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# Master memory

Presented by

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**Specialty : Mathematics**

**Option : Dynamic Systems**

# Title

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## Orthogonality of bounded linear operators

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*Defended on 28 / 06 /2022 before the jury composed of*

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# Contents

# Introduction

The main topic treated in this memory is the study of some orthogonality types, with emphasis mainly on the elements and linear operators in normed space. The theory of Birkhoff-James orthogonality [6, 11] of elements in a normed linear space was introduced by Birkhoff in [6], in order to generalize the concept of orthogonality in inner product spaces. Over the years, Birkhoff-James orthogonality has been undoubtedly established as an important concept in the study of geometry of normed linear spaces by virtue of its rich connection with several geometric properties of the space, like strict convexity, smoothness etc. [12, 11, 8]. Recently, a renewed interest has been generated towards studying the Birkhoff-James orthogonality of elements in the space of bounded linear operators between normed linear spaces [5, 20, 25]. While complete characterization of Birkhoff-James orthogonality of bounded linear operators defined on a Hilbert space [5, 18], or a finite dimensional real Banach space [24] has been obtained, the problem of characterizing Birkhoff-James orthogonality of bounded linear operators on infinite dimensional normed linear spaces remains unsolved.

This memory is organized as follows. In the preliminaries (Chapter 1) we establish the swimming of the memory. We introduce basic notions in normed spaces, Hilbert spaces, Banach spaces theory and we recall that main definition and properties of linear operator that we will use later. In the second chapter of this memory we give some orthogonality types in normed spaces, Birkhoff-James orthogonality Strongly Birkhoff-James orthogonality and at last the isosceles orthogonality type. and we establish the basics of the theory of this types of orthogonality. In the last chapter we called the Birkhoff-James orthogonality of bounded linear operators, we mention the definition and properties of Birkhoff-James orthogonality of bounded linear operator, and we give the important theorems with proof and different examples. At the end of the chapter, the relation between the different types of orthogonality has been proven and revealed.

# Chapter 1

# Preliminaries

## 1.1 Normed Spaces

**Definition 1.1.1.**

*Let $X$ be a linear space over the field $\mathbb{K}\,(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$, Then a norm on $X$ is a map $\| \,.\, \|: X \longrightarrow \mathbb{R}^+$ such that for all $x, y \in X, \lambda \in \mathbb{K}$, the following properties are satisfied:*

*i)* $\|x\| \geq 0$, *for all $x \in X$ (Nonnegative),*

*ii)* $\|\lambda x\| = \mid \lambda \mid \|x\|$, *for all $x \in X$ and $\lambda \in \mathbb{R}$ ( or $\mathbb{C}$) (Homogeneous),*

*iii)* $\|x + y\| \leq \|x\| + \|y\|$, *for all $x, y \in X$ (Triangle inequality).*

*A norm on $X$ is a semi-norm which also satisfies:*

*iv)* $\|x\| = 0$ *implies that $x = 0$ (strictly positive).*

**Definition 1.1.2.**

*Let $\|\cdot\|$ be a norm on $X$. Then the distance from $x$ to $y$ in $X$ is $d(x, y) = \|x - y\|$.*

**Example 1.1.1.**

*i) Let $X = \mathbb{R}$, and $\|x\| = |x|$, the absolute value of $x$.*

*ii) Let $X = \mathbb{C}$, and $\|x\| = |z|$, the modulus of $z$.*

*iii) Let $X = \mathbb{R}^n$ (or $X = \mathbb{C}^n$). There are three standard norm.*

*For every $(x_1, \cdots, x_n) \in X$, we have the 1-norm $\|x\|_1$, defined such that,*

$$\|x\|_1 = |x_1| + \cdots + |x_n|,$$

*the Euclidean norm $\|x\|_2$, defined such that*

$$\|x\|_2 = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{\frac{1}{2}},$$

*and the sup-norm $\|x\|_\infty$, defined such that*

$$\|x\|_\infty = \max\{|x_i| \quad 1 \leq i \leq n\}.$$

*More general, we define the $l_p$-norm ( for $p \geq 1$) by has a norm given by:*

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

**Definition 1.1.3.**

*The pair $(X, \|\cdot\|)$ is called normed linear space where $X$ is a linear space and $\|\cdot\|$ is a norm on $X$.*

**Proposition 1.1.1.**

*Suppose that $X$ is a normed linear space with respect to a norm $\|\cdot\|_1$ and $\|\cdot\|_2$. Then we say that these norms are equivalent if there exist constants $C_1, C_2 > 0$, such that:*

$$\forall x \in X; \quad C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1. \tag{1.1}$$

*Observe that equation (1.1) can be rearranged to read:*

$$\forall x \in X; \quad \frac{1}{C_2} \|x\|_2 \leq \|x\|_1 \leq \frac{1}{C_1} \|x\|_2.$$

**Definition 1.1.4.**

*Let $X$ be a normed linear space.*

*i) The unit ball $B_X$ is defined by*

$$B_X = \{x \in X : \|x\| \leq 1\}.$$

*ii) The unit sphere is defined by*

$$S_X = \{x \in X : \|x\| = 1\}.$$

**Example 1.1.2.**

*Let $X = \mathbb{R}^n$ with the Euclidean metric. Then*

$$B_\epsilon(0) = \left\{x \in X : \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} < \epsilon\right\},$$

*simply a ball of radius $\epsilon$ about the origin.*

Sequences provide a convenient tool for studying many properties of subsets of a normed linear space $X$, including open and closed sets.

**Definition 1.1.5.**

*Let $X$ be a normed linear space.*

*i) A sequence $\{x_n\}$ in $X$ converges to $x \in X$ if $\|x_n - x\| \to 0$ as $n \to \infty$.*

*ii) A sequence $\{x_n\}$ in $X$ is Cauchy if $\|x_n - x_m\| \to 0$ as $n, m \to \infty$.*

## 1.2 Banach spaces

**Definition 1.2.1.**

*A normed linear space $X$ is complete if every Cauchy sequence in $X$ converges to an element of $X$. A complete normed linear space is called a Banach space.*

**Remark 1.2.1.**

*Every finite-dimensional normed linear space is complete and hence a Banach space.*

**Example 1.2.1.**

*i) $X = \mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$ is a Banach space.*

*ii) $X = C([0,1])$ is complete with the sup-norm $\|\cdot\|_\infty$, but it is not complete with the p-norm for $1 \leq p < \infty$, then is a Banach space with the sup-norm $\|\cdot\|_\infty$ .*

*ii) $\ell_p$ is a Banach space.*

## 1.3 Hilbert spaces

Let $\mathcal{H}$ be a real vector space.

**Definition 1.3.1.**

*An inner product $\langle \cdot, \cdot \rangle$ in $\mathcal{H}$ is a mapping $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$*

*with the following properties:*

**i)** $\langle x, x \rangle \geq 0$ *for all* $x \in \mathcal{H}$ *and* $\langle x, x \rangle = 0$ *iff* $x = 0$,

**ii)** $\langle x, y\rangle = \langle y, x\rangle$ *for all* $x, y \in \mathcal{H}$,

**iii)** $\langle x + \lambda y, z\rangle = \langle x, z\rangle + |\lambda| \langle y, z\rangle$ *for all* $x, y, z \in \mathcal{H}$ *and* $\lambda \in \mathbb{R}$.

*A real pre-Hilbert space is a pair* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$.

**Example 1.3.1.**

**i)** *The Euclidean space is in particular real pre-Hilbertian space.*

**i)** *The real pre-Hilbert space is in particular complex pre-Hilbert space.*

The following inequality is fundamental.

**Proposition 1.3.1.**
*(Cauchy-Schwarz) Let* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *be a pre–Hilbert space. Then*

$$|\langle x, y\rangle| \leq \|x\| \|y\| \qquad \forall x, y \in \mathcal{H}. \tag{1.2}$$

*Moreover, equality holds iff* $x$ *and* $y$ *are linearly dependent.*

*Proof.* The conclusion is trivial if $y = 0$. So, we will suppose $y \neq 0$. In fact, to begin with, let $\|y\| = 1$. Then,

$$0 \leq \|x - \langle x, y\rangle y\|^2 = \|x\|^2 - \langle x, y\rangle^2 \tag{1.3}$$

whence the conclusion follows. In the general case, it suffices to apply the above inequality to $\frac{y}{\|y\|}$.

If $x$ and $y$ are linearly dependent, then it is clear that $|\langle x, y\rangle| = \|x\| \|y\|$.

Conversely, if $\langle x, y\rangle = \pm \|x\| \|y\|$ and $y \neq 0$, then (1.3) implies that $x$ and $y$ are linear dependent. □

**Corollary 1.3.1.**
*Let* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *be a pre-Hilbert space. Then the function* $\|\cdot\|$ *defined by* $|\langle x, x\rangle| = \|x\|^2$. *has the following properties:*

**1-** $\|x\| \geq 0$ *for all* $x \in \mathcal{H}$ *and* $\|x\| = 0$ *iff* $x = 0$,

**2-** $\|\lambda x\| = |\lambda| \|x\|$ *for any* $x \in \mathcal{H}$ *and* $\lambda \in \mathbb{R}$,

**3-** *Triangle Inequality:* $\|x+y\| \leq \|x\| + \|y\|$ *for all* $x, y \in \mathcal{H}$.

*If* $\langle ., . \rangle$ *is an inner product on* $\mathcal{H}$ *then* $\|.\|$ *is a norm on* $\mathcal{H}$, *which means that in addition to* (1) *and* (3) *above, we also have:*

**4-** $\|x\| = 0 \Rightarrow x = 0$.

*Proof.* Let $x \in \mathcal{H}$ then $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$ what's more

$$\begin{aligned} \|\lambda x\| &= \langle \lambda x, \lambda x \rangle^{\frac{1}{2}} = \left(\lambda\overline{\lambda} \langle x, x \rangle\right)^{\frac{1}{2}} = \left(|\lambda|^2 \langle x, x \rangle\right)^{\frac{1}{2}} \\ &= |\lambda| \left(\langle x, x \rangle\right)^{\frac{1}{2}} = |\lambda| \parallel x \parallel . \end{aligned}$$

Finally

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle x, y \rangle \\ &= \|x\|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|^2 \\ &= \|x\|^2 + 2\mathbf{Re} \langle x, y \rangle + \|y\|^2 \\ &= \|x\|^2 + 2 \left|\langle x, y \rangle\right| + \|y\|^2 \\ &\leq \|x\|^2 + 2 \parallel x \parallel\parallel y \parallel + \|y\|^2 \\ &= \left(\parallel x \parallel + \parallel y \parallel\right)^2 . \end{aligned}$$

Moreover

$$\|x+y\| \leq \parallel x \parallel + \parallel y \parallel .$$

The norm $\|\cdot\|$ thus defined is called the norm induced by inner product. □

**Lemma 1.3.1.**

*If* $\mathcal{H}$ *is a pre-Hilbert space, its norm satisfies the equality of the parallelogram:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \ \ \forall x, y \in \mathcal{H}.$$

*It's a simple calculation:*

$$\|x+y\|^2 + \|x-y\|^2 = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle + \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle.$$

**Remark 1.3.1.**

*It is easy to see that, in a pre-Hilbert space* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, *the function*

$$d(x, y) = \|x-y\| \qquad \forall x, y \in \mathcal{H} \tag{1.4}$$

*is a metric.*

**Definition 1.3.2.**

*A pre-Hilbert space* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ *is called an Hilbert space if it is complete with respect to the metric defined in* (1.4).

**Example 1.3.2.**

1) $\mathbb{R}$ *is a Hilbert,* $\langle x, y \rangle = xy$ *and* $\|x\| = \mid x \mid$ .

2) $\mathbb{C}^n$ *equipped with the inner product:* $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$ *is a Hilbert space.*

3) $\mathbb{R}^N$ *is a Hilbert,* ( $\langle x, y \rangle = \sum_{i=1}^{N} x_i y_i$ *and* $\|x\| = (\sum_{i=1}^{n} \mid x_i \mid^2)^{\frac{1}{2}}$).

4) $\ell_2 = \{(x_n)_{n\in\mathbb{N}}, \sum_0^\infty \mid x_n \mid^2 < \infty\}$ *is a Hilbert.*

   *We have*
$$\langle x, y \rangle = \sum_0^\infty x_i \overline{y_i} \text{ and } \|x\| = (\sum_0^\infty \mid x_n \mid^2)^{\frac{1}{2}}.$$

5) $L^2(I)$ *with* $I$ *open is a Hilbert and*
$$\langle f, g \rangle = \int_I f(t) \overline{g}(t) dt.$$

6) *The space* $C([0, 1])$ *with one of the norms:*
$$\|f\|_1 = \int_0^1 \mid f(t) \mid dt, \ \|f\|_\infty = \sup_{t\in[0,1]} \mid f(t) \mid,$$
   *is not a Hilbert space because the equality of the parallelogram is not true for these norms.*

## 1.4 Bounded linear operators

**Definition 1.4.1.**

*Let* $X, Y$ *be a linear spaces. Let* $T : X \to Y$ *be a mapping between* $X$ *into* $Y$ . *Then* $T$ *is linear if for every* $x, y \in X$ *and* $\lambda \in \mathbb{K}$.
$$\begin{cases} 1- & T(x + y) = Tx + Ty, \\ 2- & T(\lambda x) = \lambda Tx. \end{cases}$$
*The set of all linear maps* $T : X \longrightarrow Y$ *is denoted by* $\mathcal{L}(X, Y)$. *When the domain and range spaces are the same, we write* $\mathcal{L}(X, X) = \mathcal{L}(X)$.

If $X.Y$ are normed spaces, then we can define the notion of a bounded linear map. As we will see, the boundedness of a linear map is equivalent to its continuity.

**Definition 1.4.2.**

*Let $X$ and $Y$ be two normed linear spaces. A linear map $T : X \longrightarrow Y$ is bounded if there is a constant $C \geq 0$ such that*

$$\| Tx \| \leq C. \| x \| \text{ for all } x \in X. \tag{1.5}$$

*If no such constant exists, then we say that $T$ is unbounded. We denote the set of all bounded linear mapping between $X$ into $Y$ by $B(X, Y)$.*

- *If $X = Y$ then we write $B(X) = B(X, X)$.*
- *If $Y = \mathbb{K}$ then we say that $T$ is a functional. The set of all bounded linear functionals on $X$ is the dual space of $X$, and is denoted*

$$X^* = B(X, \mathbb{K}) = \{T : X \to \mathbb{K} : T \text{ is bounded and linear}\}.$$

**Definition 1.4.3.**

*If $T : X \longrightarrow Y$ is a bounded linear map, then we define the operator norm or uniform norm $\| T \|$ of $T$ by*

$$\| T \| = \inf\{C \text{ such that } \| Tx \| \leq C. \| x \|\} \quad \text{for all } X;\ x \in C, \tag{1.6}$$

*Note that $\|Tx\|$ is the norm of $Tx$ in $Y$ , while $\|x\|$ is the norm of $x$ in $X$.*
*The operator norm of $T$ is $\| T \| = \sup_{\|x\|=1} \| Tx \|$ is equivalent expressions for $\| T \|$ are:*

$$\| T \| = \sup_{\|x\| \neq 0} \frac{\| Tx \|}{\|x\|};\ \| T \| = \sup_{\|x\| \leq 1} \| Tx \|. \tag{1.7}$$

*Note that since the norm on $\mathbb{K}$ is just absolute value, the operator norm of a linear functional $T \in X^* = B(X, \mathbb{K})$ is:*

$$\|T\| = \sup_{\|x\|=1} |Tx|.$$

The next theorem shows that $B(X, Y)$ is a Banach space whenever $Y$ is a Banach space.

**Theorem 1.4.1.**

*If $X$ is a normed space and $Y$ is a Banach space, then $B(X, Y)$ is a Banach space.*

**Definition 1.4.4.**

*Let $X, Y$ be normed linear spaces, and let $T : X \to Y$ be a linear operator.*

*i) $T$ is continuous at a point $x \in X$ if $x_n \to x$ in $X$ implies $Tx_n \to Tx$ in $Y$.*

*ii) $T$ is continuous if it is continuous at every point, i.e., if $x_n \to x$ in $X$ implies $Tx_n \to Tx$ in $Y$ for every $x \in X$.*

For linear maps, boundedness is equivalent to continuity.

**Theorem 1.4.2.**

*Let $X, Y$ be normed linear spaces, and $T : X \to Y$ be a linear mapping. Then the following statements are equivalent:*

*i) $T$ is continuous at some $x \in X$.*

*ii) $T$ is continuous at $x = 0$.*

*iii) $T$ is continuous.*

*iv) $T$ is bounded.*

*Proof.* $(iii) \Rightarrow (iv)$. Suppose that $T$ is continuous but unbounded. Then $\|T\| = 1$, so there must exist $x_n \in X$ with $\|x_n\| = 1$ such that $\|Tx_n\| \geq n$. Set y$_n = x_n/n$ . Then $\|y_n - 0\| = \|y_n\| = \|x_n\| /n \to 0$, so $y_n \to 0$. Since $T$ is continuous and linear, this implies $Ty_n \to T0 = 0$, and therefore $\|Ty_n\| \to \|T0\| = 0$.
But

$$\|Ty_n\| = \frac{1}{n} \|Tx_n\| \geq \frac{1}{n}.n = 1,$$

for all $n$, which is a contradiction. Hence $T$ must be bounded.
$(iv) \Rightarrow (iii)$. Suppose that $T$ is bounded, so $\|T\| < 1$. Let $x \in X$ and $x_n \to x$. Then $\|Tx_n - Tx\| = \|T(x_n - x)\| \leq \|T\| \|x_n - x\| \to 0$,i.e., $Tx_n - Tx$. Thus $T$ is continuous. □

**Definition 1.4.5.**

*Let $T : X \to Y$ a linear operator*

*i) The kernel of $T$ denoted by $\mathcal{N}(T)$,is the set defined by*

$$\mathcal{N}(T) = \{x \in X : Tx = 0\}.$$

*ii) The range (image) of $T$ denoted by $\mathcal{R}(T)$,is the set defined by*

$$\mathcal{R}(T) = \{Tx : x \in X\}.$$

Now, we recall some definitions and propositions of matrix operators. Consider the linear space $\mathcal{M}_n(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of square $n \times n$ matrices

**Definition 1.4.6.** *A matrix norm $\|\cdot\|$ on the space of square $n\times n$ matrices in $\mathcal{M}_n(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is a norm on the linear space $\mathcal{M}_n(\mathbb{K})$ with the additional property that*

$$\|AB\| \leq \|A\| \|B\|, \quad \textit{for all } A, B \in \mathcal{M}_n(\mathbb{K}).$$

*Since $I^2 = I$, from $\|I\| = \|I^2\| \leq \|I\|^2$, we get $\|I\| \geq 1$.*

**Proposition 1.4.1.** *Let a matrix $A \in \mathcal{M}_n(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), then for every $x \in \mathbb{K}^n$, there is a real constant $C_A$, such that*

$$\|Ax\| \leq C_A \|x\|,$$

**Definition 1.4.7.** *If $\|\cdot\|$ is any norm on $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), we define the function $\|\cdot\|$ on $\mathbb{K}$ by*

$$\|A\| = \sup_{\substack{x\in\mathbb{K}^n \\ \|x\|\neq 0}} \frac{\|Ax\|}{\|x\|} = \sup_{\substack{x\in\mathbb{K}^n \\ \|x\|=1}} \|Ax\|.$$

**Proposition 1.4.2.** *For every square matrix $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{K})$, we have*

$$\|A\|_1 = \sup_{\substack{x\in\mathbb{K}^n \\ \|x\|_1=1}} \|Ax\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|,$$
$$\|A\|_\infty = \sup_{\substack{x\in\mathbb{K}^n \\ \|x\|_\infty=1}} \|Ax\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|.$$

New, we give the definition of linear compact operator and the attain its norm.

**Definition 1.4.8.**

*A linear operator* $u : X \longrightarrow Y$ *is said to be compact if* $u(B)$ *is a precompact subset of* $Y$ *for every bounded subset* $B$ *of* $X$*. An equivalent formulation is that* $u$ *is compact if and only if every bounded sequence* $(x_i)_{i=1}^{\infty}$ *in* $X$ *has a subsequence* $(x_{i_k})_{k=1}^{\infty}$ *such that* $(u(x_{i_k}))_{k=1}^{\infty}$ *converges in* $Y$*.*

*We denote by* $\mathcal{K}(X,Y)$ *the linear space of all compact linear operators from* $X$ *into* $Y$*.*

**Proposition 1.4.3.** *[22]*

*The classes* $\mathcal{K}$ *is closed Banach space.*

**Definition 1.4.9.** *[26]*

*Let* $X$ *be a normed linear space and let* $T \in B(X,Y)$*,* $T$ *is said to attain its norm at* $x_0 \in S_X$ *if* $\|Tx_0\| = \|T\|$*.*

*We let* $M_T$ *denote the set of all unit vectors in* $S_X$ *at which* $T$ *attains norm, i.e.,*

$$M_T = \{x \in S_X : \|Tx\| = \|T\|\}.$$

# Chapter 2

# Some orthogonality types

## 2.1 Orthogonality in Hilbert spaces

**Definition 2.1.1.**

*Let $\mathcal{H}$ be a Hilbert space and let $x, y \in \mathcal{H}$. We say that $x$ is orthogonal to $y$, written as $x \perp y$, if $\langle x, y \rangle = 0$.*

**Remark 2.1.1.**

*The orthogonality relation has the following properties:*

**i)** $0_{\mathcal{H}} \perp x$ *for any* $x \in \mathcal{H}$.

**ii)** $x \perp y \Rightarrow y \perp x$.

**iii)** $x \perp x \Rightarrow x = 0_{\mathcal{H}}$.

**iv)** $x \perp x_n, n \in \mathbb{N}$, *and* $\lim\limits_{n \to \infty} \|x_n - x\| = 0 \Rightarrow x_n \perp x$.

**Definition 2.1.2.**

*Consider non-empty sets $A$ and $B$ of Hilbert space $\mathcal{H}$. We say $A$ is orthogonal to $B$, denoted $A \perp B$, if for all $x \in A$ and for all $y \in B$ on $x \perp y$.*

**Theorem 2.1.1.**

*If $A$ is non-empty sets of a Hilbert space $\mathcal{H}$, then*

$$A^{\perp} = \{x, x \in \mathcal{H}, \text{ and } x \perp A\}$$

*is a closed linear subspaces of $\mathcal{H}$.*

*$A^{\perp}$ is called the orthogonal complement of $A$.*

*Proof.* First observe that $A^\perp \neq \emptyset$, because $0_\mathcal{H} \in A^\perp$. Let $x_1, x_2 \in A^\perp, \lambda_1, \lambda_2 \in \mathbb{K}$ and $y \in A$, then

$$\langle \lambda_1 x_1 + \lambda_2 x_2, y \rangle = \lambda_1 \langle x_1, y \rangle + \lambda_2 \langle x_2, y \rangle = 0,$$

so $\lambda_1 x_1 + \lambda_2 x_2 \in A^\perp$ and $A^\perp$ is a linear subspace of $\mathcal{H}$.

Let $x_0 \in \overline{A^\perp}$ then there is a sequence $\{x_n; n \in \mathbb{N}\} \subset A^\perp$ such as $\lim\limits_{n\to\infty} \|x_n - x_0\| = 0$.

We have for everything

$$y \in A : \langle x_0, y \rangle = \lim_{n\to\infty} \langle x_n, y \rangle = 0.$$

So $x_0 \in A^\perp$ and $A^\perp$ is a closed linear subspace. □

**Definition 2.1.3.**

*A non-empty set $A$ of a Hilbert space $\mathcal{H}$ is a orthogonal system if : for every $x, y \in A, x \neq y$, we have $x \perp y$.*

**Definition 2.1.4.**

*A set $E = \{e_i; i \in I\}$ of a Hilbert space $\mathcal{H}$ is a orthonormal system if:*

$$\langle e_i, e_j \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

**Example 2.1.1.**

$\mathbb{C}_n = \{(z_1, z_2, \cdots, z_n) : z_j \in \mathbb{C}, j = 1, 2, \cdots, n\}$

$$\langle (z_1, z_2, \cdots, z_n), (w_1, w_2, \cdots, z_n) \rangle = \sum_{k=1}^{n} z_k \overline{w_k},$$

is a inner product

$$e_i = (0, \cdots, 0, 1, 0, \cdots, 0), \quad e_1 = (1, \cdots, 0, 0, \cdots, 0), \quad e_n = (0, \cdots, 0, 0, \cdots, 1),$$

$\{e_i\}_i$ is an orthogonal system, effect

$$\begin{aligned} \langle e_i, e_j \rangle &= \langle (0, \cdots, 0, 1, 0, \cdots, 0), (0, \cdots, 0, 1, 0, \cdots, 0) \rangle \\ &= \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \end{aligned}$$

## 2.2 Birkhoff -Jamse orthogonality

The first orthogonality type that was defined for general normed linear spaces is probably the one called Roberts orthogonality introduced by Roberts in 1934 *see*[23].

**Definition 2.2.1.** *(Roberts)*
*Let $X$ be a normed linear space, we say that a vector $x$ is said to be Roberts orthogonal to a vector $y$ ($x \perp_R y$) if the equality $\| x + \alpha y \| = \| x - \alpha y \|$ holds for any scalar $\alpha$.*

Later, in 1935, Birkhoff introduced Birkhoff orthogonality [6], which was revealed to be the most important orthogonality defined for normed linear spaces.
In [10],[6],[12] James studied many important properties related to the notion of orthogonality. The notion of orthogonality has been studied by many mathematicians over the time, a few of them are Alonso and Soriano [1], Benítez et. al. [4], Kapoor and Prasad [15] and Partington [17].

**Definition 2.2.2.** *(Birkhoff)*
*Let $X$ be a normed linear space, we say that a vector $x$ is said to be Birkhoff-James orthogonal to a vector $y$ ($x \perp_B y$) if the inequality $\| x + \alpha y \| \geq \| x \|$ holds for any scalar $\alpha$.*

**Example 2.2.1.**
*Concider in $\ell_2(\mathbb{R}^2)$ the vectors $(1,0)$,$(0,1)$ and $(1,1)$ then $(1,0)\perp_B(0,1)$ but $(1,1) \not\perp_B (0,1)$, whereas in $\ell_\infty(\mathbb{R}^2)$ the vectors $(1,1),(1,0)$ and $(0,1)$ then $(1,1)\perp_B(1,0)$ and $(1,1)\perp_B(0,1)$.*

**Proposition 2.2.1.**
*Let $X$ be a normed linear space, $x, y$ be a vectors in $X$ and $\lambda, \mu$ be a scalars, then the following holds*

- *Non-degeneracy: $\lambda x \perp_B \mu x$ if and only if $\| \lambda\mu x \| = 0$.*
- *Homogeneity: If $\lambda x \perp_B y$, then $\lambda x \perp_B \mu y$ for any scalar $\lambda$.*
- *Continuity: Let $\{x_i\}_{i=1}^\infty$ , $\{y_i\}_{i=1}^\infty$ be two sequences such that $x = \lim\limits_{i\longrightarrow\infty} x_i$ and $y = \lim\limits_{i\longrightarrow\infty} y_i$. If $x_i \perp_B y_i$, for each $i \in \mathbb{N}$ then $x \perp_B y$.*

**Definition 2.2.3.** *[28]*

*For any two elements $x, y$ in a real normed linear space $X$, let us say that $y \in x^+$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \geq 0$. Accordingly, we say that $y \in x^-$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \leq 0$.*

**Proposition 2.2.2.** *[24]*

*Let $X$ be a real normed linear space and $x, y \in X$. Then the following are true:*

*1- Either $y \in x^+$ or $y \in x^-$.*

*2- $x \perp_B y$ if and only if $y \in x^+$ and $y \in x^-$.*

**Characterizations.** We start this section with the following important characterization of Birkhoff orthogonality.

**Theorem 2.2.1.** *[9, Corollary 2.2]*

*Let $x$ and $y$ two elements of a normed linear space $X$, then $x \perp_B (\alpha x + y)$ if and only if, there exists $f \in X^*$ satisfying $\mid f(x) \mid = \| f \| \| x \|$ such that $\alpha = -\frac{f(y)}{f(x)}$.*

The above theorem immediately yields the following corollary.

**Corollary 2.2.1.**

*Let $x$ and $y$ two elements of a normed linear space $X$, then $x \perp_B (\alpha x + y)$ if and only if there existe a non-zero $f \in X^*$ such that $\mid f(x) \mid = \| f \| \| x \|$ and $f(y) = 0$.*

**Homogeneity**. The homogeneity of Birkhoff orthogonality follows from the absolute homogeneity of the norm

**Theorem 2.2.2.**

*Birkhoff orthogonality is homogeneous in any normed linear space.*

**Existence.** Concerning existence properties of Birkhoff orthogonality we have the following results.

**Theorem 2.2.3.** *(Right existence)[9]*

*Let $X$ be a normed linear space. For any $x, y \in X$, there exists a real number $\alpha$ such that $x \perp_B (\alpha x + y)$.*
*Moreover, for all number $\alpha$ satisfies $\alpha \leq \frac{\|y\|}{\|x\|}$. If $x \perp_B (\alpha x + y)$ and $x \perp_B (\beta x + y)$ we have $x \perp_B (\gamma x + y)$ for all number $\gamma$ lying between $\alpha$ and $\beta$.*

A result similar to the one in Theorem 2.2.3 holds for the left existence.

**Theorem 2.2.4.** *(Left existence)*
*Let $X$ be a normed linear space and $x, y \in X$. Then there exists a real number $\alpha$ such that $(\alpha x + y) \perp_B x$. Moreover*

$$\| \alpha x + y \| = \inf\{\| \beta x + y \| : \beta \in \mathbb{R}\}.$$

*If $(\alpha x + y) \perp_B x$ and $(\beta x + y) \perp_B x$ then $(\gamma x + y) \perp_B x$ for any real number $\gamma$ lying between $\alpha$ and $\beta$*

**Additivity.** Now, we give the definition of the right additivity and the left additivity of Birkhoff-James orthogonality.

**Definition 2.2.4.** *(Right additivity)*
*Let $X$ be a normed linear space. We say that Birkhoff-James orthogonality is right additive in $X$ if for any $x, y, z \in X$, $x \perp_B y, x \perp_B z$ implies that $x \perp_B (y + z)$.*

**Definition 2.2.5.** *(Left additivity)*
*Let $X$ be a normed linear space. We say that Birkhoff-James orthogonality is left additive in $X$ if for any $x, y, z \in X x \perp_B z, y \perp_B z$ implies that $(x + y) \perp_B z$.*

**Remark 2.2.1.**
*In general Birkhoff-James orthogonality is not right additive, i.e., $x \perp_B y$ and $x \perp_B z$ may not imply that $x \perp_B (y + z)$*

**Example 2.2.2.**
*Let $(1, 1), (1, 0)$ and $(1, 0)$ in $\ell_\infty(\mathbb{R}^2)$, then $(1, 1) \perp_B (1, 0)$ and $(1, 1) \perp_B (0, 1)$ but $(1, 1) \not\perp_B$ $(1, 1)$.*

**Symmetry.** Birkhoff-James orthogonality is not symmetric in general i.e., $x \perp_B y$ does not imply $y \perp_B x$. In fact, we have the following theorem that characterizes inner product spaces among normed linear spaces in terms of the symmetry of Birkhoff- James orthogonality.

**Theorem 2.2.5.**
*Let $X$ be a normed linear space . $X$ is an inner product space if and only if Birkhoff-James orthogonality is symmetric in $X$ i.e., for any $x, y \in X$ $x \perp_B y$ implies that $y \perp_B x$.*

The following theorem follows easily from the Hahn-Banach theorem.

**Theorem 2.2.6.**

*Let $X$ be a normed linear space, then for any vector $x \in X$ there exists a hy-perplane $H \subset X$ such that $x \perp_B H$ i.e., $x \perp_B y$ $\forall y \in H$.*

## 2.3 Strongly Birkhoff-James orthogonality

**Definition 2.3.1.** *[19, 27]*

*Let $X$ be a normed linear space and let $x, y \in X$. We say that $x$ is strongly orthogonal to $y$ in the sense of Birkhoff-James, written as $x \perp_{SB} y$, if $\|x + \lambda y\| > \|x\|$ for all $\lambda \neq 0$.*

**Remark 2.3.1.**

*If $x \perp_{SB} y$ then $x \perp_B y$ but the converse is not true.*

**Example 2.3.1.**

*In $\ell_\infty(\mathbb{R}^2)$ the element $(1, 0)$ is orthogonal to $(0, 1)$ in the sense of Birkhoff-James but not strongly orthogonal.*

**Definition 2.3.2.**

*A finite set of elements $\{x_1, x_2, \ldots, x_n\}$ in a normed linear space $X$ is said to be a strongly orthogonal set in the sense of Birkhoff-James iff for each $i \in \{1, 2, \ldots, n\}$,*

$$\|x_i\| < \left\| x_i + \sum_{j=1, j\neq i}^{n} \lambda_i x_j \right\|,$$

*whenever not all $\lambda_j$'s are 0.*
*In addition if $\|x_i\| = 1$, for each $i$, then the set is called strongly orthonormal set in the sense of Birkhoff-James.*

Recall that a normed linear space $X$ is said to be uniformly convex iff given $\epsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in S_X$ and $\|x - y\| \geq \epsilon$ then $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$.

The number $\delta(\epsilon) = \inf\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \|x - y\| \geq \epsilon\}$ is called the modulus of convexity of $X$. A space $X$ is strictly convex iff $\delta(2) = 1$.

The concept of strictly convex and uniformly convex spaces have been extremely useful in the studies of the geometry of Banach Spaces. One may go through [3],[16],[14],[10],[11],[21],[29], for more information related to strictly convex spaces.

In this theorem given the characterization of strictly convex spaces and strongly orthogonality of th sense of Birkhoff-James.

**Theorem 2.3.1.**

*Let $X$ be a real normed linear space. If for $x, y \in X - \{0\}$, $x \perp_B y$ implies $x \perp_{SB} y$ then $X$ is strictly convex.*

*Proof.* Let the unit sphere $S_X$ contains a straight line segment i.e., there exists $\|u\| = \|v\| = 1$ with $\|tu + (1-t)v\| = 1$ $\forall t \in [0, 1]$.
Let $x = \frac{1}{2}u + \frac{1}{2}v, y = v - u$. Consider $x + \lambda y$.
If $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$ then $\|x + \lambda y\| = \|tu + (1-t)v\| = 1$ where $t = \frac{1}{2} - \lambda$. If $\lambda < -\frac{1}{2}$ then $\frac{1}{2} - \lambda > 1$ and so we can write $\frac{1}{2} - \lambda = t\alpha$ for some $t \in (0, 1)$ and $\alpha > 1$. In this case

$$\begin{aligned}
\|x + \lambda y\| &= \|t\alpha u + (1 - t\alpha)v\| \\
&= \|t\alpha u + (\alpha - t\alpha)v + (1 - \alpha)v\| \\
&\geq |\|t\alpha u + (\alpha - t\alpha)v\| - \|(\alpha - 1)v\|| \\
&= ||\alpha| - |\alpha - 1|| \\
&= 1
\end{aligned}$$

If $\lambda > \frac{1}{2}$ then $\frac{1}{2} + \lambda > 1$ and so we can write $\frac{1}{2} + \lambda = t\alpha$ for some $t \in (0, 1)$ and $\alpha > 1$. In this case

$$\begin{aligned}
\|x + \lambda y\| &= \|(1 - t\alpha)u + t\alpha v\| \\
&= \|(\alpha - t\alpha)u + t\alpha v + (1 - \alpha)u\| \\
&\geq |\|(\alpha - t\alpha)u + t\alpha v\| - \|(\alpha - 1)u\|| \\
&= ||\alpha| - |\alpha - 1|| \\
&= 1
\end{aligned}$$

Thus $\|x + \lambda y\| \leq \|x\| \, \forall \lambda$ but $\|x + \lambda_0 y\| = \|x\|$, for $\lambda_0 \in [-\frac{1}{2}, \frac{1}{2}]$. This is a contradiction to our hypothesis and so $X$ is strictly convex.

□

Conversely we have the following theorem

**Theorem 2.3.2.** *[27]*

*Let $X$ a normed linear space. Suppose $X$ is a strictly convex space and $x, y \in X - \{0\}$ with $x \perp_B y$, then $x \perp_{SB} y$.*

*Proof.* Without loss of generality we assume that there exists $x, y \in X$, $\|x\| = 1$ such that $\|x + \lambda y\| \geq 1$ $\forall \lambda$ but $\|x + \lambda_0 y\| = 1$ for some $\lambda_0 \neq 0$.

Let $0 < t < 1$, then

$$\begin{aligned} 1 &= t\|x\| + (1-t)\|x + \lambda_0 y\| &&\geq \|tx + (1-t)(x + \lambda_0 y)\| \\ 1 &\geq \|tx + (1-t)(x + \lambda_0 y)\| &&= \|x + (1-t)\lambda_0 y\| \geq 1 \end{aligned}$$

Thus $\|tx + (1-t)(x + \lambda_0 y)\| = 1$, which contradicts the fact that $X$ is strictly convex. $\square$

Characterization of a real strictly convex space.

**Theorem 2.3.3.** *[27] A real normed linear space $X$ is strictly convex iff $x, y \in S_X$ and $x \perp_B y \Rightarrow x \perp_{SB} y$.*

## 2.4 Isosceles orthogonality

In 1945, James. introduced in [10] the definition of isosceles orthogonality in normed linear spaces.

**Definition 2.4.1.** *[10]*

*Let $X$ be a normed linear space, we say that a vector $x$ is said to be isosceles orthogonal to a vector $y$ ($x \perp_I y$) if the inequality $\|x + y\| = \|x - y\|$ holds.*

**Homogeneity.**

One of the most important properties of isosceles orthogonality is that it is homogenous only in inner product spaces. We see now the origin of this result.

**Theorem 2.4.1.**

*A normed linear space $X$ is an inner product space if and only if, for any $x, y \in S_X$ and any number $\alpha$, the identity $\|\alpha x + y\| = \|x + \alpha y\|$ holds.*

*Assume that isosceles orthogonality is homogeneous.*

*Let* $x, y \in S_X$ *and* $\alpha \in \mathbb{R}$ *. Since* $x + y \perp_I x - y$*, then* $(1 + \alpha)(x + y) \perp_I (1 - \alpha)(x - y)$*, i.e.,* $\|\alpha x + y\| = \|x + \alpha y\|$.

**Theorem 2.4.2.**

*Isosceles orthogonality is homogeneous in a normed linear space if and only if this space is an inner product space.*

**Theorem 2.4.3.**

*A normed linear space* $X$ *is an inner product space if and only if there exists a number* $\alpha \neq \{0, -1, 1\}$ *such that the implication*

$$x, y \in X, x \perp_I y \Longrightarrow x \perp_I \alpha y$$

*holds.*

**Additivity.**

From the continuity of the norm it follows easily that if isosceles orthogonality is additive then it is also homogeneous. Thus, the next theorem follows directly from Theorem 2.4.2.

**Theorem 2.4.4.**

*The Isosceles orthogonality is additive in a normed linear space if and only if this space is an inner product space.*

New, we give the differences between Birkhoff orthogonality and isosceles orthogonality.

**Theorem 2.4.5.** *[2, Chapter 4 and Chapter 10]*

*Let* $X$ *be a normed linear space with unit sphere* $S_X$*. Then the following properties are equivalent:*

**i)** $x, y \in X, x \perp_I y \Rightarrow x \perp_B y,$

**ii)** $x, y \in X, x \perp_B y \Rightarrow x \perp_I y,$

**iii)** $x, y \in S_X, x \perp_I y \Rightarrow x \perp_B y,$

**iv)** $x, y \in S_X, x \perp_B y \Rightarrow x \perp_I y$,

**v)** $x, y \in S_X, x \perp_I y \Rightarrow x + y \perp_B x - y$,

**vi)** *$X$ is an inner product space.*

Chapter 3

# Characterization of orthogonality of bounded linear operators

## 3.1 Birkhoff-James orthogonality of linear operators

**Definition 3.1.1.**
*Let $X$ be a finite dimensional real normed space , for any two element $T, A \in \mathcal{L}(X)$ , $T$ is said to be orthogonal to $A$ in the sense of Birkhoff-James , written as $T \perp_B A$ if and only if*

$$\|T\| \leq \|T + \lambda A\| \quad \forall \lambda \in \mathbb{R}$$

.

**Example 3.1.1.**
*In $B(\ell_2(\mathbb{R}^2))$ Consider two operators $T$ such that $T(0,1) = (0,0)$, and the identity operator $I$ .*
*Then $I \perp_B T$ because: $I + \lambda T = (1,0)$ then, $\|I + \lambda T\| = 1$ and $\|I\| = 1$. Moreover,*

$$\|I + \lambda T\| \geq \|I\| \text{ for all } \lambda \in \mathbb{R}.$$

**Proposition 3.1.1.**
*Let $X$ is a finite dimensional normed linear space and $T, A \in \mathcal{L}(X)$ is such that $T \perp_B A$ then there exists $x \in X$ with $\|x\| = 1$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$.*

**Example 3.1.2. 1-** *Let $T, A : (\mathbb{R}^3, \|.\|_\infty) \to (\mathbb{R}^3, \|.\|_\infty)$ be given by*

$$T = \begin{pmatrix} 1 & 4 & -2 \\ 3 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 3 & 2 & 5 \\ 1 & 1 & 0 \\ 0 & 8 & 3 \end{pmatrix}.$$

*Then $T \perp_B A$ and there exists $x = (1,1,-1) \in \mathbb{R}^3$ with $\|x\|_\infty = 1$ such that $\|Tx\|_\infty = \|T\|_\infty$ and $Tx \perp_B Ax$.*

**2-** *Let $T, A : (\mathbb{R}^3, \|.\|_\infty) \to (\mathbb{R}^3, \|.\|_\infty)$ be given by*

$$T = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

*Then $T \perp_B A$ and there exists $x = (0,1,1) \in \mathbb{R}^3$ with $\|x\| = 1$ such that $\|Tx\|_\infty = \|T\|_\infty$ and $Tx \perp_B Ax$.*

In this theorem we give Birkhoff-James orthogonality of linear operators defined on a finite dimensional real Banach space.

**Theorem 3.1.1.** *[24]*
*Let $X$ be a finite dimensional real Banach space. Let $T, A \in \mathcal{L}(X)$. Then $T \perp_B A$ if and only if there exists $x, y \in M_T$ such that $Ax \in Tx^+$ and $Ay \in Ty^-$.*

Now we give the Birkhoff-James orthogonality of compact linear operators defined on a reflexive Banach spaces.

**Theorem 3.1.2.** *[28]*
*Let $X$ be a reflexive Banach space and $Y$ be any normed linear space. Then for any $T, A \in \mathcal{K}(X,Y)$, $T \perp_B A$ if and only if there exists $x, y \in M_T$ such that $Ax \in (Tx)^+$ and $Ay \in (Ty)^-$.*

*Proof.* Let us first prove the easier sufficient part.
Since $A \in (Tx)^+$, $\|T + \lambda A\| \geq \|Tx + \lambda Ax\| \geq \|Tx\| = \|T\|$ for all $\lambda \geq 0$.
Similarly $Ay \in (Ty)^-$ implies that $\|T + \lambda A\| \geq \|Ty + \lambda Ay\| \geq \|Ty\| = \|T\|$ for all $\lambda \leq 0$.
This completes the proof of the sufficient part.
Let us now prove the necessary part.

Since $T$ and $A$ are compact linear operators, $(T + \frac{1}{n}A)$ is also a compact linear operator for each $n \in \mathbb{N}$. Since $X$ is reflexive, $(T + \frac{1}{n}A)$ attains norm for each $n \in \mathbb{N}$.
Therefore, for each $n \in \mathbb{N}$, there exists $x_n \in S_X$ such that $\left\|(T + \frac{1}{n}A)x_n\right\| = \left\|T + \frac{1}{n}A\right\|$.
Since $X$ is reflexive, $B_X$ is weakly compact. Therefore $\{x_n\}$ has a weakly convergent subsequence. Without loss of generality we may assume that $\{x_n\}$ weakly converges to $x$.
Since $T$ and $A$ are compact linear operators, $Tx_n \longrightarrow Tx$ and $Ax_n \longrightarrow Ax$.
Since $T \perp_B A$ we have $\left\|T + \frac{1}{n}A\right\| \geq \|T\|$ for all $n \in \mathbb{N}$.
Therefore,

$$\left\|(T + \frac{1}{n}A)x_n\right\| = \left\|T + \frac{1}{n}A\right\| \geq \|T\| \geq \|Tx_n\| \text{ for all } n \in \mathbb{N}.$$

Letting $n \longrightarrow \infty$, we see that, $\|Tx\| \geq \|T\| \geq \|Tx\|$. This proves that $\|Tx\| = \|T\|$, i.e., $x \in M_T$.
Know,we show that $Ax \in (Tx)^+$.
For any $\lambda \geq \frac{1}{n}$ , we claim that $\|Tx_n + \lambda Ax_n\| \geq \|Tx_n\|$.
Otherwise, $Tx_n + \frac{1}{n}Ax_n = (1 - \frac{1}{n\lambda})Tx_n + \frac{1}{n\lambda}(Tx_n + \lambda Ax_n)$ gives that,

$$\begin{aligned}
\left\|T + \tfrac{1}{n}A\right\| &= \left\|Tx_n + \tfrac{1}{n}Ax_n\right\| \\
&\geq (1 - \tfrac{1}{n\lambda}) \|\|Tx_n\| + \tfrac{1}{n\lambda} \|(Tx_n + \lambda Ax_n)\| \\
&\geq (1 - \tfrac{1}{n\lambda}) \|Tx_n\| + \tfrac{1}{n\lambda} \|Tx_n\| \\
&= \|Tx_n\| \geq \|T\|,
\end{aligned}$$

a contradiction.
This completes the proof of our claim.
Now for any $\lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that $\lambda > \frac{1}{n_0}$ . So for all $n \geq n_0$, $\|Tx_n + \lambda Ax_n\| \geq \|Tx_n\|$. Therefore, letting $n \longrightarrow \infty$, we have, $\|Tx + \lambda Ax\| \geq \|Tx\|$.
This completes the proof of the fact that $Ax \in (Tx)^+$.
Similarly, considering the compact operators $T - \frac{1}{n}A$, it is now easy to see that there exists $y \in M_T$ such that $Ay \in (Ty)^-$.
This completes the proof. □

For bounded linear operators defined on a normed linear space, the situation is far more complicated since in this case the norm attainment set exemple. In the next proposition, we give a sufficient condition for Birkhoff-James orthogonality of bounded linear operators.

**Proposition 3.1.2.** *[28]*

*Let $X$ and $Y$ be normed linear spaces. Let $T, A \in B(X,Y)$. Suppose there exists two sequences $\{x_n\}$ and $\{y_n\}$ in $S_X$ satisfying the following two conditions:*

*(i) $\|Tx_n\| \longrightarrow \|T\|$ and $\|Ty_n\| \longrightarrow \|T\|$, as $n \longrightarrow \infty$*

*(ii) $Ax_n \in (Tx_n)^+$ and $Ay_n \in (Ty_n)^-$ for all $n \in \mathbb{N}$.*

*Then $T \perp_B A$.*

*Proof.* Since $Ax_n \in (Tx_n)^+$, for any $\lambda \geq 0$ we have,

$$\|T + \lambda A\| \geq \|Tx_n + \lambda Ax_n\| \geq \|Tx_n\|$$

for all $n \in \mathbb{N}$. Therefore letting $n \to \infty$, we have, $\|T + \lambda A\| \geq \|T\|$, since $\|Tx_n\| \longrightarrow \|T\|$ as $n \longrightarrow \infty$.

Similarly, $Ay_n \in (Ty_n)^-$ implies that, for any $\lambda \leq 0$,

$$\|T + \lambda A\| \geq \|Ty_n + \lambda Ay_n\| \geq \|Ty_n\|$$

for all $n \in N$. Therefore letting $n \longrightarrow \infty$, we have, $\|T + \lambda A\| \geq \|T\|$, since $\|Ty_n\| \longrightarrow \|T\|$ as $n \longrightarrow \infty$. This completes the proof of the fact that $T \perp_B A$. □

## 3.2 Strongly Birkhoff-James orthogonality of linear operators

**Definition 3.2.1.**

*Let $X$ be a finite dimensional real normed space , for any two element $T, A \in \mathcal{L}(X)$ , $T$ is said to be orthogonal to $A$ in the sense of Birkhoff-James , written as $T \perp_B A$ if and only if*

$$\|T\| \leq \|T + \lambda A\| \quad \forall \lambda \in \mathbb{R}^*$$

**Theorem 3.2.1.** *[13]*

*Let $X$ be a real reflexive strictly convex Banach space and $A \in \mathcal{K}(X)$ be injective. Then for any $T \in \mathcal{K}(X)$,*

$$T \perp_B A \Longrightarrow T \perp_{SB} A.$$

**Theorem 3.2.2.** *[13]*

*Let $X$ be a real reflexive strictly convex Banach space. Then for any $T, A \in \mathcal{K}(X)$,*

$$T \perp_B A \Longrightarrow T \perp_{SB} A \text{ or } Ax = 0 \text{ for some } x \in M_T.$$

*Proof.* Let $T, A \in \mathcal{K}(X)$ such that $T \perp_B A$. If $Ax = 0$ for some $x \in M_T$ then we are done. Assuming $Ax = 0$ for any $x \in M_T$ we show that $T \perp_{SB} A$.

We consider the following two cases:

**Case 1.** There exists $x \in M_T$ such that $Tx \perp_B Ax$.

Since $X$ is strictly convex, by Theorem 1.3.20 of Sain et al [27]. we get $Tx \perp_{SB} Ax$.

Then for all real scalar $\lambda \neq 0$, we have

$$\|T + \lambda A\| \geq \|Tx + \lambda Ax\| > \|Tx\| = \|T\|$$

which implies $T \perp_{SB} A$.

**Case 2.** There exists no $x \in M_T$ such that $Tx \perp_B Ax$.

For any $x \in M_T$ , we first show that either

$$\|Tx + \lambda Ax\| \geq \|Tx\| \ \ \forall \lambda \geq 0 \text{ or } \ \|Tx + \lambda Ax\| \geq \|Tx\| \ \ \forall \lambda \leq 0$$

.

If possible let there exist $\lambda_1 > 0$ and $\lambda_2 < 0$ such that

$$\|Tx + \lambda_1 Ax\| < \|T\| \ \text{ and } \ \|Tx + \lambda_2 Ax\| < \|T\|.$$

Now $Tx = (1 - t)(Tx + \lambda_1 Ax) + t(Tx + \lambda_2 Ax)$ where $t = \frac{\lambda_1}{\lambda_1 - \lambda_2} \in (0, 1)$.

This shows that

$$\|Tx\| \leq \|(1 - t)(Tx + \lambda_1 Ax)\| + \|t(Tx + \lambda_2 Ax)\| < \|T\|$$

a contradiction to the fact that $x \in M_T$ .

Thus for each $x \in M_T$ either

$$\|Tx + \lambda Ax\| \geq \|T\| \ \ \forall \lambda \geq 0, \text{ or } \ \|Tx + \lambda Ax\| \geq \|T\| \ \ \ \forall \lambda \leq 0.$$

Conversely, suppose that there exists $x_1, x_2 \in M_T$ such that

$$\|Tx_1 + \lambda Ax_1\| \geq \|T\| \ \ \ \forall \lambda \geq 0 \text{ and } \ \|Tx_2 + \lambda Ax_2\| \geq \|T\| \ \ \ \forall \lambda \leq 0.$$

If not, without loss of generality we may assume that

$$\forall x \in M_T, \|Tx + \lambda Ax\| \geq \|T\| \quad \forall \lambda \leq 0.$$

For each $n \in \mathbb{N}$, the operator $(T + \frac{1}{n}A)$, being compact on a reflexive normed space attains its norm. So there exists $x_n \in S_X$ such that $\left\|T + \frac{1}{n}A\right\| = \left\|(T + \frac{1}{n}A)x_n\right\|$. Using reflexivity of $X$ and we can find a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x_0$ (say) in $B_X$ weakly. Without loss of generality we assume that $x_n \rightharpoonup x_0$ weakly. Then $T, A$ being compact, $Tx_n \longrightarrow Tx_0, Ax_n \longrightarrow Ax_0$. As $T \perp_B A$ we have $\left\|T + \frac{1}{n}A\right\| \geq \|T\| \quad \forall n \in \mathbb{N}$ and so

$$\left\|Tx_n + \frac{1}{n}Ax_n\right\| \geq \|T\| \geq \|Tx_n\| \quad \forall n \in \mathbb{N}.$$

Letting $n \longrightarrow \infty$ we get $\|Tx_0\| \geq \|T\| \geq \|Tx_0\|$. So $x_0 \in M_T$ . For any $\lambda > \frac{1}{n}$ , we claim that $\|Tx_n + \lambda Ax_n\| \geq \|Tx_n\|$.

Otherwise

$$\begin{aligned}
Tx_n + \frac{1}{n}Ax_n &= (1 - \frac{1}{n\lambda})Tx_n + \frac{1}{n\lambda}(Tx_n + \lambda Ax_n) \\
\Longrightarrow \left\|Tx_n + \frac{1}{n}Ax_n\right\| &\leq (1 - \frac{1}{n\lambda}) \|Tx_n\| + \frac{1}{n\lambda} \|Tx_n + \lambda Ax_n\| \\
\Longrightarrow \left\|Tx_n + \frac{1}{n}Ax_n\right\| &< \|Tx_n\| \text{ , a contradiction.}
\end{aligned}$$

Choose $\lambda > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\lambda > \frac{1}{n_0}$ and so for all $n \geq n_0$ we get, $\|Tx_n + \lambda Ax_n\| \geq \|Tx_n\|$.

Letting $n \longrightarrow \infty$ we get $\|Tx_0 + \lambda Ax_0\| \geq \|Tx_0\|$ which holds for any $\lambda \geq 0$. This along with shows that $Tx_0 \perp_B Ax_0$, which violates hypothesis of Case 2. Hence there exists $x_1, x_2 \in M_T$ such that $\|Tx_1 + \lambda Ax_1\| \geq \|T\| \quad \forall \lambda \geq 0$ and $\|Tx_2 + \lambda Ax_2\| \geq \|T\| \quad \forall \lambda \leq 0$. As $X$ is strictly convex, we have

$$\|Tx_1 + \lambda Ax_1\| > \|T\| \quad \forall \lambda > 0 \text{ and } \|Tx_2 + \lambda Ax_2\| > \|T\| \quad \forall \lambda < 0.$$

For $\lambda > 0$,

$$\|T + \lambda A\| \geq \|(T + \lambda A)x_1\| > \|T\|$$

and for $\lambda < 0$,

$$\|T + \lambda A\| \geq \|(T + \lambda A)x_2\| > \|T\| .$$

So $T \perp_{SB} A$.

This completes the proof.

□

**Theorem 3.2.3.** *[13]*

*Let $X$ be a real normed linear space and for any $T, A \in \mathcal{K}(X)$, $T \perp_B A \Longrightarrow T \perp_{SB} A$ or $Ax = 0$ for some $x \in M_T$ . Then $X$ is strictly convex.*

Combining Theorem 3.2.2 and Theorem 3.2.3, we get the following result:

**Theorem 3.2.4.** *[13]*

*Let $X$ be a real reflexive Banach space. Then $X$ is strictly convex if and only if for any $T, A \in \mathcal{K}(X), T \perp_B A \Longrightarrow T \perp_{SB} A$ or $Ax = 0$ for some $x \in M_T$.*

**Corollary 3.2.1.** *[13]*

*Let $X$ be a real finite dimensional normed linear space. Then $X$ is strictly convex if and only if for any $T, A \in B(X), T \perp_B A \Longrightarrow T \perp_{SB} A$ or $Ax = 0$ for some $x \in M_T$.*

## 3.3 Orthogonality in $B(\mathcal{H})$

As discussed earlier, for any two bounded linear operators $T, A$ on a Hilbert space $\mathcal{H}$, $A \perp_B T$ may not imply $T \perp_B A$ and conversely. We begin with finite dimensional Hilbert space and characterize those $T \in B(\mathcal{H})$ for which $A \perp_B T \Longrightarrow T \perp_B A$ for all $A \in B(\mathcal{H})$.

**Theorem 3.3.1.** *[13]*

*Let $\mathcal{H}$ be a finite dimensional real inner-product space and $T \in B(\mathcal{H})$. Then for all $A \in B(\mathcal{H}), A \perp_B T \Longrightarrow T \perp_B A$ if and only if $M_T = S_{\mathcal{H}}$.*

**Corollary 3.3.1.** *[7]*

*Let $\mathcal{H}$ be a real finite dimensional inner-product space and $T \in \mathcal{L}(\mathcal{H})$.*

*Then for all $A \in B(\mathcal{H}), T \perp_B A \Longrightarrow A \perp_B T$ if and only if $T$ is the zero operator.*

**Theorem 3.3.2.** *[13]*

*Let $\mathcal{H}$ be a real infinite dimensional Hilbert space and $T \in \mathcal{K}(\mathcal{H})$. Then for all $A \in B(\mathcal{H}), A \perp_B T \Longrightarrow T \perp_B A$ if and only if $T$ is the zero operator.*

Likewise of isosceles orthogonality of two element we give the definition of isosceles orthogonality of two operators.

**Definition 3.3.1.**
*Let $X$ be a normed space , for any two element $T, A \in \mathcal{L}(X)$ , $T$ is said to be isosceles orthogonal to $A$ , written as $T \perp_I A$ if and only if*

$$\|T + A\| = \|T - A\|$$

The study of orthogonality of bounded linear operators is also related to the following notion of operators having disjoint support.

**Definition 3.3.2.**
*Let $\mathcal{H}$ be a real Hilbert space. Two operators $A, B \in B(\mathcal{H})$ have disjoint support if and only if $AB^* = B^*A = 0$.*

We give the relation between disjoint support, Birkhoff-James orthogonality and isosceles orthogonality in the context of bounded linear operators on a Hilbert space,

**Proposition 3.3.1.** *[7]*
*Let $A, B \in B(\mathcal{H})$, where $\mathcal{H}$ is a real or complex Hilbert space, such that $AB^* = B^*A = 0$ ( disjoint support), then the following holds:*

1. *$A \perp_B B$ and $B \perp_B A$.*
2. *$A \perp_I B$.*

*Proof.* 1. Consider $h \in S_{\mathcal{H}}$. Then for any $\in \mathbb{K}$

$$\|(A + \lambda B)h\|^2 = \|Ah\|^2 + \|\lambda Bh\|^2 + 2|\lambda|^2 Re\langle B^*Ah, h\rangle = \|Ah\|^2 + \|\lambda Bh\|^2 ,$$

where $Re(z)$ denotes the usual real part of $z \in \mathbb{K}$. Therefore,

$$\|A + \lambda B\|^2 \geq \|(A + \lambda B)h\|^2 \geq \|Ah\|^2$$

for all $h \in S_{\mathcal{H}}$,

which implies that $\|A + \lambda B\| \geq \|A\|$ for any $\lambda \in \mathbb{K}$.

Interchanging the roles of $A$ and $B$, we can obtain in a similar way that $B \perp_B A$.

2. If $A, B \in B(\mathcal{H})$ satisfy $B^*A = 0$, then for any $\lambda \in \mathbb{K}$ , we have,

$$\begin{aligned}\|A+\lambda B\|^2 &= \sup\left\{\|(A+\lambda B)h\|^2 : h \in S_{\mathcal{H}}\right\}\\ &= \sup\left\{\|Ah\|^2 + \|\lambda Bh\|^2 : h \in S_{\mathcal{H}}\right\}\\ &= \|A-\lambda B\|^2.\end{aligned}$$

This completes the proof of the second part of the proposition and establishes it completely. □

However, not every pair of operators $A, B \in B(\mathcal{H})$, such that $A \perp_B B$ or $A \perp_I B$, have disjoint support. This idea can be illustrated in the next example.

**Example 3.3.1.**

*Let $\mathcal{H}$ be the two-dimensional real Hilbert space. We consider Let* $A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *Then,*

*1-* $\|A+B\| = \|A-B\| = 4$ *and*

*2-* $A^*B = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$

*which implies that $A, B$ do not have disjoint support.*

**Theorem 3.3.3.** *[7]*

*Let $A, B \in B(\mathcal{H})$ and suppose that there exists $h_0, k_0 \in \mathcal{H}$ such that $h_0 \in M_{A+B}$ and $k_0 \in M_{A-B}$ . Then the following assertions are true.*

(i) *If $\langle Ah_0, Bh_0\rangle \leq 0$ and $\langle Ak_0, Bk_0\rangle \geq 0$, then $A \perp_I B$.*

(ii) *If $A \perp_I B$ then $\langle Ah_0, Bh_0\rangle \geq 0$ and $\langle Ak_0, Bk_0\rangle \leq 0$.*

*Proof.* 1. Assume that all the conditions of the statement are satisfied. Let $f, g : H \longrightarrow \mathbb{R}$ be given by

$$f(h) = \|(A+B)h\|^2 = \|Ah\|^2 + \|Bh\|^2 + 2\langle Ah, Bh\rangle \text{ and}$$
$$g(h) = \|(AB)h\|^2 = \|Ah\|^2 + \|Bh\|^2 - 2\langle Ah, Bh\rangle.$$

Then, $f(h) - g(h) = 4\langle Ah, Bh\rangle$. Suppose that

$$g(k0) = \|A - B\|^2 < \|A + B\|^2 = f(h0) \Longrightarrow g(h_0) \leq g(k_0) < f(h_0).$$

Thus, 0< $f(h_0) - g(h_0) = 4\langle Ah_0, Bh_0\rangle$, which is a contradiction. Hence, $g(k_0) \geq f(h_0)$. Analogously, it can be proved that $f(h_0) \geq g(k_0)$. Finally,

$$\|A + B\|^2 = f(h_0) = g(k_0) = \|AB\|^2\,,$$

which implies $A \perp_I B$.

2. We only prove the first inequality, the other can be obtained with a similar argument.

   By the real polarization formula we get

$$\begin{aligned}\langle Ah_0, Bh_0\rangle &= \frac{1}{4}[\|(A + B)h_0\|^2 - \|(A - B)h_0\|^2] \\ &\geq \frac{1}{4}[\|A + B\|^2 - \|AB\|^2] = 0.\end{aligned}$$

□

**Remark 3.3.1.**

*Suppose that in Theorem 3.3.3, $h_0$ and $k_0$ also satisfy*

$$\langle Ah_0, Bh_0\rangle = \langle Ak_0, Bk_0\rangle = 0.$$

*Then, there exists $h_1 \in S_{\mathcal{H}}$ such that $\|(A + B)h_1\| = \|(A - B)h_1\|$.*

*It can be easily proved using polarization formula and hypothesis,*

$$0 = \langle Ah_0, Bh_0\rangle = \frac{1}{4}[\|(A + B)h_0\|^2 \ \|(A - B)h_0\|^2]$$

*and this implies $\|(A - B)h_0\| = \|(A + B)h_0\| = \|(A - B)k_0\|$, where last equality is due to isosceles orthogonality between $A$ and $B$ previously proved.*

*By a similar argument, it can be proved that $\|(A + B)k_0\| = \|(A + B)h_0\|$.*

*The proof is completed by taking $h_1 \in \{h_0; k_0\}$.*

The following result combines Theorem 3.2.2 and last remark.

**Corollary 3.3.2.** *[7]*

*Let $A, B \in B(\mathcal{H})$ and suppose that there exists $h_1 \in M_{A+B} \cap M_{A-B}$ such that $\langle Ah_1, Bh_1\rangle =$ 0. Then $A \perp_I B$,*

$$\|A\|^2 + \|B\|^2 \leq \|A + B\|^2 + \|AB\|^2 \leq 2\left(\rangle\, \|A\|^2 + \|B\|^2\right),$$

*and if* $h_1 \notin N(A) \cup N(B)$ *then* $\|A+B\|^2 = \|A-B\|^2 = 2.$

*Proof.* It was proved in Theorem 3.3.3 that, under these hypothesis, $A \perp_I B$. Moreover,

$$\|A+B\|^2 = \|(A+B)h_1\|^2 = \|Ah_1\|^2 + \|Bh_1\|^2 \geq \|A\|^2 + \|B\|^2 \text{ and}$$
$$\|A-B\|^2 = \|(A-B)h_1\|^2 = \|Ah_1\|^2 + \|Bh_1\|^2 \geq \|A\|^2 + \|B\|^2.$$

Then

$$\|A+B\|^2 + \|A-B\|^2 \geq 2\left(\langle\|A\|^2 + \|B\|^2\right).$$

On the other hand,

$$\|A\|^2 + \|B\|\,2 \geq 2\max(\|A\|^2; \|B\|^2) \geq \|A+B\|^2 + \|A-B\|^2,$$

$$\|(A+B)h_1\|^2 \, \|Bh_1\|^2 \;\; \|Ah_1\|^2 \, \|Bh_1\|^2 = |\langle (A+B)h_1, Bh_1\rangle|^2 |\langle Ah_1, Bh_1\rangle|^2.$$

If we assume that $h_1 \neq N(A) \cup N(B)$, then $\|A+B\|^2 = 1 + \|Ah_1\|^2$. By symmetry we obtain that $\|A+B\|^2 = 1 + \|Bh_1\|^2$.

Now, by the Parallelogram law we get

$$\|A+B\|^2 + \|A-B\|^2 = \|(A+B)h_1\|^2 + \|(AB)h_1\|^2 = 2(\|Ah_1\|^2 + \|Bh_1\|^2).$$

It follows that $\|Bh_1\| = 1$ and $\|A+B\|^2 = 2.$ □

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