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STABILITY OF SOLUTIONS FOR A EULER-BERNOULLI BEAM EQUATION WITH MEMORY AND BOUNDARY CONTROL

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Dedication

This study is wholeheartedly dedicated to my " **beloved parents**", who have been my source of inspiration and gave me strength when i thought of giving up, who continually provide their moral, spiritual,emotional and financial support.

To that who supported me in her prayers.. To that who stayed up at night just for me.. To that who share my joys and sorrows.. To the source of tenderness and sympathy..**Deer mather**

To that man who inform me that the world is a striggle and it's weapons are science and patience..To that person who seek for my comfort and happiness..To the greatest man at world..**My father**

To my lovely brothers **Salah Eddine** and **Oussama**, may god protect them.

To my beloved sisters **Meriem** and my little **Assia** for all their love and support.

To my grand father **Ahmed** and **Amar**, may god protect them.

To my uncles and aunts.

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To my freinds **Samira**,**Narimen**, **Khalida**,**Chaima**, **Romaissa**, **Hadjer** and my partner **Roufida**

To my supervisor **Dr Benterki** and all the teachers specially **Mrs Belkacem** whose has been a constant source of knowledge and inspiration

To all that my memory remembers and I do not reveal their names in my memory.

Halima...

Dedication

Firstly I dedicated this thesis to Almighty god Elhamdoallah for the guidance,strength,power of mind,protection and skills and for giving us a healthy life all of these .
Prophet Muhammad you are our role model and example in my life .
this study is wholeheartedly dedicated to our beloved parents(Dr mohamadi Seddik and MRS mohamadi fouzia),who have been our source of inspiration and gave us strength of not giving up,who continually provide their moral,spiritual,emotional,and financial support..
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CONTENTS

INTRODUCTION

Over the past years, the flexible beam structure is widely used in modern engineering because of its advantages (e.g. light weight, low energy consumption, etc.), and its control problem becomes one of the hot research topics. A large number of systems can be modeled as mechanical flexible systems such as telephone wires, conveyor belts, crane cables, helicopter blades, robotic arms, mooring lines, marine risers, and so on.

However, unwanted vibrations due to the flexibility property and the time-varying disturbances restrict the utility of these flexible systems in different engineering applications. If the flexible beam system cannot be well controlled, the vibration will not only affect the accuracy and efficiency of the system, but also accelerate the equipment fatigue damage, seriously shorten the service life of the materials, and bring production safety risk and economic loss. Therefore, it is very important to effectively control flexible beam systems. Flexible beam systems and their vibration suppression have received great attention in the literatures. Boundary control has several merits for vibration suppression of the flexible beam systems. In this work we study the decay rates for the solutions to the mixed problem for Euler–Bernoulli beam equation with memory term.

$$\begin{cases} y_{tt} + y_{xxxx} - \int_0^t g(t-s) y_{xxxx}(x,s)ds = 0 & in \quad \Omega \times \mathbb{R}^+ \\ y(0,t) = 0 \quad , \quad y_{xx}(0,t) = y_{xx}(1,t) = 0 & \forall t > 0 \\ y_{xxx}(1,t) - \int_0^t g(t-s) y_{xxx}(1,s)ds = 0 \end{cases} \tag{1}$$

$$\begin{cases} \rho y_{tt}(x,t) + EI y_{xxxx}(x,t) - T y_{xx}(x,t) - EI \int_0^t g(t-s) y_{xxxx}(x,s)ds = 0 & in \quad \Omega \times \mathbb{R}^+, \\ y_{xx}(0,t) = y_{xx}(L,t) = y(0,t) = 0 \qquad \forall t > 0, \\ -EI y_{xxx}(L,t) + T y_x(L,t) + EI \int_0^t g(t-s) y_{xxx}(L,s)ds = U(t) \qquad \forall t > 0, \\ U(t) = 0. \end{cases} \tag{2}$$

Since the pioneer works of Dafermos in 1970, where the general decay was discussed, problems related to viscoelasticity have attracted a great deal of attention and many results of existence and long-time behavior have been established. Global existence and uniform decay of solutions have been discussed for similar problems by different authors. It seems that all started with kernels of the form $h(t) = e^{-\beta t}, \beta > 0$, then kernels satisfying $-\xi_1 h(t) \leq h'(t) \leq -\xi_2 h(t), \forall t \geq 0$ for some positive constants ξ_1 and ξ_2 together with some conditions on the second derivative. Later these conditions have been relaxed to only $h'(t) \leq -\xi h(t)$, for all $t \geq 0$ and some $\xi > 0$. Recently, the constant ξ has been replaced by a function of time $\xi(t)$. This has allowed the authors to derive decay results of other types than just exponential or polynomial type. In our work, we would like to prove that the energy decays at a little bit slower rate than $\gamma(t)^{-1}$. In this respect we recall that when $\gamma(t)$ is a polynomial this result is somehow in agreement with the finding of Fabrizio and Polidoro, where the authors proved that the degree for the decay of the energy is smaller than or equal to the degree of the decay of $h(t)$. We mention here that a mere replacement of the exponential function by an arbitrary function $\gamma(t)$ does not prove the result without further conditions on $\gamma(t)$.

our work consists of two chapters:

Chapter 1:

this chapter presents the preliminary concepts(Inequalities of Cauchy schwartz, Young, Holder's and poincaré).

Chapter 2:present

On this chapter we study three problems: *first one*: Arbitrary decays in linear viscoelasticity, *the second one*, Arbitrary decay for a Euler-Bernoulli Beam Equation with Memory , *The third one*, Stabilization of a viscoelastic Euler-Bernoulli beam .

CHAPTER 1

PRELIMINARY CONCEPTS

1.1 Inequalities

The Cauchy-Schwartz inequality is an elementary inequality and at the same time a powerful inequality, which can be stated as follows:
Cauchy-Schwarz inequality :

$$|x.y| \leqslant \|x\|\|y\|, \quad (x, y \in \mathbb{R}^n)$$

Lemma 1 *[5] Let $\varphi(x,t) \in \mathbb{R}$ be a function defined on $x \in [0,L]$ and $t \in [0,\infty)$ that satisfies the boundary condition*

$$\varphi(0,t) = 0, \forall t \in [0,\infty) \tag{1.1}$$

then the following inequality holds

$$\varphi^2(x,t) \leq L \int_0^L (\varphi'(x,t))^2 dx \quad \forall x \in [0,L] \tag{1.2}$$

If in addition to (1.1), the function $\varphi(x,t)$ satisfies the boundary condition

$$\varphi'(0,t) = 0, \quad \forall t \in [0,\infty), \tag{1.3}$$

then the following inequality also holds

$$(\varphi'(x,t))^2 \leq L \int_0^L (\varphi''(x,t))^2 dx \quad \forall x \in [0,L] \tag{1.4}$$

Young inequality

Theorem 1.1.1

Let $1 < p, q < \infty$, $\quad \dfrac{1}{p} + \dfrac{1}{q} = 1,$
then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b > 0.$$

Holder's inequality: For Ω an open subset of $\mathbb{R}^n$. Assume $1 < p, q < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then, if $u \in L^p(\Omega),\ v \in L^q(\Omega)$

$$\int_\Omega \mid uv \mid dx \leqslant \|u\|_p \|v\|_q$$

Poincaré inequality

Let I=(a,b) be an open interval, possibly unbounded, and let $p \in \mathbb{R}$ with $1 \leqslant p \leqslant \infty$

The sobolev espace $W^{1,p}(I)$ is defined to be

$$W^{1,p}(I) = \{u \in L^p(I); \exists g \in L^p(I) \ \ such\ that\ \int_I u\varphi' = -\int_I g\varphi \ \ \forall\varphi \in C_c^1(I)\}$$

We set

$$H^1(I) = W^{1,2}(I)$$

For $u \in W^{1,p}(I)$ we denote $u' = g$.

Given $1 \leqslant p < \infty$ denote $W_0^{1,p}(I)$ the closure of $C_c^1(I)$ in $W^{1,p}(I)$.Set

$$H_0^1(I) = W_0^{1,2}(I)$$

.

The space $W_0^{1,p}(I)$ is equipped with the norm of $W^{1,p}(I)$, and the space H_0^1 is equipped with the scalar product of H^1.

Theorem 1.1.2

Let p, so that $1 \leq p < \infty$ and Ω is a bounded, open subset of $\mathbb{R}^n$. Then, there exists a constant c, depending only on Ω and p, so that, for every function u of the sobolev space $H_0^1(\Omega)$ of zero-trace functions,

$$\|u\|_{L^q(\Omega)} \leq c\|Du\|_{L^p(\Omega)}.$$

for each $q \in [0, p']$, such that $q < p'$ the constant c depending only on p, q, n and Ω.
such that D is the differential of u .

1.2 Lyapunov Exponential stability theorem

Suppose there is a function $\mathbf{V}$ and constant $\alpha > 0$ such that

- $\mathbf{V}$ is positive definite.
- $\dot{\mathbf{V}}(z) \leq -\alpha\mathbf{V}(z)$ for all z

Then, there is an $M > 0$ such that every trajectory of $\dot{x} = f(x)$ satisfies

$$\|x(t)\| \leq Me^{-\frac{\alpha t}{2}}\|x(0)\|.$$

This is called Global Exponential Stability.

$\dot{\mathbf{V}}(z) \leq -\alpha\mathbf{V}(z)$ gives guaranteed minimum dissipation rate, proportional to energy.

1.3 Converse Lyapunov global exponential stability theorem

Suppose there is $\beta > 0$ and $M > 0$ such that each trajectory of $\dot{x} = f(x)$ satisfies

$$\|x(t)\| \leq Me^{-\beta t}\|x(0)\| \qquad \text{for all } t \geq 0.$$

Then, there is a Lyapunov function that proves the system is exponencially stable there is function $\mathbf{V} : \mathbb{R}^n \to \mathbb{R}$ and contant $\alpha > 0$ such that

- $\mathbf{V}$ is positive definite.
- $\dot{\mathbf{V}}(z) \leq -\alpha \mathbf{V}(z)$ for all z.

CHAPTER 2

STABILITY SOLUTIONS FOR A EULER-BERNOULLI BEAM EQUATION

2.1 Arbitrary decays in linear viscoelasticity

We shall consider the following wave equation with a viscoelastic damping term:

$$\begin{cases} y_{tt} - \Delta y + \int_0^t g(t-s)\Delta y(s)ds = 0, \ \ in \quad \Omega \times \mathbb{R}^+ \\ y = 0, on \quad \Gamma \times \mathbb{R}^+ \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), \ \ in\, \Omega \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smoth boundary $\Gamma = \partial\Omega$, such that $\Delta y = \sum_{i=0}^{n} \frac{\partial^2 y_i}{\partial {y_i}^2}$.

Theorem 2.1.1 *Let* $(y_0, y_1) \in H^2(\Omega) \bigcap H_0^1(\Omega) x H_0^1(\Omega)$ *and* $g(t)$ *be a nonnegative summable kernel.*
Then ther exists a unique regular solution y *to problem* (2.1) *such that*

$$y \in L^\infty_{loc}(0,\infty; H^2(\Omega) \bigcap H_0^1(\Omega)), y_t \in L^\infty_{loc}(0,\infty; H_0^1(\Omega)), y_{tt} \in L^\infty_{loc}(0,\infty; L^2(\Omega))$$

If $(y_0, y_1) \in H_0^1(\Omega) x L^2(\Omega)$*, then there exists a unique regular solution* y *satisfying*

$$y \in C([0,\infty); H_0^1(\Omega)) \bigcap C^1([0,\infty), L^2(\Omega)).$$

Multiplying (2.1) by y_t and integrating ouver Ω , we get

$$\int_\Omega y_t y_{tt} dx - \int_\Omega y_t \Delta y dx + \int_\Omega y_t \int_0^t g(t-s)\Delta y(s)ds = 0$$

$$\begin{aligned} I_1 &= \int_\Omega y_t y_{tt} dx \\ &= \frac{1}{2}\int_\Omega \frac{d}{dt}(y_t)^2 dx = \frac{1}{2}\frac{d}{dt}\|y_t\|_2^2 \end{aligned} \tag{2.2}$$

$$\begin{aligned} I_2 &= -\int_\Omega y_t \Delta y dx \\ &= \int_\Omega \nabla y_t \nabla y dx = \frac{1}{2}\frac{d}{dt}\|\nabla y\|_2^2 \end{aligned} \tag{2.3}$$

$$\begin{aligned} I_3 &= \int_\Omega y_t \int_0^t g(t-s)\Delta y(s) ds \\ &= -\int_\Omega \nabla y_t \int_0^t g(t-s)\nabla y(s) ds \end{aligned} \tag{2.4}$$

Combining (2.2),(2.3) and (2.4), the result will be

$$\frac{1}{2}\frac{d}{dt}\{\|y_t\|_2^2 + \|\nabla y\|_2^2\} = \int_\Omega \nabla y_t \int_0^t g(t-s)\nabla y(s) ds dx$$

We define the (classical) energy by

$$E(t) = \frac{1}{2}\{\|y_t\|_2^2 + \|\nabla y\|_2^2\} \tag{2.5}$$

Then by equation (2.1) it is easy to see that

$$E'(t) = \int_\Omega \nabla y_t \int_0^t g(t-s)\nabla y(s) ds dx$$

Observe that $E'(t)$ is of an unknown sign and that

$$2\int_\Omega \nabla y_t \int_0^t g(t-s)\nabla y(s) ds dx = \int_\Omega g'\Box\nabla y(t) dx - g(t)\|\nabla y\|_2^2$$

$$-\frac{d}{dt}\{\int_\Omega g\Box\nabla y(t) dx - (\int_0^t g(s) ds)\|\nabla y\|_2^2\} \tag{2.6}$$

where

$$g\Box v(t) := \int_0^t g(t-s)|v(t)-v(s)|^2 ds$$

and $\|.\|$ denotes the norm in $L^2(\Omega)$.Therefore,if we modify $E(t)$ to

$$\varepsilon(t) := \frac{1}{2}\{\|y_t\|_2^2 + (1-\int_0^t g(s)ds)\|\nabla y\|_2^2 + \int_\Omega g\Box\nabla y(t) dx\} \tag{2.7}$$

we obtain

$$\varepsilon'(t) = \frac{1}{2}(\int_\Omega g'\Box\nabla y(t) dx - g(t)\|\nabla y\|_2^2). \tag{2.8}$$

We assume that the kernel is such that

$$1 - \int_0^{+\infty} g(s) ds = 1 - k > 0.$$

Hence,if $g' \leq 0$,it follows that $\varepsilon(t)$ is nonincreasing and bounded above uniformly by $\varepsilon(0) = E(0)$.
Next, we define the standard functionals,

$$\psi_1(t) := \int_\Omega y_t y dx,$$

$$\psi_2(t) := -\int_\Omega y_t \int_0^t g(t-s)(y(t)-y(s))dsdx,$$

and the new one

$$\psi_3(t) := \int_\Omega \int_0^t G_\gamma(t-s)|\nabla y(s)|^2 dsdx,$$

where

$$G_\gamma(t) := \gamma(t)^{-1}\int_t^\infty g(s)\gamma(s)ds.$$

A similar functional to $G_\gamma(t)$ has been used in the theory of population dynamics where the problems are of Volterra type.The modified energy we will work with is

$$L(t) := \varepsilon(t) + \Sigma_{i=1}^3 \lambda_i \psi_i(t) \tag{2.9}$$

for some $\lambda_i > 0, i = 1,2,3$ to be determined.

2.1.1 Equivalence between $L(t)$ and $E(t) + \psi_3(t)$

The first result tells us that $L(t)$ and $\varepsilon(t) + \psi_3(t)$ are equivalent.

Proposition 1 *There exist $\rho_i > 0, i = 1,2$ such that*

$$\rho_1[\varepsilon(t) + \psi_3(t)] \leq L(t) \leq \rho_2[\varepsilon(t) + \psi_3(t)]$$

for all $t \geq 0$ and small $\lambda_i, i = 1,2$.

proof 1 *By the inequalities*
Applying Cauchy Schwartz, Young's inequality and Poincare we get

$$\begin{aligned}
\psi_1(t) &= \int_\Omega y_t y dx \\
\psi_1(t) &\leq (\int_\Omega (y_t)^2 dx)^{\frac{1}{2}} (\int_\Omega (y)^2 dx)^{\frac{1}{2}} \\
&\leq \|y_t\|_2 \|y\|_2 \\
&\leq \frac{1}{2}\|y_t\|_2^2 + \frac{c_p}{2}\|\nabla y\|_2^2
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
\psi_2(t) &= -\int_\Omega y_t \int_0^t g(t-s)(y(t)-y(s))dsdx \\
&\leq \|y_t\|(\int_\Omega (\int_0^t g(t-s)(y(t)-y(s))ds)^2 dx)^{\frac{1}{2}} \\
\int_\Omega (\int_0^t g(t-s)(y(t)-y(s))ds)^2 dx &\leq \int_\Omega (\int_0^t g(t-s)ds)(\int_0^t g(t-s)(y(t)-y(s))^2 ds)dx \\
&\leq c_p \int_0^t g(s)ds(\int_\Omega \int_0^t g(t-s)(\nabla y(t) - \nabla y)^2 ds)dx \\
&\leq c_p k \int_\Omega g\Box\nabla y dx
\end{aligned}$$

we get

$$\begin{aligned}\psi_2(t) &\leq \|y_t\|(kc_p\int_\Omega g\Box\nabla y dx)^{\frac{1}{2}}\\ &\leq \frac{1}{2}\|y_t\|_2^2+\frac{c_p}{2}k\int_\Omega(g\Box\nabla y)dx \end{aligned} \tag{2.11}$$

where c_p is the poincare constant, we have

$$L(t)\leq\frac{1}{2}\|y_t\|_2^2+\frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla y\|_2^2+\frac{1}{2}\int_\Omega g\Box\nabla y(t)dx+\lambda_1(\frac{1}{2}\|y_t\|_2^2$$

$$+\frac{c_p}{2}\|\nabla y\|_2^2)+\lambda_2(\frac{1}{2}\|y_t\|_2^2+\frac{c_p}{2}k\int_\Omega(g\Box\nabla y)dx)+\lambda_3\psi_3$$

simplicity, we have

$$L(t)\leq\frac{1}{2}(1+\lambda_1+\lambda_2)\|y_t\|_2^2+\frac{1}{2}(1-\int_0^t g(s)ds+\lambda_1c_p)\|\nabla y\|_2^2$$

$$+\frac{1}{2}(1+\lambda_2c_pk)\int_\Omega(g\Box\nabla y)dx+\lambda_3\psi_3(t)$$

On the other hand,

$$2L(t)\geq\|y_t\|_2^2+(1-\int_0^t g(s)ds)\|\nabla y\|_2^2+\int_\Omega g\Box\nabla y(t)dx-\lambda_1(\|y_t\|_2^2$$

$$+c_p\|\nabla y\|_2^2)-\lambda_2(\|y_t\|_2^2+c_pk\int_\Omega(g\Box\nabla y)dx)+2\lambda_3\psi_3$$

simplicity, we have

$$2L(t)\geq(1-\lambda_1-\lambda_2)\|y_t\|_2^2+(1-k-\lambda_1c_p)\|\nabla y\|_2^2$$

$$+(1-\lambda_2c_pk)\int_\Omega(g\Box\nabla y)dx+2\lambda_3\psi_3(t)$$

Therefore, $\rho_1[\varepsilon(t)+\psi_3(t)]\leq L(t)\leq\rho_2[\varepsilon(t)+\psi_3(t)]$ for some constantn $\rho_i>0, i=1,2$ and small $\lambda_i, i=1,2$ such that

$$\begin{cases}1-\lambda_1-\lambda_2>0\\ 1-\lambda_2c_pk>0\end{cases}\Rightarrow\begin{cases}\lambda_2<1-\lambda_1\\ \lambda_2<\frac{1}{c_pk}\end{cases}\Rightarrow\lambda_2<\min\{\frac{1}{c_pk},1-\lambda_1\}$$

$$\begin{cases}1-k-\lambda_1c_pk>0\\ 1-\lambda_1>0\end{cases}\Rightarrow\Big\{\lambda_1<\frac{1-k}{c_p}\lambda_1<1\quad\Rightarrow\lambda_1<\min\{1,\frac{(1-k)}{c_p}\}$$

The following inequality will be used repeatedly in the sequal.

Lemma 2

$$ab\leq\delta a^2+\frac{b^2}{4\delta}, a,b\in\mathbb{R},\delta>0.$$

Our next result is a simple identity which gives a better estimate for

$$\int_\Omega\nabla y\int_0^t g(t-s)\nabla y(s)dsdx$$

.

Lemma 3 *We have, for $t \geq 0$,*

$$\int_\Omega \nabla y \int_0^t g(t-s)\nabla y(s)dsdx$$

$$= \frac{1}{2}(\int_0^t g(s)ds)\|\nabla y\|_2^2 + \frac{1}{2}\int_0^t g(t-s)\|\nabla y(s)\|_2^2 ds - \frac{1}{2}\int_\Omega (g\Box\nabla y)dx. \tag{2.12}$$

The proof is straightforward.
This identity is better in our case than the following ones:

$$\int_\Omega \nabla y \int_0^t g(t-s)\nabla y(s)dsdx$$

$$\leq \delta\|\nabla y\|_2^2 + \frac{1}{4\delta}(\int_0^t g(s)ds)\int_0^t g(t-s)\|\nabla y(s)\|_2^2 ds, t \geq 0, \delta > 0$$

or

$$\int_\Omega \bigtriangledown y \int_0^t g(t-s) \bigtriangledown y(s)dsdx$$

$$= -\int_\Omega \bigtriangledown y \int_0^t g(t-s)(\bigtriangledown y(t) - \bigtriangledown y(s))dsdx + \int_\Omega \bigtriangledown y \int_0^t g(t-s)(\bigtriangledown y(t))dsdx$$

$$= -\int_\Omega \bigtriangledown y \int_0^t g(t-s)(\bigtriangledown y(t) - \bigtriangledown y(s))dsdx + (\int_0^t g(s)ds)\| \bigtriangledown y\|^2$$

$$\leq (\delta + (\int_0^t g(s)ds))\| \bigtriangledown y\|^2 + \frac{1}{4\delta}(\int_0^t g(s)ds))\int_\Omega (g\Box \bigtriangledown y)dx \qquad , t \geq 0, \delta > 0$$

Which were used in almost all the previous works to estimate the term

$$\int_\Omega \bigtriangledown y \int_0^t g(t-s) \bigtriangledown y(s)dsdx$$

that appears in the proofs (see the first relation in the proof of Theorem 2.1 below).our relation (2.12) provides us with the negative term $\frac{1}{2}\int_\Omega (g\Box \bigtriangledown y)dx$ which will be of great help in canceling a similar undesirable term

2.1.2 Asymptotic Behavior

In this section we state and prove our result.But first we introduce the following notation . For every measurable set $\mathcal{A} \subset \mathbb{R}^+$,we define the probability measure $\hat{g}$ by

$$\hat{g}(\mathcal{A}) := \frac{1}{k}\int_{\mathcal{A}} g(s)ds \tag{2.13}$$

The flatness set and the flatness rate of g are defined by

$$\mathcal{F}_g := s \in \mathbb{R}^+ : g(s) > 0 \, and \, g'(s) = 0 \tag{2.14}$$

and

$$\mathcal{R}_g := \hat{g}(\mathcal{F}_g),$$

respetively.

Our assumptions on the kernel g(t) are the following:
(H1)$g(t) \leq 0$ for all $t \leq 0$ and $0 < k = \int_0^\infty g(s)ds < 1$.
(H2)$g'(t) \leq 0$ for almost all $t > 0$.
(H3)There exists a nondecreasing function $\gamma(t) > 0$ such that $\frac{\gamma'(t)}{\gamma(t)} =: \eta(t)$ is a decreasing function and $\int_0^\infty g(s)\gamma(s)ds < +\infty$
Let $t_* > 0$ be a number such that $\int_0^{t_*} g(s)ds = g_* > 0$.For simplicity,we consider kernels continuous and differentiable a.e

Theorem 2.1.2 *Assume that the hypotheses (H1)-(H3)and $\mathcal{R}_g < \frac{1}{4}$ hold.*
If $G_\gamma(0) < [(8-k)g_* - 3k]/4,\ g_* > \frac{3k}{(8-k)}$, then, there exist positive constants C and v such that

$$E(t) \leq C\gamma(t)^{-v},\ \ t > 0$$

proof 2 *The methode of proof consists in showing an inequality of the form*
$L'(t) \leq -C\eta(t)L(t),\ \ t \geq 0$ for some positive constant C . An integration of this inequality gives us the decay for L(t) .
Then an application of Proposition 1 implie the sought relation for $\varepsilon(t)$ and thereafter for $E(t)$.
A differentiation of $\psi_1(t)$ with respect to t along trajectoires of (2.1) gives

$$\psi_1(t) = \int_\Omega y_t y dx$$

$$\psi_1'(t) = \int_\Omega y_{tt} y dx + \|y_t\|^2$$

replacing y_{tt} by other terms in problem we get

$$\psi_1'(t) = \int_\Omega \triangle\, y y dx - \int_\Omega y \int_0^t g(t-s) \triangle\, y(x,s) ds dx + \|y_t\|^2$$

$$\psi_1'(t) := \|y_t\|^2 + \|\bigtriangledown y\|^2 - \int_\Omega \bigtriangledown y \int_0^t g(t-s) \bigtriangledown y(s) ds dx$$

and by Lemma 3 (identify (2.12)) we obtain

$$\psi_1'(t) \leq \|y_t\|^2 - (1 - \frac{k}{2})\|\bigtriangledown y\|^2 + \frac{1}{2}\int_0^t g(t-s)\|\bigtriangledown y(s)\|^2 ds - \frac{1}{2}\int_\Omega (g\Box \bigtriangledown y) dx \qquad (2.15)$$

For $\psi_2(t)$ we have

$$\psi_2'(t) = -\int_\Omega y_{tt} \int_0^t g(t-s)(y(t) - y(s)) ds dx$$

$$-\int_\Omega y_t [\int_0^t g'(t-s)(y(t) - y(s)) ds + y_t \int_0^t g(s) ds] dx$$

or

$$\psi_2'(t) = -\int_\Omega [(1 - \int_0^t g(s)ds) \triangle y + \int_0^t g(t-s)(\triangle y(t) - \triangle y(s)) ds]$$

$$\times \int_0^t g(t-s)(y(t)-y(s))dsdx - (\int_0^t g(s)ds)\|y_t\|^2$$
$$- \int_\Omega y_t \int_0^t g'(t-s)(y(t)-y(s))dsdx$$

Therefore,

$$\begin{aligned}\psi_2'(t) &= (1-\int_0^t g(s)ds)\int_\Omega \nabla y \int_0^t g(t-s)(\nabla y(t)-\nabla y(s))dsdx \\ &+ \int_\Omega \mid \int_0^t g(t-s)(\nabla y(t)-\nabla y(s))ds \mid^2 dx - (\int_0^t g(s)ds)\|y_t\|^2 \\ &- \int_\Omega y_t \int_0^t g'(t-s)(y(t)-y(s))dsdx \qquad (2.16)\end{aligned}$$

Now we proceed to estimate three terms in the right hand side of (2.16).For all measurable sets $\mathcal{A}$ *and* $\mathcal{F}$ *such that* $\mathcal{A}=\mathbb{R}^+\backslash\mathcal{F}$*,we have*

$$\int_\Omega \nabla y \int_0^t g(t-s)(\nabla y(t)-\nabla y(s))dsdx$$

$$\begin{aligned}&= \int_\Omega \nabla y \int_{\mathcal{A}\cap[0,t]} g(t-s)(\nabla y(t)-\nabla y(s))dsdx \\ &+ \int_\Omega \nabla y \int_{\mathcal{F}\cap[0,t]} g(t-s)(\nabla y(t)-\nabla y(s))dsdx \qquad (2.17)\end{aligned}$$

$$\leq \int_\Omega \nabla y \int_{\mathcal{A}\cap[0,t]} g(t-s)(\nabla y(t)-\nabla y(s))dsdx$$
$$+(\int_{\mathcal{F}\cap[0,t]} g(s)ds) \parallel \nabla y \parallel^2 - \int_\Omega \nabla y \int_{\mathcal{F}\cap[0,t]} g(t-s) \nabla y(s)dsdx$$

To simplify notation let us denote $\mathcal{B}_t := \mathcal{B}\cap[0,t]$ *. Using the Lemma 2 ,it is easy to see that for* $\delta_1>0$

$$\int_\Omega \nabla y \int_{\mathcal{A}_t} g(t-s)(\nabla y(t)-\nabla y(s))dsdx$$
$$\leq \delta_1\|\nabla y\|^2 + \frac{k}{4\delta_1}\int_\Omega\int_{\mathcal{A}_t} g(t-s) \mid \nabla y(t)-\nabla y(s) \mid^2 dsdx$$

and

$$\int_\Omega \nabla y \int_{\mathcal{F}_t} g(t-s) \nabla y(s)dsdx$$
$$\leq \frac{1}{2}(\int_{\mathcal{F}_t} g(s)ds) \parallel \nabla y\|^2 + \frac{1}{2}\int_{\mathcal{F}_t} g(t-s)\|\nabla y(s)\|^2 ds$$

Therefore,(2.17) becomes

$$\int_\Omega \triangledown y \int_0^t g(t-s)(\triangledown y(t)-\triangledown y(s))dsdx$$

$$\leq \delta_1 \| \triangledown y\|^2 + \frac{k}{4\delta_1}\int_\Omega \int_{\mathcal{A}_t} g(t-s) \mid \triangledown y(t)-\triangledown y(s) \mid^2 dsdx \tag{2.18}$$

$$+\frac{3}{2}k\hat{g}(\mathcal{F})\| \triangledown y\|^2 + \frac{1}{2}\int_{\mathcal{F}_t} g(t-s)\| \triangledown y(s)\|^2 ds$$

Where $\hat{g}$ is defined in (2.13).The second term in the right hand side of (2.16) satisfies the relation

$$\int_\Omega \mid \int_0^t g(t-s)(\triangledown y(t)-\triangledown y(s))ds \mid^2 dx$$

$$\leq (1+\frac{1}{\delta_2})k\int_\Omega \int_{\mathcal{A}_t} g(t-s) \mid \triangledown y(t)-\triangledown y(s) \mid^2 dsdx \tag{2.19}$$

$$+(1+\delta_2)k\hat{g}(\mathcal{F})\int_\Omega \int_{\mathcal{F}_t} g(t-s) \mid \triangledown y(t)-\triangledown y(s) \mid^2 dsdx, \delta_2>0$$

The third term is istimated using Lemma 2 as follows:

$$I_1 = \int_\Omega y_t \int_0^t g'(t-s)(y(t)-y(s))dsdx$$

Applying Poincaré's inequality , there exist $c_p>0$, such that

$$I_1 \leq \int_\Omega y_t (c_p \int_0^t g'(t-s)(\triangledown y(t)-\triangledown y(s))ds)dx \tag{2.20}$$

Appliying Cauchy Schwartz and Young's inequality, we get

$$\begin{aligned} I_1 &\leq (\int_\Omega (y_t)^2 dx)^{\frac{1}{2}} (c_p \int_\Omega (\int_0^t g'(t-s)(\triangledown y(t)-\triangledown y(s))ds)^2 dx)^{\frac{1}{2}} \\ &= (\int_\Omega (y_t)^2 dx)^{\frac{1}{2}} (c_p \int_\Omega (\int_0^t |g'|^{\frac{1}{2}}|g'|^{\frac{1}{2}}(\triangledown y(t)-\triangledown y(s))ds)^2 dx)^{\frac{1}{2}} \\ &\leq (\int_\Omega (y_t)^2 dx)^{\frac{1}{2}} (c_p \int_\Omega \int_0^t |g'|ds \int_0^t |g'|(\triangledown y(t)-\triangledown y(s))^2 dsdx)^{\frac{1}{2}} \end{aligned} \tag{2.21}$$

$$\leq (\int_\Omega (y_t)^2 dx)^{\frac{1}{2}} (c_p g(0)\int_\Omega |g'|\Box \triangledown y dx)^{\frac{1}{2}}$$

$$\leq \delta_3 \|y_t\|^2 + \frac{c_p}{4\delta_3}(\int_0^t \mid g'(s) \mid ds)\int_\Omega (\mid g' \mid \Box \triangledown y)dx$$

$$\leq \delta_3 \|y_t\|^2 - \frac{c_p}{4\delta_3} g(0)\int_\Omega (g'\Box \triangledown y)dx \tag{2.22}$$

for any $\delta_3>0$.Taking into account (2.17)-(2.22) in (2.16) we obtain

$$\begin{aligned}
\psi_2'(t) &\leq (1-g_*)[\delta_1+\frac{3}{2}k\hat{g}(\mathcal{F})]\| \bigtriangledown y\|^2+(\delta_3-g_*)\|y_t\|^2\\
&+ [(1-g_*)\frac{k}{4\delta_1}+(1+\frac{1}{\delta_2})k]\int_\Omega\int_{\mathcal{A}_t} g(t-s) \mid \bigtriangledown y(t)-\bigtriangledown y(s)\mid^2 dsdx\\
&+ \frac{1}{2}(1-g_*)\int_{\mathcal{F}_t} g(t-s)\| \bigtriangledown y(s)\|^2 ds-\frac{C_p}{4\delta_3}g(0)\int_\Omega (g'\Box\bigtriangledown y)dx\\
&+ (1+\delta_2)k\hat{g}(\mathcal{F})\int_\Omega\int_{\mathcal{F}_t} g(t-s)\mid \bigtriangledown y(t)-\bigtriangledown y(s)\mid^2 dsdx \qquad (2.23)
\end{aligned}$$

Further,a differentiation of $\psi_3(t)$ yields

$$\begin{aligned}
{\psi_3}'(t) &= (\int_0^1\int_0^t G_\gamma(t-s)|\bigtriangledown y(s)|^2 dsdx)'\\
&= \int_0^1 G_\gamma(0)|\bigtriangledown y(s)|^2+\int_0^t {G_\gamma}'(t-s)|\bigtriangledown y(s)|^2 dsdx\\
&= \int_0^1 G_\gamma(0)|\bigtriangledown y(s)|^2-\int_0^1\int_0^t \frac{\gamma'(t-s)}{\gamma(t-s)}G_\gamma(t-s)|\bigtriangledown y(s)|^2 dsdx\\
&- \int_0^1\int_0^t g(t-s)|\bigtriangledown y(s)|^2 dsdx\\
&= G_\gamma(0)\|\bigtriangledown y\|_2^2-\int_0^t \frac{\gamma'(t-s)}{\gamma(t-s)}G_\gamma(t-s)\|\bigtriangledown y(s)\|_2^2 ds\\
&- \int_0^t g(t-s)\|\bigtriangledown y(s)\|_2^2 ds\\
&\leq G_\gamma(0)\|\bigtriangledown y\|_2^2-\eta(t)\int_0^t G_\gamma(t-s)\|\bigtriangledown y(s)\|_2^2 ds-\int_0^t g(t-s)\|\bigtriangledown y(s)\|_2^2 ds \quad (2.24)
\end{aligned}$$

Where we have used the fact that $\frac{\gamma'(t)}{\gamma(t)}=\eta(t)$ is a nonincreasing function, and we define

$$G_\gamma'(t-s)=-\frac{\gamma'(t-s)}{\gamma(t-s)}G_\gamma(t-s)-g(t-s)$$

.

Taking into account the estimations $E'(t)$,2.15,(2.23) and (2.24) ,we see that

$$\begin{aligned}
L'(t) &\leq (\frac{1}{2}-\frac{C_p}{4\delta_3}g(0)\lambda_2)\int_\Omega (g'\Box\bigtriangledown y)dx+[\lambda_1+(\delta_3-g_*)\lambda_2]\|y_t\|^2\\
&+ \{\lambda_2(1-g_*)[\delta_1+\frac{3}{2}k\hat{g}(\mathcal{F})]+\lambda_3 H_\gamma(0)-\lambda_1(1-\frac{k}{2})\}\|\bigtriangledown y\|^2\\
&+ (\frac{\lambda_1}{2}-\lambda_3)\int_0^t g(t-s)\|\bigtriangledown y(s)\|_2^2 ds-\frac{\lambda_1}{2}\int_\Omega (g\Box\bigtriangledown y)dx\\
&+ \lambda_2 k[\frac{1-g_*}{4\delta_1}+1+\frac{1}{\delta_2}]\int_\Omega\int_{\mathcal{A}_t} g(t-s)\mid \bigtriangledown y(t)-\bigtriangledown y(s)\mid^2 dsdx\\
&+ (1+\delta_2)k\hat{h}(\mathcal{F})\lambda_2\int_\Omega\int_{\mathcal{F}_t} g(t-s)\mid \bigtriangledown y(t)-\bigtriangledown y(s)\mid^2 dsdx\\
&+ \frac{\lambda_2}{2}(1-g_*)\int_{\mathcal{F}_t} g(t-s)\|\bigtriangledown y(s)\|^2 ds-\lambda_3\eta(t)\psi_3(t) \qquad (2.25)
\end{aligned}$$

We select $\gamma_2\leq\frac{\delta_3}{c_p g(0)}$ so that

$$\tfrac{1}{2}-\tfrac{c_p}{4\delta_3}g(0)\gamma_2\geq\tfrac{1}{4}$$

and introduce the sets

$$\mathcal{A}_n = \{s \in \mathbb{R}^+ : ng'(s) + g(s) \leq 0\}, n \in \mathbb{N}.$$

Observe that

$$\bigcup_n \mathcal{A}_n = \mathbb{R}^+ \backslash \{\mathcal{F}_h \bigcup \mathcal{N}_h\},$$

Where $\mathcal{N}_h$ is the nullset where g' is not defined and $\mathcal{F}_h$ is as in (2.34) .
Furthermore, if we denote $\mathcal{F}_n := \mathbb{R}^+ \backslash \mathcal{A}_n$, then $\lim_{n\to\infty} \hat{g}(\mathcal{F}_n) = \hat{g}(\mathcal{F}_h)$ because $\mathcal{F}_{n+1} \subset \mathcal{F}_n$ for all n and $\bigcap_n = \mathcal{F}_h \bigcup \mathcal{N}_h$.In 2.25,we take $\mathcal{A} := \mathcal{A}_n$ and $\mathcal{F} := \mathcal{F}_n$ and$\lambda_1 = (g_* - \epsilon)\lambda_2$ for some small $\epsilon > 0$.It follows that

$$\begin{aligned} L'(t) &\leq \lambda_2(\delta_3 - \epsilon)\|y_t\|_2^2 \\ &+ [\lambda_2(1-g_*)(\delta_1 + \frac{3}{2}k\hat{g}(\mathcal{F}_n)) + \lambda_3 H_\gamma(0) - (g_* - \epsilon)\lambda_2(1 - \frac{k}{2})]\| \bigtriangledown y\|_2^2 \\ &+ [\lambda_2(\frac{g_* - \epsilon}{2}) - \lambda_3] \int_0^t g(t-s)\| \bigtriangledown y(s)\|_2^2 ds - \frac{\lambda_2(g_* - \epsilon)}{2} \int_\Omega g \Box \bigtriangledown y dx \\ &+ [\lambda_2 k(\frac{1-g_*}{4\delta_1} + 1 + \frac{1}{\delta_2}) - \frac{1}{4n}] \int_\Omega \int_{\mathcal{A}_{nt}} g(t-s)(\bigtriangledown y(t) - \bigtriangledown y(s))^2 ds dx \\ &+ \lambda_2(1+\delta_2)k\hat{g}(\mathcal{F}_n) \int_0^1 \int_{\mathcal{F}_{nt}} g(t-s)(\bigtriangledown y(t) - \bigtriangledown y(s))^2 ds dx \\ &+ \frac{\lambda_2(1-g_*)}{2} \int_{\mathcal{F}_{nt}} g(t-s)\| \bigtriangledown y(s)\|_2^2 ds - \lambda_3 \eta(t) \int_0^t G_\gamma(t-s)\| \bigtriangledown y\|_2^2 ds \end{aligned} \tag{2.26}$$

We have

$$(*) \begin{cases} \delta_3 - \epsilon < 0 & (2.27) \\ \lambda_2(1-g_*)(\delta_1 + \frac{3}{2}k\hat{g}(\mathcal{F}_n)) + \lambda_3 G_\gamma(0) - (g_* - \epsilon)\lambda_2(\delta + (1-\delta))(1 - \frac{k}{2}) < 0 & (2.28) \\ \lambda_2(\frac{g_* - \epsilon}{2}) - \lambda_3 + \frac{\lambda_2(1-g_*)}{2} < 0 & (2.29) \\ \lambda_2(1+\delta_2)k\hat{g}(\mathcal{F}_n) - \frac{(g_* - \epsilon)}{2}\lambda_2 < 0 & (2.30) \\ \lambda_2 k(\frac{1-g_*}{4\delta_1} + 1 + \frac{1}{\delta_2}) - \frac{1}{4n} < 0 & (2.31) \end{cases}$$

from the relatin (2.27) *we choose δ_3, ϵ small*

$$\delta_3 - \epsilon < 0 \Longrightarrow \delta_3 < \epsilon$$

and from (2.28) *for a small ϵ and large values of n and t, we see that if $\hat{g}(\mathcal{F}_n) < \frac{1}{4}$, we have*

$$\frac{3}{2}(1-g_*)k\hat{g}(\mathcal{F}_n) - \delta(g_* - \epsilon)(1 - \frac{k}{2}) < 0$$

$$\frac{3}{2}k(1-g_*)\hat{g}(\mathcal{F}_n) < \delta(g_* - \epsilon)(1 - \frac{k}{2}) < \delta g_*(1 - \frac{k}{2})$$

$$\delta > (1-g_*)\frac{3k\hat{g}(\mathcal{F}_n)}{g_*(2-k)}$$

$$\frac{3k(1-g_*)}{g_*(2-k)}\frac{1}{4} = \frac{3}{4}\frac{k(1-g_*)}{g_*(2-k)}$$

$$\delta = \frac{3}{4}\frac{k(1-g_*)}{g_*(2-k)}$$

Note that $\delta < 1$. For the remaining $1-\delta$ we require that λ_2 and λ_3 satisfy

$$\lambda_2 k(\frac{1-g_*}{4\delta_1} + 1 + \frac{1}{\delta_2}) < \frac{1}{4n}$$

and

$$\lambda_2(\frac{g_*-\epsilon}{2} + \frac{1-g_*}{2}) < \lambda_3 < \lambda_2\frac{(1-\delta)(g_*-\epsilon)(2-k)}{2G_\gamma(0)}$$

We have $g_* - \delta < g_*$ so

$$\lambda_2(\frac{g_*}{2} + \frac{1-g_*}{2}) < \lambda_3 < \lambda_2\frac{(1-\delta)g_*(2-k)}{2G_\gamma(0)}$$

$$\begin{aligned}(1-\delta) &= 1 - \frac{3}{4}\frac{k(1-g_*)}{g_*(2-k)}\\ &= \frac{4g_*(2-k) - 3k(1-g_*)}{4g_*(2-k)}\end{aligned}$$

So, we get

$$(g_* + 1 - g_*)\frac{\lambda_2}{2} < \lambda_3 < \frac{(8-k)g_* - 3k}{8G_\gamma(0)}\lambda_2$$

This is possible if $G_\gamma(0) < [g_*(8-k) - 3k]/4$and $g_* > 3k/(8-k)$.
Finally, we choose δ_1, ϵ small and $\delta_3 < \epsilon$

These choices together with (2.26) *lead to*

$$L'(t) \leq -C_1\varepsilon(t) - \lambda_3\eta(t)\phi_3(t), t \geq t_*$$

for some positive constant C_1.As $\eta(t)$ is decreasing, we have $\eta(t) \leq C_1$ after some $\hat{t} \geq t_*$. The right hand side inequality in Proposition 1 implies that

$$L'(t) \leq -C_2\eta(t)L(t), t \geq \hat{t} \tag{2.32}$$

for some positive constant C_2. An integration of (2.32) *yields*

$$\int_{\hat{t}}^{t}\frac{L'(s)}{L(s)}ds \leq \int_{\hat{t}}^{t} -C_2\eta(s)ds$$

$$lnL(t) - lnL(\hat{t}) \leq \int_{\hat{t}}^{t} -C_2\eta(s)ds$$

$$L(t) \leq e^{-C_2\int_{\hat{t}}^{t}\eta(s)ds}L(\hat{t}), t \geq \hat{t}.$$

and the left hand side inequality in Proposition 1 gives

$$\rho_1[\varepsilon(t) + \phi_3(t)] \leq e^{-C_2\int_{\hat{t}}^{t}\eta(s)ds}L(\hat{t}), t \geq \hat{t}.$$

Therefore by the definitions of $E(t)$ and $\varepsilon(t)$,the continuity of $E(t)$ and boundedness of the interval[0,$\hat{t}$] we infer that $E(t) \leq C/\gamma(t)^{\vartheta}, t > 0$
for some positive constants C and ϑ

2.2 Arbitrary decay for a Euler-Bernoulli Beam Equation with Memory

The main purpose of this work is to study the asymptotic behavior of the solutions of viscoelastic Euler-Bernoulli Beam Equation with boundary condition of memory type. For this, we consider the following initial boundary- value problem:

$$\begin{cases} y_{tt} + y_{xxxx} - \int_0^t g(t-s)y_{xxxx}(x,s)ds = 0 & in \quad [0,1]\times\mathbb{R}^+ \\ y(0,t) = 0 \quad , \quad y_{xx}(0,t) = y_{xx}(1,t) = 0 & \forall t > 0 \\ y_{xxx}(1,t) - \int_0^t g(t-s)y_{xxx}(1,s)ds = 0 \end{cases} \tag{2.33}$$

we note $\|.\|$ the $L^2([0,1])$ norme .

Multiplying (2.33) by y_t and integrating over $[0,1]$ yield

$$\int_0^1 y_t y_{tt} dx + \int_0^1 y_t y_{xxxx} dx - \int_0^1 y_t \int_0^t g(t-s) y_{xxxx}(x,s) ds dx = 0$$

For the first integral we have

$$\begin{aligned} \int_0^1 y_t y_{tt} dx &= \frac{1}{2}\int_0^1 \frac{d}{dt}(y_t)^2 dx \\ &= \frac{1}{2}\frac{d}{dt}\|y_t\|^2. \end{aligned} \tag{2.34}$$

Integration by parts the second integral we obtain

$$\begin{aligned} \int_0^1 y_t y_{xxxx} dx &= y_t(1,t)y_{xxx}(1,t) - \int_0^1 y_{tx} y_{xxx} dx \\ &= y_t(1,t)y_{xxx}(1,t) + \frac{1}{2}\frac{d}{dt}\|y_{xx}\|^2. \end{aligned} \tag{2.35}$$

For the last integral we have

$$\begin{aligned} -\int_0^1 y_t \int_0^t g(t-s)y_{xxxx}(x,s)dsdx &= -y_t(1,t)\int_0^t g(t-s)y_{xxx}(1,s)ds \\ &+ \int_0^1 y_{tx}\int_0^t g(t-s)y_{xxx}(x,s)dsdx \\ &= -y_t(1,t)\int_0^t g(t-s)y_{xxx}(1,s)ds \\ &- \int_0^1 y_{txx}\int_0^t g(t-s)y_{xx}(x,s)dsdx \\ &= -y_t(1,t)\int_0^t g(t-s)y_{xxx}(1,s)ds \\ &- \frac{1}{2}\int_0^1 g'\Box y_{xx}(x,t)dx + \frac{1}{2}g(t)\|y_{xx}\|^2 \\ &+ \frac{1}{2}\frac{d}{dt}\{\int_0^1 g\Box y_{xx}(x,t)dx - \int_0^t g(s)ds\|y_{xx}\|^2\} \end{aligned} \tag{2.36}$$

combining (2.34), (2.35) and (2.36), the result will be

$$\frac{1}{2}\frac{d}{dt}\{\|y_t\|^2 + \|y_{xx}\|^2 + \int_0^1 g\Box y_{xx}(x,t)dx - (\int_0^t g(s)ds)\|y_{xx}\|^2\}$$
$$+y_t(1,t)[y_{xxx}(1,t) - \int_0^t g(t-s)y_{xxx}(1,s)ds] - \frac{1}{2}\int_0^1 g'\Box y_{xx}(x,t)dx + \frac{1}{2}g(t)\|y_{xx}\|^2 = 0$$

Taking into account the boundary conditions, it results that

$$\frac{1}{2}\frac{d}{dt}\{\|y_t\|^2 + (1 - \int_0^t g(s)ds)\|y_{xx}\|^2 + \int_0^1 g\Box y_{xx}(x,t)dx\}$$

$$= \frac{1}{2}\int_0^1 g'\Box y_{xx}(x,t)dx - \frac{1}{2}g(t)\|y_{xx}\|^2$$

We define the energy E(t) of problem (2.33) by

$$E(t) = \frac{1}{2}[\|y_t\|^2 + (1 - \int_0^t g(s)ds)\|y_{xx}\|^2 + \int_0^1 g\Box y_{xx}(x,t)dx]$$

Then, the derivative of the energy is given by

$$E'(t) = \frac{1}{2}\int_0^1 g'\Box y_{xx}(x,t)dx - \frac{1}{2}g(t)\|y_{xx}\|^2 \tag{2.37}$$

We then define the modified energy by

$$L(t) = E(t) + \lambda_1\phi_1(t) + \lambda_2\phi_2(t) + \lambda_3\phi_3(t) \tag{2.38}$$

where

$$\phi_1(t) = \int_0^1 y_t y dx$$

$$\phi_2(t) = -\int_0^1 y_t \int_0^t g(t-s)(y(t) - y(s))dsdx$$

$$\phi_3(t) = \int_0^1 \int_0^t G_\gamma(t-s)|y_{xx}(s)|^2 dsdx$$

with $G_\gamma(t) = \gamma(t)^{-1}\int_t^{+\infty} g(s)\gamma(s)ds$

2.2.1 Equivalence between $L(t)$ and $E(t) + \phi_3(t)$:

We have the following lemma:

Lemma 4 *There exist $\rho_i > 0$, $i = 1, 2$ such that*

$$\rho_1[E(t) + \phi_3(t)] \le L(t) \le \rho_2[E(t) + \phi_3(t)] \tag{2.39}$$

for all $t \ge 0$ and small $\lambda_i, i = 1, 2$.

proof 3 *Applying Cauchy Schwartz, Young and Poincaré's inequality we get*

$$\begin{aligned}
\phi_1(t) &= \int_0^1 y_t y dx \\
&\le \int_0^1 |y_t||y|dx \\
&\le \le \frac{1}{2}\|y_t\|^2 + \frac{1}{2}\|y\|^2 \\
\phi_1(t) &\le \frac{1}{2}\|y_t\|^2 + \frac{1}{2}C_p^2\|y_{xx}\|^2.
\end{aligned} \tag{2.40}$$

$$\begin{aligned}
\phi_2(t) &= -\int_0^1 y_t \int_0^t g(t-s)(y(t)-y(s))dsdx \\
&\leq \|y_t\|(\int_0^1 (\int_0^t g(t-s)(y(t)-y(s))ds)^2 dx)^{\frac{1}{2}} \\
(\int_0^t g(t-s)(y(t)-y(s))ds)^2 &\leq \int_0^t g(s)ds \int_0^t g(t-s)(y(t)-y(s))^2 ds \\
\int_0^1 (\int_0^t g(t-s)(y(t)-y(s))ds)^2 dx &\leq C_p^2 \int_0^t g(s)ds \int_0^1 \int_0^t g(t-s)(y_{xx}(t)-y_{xx})^2 dsdx \\
&\leq C_p^2 k \int_0^1 g\Box y_{xx} dx.
\end{aligned}$$

$$\phi_2(t) \leq \frac{1}{2}\|y_t\|^2 + \frac{kC_p^2}{2}\int_0^1 g\Box y_{xx}dx \tag{2.41}$$

where C_p is the Poincaré constant.

Replacing (2.40), (2.41) *in the modified energy* (2.38) *we get*

$$L(t) \leq \frac{1}{2}(1+\lambda_1+\lambda_2)\|y_t\|^2 + \frac{1}{2}(1-\int_0^t g(s)ds + \lambda_1 C_p^2)\|y_{xx}\|^2 + \frac{1}{2}(1+\lambda_2 C_p^2 k)\int_0^1 g\Box y_{xx}dx + \lambda_3\phi_3(t)$$

On the other hand,

$$(1-\lambda_1-\lambda_2)\|y_t\|^2 + (1-k-\lambda_1 C_p^2)\|y_{xx}\|^2 + (1-\lambda_2 C_p^2 k)\int_0^1 g\Box y_{xx}dx + 2\lambda_3\phi_3(t) \leq 2L(t)$$

$$\begin{cases} 1-\lambda_1-\lambda_2 > 0 \\ 1-k-\lambda_1 C_p^2 > 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 < 1-\lambda_2 \\ \lambda_1 < \frac{1-k}{C_p^2} \end{cases} \quad \text{we get } \lambda_1 < min\{1-\lambda_2, \tfrac{1-k}{C_p^2}\}$$

$$\begin{cases} 1-\lambda_2 C_p^2 k > 0 \\ 1-\lambda_2 > 0 \end{cases} \Rightarrow \begin{cases} \lambda_2 < \frac{1}{C_p^2 k} \\ \lambda_2 < 1 \end{cases} \quad \text{we get } \lambda_2 < min\{1, \tfrac{1}{C_p^2 k}\}$$

Therefore

$$\rho_1(E(t)+\phi_3(t)) \leq L(t) \leq \rho_2(E(t)+\phi_3(t))$$

for some constant $\rho_1 > 0, i = 1, 2$

2.2.2 Asymptotyc Behavior

In this section we state and prove our result. To this end we need some notation. For every measurable set $\mathcal{A} \subset \mathbb{R}^+$, we define $\widehat{g}$ by

$$\widehat{g}(\mathcal{A}) = \frac{1}{k}\int_{\mathcal{A}} g(s)ds \tag{2.42}$$

where $\mathcal{A}_t = \mathcal{A}\bigcap[0,t]$. The flatness set and the flatness rate of g are defined by

$$\mathcal{F}_g = \{s \in \mathbb{R}^+ : g(s) > 0 \, and \, g'(s) = 0\} \tag{2.43}$$

and

$$\mathcal{R}_g = \widehat{g}(\mathcal{F}_g),$$

respectively.
For relaxation function g, we assume the following:
(A1) $g : [0, +\infty) \longrightarrow \mathbb{R}^+$ are nonincreasing C^1 functions satisfying

$$g(0) > 0,\ 0 < k = \int_0^{+\infty} g(s)ds < 1$$

.
(A2) $g'(t) \leqslant 0$ for almost $t > 0$.
(A3) There exists a nondecreasing function $\gamma(t) > 0$ such that $\dfrac{\gamma'(t)}{\gamma(t)} =: \eta(t)$ is a decreasing function and $\int_0^{+\infty} \gamma(s)g(s)ds < +\infty$.

Let $t_\star > 0$ be a number such that $\int_0^{+\infty} g(s)ds = g_\star$.

Theorem 2.2.1 *Assume that the hypotheses (A1)–(A3) and $\mathcal{R}_g < \frac{1}{4}$ hold.*
If $G_\gamma(0) < \frac{4g_*(2-k)-3k}{2}$, $g_* > 3k/4(2-k)$, then, there exist positive constants C and ν such that

$$E(t) \leq C\gamma(t)^{-\nu}, \quad t \geq 0.$$

proof 4

$$\phi_1(t) = \int_0^1 y_t y dx$$

A differentiation of $\phi_1(t)$ with respect to t gives

$$\phi_1'(t) = \int_0^1 y_{tt} y dx + \|y_t\|^2$$

Using (2.33) *we get*

$$\begin{aligned}
\phi_1'(t) &= -\int_0^1 y_{xxxx} y dx + \int_0^1 y \int_0^t g(t-s) y_{xxxx}(x,s) ds dx + \|y_t\|^2 \\
&= -y_{xxx}(1,t)y(1,t) + \int_0^1 y_{xxx} y_x dx + y(1,t)\int_0^t g(t-s) y_{xxx}(1,s) ds \\
&- \int_0^1 y_x \int_0^t g(t-s) y_{xxx}(x,s) ds dx + \|y_t\|^2 \\
\phi_1'(t) &\leq (-1 + \frac{k}{2})\|y_{xx}\|^2 + \|y_t\|^2 \\
&+ \frac{1}{2}\int_0^t g(t-s)\|y_{xx}(s)\|^2 ds - \frac{1}{2}\int_0^1 g\Box y_{xx} dx \qquad (2.44)
\end{aligned}$$

$$\phi_2(t) = -\int_0^1 y_t \int_0^t g(t-s)(y(t)-y(s)) ds dx$$

$$\begin{aligned}
\phi_2'(t) &= -\int_0^1 y_{tt}\int_0^t g(t-s)(y(t)-y(s)) ds dx - \int_0^1 y_t \frac{d}{dt}(\int_0^t g(t-s)(y(t)-y(s)) ds) dx \\
&= -\int_0^1 y_{tt}\int_0^t g(t-s)(y(t)-y(s)) ds dx - \int_0^1 y_t \int_0^t g'(t-s)(y(t)-y(s)) ds dx \\
&- \int_0^1 y_t^2 \int_0^t g(t-s) ds dx \qquad (2.45)
\end{aligned}$$

Replacing y_{tt} by other terms in problem (2.33) *we get*

$$\begin{aligned}
I_1 &= \int_0^1 y_{xxxx}\int_0^t g(t-s)(y(t)-y(s)) ds dx \\
&- \int_0^1 (\int_0^t g(t-s) y_{xxxx}(x,s) ds) \int_0^t g(t-s)(y(t)-y(s)) ds dx \qquad (2.46)
\end{aligned}$$

$$\beta_1 = \int_0^1 y_{xxxx} \int_0^t g(t-s)(y(t)-y(s))dsdx$$

Integrating by parts and using the boundary condtions we get

$$\begin{aligned}
\beta_1 &= y_{xxx}(1,t)\int_0^t g(t-s)(y(1,t)-y(1,s))ds - \int_0^1 y_{xxx}\int_0^t g(t-s)(y_x(t)-y_x(s))dsdx \\
&= y_{xxx}(1,t)\int_0^t g(t-s)(y(1,t)-y(1,s))ds + \int_0^1 y_{xx}\int_0^t g(t-s)(y_{xx}(t)-y_{xx}(s))dsdx \\
&\leq y_{xxx}(1,t)\int_0^t g(t-s)(y(1,t)-y(1,s))ds + \left(\delta_2 + \frac{3}{2}k\hat{g}(\mathcal{F})\right)\|y_{xx}\|^2 \\
&+ \frac{k}{4\delta_2}\int_0^1\int_{\mathcal{A}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx + \frac{1}{2}\int_{\mathcal{F}_t} g(t-s)\|y_{xx}(s)\|^2ds \qquad (2.47)
\end{aligned}$$

$$\begin{aligned}
\beta_2 &= -\int_0^1 \left(\int_0^t g(t-s)y_{xxxx}(x,s)ds\right)\int_0^t g(t-s)(y(t)-y(s))dsdx \\
\beta_2 &= -\int_0^t g(t-s)y_{xxx}(1,s)ds\int_0^t g(t-s)(y(1,t)-y(1,s))ds \\
&+ \int_0^1\left(\int_0^t g(t-s)y_{xxx}(x,s)ds\right)\int_0^t g(t-s)(y_x(t)-y_x(s))dsdx \\
&= -\int_0^t g(t-s)y_{xxx}(1,s)ds\int_0^t g(t-s)(y(1,t)-y(1,s))ds \\
&- \int_0^1\left(\int_0^t g(t-s)y_{xx}(x,s)ds\right)\int_0^t g(t-s)(y_{xx}(t)-y_{xx}(s))dsdx \\
&= -\int_0^t g(t-s)y_{xxx}(1,s)ds\int_0^t g(t-s)(y(1,t)-y(1,s))ds \\
&+ \int_0^1\left[\int_0^t g(t-s)(y_{xx}(x,t)-y_{xx}(x,s))ds\right) \\
&- \int_0^t g(t-s)y_{xx}(x,t)ds\Big]\Big[\int_0^t g(t-s)(y_{xx}(t)-y_{xx}(s))ds\Big]dx \\
&= -\int_0^t g(t-s)y_{xxx}(1,s)ds\int_0^t g(t-s)(y(1,t)-y(1,s))ds \\
&+ \int_0^1\left(\int_0^t g(t-s)(y_{xx}(t)-y_{xx}(s))ds\right)^2dx \\
&- \left(\int_0^t g(s)ds\right)^2\|y_{xx}\|^2 + k\int_0^1 y_{xx}(t)\int_0^t g(t-s)y_{xx}(s)dsdx \\
\beta_2 &\leq -\int_0^t g(t-s)y_{xxx}(1,s)ds\int_0^t g(t-s)(y(1,t)-y(1,s))ds \\
&+ \left(1+\frac{1}{\delta_3}\right)k\int_0^1\int_{\mathcal{A}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx \\
&+ (1+\delta_3)k\hat{g}(\mathcal{F})\int_0^1\int_{\mathcal{F}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx - g_*k\|y_{xx}\|_2^2 + k\delta_4\|y_{xx}\|_2^2 \\
&+ \frac{k^2}{4\delta_4}\int_0^t g(t-s)\|y_{xx}(s)\|_2^2ds \qquad (2.48)
\end{aligned}$$

Using the boundary conditions, taking account (2.47) *and* (2.48), *the result will be*

$$\begin{aligned} I_1 &\leq k\left(\frac{1}{4\delta_2}+1+\frac{1}{\delta_3}\right)\int_0^1\int_{\mathcal{A}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx \\ &+ (\frac{3}{2}k\hat{g}(\mathcal{F})+k\delta_4+\delta_2)\|y_{xx}\|^2+\frac{1}{2}\int_{\mathcal{F}_t} g(t-s)\|y_{xx}(s)\|^2ds \\ &+ (1+\delta_3)k\hat{g}(F)\int_0^1\int_{F_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx \\ &+ \frac{k^2}{4\delta_4}\int_0^t g(t-s)\|y_{xx}(s)\|^2ds. \end{aligned} \tag{2.49}$$

We have

$$\begin{aligned} I_2 &= -\int_0^1 y_t^2\int_0^t g(t-s)dsdx \\ &= -\int_0^t g(s)ds\int_0^1 y_t^2dx \\ &= -g_*\|y_t\|^2 \end{aligned} \tag{2.50}$$

and

$$I_3 = -\int_0^1 y_t\int_0^t g'(t-s)(y(t)-y(s))dsdx$$

Appliying Cauchy Schwartz, Poincaré and Young's inequalities, we obtain

$$\begin{aligned} I_3 &\leq (\int_0^1(-y_t)^2dx)^{\frac{1}{2}}({c_p}^2\int_0^1(\int_0^t g'(t-s)(y_{xx}(t)-y_{xx}(s))ds)^2dx)^{\frac{1}{2}} \\ &= (\int_0^1(-y_t)^2dx)^{\frac{1}{2}}({c_p}^2\int_0^1(\int_0^t |g'|^{\frac{1}{2}}|g'|^{\frac{1}{2}}(y_{xx}(t)-y_{xx}(s))ds)^2dx)^{\frac{1}{2}} \\ &\leq (\int_0^1(-y_t)^2dx)^{\frac{1}{2}}({c_p}^2\int_0^1\int_0^t|g'|ds\int_0^t|g'|(y_{xx}(t)-y_{xx}(s))^2dsdx)^{\frac{1}{2}} \\ &\leq (\int_0^1(-y_t)^2dx)^{\frac{1}{2}}({c_p}^2g(0)\int_0^1|g'|\Box y_{xx}dx)^{\frac{1}{2}} \\ &\leq \delta_5\|y_t\|_2^2-\frac{{c_p}^2}{4\delta_5}g(0)\int_0^1 g'\Box y_{xx}dx \end{aligned} \tag{2.51}$$

Taking account (2.49), (2.50) *and* (2.51)

$$\begin{aligned} {\phi_2}'(t) &\leq (-g_*+\delta_5)\|y_t\|^2+(\delta_2+\frac{3}{2}k\hat{g}(\mathcal{F})+k\delta_4))\|y_{xx}\|^2 \\ &+ (\frac{k}{4\delta_2}+(1+\frac{1}{\delta_3})k)\int_0^1\int_{\mathcal{A}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx+\frac{1}{2}\int_{\mathcal{F}_t} g(t-s)\|y_{xx}(s)\|^2ds \\ &+ (1+\delta_3)k\hat{g}(\mathcal{F})\int_0^1\int_{\mathcal{F}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx+\frac{k^2}{4\delta_4}\int_0^t g(t-s)\|y_{xx}(s)\|^2ds \\ &- \frac{{c_p}^2}{4\delta_5}g(0)\int_0^1 g'\Box y_{xx}dx. \end{aligned} \tag{2.52}$$

Further, a differentiation of $\phi_3(t)$ yields

$$\begin{aligned}
{\phi_3}'(t) &= \int_0^1 G_\gamma(0)|y_{xx}(t)|^2dx + \int_0^1\int_0^t {G_\gamma}'(t-s)|y_{xx}(s)|^2dsdx \\
&= \int_0^1 G_\gamma(0)|y_{xx}(t)|^2dx - \int_0^1\int_0^t \frac{\gamma'(t-s)}{\gamma(t-s)}G_\gamma(t-s)|y_{xx}(s)|^2dsdx \\
&- \int_0^1\int_0^t g(t-s)|y_{xx}(s)|^2dsdx \\
&= G_\gamma(0)\|y_{xx}\|^2 - \int_0^t \frac{\gamma'(t-s)}{\gamma(t-s)}G_\gamma(t-s)\|y_{xx}(s)\|^2ds \\
&- \int_0^t g(t-s)\|y_{xx}(s)\|^2ds \\
&\leq G_\gamma(0)\|y_{xx}\|^2 - \eta(t)\int_0^t G_\gamma(t-s)\|y_{xx}(s)\|^2ds - \int_0^t g(t-s)\|y_{xx}(s)\|^2ds \quad (2.53)
\end{aligned}$$

Where we have used the fact that $\frac{\gamma'(t)}{\gamma(t)} = \eta(t)$ is a nonincreasing function.
Therefore, by gathering (2.37), (2.44), (2.52) *and* (2.53), *we obtain*

$$\begin{aligned}
L'(t) &\leq E'(t) + \lambda_1\phi_1'(t) + \lambda_2\phi_2'(t) + \lambda_3\phi_3'(t) \\
&\leq \frac{1}{2}\int_0^1 g'\Box y_{xx}dx - \frac{1}{2}g(t)\|y_{xx}\|^2 - \lambda_1(1-\frac{1}{2}k)\|y_{xx}\|^2 + \frac{\lambda_1}{2}\int_0^t g(t-s)\|y_{xx}(s)\|^2ds \\
&- \frac{\lambda_1}{2}\int_0^1 g\Box y_{xx}dx + \lambda_1\|y_t\|^2 + \lambda_2(-g_* + \delta_5)\|y_t\|^2 + \lambda_2(\delta_2 + \frac{3}{2}k\hat{g}(\mathcal{F}) + k\delta_4)\|y_{xx}\|^2 \\
&+ \lambda_2(\frac{k}{4\delta_2} + (1+\frac{1}{\delta_3})k)\int_0^1\int_{\mathcal{A}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx + \frac{\lambda_2}{2}\int_{\mathcal{F}_t} g(t-s)\|y_{xx}(s)\|^2ds \\
&+ \lambda_2(1+\delta_3)k\hat{g}(\mathcal{F})\int_0^1\int_{\mathcal{F}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx + \lambda_2\frac{k^2}{4\delta_4}\int_0^t g(t-s)\|y_{xx}(s)\|^2ds \\
&- \lambda_2\frac{{c_p}^2}{4\delta_5}g(0)\int_0^1 g'\Box y_{xx}dx + \lambda_3G_\gamma(0)\|y_{xx}\|^2 - \lambda_3\eta(t)\int_0^t G_\gamma(t-s)\|y_{xx}(s)\|^2ds \\
&- \lambda_3\int_0^t g(t-s)\|y_{xx}(s)\|^2ds
\end{aligned} \quad (2.54)$$

After some simplification, we get

$$\begin{aligned}
L'(t) &\leq [\frac{1}{2} - \lambda_2\frac{{c_p}^2}{4\delta_5}g(0)]\int_0^1 g'\Box y_{xx}dx + [\lambda_1 + \lambda_2(-g_* + \delta_5)]\|y_t\|^2 \\
&+ [-\lambda_1 + \frac{\lambda_1}{2}k + \lambda_2(\delta_2 + \frac{3}{2}k\hat{g}(\mathcal{F}) + k\delta_4) + \lambda_3G_\gamma(0)]\|y_{xx}\|^2 \\
&+ [\frac{\lambda_1}{2} + \lambda_2\frac{k^2}{4\delta_4} - \lambda_3]\int_0^t g(t-s)\|y_{xx}(s)\|^2ds - \frac{\lambda_1}{2}\int_0^1 g\Box y_{xx}dx \\
&+ \lambda_2[(\frac{k}{4\delta_2} + (1+\frac{1}{\delta_3})k]\int_0^1\int_{\mathcal{A}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx + \frac{\lambda_2}{2}\int_{\mathcal{F}_t} g(t-s)\|y_{xx}(s)\|^2ds \\
&+ \lambda_2(1+\delta_3)k\hat{g}(\mathcal{F})\int_0^1\int_{\mathcal{F}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx \\
&- \lambda_3\eta(t)\int_0^t G_\gamma(t-s)\|y_{xx}(s)\|^2ds. \quad (2.55)
\end{aligned}$$

We select $\lambda_2 \leq \frac{\delta_5}{{c_p}^2 g(0)}$ so that

$$\tfrac{1}{2}-\tfrac{{c_p}^2}{4\delta_5}g(0)\lambda_2\geq\tfrac{1}{4}$$

and introduce the sets

$$\mathcal{A}_n=\{s\in\mathbb{R}^+ : ng'(s)+g(s)\leq 0\}, n\in\mathbb{N}.$$

Observe that

$$\bigcup_n \mathcal{A}_n=\mathbb{R}^+\setminus\{\mathcal{F}_g\bigcup\mathcal{N}_g\},$$

Where $\mathcal{N}_g$ is the nullset where g' is not defined and $\mathcal{F}_g$ is as in. Furthermore, if we denote $\mathcal{F}_n:=\mathbb{R}^+\backslash\mathcal{A}_n$, then $\lim_{n\to\infty}\hat{g}(\mathcal{F}_n)=\hat{g}(\mathcal{F}_g)$ because $\mathcal{F}_{n+1}\subset\mathcal{F}_n$ for all n and $\bigcap\limits_n=\mathcal{F}_g\bigcup\mathcal{N}_g$.
In (2.53), we take $\mathcal{A}:=\mathcal{A}_n$ and $\mathcal{F}:=\mathcal{F}_n$ and $\lambda_1=(g_*-\epsilon)\lambda_2$ for some small $\epsilon>0$.
It follows that

$$\begin{aligned}
L'(t) &\leq (\delta_5-\epsilon)\lambda_2\|y_t\|^2\\
&+ [\lambda_2(\delta_2+\frac{3}{2}k\hat{g}(\mathcal{F}_n)+k\delta_4)+\lambda_3G_\gamma(0)-(g_*-\epsilon)\lambda_2(1-\frac{k}{2})]\|y_{xx}\|^2\\
&+ [\lambda_2(\frac{g_*-\epsilon}{2}+\frac{k^2}{4\delta_4})-\lambda_3]\int_0^t g(t-s)\|y_{xx}\|^2ds-\frac{\lambda_2(g_*-\epsilon)}{2}\int_0^1 g\Box y_{xx}dx\\
&+ [\lambda_2k(\frac{1}{4\delta_2}+1+\frac{1}{\delta_3})-\frac{1}{4n}]\int_0^1\int_{\mathcal{A}_{nt}} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx\\
&+ \frac{\lambda_2}{2}\int_{\mathcal{F}_{nt}} g(t-s)\|y_{xx}(s)\|^2ds+\lambda_2(1+\delta_3)k\hat{g}(\mathcal{F}_n)\int_0^1\int_{\mathcal{F}_{nt}} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx\\
&- \lambda_3\eta(t)\int_0^t G_\gamma(t-s)\|y_{xx}(s)\|^2ds \qquad (2.56)
\end{aligned}$$

We have

$$(*)\begin{cases}
\delta_5-\epsilon<0 & (2.57)\\
\frac{3}{2}k\hat{g}(\mathcal{F}_n)<\delta(g_*-\epsilon)(1-\frac{k}{2}) & (2.58)\\
\lambda_3G_\gamma(0)-\lambda_2(-\delta_2-k\delta_4+(1-\delta)(g_*-\epsilon)(1-\frac{k}{2}))<0 & (2.59)\\
\lambda_2(\frac{g_*-\epsilon}{2}+\frac{k^2}{4\delta_4})-\lambda_3+\frac{\lambda_2}{2}<0 & (2.60)\\
(1+\delta_3)k\hat{g}(\mathcal{F}_n)-\frac{(g_*-\epsilon)}{2}<0 & (2.61)\\
\lambda_2k(\frac{1}{4\delta_2}+1+\frac{1}{\delta_3})-\frac{1}{4n}<0 & (2.62)
\end{cases}$$

from the relatin (2.57) we deduce

$$\delta_5-\epsilon<0\Longrightarrow\delta_5<\epsilon$$

and from (2.58) for a small ϵ and large values of n and t, we see that if $\hat{g}(\mathcal{F}_n)<\frac{1}{4}$, we have

$$\frac{3}{2}k\hat{g}(\mathcal{F}_n)-\delta(g_*-\epsilon)(1-\frac{k}{2})<0$$

$$\frac{3}{2}k\hat{g}(\mathcal{F}_n)<\delta(g_*-\epsilon)(1-\frac{k}{2})$$

with

$$\delta=\frac{3}{4}\frac{k}{g_*(2-k)}.$$

Note that $\delta < 1$. For the remaining $1-\delta$ we require that λ_2 and λ_3 satisfy

$$\lambda_2 k(\frac{1}{4\delta_2} + 1 + \frac{1}{\delta_3}) < \frac{1}{4n}$$

and

$$\lambda_2(\frac{g_* - \epsilon}{2} + \frac{k^2}{4\delta_4} + \frac{1}{2}) < \lambda_3 < \lambda_2 \frac{(1-\delta)(g_* - \epsilon)(2-k)}{2G_\gamma(0)}$$

$$\lambda_2(\frac{g_*}{2} + \frac{k^2}{4\delta_4} + \frac{1}{2}) < \lambda_3 < \lambda_2 \frac{(1-\delta)g_*(2-k)}{2G_\gamma(0)}$$

$$(1-\delta) = 1 - \frac{3}{4}\frac{k}{g_*(2-k)}$$

$$= \frac{4g_*(2-k) - 3k}{4g_*(2-k)}$$

So, we get

$$\frac{\lambda_2}{2} < (g_* + \frac{k^2}{2\delta_4} + 1)\frac{\lambda_2}{2} < \lambda_3 < \frac{4g_*(2-k) - 3k}{4G_\gamma(0)}\lambda_2$$

This is possible if $G_\gamma(0) < \frac{4g_*(2-k)-3k}{2}$ and $g_* > 3k/4(2-k)$.

These choices together with (2.55) lead to

$$L'(t) \leq -C_1\varepsilon(t) - \lambda_3\eta(t)\phi_3(t), t \geq t_*$$

for some positive constant C_1. As $\eta(t)$ is decreasing, we have $\eta(t) \leq C_1$ for all $\hat{t} \geq t_*$. The right hand side inequality in Lemma 4 implies that

$$L'(t) \leq -C_2\eta(t)L(t), t \geq \hat{t} \tag{2.63}$$

for some positive constant C_2. An integration of (2.63) yields

$$\int_{\hat{t}}^{t} \frac{L'(s)}{L(s)} ds \leq \int_{\hat{t}}^{t} -C_2\eta(s)ds$$

$$lnL(t) - lnL(\hat{t}) \leq \int_{\hat{t}}^{t} -C_2\eta(s)ds$$

$$L(t) \leq e^{-C_2\int_{\hat{t}}^{t}\eta(s)ds}L(\hat{t}), t \geq \hat{t}.$$

Then using the left hand side inequality in (1.7), we get

$$\rho_1[\varepsilon(t) + \phi_3(t)] \leq e^{-C_2\int_{\hat{t}}^{t}\eta(s)ds}L(\hat{t}), t \geq \hat{t}.$$

By virtue of the continuity and boundedness of $E(t)$ in the interval $[0, \hat{t}]$, we conclude that

$$E(t) \leq C\gamma(t)^{-\nu}(t), t \geq 0$$

for some positive constants C and ν .

2.3 Stabilization of a viscoelasic Euler-Bernoulli beam

The third problem that we have stadied in the following form

$$\begin{cases}\rho y_{tt}(x,t) + EIy_{xxxx}(x,t) - Ty_{xx}(x,t) - EI\int_0^t g(t-s)y_{xxxx}(x,s)ds = 0 \qquad in \quad \Omega\times\mathbb{R}^+,\\ y_{xx}(0,t) = y_{xx}(L,t) = y(0,t) = 0 \qquad \forall t>0,\\ -EIy_{xxx}(L,t) + Ty_x(L,t) + EI\int_0^t g(t-s)y_{xxx}(L,s)ds = U(t) \qquad \forall t>0,\\ U(t) = 0.\end{cases} \tag{2.64}$$

Where $\Omega = [0,L]$, $\|.\|$ is the norme of $L^2(\Omega)$ and ρ, EI and T are positive constants .

Multiplying (2.64) by y_t and integrating ouver Ω, we get

$$\underbrace{\rho\int_0^L y_t y_{tt}dx}_{I_1} + \underbrace{EI\int_0^L y_t y_{xxxx}dx}_{I_2} - \underbrace{T\int_0^L y_t y_{xx}dx}_{I_3} - \underbrace{EI\int_0^L y_t\int_0^t g(t-s)y_{xxxx}(x,s)dsdx}_{I_4} = 0$$

$$\begin{aligned} I_1 = \rho\int_0^L y_t y_{tt}dx &= \rho\int_0^L \frac{1}{2}\frac{d}{dt}(y_t)^2 dx\\ &= \frac{\rho}{2}\frac{d}{dt}\int_0^L {y_t}^2 dx\\ &= \frac{\rho}{2}\frac{d}{dt}\|y_t\|^2 \end{aligned} \tag{2.65}$$

$$\begin{aligned} I_2 &= EI\int_0^L y_t y_{xxxx}dx\\ &= EIy_t(L,t)y_{xxx}(L,t) - EI\int_0^L y_{tx}y_{xxx}dx\\ &= EIy_t(L,t)y_{xxx}(L,t) + EI\int_0^L y_{txx}y_{xx}dx\\ &= EIy_t(L,t)y_{xxx}(L,t) + \frac{EI}{2}\frac{d}{dt}\|y_{xx}\|^2 \end{aligned} \tag{2.66}$$

$$\begin{aligned} I_3 &= -T\int_0^L y_t y_{xx}dx\\ &= -Ty_t(L,t)y_x(L,t) + T\int_0^L y_{tx}y_x dx\\ &= -Ty_t(L,t)y_x(L,t) + \frac{T}{2}\frac{d}{dt}\|y_x\|^2 \end{aligned} \tag{2.67}$$

$$\begin{aligned} I_4 &= -EI\int_0^L y_t\int_0^t g(t-s)y_{xxxx}(x,s)dsdx\\ &= -EIy_t(L,t)\int_0^t g(t-s)y_{xxx}(L,s)ds + EI\int_0^L y_{tx}\int_0^t g(t-s)y_{xxx}(x,s)dsdx\\ &= -EIy_t(L,t)\int_0^t g(t-s)y_{xxx}(L,s)ds - EI\int_0^L y_{txx}\int_0^t g(t-s)y_{xx}(x,s)dsdx, \end{aligned}$$

Using the equation (2.6) we get

$$\begin{aligned} I_4 &= -EIy_t(L,t)\int_0^t g(t-s)y_{xxx}(L,s)ds - \frac{EI}{2}\int_0^L g'\Box y_{xx}dx + \frac{EI}{2}g(t)\|y_{xx}\|^2\\ &+ \frac{EI}{2}\frac{d}{dt}\int_0^L g\Box y_{xx}dx - \frac{EI}{2}\frac{d}{dt}(\int_0^t g(s)ds)\|y_{xx}\|^2 \end{aligned} \tag{2.68}$$

Combining (2.65), (2.66), (2.67) and (2.68), we obtain

$$\frac{1}{2}\frac{d}{dt}\{\rho\|y_t\|^2 + EI\left(1-\int_0^t g(s)ds\right)\|y_{xx}\|^2 + T\|y_x\|^2 + EI\int_0^L g\Box y_{xx}dx\}$$
$$-\frac{EI}{2}(\int_0^L g'\Box y_{xx}dx - g(t)\|y_{xx}\|^2) + EIy_t(L,t)y_{xxx}(L,t) - Ty_t(L,t)y_x(L,t)$$
$$-EIy_t(L,t)\int_0^t g(t-s)y_{xxx}(L,s)ds = 0.$$

Then

$$\frac{1}{2}\frac{d}{dt}\{\rho\|y_t\|^2 + EI\left(1-\int_0^t g(s)ds\right)\|y_{xx}\|^2 + T\|y_x\|^2 + EI\int_0^L g\Box y_{xx}dx\}$$
$$= \frac{EI}{2}(\int_0^L g'\Box y_{xx}dx - g(t)\|y_{xx}\|^2) + y_t(L,t)U(t)$$

We define the energy $E(t)$ of problem (2.64) by

$$E(t) = \frac{1}{2}\left[\rho\|y_t\|^2 + EI\left(1-\int_0^t g(s)ds\right)\|y_{xx}\|^2 + T\|y_x\|^2 + EI\int_0^L g\Box y_{xx}dx\right]$$

The derivation of the energy is given by

$$E'(t) = \frac{EI}{2}[\int_0^L g'\Box y_{xx}dx - g(t)\|y_{xx}\|^2] + y_t(L,t)U(t)$$

where

$$g\Box y_{xx} = \int_0^t g(t-s)(y_{xx}(t)-y_{xx}(s))^2 ds$$

and

$$g'\Box y_{xx} = \int_0^t g'(t-s)(y_{xx}(t)-y_{xx}(s))^2 ds.$$

We assume that the kernel is such that

$$1-\int_0^{+\infty} g(s)ds = 1-k > 0.$$

Next, we define the standard functionals

$$\varphi_1(t) = \rho\int_0^L y_t y dx,$$

$$\varphi_2(t) = -\rho\int_0^L y_t\int_0^t g(t-s)(y(t)-y(s))dsdx,$$

and the new one

$$\varphi_3(t) = \int_0^L\int_0^t G_\gamma(t-s)(EI|y_{xx}(s)|^2 + T|y_x(s)|^2)dsdx,$$

where

$$G_\gamma(t) = \gamma(t)^{-1} \int_t^\infty g(s)\gamma(s)ds.$$

The modified energy we will work with is

$$L(t) = E(t) + \sum_{i=1}^{3} \lambda_i \varphi_i(t)$$

for some $\lambda_i > 0$,$i = 1, 2, 3$ to be determined.

2.3.1 Equivalence between $L(t)$ and $E(t) + \varphi_3(t)$:

The first result tells us that $L(t)$ and $E(t) + \varphi_3(t)$ are equivalent

Proposition 2 *There exist $c_i > 0$, $i = 1, 2$ such that*

$$c_1[E(t) + \varphi_3(t] \leq L(t) \leq c_2[E(t) + \varphi_3(t]$$

for all $t \geq 0$ and small $\lambda_i, i = 1, 2$.

proof 5

$$\varphi_1(t) = \rho \int_0^L y_t y dx$$

$$\begin{aligned}\varphi_1(t) &\leq \rho(\int_0^L {y_t}^2 dx)^{\frac{1}{2}} (\int_0^L y^2 dx)^{\frac{1}{2}} \\ &\leq \rho \|y_t\| \|y\| \\ &\leq \frac{\rho}{2} \|y_t\|^2 + \frac{\rho}{2} \|y\|^2\end{aligned}$$

Using poincare's inequality, we get

$$\begin{aligned}\varphi_1(t) \quad &\leq \tfrac{\rho}{2} \|y_t\|^2 + \tfrac{\rho c_p}{2} \|y_x\|^2 \\ &\leq \tfrac{\rho}{2} \|y_t\|^2 + \tfrac{\rho c_p}{T} \tfrac{T}{2} \|y_x\|^2 \\ \varphi_1(t) \quad &\leq C_1 (\tfrac{\rho}{2} \|y_t\|^2 + \tfrac{T}{2} \|y_x\|^2)\end{aligned} \tag{2.69}$$

with $C_1 = \max(1, \frac{\rho c_p}{T})$. For $\varphi_2(t)$

$$\begin{aligned}\varphi_2(t) \quad &= \quad -\rho \int_0^L y_t \int_0^t g(t-s)(y(t) - y(s))dsdx \\ &\leq \quad \rho \|y_t\| \left(\int_0^L (\int_0^t g(t-s)(y(t) - y(s))ds)^2 dx \right)^{\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}\int_0^L (\int_0^t g(t-s)(y(t) - y(s))ds)^2 dx \quad &\leq \quad \int_0^t g(t-s)ds \int_0^L (\int_0^t g(t-s)(y(t) - y(s))^2 ds)dx \\ &\leq \quad c_p^2 \int_0^t g(s)ds (\int_0^L \int_0^t g(t-s)(y_{xx}(t) - y_{xx})^2 ds)dx \\ &\leq \quad c_p^2 k \int_0^L g \Box y_{xx} dx\end{aligned}$$

Then

$$\begin{aligned}
\varphi_2(t) &\leq \rho\|y_t\|(kc_p^2\int_0^L g\Box y_{xx}dx)^{\frac{1}{2}} \\
&\leq \frac{\rho}{2}\|y_t\|^2 + kc_p^2\frac{\rho}{2}\int_0^L g\Box y_{xx}dx \\
\varphi_2(t) &\leq \frac{\rho}{2}\|y_t\|^2 + \frac{\rho k c_p^2}{EI}\frac{EI}{2}\int_0^L g\Box y_{xx}dx \\
&\leq C_2(\frac{\rho}{2}\|y_t\|^2 + \frac{EI}{2}\int_0^L g\Box y_{xx}dx) \qquad (2.70)
\end{aligned}$$

with $C_2 = \max(1, \frac{\rho {c_p}^2}{EI}k)$

where c_p *is the poincaré constant, we have*

$$\begin{aligned}
L(t) &\leq \frac{\rho}{2}\|y_t\|_2^2 + \frac{EI}{2}(1-\int_0^t g(s)ds)\|y_{xx}\|^2 + \frac{T}{2}\|y_x\|^2 + \frac{EI}{2}\int_0^L g\Box y_{xx}dx + \lambda_1 C_1\frac{\rho}{2}\|y_t\|^2 \\
&+ \lambda_1 C_1\frac{T}{2}\|y_x\|^2 + \lambda_2 C_2\frac{\rho}{2}\|y_t\|^2 + \lambda_2 C_2\frac{EI}{2}\int_0^L g\Box y_{xx}dx + \lambda_3\varphi_3(t) \\
L(t) &\leq \frac{\rho}{2}(1+\lambda_1 C_1+\lambda_2 C_2)\|y_t\|^2 + \frac{T}{2}(1+\lambda_1 C_1)\|y_x\|^2 + \frac{EI}{2}(1-\int_0^t g(s)ds)\|y_{xx}\|_2^2 \\
&+ \frac{EI}{2}(1+\lambda_2 C_2)\int_0^L g\Box y_{xx}dx + \lambda_3\varphi_3(t) \qquad (2.71)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
2L(t) &\geq \rho(1-\lambda_1 C_1-\lambda_2 C_2)\|y_t\|^2 + T(1-\lambda_1 C_1)\|y_x\|^2 + EI(1-k)\|y_{xx}\|^2 \\
&+ EI(1-\lambda_2 C2)\int_0^L g\Box y_{xx}dx + 2\lambda_3\varphi_3(t). \qquad (2.72)
\end{aligned}$$

Therefore, $C_1[E(t)+\varphi_3(t)] \leq L(t) \leq C_2[E(t)+\varphi_3(t)]$ *for some constant* $C_i > 0, i = 1,2$ *and small* λ_i, $i = 1,2$ *such that*

$$\begin{cases} 1-\lambda_1 C_1-\lambda_2 C_2 > 0 \\ 1-\lambda_1 C_1 > 0 \\ 1-\lambda_2 C_2 > 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 < \frac{1}{C_1} \\ \lambda_2 < \frac{1}{C_2} \text{ and } \lambda_2 < \frac{1-\lambda_1 C_1}{C_2} \end{cases} \Rightarrow \begin{cases} \lambda_1 < \frac{1}{C_1} \\ \lambda_2 = \min\{\frac{1}{C_2}, \frac{1-\lambda_1 C_1}{C_2}\} \end{cases}$$

2.3.2 Asymptotic Behavior

In this section we state and prove our result. But first we introduce the following notation. For every measurable set $\mathcal{A} \subset \mathbb{R}^+$, we define the probability measure $\hat{g}$ by

$$\hat{g}(\mathcal{A}) = \frac{1}{k}\int_{\mathcal{A}} g(s)ds.$$

The flatness set and the flatness rate of g are defined by

$$\mathcal{F}_g = \{s \in \mathbb{R}^+ : g(s) > 0 \text{ and } g'(s) = 0\}$$

and

$$\mathcal{R}_g = \hat{g}(\mathcal{F}_g),$$

respectively.
Our assumptions on the kernel $g(t)$ are the following:
$(H_1) g(t) \geq 0$ for all $t \geq 0$ and $0 < k = \int_0^{+\infty} g(s)ds < 1$.
$(H_2) g'(t) \leq 0$ for all t¿0.
(H_3) There exists a nondecreasing function $\gamma(t) > 0$ such that $\frac{\gamma'(t)}{\gamma(t)} = \eta(t)$ is a decreasing function and $\int_0^{+\infty} g(s)\gamma(s)ds < +\infty$.
Let $t_* > 0$ be a number such that $\int_0^{t_*} g(s)ds = g_* > 0$. For simplicity, we consider kernels continuous everywhere and differentiable.

A differentiation of $\varphi_1(t)$ with respect to t. Applying Cauchy Schwartz, Young's inequality, Poincare and lemma 3 gives

$$\varphi_1'(t) = (\int_0^L \rho y_t y dx)'$$

$$\varphi_1'(t) = \rho \int_0^L y_{tt} y dx + \rho \int_0^L {y_t}^2 dx$$

$$\varphi_1'(t) = \rho \int_0^L y_{tt} y dx + \rho \|y_t\|^2$$

Replacing y_{tt} by other terms in problem (2.64) we get

$$\begin{aligned}
\varphi_1'(t) &= -EI \int_0^L y_{xxxx} y dx + T \int_0^L y_{xx} y dx + EI \int_0^L y \int_0^t g(t-s) y_{xxxx}(x,s) ds dx + \rho \|y_t\|^2 \\
&= -EI y_{xxx}(L,t) y(L,t) + EI \int_0^L y_{xxx} y_x dx + T y_x(L,t) y(L,t) - T \int_0^L {y_x}^2 dx \\
&+ EI y(L,t) \int_0^t g(t-s) y_{xxx}(L,s) ds - EI \int_0^L y_x \int_0^t g(t-s) y_{xxx}(x,s) ds dx + \rho \|y_t\|^2 \\
&= -EI y_{xxx}(L,t) y(L,t) - EI \|y_{xx}\|^2 + T y_x(L,t) y(L,t) - T \|y_x\|^2 \\
&+ EI y(L,t) \int_0^t g(t-s) y_{xxx}(L,s) ds + EI \int_0^L y_{xx} \int_0^t g(t-s) y_{xx}(x,s) ds dx + \rho \|y_t\|^2 \\
\varphi_1'(t) &= EI(\frac{k}{2} - 1) \|y_{xx}\|^2 - T \|y_x\|^2 + \rho \|y_t\|^2 - \frac{EI}{2} \int_0^L g \Box y_{xx} dx + \frac{EI}{2} \int_0^t g(t-s) \|y_{xx}(s)\|^2 ds \\
&- EI y_{xx}(L,t) y(L,t) + T y_x(L,t) y(L,t) + EI y(L,t) \int_0^t g(t-s) y_{xxx}(L,s) ds \qquad (2.73)
\end{aligned}$$

$$\varphi_2'(t) = (-\rho \int_0^L y_t \int_0^t g(t-s)(y(t) - y(s)))' ds dx$$

$$
\begin{aligned}
\varphi_2'(t) &= -\rho\int_0^L y_{tt}\int_0^t g(t-s)(y(t)-y(s))dsdx - \rho\int_0^L y_t(\int_0^t g(t-s)(y(t)-y(s))ds)'dx \\
&= -\rho\int_0^L y_{tt}\int_0^t g(t-s)(y(t)-y(s))dsdx - \rho\int_0^L y_t\int_0^t g'(t-s)(y(t)-y(s))dsdx \\
&- \rho\int_0^L y_t^2\int_0^t g(t-s)dsdx
\end{aligned}
\tag{2.74}
$$

replacing y_{tt} by other terms in problem (2.64) we get

$$
\varphi_2'(t) = \underbrace{EI\int_0^L y_{xxxx}\int_0^t g(t-s)(y(t)-y(s))dsdx}_{J_1}\underbrace{-T\int_0^L y_{xx}\int_0^t g(t-s)(y(t)-y(s))dsdx}_{J_2}
$$

$$
\underbrace{-EI\int_0^L(\int_0^t g(t-s)y_{xxxx}(x,s)ds)\int_0^t g(t-s)(y(t)-y(s))dsdx}_{J_3}
$$

$$
\begin{aligned}
J_1 &= EI\int_0^L y_{xxxx}\int_0^t g(t-s)(y(t)-y(s))dsdx \\
J_1 &= EIy_{xxx}(L,t)\int_0^t g(t-s)(y(L,t)-y(L,s))ds - EI\int_0^L y_{xxx}\int_0^t g(t-s)(y_x(t)-y_x(s))dsdx \\
&= EIy_{xxx}(L,t)\int_0^t g(t-s)(y(L,t)-y(L,s))ds + EI\int_0^L y_{xx}\int_0^t g(t-s)(y_{xx}(t)-y_{xx}(s))dsdx \\
J_1 &\le EIy_{xxx}(L,t)\int_0^t g(t-s)(y(L,t)-y(L,s))ds + EI\delta_1\|y_{xx}\|^2 \\
&+ EI\frac{k}{4\delta_1}\int_0^L\int_{\mathcal{A}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx + EI\frac{3}{2}k\hat{g}(\mathcal{F})\|y_{xx}(s)\|^2 \\
&+ \frac{EI}{2}\int_{\mathcal{F}_t} g(t-s)\|y_{xx}(s)\|^2ds
\end{aligned}
\tag{2.75}
$$

$$
\begin{aligned}
J_2 &= -T\int_0^L y_{xx}\int_0^t g(t-s)(y(t)-y(s))dsdx \\
&= -Ty_x(L,t)\int_0^t g(t-s)(y(L,t)-y(L,s))ds + T\int_0^t g(s)ds\|y_x\|^2 \\
&- T\int_0^L y_x\int_0^t g(t-s)y_x(s)dsdx \\
J_2 &\le -Ty_x(L,t)\int_0^t g(t-s)(y(L,t)-y(L,s))ds + T(\int_0^t g(s)ds+\delta_2)\|y_x\|^2 \\
&+ T\frac{k}{4\delta_2}\int_0^t g(t-s)\|y_x(s)\|^2ds
\end{aligned}
\tag{2.76}
$$

$$
\begin{aligned}
J_3 &= -EI\int_0^L(\int_0^t g(t-s)y_{xxxx}(x,s)ds)\int_0^t g(t-s)(y(t)-y(s))dsdx \\
J_3 &= -EI\int_0^t g(t-s)y_{xxx}(L,s)ds\int_0^t g(t-s)(y(L,t)-y(L,s))ds \\
&+ EI\int_0^L(\int_0^t g(t-s)y_{xxx}(x,s)ds)\int_0^t g(t-s)(y_x(t)-y_x(s))dsdx \\
J_3 &= -EI\int_0^t g(t-s)y_{xxx}(L,s)ds\int_0^t g(t-s)(y(L,t)-y(L,s))ds \\
&- EI\int_0^L(\int_0^t g(t-s)y_{xx}(x,s)ds)\int_0^t g(t-s)(y_{xx}(t)-y_{xx}(s))dsdx \\
&= -EI\int_0^t g(t-s)y_{xxx}(L,s)ds\int_0^t g(t-s)(y(L,t)-y(L,s))ds \\
&+ EI\int_0^L[\int_0^t g(t-s)(y_{xx}(x,t)-y_{xx}(x,s))ds) \\
&- EI\int_0^t g(t-s)y_{xx}(x,t)ds][\int_0^t g(t-s)(y_{xx}(t)-y_{xx}(s))ds]dx \\
&\leq -EI\int_0^t g(t-s)y_{xxx}(L,s)ds\int_0^t g(t-s)(y(L,t)-y(L,s))ds \\
&+ EI\int_0^L(\int_0^t g(t-s)(y_{xx}(t)-y_{xx}(s))ds)^2dx - EI(\int_0^t g(s)ds)^2\|y_{xx}\|^2 \\
&+ EIk\int_0^L y_{xx}(t)\int_0^t g(t-s)y_{xx}(s)dsdx \\
J_3 &\leq -EI\int_0^t g(t-s)y_{xxx}(L,s)ds\int_0^t g(t-s)(y(L,t)-y(L,s))ds \\
&+ EI(1+\frac{1}{\delta_3})k\int_0^L\int_{\mathcal{A}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx \\
&+ EI(1+\delta_3)k\hat{g}(\mathcal{F})\int_0^L\int_{\mathcal{F}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx + EIk\delta_4\|y_{xx}\|^2 \\
&+ EI\frac{k^2}{4\delta_4}\int_0^t g(t-s)\|y_{xx}(s)\|^2ds
\end{aligned} \tag{2.77}
$$

$$
J_4 = -\rho\int_0^L y_t\int_0^t g'(t-s)(y(t)-y(s))dsdx,
$$

Appliying Cauchy Schwartz inequality, we get

$$
J_4 \leq \rho(\int_0^L (y_t)^2dx)^{\frac{1}{2}}\int_0^L(\int_0^t g'(t-s)(y(t)-y(s))ds)^2dx)^{\frac{1}{2}}
$$

by Poincaré's inequality, there exist $c_p > 0$ such that

$$
\begin{aligned}
J_4 &\leq \rho(\int_0^L(-y_t)^2dx)^{\frac{1}{2}}(c_p{}^2\int_0^L(\int_0^t g'(t-s)(y_{xx}(t)-y_{xx}(s))ds)^2dx)^{\frac{1}{2}} \\
&\leq \rho(\int_0^L(-y_t)^2dx)^{\frac{1}{2}}(c_p{}^2\int_0^L(\int_0^t |g'|^{\frac{1}{2}}|g'|^{\frac{1}{2}}(y_{xx}(t)-y_{xx}(s))ds)^2dx)^{\frac{1}{2}} \\
&\leq \rho(\int_0^L(-y_t)^2dx)^{\frac{1}{2}}(c_p{}^2\int_0^L\int_0^t|g'|ds\int_0^t|g'|(y_{xx}(t)-y_{xx}(s))^2dsdx)^{\frac{1}{2}} \\
&\leq \rho(\int_0^L(-y_t)^2dx)^{\frac{1}{2}}(c_p{}^2g(0)\int_0^L|g'|\Box y_{xx})^{\frac{1}{2}} \\
&\leq \rho\delta_5\|y_t\|^2 - \rho\frac{c_p{}^2}{4\delta_5}g(0)\int_0^L g'\Box y_{xx}dx.
\end{aligned} \tag{2.78}
$$

Then

$$\begin{aligned} J_5 &= -\rho \int_0^L y_t^2 \int_0^t g(t-s)dsdx \\ &= -\rho \int_0^t g(s)ds \int_0^L y_t^2 dx \\ &= -\rho \int_0^t g(s)ds \|y_t\|^2 \end{aligned} \tag{2.79}$$

Combining (2.75), (2.76), (2.77), (2.78), and (2.79), we obtain

$$\begin{aligned} {\varphi_2}'(t) &\leq \rho(-\int_0^t g(s)ds + \delta_5)\|y_t\|^2 + EI(\delta_1 + \frac{3}{2}k\hat{g}(\mathcal{F}) + k\delta_4)\|y_{xx}\|^2 + T(\int_0^t g(s)ds + \delta_2)\|y_x\|^2 \\ &+ EI(\frac{k}{4\delta_1} + (1+\frac{1}{\delta_3})k)\int_0^L \int_{\mathcal{A}_t} g(t-s)(y_{xx}(t) - y_{xx}(s))^2 dsdx \\ &+ EI(1+\delta_3)k\hat{g}(\mathcal{F})\int_0^L \int_{\mathcal{F}_t} g(t-s)(y_{xx}(t) - y_{xx}(s))^2 dsdx \\ &+ T\frac{k}{4\delta_2}\int_0^t g(t-s)\|y_x(s)\|^2 ds - \rho\frac{{c_p}^2}{4\delta_5}g(0)\int_0^L g'\Box y_{xx}dx + \frac{EI}{2}\int_{\mathcal{F}_t} g(t-s)\|y_{xx}(s)\|^2 ds \\ &+ EI\frac{k^2}{4\delta_4}\int_0^t g(t-s)\|y_{xx}(s)\|^2 ds - U(t)\int_0^t g(t-s)[y(L,t) - y(L,s)]ds \end{aligned} \tag{2.80}$$

Further, a differentiation of $\varphi_3(t)$ yields

$$\begin{aligned} {\varphi_3}'(t) &= \int_0^L G_\gamma(0)[EI|y_{xx}(s)|^2 + T|y_x(s)|^2] + \int_0^t {G_\gamma}'(t-s)[EI|y_{xx}(s)|^2 + T|y_x(s)|^2]dsdx \\ &= \int_0^L G_\gamma(0)[EI|y_{xx}(s)|^2 + T|y_x(s)|^2] - \int_0^L \int_0^t \frac{\gamma'(t-s)}{\gamma(t-s)} G_\gamma(t-s)[EI|y_{xx}(s)|^2 \\ &+ T|y_x(s)|^2]dsdx - \int_0^L \int_0^t G_\gamma(t-s)[EI|y_{xx}(s)|^2 + T|y_x(s)|^2]dsdx \\ &= G_\gamma(0)[EI\|y_{xx}\|^2 + T\|y_x\|^2] - \int_0^t \frac{\gamma'(t-s)}{\gamma(t-s)} G_\gamma(t-s)[EI\|y_{xx}(s)\|^2 + T\|y_x(s)\|^2]ds \\ &- \int_0^t g(t-s)[EI\|y_{xx}(s)\|^2 + T\|y_x(s)\|^2]ds \\ {\varphi_3}'(t) &\leq EIG_\gamma(0)\|y_{xx}\|^2 + TG_\gamma(0)\|y_x\|^2 - EI\eta(t)\int_0^t G_\gamma(t-s)\|y_{xx}(s)\|^2 ds \\ &- T\eta(t)\int_0^t G_\gamma(t-s)\|y_x(s)\|^2 ds - EI\int_0^t g(t-s)\|y_{xx}(s)\|^2 ds \\ &- T\int_0^t g(t-s)\|y_x(s)\|^2 ds \end{aligned} \tag{2.81}$$

Where we have used the fact that $\frac{\gamma'(t)}{\gamma(t)} = \eta(t)$ is a nonincreasing function, and we define $G'_\gamma(t-s) = -\frac{\gamma'(t-s)}{\gamma(t-s)}G_\gamma(t-s) - g(t-s)$

Taking into account the estimations $E'(t)$, (2.73), (2.80) and (2.81), we see that

$$\begin{aligned}
L'(t) &\leq E'(t)+\lambda_1\varphi_1'(t)+\lambda_2\varphi_2'(t)+\lambda_3\varphi_3'(t)\\
&\leq [y_t(L,t)+\lambda_1 y(L,t)-\lambda_2\int_0^t g(t-s)[y(L,t)-y(L,s)]ds]U(t)\\
&+ EI[\lambda_1[\frac{k}{2}-1]-\frac{1}{2}g(t)+\lambda_2[\delta_1+\frac{3}{2}k\hat{g}(\mathcal{F})+\delta_4 k]+\lambda_3 G_\gamma(0)]\|y_{xx}\|^2\\
&+ T[-\lambda_1+\lambda_2(k+\delta_2)+\lambda_3 G_\gamma(0)]\|y_x\|^2\\
&+ \rho[\lambda_1+\lambda_2(\delta_5-g_*)]\|y_t\|^2+EI(\frac{\lambda_1}{2}+\frac{\lambda_2 k^2}{4\delta_4}-\lambda_3)\int_0^t g(t-s)\|y_{xx}(s)\|^2ds\\
&+ T(\lambda_2\frac{k}{4\delta_2}-\lambda_3)\int_0^t g(t-s)\|y_x(s)\|^2ds-\lambda_1\frac{EI}{2}\int_0^L g\Box y_{xx}dx\\
&+ (\frac{EI}{2}-\lambda_2\rho\frac{C_p^2}{4\delta_5}g(0))\int_0^L g'\Box y_{xx}dx+\lambda_2\frac{EI}{2}\int_{\mathcal{F}_t} g(t-s)\|y_{xx}(s)\|^2ds\\
&+ \lambda_2 EI(1+\delta_3)k\hat{g}(\mathcal{F})\int_0^L\int_{\mathcal{F}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx\\
&+ \lambda_2 EI(\frac{k}{4\delta_1}+(1+\frac{1}{\delta_3})k)\int_0^L\int_{\mathcal{A}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx\\
&- \lambda_3 EI\eta(t)\int_0^t G_\gamma(t-s)\|y_{xx}(s)\|^2ds-T\lambda_3\eta(t)\int_0^t G_\gamma(t-s)\|y_x(s)\|^2ds \qquad (2.82)
\end{aligned}$$

After some simplification, we get

$$\begin{aligned}
L'(t) &\leq (\frac{EI}{2}-\lambda_2\rho\frac{C_p^2}{4\delta_5}g(0))\int_0^L g'\Box y_{xx}dx-\lambda_1\frac{EI}{2}\int_0^L g\Box y_{xx}dx\\
&+ EI[\lambda_1[\frac{k}{2}-1]+\lambda_2[\delta_1+\frac{3}{2}k\hat{g}(\mathcal{F})+\delta_4 k]+\lambda_3 G_\gamma(0)]\|y_{xx}\|^2\\
&+ T[-\lambda_1+\lambda_2(k+\delta_2)+\lambda_3 G_\gamma(0)]\|y_x\|^2+\rho[\lambda_1+\lambda_2(\delta_5-g_*)]\|y_t\|^2\\
&+ EI[\frac{\lambda_2 k^2}{4\delta_4}-\lambda_3+\frac{\lambda_1}{2}]\int_0^t g(t-s)\|y_{xx}(s)\|^2ds+EI\frac{\lambda_2}{2}\int_{\mathcal{F}_t} g(t-s)\|y_{xx}(s)\|^2ds\\
&+ T(\lambda_2\frac{k}{4\delta_2}-\lambda_3)\int_0^t g(t-s)\|y_x(s)\|^2ds\\
&+ \lambda_2 EI(1+\delta_3)k\hat{g}(\mathcal{F})\int_0^L\int_{\mathcal{F}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx\\
&+ \lambda_2 EI(\frac{k}{4\delta_1}+(1+\frac{1}{\delta_3})k)\int_0^L\int_{\mathcal{A}_t} g(t-s)(y_{xx}(t)-y_{xx}(s))^2dsdx\\
&- \lambda_3 EI\eta(t)\int_0^t G_\gamma(t-s)\|y_{xx}(s)\|^2ds-T\lambda_3\eta(t)\int_0^t G_\gamma(t-s)\|y_x(s)\|^2ds \qquad (2.83)
\end{aligned}$$

We select $\lambda_2 \leq \frac{\delta_5}{{c_p}^2 g(0)}$ so that

$$\tfrac{EI}{2}-\tfrac{{c_p}^2}{4\delta_5}g(0)\rho\lambda_2 \geq \tfrac{-\rho+2EI}{4}$$

and introduce the sets

$$\mathcal{A}_n=\{s\in\mathbb{R}^+ : ng'(s)+g(s)\leq 0\}, n\in\mathbb{N}.$$

Observe that

$$\bigcup_n \mathcal{A}_n=\mathbb{R}^+\backslash\{\mathcal{F}_g\bigcup\mathcal{N}_g\},$$

Where $\mathcal{N}_g$ is the nullset where g' is not defined and $\mathcal{F}_g$ is as in. Furthermore, if we denote $\mathcal{F}_n := \mathbb{R}^+ n\mathcal{A}_n$, then $\lim_{n\to\infty}\hat{g}(\mathcal{F}_n) = \hat{g}(\mathcal{F}_g)$ because $\mathcal{F}_{n+1} \subset \mathcal{F}_n$ for all n and $\bigcap_n = \mathcal{F}_g \bigcup \mathcal{N}_g$.
In (2.53), we take $\mathcal{A} := \mathcal{A}_n$ and $\mathcal{F} := \mathcal{F}_n$ and $\lambda_1 = (g_* - \epsilon)\lambda_2$ for some small $\epsilon > 0$. It follows that

$$\begin{aligned}
L'(t) &\leq \rho(\delta_5 - \epsilon)\lambda_2\|y_t\|^2 \\
&+ EI[\lambda_2[\delta_1 + \frac{3}{2}k\hat{g}(\mathcal{F}_{nt}) + \delta_4 k] + \lambda_3 G_\gamma(0) + \lambda_2(g_* - \epsilon)[\frac{k}{2} - 1]]\|y_{xx}\|^2 \\
&+ T[-(g_* - \epsilon)\lambda_2 + \lambda_2(k + \delta_2) + \lambda_3 G_\gamma(0)]\|y_x\|^2 \\
&+ EI[\frac{\lambda_2 k^2}{4\delta_4} - \lambda_3 + \lambda_2\frac{(g_* - \epsilon)}{2}]\int_0^t g(t-s)\|y_{xx}(s)\|^2 ds + EI\frac{\lambda_2}{2}\int_{\mathcal{F}_{nt}} g(t-s)\|y_{xx}(s)\|^2 ds \\
&- EI\lambda_2\frac{(g_* - \epsilon)}{2}\int_0^L g\Box y_{xx} dx + T(\lambda_2\frac{k}{4\delta_2} - \lambda_3)\int_0^t g(t-s)\|y_x(s)\|^2 ds \\
&+ [\lambda_2 k EI(\frac{1}{4\delta_1} + 1 + \frac{1}{\delta_3}) - \frac{1}{4n}]\int_0^L\int_{\mathcal{A}_{nt}} g(t-s)(y_{xx}(t) - y_{xx}(s))^2 ds dx \\
&+ \lambda_2 EI(1 + \delta_3)k\hat{g}(\mathcal{F}_n)\int_0^L\int_{\mathcal{F}_{nt}} g(t-s)(y_{xx}(t) - y_{xx}(s))^2 ds dx \\
&- \lambda_3 EI\eta(t)\int_0^t G_\gamma(t-s)\|y_{xx}(s)\|^2 ds - T\lambda_3\eta(t)\int_0^t G_\gamma(t-s)\|y_x(s)\|^2 ds \qquad (2.84)
\end{aligned}$$

We have

$$(*)\begin{cases}
\delta_5 - \epsilon < 0 & (2.85)\\
\frac{3}{2}k\hat{g}(\mathcal{F}_n) + \delta(g_* - \epsilon)(-1 + \frac{k}{2}) < 0 & (2.86)\\
\lambda_2 k EI(\frac{1}{4\delta_1} + 1 + \frac{1}{\delta_3}) - \frac{1}{4n} < 0 & (2.87)\\
\lambda_3 G_\gamma(0) - \lambda_2[-\delta_1 - k\delta_4 - (1-\delta)(g_* - \epsilon)(\frac{k}{2} - 1)] < 0 & (2.88)\\
\lambda_2(\frac{1}{2} + \frac{k^2}{4\delta_4} + \frac{(g_* - \epsilon)}{2}) - \lambda_3 < 0 & (2.89)\\
-(g_* - \epsilon)\lambda_2 + \lambda_2(k + \delta_2) + \lambda_3 G_\gamma(0) < 0 & (2.90)\\
T(\lambda_2\frac{k}{4\delta_2} - \lambda_3) < 0 & (2.91)\\
\lambda_2 EI[(1 + \delta_3)k\hat{g}(\mathcal{F}_n) - \frac{(g_* - \epsilon)}{2}] < 0 & (2.92)
\end{cases}$$

from the relatin (2.85) we deduce

$$\delta_5 - \epsilon < 0 \Longrightarrow \delta_5 < \epsilon$$

and from (2.86) for a small ϵ and large values of n and t, we see that if $\hat{g}(\mathcal{F}_n) < \frac{1}{4}$, we have

$$\frac{3}{2}k\hat{g}(\mathcal{F}_n) < \delta(g_* - \epsilon)(1 - \frac{k}{2})$$

with

$$\delta = \frac{3}{4}\frac{k}{g_*(2-k)}.$$

Note that $\delta < 1$. For the remaining $1 - \delta$ we require that λ_2 and λ_3 satisfy

$$\lambda_1 k EI(\frac{1}{4\delta_1} + 1 + \frac{1}{\delta_3}) < \frac{1}{4n}$$

and from (2.88) and (2.90)

$$\lambda_2(\frac{k^2}{4\delta_5} + \frac{(g_* - \epsilon)}{2} + \frac{1}{2}) < \lambda_3 < \lambda_2 \frac{(1-\delta)(g_* - \epsilon)(1 - \frac{k}{2})}{G_\gamma(0)}$$

$$(1 - \delta) = 1 - \frac{3}{4}\frac{k}{g_*(2-k)}$$

$$= \frac{4g_*(2-k) - 3k}{4g_*(2-k)}$$

so, we get

$$\frac{\lambda_2}{2} < \lambda_2(\frac{k^2}{4\delta_5} + \frac{1}{2} + \frac{g_* - \epsilon}{2}) < \lambda_3 < \frac{4g_*(2-k) - 3k}{8G_\gamma(0)}\lambda_2$$

This is possible if

$$G_\gamma(0) < \frac{[4g_*(2-k) - 3k]}{4}$$

and $g_* > 3k/4(2-k)$. These choices together with (2.83) lead to

$$L'(t) \leq -C_1 E(t) - \lambda_3 \eta(t)\phi_3(t), t \geq t_*$$

for some positive constant C_1. As $\eta(t)$ is decreasing, we have $\eta(t) \leq C_1$ for all $\hat{t} \geq t_*$. The right hand side inequality in Lemma 4 implies that

$$L'(t) \leq -C_2 \eta(t) L(t), t \geq \hat{t} \tag{2.93}$$

for some positive constant C_2. An integration of (2.93) yields

$$\int_{\hat{t}}^{t} \frac{L'(s)}{L(s)} ds \leq \int_{\hat{t}}^{t} -C_2 \eta(s) ds$$

$$lnL(t) - lnL(\hat{t}) \leq \int_{\hat{t}}^{t} -C_2 \eta(s) ds$$

$$L(t) \leq e^{-C_2 \int_{\hat{t}}^{t} \eta(s) ds} L(\hat{t}), t \geq \hat{t}.$$

Then using the left hand side inequality in (1.7), we get

$$\rho_1[E(t) + \phi_3(t)] \leq e^{-C_2 \int_{\hat{t}}^{t} \eta(s) ds} L(\hat{t}), t \geq \hat{t}.$$

By virtue of the continuity and boundedness of $E(t)$ in the interval $[0, \hat{t}]$, we conclude that

$$E(t) \leq C\gamma(t)^{-\nu}(t), t \geq 0$$

for some positive constants C and ν .

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