(y))U+A_3yU_y)}=\frac{dy}{y((bxy+g(x))U-xA_3U_x},$$

then

$$\int_0^T div(\Gamma)dt = \oint_\Gamma \frac{(axy+f(y))U_x}{A_3yU_y}dx + \oint_\Gamma \frac{(bxy+g(x))U_y}{A_3xU_y}dx.$$

Since $U(x,y)=0$ is algebraic curve, then we have $U_x(x,y)dx+U_y(x,y)dy=0$, thus $U_x(x,y)dx=-U_y(x,y)dy$, and we get

$$\begin{aligned}\int_0^T div(\Gamma)dt &= \oint_\Gamma \frac{bxy+g(x)}{A_3x}dx - \oint_\Gamma \frac{axy+f(y)}{A_3y}dy,\\ &= \oint_\Gamma \frac{b}{A_3}y+\frac{g(x)}{A_3x}dx - \oint_\Gamma \left(\frac{a}{A_3}x+\frac{f(y)}{A_3y}\right)dy,\end{aligned}$$

By applying Green's formula

$$\begin{aligned}\int_0^T div(\Gamma)dt &= \int\int_{int(\Gamma)} -\frac{d}{dx}\left(\frac{a}{A_3}x+\frac{f(y)}{A_3y}\right) - \frac{d}{dy}\left(\frac{b}{A_3}y+\frac{g(x)}{A_3x}\right)dydx,\\ &= -\frac{a+b}{A_3}\int\int_{int(\Gamma)} dydx,\end{aligned}$$

where $int(\Gamma)$ denotes the interior of Γ.
It is well known that according to [Theorem 1.9.1 Chapter 1], the periodic orbit is a hyperbolic limit cycle if and only if

$$\int_0^T div(\Gamma)dt \neq 0,$$

since $a+b\neq 0$ then $\int\int_{int(\Gamma)} dydx \neq 0$ thus $\int_0^T div(\Gamma)dt \neq 0$.
Consequently, if the ovals of $U=0$ be a periodic solutions of the system (2.8) in Ω, then are a hyperbolic limit cycles.

Example 2.4.1 *We pose* $a=1, b=1, A_3=1, f(y)=y, g(x)=x$ *and* $U(x,y)=(x-2)^2+(y-2)^2-1$, *system (2.8) become*

$$\begin{cases}\dot{x}=x(3y+3xy-3x^2y+x^3y-2y^2-4xy^2+y^3+xy^3),\\ \dot{y}=y(11x-6x^2+x^3+3xy-4x^2y+x^3y-3xy^2+xy^3),\end{cases} \tag{2.9}$$

the curve $U(x,y)=0$ *is an invariant algebraic curve of the differential system (2.9) with the cofactor*

$$K(x,y)=x(xy+y)(-4+2x)+y(xy+x)(-4+2y).$$

Then, we have

$$\begin{aligned}\int_0^T div(\Gamma)dt &= \int_0^T K(x,y)dt\\ &= \int_0^T (x(xy+y)(-4+2x)+y(xy+x)(-4+2y))dt,\\ &= \oint_\Gamma \frac{x(xy+y)(-4+2x)}{xy(-4+2y)}dx + \oint_\Gamma \frac{y(xy+x)(-4+2y)}{xy(-4+2y)}dx,\\ &= \oint_\Gamma \frac{xy+x}{x}dx - \oint_\Gamma \frac{xy+y}{y}dy,\\ &= \oint_\Gamma y+1dx - \oint_\Gamma x+1dy,\end{aligned}$$

By applying Green's formula

$$\begin{aligned}\int_0^T div(\Gamma)dt &= \int\int_{int(\Gamma)} -\frac{d}{dx}(x+1) - \frac{d}{dy}(y+1)dydx,\\ &= -2\int\int_{int(\Gamma)} dydx,\end{aligned}$$

Thus $\int_0^T div(\Gamma)dt \neq 0$. *Then* $U(x,y)=0$ *is a hyperbolic limit cycle of this system and this limits cycle surrounds a stable focus.*

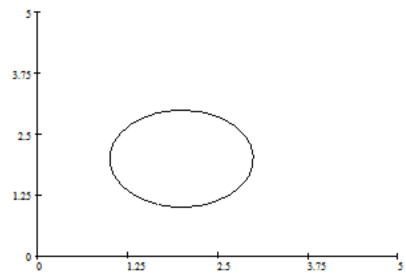


Figure 2.4: The ovals of the algebraic curve U of system (2.9)

Example 2.4.2 *We pose* $a=2, b=2, A_3=1, f(y)=2y, g(x)=2x$ *and* $U(x,y)=(y-2)^4-2(y-2)^2-3(x-2)^2+(x-2)^4+\frac{5}{2}$, *system (2.8) become*

$$\begin{cases}\dot{x} = x\left(\begin{array}{c}5y-11xy+2x^2y+26x^3y-14x^4y+2x^5y-4y^2-\\ 48xy^2+20y^3+44xy^3-12y^4-16xy^4+2y^5+2xy^5\end{array}\right),\\ \dot{y} = y\left(\begin{array}{c}49x-82x^2+66x^3-20x^4+2x^5-19xy-40x^2y+\\ 42x^3y-16x^4y+2x^5y-4xy^2+28xy^3-14xy^4+2xy^5\end{array}\right),\end{cases} \tag{2.10}$$

the curve $U(x,y)=0$ *is an invariant algebraic curve of the differential system (2.10) withe the cofactor*

$$K(x,y) = 2x(xy+y)(-20+42x-24x^2+4x^3)+2y(xy+x)(-24+44y-24y^2+4y^3).$$

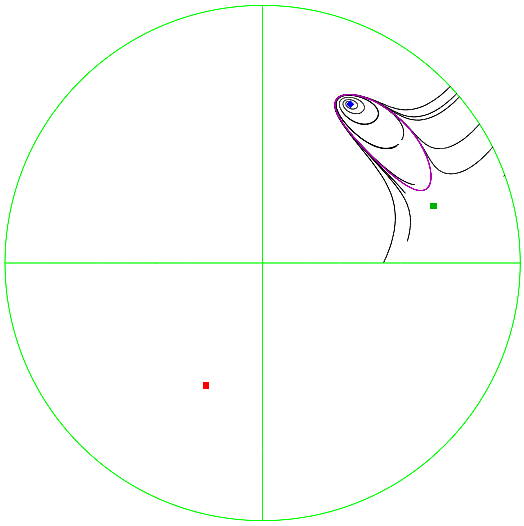


Figure 2.5: Limit cycles on the Poincare disc of System (2.9).

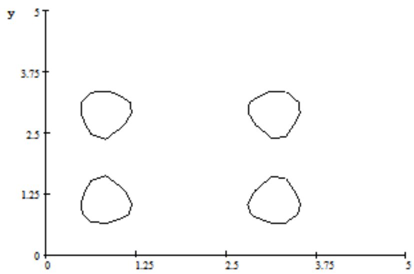


Figure 2.6: The ovals of the algebraic curve U of system (2.10).

Then, we have $\int_0^T div(\Gamma)dt = -4\int\int_{int(\Gamma)} dydx \neq 0$. *Consequently, system (2.10) admits four hyperbolic limit cycles represented by the curve*

$$U(x,y) = (y-2)^4 - 2(y-2)^2 - 3(x-2)^2 + (x-2)^4 + \frac{5}{2} = 0,$$

and each limit cycle surrounds a stable focus.

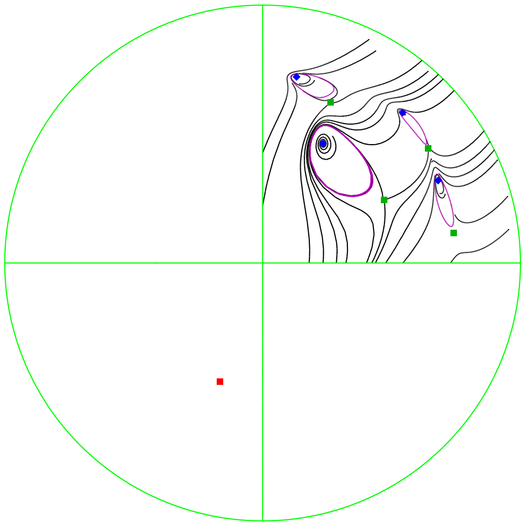


Figure 2.7: Limit cycles on the Poincare disc of System (2.10).

2.4.2 Class 2

In this subsection we consider the polynomial differential system (2.3) where $F = ayp(x) + f(y)$, $G = bxq(y) + g(x)$ such that $ap'(x) + bq'(y) \neq 0$ and $A_3 \in \mathbb{R}^*$, then System (2.3) can be written in the form

$$\begin{cases} \dot{x} = x((ayp(x) + f(y))U(x,y) + yA_3U_y(x,y)), \\ \dot{y} = y((bxq(y) + g(x))U(x,y) - xA_3U_x(x,y)) \end{cases} \tag{2.11}$$

As a main result, we have

Theoreme 2.4.2 *The system (2.11) admits as limit cycles the ovals of the algebraic curve $U = 0$ in the quadrant $\Omega = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$.*

Proof: We show that $U(x,y)$ is an invariant algebraic curve of the differential system (2.11). We have

$$\begin{aligned} \frac{dU(x,y)}{dt} &= \dot{x}U_x(x,y) + \dot{y}U_y(x,y), \\ &= x((ap(x)y + f(y))U(x,y) + yA_3U_y(x,y))U_x(x,y) \\ &\quad + y((bxq(y) + g(x))U(x,y) - xA_3U_x(x,y))U_y(x,y), \\ &= (x(ap(x)y + f(y))U_x + y(bxq(y) + g(x))U_y)U(x,y). \end{aligned}$$

Therefor, the curve defined by $U(x,y) = 0$ is invariant curve for system (2.8) withe associated cofactor

$$K(x,y) = x(ap(x)y + f(y))U_x + y(bxq(y) + g(x))U_y$$

It is clear that $\Gamma : \{(x,y) : U(x,y) = 0\}$ is a trajectory of(2.11).
In the other hand, we have

$$\begin{aligned}\int_0^T div(\Gamma)dt &= \int_0^T K(x,t)dt \\ &= \int_0^T (x(ayp(x)+f(y))U_x + y(bxq(y)+g(x))U_y)dt, \\ &= \int_0^T x(ayp(x)+f(y))U_x dt + \int_0^T y(bxq(y)+g(x))U_y dt,\end{aligned}$$

we have

$$dt = \frac{dx}{x((ayp(x)+f(y))U + A_3yU_y)} = \frac{dy}{y((bxq(y(+g(x))U - xA_3U_x}.$$

Thus

$$dt = \frac{dx}{A_3xyU_y} = -\frac{dy}{yxA_3U_x}.$$

So

$$\begin{aligned}\int_0^T div(\Gamma)dt &= \oint_\Gamma \frac{(ayp(x)+f(y))U_x}{A_3yU_y}dx + \oint_\Gamma \frac{(bxq(y)+g(x))U_y}{A_3xU_y}dx, \\ &= \oint_\Gamma \frac{bxq(y)+g(x)}{A_3x}dx - \oint_\Gamma \frac{ayp(x)+f(y)}{A_3y}dy, \\ &= \oint_\Gamma \left(\frac{bq(y)}{A_3} + \frac{g(x)}{A_3x}\right)dx - \oint_\Gamma \left(\frac{ap(x)}{A_3} + \frac{f(y)}{A_3y}\right)dy,\end{aligned}$$

By applying Green's formula

$$\begin{aligned}\int_0^T div(\Gamma)dt &= \int\int_{int(\Gamma)} -\frac{d}{dx}\left(\frac{ap(x)}{A_3} + \frac{f(y)}{A_3y}\right) - \frac{d}{dy}\left(\frac{bq(y)}{A_3} + \frac{g(x)}{A_3x}\right)dydx, \\ &= -\int\int_{int(\Gamma)} (ap'(x) + bq'(y))\,dydx,\end{aligned}$$

where $int(\Gamma)$ denotes the interior of Γ.
Since $ap'(x) + bq'(y) \neq 0$, thus $\int_0^T div(\Gamma)dt \neq 0$. Consequently, the ovals of $U = 0$ be a periodic solutions of the system, then are a hyperbolic limit cycles.

Example 2.4.3 *We pose* $a = 1, b = 2, A_3 = 1, f(y) = y+1,\ g(x) = x-1,\ p(x) = x^3$, $q(y) = y^3$ *and* $U(x,y) = (x-2)^2 + (y-2)^2 - 1$. *System (2.11) become*

$$\begin{cases} \dot{x} = x\begin{pmatrix} 7 - 4x + x^2 - y - 4xy + x^2y + 7x^3y - 4x^4y + x^5y \\ -y^2 - 4x^3y^2 + y^3 + x^3y^3 \end{pmatrix}, \\ \dot{y} = y\begin{pmatrix} -7 + 15x - 7x^2 + x^3 + 4y - 4xy - y^2 + xy^2 + 14xy^3 \\ -8x^2y^3 + 2x^3y^3 - 8xy^4 + 2xy^5 \end{pmatrix}. \end{cases} \tag{2.12}$$

The curve $U(x,y) = 0$ *is an invariant algebraic curve of the differential system (2.12) withe the cofactor*

$$k(x,y) = x(yx^3 + y + 1)(-4 + 2x) + y(2xy^3)(-4 + 2y).$$

We have

$$\int_0^T div(\Gamma)dt = \int_0^T K(x,y)dt = -\int\int_{int(\Gamma)} \left(3x^2 + 6y^2\right) dydx$$

Notice that $3x^2 + 6y^2 > 0$*, so* $\int_0^T div(\Gamma)dt \neq 0$*. Then* $U(x,y) = 0$ *is a hyperbolic limit cycle of this system. Moreover this limits cycle surrounds a stable focus.*

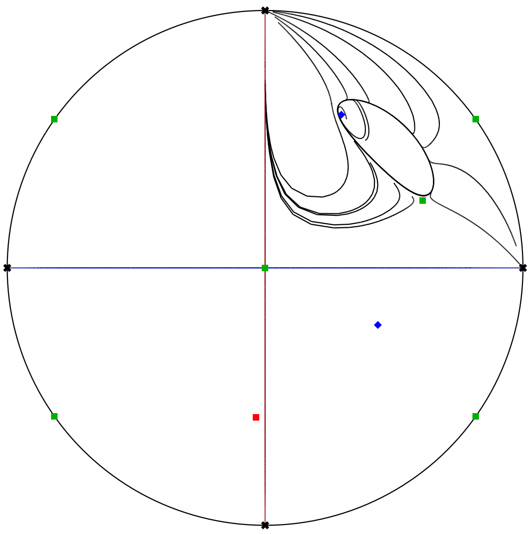


Figure 2.8: Limit cycles on the Poincare disc of System (2.12).

Example 2.4.4 *We pose* $a = 2, b = 2, A_3 = 2, f(y) = 2y$*,* $g(x) = 2x$*,* $p(x) = x^3 + 1$*,* $q(y) = y^3 + 1$ *and* $U(x,y) = (y-2)^4 - 2(y-2)^2 - 3(x-2)^2 + (x-2)^2 + \frac{1}{2}$*, system (2.11) become*

$$\begin{cases} \dot{x} = x\left(\begin{array}{c} 2y - 80xy + 84x^2y - 7x^3y - 36x^4y + 42x^5y - 16x^6y + 2x^7y \\ -8y^2 - 48x^3y^2 + 40y^3 + 44x^3y^3 - 24y^4 - 16x^3y^4 + 4y^5 + 2x^3y^5 \end{array}\right), \\ \dot{y} = y\left(\begin{array}{c} 90x - 164x^2 + 132x^3 - 40x^4 + 4x^5 - 96xy + 88xy^2 - 7xy^3 - 40 \\ x^2y^3 + 42x^3y^3 - 16x^4y^3 + 2x^5y^3 - 44xy^4 + 44xy^5 - 16xy^6 + 2xy^7 \end{array}\right), \end{cases} \tag{2.13}$$

The curve $U(x,y) = 0$ *is an invariant algebraic curve of the differential system (2.13) withe the cofactor*
$k(x,y) = -176xy + 168x^2y - 96x^3y - 24x^4y + 84x^5y - 48x^6y + 8x^7y + 176xy^2 -$

$96xy^3 - 32xy^4 + 88xy^5 - 48xy^6 + 8xy^7$.
Then, we have

$$\int_0^T div(\Gamma)dt = -6 \int\int_{int(\Gamma)} (x^2 + y^2)dydx \neq 0.$$

Then, system (2.13) admits two hyperbolic limit cycles represented by the curve

$$U(x, y) = (y - 2)^4 - 2(y - 2)^2 - 3(x - 2)^2 + (x - 2)^4 + \frac{1}{2} = 0.$$

This limist cycles surrounds a unstable focus.

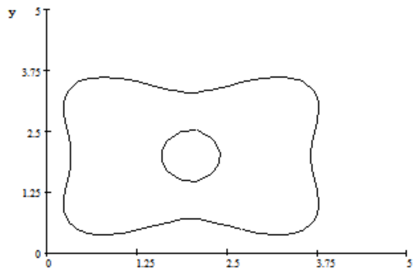


Figure 2.9: The ovals of the algebraic curve U of system (2.13).

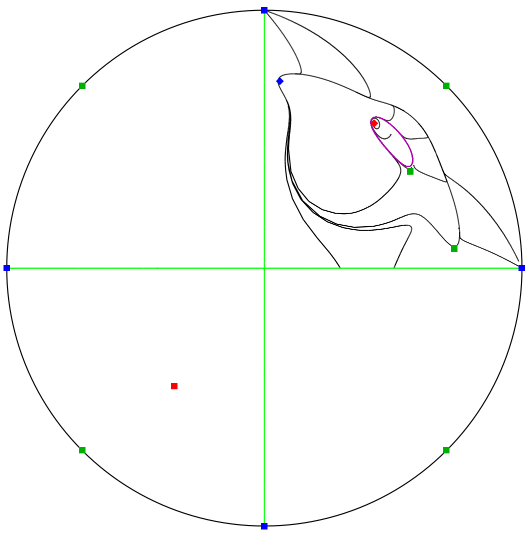


Figure 2.10: Limit cycles on the Poincare disc of system (2.13).

2.4.3 Class 3

In this subsection we consider the polynomial differential system (2.3) where $F = U_x + wA_3(x,y) + wy(A_3)_y$, $G = U_y - wA_3(x,y) - wx(A_3)_x$, then system (2.3) can be written in the form

$$\begin{cases} \dot{x} = x\left(\left(U_x\left(x,y\right) + wA_3\left(x,y\right) + wy(A_3)_y\left(x,y\right)\right)U\left(x,y\right) + yA_3\left(x,y\right)U_y\left(x,y\right)\right), \\ \dot{y} = y\left(\left(U_y\left(x,y\right) - wA_3\left(x,y\right) - wx(A_3)_x\left(x,y\right)\right)U\left(x,y\right) - xA_3\left(x,y\right)U_x\left(x,y\right)\right), \end{cases} \tag{2.14}$$

where w is a real constant and A_3 is a polynomial functions of $\mathbb{R}^2$ in $\mathbb{R}$.

As a main result, we have

Theoreme 2.4.3 *The system (2.14) admits as limit cycles the ovals of the algebraic curve $U = 0$ in the quadrant $\Omega = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$.*

Proof We have to show that $U(x,y) = 0$ is an invariant algebraic curve of the differential system (2.14). Indeed, a direct calculation shows that

$$\begin{aligned} K(x,y) &= \frac{x\left(\left(U_x + wA_3 + wy(A_3)_y\right)U + A_3yU_y\right)U_x}{U} \\ &\quad + \frac{y\left(\left(U_y - wA_3 - wx(A_3)_x\right)U - A_3xU_x\right)U_y}{U} \\ &= xU_x^2 + yU_y^2 + A_3wxU_x - A_3wyU_y + wxyU_x(A_3)_y - wxyU_y(A_3)_x. \end{aligned}$$

Therefore, the curve defined by $U(x,y) = 0$, is a invariant curve of (2.14) with the associated cofactor

$$K(x,y) = xU_x^2 + yU_y^2 + A_3wxU_x - A_3wyU_y + wxyU_x(A_3)_y - wxyU_y(A_3)_x.$$

It is clear that $\Gamma : \{(x,y) : U(x,y) = 0\}$ is a trajectory of (2.14).
In the other hand, we have

$$\begin{aligned} div\left(x,y\right) &= \frac{d}{dx}\left(x\left(FU + yA_3U_y\right)\right) + \frac{d}{dy}\left(y\left(GU - xA_3U_x\right)\right) \\ &= \left(FxU_x + GyU_y - A_3xU_x + A_3yU_y - xyU_x(A_3)_y + xyU_y(A_3)_x\right) \\ &\quad + \left(F + G + xF_x + yG_y\right)U. \end{aligned}$$

If we denote by T the period of this solution. So we have successively,

$$\int_0^T div(x,y)dt = \lambda \int_0^T div(x,y)dt + (1-\lambda)\int_0^T K(x,y)dt.$$

for all $\lambda \in \mathbb{R}$. Since

$$\begin{aligned} \lambda div(x,y) + (1-\lambda)K(x,y) &= \lambda\left(F + G + xF_x + yG_y\right)U + x\left(F - A_3\lambda - \lambda y(A_3)_y\right)U_x \\ &\quad + y\left(G + A_3\lambda + \lambda x(A_3)_x\right)U_y \end{aligned}$$

then

$$\int_0^T div(x,y)dt = \int_0^T x\left(F - A_3\lambda - \lambda y(A_3)_y\right)U_x + y\left(G + A_3\lambda + \lambda x(A_3)_x\right)U_y dt.$$

Furthermore, since $F(x,y) = U_x + wA_3 + wy(A_3)_y$ and $G(x,y) = U_y - wA_3 - wx(A_3)_x$, then

$$\begin{aligned}\int_0^T div(x,y)dt &= \int_0^T x\left(U_x + (w-\lambda)A_3 + y(w-\lambda)(A_3)_y\right)U_x dt \\ &+ \int_0^T y\left(U_y - (w-\lambda)A_3 - x(w-\lambda)(A_3)_x\right)U_y dt.\end{aligned}$$

If we set $\lambda = w$, we will have

$$\int_0^T div(x,y)dt = \int_0^T xU_x^2(x,y) + yU_y^2(x,y)\,dt \neq 0.$$

Consequently, if the ovals of $U = 0$ be a periodic solutions of the system, then are a hyperbolic limit cycles.

Example 2.4.5 *We pose* $F(x,y) = 3x - 4 + 2y, G(x,y) = y - 4 - 2x, w = 1, A_3 = x + y$ *and* $U(x,y) = (x-2)^2 + (y-2)^2 - 1$, *system* (2.14) *become*

$$\begin{cases}\dot{x} = x(-28 + 37x - 16x^2 + 3x^3 + 30y - 24xy + 2x^2y - 16y^2 + 5xy^2 + 4y^3), \\ \dot{y} = y(-28 + 2x + 8x^2 - 4x^3 + 23y + 8xy - x^2y - 8y^2 - 2xy^2 + y^3).\end{cases} \tag{2.15}$$

We have

$$\begin{aligned}\frac{dU}{dt} &= \dot{x}U_x + \dot{y}U_y \\ &= (x(-4+2x)^2 + y(-4+2y)^2 + x(x+y)(-4+2x) - y(x+y)(-4+2y) \\ &\quad + xy(-4+2x) - xy(-4+2y))(7 - 4x + x^2 - 4y + y^2),\end{aligned}$$

then the cofactor is

$$K(x,y) = x(-4+2x)((-4+2x) + x + 2y) + y(-4+2y)((-4+2y) - 2x - y).$$

On the other hand

$$div(x.y) = x(-4+2x)((-4+2x) + 3y) + y(-4+2y)((-4+2y) - x).$$

So we have

$$\int_0^T div(x,y)dt = \lambda\int_0^T div(x,y)dt + (1-\lambda)\int_0^T K(x,y)dt.$$

then

$$\int_0^T div(x,y)dt = \int_0^T x((-4+2x)+(1-\lambda)(x+y)+y(1-\lambda))(-4+2x)dt + \int_0^T y(-4+2y)-(1-\lambda)(x+y)-x(1-\lambda))(-4+2y)dt,$$

If we take $\lambda = 1$ *, we will have*

$$\int_0^T div(x,y)dt = \int_0^T x(-4+2x)^2 + y(-4+2y)^2 dt \neq 0.$$

Consequently , the system has a hyperbolic limit cycle given by $U(x,y) = 0$*, moreover this limits cycle surrounds a stable focus.*

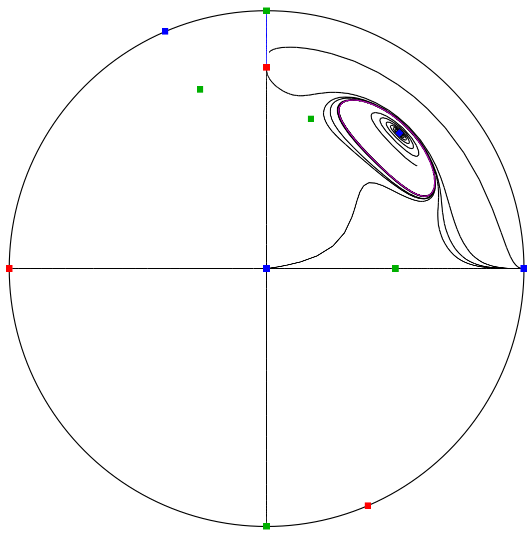


Figure 2.11: Limit cycles on the Poincare disc of System (2.15).

Example 2.4.6 *We pose* $w = 1, A_3 = x$ *and*

$$U(x,y) = ((y-2)^2 + (x-2)^2 - 1)^2 + 3(y-2)(x-2)^2 - (y-2)^3,$$

$F(x,y) = -32+49x-24x^2+4x^3+20y-10xy-8y^2+4xy^2$, $G(x,y) = -56+18x-$

$5x^2 + 72y - 16xy + 4x^2y - 27y^2 + 4y^3$, *system* (2.14) *become*

$$\begin{cases} \dot{x} = x \begin{pmatrix} -1056 + 2641x - 3128x^2 + 2332x^3 - 1128x^4 + 337x^5 - 56x^6 + 4x^7 \\ +2452y - 4410xy + 3304x^2y - 1354x^3y + 300x^4y - 30x^5y - 2536y^2 \\ +3440xy^2 - 1956x^2y^2 + 648x^3y^2 - 120x^4y^2 + 12x^5y^2 + 1456y^3 \\ -1372xy^3 + 456x^2y^3 - 76x^3y^3 - 500y^4 + 351xy^4 - 72x^2y^4 + 12x^3y^4 \\ +92y^5 - 46xy^5 - 8y^6 + 4xy^6 \end{pmatrix}, \\ \dot{y} = y \begin{pmatrix} -1848 + 2386x - 2053x^2 + 992x^3 - 296x^4 + 54x^5 - 5x^6 + 5512y \\ -4960xy + 3272x^2y - 1268x^3y + 321x^4y - 48x^5y + 4x^6y - 6939y^2 \\ +4296xy^2 - 1980x^2y^2 + 448x^3y^2 - 57x^4y^2 + 4740y^3 - 1982xy^3 \\ +692x^2y^3 - 96x^3y^3 + 12x^4y^3 - 1900y^4 + 458xy^4 - 115x^2y^4 + 459y^5 - \\ 48xy^5 + 12x^2y^5 - 63y^6 + 4y^7 \end{pmatrix}. \end{cases} \tag{2.16}$$

The curve $U(x, y) = 0$ *is an invariant algebraic curve of the differential system (2.16) withe the cofactor*
$k(x, y) = x^2(-32 + 48x - 24x^2 + 4x^3 + 20y - 10xy - 8y^2 + 4xy^2) + x(-32 + 48x - 24x^2 + 4x^3 + 20y - 10xy - 8y^2 + 4xy^2)^2 - 2xy(-56 + 20x - 5x^2 + 72y - 16xy + 4x^2y - 27y^2 + 4y^3) + y(-56 + 20x - 5x^2 + 72y - 16xy + 4x^2y - 27y^2 + 4y^3)^2$.

Then, we have
$\int_0^T div(\Gamma)dt = \int_0^T (x(-32 + 48x - 24x^2 + 4x^3 + 20y - 10xy - 8y^2 + 4xy^2)^2 + y(-56 + 20x - 5x^2 + 72y - 16xy + 4x^2y - 27y^2 + 4y^3)^2)dt \neq 0$.

Then, system (2.16) admits three hyperbolic limit cycles represented by the curve $U(x, y) = ((y - 2)^2 + (x - 2)^2 - 1)^2 + 3(y - 2)(x - 2)^2 - (y - 2)^3$
Notice that one of these three limit cycles surrounds a stable node and the two others surrounds a two stable focus.

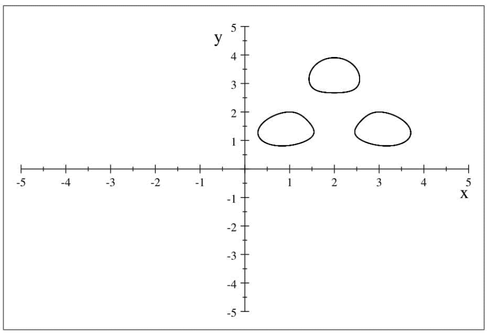


Figure 2.12: The ovals of the algebraic curve U of system (2.16).

Proposition 2.4.1 *If* $F(x, y) = U_x(x, y)$ *and* $G(x, y) = U_y(x, y)$ *(i.e.* $w = 0$*), then the system* (2.14) *admits the ovals of the algebraic curve* $U = 0$ *in the quadrant* Ω *as a limit cycles .*

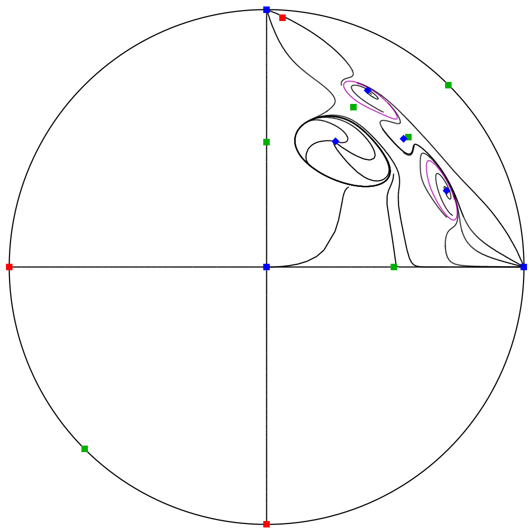


Figure 2.13: Limit cycles on the Poincare disc of system (2.16).

Proof: If we take $F(x,y) = U_x(x,y)$ and $G(x,y) = U_y(x,y)$(i.e. $w = 0$) in the system (2.14), then

$$\begin{cases} \dot{x} = x\left(U_x(x,y)\,U(x,y) + yA_3(x,y)\,U_y(x,y)\right), \\ \dot{y} = y\left(U_y(x,y)\,U(x,y) - xA_3(x,y)\,U_x(x,y)\right), \end{cases} \tag{2.17}$$

a direct calculation shows that

$$\begin{aligned} K(x,y) &= \frac{x\left(\left(U_x\right)U + A_3yU_y\right)U_x + y\left(\left(U_y\right)U - A_3xU_x\right)U_y}{U} \\ &= xU_x^2(x,y) + yU_y^2(x,y), \end{aligned}$$

therefore $U(x,y) = 0$ is an invariant algebraic curve of the differential system (2.17). Moreover we have

$$\int_0^T K(x,y)dt = \int_0^T xU_x^2(x,y) + yU_y^2(x,y)dt \neq 0.$$

i.e. the ovals of $U(x,y) = 0$ are algebraic limit cycles of a system (2.17).

Example 2.4.7 *We pose* $w = 0, A_3 = x + y$ *and* $U(x,y) = (x-2)^2 + (y-2)^2 - 1$ $F(x,y) = 2x - 4, G(x,y) = 2y - 4$, *system* (2.14) *become*

$$\begin{cases} \dot{x} = x(-28 + 30x - 12x^2 + 2x^3 + 16y - 12xy - 8y^2 + 4xy^2 + 2y^3), \\ \dot{y} = y(-28 + 16x - 2x^3 + 30y - 4xy - 12y^2 + 2y^3), \end{cases} \tag{2.18}$$

The curve $U(x,y) = 0$ *is an invariant algebraic curve of the differential system (2.18) withe the cofactor*

$$K(x,y) = x(-4 + 2x) + y(-4 + 2y).$$

Therefor

$$\int_0^T K(x,y)dt = \int_0^T xU_x^2(x,y) + yU_y^2(x,y)dt \neq 0.$$

i.e. $U(x,y) = 0$ *is an algebraic limit cycle of a system and this limit cycle surrounds a stable focus.*

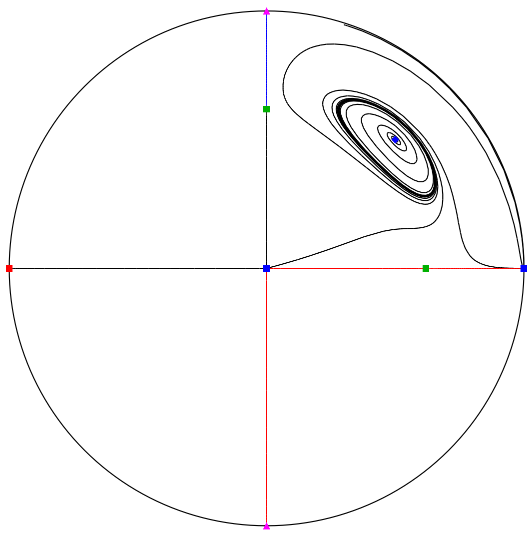


Figure 2.14: Limit cycles on the Poincare disc of System (2.18)

Example 2.4.8 *We pose* $w = 0, A_3 = x$ *and*

$$U(x,y) = ((y-2)^2 + (x-2)^2 - 1)^2 + 3(y-2)(x-2)^2 - (y-2)^3$$

Then $F(x,y) = -32 + 48x - 24x^2 + 4x^3 + 20y - 10xy - 8y^2 + 4xy^2$, $G(x,y) = -56 + 20x - 5x^2 + 72y - 16xy + 4x^2y - 27y^2 + 4y^3$, *system* (2.14) *become*

$$\begin{cases} \dot{x} = x \begin{pmatrix} -1056 + 2608x - 3096x^2 + 2308x^3 - 1120x^4 + 336x^5 - 56x^6 + 4x^7 \\ +2452y - 4354xy + 3284x^2y - 1349x^3y + 300x^4y - 30x^5y - 2536y^2 \\ +3404xy^2 - 1948x^2y^2 + 646x^3y^2 - 120x^4y^2 + 12x^5y^2 + 1456y^3 \\ -1363xy^3 + 456x^2y^3 - 76x^3y^3 - 500y^4 + 350xy^4 - 72x^2y^4 + 12x^3y^4 \\ +92y^5 - 46xy^5 - 8y^6 + 4xy^6 \end{pmatrix}, \\ \dot{y} = y \begin{pmatrix} -1848 + 2452x - 2117x^2 + 1040x^3 - 312x^4 + 56x^5 - 5x^6 + 5512y \\ -5072xy + 3312x^2y - 1278x^3y + 321x^4y - 48x^5y + 4x^6y \\ -6939y^2 + 4368xy^2 - 1996x^2y^2 + 452x^3y^2 - 57x^4y^2 + 4740y^3 \\ -2000xy^3 + 692x^2y^3 - 96x^3y^3 + 12x^4y^3 - 1900y^4 + 460xy^4 \\ -115x^2y^4 + 459y^5 - 48xy^5 + 12x^2y^5 - 63y^6 + 4y^7 \end{pmatrix}, \end{cases} \tag{2.19}$$

The curve $U(x, y) = 0$ is an invariant algebraic curve of the differential system (2.19) withe the cofactor

$K(x, y) = x(-32 + 48x - 24x^2 + 4x^3 + 20y - 10xy - 8y^2 + 4xy^2) + y(-56 + 20x - 5x^2 + 72y - 16xy + 4x^2y - 27y^2 + 4y^3)$.

Then, we have

$$\int_0^T div(\Gamma)dt = \int_0^T xU^2(x, y) + yU^2(x, y)dt \neq 0.$$

Then, system (2.19) admits three hyperbolic limit cycles represented by the curve

$$U(x, y) = ((y - 2)^2 + (x - 2)^2 - 1)^2 + 3(y - 2)(x - 2)^2 - (y - 2)^3 = 0.$$

One of theses three limit cycles surrounds a stable node and the other two limit cycles surrounds a stable focus.

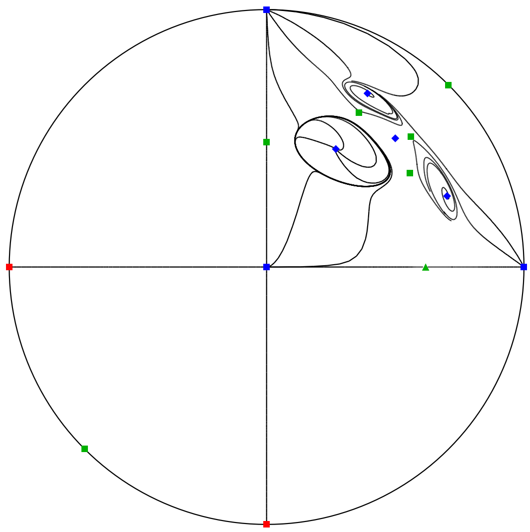


Figure 2.15: Limit cycles on the Poincare disc of system (2.19)

Chapter 3

Explicit non-algebraic limit cycle of kolmogorov systems

3.1 Introduction

For a given system, it is very difficult to know the number of limit cycles and to know if they are algebraic or not, as well as the determination of their explicit expressions. Many researchers have been interested in studying Kolmogorov systems, in particular, integrability and the existence of limit cycles. To our knowledge, all the explicit expressions of the limit cycles to our days were only algebraic (see works [1],[3],[4] are interested in studying Kolmogorov systems. It was only after 2006 that it became possible to find explicit expressions of cycles non algebraic limits. After the Odani work [12], where it has been proved that the limit cycle appearing in the Van der Pol equation is not algebraic without giving an explicit expression, several articles have been published presenting differential systems polynomials for which non-algebraic limit cycles exist and are explicitly determined see [2] interested in studying Kolmogorov systems.

In this chapter we are interested in studying the integrability and the limit cycles of the 2-dimensional Kolmogorov systems of the form

$$\begin{cases} \dot{x} = x \begin{pmatrix} y(x-\alpha)(xy + (a+2c)(x-\alpha)^2 + (a-2c)(y-\beta)^2) \\ +((x-\alpha)^2 + (y-\beta)^2)((x-\alpha)^2 - y^2 + \beta^2) \end{pmatrix}, \\ \dot{y} = y \begin{pmatrix} x(y-\beta)(xy + (a+2c)(x-\alpha)^2 + (a-2c)(y-\beta)^2) \\ +((x-\alpha)^2 + (y-\beta)^2)(x^2 - \alpha^2 - (y-\beta)^2) \end{pmatrix}, \end{cases} \tag{3.1}$$

where a, c, α and β are real constants. Moreover, we determine sufficient conditions for a polynomial differential system to possess an explicit non-algebraic or algebraic limit cycle. Concrete examples exhibiting the applicability of our result are introduced.

3.2 Main result

Theoreme 3.2.1 *Consider polynomial differential system (3.1). Then the following statements hold.*
If $\beta > 0, a < 0, \alpha > 0, |2a+1| > 4|c|, 2a+1 < 0$, *then the system (3.1) has an explicit limit cycle, given in polar coordinates* (r, θ) *where* $x = r\cos(\theta) + \alpha, y = r\sin(\theta) + \beta$

$$r(\theta, r_*) = \frac{\rho(\theta, r_*)g(\theta) + \sqrt{(\rho(\theta, r_*)g(\theta))^2 + 2\alpha\beta\rho(\theta, r_*)(2 - \rho(\theta, r_*)\sin(2\theta)}}{2 - \rho(\theta, r_*)\sin(2\theta)},$$

where
$\rho(\theta, r_*) = e^{a\theta + c\sin(2\theta)}(\frac{r_*^2}{\beta(r_* + \alpha)} + f(\theta)), g(\theta) = \beta cos(\theta) + \alpha\sin(\theta)$,
$f(\theta) = \int_0^\theta e^{-as - c\sin(2s)}ds$ *and*
$r_* = \left(\frac{-e^{\pi a}}{2e^{2\pi a} - 2}\right)(\beta e^{\pi a}f(2\pi) + \sqrt{(\beta e^{\pi a}f(2\pi))^2 - 4\alpha\beta(e^{2\pi a} - 1)f(2\pi)})$.
Moreover this limit cycle is stable hyperbolic limit cycle and:
i) if $c \neq 0$, *this limit cycle is non algebraic.*
ii) if $c = 0$, *this limit cycle is algebraic and given in Cartesian coordinates* (x, y) *by*

$$(x - \alpha)^2 + (y - \beta)^2 + \frac{1}{a}xy = 0.$$

3.3 proof of main results

Of all it is easy to testify $(\alpha, \beta), \alpha > 0, \beta > 0$ is positive equilibrium point of system (3.1), through the transformation $x = X + \alpha, y = Y + \beta$, we still denote X, Y by x, y for convenience, the point (α, β) will be moved into $(0, 0)$ of the new system and the system (3.1) will be changed into following system:

$$\begin{cases} \dot{x} = (x + \alpha)\begin{pmatrix} x(x + \alpha)(y + \beta)^2 + (x^2 + y^2)(x^2 - y^2 - 2y\beta) \\ +x(y + \beta)(ax^2 + ay^2 + 2c(x^2 - y^2)) \end{pmatrix}, \\ \dot{y} = (y + \beta)\begin{pmatrix} y(x + \alpha)^2(y + \beta) + (x^2 + y^2)(x^2 - y^2 + 2x\alpha) \\ +y(x + \alpha)(ax^2 + ay^2 + 2c(x^2 - y^2)) \end{pmatrix}, \end{cases} \tag{3.2}$$

Clearly the equilibrium point (0.0) of the differential system (3.2) is an unstable node because its eigenvalues are $\alpha^2\beta^2$ with multiplicity for more details see for instance [9].
Notice that system (3.1) has an periodic orbit contained in the open quadrant $(x > 0, y > 0)$ if and if system (3.2) has an periodic orbit contained in the quadrant $(x > -\alpha, y > -\beta)$.

Lemma 3.3.1 *System* (3.1) *has the first integral of the form*

$$H(x, y) = \frac{(x - \alpha)^2 + (y - \beta)^2}{xy}e^{-(a\arctan\frac{y-\beta}{x-\alpha} + c\sin 2\arctan\frac{y-\beta}{x-\alpha})} - \int_0^{\arctan\frac{y-\beta}{x-\alpha}} e^{-as - c\sin 2s}ds.$$

Proof The polynomial differential system (3.2) in polar coordinates becomes

$$\begin{cases} \dot{r} = f_1(\theta)r^5 + f_2(\theta)r^4 + f_3(\theta)r^3 + f_4 r^2 + \alpha^2\beta^2 r, \\ \dot{\theta} = g(\theta)r^3 + 2\alpha\beta r^2, \end{cases} \tag{3.3}$$

where

$g(\theta) = \beta\cos\theta + \alpha\sin\theta,$

$f_1(\theta) = \cos^4\theta - \sin^4\theta + \cos^2\theta\sin^2\theta + (a-2c)\cos\theta\sin^3\theta + (a+2c)\cos^3\theta\sin\theta,$

$f_2(\theta) = (\alpha + \alpha\beta + 2c\beta)\cos^3\theta + (a\alpha - 2c\alpha - \beta)\sin^3\theta + (\beta + a\alpha + 2c\alpha)\cos^2(\theta)\sin\theta + (3\alpha + a\beta - 2c\beta)\cos\theta\sin^2\theta,$

$f_3(\theta) = \beta^2\cos 2\theta + \alpha^2\sin^2\theta + 4\alpha\beta\cos\theta\sin\theta + 2c\alpha\beta\cos 2\theta + a\alpha\beta,$

$f_4(\theta) = 2\alpha g(\theta).$

In the open quadrant $(x > -\alpha, y > -\beta)$ we have $(r\cos\theta > -\alpha, r\sin\theta > -\beta)$, therefor $2\alpha\beta + \beta r\cos\theta + \alpha r\sin\theta > 0$ for all $\theta \in \mathbb{R}$ so $\dot{\theta} > 0$ for all $\theta \in \mathbb{R}$.
The differential system (3.3) where $g(\theta)r^3 + 2\alpha\beta r^2 \neq 0$ can be written as the equivalent differential equation

$$\frac{dr}{d\theta} = \frac{f_1(\theta)r^4 + f_2(\theta)r^3 + f_3(\theta)r^2 + f_4(\theta)r + (\alpha^2\beta^2)}{g(\theta)r^2 + 2\alpha\beta r}. \tag{3.4}$$

Note that since $\dot{\theta}$ is positive for all t, the orbit $r(\theta)$ of the differential equation (3.4) has preserved their orientation with respect to the orbits $(r(t), \theta(t))$ or $(x(t), y(t))$ the differential systems (3.3) or (3.2).
Via the change of variables $\rho = \dfrac{r^2}{(r\cos\theta + \alpha)(r\sin\theta + \beta)}$, we have

$$\begin{aligned} \frac{d\rho}{d\theta} &= \frac{2}{(\alpha + r\cos\theta)^2}\frac{2r\alpha\beta + r^2\beta\cos\theta + r^2\alpha\sin\theta}{2\beta^2 - r^2\cos 2\theta + r^2 + 4r\beta\sin\theta}\frac{dr}{d\theta} \\ &+ 2\frac{r^3}{(\alpha + r\cos\theta)^2}\frac{-2r\cos^2(\theta) - \alpha\cos\theta + r\beta\sin\theta}{2\beta^2 - r^2\cos 2\theta + r^2 + 4r\beta\sin\theta}. \end{aligned}$$

and the differential equation (3.4) becomes the linear differential equation

$$\frac{d\rho}{d\theta} = (a + 2c\cos 2\theta)\rho + 1. \tag{3.5}$$

The general solution of linear equation (3.5) is

$$\rho(\theta, h) = e^{a\theta + c\sin 2\theta}\left(h + \int_0^\theta e^{-as - c\sin 2s} ds\right).$$

where $h \in \mathbb{R}$.
Consequently, the implicit form of the solution of the differential equation (3.4) is

$$F(\theta, r) = r^2 - (r\cos\theta + \alpha)(r\sin\theta - \beta)\rho(\theta, h) = 0.$$

By passing to Cartesian coordinates we deduce the first integral is

$$H(x, y) = \frac{x^2 + y^2}{(x+\alpha)(y+\beta)} e^{-(a \arctan\frac{y}{x} + c\sin 2\arctan\frac{y}{x})} - \int_0^{\arctan\frac{y}{x}} e^{-as - c\sin 2s} ds.$$

Going back through the changes of variables we obtain the first integral of the lemma

$$H(x, y) = \frac{(x-\alpha)^2 + (y-\beta)^2}{xy} e^{-(a \arctan\frac{y-\beta}{x-\alpha} + c\sin 2\arctan\frac{y-\beta}{x-\alpha})} - \int_0^{\arctan\frac{y-\beta}{x-\alpha}} e^{-as - c\sin 2s} ds.$$

Lemma 3.3.2 *if* $\beta > 0, a < 0, \alpha > 0, |2a+1| > |4c|, 2a+1 < 0$ *the following statement hold*

$$0 < \rho(\theta) = e^{a\theta + c\sin 2\theta}\left(\frac{e^{2a\pi}}{1 - e^{2a\pi}} f(2\pi) + \int_0^\theta e^{-as - c\sin 2s} ds\right) < 2,$$

where $f(2\pi) = \int_0^{2\pi} e^{-as - c\sin 2s} ds$.

Proof: Let $\varphi(\theta) = 2e^{-(a\theta + c\sin 2\theta)} - f(\theta)$, we prove that $0 < \rho(\theta) < 2$ let $\varphi(\theta) = 2e^{-(a\theta + c\sin 2\theta)} - f(\theta)$, then

$$\frac{d\varphi}{d\theta} = -(2a + 1 + 4c\cos 2\theta) e^{-(a\theta + c\sin 2\theta)}$$

Since $a < 0, |2a+1| > 4|c|, 2a+1 < 0$ it follow $2a + 1 + 4c\cos 2\theta < 2a + 1 + 4|c| < 0, -(2a + 1 + 4c\cos 2\theta) > 0$, hence the function $\theta \mapsto \varphi(\theta)$ is strictly rising withe

$$\varphi(0) = 2 < 2e^{-a\theta - c\sin 2\theta} - f(\theta) < 2e^{-2a\pi} - f(2\pi) = \varphi(2\pi). \tag{3.6}$$

for all $\theta \in [0; 2\pi]$.
Taking into account (3.6), we have $2 < 2e^{-a\theta - c\sin 2\theta} - f(\theta)$, then $f(\theta) < 2e^{-a\theta - c\sin 2\theta} - 2$ and

$$\begin{aligned}\rho(\theta) &= e^{a\theta + c\sin 2\theta}\left(\frac{e^{2a\pi}}{1 - e^{2a\pi}} f(2\pi) + f(\theta)\right) \\ &< e^{a\theta + c\sin 2\theta}\left(\frac{e^{2a\pi}}{1 - e^{2a\pi}} f(2\pi) + 2e^{-a\theta - c\sin 2\theta} - 2\right).\end{aligned}$$

from (3.6), we have $2 < 2e^{-2a\pi} - f(2\pi)$, then $f(2\pi) < 2e^{-2a\pi} - 2$, therefore

$$\begin{aligned}\rho(\theta) &= e^{a\theta + c\sin 2\theta}\left(\frac{e^{2a\pi}}{1 - e^{2a\pi}} f(2\pi) + f(\theta)\right) \\ &< e^{a\theta + c\sin 2\theta}\left(\frac{e^{2a\pi}}{1 - e^{2a\pi}} (2e^{-2a\pi} - 2) + 2e^{-a\theta - c\sin 2\theta} - 2\right) \\ &< e^{a\theta + c\sin 2\theta}(2 + 2e^{-a\theta - c\sin 2\theta} - 2) = 2.\end{aligned}$$

We have $\dfrac{df}{d\theta} = e^{a\theta+csin2\theta} > 0$, thus $0 < f(\theta) < f(2\pi)$.
Moreover, we have

$$\begin{aligned}\rho(\theta) &= e^{a\theta+c\sin 2\theta}(\frac{e^{2a\pi}}{1-e^{2a\pi}}f(2\pi)+f(\theta)) \\ &> e^{a\theta+c\sin 2\theta}(\frac{e^{2a\pi}}{1-e^{2a\pi}}f(\theta)+f(\theta)) \\ &> \frac{e^{a\theta+c\sin 2\theta}}{1-e^{2a\pi}}f(\theta) > 0.\end{aligned}$$

Because $a < 0$ and $0 < f(\theta) < f(2\pi)$.

3.3.1 Proof of Theorem 3.2.1

First we prove that the system (3.1) has an explicit limit cycle.
Notice the system (3.2) has a periodic orbit if and only is equation (3.4) has a strictly positive 2π- periodic solution $r(\theta)$. This, moreover, this is equivalent to the existence of a solution of (3.4) that satisfies $r(0;r_*) = r(2\pi, r_*)$ and $r(\theta, r_*) > 0$ for any θ in $[0, 2\pi]$. we remark that the solution $r(\theta, r_0)$ of the differential equation (3.4) such that $r(\theta, r_0) = r_0 > 0$, we have

$$F(0, r_0) = r_0^2 - \beta(r+\alpha)h = 0,$$

therefore $h = \dfrac{r_0^2}{\beta(r_0+\alpha)}$.
Corresponds to the value h provided a rewriting of the implicit form of the solution of the differential equation (3.4) as,

$$F(\theta, r) = r^2 - (r\cos\theta+\alpha)(r\sin\theta+\beta)e^{a\theta+c\sin 2\theta}\left(\frac{r_0^2}{\beta(r_0+\alpha)}+f(\theta)\right) = 0.$$

A periodic solution of system (3.3) must satisfy the condition $r(2\pi, r_0) = r(0, r_0)$.
Since, $r^2(\theta, r_0) = (r\cos\theta+\alpha)(r\sin\theta+\beta)\rho(\theta, h)$. Where $\rho(\theta, h), \cos\theta$ and $\sin\theta$ are 2π periodic functions.
then $r(2\pi, r_0) = r(0, r_0)$ if and only if $\rho(2\pi, r_0) = \rho(0, r_0)$ with
$\rho(0, r_0) = \dfrac{r_0^2}{\beta(r_0+\alpha)}$ and $\rho(2\pi, r_0) = e^{2a\pi}\left(\dfrac{r_0^2}{\beta(r_0+\alpha)}+f(2\pi)\right)$,
so we have

$$\frac{r_0^2}{\beta(r_0+\alpha)} = e^{2a\pi}\left(\frac{r_0^2}{\beta(r_0+\alpha}+f(2\pi)\right),$$

therefore

$$\frac{r_0^2}{\beta(r_0+\alpha)} = \frac{e^{2a\pi}}{1-e^{2a\pi}}f(2\pi). \tag{3.7}$$

implies that

$$\rho(\theta) = e^{a\theta+c\sin 2\theta}\left(\frac{e^{2a\pi}}{1-e^{2a\pi}}f(2\pi)+f(\theta)\right). \tag{3.8}$$

Where $f(\theta) = \int_0^\theta e^{-as-c\sin 2s}$ and $f(2\pi) = \int_0^{2\pi} e^{-as-c\sin 2s}ds$.
Then the implicit form of the solution system (3.4) such that $r(2\pi, r_0) = r(0, r_0)$ is given by

$$F(\theta, r) = r^2 - (r\cos\theta + \alpha)(r\sin\theta + \beta)\rho(\theta) = 0. \tag{3.9}$$

From (3.9) we can obtain a two different values of $r(\theta)$.after distribution this equation we obtain

$$r^2\left(1 - \rho(\theta)\frac{\sin 2\theta}{2}\right) - rg(\theta)\rho(\theta) - \alpha\beta\rho(\theta) = 0.$$

The discriminant of this quadratic equation is:

$$\Delta = (g(\theta)(\rho(\theta))^2 + 2\alpha\beta(2 - \sin 2\theta)\rho(\theta).$$

Since $\alpha > 0, \beta > 0, 0 < \rho(\theta) < 2$, for all $\theta \in [0; 2\pi]$ it follows that $2 - \rho(\theta)\sin 2\theta > 0$, so $\Delta > 0$.
We have a two different values of $r(\theta)$ the first value

$$r_1 = \frac{g(\theta)\rho(\theta) - \sqrt{(g(\theta)\rho(\theta))^2 + 2\alpha\beta\rho(\theta)(2 - \rho(\theta)\sin 2\theta)}}{2 - \rho(\theta)\sin 2\theta}.$$

Since $-(\alpha + \beta) < g(\theta) < (\alpha + \beta), 0 < \rho(\theta) < 2$ for all $\theta \in [0; 2\pi]$ it follows that $2 - \rho(\theta)\sin 2\theta > 0$, therefore

$$(g(\theta)\rho(\theta))^2 + 2\alpha\beta\rho(\theta)(2 - \rho(\theta)\sin 2\theta) > (g(\theta)\rho(\theta))^2,$$

then

$$\sqrt{(g(\theta)\rho(\theta))^2 + 2\alpha\beta\rho(\theta)(2 - \rho(\theta)\sin 2\theta)} > |(g(\theta)\rho(\theta))|,$$

thus

$$-\sqrt{(g(\theta)\rho(\theta))^2 + 2\alpha\beta\rho(\theta)(2 - \rho(\theta)\sin 2\theta)} < -(g(\theta)\rho(\theta)),$$

so

$$(g(\theta)\rho(\theta)) - \sqrt{(g(\theta)\rho(\theta))^2 + 2\alpha\beta\rho(\theta)(2 - \rho(\theta)\sin 2\theta)} < 0.$$

Consequently we have $r_1 < 0$ and we do not consider this cases, the second value is positive

$$r(\theta, r_0) = \frac{g(\theta)\rho(\theta) + \sqrt{(g(\theta)\rho(\theta))^2 + 2\alpha\beta\rho(\theta)(2 - \rho(\theta)\sin 2\theta)}}{2 - \rho(\theta)\sin 2\theta},$$

where $\rho(\theta) = e^{a\theta + c\sin 2\theta}\left(\frac{e^{2a\pi}}{1 - e^{2a\pi}}f(2\pi) + f(\theta)\right)$.
From (3.7) we can obtain a two different values of r_0. After some distribution this equality we obtain the following equation:

$$r_0^2 - r_0\left(\beta\frac{e^{2a\pi}}{1 - e^{2a\pi}}f(2\pi)\right) - \alpha\beta\frac{e^{2a\pi}}{1 - e^{2a\pi}}f(2\pi) = 0.$$

The discriminant of this quadratic equation is:

$$\Delta = \left(\frac{e^{2a\pi}}{1-e^{2a\pi}} f(2\pi)\right)^2 + 4\alpha\beta\frac{e^{2a\pi}}{1-e^{2a\pi}} f(2\pi).$$

Since $a < 0, \alpha > 0, \beta > 0$ and $f(\theta) > 0$ for all $\theta \in \mathbb{R}$ there is $\frac{e^{2a\pi}}{1-e^{2a\pi}} > 0$, thus $\Delta > 0$. We have a two different values of $r(\theta)$; the first value is

$$r_{*1} = \frac{-e^{a\pi}}{2e^{2a\pi}-2}\left(\beta f(2\pi)e^{a\pi} - \sqrt{(\beta e^{a\pi} f(2\pi))^2 - 4\alpha\beta(e^{2a\pi}-1)f(2\pi)}\right).$$

Since $a < 0, \alpha > 0, \beta > 0$ and $f(\theta) > 0$ for all $\theta \in \mathbb{R}$ we have, $\frac{e^{a\pi}}{2e^{2a\pi}-2} < 0$, then

$$-4\alpha\beta(e^{2a\pi}-1)f(2\pi) > 0,$$

thus

$$(\beta e^{a\pi} f(2\pi))^2 - 4\alpha\beta(e^{2a\pi}-1) > (\beta e^{a\pi} f(2\pi))^2,$$

and

$$\beta f(2\pi)e^{a\pi} - \sqrt{(\beta e^{a\pi} f(2\pi))^2 - 4\alpha\beta(e^{2a\pi}-1)f(2\pi)} < 0,$$

thus

$$\frac{-e^{a\pi}}{2e^{2a\pi}-2}\left(\beta e^{a\pi} f(2\pi) - \sqrt{\beta e^{a\pi} f(2\pi))^2 - 4\alpha\beta(e^{2a\pi}-1)}\right) < 0.$$

Then $r_{*1} < 0$, and we do not consider this case. We only take into consideration the following value $r_* > 0$

$$r_* = \frac{-e^{a\pi}}{2e^{2a\pi}-2}\left(\beta f(2\pi)e^{a\pi} + \sqrt{(\beta e^{a\pi} f(2\pi))^2 - 4\alpha\beta(e^{2a\pi}-1)f(2\pi)}\right).$$

After the substitution of these value r_* into $r(\theta, r_0)$ we obtain

$$r(\theta, r_*) = \frac{\rho(\theta, r_*)g(\theta) + \sqrt{(\rho(\theta, r_*)g(\theta))^2 + 2\alpha\beta\rho(\theta, r_*)(2-\rho(\theta, r_*)\sin(2\theta))}}{2-\rho(\theta, r_*)\sin(2\theta)}, \tag{3.10}$$

where $\rho(\theta, r_*) = e^{a\theta + c\sin 2\theta}(\dfrac{r_*^2}{\beta(r_*+\alpha)} + f(\theta))$, and the implicit form of the solution of the differential equation (3.4) can be written as

$$F(r, \theta) = r^2 - (r\cos\theta + \alpha)(r\sin\theta + \beta)e^{a\theta + c\sin 2\theta}\left(\frac{r_*^2}{\beta(r_*+\alpha)} + f(\theta)\right) = 0. \tag{3.11}$$

To show that it is a periodic solution of system (3.2) and is located in the quadrant $(x > -\alpha, y > -\beta)$, we have to show that:
1) On the (3.11) does not exist any singular point.
2) The function $\theta \mapsto r(\theta, r_*)$ is 2π periodic.
3) The orbit (3.11) do not intersect the straight lines $r\cos\theta + \alpha = 0, r\sin\theta + \beta = 0$.
4) The function (3.10) is positive.

1) **Prove that there is no singular point in (3.11)**. In particularly we prove that the straight lines $rg(\theta)+2\alpha\beta$ does not intersect the orbit (3.11).
We prove that the system

$$\begin{cases} r^2-(r\cos\theta+\alpha)(r\sin\theta+\beta)e^{a\theta+c\sin 2\theta}(\dfrac{r_*^2}{\beta(r_*+\alpha)}+f(\theta))=0, \\ r(\beta\cos\theta+\alpha\sin\theta)+2\alpha\beta=0, \end{cases} \tag{3.12}$$

has no solution.
In the second equation implies that $r\sin\theta+\beta=-\frac{\beta}{\alpha}(r\cos\theta+\alpha)$ then the first equation, can be can be written as

$$r^2=-\frac{\beta}{\alpha}(r\cos\theta+\alpha)^2e^{a\theta+c\sin 2\theta}(\frac{r_*^2}{\beta(r_*+\alpha)}+f(\theta)).$$

Since $a<0, \alpha>0, \beta>0, e^{a\theta+c\sin 2\theta}\left(\dfrac{r_*^2}{\beta(r_*+\alpha)}+f(\theta)\right)>0$ and $-\frac{\beta}{\alpha}(r\cos\theta+\alpha)^2<0$ so $r^2<0$ for all $\theta\in\mathbb{R}$ which is a contradiction , then the (3.12) has no solution.

2) **Periodicity of solution $\mathbf{r}(\theta,\mathbf{r_0})$**. Prove that $\theta\mapsto r(\theta,r_*)$ is 2π periodic

$$r(\theta,r_*)=\frac{\rho(\theta,r_*)g(\theta)+\sqrt{(\rho(\theta,r_*)g(\theta))^2+2\alpha\beta\rho(\theta,r_*)(2-\rho(\theta,r_*)\sin 2\theta)}}{2-\rho(\theta,r_*)\sin 2\theta}.$$

the function $g(\theta)$ is 2π-periodic it follows that $r(\theta,r_*)$ is 2π-periodic if $\rho(\theta,r_*)$ is periodic and we have

$$\begin{aligned} \rho(\theta+2\pi) &= e^{a(\theta+2\pi)+c\sin 2(\theta+2\pi)}\left(\frac{r_*^2}{\beta(r_*+\alpha)}+f(\theta+2\pi)\right) \\ &= e^{a(\theta+2\pi)+c\sin 2\theta}\left(\frac{e^{2a\pi}}{1-e^{2a\pi}}f(2\pi)+f(\theta+2\pi)\right). \end{aligned} \tag{3.13}$$

In other hand, we have

$$\begin{aligned} f(\theta+2\pi) &= \int_0^{\theta+2\pi} e^{-as-c\sin 2s}ds \\ &= \int_0^{2\pi} e^{-as-c\sin 2s}ds+\int_{2\pi}^{\theta+2\pi} e^{-as-c\sin 2s}ds \\ &= f(2\pi)+\int_{2\pi}^{\theta+2\pi} e^{-as-c\sin 2s}ds. \end{aligned}$$

We make to change of variable $\mu=s-2\pi$ in the integral $\int_{2\pi}^{\theta+2\pi} e^{-as-c\sin 2s}ds$ we get

$$\begin{aligned} f(\theta+2\pi) &= f(2\pi)+\int_0^{\theta} e^{-a(\mu+2\pi)-c\sin 2\mu} \\ &= f(2\pi)+e^{-2a\pi}f(\theta). \end{aligned} \tag{3.14}$$

We replace $f(\theta+2\pi)$ by (3.14) in (3.13).

$$\begin{aligned}\rho(\theta+2\pi, r_*) &= e^{a(\theta+2\pi)+c\sin 2\theta}\left(\frac{e^{2a\pi}}{1-e^{2a\pi}}f(2\pi)+f(2\pi)+e^{-2a\pi}f(\theta)\right)\\ &= e^{a\theta+c\sin 2\theta}\left(\frac{e^{2a\pi}}{1-e^{2a\pi}}f(2\pi)+f(\theta)\right)=\rho(\theta, r_*).\end{aligned}$$

We obtain that $\rho(\theta+2\pi, r_*)=\rho(\theta, r_*)$, since $g(\theta)$ and $\rho(\theta, r_*)$ are 2π periodic functions, then $r(\theta, r_*)$ is also 2π periodic function.

3) **Show that the periodic solution of system (3.2) is located in the quadrant in** $(x>-\alpha, y>-\beta)$, we have to show that the orbit (3.11) do not intersect the straight lines $r\cos\theta+\alpha=0, r\sin\theta+\beta=0$.
the orbit (3.11) can be written as

$$r^2=(r\cos\theta+\alpha)(r\sin\theta+\beta)e^{a\theta+c\sin 2\theta}\left(\frac{r_*^2}{\beta(r_*+\alpha)}+f(\theta)\right). \tag{3.15}$$

If substituting $r\cos\theta+\alpha=0$ in (3.15), we get $r^2=0$ contradiction, agene if substituting $r\cos\theta+\beta=0$ in (3.15), we get $r^2=0$ contradiction. This implies that the orbit (3.11) do not intersect the straight line $r\cos\theta+\alpha=0$ and $r\cos\theta+\alpha=0$ so the orbit (3.11) is located in the quadrant in $(x>-\alpha, y>-\beta)$.

4) **The strict positivity of $\mathbf{r}(\theta, \mathbf{r_0})$.**
We prove that $r(\theta, r_*)>0$, since $a<0, \alpha>0, \beta>0$ and $|g(\theta)|<\alpha+\beta$ for all $\theta\in\mathbb{R}$ then,

$$2\alpha\beta\rho(\theta, r_*)(2-\rho(\theta, r_*)\sin 2\theta)>0,$$

and

$$(\rho(\theta, r_*)g(\theta))^2+2\alpha\beta\rho(\theta, r_*)(2-\rho(\theta, r_*)\sin 2\theta)>(\rho(\theta, r_*)g(\theta))^2,$$

thus

$$\rho(\theta, r_*)g(\theta)+\sqrt{(\rho(\theta, r_*)g(\theta))^2+2\alpha\beta\rho(\theta, r_*)(2-\rho(\theta, r_*)\sin 2\theta)}>0,$$

Therefore,

$$r(\theta, r_*)=\frac{\rho(\theta, r_*)g(\theta)+\sqrt{(\rho(\theta, r_*)g(\theta))^2+2\alpha\beta\rho(\theta, r_*)(2-\rho(\theta, r_*)\sin(2\theta)}}{2-\rho(\theta, r_*)\sin(2\theta)}>0.$$

In order to prove that the periodic orbit is hyperbolic limit cycle. For this aim, we introduce the Poincare return map of (3.10) is given by

$$r_0\to P(2\pi,\gamma)=r(2\pi,\gamma)=\frac{1}{2}\left(\beta\rho(2\pi,\gamma)+\sqrt{(\beta\rho(2\pi,\gamma))^2+4\alpha\beta\rho(2\pi,\gamma)}\right),$$

with $\rho(2\pi,\gamma) = e^{2a\pi}\left(\frac{\gamma^2}{\beta(\gamma+\alpha)} + \int_0^{2\pi} e^{-as-c\sin 2sds}\right)$.

1) A periodic orbit of system (3.2) is hyperbolic limit cycle if and only if

$$\left.\frac{dr(2\pi,\gamma)}{d\gamma}\right|_{\gamma=r_*} \neq 1$$

in

$$r_* = \frac{-e^{\pi a}}{2e^{2\pi a}-2}\left(\beta e^{\pi a} f(2\pi) + \sqrt{(\beta e^{\pi a} f(2\pi))^2 - 4\alpha\beta(e^{2\pi a}-1)f(2\pi)}\right).$$

(see [9]).
We remark that $g(2\pi) = \beta$

$$r(2\pi,\gamma) = f(\rho(2\pi,\gamma)) = \frac{\beta\rho(2\pi,\gamma) + \sqrt{(\beta\rho(2\pi,\gamma))^2 + 4\alpha\beta\rho(2\pi,\gamma)}}{2}$$

and

$$\frac{d}{d\gamma} e^{2a\pi}\left(\frac{\gamma^2}{\beta(\gamma+\alpha)} + \int_0^{2\pi} e^{-as-c\sin 2sds}\right).$$

$$\begin{aligned}
\frac{d}{d\gamma}\left(r(2\pi,\gamma)\right) &= \frac{d}{d\gamma}\rho(2\pi,\gamma).\frac{d}{d\rho}f(\rho(2\pi,\gamma)) \\
&= \frac{e^{2\pi a}}{2}\frac{\gamma\left(2\alpha+\gamma\right)}{\left(\alpha+\gamma\right)^2}\frac{\left(2\alpha + \beta\rho(2\pi,\gamma) + \sqrt{\beta\rho(2\pi,\gamma)\left(4\alpha+\beta\rho(2\pi,\gamma)\right)}\right)}{\sqrt{\beta\rho(2\pi,\gamma)\left(4\alpha+\beta\rho(2\pi,\gamma)\right)}}.
\end{aligned}$$

Since $\rho(2\pi,\gamma) = e^{2a\pi}\left(\frac{\gamma^2}{\beta(\gamma+\alpha)} + f\left(2\pi\right)\right)$, then

$$\frac{d}{d\gamma}\left(r(2\pi,\gamma)\right) = \frac{e^{2\pi a}}{2}\frac{\gamma\left(2\alpha+\gamma\right)}{\left(\alpha+\gamma\right)^2}\left(\frac{\left(2\alpha + \beta e^{2a\pi}\left(\frac{\gamma^2}{\beta(\gamma+\alpha)} + f(2\pi)\right)\right)}{\sqrt{\beta e^{2a\pi}\left(\frac{\gamma^2}{\beta(\gamma+\alpha)} + f(2\pi)\right)\left(4\alpha + \beta e^{2a\pi}\left(\frac{\gamma^2}{\beta(\gamma+\alpha)} + f(2\pi)\right)\right)}} + 1\right)$$

Taking into account $\dfrac{r_0^2}{\beta(r_0+\alpha)} = \dfrac{e^{2a\pi}}{1-e^{2a\pi}} f(2\pi)$, we have

$$\begin{aligned}
\frac{d}{d\gamma}\left(r(2\pi,\gamma)\right) &= \frac{\gamma e^{2\pi a}\left(2\alpha+\gamma\right)}{2\left(\alpha+\gamma\right)^2}\left(\frac{\left(2\alpha + \beta e^{2a\pi}\left(-\frac{f(2\pi)}{e^{2\pi a}-1}\right)\right)}{\sqrt{\beta e^{2a\pi}\left(-\frac{f(2\pi)}{e^{2\pi a}-1}\right)\left(4\alpha + \beta e^{2a\pi}\left(-\frac{f(2\pi)}{e^{2\pi a}-1}\right)\right)}} + 1\right) \\
&= \frac{\gamma}{2}\frac{e^{2\pi a}\left(2\alpha+\gamma\right)}{\left(\alpha+\gamma\right)^2}\left(\frac{\left(2\alpha + \beta e^{2a\pi}\left(-\frac{f(2\pi)}{e^{2\pi a}-1}\right)\right)}{\sqrt{\frac{f(2\pi)\beta e^{2\pi a}}{(e^{2\pi a}-1)^2}\left(4\alpha - 4\alpha e^{2\pi a} + f(2\pi)\beta e^{2\pi a}\right)}} + 1\right).
\end{aligned}$$

Since $a < 0$, then

$$\frac{d}{d\gamma}\left(r(2\pi,\gamma)\right) = \frac{1}{2}\gamma\frac{e^{\pi a}\left(2\alpha+\gamma\right)}{\left(\alpha+\gamma\right)^2}\left(\frac{2\alpha - 2\alpha e^{2\pi a} + f\beta e^{2\pi a}}{\sqrt{f\beta\left(4\alpha - 4\alpha e^{2\pi a} + f\beta e^{2\pi a}\right)}} + 1\right)$$

and

$$\begin{aligned}\left.\frac{dr(2\pi,\gamma)}{d\gamma}\right|_{\gamma=r_*} &= \frac{e^{\pi a}}{2}\left.\left(\frac{\gamma\left(2\alpha+\gamma\right)}{\left(\alpha+\gamma\right)^2}\right)\right|_{\gamma=r_*}\left(\frac{2\alpha-2\alpha e^{2\pi a}+f(2\pi)\beta e^{2\pi a}}{\sqrt{f(2\pi)\beta\left(4\alpha-4\alpha e^{2\pi a}+f(2\pi)\beta e^{2\pi a}\right)}}+1\right),\\ &= e^{2a\pi}<1.\end{aligned}$$

Consequently the limit cycle of the differential equation (3.4) is hyperbolic and stable, for more details see (See [9]).
On the other hand, by system (3.3) we have

$$\begin{aligned}\dot{\theta} &= g_1(\theta)r^3+g_2(\theta)r^2\\ &= \left(2\alpha\beta+r\beta\cos\theta+r\alpha\sin\theta\right)r^2.\end{aligned}$$

Since $\alpha>0, \beta>0, -\left(\alpha+\beta\right)<\beta\cos\theta+\alpha\sin\theta<\alpha+\beta$ and $0<r<\min\left\{\alpha,\beta\right\}$, then $\left(2\alpha\beta+r\beta\cos\theta+r\alpha\sin\theta\right)>0$ for all $\theta\in\mathbb{R}$, hence $\dot{\theta}>0$. Consequently, this limit cycle is a stable and hyperbolic for the differential system (3.2), because the orbits $r(\theta)$ of (3.4) have the same orientation with respect to those $(x(t),y(t))$.

Proof of statement (i) of Theorem 3.2.1 If $c\neq 0$ Clearly the curve $(r(\theta)\cos\theta, r(\theta)\sin\theta)$ in the (x,y) plane with

$$F(r,\theta)=r^2-(r\cos\theta+\alpha)(r\sin\theta+\beta)e^{a\theta+c\sin 2\theta}\left(\frac{r_*^2}{\beta(r_*+\alpha)}+f\left(\theta\right)\right)=0 \quad (3.16)$$

is not algebraic, due to the expression $e^{a\theta+c\sin 2\theta}\left(\frac{r_*^2}{\beta(r_*+\alpha)}+f\left(\theta\right)\right)$. More precisely, in Cartesian coordinates the curve defined by this limit cycle is

$$x^2+y^2-(x+\alpha)(y+\beta)e^{a\arctan\frac{y}{x}+c\sin 2\left(\arctan\frac{y}{x}\right)}\left(\frac{r_*^2}{\beta(r_*+\alpha)}+\int_0^{\arctan\frac{y}{x}}e^{-as-c\sin 2s}ds\right)=0.$$

If the limit cycle is algebraic this curve must be given by a polynomial, but a polynomial $F(x,y)$ in the variables x and y satisfies that there is a positive integer n such that $\frac{\partial^n F}{\partial x^n}=0$ and this is not the case because in the derivative $\frac{\partial F}{\partial x}$ appears again the expression

$$e^{a\arctan\frac{y}{x}+c\sin 2\left(\arctan\frac{y}{x}\right)}\left(\frac{r_*^2}{\beta(r_*+\alpha)}+\int_0^{\arctan\frac{y}{x}}e^{-as-c\sin 2s}ds\right)$$

which already appears in $F(x,y)$, and this expression will appear in the partial derivative at any order.

Proof of statement (ii) of Theorem 3.2.1 If we take $c = 0$ in (3.8),we obtain

$$\rho(\theta) = e^{a\theta}\left(\frac{e^{2a\pi}}{1-e^{2a\pi}}f(2\pi) + \int_0^\theta e^{-as}ds\right),$$

then

$$\begin{aligned}\rho(\theta) &= e^{a\theta}(\frac{e^{2a\pi}}{1-e^{2a\pi}}(\frac{-1}{a}e^{-2a\pi}+\frac{1}{a}) + \frac{-1}{a}e^{-a\theta}+\frac{1}{a}\\ &= \frac{1}{a}e^{a\theta}(\frac{e^{2a\pi}}{1-e^{2a\pi}}(1-e^{-2a\pi}) - e^{-a\theta}+1)\\ &= \frac{1}{a}e^{a\theta}(\frac{e^{2a\pi}-1}{1-e^{2a\pi}} - e^{-a\theta}+1)\\ &= \frac{1}{a}e^{a\theta}(-1-e^{-a\theta}+1).\end{aligned}$$

So we obtain

$$\rho(\theta) = -\frac{1}{a}.$$

After the the changes of variables in (3.11) we obtain

$$r^2 + \frac{1}{a}(r\cos\theta+\alpha)(r\sin\theta+\beta) = 0.$$

By passing to Cartesian coordinates (X, Y) we obtain

$$X^2+Y^2+\frac{1}{a}(X+\alpha)(Y+\beta) = 0.$$

By means of the changes of coordinates $X = x-\alpha, Y = y-\beta$ we obtain

$$(x-\alpha)^2+(y-\beta)^2+\frac{1}{a}xy = 0.$$

3.4 Examples

The following examples illustrate our result.

Example 3.4.1 *When $a = -2, \alpha = \beta = 2, c = \frac{1}{2}$, system (3.1) reads*

$$\begin{cases}\dot{x} = x\left(\begin{array}{c} y(x-2)(xy-(x-2)^2-3(y-2)^2)\\ +((x-2)^2+(y-2)^2)((x-2)^2-y^2+4)\end{array}\right),\\ \dot{y} = y\left(\begin{array}{c} x(y-2)(xy-(x-2)^2-3(y-2)^2)\\ +((x-2)^2+(y-2)^2)(x^2-4-(y-2)^2)\end{array}\right),\end{cases} \tag{3.17}$$

then we have $f(2\pi) = 143375 > 0$ and $r_* = 1.00741$ it is easy to verify that all condition of Theorem 3.2.1 are satisfied, we conclude that system has non algebraic limit cycle shown on the Poincare disc

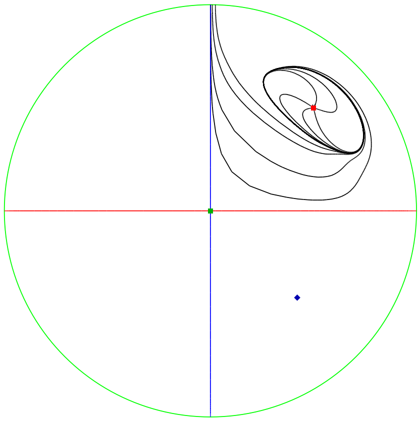


Figure 3.1: Limit Cycles of System (3.17)

Example 3.4.2 *When $a = -1, \alpha = 1, \beta = 2, c = 0$, system (3.1) reads*

$$\begin{cases} \dot{x} = x \begin{pmatrix} y(x-1)(xy-(x-1)^2-(y-2)^2) \\ +((x-1)^2+(y-2)^2)((x-1)^2-y^2+4) \end{pmatrix}, \\ \dot{y} = y \begin{pmatrix} x(y-2)(xy-(x-1)^2-(y-2)^2) \\ +((x-1)^2+(y-2)^2)(x^2-1-(y-2)^2) \end{pmatrix}, \end{cases} \tag{3.18}$$

then we have $f(2\pi) = 534.492 > 0$ and $r_* = 2.65305$ it is easy to verify that all condition of Theorem 3.2.1 are satisfied, we conclude that system has algebraic limit cycle shown on the Poincare disc

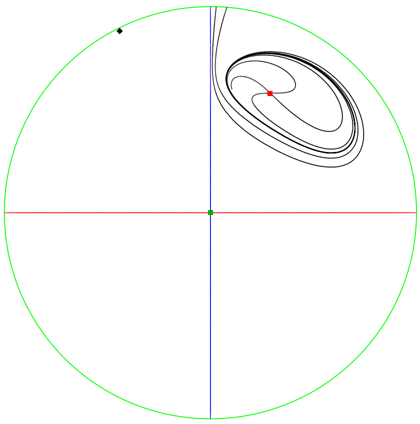


Figure 3.2: Limit Cycles of System (3.18)

Conclusion

In our work we were interested in the study of the limit cycles of certain classes of Kolmogorov's nonlinear planar differential system is to know the existence of limit cycles. Moreover, if there exists we distinguish when it is algebraic or not.

In the first chapter we presented some basics concerning the qualitative theori differential systems, especially planar differential systems.

In the second chapter, we studied some classes of kolmogorov system. In a first part, we prove that when these kolmogorov differential system of degree $j = k + 1$ has an irreducible invariant algebraic of degree k has no limit cycles , although this system admits a family of periodic solutions. In a second part, In the second part, we studied three classes of the kolmogorov system and give sufficient conditions for the existence of hyperbolic algebraic limit cycles.

As for the third chapter, we have built a class of differential systems kolmogorov to show that it admits a limit cycle under appropriate conditions. Using the writing of the system in polar coordinates , we obtained the explicit expression of the first integral, which led us to a non-algebraic limit cycle.

Bibliography

[1] A. Bendjeddou, A. Berbache and R. Cherfa , A class of Kolmogorov system with exact algebraic limit cycle.International Journal of Differential Equations and Applications, V14 no 3, 2015, 159-165.

[2] A. Bendjeddou, M. Grazem, A Class of quintic Kolmogorov Systems with Explicit Non-algebraic Limit Cycle. Journal of siberian federal university. Mathematics and Physics 2019, 12(3), 285–297.

[3] S. Benyoucef and A. Bendjeddou, Kolmogorov System with Explicit Hyperbolic Limit Cycle. Journal of Siberian Federal University. Mathematics and Physics ,Vol. 10, No 2, 216–222.

[4] S. Benyoucef and A. Bendjeddou, A class of Kolmogorov system with exact limit cycles. Int. J. Pur. App. Math., Vol. 103 (4) (2015), 439-451.

[5] A. Canada, P. Drabek and A. Fonda , *Handbook of differential equations Ordinary differential equations*, Elsevier, 2004.

[6] C. Christopher, J. Llibre, C. Pantazi and X, Zhang, Darboux integrability and invariant algebraic curves for planar polynomial systems, J. Phys. A: Math. Gen. 35 (2002), 2457–2476 .

[7] G. Darboux, Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré. Bulletin des sciences mathématiques et astronomiques 2eme série,tome 2,no 1 (1878), p. 60-96, p.123-144, p. 151-200.

[8] H. Dulac, Sur les cycles limites. Bulletin de la S. M. F, Numdam, tome 51 (1923), p. 45-188.

[9] F. Dumortier, J. Llibreand and J. Artés, Qualitative Theory of Planar Differential Systems, Springer, 2006.

[10] A. Ferragut and J. Llibre, *Invariant curves and integrability of planar C^r differential systems.* Rocky mountain journal of mathematics, volume 48,number 5,2018.

[11] D. Hilbert, *Mathematische Problem (lecture)*. Second Internat. Congress Math. Paris, 1900, Nachr. Ges. Wiss. Göttingen Math. – Phys. Kl. (1900).

[12] K. Odani, The limit cycle of the van der Pol equation is not algebraic, J. of Diff. Equations. 115(1995), 146–152.

[13] L. Perko. *Diferential equations and dynamical systems*, Third edition.Texts in Applied Mathematics, Vol. 7. Springer-Verlag, New York, 2001.

[14] H. Poincare, Mémoire sur les courbes définies par une équation différentielle. Journal de mathématiques pures et appliquées 3e série (III), tome 8 (1882), 251-296.

[15] H. Poincare, Mémoire sur les courbes définies par une équation différentielle. Journal de mathématiques pures et appliquées 3e série (III), tome 7 (1881), 375-422.

[16] H. Poincare, Mémoire sur les courbes définies par une équation différentielle. Journal de mathématiques pures et appliquées 3e série (IV), tome 1 (1885), 167-244.

[17] H. Poincare, Mémoire sur les courbes définies par une équation différentielle. Journal de mathématiques pures et appliquées 3e série (IV), tome 2 (1886), 151-217.