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Periodic orbits of differential systems via averaging theory

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Contents

List of Figures

Introduction

It is usually referred to us that the language of nature and science is mathematics. Ever since humanity became aware of the world around it, there has been an absolute need to understand the rules behind natural phenomena, and no one can ignore the main role that mathematics plays in this context.

Mathematics help to understand what we see and to make predictions about what will happen in a precise language. In fact, differential equations have shown to be one of the most powerful tools to model the reports between phenomena of the reality in which we are living, not only to describe the laws of nature but also, to explain the behavior of some social processes. Although initially considered as simple modeling tools, ordinary differential equations have given rise to a whole theory of their own.

The birth of differential equations is essentially related to the evolution of infinitesimal calculus in the 17th century by I. Newton (1642-1727) and G. W. Leibnitz (1646-1716).

During the 18th century, the study of differential equations was supported by L. Euler (1707-1783), who was interested in solving certain mechanical problems, as well as by the two French mathematicians J. L. Lagrange (1736-1813) and P. S. Laplace (1749-1827), who introduced, among others, the notion of partial differential equations.

Differential equations whether ordinary or partial can be classify them either linear or nonlinear. In general; there can be a system of differential equation is called a differential system. This system can be either linear or nonlinear.

The nonlinear differential systems constitute a very important branch of differential systems. From a technical point of view, nonlinear differential systems play an essential role in control systems. This is because, in reality, all systems are non-linear in nature.

In mathematics, a nonlinear system does not fulfill the superposition rule, where its output is not directly proportional to its input.

Nonlinear problems are of great interest in many fields, including engineering, biology, physics, mathematics, and many other fields. Nonlinear dynamical systems, which describe changes in variables over time, can appear chaotic, unpredictable, in contrast to much simpler linear systems.

If differential equations are nonlinear, their solutions cannot easily and sometimes are impossible to be given in terms of known functions. Therefore numerical and asymptotic techniques and methods are used to obtain approximations of the solutions. An analogous method is provided by the qualitative theory of differential equations, which seeks to find properties of the solutions without actually solving the equations.

Two classical and difficult problems of the qualitative theory of planar polynomial differential systems are the characterization of their centers, and the study of their cyclicity, i.e. how many limit cycles can bifurcate from a center when we perturb it inside a given class of polynomial differential systems. Of course, this kind of bifurcation is called in the literature a Hopf bifurcation. In our thesis we are interested to planar polynomial differential systems of the form

$$\dot{x} = -y, \quad \dot{y} = x + Q_n(x, y), \tag{1}$$

where Q_n is a real homogeneous polynomial of degree n. These kind of systems are called *Kukles homogeneous diffrential systems*.

In the literature we find many works studying kukles differential systems. In 1999, Volokitin and Ivanov [71] were the first who studied the problem of center-focus for kukles differential systems (1) for $n \geqslant 2$, where they conjectured that the origin of these systems is a center if and only if systems (1) are symmetric with respect to one of the coordinates axes, and they proved this conjecture for $n = 2$ and $n = 3$. In 2002 Giné [33] proved this conjecture for $n = 4$ and $n = 5$. In 2015 Giné et all [34] proved that the conjecture is true for all $n \geq 2$.

For the global phase portraits of Kukles differential systems (1) we can cite the following works: Vulpe [73] studied the global phase portraits for all center quadratic differential systems and since systems (1) for $n = 2$ is a particular case of these systems, we know that their phase portraits are studied. Buzzi et al. [20], Malkin [60], Vulpe and Sibirskij [74] and Żołądek [75, 76] classified the global phase portraits of cubic polynomial differential systems with a symmetry with respect to a straight line. Benterki and Llibre [5] provided the global phase portraits of systems (1) for $n = 4$.

Llibre and Silva [51, 52] classified the phase portraits of the systems (1) with $n = 5, 6$, and the global phase portraits for the case $n = 7$ was studied by Benterki and Llibre [12].

One of the best methods which allowing to study the number of limit cycles of a differential systems is the averaging method study which is a classical tool allowing us to study the dynamics of non-linear differential system with periodical orbits. The method of averaging has a long history starting with the classical works of Lagrange and Laplace, who provided an intuitive justification of the method. The first formalization of this theory was done in 1928 by Fatou [27]. Important practical and theoretical contributions to the averaging theory were made in 1930 by Bogoliubov– Krylov [17], in 1945 by Bogoliubov [16], and by Bogoliubov–Mitropolsky.

The second part of the sixeteenth Hilber problem also arises for the discontinuous differential systems, where the first work studying the discontinuous piecewise linear differential systems in the plane is due to Andronov, Vitt and Khaikin in [1]. Later on these systems became a topic of great interest in the mathematical community due to their applications for modeling real phenomena, see for instance the books [14, 68] and references quoted there.

To determine the non–existence, the existence of limit cycles and their number is one of the big problems in the qualitative theory of the planar differential systems, and in particular of the planar discontinuous piecewise linear differential systems separated by a curve. In this work we are considering that a *crossing limit cycle* is a periodic orbit isolated in the set of all periodic orbits of the system which has exactly two points on the discontinuity curve.

The problem of finding the best upper bound for the maximum number of limit cycles that a family of piecewise linear differential systems in the plane separated by a straight line can have, has been studied by many authors recently, see for instance [2, 25, 32, 69]. Lum and Chua[57, 58] in 1990 conjectured that the continuous (but non–smooth) piecewise linear systems in the plane separated by one straight line have at most one limit cycle. This conjecture was proved by Freire et al [31] in 1998, for a shorter proof see [46]. Han and Zhang [35] in 2010 conjectured that discontinuous piecewise linear differential systems in the plane separated by a straight line have at most two crossing limit cycles. Huan and Yang [37] in 2012 provided a negative answer to this conjecture exhibiting a numerical example with three crossing limit cycles. Llibre and Ponce in [47] proved the existence of these three limit cycles analytically. Nowadays it remains as an open problem to know if three is the maximum number of crossing limit cycles that this class of systems can have.

In the article [44] the authors considered the problem of Lum and Chua restricted to the class of discontinuous piecewise linear differential centers in the plane separated by a straight line, and they proved that those systems has no crossing limit cycles. But in [45, 54] were studied planar discontinuous piecewise linear differential centers with a curve of discontinuity different from a straight line, and then those systems which can exhibit crossing limit cycles. For this reason it is interesting to study the role which plays the shape of the discontinuity curve in the number of crossing limit cycles that planar discontinuous piecewise linear differential centers can have.

This work consists of five chapters, where in the first chapter we briefly present some of the basic concepts, definitions and results used through this thesis.

In the second chapter, we solve the second part of 16–th Hilbert problem for a discontinuous piecewise linear differential systems fromed by centers separated by irreducible cubic curves .

In Chapter three, we solve the second part of 16–th Hilbert problem for a discontinuous piecewise linear differential systems fromed by centers or Hamiltonian without equilibria separated by irreducible cubics. The main goal was providing the maximum number of crossing limit cycles of two different families of discontinuous piecewise linear differential systems. More precisely we prove that the systems formed by two zones, where, in one zone we define a linear center and in the second zone we define a Hamiltonian system without equilibria which can exhibit three crossing limit cycles having two intersection points with the cubic of separation.

In the fourth chapter we provided the exact upper bound of limit cycles for linear piecewise differential systems, formed by linear Hamiltonian systems without equilibrium and separated by two concentric circles. Furthermore, we proved that our result is reached by giving some systems having exactly one, two or three limit cycles.

Finally which concerns the fifth chapter we are intersecting in providing all the global phase portraits of the generalized kukles differential systems

$$\dot{x} = -y, \quad \dot{y} = x + ax^8 + bx^6y^2 + cx^4y^4 + dx^2y^6 + ey^8,$$

symmetric with respect to the x–axis, with $a^2 + b^2 + c^2 + d^2 + e^2 \neq 0$, and by using the averaging theory up to seven order, we give the upper bounds of limit cycles which can bifurcate from its center when we perturb it inside the class of all polynomial differential systems of degree 8.

Chapter 1

Some Basic Concepts on Qualitative Theory of Ordinary Differential Equations

This chapter is devoted to give basic ideas and results from the qualitative theory of ordinary differential equations, which are a general background and important for the developing of this thesis. Here is presented some existence results related with our work, furthermore are presented theorems and techniques which will be helpful in the developing of the results.

Section 1.1 Vector Fields and Flows

Definition 1.1 (Vector fields)
Let $\mathcal{D}$ be an open subset in $\mathbb{R}^n$. We define a vector field of class $\mathcal{C}^r$ on $\mathcal{D}$ as a $\mathcal{C}^r$ map $\mathcal{X} : \mathcal{D} \longrightarrow \mathbb{R}^n$, where $\mathcal{X}(x)$ is meant to represent the free part of a vector attached at the point $x \in \mathcal{D}$. Here the r of $\mathcal{C}^r$ denotes a positive integer or $+\infty$. The graphical representation of a vector field on the plane consists in drawing a number of well chosen vectors $(x, \mathcal{X}(x))$ as in Figure 1.1. En lever of [24]

Remark 1 *Integrating a vector field means that we look for curves $x(t)$, with t belonging*

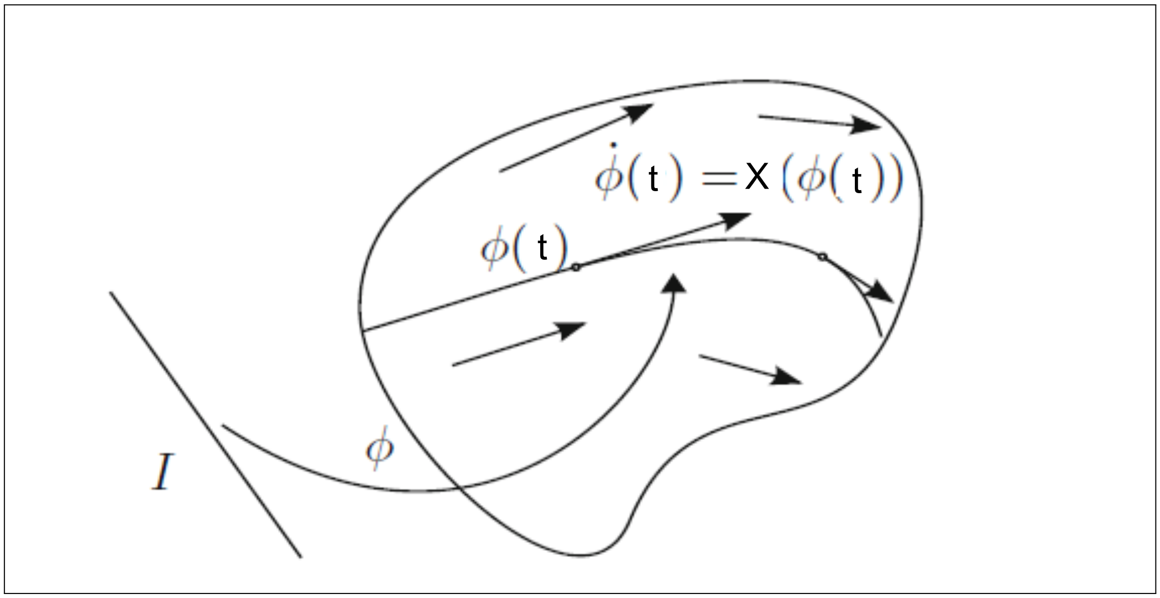


Figure 1.1: Vector fields.

to some interval in $\mathbb{R}$*, that are solutions of the differential equation*

$$\dot{x}(t) = \mathcal{X}(x(t)). \tag{1.1}$$

where $x \in \mathcal{D}$*, and* $\dot{x}$ *denotes* dx/dt*. The variables* x *and* t *are called the dependent variable and the independent variable of the differential equation (1.1), respectively. Usually* t *is also called the time.*

If $\mathcal{X} = \mathcal{X}(x)$ *does not depend on* t*, we say that the differential equation (1.1) is autonomous.*

Definition 1.2 (Flow)
Let $\mathcal{X}$ *be the vector field defned in (1.1). The flow is a* C^1 *function*

$$\phi : \mathbb{R} \times \mathcal{D} \to \mathcal{D},$$

where $\mathcal{D}$ *is an open subset of* $\mathbb{R}^n$ *and if* $\phi_t(x) = \phi(t, x)$*, then* ϕ_t *satisfies*

(i) $\phi_0(x) = \phi(x) \quad \forall x \in \mathcal{D}$,

(ii) $\dfrac{d\phi}{dt}(x) = \mathcal{X}(\phi_t(x))$,

(iii) $\phi_t \circ \phi_s(x) = \phi_{t+s}(x) \quad \forall s, t \in \mathbb{R}$ *and* $x \in \mathcal{D}$. *En lever of [64]*

Remark 2 *The same properties are preserve for a linear system which has the flow* $\phi_t = e^{At}$ *defined from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$.

Section 1.2 Discontinuous Vector Fields

Definition 1.3 (Piecewise linear differential systems.) *A differential system defined in* $\mathbb{R}^2$ *is a piecewise linear differential system (PWLS) in* $\mathbb{R}^2$ *if there exists a set of* 3*–tuples* $\{(A_i, B_i, R_i)\}_{i\in I}$ *where* A_i *is a* 2×2 *real matrix;* $B_i \in \mathbb{R}^2$ *and* R_i *are connected and open regions in* $\mathbb{R}^2$ *separated by a discontinuity manifold* Σ. *These regions satisfy* $R_i \cap R_j = \emptyset$ *for* $i \neq j$ *and* $\cup_{i\in I} R_i \cup \Sigma = \mathbb{R}^2$; *and* $A_i x + B_i$ *is the vector field in* R_i *when* $x \in R_i$.

The switching manifold or discontinuity manifold Σ is described as

$$\Sigma = \{(x,y) \in \mathbb{R}^2 : H(x,y) = 0\},$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a C^r function, $r \geq 1$, and $(0,0)$ is a regular value of H.

Under these conditions, a Piecewise Linear Differential Systems can be written as follows

$$Z(p) = X_i(p) = A_i p + B_i, \;\; p \in R_i, \; i \in \{1, \dots, n\}, \tag{1.2}$$

where each vector field X_i is smooth, and defines a smooth flow $\varphi X_i(p,t)$ within any open set $U \subset R_i$. In particular, each flow $\varphi X_i(p,t)$ is well defined on both sides of the boundary ∂R_i.

We note that Definition 1.3 does not specify a rule for the evolution of the dynamics within a discontinuity set. This depends basically of the behavior of the vector fields $X_{(i-1)}$ and X_i close to the discontinuity manifold for $i \in \{1,2,3,\dots,n\}$ where we denote that $i-1 = n$ when $i = 1$. Here we namely $\Sigma_{(i-1)i} = \overline{R}_{(i-1)} \cap \overline{R}_i \subset \Sigma$.

To extend the evolution of the dynamics on the discontinuity manifold Σ, we divide each $\Sigma_{(i-1)i}$ in three regions. See Figure 1.2.

- Crossing region: $\Sigma^c_{(i-1)i} = \{P \in \Sigma_{(i-1)i} \mid X_{(i-1)}H(P) \cdot X_i H(P) > 0\}$,

- Escaping region: $\Sigma^e_{(i-1)i} = \{P \in \Sigma_{(i-1)i} \mid X_{(i-1)}H(P) < 0 \; and \; X_i H(P) > 0\}$,
- Sliding region: $\Sigma^s_{(i-1)i} = \{P \in \Sigma_{(i-1)i} \mid X_{(i-1)}H(P) > 0 \; and \; X_i H(P) < 0\}$.

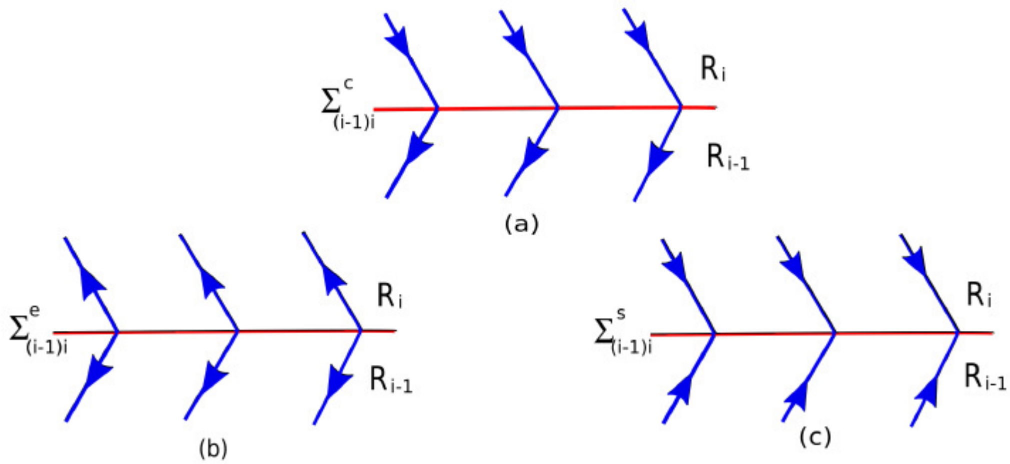


Figure 1.2: Crossing (a), escaping (b) and sliding (c).

Section 1.3 Singular Points

Let $f : \mathcal{D} \to \mathbb{R}^n$ be a continuous function in an open set $\mathcal{D} \subset \mathbb{R}^n$ and consider the autonomous equation

$$\dot{x} = f(x). \tag{1.3}$$

The set $\mathcal{D}$ is called the phase space of the differntial equation (1.3).

Definition 1.4 (Singular points)
A point $x_0 \in \mathbb{R}^n$ is called a singular point (or an equilibrium point) of (1.3) if $f(x_0) = 0$. If a singular point has a neighborhood that does not contain any other singular points, then that singular point is called an isolated singularity.

1.3.1 Types of singular points

Definition 1.5 *Let p be a singular point of a planar $\mathcal{C}^r$ vector field $\mathcal{X} = (P, Q)$. In general the study of the local behavior of the flow near p is quite complicated. Already the linear systems show different classes, even for local topological equivalence.*

We say that

$$D\mathcal{X}(p) = \begin{pmatrix} \dfrac{\partial P}{\partial x}(p) & \dfrac{\partial P}{\partial y}(p) \\ \dfrac{\partial Q}{\partial x}(p) & \dfrac{\partial Q}{\partial y}(p) \end{pmatrix},$$

is the linear part of the vector field $\mathcal{X}$ at the singular point p.

We classify the singular points of a planar differential system in *hyperbolic*, *semi-hyperbolic*, *nilpotent* and *linearly zero*.

The *hyperbolic* ones are the singular points such that the linear part of the differential system has eigenvalues with nonzero real part, see for instance Theorem 2.15 of [24] for the classification of their local phase portraits.

The *semi-hyperbolic* points are the ones having a unique eigenvalue equal to zero, their phase portraits are characterized in Theorem 2.19 of [24].

The *nilpotent* singular points have both zero eigenvalues but their linear part is not identically zero.

Finally the *linearly zero* singular points are the ones such that their linear part is identically zero, and their local phase portraits must be studied using the change of variables called Blow-ups, see for instance chapter 2 and 3 of [24].

Section 1.4 Phase Portrait of a Vector Fields

Although it is often impossible (or very difficult) to determine explicitly the solutions of a differential equation, it is still important to obtain information about these solutions, at least of qualitative nature. To a considerable extent, this can be done describing the phase portrait of the differential equation.

EXEMPLE 1.1 *Let us construct the phase portrait for the equation*

$$\ddot{x} = x^3 - x, \tag{1.4}$$

Equation (1.4) can be written as a Newtonian system $\dot{x} = y, \quad \dot{y} = x^3 - x$.

The differential equation for the phase portraits is

$$\frac{dy}{dx} = \frac{x^3 - x}{y}. \tag{1.5}$$

This equation is separable, leading to

$$\int y dy = \int x^3 - x dx,$$

or $\frac{1}{2}y^2 = \frac{1}{4}x^4 - \frac{1}{2}x^2 + C$, *where* C *is the parameter of the phase portraits. Therefore the equation of the phase portraits is*

$$y(x) = \pm\sqrt{\frac{1}{2}x^4 - x^2 + 2C}.$$

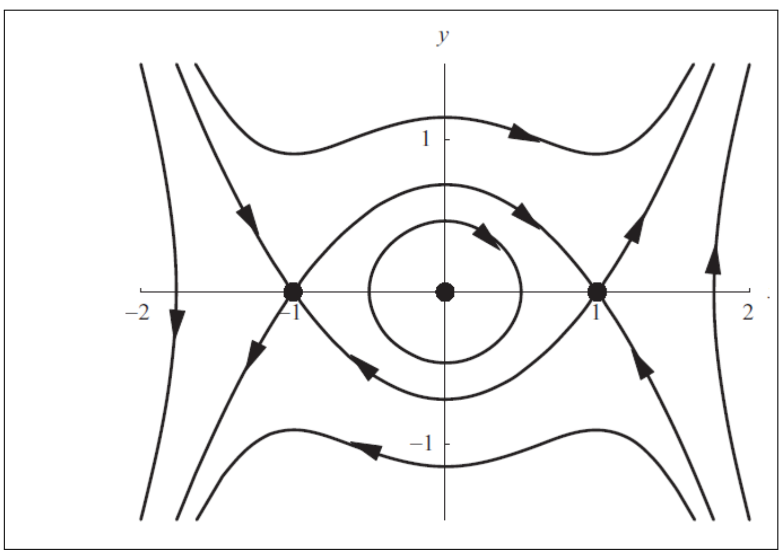


Figure 1.3: The phase portrait for $\ddot{x} = x^3 - x$.

Definition 1.6 (Periodic solutions) *A solution $\varphi(t,x)$ of (1.3) is periodic if there exists a finite time $T > 0$ such that $\varphi(t+T,x) = \varphi(t,x)$ for all $t \in \mathbb{R}$. The minimal T for which the solution $\varphi(t,x)$ of (1.3) is periodic is called the period.*

Definition 1.7 (Limit cycle) *A limit cycle of a planar vector field given by (1.1) is an isolated periodic trajectory. In other word, a periodic trajectory of a vector field is a limit cycle, if it has an annular neighborhood free from other periodic trajectories.*

Section 1.5 Poincaré Compactification.

In this subsection we give some basic results which are necessary for studying the behavior of the trajectories of a planar polynomial differential systems near infinity. Let $\mathcal{X}(x,y) = (P(x,y), Q(x,y))$ represent a vector field to each system which we are going to study its phase portraits, then for doing this we use the so called a Poincaré compactification.

We consider the Poincaré sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$, and we define the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ (with $T_{(0,0,1)}\mathbb{S}^2$ the tangent space of $\mathbb{S}^2$ at the point $(0,0,1)$), such that for each point $q \in T_{(0,0,1)}\mathbb{S}^2$, $T_{(0,0,1)}\mathbb{S}^2(q)$ associates the two intersection points of the straight line which connects the point q and $(0,0)$). The equator $\mathbb{S}^1 = \{(x,y,z) \in \mathbb{S}^2 : z = 0\}$ represent the infinity points of $\mathbb{R}^2$. In summary we get a vector field $\mathcal{X}'$ defined in $\mathbb{S}^2 \setminus \mathbb{S}^1$, which is formed by to symmetric copies of $\mathcal{X}$, and we prolong it to a vector field $p(\mathcal{X})$ on $\mathbb{S}^2$. By studiying the dynamics of $p(\mathcal{X})$ near $\mathbb{S}^1$ we get the dynamics of $\mathcal{X}$ at infinity. We need to do the calculations on the Poincaré sphere near the local charts $U_i = \{Y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{Y \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1,2,3$; with the associated diffeomorphisms $F_i : U_i \longrightarrow \mathbb{R}^2$ and $G_i : V_i \longrightarrow \mathbb{R}^2$ for $i = 1,2,3$. After a rescaling in the independent variable in the local chart (U_1, F_1) the expression of $p(\mathcal{X})$ is

Figure 1.4: The local charts in the Poincaré sphere.

$$\dot{u} = v^n\left[-uP\left(\frac{1}{v},\frac{u}{v}\right) + Q\left(\frac{1}{v},\frac{u}{v}\right)\right], \qquad \dot{v} = -v^{n+1}P\left(\frac{1}{v},\frac{u}{v}\right),$$

in the local chart (U_2, F_2) the expression of $p(\mathcal{X})$ is

$$\dot{u} = v^n \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \quad \dot{v} = -v^{n+1} Q\left(\frac{u}{v}, \frac{1}{v}\right),$$

and for the local chart (U_3, F_3) the expression of $p(\mathcal{X})$ is

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v).$$

Note that for studying the singular points at infinity we only need to study the infinite singular points of the chart U_1 and the origin of the chart U_2, because the singular points at infinity appear in pairs diametrally opposite.

For more details on the Poincaré compactification see Chapter 5 of [24].

1.5.1 Phase portraits on the Poincaré disc

In this subsection we shall see how to characterize the global phase portraits in the Poincaré disc of polynomial differential systems. We shall determine the local phase portrait at all its finite and infinite singular points, then we have to study the properties of its separatrices, where a *separatrix* of $p(\mathcal{X})$ is an orbit which is either a singular point, or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a singular point. Neumann [62] proved that the set formed by all separatrices of $p(\mathcal{X})$; denoted by $S(p(\mathcal{X}))$ is closed. We denote by S the number of separatrices.

The open connected components of $\mathbb{D}^2 \setminus S(p(\mathcal{X}))$ are called *canonical regions* of $p(\mathcal{X})$: We define a *separatrix configuration* as a union of $S(p(\mathcal{X}))$ plus one solution chosen from each canonical region. Two separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are said to be *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S(p(\mathcal{X}))$ into the trajectories of $S(p(\mathcal{Y}))$. The following result is due to Markus [61], Neumann [62] and Peixoto [63].

Theorem 1.1 *[10]*
The phase portraits in the Poincaré disc of the two compactified polynomial differential systems $p(\mathcal{X})$ and $p(\mathcal{Y})$ are topologically equivalent if and only if their separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are topologically equivalent.

This theorem implies that once separatrix configurations of a vector field in the Poincaré disc is determined, the global phase portrait of that vector field is obtained up to topological equivalence.

Similarly a parabolic sector, a hyperbolic sector and an elliptic sector are defined in the standard way.

Definition 1.8 *A sector which is topologically equivalent to the sector shown in Figure 1.5(a) is called a hyperbolic sector. A sector which is topologically equivalent to the sector shown in Figure 1.5(b) is called a parabolic sector. And a sector which is topologically equivalent to the sector shown in Figure 1.5(c) is called an elliptic sector.*

□

Figure 1.5: (a) A hyperbolic sector. (b) A parabolic sector. (c) An elliptic sector.

Section 1.6 The Averaging Theory Up to Seventh Order

To study the limit cycles that can bifurcate from the periodic orbits of centers, we can use one of the three following techniques; Abelian integral; averaging theory; or Melnikov function, which produce the same results in the plane.

The averaging theory is fundamental to our study, so we introduce the main result in order to apply it, see [43].

Theorem 1.2 *[43]*
Consider the differential system

$$\dot{x} = \sum_{i=1}^{7} \varepsilon^i F_i(t,x) + \varepsilon^{i+1} R(t,x,\varepsilon), \tag{1.6}$$

where $F_i : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ *for* $i = 1, \cdots, 7$, *and* $R : \mathbb{R} \times \mathcal{D} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ *are* $T-$*periodic in the first variable and continuous functions,* $\mathcal{D}$ *is an open interval of* $\mathbb{R}^n$, *and* ε *a small parameter. Assume that the following hypotheses* (i) *and* (ii) *hold:*

(a) $F_1(t,.) \in \mathcal{C}^6(\mathcal{D})$, $F_2(t,.) \in \mathcal{C}^5(\mathcal{D})$, $F_3(t,.) \in \mathcal{C}^4(\mathcal{D})$, $F_4(t,.) \in \mathcal{C}^3(\mathcal{D})$, $F_5(t,.) \in \mathcal{C}^2(\mathcal{D})$, $F_6(t,.) \in \mathcal{C}^1(\mathcal{D})$ *and* $F_7(t,.) \in \mathcal{C}^0(\mathcal{D})$ *for all* $t \in \mathbb{R}$, F_1, F_2, F_3, F_4, F_5, F_6, F_7, R, $\partial_x^{(j_1-1)} F_1$, $\partial_x^{(j_2-1)} F_2$, $\partial_x^{(j_3-1)} F_3$, $\partial_x^{(j_4-1)} F_4$, $\partial_x^{(j_5-1)} F_5$ *and* $\partial_x F_6$, *for* $j_1 = \{2, \cdots, 7\}$, $j_2 = \{2, \cdots, 6\}$, $j_3 = \{2, \cdots, 5\}$, $j_4 = \{2, \cdots, 4\}$ *and* $j_5 = \{2, 3\}$, *are locally Lipschitz with respect to* x, *and* R *is six times differentiable with respect to* ε.

For $i = 1, 2, \ldots, 7$ *we define the averaging function* $f_i : \mathcal{D} \to \mathbb{R}$ *of order* i *as*

$$f_i(z) = \frac{y_i(T,z)}{i!}, \tag{1.7}$$

where $y_i : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ *for* $i = 1, 2, \ldots, 7$ *are defined recurrently by the following integral* *equations*

$$y_i(t,z) = i!\int_0^t \Big(F_i(s,\varphi(s,z)) + \sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}.\partial^L F_{i-l}(s,\varphi(s,z))\prod_{j=1}^{l} y_j(s,z)^{b_j}\Big)ds.$$

Here $\partial^L G(\phi,z)$ denotes the derivative of order L of a function G with respect to the variable z, and S_l is the set of all $l-$uples of non-negative integers $(b_1, b_2, \dots, b_l)$ satisfying $b_1+2b_2+\cdots+lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$.

(b) For $V \in \mathcal{D}$ an open and bounded set, and for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ there exists a_ε such that $f_1(a_\varepsilon) + \varepsilon f_2(a_\varepsilon) + \varepsilon^2 f_3(a_\varepsilon) + \varepsilon^3 f_4(a_\varepsilon) + \varepsilon^4 f_5(a_\varepsilon) + \varepsilon^5 f_6(a_\varepsilon) + \varepsilon^6 f_7(a_\varepsilon) = 0$ and

$$d_B\Big(f_1(a_\varepsilon) + \varepsilon f_2(a_\varepsilon) + \varepsilon^2 f_3(a_\varepsilon) + \varepsilon^3 f_4(a_\varepsilon) + \varepsilon^4 f_5(a_\varepsilon) + \varepsilon^5 f_6(a_\varepsilon) + \varepsilon^6 f_7(a_\varepsilon), V, a_\varepsilon\Big) \neq 0. \quad (1.8)$$

Then for $|\varepsilon| > 0$ sufficiently small there exists a $T-$periodic solution $\varphi(\cdot,\varepsilon)$ of the system (1.6) such that $\varphi(0,\varepsilon) = a_\varepsilon$.

The expression (1.8) means that the Brouwer degree of the function $f_1 + \varepsilon f_2 + \varepsilon^2 f_3 + \varepsilon^3 f_4 + \varepsilon^4 f_5 + \varepsilon^5 f_6 + \varepsilon^6 f_7(a_\varepsilon) : V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $f_1+\varepsilon f_2+\varepsilon^2 f_3+\varepsilon^3 f_4+\varepsilon^4 f_5+\varepsilon^5 f_6 + \varepsilon^6 f_7(a_\varepsilon)$ at a_ε is not zero.

If f_1 is not identically zero, then the zeros of $f_1 + \varepsilon f_2 + \varepsilon^2 f_3 + \varepsilon^3 f_4 + \varepsilon^4 f_5 + \varepsilon^5 f_6 + \varepsilon^6 f_7$ are mainly the zeros of f_1 for ε sufficiently small. In this case, the previous result provides the averaging theory of first order.

If f_1 is identically zero and f_2 is not identically zero, then the zeros of $f_2 + \varepsilon f_3 + \varepsilon^2 f_4 + \varepsilon^3 f_5 + \varepsilon^4 f_6 + \varepsilon^5 f_7$ are mainly the zeros of F_2 for ε sufficiently small. In this case, the previous result provides the averaging theory of second order.

If f_1 and f_2 are both identically zero and f_3 is not identically zero, then the zeros of $f_3 + \varepsilon f_4 + \varepsilon^2 f_5 + \varepsilon^3 f_6 + \varepsilon^4 f_7$ are mainly the zeros of f_3 for ε sufficiently small. In this case, the previous result provides the averaging theory of third order.

If f_1, f_2 and f_3 are identically zero and f_4 is not identically zero, then the zeros of $f_4 + \varepsilon f_5 + \varepsilon^2 f_6 + \varepsilon^3 f_7$ are mainly the zeros of f_4 for ε sufficiently small. In this case, the previous result provides the averaging theory of fourth order.

If f_1, f_2 , f_3 and f_4 are identically zero and f_5 is not identically zero, then the zeros of

$f_5 + \varepsilon f_6 + \varepsilon^2 f_7$ are mainly the zeros of f_5 for ε sufficiently small. In this case, the previous result provides the averaging theory of fifth order.

If f_1, f_2 , f_3 and f_4 and f_5 are identically zero and f_6 is not identically zero, then the zeros of $f_6 + \varepsilon f_7$ are mainly the zeros of f_6 for ε sufficiently small. In this case, the previous result provides the averaging theory of sixth order.

If f_1, f_2 , f_3 , f_4, f_5 and f_6 are identically zero and f_7 is not identically zero, then the zeros of f_7 is mainly the zeros of f_7 for ε sufficiently small. In this case, the previous result provides the averaging theory of seventh order. For more details of this theory.

To know the number of zeros of a real polynomial, we are going to use the following Theorem.

Theorem 1.3 *[Descartes Theorem]*

Consider the real polynomial $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r}$ with $0 \le i_1 < i_2 < \cdots < i_r$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, \cdots, r\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m, then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $r-1$ positive real roots.

Exemple 1.2 *Consider the Van der Pol differential equation $\ddot{x} + x = \epsilon(1 - x^2)\dot{x}$ which can be written as the differential system*

$$\dot{x} = y, \quad \dot{y} = -x + \epsilon(1 - x^2)y. \tag{1.9}$$

In polar coordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$, this system becomes

$$\begin{aligned} \dot{r} &= \epsilon r(1 - r^2\cos^2\theta)\sin^2\theta, \\ \dot{\theta} &= -1 + \epsilon\cos\theta(1 - r^2\cos^2\theta)\sin\theta, \end{aligned}$$

or, equivalently,

$$\frac{dr}{d\theta} = -\epsilon r(1 - r^2 cos^2\theta)sin^2\theta + O(\epsilon^2).$$

In order to apply the averaging theory of first order, we take $x = r$, $t = \theta$, $T = 2\pi$ and

$$F(t,x) = -r(1 - r^2\cos^2\theta)\sin^2\theta,$$

we get that

$$f^0(r) = \frac{1}{2\pi}\int_0^{2\pi} r(1 - r^2\cos^2\theta)\sin^2\theta d\theta = \frac{1}{8}r(r^2 - 4).$$

The unique positive root of $f^0(r)$ is $r = 2$. Since $(df^0/dr)(2) = 1$, it follows that system (1.9) has, for $|\epsilon|$ sufficiently small, a limit cycle bifurcating from the periodic orbit of radius 2 of the unperturbed system (1.9) with $\epsilon = 0$. Moreover, since $(df^0/dr)(2) = 1 > 0$, this limit cycle is unstable.

Chapter 2

The Limit Cycles of Discontinuous Piecewise Linear Differential Systems Formed by Centers and Separated by Irreducible Cubic Curves

The objective of this chapter is to solve the 16th Hilbert problem extended to the limit cycles of the discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubics curves, when these limit cycles intersect the cubics in four points. Not that, if the limit cycles intersect the cubics in two points this problem has been solved in [11].

Using the first integrals of the linear centers, which are quadratic functions in the cartesian coordinates (x, y) of the plane, and the expression of the irreducible cubics, we shall obtain a set of polynomial equations whose solutions provide the limit cycles of the discontinuous piecewise linear differential systems that we consider.
Studying the solutions of these polynomial equations we obtain the exact maximum number of such limit cycles, which vary with the different irreducible cubics. Moreover, from such polynomial equations we can also describe the possible different configurations of limit cycles for each cubic.

Section 2.1 Classification of the Irreducible Cubics Curves

A Cubic Curves is the set of points $(x, y) \in \mathbb{R}^2$ satisfying $P(x, y) = 0$ for some polynomial $P(x, y)$ of degree three. This cubic is irreducible (respectively reducible) if the polynomial $P(x, y)$ is irreducible (respectively reducible) in the ring of all real polynomials in the variables x and y.

A point (x_0, y_0) of a cubic $P(x, y) = 0$ is singular if $P_x(x_0, y_0) = 0$ and $P_y(x_0, y_0) = 0$. A cubic curves is singular if it has some singular point, as usual here P_x and P_y denote the partial derivatives of P with respect to the variables x and y respectively.

A flex of an algebraic curve Γ is a point p of Γ such that Γ is nonsingular at p and the tangent at p intersects Γ at least three times.

The next theorem characterizes all the irreducible algebraic cubic curves.

Theorem 2.1 *The following statements classify all the irreducible algebraic cubic curves.*

(a) A cubic is nonsingular and irreducible if and only if it can be transformed with an affine transformation into one of the following two curves

$$c_1 = c_1(x, y) = y^2 - x(x^2 + bx + 1) = 0 \quad \textit{with } b \in (-2, 2), \textit{ or}$$
$$c_2 = c_2(x, y) = y^2 - x(x-1)(x-r) = 0 \quad \textit{with } r > 1.$$

(b) A cubic is singular and irreducible if and only if it can be transformed with an affine transformation into one of the following three curves:

$$c_3 = c_3(x, y) = y^2 - x^3 = 0, \qquad \textit{or}$$
$$c_4 = c_4(x, y) = y^2 - x^2(x-1) = 0, \quad \textit{or}$$
$$c_5 = c_5(x, y) = y^2 - x^2(x+1) = 0.$$

Statement (a) of Theorem 2.1 is proved in Theorem 8.3 of the book [15] under the additional assumption that the cubic has a flex, but in section 12 of that book it is shown that every nonsingular irreducible cubic curve has a flex. While statement (b) of Theorem 2.1 follows directly from Theorem 8.4 of [15].

Crossing Limit Cycles

For $k = 1, \cdots, 5$ let C_k be the class of planar discontinuous piecewise linear differential systems formed by centers and separated by the irreducible cubic curve $c_k(x,y) = 0$, or simply the irreducible cubic curve c_k.

Figures 2.1 and 2.2 show the different regions separated by the cubic curves c_i, with $i = 1 \dots 5$.

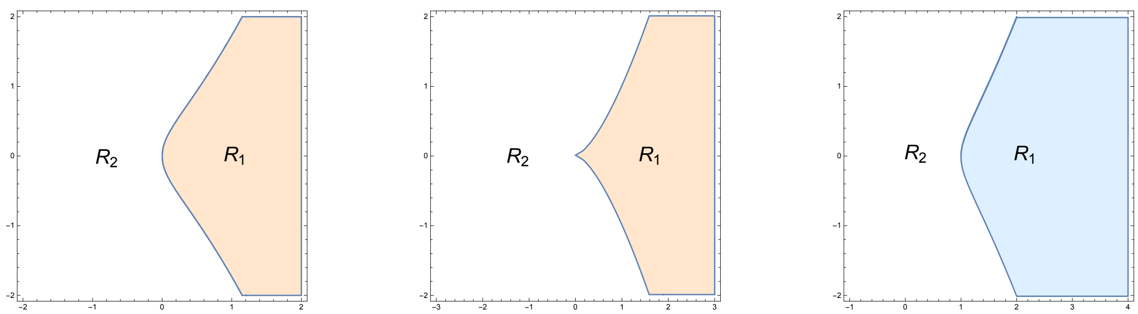


Figure 2.1: The two regions R_1 and R_2 of the plane separated by the curves c_1 on the left, c_3 on the middle and c_4 on the right.

Figure 2.2: The three regions R_1, R_2 and R_3 of the plane separated by the curves c_2 on the left and c_5 on the right.

The objective of this chapter is to give the maximum number $\mathcal{N}$ of crossing limit cycles for the planar discontinuous piecewise linear differential centers which intersect the irreducible cubic curves c_i, with $i = 1 \dots 5$, and have four points of intersection with the cubic of separation. Firstly, we give $\mathcal{N}$ when the crossing limit cycles intersects the cubic curves c_2 and c_5 in four points in three regions. Secondly, we give $\mathcal{N}$ when the crossing limit cycles intersect c_i with $i = 1 \dots 5$, in four points in two different zones.

The following lemma provides a normal form for an arbitrary linear differential system having a center.

LEMMA 2.1 *Through a linear change of variables and a rescaling of the independent variable every center in $\mathbb{R}^2$ can be written*

$$\dot{x} = -b_2 x - \frac{4b_2^2 + \omega^2}{4} y + d, \quad \dot{y} = x + b_2 y + c, \ \ with \ \ \omega > 0.$$

The first integral of this linear differential system is

$$H(x,y) = 4(x + b_2 y)^2 + 8(cx - dy) + y^2\omega^2.$$

For a proof of this lemma, see [54].

We denote by LC the limit cycles.

Section 2.2 LC in three regions intersecting c_i, $i = 2,5$ in four points

In this subsection we are interessted to provide lower bounds for the maximum number of crossing limit cycles of piecewise linear differential centers separated by the irreducible cubic curves c_2 and c_5 , having four points of intersection with the cubic of separation. We note that such piecewise differential systems are formed by three pieces in each one there is a linear differential center.

2.2.1 Statement of the first main result

We study the crossing limit cycles contained in three regions of the discontinuous piecewise linear centers in the classes C_2 or C_5 and having four points of intersection with the cubic of separation. The notation (C_k^i) indicate the configuration number i of crossing limit cycle for the class C_k, where $i \in \mathbb{N}^*$ and $k = 1,\ldots,5$.

Then our first main result is the following.

Theorem 2.2 *The following statements hold.*

(a) There are systems in C_2 and in C_5 exhibiting exactly one crossing limit cycle which intersects c_2 or c_5 in four points. The class C_2 has one possible configuration see Figure 2.3, while the class C_5 has two possible configurations, see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure 2.4;

(b) there are systems in C_2 and in C_5 exhibiting exactly two crossing limit cycles which intersect c_2 or c_5 in four points. The class C_2 has one possible configuration see Figure 2.5, while the class C_5 has three possible configurations, see $(\mathcal{C}_5^1)$ of Figure 2.5, and $(\mathcal{C}_5^2)$ and $(\mathcal{C}_5^3)$ of Figure 2.6;

(c) there are systems in C_2 and in C_5 exhibiting exactly three crossing limit cycles which intersect c_2 or c_5 in four points. The class C_2 has one possible configuration see Figure 2.7, while the class C_5 has four possible configurations, see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure 2.8, and $(\mathcal{C}_5^3)$ and $(\mathcal{C}_5^4)$ of Figure 2.9;

(d) there are systems in C_2 and in C_5 exhibiting exactly four crossing limit cycles which intersect c_2 or c_5 in four points. The class C_2 has one possible configuration $(\mathcal{C}_2)$ of Figure 2.10, and we give four configurations of the class C_5, see (C_5^1) of Figure 2.10, (C_5^2) and (C_5^3) of Figure 2.11, and (C_5^4) of Figure 2.12.

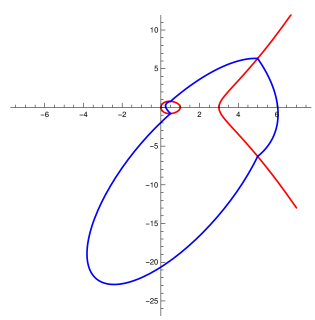
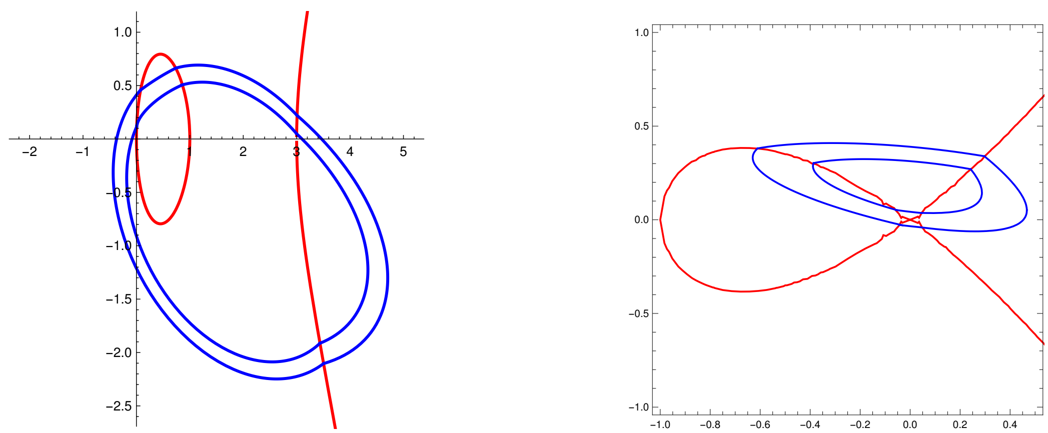


Figure 2.3: The unique limit cycle of the discontinuous piecewise linear differential system (2.1) contained in three zones.

$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

Figure 2.4: The unique limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (2.3)–(2.4), and $(\mathcal{C}_5^2)$ for (2.6)–(2.7) contained in three zones.

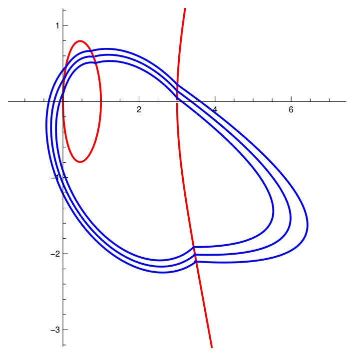


$(\mathcal{C}_2)$ $(\mathcal{C}_5^1)$

Figure 2.5: The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2)$ for (2.8) and $(\mathcal{C}_5^1)$ for (2.10) contained in three zones.

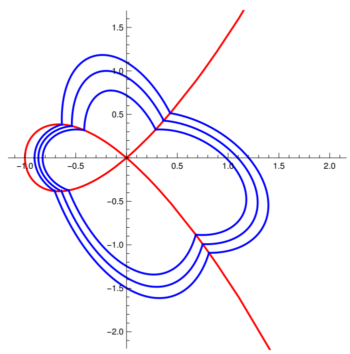
$(\mathcal{C}_5^2)$ $(\mathcal{C}_5^3)$

Figure 2.6: The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^2)$ for (2.11) and $(\mathcal{C}_5^3)$ for (2.12) contained in three zones.

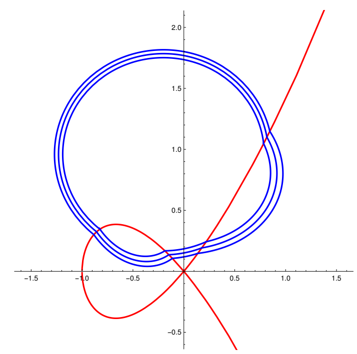


$(\mathcal{C}_2)$

Figure 2.7: The three limit cycles of the discontinuous piecewise linear differential system (2.14) contained in three zones.

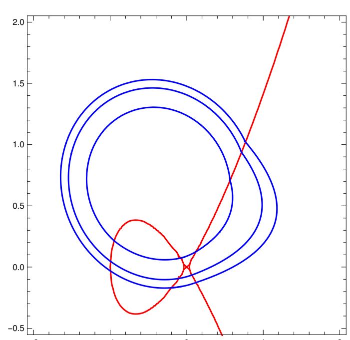


$(\mathcal{C}_5^1)$

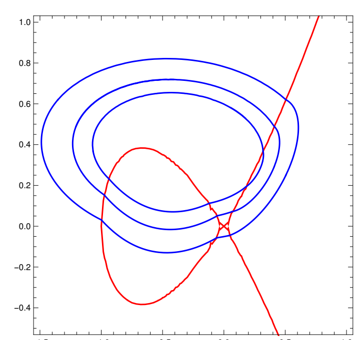


$(\mathcal{C}_5^2)$

Figure 2.8: The three limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (2.15) and $(\mathcal{C}_5^2)$ for (2.16) contained in three zones.



$(\mathcal{C}_5^3)$



$(\mathcal{C}_5^4)$

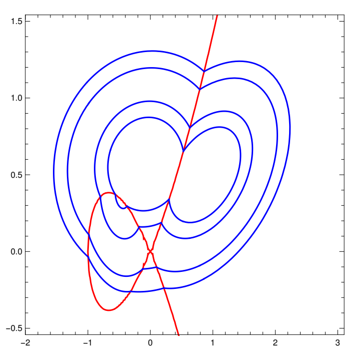
Figure 2.9: The three limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^3)$ for (2.18), and $(\mathcal{C}_5^4)$ for (2.20) contained in three zones.

$(\mathcal{C}_2)$ $(\mathcal{C}_5^1)$

Figure 2.10: The four limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2)$ for (2.21), and $(\mathcal{C}_5^1)$ for (2.22) contained in three zones.

$(\mathcal{C}_5^2)$ $(\mathcal{C}_5^3)$

Figure 2.11: The four limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^2)$ for (2.23), and $(\mathcal{C}_5^3)$ for (2.24) contained in three zones.



$(\mathcal{C}_5^4)$

Figure 2.12: The four limit cycles of the discontinuous piecewise linear differential system (2.25) contained in three zones.

2.2.2 Proof of the first main result

Proof. **(Proof of statement (a) of Theorem 2.2.)** *First we prove the statement for the class C_2. We consider the linear differential centers*

$$
\begin{aligned}
&\dot{x} = -\frac{x}{4} - \frac{5y}{16} + \frac{5}{4}, \quad \dot{y} = x + \frac{y}{4} + \frac{1}{2}, && in\ R_1,\\
&\dot{x} = \frac{1}{56}(14x - 7y - 79), \quad \dot{y} = x - \frac{y}{4} - \frac{211}{64}, && in\ R_2,\\
&\dot{x} = \dot{x} = -x - \frac{5y}{4} + \frac{1}{2}, \quad \dot{y} = x + y - 2, && in\ R_3,
\end{aligned}
\tag{2.1}
$$

with their corresponding first integrals

$$
\begin{aligned}
H_1(x,y) &= 4\left(x + \frac{y}{4}\right)^2 + 8\left(\frac{x}{2} - \frac{5y}{4}\right) + y^2,\\
H_2(x,y) &= 4x^2 + x\left(-2y - \frac{211}{8}\right) + \frac{1}{14}y(7y + 158),\\
H_3(x,y) &= 4(x+y)^2 + 8\left(-2x - \frac{y}{2}\right) + y^2,
\end{aligned}
$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.1) has exactly one crossing limit cycle, because the system of equations

$$
\begin{aligned}
&H_1(\alpha,\beta) - H_1(\gamma,\delta) = 0,\\
&H_2(\alpha,\beta) - H_2(f,g) = 0,\\
&H_2(\gamma,\delta) - H_2(h,k) = 0,\\
&H_3(f,g) - H_3(h,k) = 0,\\
&\beta^2 - \alpha(\alpha-1)(\alpha-3) = 0, \quad \delta^2 - \gamma(\gamma-1)(\gamma-3) = 0,\\
&g^2 - f(f-1)(f-3) = 0, \quad k^2 - h(h-1)(h-3) = 0,
\end{aligned}
\tag{2.2}
$$

has the unique real solution $(\alpha,\beta,\gamma,\delta,f,g,h,k) = (5, -2\sqrt{10}, 5, 2\sqrt{10}, 1/2, -\sqrt{5}/2\sqrt{2}, 1/2,$ $\sqrt{5}/2\sqrt{2})$, and this limit cycle is shown in Figure 2.3. This completes the proof of statement (a) for the class C_2.

Now we prove the existence of two different configurations of one crossing limit cycle for the *class C_5. For the first possible configuration we consider the linear differential centers*

$$\dot{x} = -\frac{x}{8} - \frac{17y}{64} + \frac{3}{8}, \quad \dot{y} = x + \frac{y}{8} - 1, \quad in\ R_1,$$
$$\dot{x} = -x - 10y - \frac{1}{2}, \quad \dot{y} = x + y - 2, \qquad in\ R_2, \tag{2.3}$$

with their corresponding first integrals

$$H_1(x,y) = 4\left(x + \frac{y}{8}\right)^2 + 8\left(-x - \frac{3y}{8}\right) + y^2,$$
$$H_2(x,y) = 4(x+y)^2 + 8\left(\frac{5}{12}\left(-3\sqrt{2} + \frac{81}{50} - \frac{288}{25\sqrt{5}}\right)x - y\right) + 16y^2,$$

respectively.

In the region R_3 we consider the linear center

$$\dot{x} = \frac{x}{4} - \frac{101y}{16} + \frac{3800\sqrt{2} + 3584\sqrt{5} + 2203113}{320\left(95\sqrt{2} + 56\sqrt{5} - 4380\right)},$$
$$\dot{y} = x - \frac{y}{4} + \frac{-448604\sqrt{2} - 315662\sqrt{5} + 48\sqrt{10} + 22560621}{160\left(95\sqrt{2} + 56\sqrt{5} - 4380\right)}, \tag{2.4}$$

with its first integral

$$\begin{aligned} H_3(x,y) = \quad & \frac{1}{40(95\sqrt{2} + 56\sqrt{5} - 4380)} \\ & (160(95\sqrt{2} + 56\sqrt{5} - 4380)x^2 + x(-80(95\sqrt{2} + 56\sqrt{5} - 4380)y \\ & -897208\sqrt{2} - 631324\sqrt{5} + 96\sqrt{10} + 45121242) + y(1010(95\sqrt{2} \\ & +56\sqrt{5} - 4380)y - 3800\sqrt{2} - 3584\sqrt{5} - 2203113). \end{aligned}$$

The discontinuous piecewise linear differential system (2.3)–(2.4) has exactly one crossing limit cycle, because the system of equations

$$\begin{aligned} & H_1(\alpha,\beta) - H_1(\gamma,\delta) = 0, \\ & H_2(\alpha,\beta) - H_2(f,g) = 0, \\ & H_2(\gamma,\delta) - H_2(h,k) = 0, \\ & H_3(f,g) - H_3(h,k) = 0, \\ & \beta^2 - \alpha^2(\alpha+1) = 0, \quad \delta^2 - \gamma^2(\gamma+1) = 0, \quad g^2 - f^2(f+1) = 0, \quad k^2 - h^2(h+1) = 0, \end{aligned} \tag{2.5}$$

has the unique real solution $(\alpha,\beta,\gamma,\delta,f,g,h,k) = (3,-6,3,6,-1/2,-1/(2\sqrt{2}),-4/5,4/(5\sqrt{5}))$, *see* $(\mathcal{C}_5^1)$ *of Figure* 2.4.

For the second configuration, we consider the linear differential centers

$$\begin{aligned} \dot{x} &= \frac{x}{7} - \frac{785y}{49} + \frac{70\sqrt{50-8\sqrt{6}}+40083}{490\left(\sqrt{2}-2\sqrt{3}\right)}, \quad \dot{y} = x - \frac{y}{7} + \frac{1}{5}, \quad in\ R_1, \\ \dot{x} &= \frac{x}{8} - \frac{577y}{64} + \frac{-10025\sqrt{3}-29787}{9216}, \quad \dot{y} = x - \frac{y}{8} - \frac{1}{3}, \quad in\ R_2, \end{aligned} \tag{2.6}$$

with their corresponding first integrals

$$\begin{aligned} H_1(x,y) &= 4\left(x-\frac{y}{7}\right)^2 + 8\left(\frac{x}{5} + \frac{\left(70\sqrt{2}-280\sqrt{3}-40083\right)y}{490\left(\sqrt{2}-2\sqrt{3}\right)}\right) + 64y^2, \\ H_2(x,y) &= 4\left(x-\frac{y}{8}\right)^2 + 8\left(\frac{\left(10025\sqrt{3}+29787\right)y}{9216} - \frac{x}{3}\right) + 36y^2, \end{aligned}$$

respectively. In the region R_3 *we consider the linear differential center*

$$\begin{aligned} \dot{x} &= -\frac{3x}{7} - \frac{373y}{1764} + \frac{-23556784\sqrt{2}-26814823\sqrt{3}+14402472\sqrt{6}+83393133}{80960544}, \\ \dot{y} &= x + \frac{3y}{7} + \frac{-4861464\sqrt{2}+3092483\sqrt{3}+1548288\sqrt{6}-1763763}{56448\left(88\sqrt{2}-91\sqrt{3}+15\right)}, \end{aligned} \tag{2.7}$$

with its first integral

$$H_3(x,y) = 4\left(x+\frac{3y}{7}\right)^2 + \frac{y^2}{9} + \frac{A}{7056\left(88\sqrt{2}-91\sqrt{3}+15\right)\left(16\sqrt{3}-3\right)}.$$

Where

$$\begin{aligned} A = \ & 88902216\sqrt{2}-37497657\sqrt{3}-82428288\sqrt{6}+153730473)x - 8(7597532\sqrt{3} \\ & +266112\sqrt{6(259-32\sqrt{3})}-23982207)y. \end{aligned}$$

The real solution of the system of equations (2.5) *with the values of* $H_i(x,y)$ *with* $i = 1,2,3$ *given for this second configuration is* $(\alpha,\beta,\gamma,\delta,f,g,h,k) = (1,-\sqrt{2},2,-2\sqrt{3},-1/4,-\sqrt{3}/8,-3/4,-3/8)$, *then the discontinuous piecewise linear differential system* (2.6)–(2.7) *has one crossing limit cycle* $(\mathcal{C}_5^2)$ *of Figure* 2.4. *This completes the proof of statement (a) for the class* C_5. ■

Proof. **(Proof of statement (b) of Theorem 2.2.)** *First we prove the statement for the class*

C_2. We consider the linear differential centers

$$\begin{aligned}
&\dot{x} = -x - 5y - 1.8035.., \quad \dot{y} = x + y - 0.664282.., && in\ R_1,\\
&\dot{x} = -\frac{x}{2} - \frac{10y}{4} - 1, \quad \dot{y} = x + \frac{y}{2} - \frac{3}{2}, && in\ R_2,\\
&\dot{x} = -2x - 8y + 0.837903.., \quad \dot{y} = x - 2y - 0.169396.., && in\ R_3,
\end{aligned} \tag{2.8}$$

with their corresponding first integrals

$$\begin{aligned}
&H_1(x,y) = 4(x+y)^2 + 8(1.8035..y - 0.664282..x) + 16y^2,\\
&H_2(x,y) = 4\left(x + \frac{y}{2}\right)^2 + 8\left(y - \frac{3x}{2}\right) + 9y^2,\\
&H_3(x,y) = 4(x-2y)^2 + 8(-0.169396..x - 0.837903..y) + 16y^2,
\end{aligned} \tag{2.9}$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.8) has exactly two crossing limit cycles, because the system of equations (2.2) has the two real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (3.00039.., 0.0485802.., 3.43695.., -1.91305.., 0.860569.., 0.506666.., 0.00442503.., 0.114878..)$ *and* $(\alpha_2,\ \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (3.00805.., 0.220494.., 3.50419.., -2.1034.., 0.735716.., 0.663523.., 0.0779996.., 0.458408..)$, *see* $(\mathcal{C}_2)$ *of Figure 2.5. This completes the proof of statement (b) for the class* C_2.

Now we prove the existence of three different configurations of two crossing limit cycles for the class C_5. *To obtain the first configuration we consider the linear differential centers*

$$\begin{aligned}
&\dot{x} = -1.77296..x - 3.39338..y + 1, \quad \dot{y} = x + 1.77296..y - 0.15026, && in\ R_1,\\
&\dot{x} = -1.77845..x - 9.41287..y + 2, \quad \dot{y} = x + 1.77845..y - 0.640432, && in\ R_2,\\
&\dot{x} = -\frac{3x}{2} - \frac{45y}{4} + 2, \quad \dot{y} = x + \frac{3y}{2} - 0.3, && in\ R_3,
\end{aligned} \tag{2.10}$$

with their corresponding first integrals

$$\begin{aligned}
&H_1(x,y) = 4x^2 + 14.1837..xy - 1.20208..x + 13.5735..y^2 - 8y,\\
&H_2(x,y) = 4x^2 + 14.2276..xy - 5.12346..x + 37.6515..y^2 - 16y,\\
&H_3(x,y) = 4x^2 + 12xy - \frac{12x}{5} + 45y^2 - 16y,
\end{aligned}$$

respectively. The system of equations (2.5) has the two real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) =$

$(0.297668..,0.33909..,0.0389395..,-0.0396904..,-0.610209..,0.380973..,-0.0300808..,$ $-0.0296249..)$ *and* $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (0.242325..,0.270095..,0.0366105..,0.0372747..,$ $-0.386234..,0.302588..,-0.0530128..,0.0515885..)$. *Then the discontinuous piecewise linear differential system formed by the linear differential centers* (2.10) *has exactly two crossing limit cycles shown in* $(\mathcal{C}_5^1)$ *of Figure* 2.5.

For the second configuration we consider the linear differential centers

$$\begin{aligned}
&\dot{x} = -\frac{6x}{5} - \frac{369y}{100} + 0.152456, \quad \dot{y} = x + \frac{6y}{5} - 0.365572, \quad in\ R_1,\\
&\dot{x} = -y - 0.251587, \quad \dot{y} = x + 1.93017, \quad in\ R_2,\\
&\dot{x} = -\frac{x}{5} - \frac{29y}{100}, \quad \dot{y} = x + \frac{y}{5}, \quad in\ R_3,
\end{aligned} \tag{2.11}$$

with their corresponding first integrals

$$\begin{aligned}
H_1(x,y) &= 4\left(x + \frac{6y}{5}\right)^2 + 8(-0.365572x - 0.152456y) + 9y^2,\\
H_2(x,y) &= 4x^2 + 8(1.93017x + 0.251587y) + 4y^2,\\
H_3(x,y) &= 4\left(x + \frac{y}{5}\right)^2 + y^2,
\end{aligned}$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.11) *has exactly two crossing limit cycles, due to the fact that the system of equations* (2.5) *has the two real solutions* $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (0.286549..,0.325022..,0.400999..,$ $-0.474638..,-0.416749..,0.318275..,-0.313035..,-0.259454..)$ *and* $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,$ $k_2) = (0.484398..,0.590171..,0.681486..,-0.883698..,-0.719407..,0.381077..,-0.569542..,$ $-0.373673..)$, *see* $(\mathcal{C}_5^2)$ *of Figure* 2.6.

Finally to obtain the third configuration we consider the linear differential centers

$$\begin{aligned}
&\dot{x} = -2x - \frac{25y}{4} + 6.3854.., \quad \dot{y} = x + 2y + 0.134162, \quad in\ R_1,\\
&\dot{x} = -\frac{x}{2} - \frac{5y}{2} + 0.902952.., \quad \dot{y} = x + \frac{y}{2} + 0.234228, \quad in\ R_2,\\
&\dot{x} = \frac{3}{2} - \frac{25y}{16}, \quad \dot{y} = x + \frac{1}{5}, \quad in\ R_3,
\end{aligned} \tag{2.12}$$

and their corresponding first integrals

$$
\begin{aligned}
H_1(x,y) &= 4(x+2y)^2 + 8(0.134162..x - 6.3854..y) + 9y^2,\\
H_2(x,y) &= 4\left(x+\frac{y}{2}\right)^2 + 8(0.234228..x - 0.902952..y) + 9y^2,\\
H_3(x,y) &= 4x^2 + 8\left(\frac{x}{5}+\frac{3}{2}\right) + \frac{25y^2}{4},
\end{aligned}
$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.12) *has exactly two crossing limit cycles, due to fact that the system of equations* (2.5) *has the two real solutions* $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (0.844854..,1.14753..,0.137908..,0.14711..,$ $-0.884123..,0.300962..,-0.112734..,0.106189..)$ *and* $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (0.783948..,$ $1.04708..,0.219822..,0.242783..,-0.824415..,0.345453..,-0.186379..,0.168115..)$, *see* $(\mathcal{C}_5^3)$ *of Figure 2.6. This completes the proof of statement (b) for the class* C_5. ■

Proof. **(Proof of statement (*c*) of Theorem 2.2.)** *First we prove the statement for the class* C_2. *We consider the linear differential centers*

$$
\begin{aligned}
&\dot{x} = -3.95435..x - 19.6368..y - 6.39482.., \quad \dot{y} = x + 3.95435..y + 4.13635.., \quad in\ R_1,\\
&\dot{x} = -\frac{x}{2} - \frac{5y}{2} - 1, \quad \dot{y} = x + \frac{y}{2} - \frac{3}{2}, \quad in\ R_2,
\end{aligned}
\tag{2.13}
$$

$$
\dot{x} = -0.24134..x - 1.00852..y - 0.35835.., \quad \dot{y} = x + 0.24134..y \quad in\ R_3,
\tag{2.14}
$$

with the corresponding first integrals

$$
\begin{aligned}
H_1(x,y) &= 4(x+3.95435..y)^2 + 8(4.13635..x + 6.39482..y) + 16y^2,\\
H_2(x,y) &= 4\left(x+\frac{y}{2}\right)^2 + 8\left(y - \frac{3x}{2}\right) + 9y^2,\\
H_3(x,y) &= 4(x+0.241343..y)^2 + 8(0.358353..y - 0.869754..x) + 4y^2,
\end{aligned}
$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.14) *has exactly three crossing limit cycles, because the system of equations* (2.2) *has the three real solutions* $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (3.00039..,0.0485802..,3.43695..,-1.91305..,$

$0.860569.., 0.506666.., 0.00442503.., 0.114878..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (3.00318.., 0.138419.., 3.47181.., -2.01219.., 0.803882.., 0.588414.., 0.0319264.., 0.302877..)$ *and* $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (3.00805.., 0.220494.., 3.50419.., -2.1034.., 0.735716.., 0.663523.., 0.0779996.., 0.458408..)$, *see* $(\mathcal{C}_2)$ *of Figure 2.7. This complets the proof of statement* (c) *for the* C_2.

Now we prove the existence of four different configurations of three crossing limit cycles for the class C_5. *For the first configuration we consider the linear differential centers*

$$
\begin{aligned}
&\dot{x} = -0.345578..x - 1.11942..y - 0.128163.., \quad \dot{y} = x + 0.345578..y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -0.440337, \quad in\ R_1, \\
&\dot{x} = -0.0923038..x - 1.00852..y - 0.0805185.., \quad \dot{y} = x + 0.0923038..y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +0.46371, \quad in\ R_2, \\
&\dot{x} = -\frac{x}{5} - \frac{29y}{100}, \quad \dot{y} = x + \frac{y}{5}, \qquad\qquad in\ R_3,
\end{aligned}
\tag{2.15}
$$

with their corresponding first integrals

$$
\begin{aligned}
H_1(x,y) &= 4(x + 0.345578..y)^2 + 8(0.128163..y - 0.440337..x) + 4y^2, \\
H_2(x,y) &= 4(x + 0.0923038..y)^2 + 8(0.46371..x + 0.0805185..y) + 4y^2, \\
H_3(x,y) &= 4\left(x + \frac{y}{5}\right)^2 + y^2,
\end{aligned}
$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.15) *has exactly three crossing limit cycles, because the system of equations* (2.5) *has the three real solutions* $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.286549.., 0.325022.., 0.681486.., -0.883698.., -0.416749.., 0.318275.., -0.569542.., -0.373673..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.366248.., 0.428095.., 0.749792.., -0.991822.., -0.538507.., 0.365825.., -0.639545.., -0.383969..)$ *and* $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.429988.., 0.514188.., 0.81153.., -1.09226.., -0.636378.., 0.383743.., -0.706638.., -0.382736..)$, *see* $(\mathcal{C}_5^1)$ *of Figure 2.8.*

For the second configuration. We consider the linear differential centers

$$\begin{aligned}
&\dot{x} = -0.121241..x - 2.2647..y + 1.93257.., \quad \dot{y} = x + 0.121241..y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +0.00611219, \quad in\ R_1, \\
&\dot{x} = -0.217737..x - 1.04741..y + 0.515416.., \quad \dot{y} = x + 0.217737y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +0.350398, \quad in\ R_2, \\
&\dot{x} = \frac{3}{2} - \frac{25y}{16}, \quad \dot{y} = x + \frac{1}{5}, \qquad\qquad\qquad in\ R_3,
\end{aligned} \tag{2.16}$$

with their corresponding first integrals

$$\begin{aligned}
H_1(x,y) &= 4(x+2y)^2 + 8(0.134162..x - 6.3854..y) + 9y^2, \\
H_2(x,y) &= 4(x+0.217737..y)^2 + 8(0.350398..x - 0.515416..y) + 4y^2, \\
H_3(x,y) &= 4x^2 + 8\left(\frac{x}{5} - \frac{3y}{2}\right) + \frac{25y^2}{4},
\end{aligned}$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.16) has exactly three crossing limit cycles, because the system of equations (2.5) has the three real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.844854.., 1.14753.., 0.137908.., 0.14711.., -0.884123.., 0.300962.., -0.112734.., 0.106189..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.816222.., 1.1.., 0.177162.., 0.192216.., -0.856934.., 0.324128.., -0.147429.., 0.136128..)$, *and* $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.783948.., 1.04708.., 0.219822.., 0.242783.., -0.824415.., 0.345453.., -0.186379.., 0.168115..)$, *see* $(\mathcal{C}_5^2)$ *of Figure 2.8.*

For the third configuration we consider the linear differential centers

$$\begin{aligned}
&\dot{x} = -6.11391..x - 39.6298..y + 26.1607.., \quad \dot{y} = x + 6.11391..y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +6.97061, \quad in\ R_1,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
&\dot{x} = 0.0191156..x - 1.56287..y + 1.64182.., \quad \dot{y} = x - 0.0191156..y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +0.287842, \quad in\ R_2, \\
&\dot{x} = -\frac{x}{10} - \frac{113y}{50} + \frac{3}{2}, \quad \dot{y} = x + \frac{y}{10} + \frac{3}{10}, \qquad in\ R_3,
\end{aligned} \tag{2.18}$$

with their corresponding first integrals

$$
\begin{aligned}
H_1(x,y) &= 4x^2 + 14.1837..xy - 1.20208..x + 13.5735..y^2 - 8y,\\
H_2(x,y) &= 4x^2 + 14.2276..xy - 5.12346..x + 37.6515..y^2 - 16y,\\
H_3(x,y) &= 4x^2 + \frac{4xy}{5} + \frac{12x}{5} + \frac{226y^2}{25} - 12y,
\end{aligned}
$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.18) has exactly three crossing limit cycles, because the system of equations (2.5) has the three real solutions $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (0.566406..,0.708892..,0.117204..,0.123882..,$ $-0.948744..,0.214793..,-0.0766687..,0.0736711..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (0.715789..,$ $0.937598..,0.0641072..,-0.0661301..,-0.997735..,0.0474848..,-0.097471..,-0.0925994..)$ *and* $(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = (0.767868..,1.02097..,0.11983..,-0.12681..,-0.999038..,$ $-0.0309942..,-0.187296..,-0.168847)$, *see* $(\mathcal{C}_5^3)$ *of Figure 2.9.*

Finally we give an example for the fourth configuration. In the example we consider the linear differential center in systems

$$
\begin{aligned}
&\dot{x} = 0.69208..x - 1.47897..y + 0.293478.., \quad \dot{y} = x - 0.69208..y\\
&\qquad +0.00888246, \quad in\ R_1,\\
&\dot{x} = 0.242453..x - 0.308783..y + 1.11385.., \quad \dot{y} = x - 0.242453..y\\
&\qquad +0.433519, \quad in\ R_2,
\end{aligned}
\tag{2.19}
$$

$$
\dot{x} = -\frac{x}{5} - \frac{629y}{100} + \frac{23}{10}, \quad \dot{y} = x + \frac{y}{5} + \frac{3}{10}, \quad in\ R_3,
\tag{2.20}
$$

with their corresponding first integrals

$$
\begin{aligned}
H_1(x,y) &= 4x^2 - 5.53664..xy + 0.0710597..x + 5.9159..y^2 - 2.34783..y,\\
H_2(x,y) &= 4x^2 - 1.93962..xy + 3.46816..x + 1.23513..y^2 - 8.91079..y,\\
H_3(x,y) &= 4x^2 + \frac{8xy}{5} + \frac{12x}{5} + \frac{629y^2}{25} - 18.4y,
\end{aligned}
$$

respectively.

The system of equations (2.5) has the three real solutions $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) =$ $(0.319705..,0.367272..,0.17682..,0.191816..,-0.938407..,0.232893..,-0.119654..,$

0.112267..), $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (0.411196..,0.488475..,0.0696268..,0.07201..,$ $-0.975404..,0.152973..,-0.0528588..,0.0514428..)$ *and* $(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) =$ $(0.505367..,0.620052..,0.0444858..,-0.0454646..,-0.999041..,0.0309382..,-0.0582745..,$ $-0.0565511..)$. *Hence the discontinuous piecewise linear differential system formed by the linear differential centers* (2.20) *has exactly three crossing limit cycles shown in* $(\mathcal{C}_5^4)$ *of Figure* 2.9. *This completes the proof of statement (c) for the class* C_5. ■

Proof. (Proof of statement (*d*) of Theorem 2.2.) *First we prove the statement for the class* C_2. *We consider the linear differential centers*

$$
\begin{aligned}
&\dot{x} = -0.241343..x - 1.00852..y - 0.358353.., \quad \dot{y} = x + 0.241343..y, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -0.869754, \, in\ R_1, \\
&\dot{x} = -\frac{x}{2} - \frac{5y}{2} - 1, \quad \dot{y} = x + \frac{y}{2} - \frac{3}{2}, \qquad\qquad in\ R_2, \\
&\dot{x} = -\frac{x}{5} - \frac{629y}{100} + \frac{23}{10}, \quad \dot{y} = x + \frac{y}{5} + \frac{3}{10}, \qquad in\ R_3,
\end{aligned}
\tag{2.21}
$$

and the corresponding first integrals

$$
\begin{aligned}
H_1(x,y) &= 4(x + 3.95435..y)^2 + 8(4.13635..x - 6.39482..y) + 16y^2, \\
H_2(x,y) &= 4\left(x + \frac{y}{2}\right)^2 + 8\left(y - \frac{3x}{2}\right) + 9y^2, \\
H_3(x,y) &= 4(x + 0.241343..y)^2 + 8(0.358353..y - 0.869754..x) + 4y^2,
\end{aligned}
$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.21) *has exactly four crossing limit cycles, because the system of equations* (2.2) *has the four real solutions* $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (3.00039..,0.04858..,3.4369..,-1.91305..,0.86056..,$ $0.506666..,0.00442503..,0.114878..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (3.00318..,0.138419..,$ $3.47181..,-2.01219..,0.803882..,0.588414..,0.0319264..,0.302877..)$, $(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,$ $h_3,k_3) = (3.00805..,0.22049..,3.50419..,-2.1034..,0.73571..,0.66352..,0.07799..,0.4584..)$ *and* $(\alpha_4,\beta_4,\gamma_4,\delta_4,f_4,g_4,h_4,k_4) = (3.0181..,0.33254..,3.5491..,-2.2288..,0.5933..,0.76203..,$ $0.192725..,0.66088..)$, *see* $(\mathcal{C}_2)$ *of Figure* 2.10.

Now we give four different configurations of four limit cycles for the class C_5. *To obtain the* *first configuration we consider the linear differential centers*

$$
\begin{aligned}
&\dot{x} = 0.897851..x - 2.12174..y - 0.62272.., \quad \dot{y} = x - 0.897851..y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad -0.620434, \quad in\ R_1, \\
&\dot{x} = -0.225709..x - 1.26066..y - 0.265909.., \quad \dot{y} = x + 0.225709..y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad +0.334228, \quad in\ R_2, \\
&\dot{x} = \frac{x}{10} - \frac{17y}{450}, \quad \dot{y} = x - \frac{y}{10}, \qquad\qquad in\ R_3,
\end{aligned}
\tag{2.22}
$$

and their corresponding first integrals

$$
\begin{aligned}
H_1(x,y) &= 4(x - 0.897851..y)^2 + 8(0.62272..y - 0.620434..x) + 5.2624..y^2, \\
H_2(x,y) &= 4(x + 0.225709..y)^2 + 8(0.334228..x + 0.265909..y) + 4.83887..y^2, \\
H_3(x,y) &= 4\left(x - \frac{y}{10}\right)^2 + \frac{y^2}{9},
\end{aligned}
$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.22) *has exactly four crossing limit cycles, because the system of equations* (2.5) *has the four real solutions* $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.2756.., 0.31138.., 0.4902.., 0.200142.., -0.593285.., -0.378363.., -0.598415.., -0.22775..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.391233.., 0.461461.., 0.535875.., -0.664112.., -0.324107.., 0.266457.., -0.647454.., -0.38443..)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.480359.., 0.584453.., 0.57772.., -0.72566.., -0.399001.., 0.30932.., -0.6967.., -0.383691..)$ *and* $(\alpha_4, \beta_4, \gamma_4, \delta_4, f_4, g_4, h_4, k_4) = (0.55579.., 0.69324.., 0.61654.., -0.78389.., -0.46287.., 0.339233.., -0.742.., -0.376889..)$, *see* $(\mathcal{C}_5^1)$ *of Figure 2.10.*

Now we give an example for the second configuration, and we consider the linear differential centers

$$
\begin{aligned}
&\dot{x} = 0.679261..x - 20.2696..y + 7.31483.., \quad \dot{y} = x - 0.679261..y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad -0.688849, \quad in\ R_1, \\
&\dot{x} = -0.0259413..x - 3.37607..y + 1.05365.., \quad \dot{y} = x + 0.0259413..y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad +0.350162, \quad in\ R_2, \\
&\dot{x} = \frac{2x}{5} - \frac{269569y}{40000} + \frac{14}{5}, \quad \dot{y} = x - \frac{2y}{5} + \frac{17}{100}, \qquad in\ R_3,
\end{aligned}
\tag{2.23}
$$

and their corresponding first integrals

$$H_1(x,y) = 4x^2 - 5.43409..xy - 5.5108..x + 81.0784..y^2 - 58.5187..y,$$
$$H_2(x,y) = 4x^2 + 0.20753..xy + 2.8013..x + 13.5043..y^2 - 8.4292..y,$$
$$H_3(x,y) = 4x^2 - \frac{16xy}{5} + \frac{34x}{25} + \frac{269569y^2}{10000} - \frac{112y}{5},$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.23) *has exactly four crossing limit cycles, because the system of equations* (2.5) *has the four real solutions* $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (0.448209..,0.539381..,0.227449..,0.251992..,$ $-0.519234..,0.360023..,-0.268115..,0.229373..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (0.53821..,$ $0.667513..,0.117618..,0.124343..,-0.7962..,0.359438..,-0.116994..,0.109938..)$, $(\alpha_3,\beta_3,$ $\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = (0.595573..,0.752304..,0.0393207..,0.0400863..,-0.938909..,$ $0.232066..,-0.0361079..,0.0354501..)$, *and* $(\alpha_4,\beta_4,\gamma_4,\delta_4,f_4,g_4,h_4,k_4) = (0.741939..,$ $0.979229..,0.187029..,-0.20377..,-0.989482..,-0.101478..,-0.264651..,-0.226945..)$, *see* $(\mathcal{C}_5^2)$ *of Figure 2.11.*

For the third configuration we consider the linear differential centers

$$\begin{aligned} &\dot{x} = 0.216672..x - 0.113954..y + 0.0710259.., && \dot{y} = x - 0.216672..y \\ & && \quad -0.125477, \quad in\ R_1, \\ &\dot{x} = 0.469674..x - 0.937777..y + 0.544803.., && \dot{y} = x - 0.469674..y \\ & && \quad +0.465659, \quad in\ R_2, \\ &\dot{x} = \frac{9x}{20} - \frac{1681y}{400} + \frac{193}{100}, && \dot{y} = x - \frac{9y}{20} + \frac{17}{100}, \quad in\ R_3, \end{aligned} \tag{2.24}$$

with their corresponding first integrals

$$H_1(x,y) = 4x^2 - 1.73337..xy - 1.00381..x + 0.455816..y^2 - 0.568207..y,$$
$$H_2(x,y) = 4x^2 - 3.75739..xy + 3.72527..x + 3.75111..y^2 - 4.35843..y,$$
$$H_3(x,y) = 4x^2 - \frac{18xy}{5} + \frac{34x}{25} + \frac{1681y^2}{100} - \frac{386y}{25},$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.24) *has exactly four crossing limit cycles, because the system of equations* (2.5) *has four real* *solutions* $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (0.649616..,0.83435..,0.0409878..,0.0418194..,$ $-0.844622..,0.332933..,-0.0357296..,0.0350855..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (0.71307..,$ $0.933299..,0.109221..,-0.115031..,-0.966853..,0.176027..,-0.148221..,-0.136796..)$, $(\alpha_3,$ $\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = (0.80866..,1.08755..,0.20814..,-0.22878..,-0.99368..,-0.07898..,$ $-0.328436..,-0.26915..)$, *and* $(\alpha_4,\beta_4,\gamma_4,\delta_4,f_4,g_4,h_4,k_4) = (0.915369..,1.26684..,0.286357..,$ $-0.324779..,-0.831632..,-0.341241..,0.5652..,-0.372689..)$, *see* $(\mathcal{C}_5^3)$ *of Figure 2.11.*

Finally for the fourth configuration we consider the linear differential centers

$$\begin{aligned}
&\dot{x} = 0.754941..x - 3.40524..y + 1.04524.., \quad \dot{y} = x - 0.754941..y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -0.492103, \quad in\ R_1,\\
&\dot{x} = -0.110936..x - 1.51937..y + 0.51995.. \quad \dot{y} = x + 0.110936..y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +0.40335, \quad in\ R_2,\\
&\dot{x} = \frac{7x}{5} - \frac{149y}{25} + \frac{23}{10}, \quad \dot{y} = x - \frac{7y}{5} + \frac{3}{20}, \qquad\qquad in\ R_3,
\end{aligned} \tag{2.25}$$

with their corresponding first integrals

$$\begin{aligned}
H_1(x,y) &= 4x^2 - 6.03952..xy - 3.93682..x + 13.621..y^2 - 8.36195..y,\\
H_2(x,y) &= 4x^2 + 0.88749..xy + 3.2268..x + 6.07749..y^2 - 4.1596..y,\\
H_3(x,y) &= 4x^2 - \frac{28xy}{25} + \frac{6x}{5} + \frac{10049y^2}{625} - \frac{92y}{5},
\end{aligned}$$

respectively.

The discontinuous piecewise linear differential system formed by the linear differential centers (2.25) *has exactly four crossing limit cycles, due to the fact that the system of equations* (2.5) *has four real solutions* $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (0.527644..,0.652157..,0.298869..,$ $0.340614..,-0.560417..,0.371562..,-0.373144..,0.295434..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2)$ $= (0.631385..,0.806441..,0.173646..,0.18812..,-0.800529..,0.357534..,-0.177469..,$ $0.160953..)$, $(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = (0.789235..,1.0557..,0.096985..,-0.101579..,$ $-0.987466..,0.110553..,-0.121412..,-0.113803..)$, *and* $(\alpha_4,\beta_4,\gamma_4,\delta_4,f_4,g_4,h_4,k_4) =$ $(0.860054..,1.17297..,0.216049..,-0.238247..,-0.998835..,-0.0340962..,-0.31406..,$ $-0.260109..)$, *see* $(\mathcal{C}_5^4)$ *of Figure 2.12. This completes the proof of statement (d).* ■

Section 2.3 Crossing LC in two regions intersecting c_i in four points

Now we give our second main result which provides information on the number of crossing limit cycles of the discontinuous piecewise linear differential systems formed by two centers and intersect the cubic curves c_i, with $i = 1 \ldots 5$ in four points.

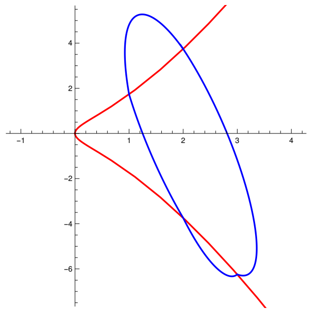
2.3.1 Statement of the second main result

We study the crosssing limit cycles contained only in two regions of the discontinuous piecewise linear centers in the class C_k for $k = 1 \ldots 5$, and having four intersection points with the cubic of separation. Then our second main result is the following.

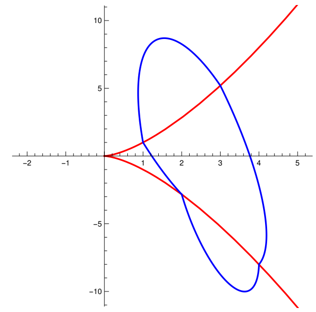
Theorem 2.3 *The following statements hold.*

(a) There are systems in C_k exibiting exactly one crossing limit cycle intersecting the cubic curves c_i in four points. The classes C_1, C_3 and C_4 have one possible configuration, see $(\mathcal{C}_1)$, $(\mathcal{C}_3)$ of Figure 2.13 and $(\mathcal{C}_4)$ of Figure 2.15, respectively. The classes C_2 and C_5 have two possible different configurations, see $(\mathcal{C}_2^1)$ and $(\mathcal{C}_2^2)$ of Figure 2.14, and $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure 2.16.

(b) There are systems in C_k exibiting exactly two crossing limit cycles intersecting the cubic curves c_i in four points. The classes C_1, C_3 and C_4 have one possible configuration, see $(\mathcal{C}_1)$, $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$ of Figure 2.17, respectively. The classes C_2 and C_5 have two possible different configurations see $(\mathcal{C}_2^1)$ and $(\mathcal{C}_2^2)$ of Figure 2.18 for the class C_2, and $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure 2.19 for the class C_5.

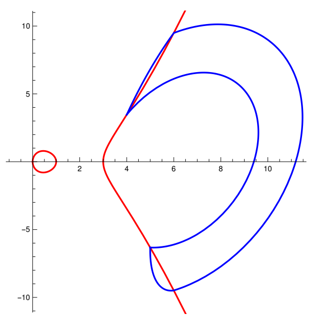


$(\mathcal{C}_1)$

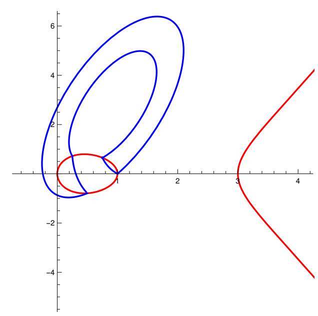


$(\mathcal{C}_3)$

Figure 2.13: The unique crossing limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_1)$ for (2.26)–(2.27), and $(\mathcal{C}_3)$ for (2.33)–(2.34) contained in two zones.

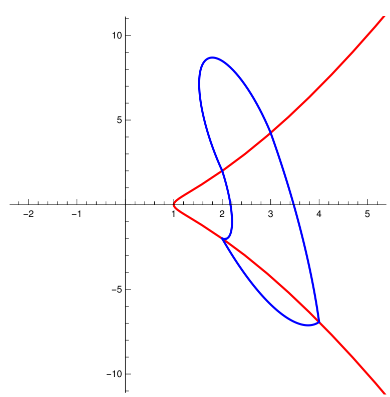


$(\mathcal{C}_2^1)$



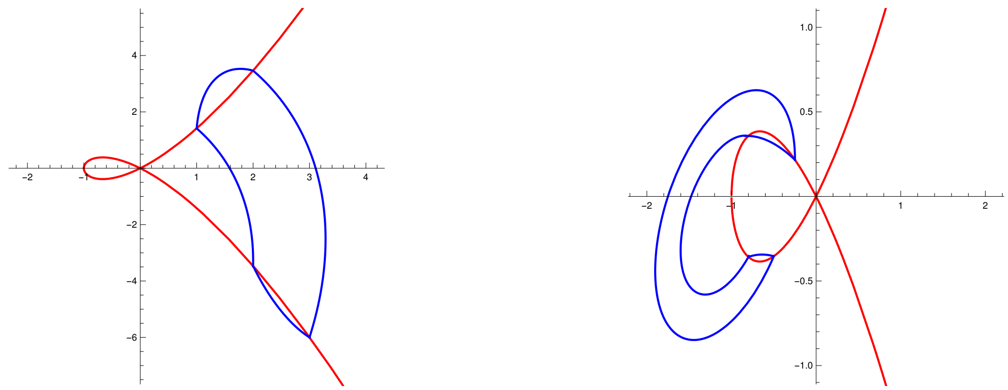
$(\mathcal{C}_2^2)$

Figure 2.14: The unique crossing limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_2^1)$ for (2.29)–(2.30), and $(\mathcal{C}_2^2)$ for (2.31)–(2.32) contained in two zones.



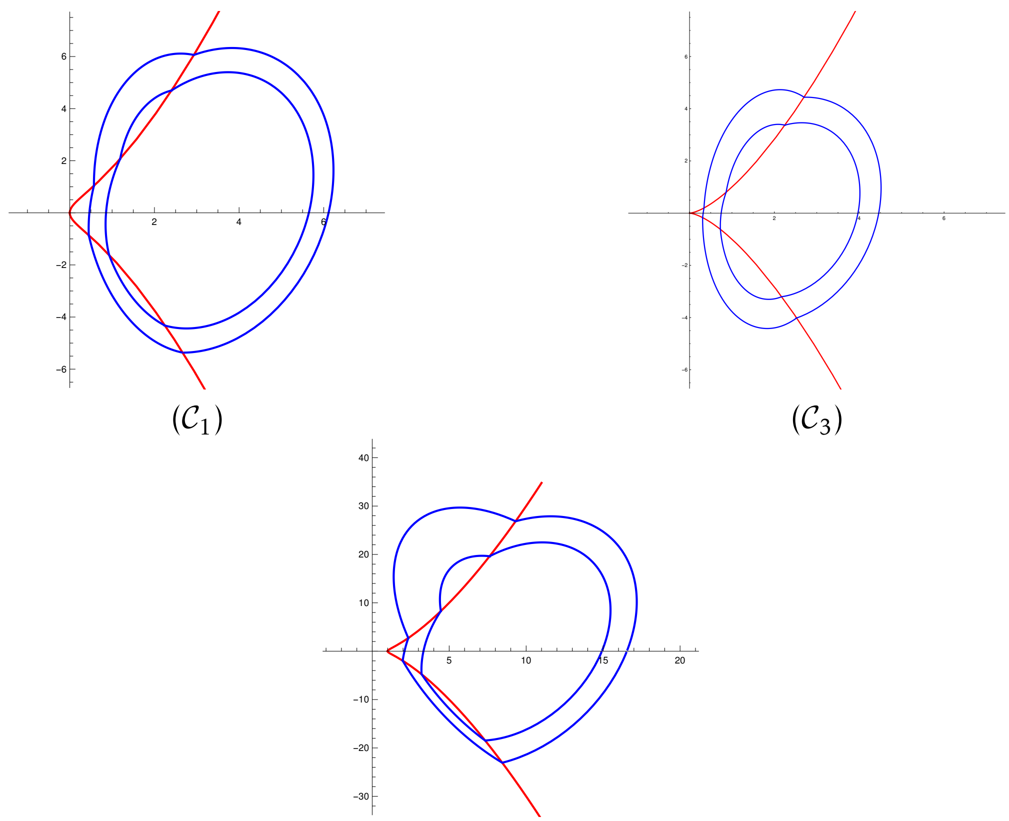
$(\mathcal{C}_4)$

Figure 2.15: The unique crossing limit cycle of the discontinuous piecewise linear differential system (2.35)–(2.36) contained in two zones.



$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

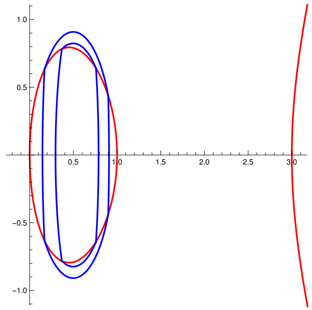
Figure 2.16: The unique crossing limit cycle of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (2.37)–(2.38), and $(\mathcal{C}_5^2)$ for (2.39)–(2.40) contained in two zones.



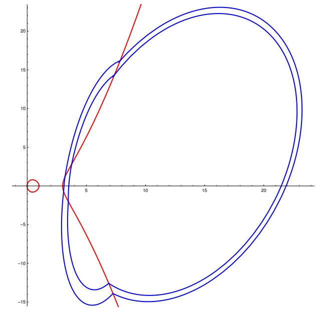
$(\mathcal{C}_1)$ $(\mathcal{C}_3)$

$(\mathcal{C}_4)$

Figure 2.17: The two crossing limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_1)$ for (2.41), $(\mathcal{C}_3)$ for (2.45), and $(\mathcal{C}_4)$ for (2.46) contained in two zones.

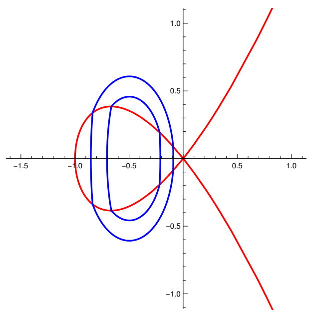


$(\mathcal{C}_2^1)$

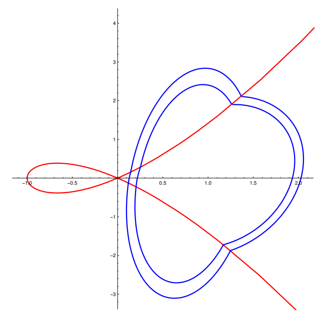


$(\mathcal{C}_2^2)$

Figure 2.18: The two crossing limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2^1)$ for (2.43), and $(\mathcal{C}_2^2)$ for (2.44)contained in two zones.



$(\mathcal{C}_5^1)$



$(\mathcal{C}_5^2)$

Figure 2.19: The two crossing limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (2.47), and $(\mathcal{C}_5^2)$ for (2.48) contained in two zones.

2.3.2 Proof of the main result

Proof. **(Proof of statement (a) of Theorem 2.3.)** *First we prove the statement for the class C_1. We consider the first linear differential center in the region R_1*

$$\begin{aligned}
\dot{x} &= -\frac{x}{6} - \frac{25y}{576} - \frac{-288\sqrt{13} + \dfrac{751}{\sqrt{3}} + 96}{576(\sqrt{13}-1)},\\
\dot{y} &= x + \frac{y}{6} - \frac{6009\sqrt{13} + 576\sqrt{14} + 1152\sqrt{39} - 1502\sqrt{42} + 576\sqrt{182} - 10515}{3456(\sqrt{13}-1)},
\end{aligned} \tag{2.26}$$

this differential system has the first integral

$$H_1(x,y) = \frac{1}{432(\sqrt{13}-1)}(1728(\sqrt{13}-1)x^2 - x(-576(\sqrt{13}-1)y + 6009\sqrt{13}$$
$$+576\sqrt{14} + 1152\sqrt{39} - 1502\sqrt{42} + 576\sqrt{182} - 10515) + y(75(\sqrt{13}$$
$$-1)y + 2(751\sqrt{3} - 864\sqrt{13} + 288))).$$

The second linear differential center in the region R_2 is

$$\begin{aligned} \dot{x} &= -\frac{x}{7} - \frac{113y}{3136} + \frac{\left(\sqrt{13}+1\right)\left(-187\sqrt{3}+192\sqrt{13}-64\right)}{5376}, \\ \dot{y} &= x + \frac{y}{7} + \frac{B}{75264}, \end{aligned} \tag{2.27}$$

with $B = \left(\sqrt{13}+1\right)\left(-10651\sqrt{13} + 896\sqrt{14} - 1792\sqrt{39} - 2618\sqrt{42} + 896\sqrt{182} + 18505\right)$.

This system has the first integral

$$H_2(x,y) = \frac{1}{784(\sqrt{13}-1)}(3136(\sqrt{13}-1)x^2 + y(113(\sqrt{13}-1)y + 14(187\sqrt{3}$$
$$-192\sqrt{13} + 64)) + x(896(\sqrt{13}-1)y - 10651\sqrt{13} + 896\sqrt{14}$$
$$-1792\sqrt{39} - 2618\sqrt{42} + 896\sqrt{182} + 18505)).$$

For the piecewise linear differential system (2.26)–(2.27) *the unique real solution of the system of equations*

$$\begin{aligned} &H_1(\alpha_1,\beta_1) - H_1(\gamma_1,\delta_1) = 0, \\ &H_1(\alpha_2,\beta_2) - H_1(\gamma_2,\delta_2) = 0, \\ &H_2(\alpha_1,\beta_1) - H_2(\alpha_2,\beta_2) = 0, \\ &H_2((\gamma_1,\delta_1) - H_2(\gamma_2,\delta_2) = 0, \\ &c_i(\alpha_1,\beta_1) = 0, \quad c_i(\alpha_2,\beta_2) = 0, \\ &c_i(\gamma_1,\delta_1) = 0, \quad c_i(\gamma_2,\delta_2) = 0, \end{aligned} \tag{2.28}$$

when $i = 1$, *is* $(\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1,\delta_1,\gamma_2,\delta_2) = (1,\sqrt{3},2,-\sqrt{14},2,\sqrt{14},3,-\sqrt{39})$.

Now we prove the statement for the class C_2. We consider the first linear differential center in the region R_1

$$\dot{x} = \frac{x}{6} - \frac{5y}{18} - 1, \quad \dot{y} = x - \frac{y}{6} + \frac{1}{18}\left(12\sqrt{3} + 6\sqrt{10} - 151\right), \tag{2.29}$$

this differential system has the first integral

$$H_1(x,y) = \frac{2}{9}(18x^2 + 2x(-3y + 12\sqrt{3} + 6\sqrt{10} - 151) + y(5y + 36)).$$

The second linear differential center in the region R_2 is

$$\begin{aligned}\dot{x} &= \frac{x}{6} - \frac{13y}{144} + \frac{-82\sqrt{3} - 205\sqrt{10} - 96\sqrt{30} - 5664}{4896},\\ \dot{y} &= x - \frac{y}{6} + \frac{-432\sqrt{10} + \sqrt{30\left(3936\sqrt{10} + 24721\right)} - 17964}{2448},\end{aligned} \tag{2.30}$$

this system has the first integral

$$\begin{aligned}H_2(x,y) = \;& \frac{1}{36(2\sqrt{3} - 5\sqrt{10})}(144(2\sqrt{3} - 5\sqrt{10})x^2 - 4x(24\sqrt{3}y - 60\sqrt{10}y\\ &+1117\sqrt{3} - 574 - 2649\sqrt{10} + 96\sqrt{30} - 720) + y(26\sqrt{3}y - 65\sqrt{10}y\\ &+384\sqrt{3} - 1632\sqrt{10})).\end{aligned}$$

The unique real solution of the system of equations (2.28) *for $i = 2$, for the piecewise linear differential system* (2.29)–(2.30) *is* $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (4, 2\sqrt{3}, 5, -2\sqrt{10}, 6, 3\sqrt{10}, 6, -3\sqrt{10})$.

For the second configuration of the class C_2, we consider the linear differential center in the region R_2

$$\begin{aligned}\dot{x} &= \frac{x}{5} - \frac{41y}{400} + \frac{1440\sqrt{3} - 640\sqrt{10} - 160\sqrt{33} + 2503}{3200\left(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}\right)},\\ \dot{y} &= x - \frac{y}{5} + \frac{160\sqrt{10} + \dfrac{2503\sqrt{10} + 960\sqrt{30} - 3200}{-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}} - 4390}{6400},\end{aligned} \tag{2.31}$$

this differential system has the first integral

$$\begin{aligned}H_2(x,y) = \;& \frac{1}{800(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})}(3200(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})x^2 + x(13170\sqrt{3}\\ &-6277\sqrt{10} + 480\sqrt{30} - 4390\sqrt{33} + 160\sqrt{330} - 1280(-3\sqrt{3} + 2\sqrt{10}+\\ &\sqrt{33})y) + 2y(164(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33})y - 1440\sqrt{3} + 640\sqrt{10} + 160\sqrt{33}\\ &-2503)).\end{aligned}$$

The second linear differential center in the region R_3 is

$$\begin{aligned}\dot{x} &= \frac{x}{5} - \frac{13y}{100} + \frac{360\sqrt{3} - 160\sqrt{10} - 40\sqrt{33} + 579}{800\left(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}\right)},\\ \dot{y} &= x - \frac{y}{5} - \frac{-3873\sqrt{3} + 4898\sqrt{10} + 1440\sqrt{11} + 480\sqrt{30} + 2449\sqrt{33}}{3200\left(-3\sqrt{3} + 2\sqrt{10} + \sqrt{33}\right)},\end{aligned} \tag{2.32}$$

this system has the first integral

$$H_1(x,y) = \frac{1}{400(-3\sqrt{3}+2\sqrt{10}+\sqrt{33})}1600(-3\sqrt{3}+2\sqrt{10}+\sqrt{33})x^2 - x(640(-3\sqrt{3} \\ +2\sqrt{10}+\sqrt{33})y - 3873\sqrt{3}+4898\sqrt{10}+1440\sqrt{11}+480\sqrt{30}+2449\sqrt{33}) \\ +4y\left(52\left(-3\sqrt{3}+2\sqrt{10}+\sqrt{33}\right)y - 360\sqrt{3}+160\sqrt{10}+40\sqrt{33}-579\right).$$

The unique real solution of the system of equations (2.28) *for $i=2$, for the piecewise linear differential system* (2.31)–(2.32) *is* $(\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1,\delta_1,\gamma_2,\delta_2) = (1/4, \sqrt{33}/8, 1/2, -\sqrt{5}/(2\sqrt{2}), 3/4, 3\sqrt{3}/8, 1, 0)$.

We prove the statement for the class C_3. We consider the first linear differential center in the region R_1

$$\dot{x} = -\frac{x}{6} - \frac{25y}{576} + \frac{-128\sqrt{2}+288\sqrt{3}+483}{-384\sqrt{2}+576\sqrt{3}+1344}, \\ \dot{y} = x + \frac{y}{6} + \frac{-96618\sqrt{2}+9\sqrt{3(58745128\sqrt{2}+89093337)}-500192}{192384}, \tag{2.33}$$

its first integral is

$$H_1(x,y) = \frac{1}{144(2\sqrt{2}-3\sqrt{3}-7)}(576(2\sqrt{2}-3\sqrt{3}-7)x^2 + x(192(2\sqrt{2}-3\sqrt{3} \\ -7)y - 3770\sqrt{2}+6861\sqrt{3}+1152\sqrt{6}+14875) + y(25(2\sqrt{2}-3\sqrt{3} \\ -7)y - 768\sqrt{2}+1728\sqrt{3}+2898).$$

The second linear differential center in the region R_2 is

$$\dot{x} = -\frac{x}{9} - \frac{145y}{5184} + \frac{134932\sqrt{2}-69489\sqrt{3}-77682\sqrt{6}+250949}{577152}, \\ \dot{y} = x + \frac{y}{9} + \frac{-23197\sqrt{2}+45834\sqrt{3}-1728\sqrt{6}+79814}{5184(2\sqrt{2}-3\sqrt{3}-7)}, \tag{2.34}$$

this system has the first integral

$$\begin{aligned} H_2(x,y) = & \frac{1}{648(-2\sqrt{2}-18\sqrt{3}+6\sqrt{6}-20)}(5184(-\sqrt{2}-9\sqrt{3}+3\sqrt{6}-10)x^2 \\ & +x(1152(-\sqrt{2}-9\sqrt{3}+3\sqrt{6}-10)y+7645\sqrt{2}+193608\sqrt{3}+332692 \\ & -67863\sqrt{6})+y(145(-\sqrt{2}-9\sqrt{3}+3\sqrt{6}-10)y+2304\sqrt{2}+10755\sqrt{3} \\ & -6912\sqrt{6}+41343)). \end{aligned}$$

The discontinuous piecewise linear differential centers (2.33)–(2.34) has one limit cycle because the system of equations (2.28) for $i=3$ has the unique real solution $(\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1,\delta_1,\gamma_2,\delta_2)=(1,1,2,-2\sqrt{2},3,3\sqrt{3},4,-8)$. This limit cycle is shown in $(\mathcal{C}_3)$ Figure 2.13.

Now we prove the statement for the class C_4. We consider the linear differential center in the region R_1

$$\dot{x}=-\frac{x}{9}-\frac{181y}{8100}+\frac{2}{9},\quad \dot{y}=x+\frac{y}{9}+\frac{\sqrt{2}}{3}-\frac{2071}{540}+\frac{8}{3\sqrt{3}}, \tag{2.35}$$

this differential system has the first integral

$$H_1(x,y)=4x^2+\frac{2}{135}x\left(60y+180\sqrt{2}+480\sqrt{3}-2071\right)+\frac{y(181y-3600)}{2025}.$$

The second linear differential center in the region R_2 is

$$\begin{aligned} \dot{x} &= -\frac{x}{8}-\frac{y}{32}+\frac{66\sqrt{2}-62\sqrt{3}-18\sqrt{6}+525}{1128}, \\ \dot{y} &= x+\frac{y}{8}+\frac{-600\sqrt{6}+8\sqrt{6\left(232\sqrt{6}+2555\right)}-12627}{4512}, \end{aligned} \tag{2.36}$$

with its first integral

$$\begin{aligned} H_2(x,y) = & \frac{1}{8(-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)}32(-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)x^2+ \\ & 2x(4(-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)y+1521\sqrt{2}+556\sqrt{3}-666\sqrt{6} \\ & -2456)+y((-15\sqrt{2}-4\sqrt{3}+6\sqrt{6}+24)y-8(-51\sqrt{2}-16\sqrt{3} \\ & +24\sqrt{6}+76)). \end{aligned}$$

In this case the unique real solution of the system of equations (2.28) for $i=4$, is $(\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1,\delta_1,\gamma_2,\delta_2)=(2,2,2,-2,3,3\sqrt{2},4,-4\sqrt{3})$. Then the disscontinuous piecewise differential *system* (2.35)–(2.36) *has one limit cycle shown in Figure* 2.15

We prove the statement for the first configuration of the class C_5. *We consider the linear differential center in the region* R_1

$$\begin{aligned}\dot{x} &= \frac{x}{6} - \frac{5y}{18} + \frac{1}{612}\left(-89\sqrt{2} - 636\right),\\ \dot{y} &= x - \frac{y}{6} + \frac{1}{306}(432\sqrt{3} + 178\sqrt{3(\sqrt{3}+2)} - 795),\end{aligned} \tag{2.37}$$

which has the first integral

$$\begin{aligned}H_1(x,y) = \quad & \frac{1}{9(\sqrt{2}-6)}(36(\sqrt{2}-6)x^2 - 4x(3(\sqrt{2}-6)y + 141\sqrt{2} + 142\sqrt{3}\\ & +6\sqrt{6} - 312) + 2y(5(\sqrt{2}-6)y + 6\sqrt{2} - 214)).\end{aligned}$$

The second linear differential center in the region R_3

$$\begin{aligned}\dot{x} &= -\frac{x}{6} - \frac{13y}{144} + \frac{53\sqrt{2} + 1134}{4896},\\ \dot{y} &= x + \frac{y}{6} + \frac{-114\sqrt{2} + 91\sqrt{3} - 24\sqrt{6} + 843}{72(\sqrt{2}-6)},\end{aligned} \tag{2.38}$$

its first integral is

$$\begin{aligned}H_2(x,y) = \quad & \frac{1}{36(\sqrt{2}-6)}(144(\sqrt{2}-6)x^2 - 4x(-12(\sqrt{2}-6)y + 114\sqrt{2} - 91\sqrt{3}\\ & +24\sqrt{6} - 843) + y(13(\sqrt{2}-6)y - 48\sqrt{2} + 394)).\end{aligned}$$

For this piecewise linear differential centers the unique real solution of system (2.28) *when* $i = 5$ *is* $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (1, \sqrt{2}, 2, -2\sqrt{3}, 2, 2\sqrt{3}, 3, -6)$. *Hence, the discontinuous piecewise linear differential system* (2.37)–(2.38) *has a unique crossing limit cycle, see* ($\mathcal{C}_5^1$) *of Figure* 2.16.

Finally we prove the statement for the second configuration of the class C_5 . *In the region* R_2

we consider the linear center

$$\dot{x} = -\frac{x}{4} - \frac{y}{8} + \frac{-22000\sqrt{2} - 3000\sqrt{3} + 34816\sqrt{5} + 24585}{640(275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5}},$$
$$\dot{y} = x + \frac{y}{4} + \frac{2400\sqrt{2} - 2200\sqrt{3} - \dfrac{4125(149\sqrt{11 - 4\sqrt{6}} + 200)}{275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5}} + 62593}{108800}, \tag{2.39}$$

its first integral is

$$\begin{aligned} H_2(x,y) = & \frac{1}{800(275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5})}(3200(275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5})x^2 \\ & + x(1600(275\sqrt{2} + 75\sqrt{3} - 272\sqrt{5})y + 940225\sqrt{2} + 312300\sqrt{3} \\ & - 1001488\sqrt{5} - 25000\sqrt{6} - 38400\sqrt{10} + 35200\sqrt{15}) + 10y(40(275\sqrt{2} \\ & + 75\sqrt{3} - 272\sqrt{5})y + 22000\sqrt{2} + 3000\sqrt{3} - 34816\sqrt{5} - 24585)). \end{aligned}$$

In the region R_3 we consider the linear differential center

$$\dot{x} = \frac{x}{2} - \frac{5y}{4} + \frac{2}{5}, \quad \dot{y} = x - \frac{y}{2} + \frac{1}{640}(96\sqrt{2} + 88\sqrt{3} + 365), \tag{2.40}$$

this system has the first integral

$$H_1(x,y) = 4x^2 + \frac{1}{80}x\left(-320y + 96\sqrt{2} + 88\sqrt{3} + 365\right) + \frac{1}{5}y(25y - 16).$$

For this piecewise linear differential centers the unique real solution of system (2.28) when $i = 5$ is $(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_1, \gamma_2, \delta_2) = (-1/4, \sqrt{3}/8, -4/5, 4/(5\sqrt{5}), -1/2, -1/(2\sqrt{2}), -4/5, -4/(5\sqrt{5})$. Then the discontinuous piecewise linear differential system (2.39)–(2.40) has exactly one crossing limit cycle, see $(\mathcal{C}_5^2)$ of Figure 2.16. This completes the proof of statement (a) of Theorem 2.3. ∎

Proof. (Proof of statement (b) of Theorem 2.3.) *First we prove the statement for class C_1. We consider the linear differential center in systems*

$$\begin{aligned} &\dot{x} = -0.0327708x - 0.167012y + 0.202324, \quad \dot{y} = x + 0.0327708y \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2.82026, \quad in\ R_1, \\ &\dot{x} = \frac{x}{10} - \frac{13y}{50} - \frac{1}{5}, \quad \dot{y} = x - \frac{y}{10} - \frac{16}{5}, \qquad\qquad in\ R_2, \end{aligned} \tag{2.41}$$

with their corresponding first integrals

$$H_1(x,y) = 4(x + 0.0327708y)^2 + 8(-2.82026x - 0.202324y) + 0.663752y^2,$$
$$H_2(x,y) = 4\left(x - \frac{y}{10}\right)^2 + 8\left(\frac{y}{5} - \frac{16x}{5}\right) + y^2,$$

respectively.

For the piecewise linear differential system (2.41) *the real solutions of the system of equations*

$$\begin{aligned} &H_1(\alpha_i,\beta_i) - H_1(\gamma_i,\delta_i) = 0,\\ &H_1(f_i,g_i) - H_1(h_i,k_i) = 0,\\ &H_2(\alpha_i,\beta_i) - H_1(f_i,g_i) = 0,\\ &H_2(\gamma_i,\delta_i) - H_1(h_i,k_i) = 0,\\ &c_s(\alpha_i,\beta_i) = c_s(\gamma_i,\delta_i) = 0,\ c_s(f_i,g_i) = c_s(h_i,k_i) = 0 \quad i = 1,2, \end{aligned} \tag{2.42}$$

when $s = 1$, *is* $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = (0.571172..,1.04103..,2.93153..,6.0596..,$ $0.449711..,-0.861917..,2.66907..,-5.36724..)$ *and* $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = (1.19066..,$ $2.07275..,2.40283..,4.69567..,0.928007..,-1.60885..,2.25189..,-4.32923..)$.

Hence this piecewise linear differential centers have exactly two crossing limit cycles, see $(\mathcal{C}_1)$ *of Figure* 2.17.

Now we prove the statement for the class C_2. *We consider the linear differential center in systems*

$$\begin{aligned} &\dot{x} = -\frac{y}{25}, \quad \dot{y} = x - \frac{1}{2}, && in\ R_1,\\ &\dot{x} = -0.252669..y, \quad \dot{y} = x - 0.498272, && in\ R_2, \end{aligned} \tag{2.43}$$

their corresponding first integrals are

$$H_1(x,y) = 4x^2 - 4x + \frac{4y^2}{25},$$
$$H_2(x,y) = 4x^2 - 3.98617..x + 1.01067..y^2,$$

respectively.

The real solutions of system (2.42) *when* $s = 2$ *are* $(0.898773..,0.43723..,0.16978..,0.63161..,$ $0.898773..,-0.43723..,0.169784..,-0.631617..)$ *and* $(0.758688..,0.640578..,0.368993..,$ $0.782685..,0.758688..,-0.640578..,0.368993..,-0.782685..)$. *Hence the discontinuous piecewise linear differential system formed by centers* (2.43) *has two crossing limit cycles, see* $(\mathcal{C}_2^1)$

of Figure 2.18.

For the second configuration of the class C_2 we consider the linear differential center in systems

$$\begin{aligned} &\dot{x}=\frac{x}{10}-\frac{17y}{80}-\frac{1}{10}, \quad \dot{y}=x-\frac{y}{10}-\frac{11}{10}, && in\ R_1,\\ &\dot{x}=0.057143..x-0.0550264..y-0.0521176.., \quad \dot{y}=x-0.057143..y\\ &\qquad -0.805757, && in\ R_2, \end{aligned} \tag{2.44}$$

and the corresponding first integrals

$$\begin{aligned} H_1(x,y)&=4\left(x-\frac{y}{10}\right)^2+8\left(\frac{y}{10}-\frac{11x}{10}\right)+\frac{81y^2}{100},\\ H_2(x,y)&=4(x-0.057143..y)^2+8(0.0521176..y-0.805757..x)+0.207044..y^2, \end{aligned}$$

respectively.

The real solutions of system (2.42) for these centers are $(0.253068.., 0.283285.., 1.26272.., 1.89942.., 0.204637.., -0.224601.., 1.16927.., -1.72215..)$ *and* $(0.145203.., 0.155388.., 1.36881.., 2.10673.., 0.119104.., -0.125997.., 1.24855.., -1.87223..)$. *Hence the discontinuous piecewise linear differential system formed by centers (2.44) has two crossing limit cycles, see $(\mathcal{C}_2^2)$ of Figure 2.18.*

For the class C_3 we consider the linear differential center in systems

$$\begin{aligned} &\dot{x}=0.0333015..x-0.132045..y-0.0410184.., \quad \dot{y}=x-0.0333015..y\\ &\qquad -1.97694, && in\ R_1,\\ &\dot{x}=0.1x-0.26y-0.2, \quad \dot{y}=x-0.1y-2.3, && in\ R_2, \end{aligned} \tag{2.45}$$

and the corresponding first integrals

$$\begin{aligned} H_1(x,y)&=4(x-0.0333015..y)^2+8(0.0410184..y-1.97694..x)+0.523744..y^2,\\ H_2(x,y)&=4(x-0.1y)^2+8(0.2y-2.3x)+y^2, \end{aligned}$$

respectively.

The real solutions of system (2.42) when $s=3$ are $(0.340737.., 0.198898.., 2.70162.., 4.44056.., 0.308678.., -0.171498.., 2.52616.., -4.01504..)$ *and* $(0.862714.., 0.801309.., 2.24826.., 3.37109.., 0.727481.., -0.620487.., 2.17185.., -3.2007..)$. *Hence the discontinuous*

piecewise linear differential system (2.45) *has two crossing limit cycles, see* ($\mathcal{C}_3$) *of Figure* 2.17*.*
We prove the statement for the class C_4*. We consider the linear differential center in systems*

$$\begin{aligned} &\dot{x} = -0.142331..x - 0.0908868y + 1.59716.., && in\ R_1, \\ &\dot{y} = x + 0.142331..y - 9.92891, \quad \dot{y} = x - \frac{y}{10} - \frac{44}{5}, && in\ R_2, \end{aligned} \tag{2.46}$$

with their corresponding first integrals

$$\begin{aligned} H_1(x,y) &= 4(x + 0.142331..y)^2 + 8(-9.92891..x - 1.59716..y) + 0.282515..y^2, \\ H_2(x,y) &= 4\left(x - \frac{y}{10}\right)^2 + 8\left(\frac{7y}{10} - \frac{44x}{5}\right) + \frac{9y^2}{25}, \end{aligned}$$

respectively.
The two real solutions of system (2.42) *when* $s = 4$ *are* $(2.34486.., 2.71929.., 9.31289.., 26.851.., 1.97922.., -1.95854.., 8.44735.., -23.0527..)$ *and* $(4.45798.., 8.2899.., 7.62439.., 19.6236.., 3.20672.., -4.7636.., 7.33329.., -18.455..)$*. Hence the discontinuous piecewise linear differential system* (2.46) *has two crossing limit cycles, see* ($\mathcal{C}_4$) *of Figure* 2.17*.*
For the first configuration of the class C_5*, we consider the linear differential centers*

$$\begin{aligned} &\dot{x} = -0.453205..y, \quad \dot{y} = x + 0.497252, && in\ R_1, \\ &\dot{x} = -\frac{y}{10}, \quad \dot{y} = x + \frac{1}{2}, && in\ R_2, \end{aligned} \tag{2.47}$$

and their corresponding first integrals

$$\begin{aligned} H_1(x,y) &= 4x^2 + 3.97802x + 1.81282y^2, \\ H_2(x,y) &= 4x^2 + 4x + \frac{2y^2}{5}, \end{aligned}$$

respectively.
The two real solutions of system (2.42) *when* $s = 5$ *are* $(-0.0927079.., 0.088306.., -0.837019.., 0.337912.., -0.0927079.., -0.088306.., -0.837019.., -0.337912..)$ *and* $(-0.217826.., 0.192647.., -0.663868.., 0.38489.., -0.217826.., -0.192647.., -0.663868.., -0.38489..)$*. Hence, the discontinuous piecewise linear differential formed* (2.47) *has two crossing limit cycles, see* ($\mathcal{C}_5$) *of Figure* 2.19*.*
Finally we prove the statement for the second configuration of the class C_5*, where we consider*

the linear differential centers

$$
\begin{aligned}
&\dot{x} = \frac{x}{10} - \frac{17y}{80} - \frac{1}{10}, \quad \dot{y} = x - \frac{y}{10} - \frac{11}{10}, \qquad\qquad in\ R_1,\\
&\dot{x} = 0.057143..x - 0.0550264..y - 0.0521176.., \quad \dot{y} = x - 0.057143..y\\
&\qquad\qquad\qquad\qquad -0.805757, \quad in\ R_3,
\end{aligned}
\tag{2.48}
$$

and their corresponding first integrals

$$
\begin{aligned}
H_1(x,y) &= 4\left(x - \frac{y}{10}\right)^2 + 8\left(\frac{y}{10} - \frac{11x}{10}\right) + \frac{81y^2}{100},\\
H_2(x,y) &= 4(x - 0.057143..y)^2 + 8(0.0521176..y - 0.805757..x) + 0.207044..y^2,
\end{aligned}
$$

respectively.

The two real solutions of system (2.42) *when* $s = 5$ *are* (0.253068.., 0.283285.., 1.26272.., 1.89942.., 0.204637.., −0.224601.., 1.16927.., −1.72215..) *and* (0.145203.., 0.155388.., 1.36881.., 2.10673.., 0.119104.., −0.125997.., 1.24855.., −1.87223..).

Hence the discontinuous piecewise linear differential system (2.48) *has two crossing limit cycles, see* ($\mathcal{C}_5^2$) *of Figure* 2.19. *This completes the proof of statement (b) for the class* C_5. ■

Chapter 3

Limit Cycles of Discontinuous Piecewise Linear Differential Systems Formed by Centers or Hamiltonian Without Equilibria Separated by Irreducible Cubics

This chapter is devoted to study the limit cycles of two families of planar linear differential systems, the first one is Hamiltonian without equilibrium points and the second one is a family of centers.

The following lemma provides a normal form for an arbitrary linear Hamiltonians system without equilibrium points.

LEMMA 3.1 *An arbitrary linear differential Hamiltonian system in $\mathbb{R}^2$ without singular points can be written as*

$$\mathcal{X}_i(x,y) = (-\lambda_i b_i x + b_i y + \mu_i, -\lambda_i^2 b_i x + \lambda_i b_i y + \sigma_i),$$

where $\sigma_i \neq \lambda_i \mu_i$ and $b_i \neq 0$ for $i = 1 \ldots 4$. The Hamiltonian function associated to the

Hamiltonian vector field $\mathcal{X}_i$ is

$$H_i(x,y) = (-\lambda_i^2 b_i/2)x^2 + \lambda_i b_i xy - (b_i/2)y^2 + \sigma_i x - \mu_i y.$$

For a proof of this lemma, see [30].

Section 3.1 LC of Discontinuous PWLS Intersecting the Curve c_i, $i = 1,..,5$ in Two Points

We denote by $\mathcal{F}_1$ the family of discontinuous piecewise differential systems separated by a cubic curve c_k for $k = 1,\ldots,5$ contained in two regions. We note that such piecewise systems formed by two pieces, in one piece we define a linear differential center, and in the other piece we define a linear Hamiltonian system without equilibrium point.

Our first objective in this chapter is to provide the maximum number of crossing limit cycles of the family $\mathcal{F}_1$ which intersect the irreducible cubic curves $c_i = 0$, with $i = 1\ldots5$ in two points.

3.1.1 Statement of the first main result

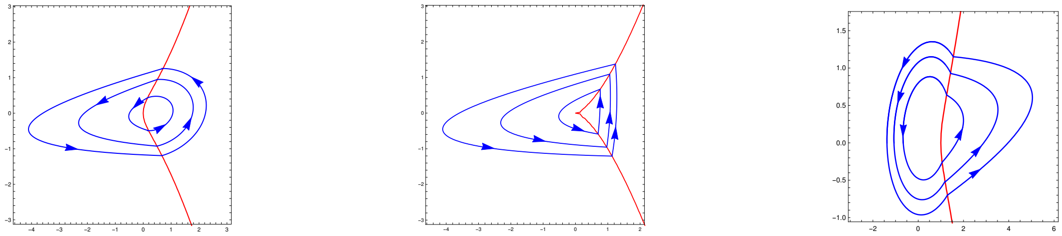
Our first main result is the following.

Theorem 3.1 *The maximum number of crossing limit cycles of discontinuous piecewise differential systems of the family $\mathcal{F}_1$ which intersect the cubic curve $c_k = 0$, for $k = 1,\ldots,5$ in two points is three. This maximum is reached in all cases.*

(i) For the classes C_1, C_3 and C_4, see Examples 1, 2 and 3, respectively.

(ii) For the class C_2, see Examples 4.1, 4.2, 4.3, 4.4, 4.5, 4.6 *and* 4.7.

(iii) For the class C_5, see Examples 5.1, 5.2, 5.3, 5.4, 5.5, 5.6 *and* 5.7.

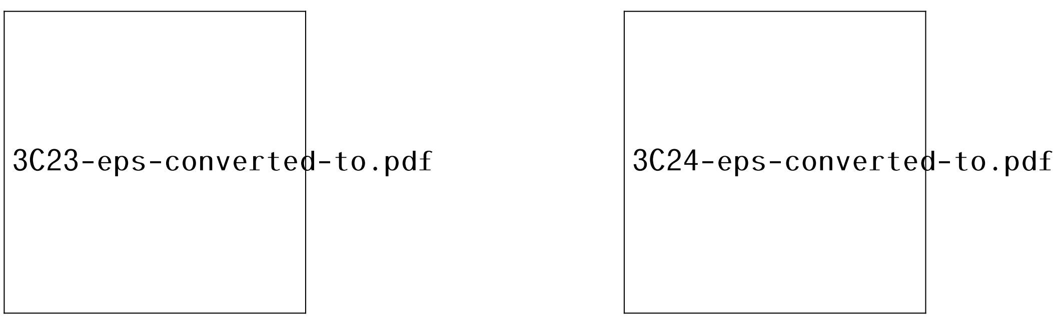


(C_1) (C_3) (C_4)

Figure 3.1: The three limit cycles of the discontinuous piecewise differential systems (3.6)–(3.7), (3.8)–(3.9), and (3.11)–(3.12).

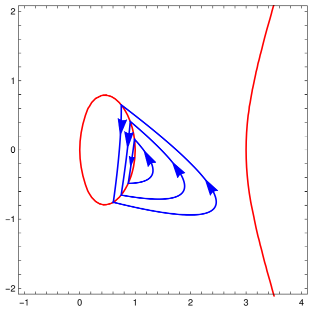
(C_2^1) (C_2^2)

Figure 3.2: The three limit cycles of the discontinuous piecewise differential system (C_2^1) for (3.14)–(3.15), and (C_2^2) for (3.16)–(3.17).

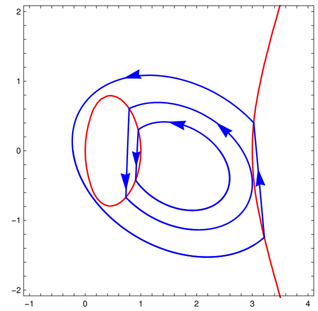


(C_2^3) (C_2^4)

Figure 3.3: The three limit cycles of the discontinuous piecewise differential system (C_2^3) for (3.18)–(3.19), and (C_2^4) for (3.20)–(3.21).

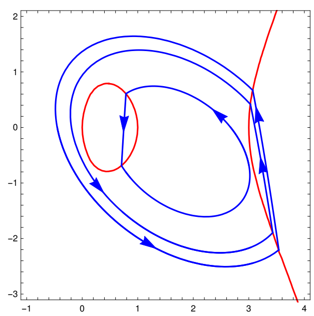


(C_2^5)



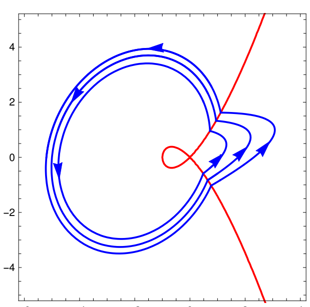
(C_2^6)

Figure 3.4: The three limit cycles of the discontinuous piecewise differential system (C_2^5) for (3.22)–(3.23), and (C_2^6) for (3.24)–(3.25).

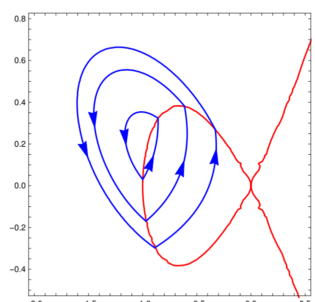


(C_2^7)

Figure 3.5: The three limit cycles of the discontinuous piecewise differential system (3.26)–(3.27).

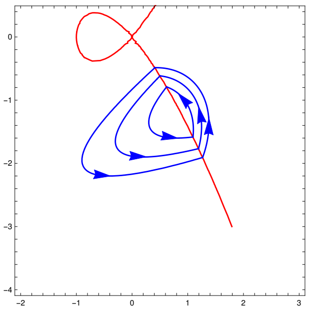


(C_5^1)

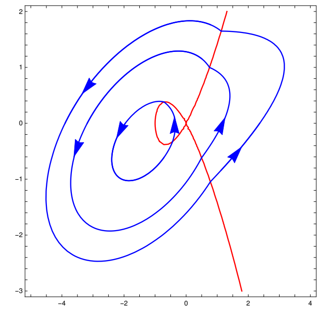


(C_5^2)

Figure 3.6: The three limit cycles of the discontinuous piecewise differential system (C_5^1) for (3.29)–(3.30), and (C_5^2) for (3.31)–(3.32).

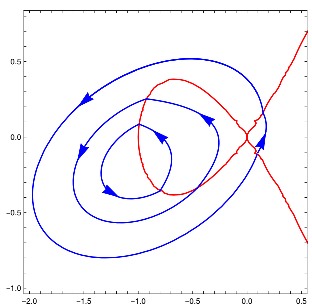


(C_5^3)

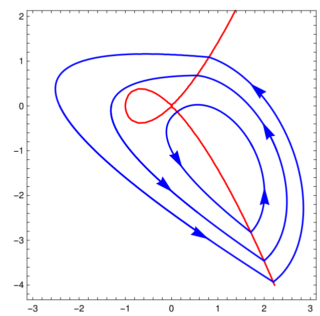


(C_5^4)

Figure 3.7: The three limit cycles of the discontinuous piecewise differential system (C_5^3) for (3.33)–(3.34), and (C_5^4) for (3.35)–(3.36).

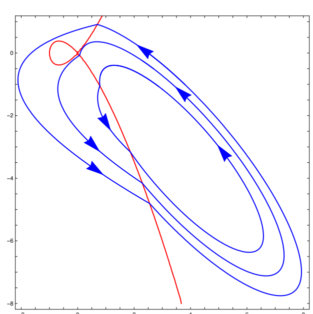


(C_5^5)



(C_5^6)

Figure 3.8: The three limit cycles of the discontinuous piecewise differential system (C_5^5) for (3.37)–(3.38), and (C_5^6) for (3.39)–(3.40).



(C_5^7)

Figure 3.9: The three limit cycles of the discontinuous piecewise differential systems (3.41)–(3.42).

3.1.2 Proof of the first main result

We shall prove that the maximum number of limit cycles of systems of family $\mathcal{F}_1$ intersecting the cubic curve c_3, in two points is three. By a similar way we prove the statement for the other four cubic curves.

We consider the discontinuous piecewise linear Hamiltonian system such that in the region $R_{13} = \{(x,y) : y^2 - x^3 \geq 0\}$ is defined as

$$\dot{x} = -\lambda b_1 x + b_1 y + \mu, \quad \dot{y} = -\lambda^2 b_1 x + \lambda b_1 y + \sigma, \tag{3.1}$$

with $b_1 \neq 0$ and $\sigma \neq \lambda\mu$. This system has the first integral

$$H_1(x,y) = -(\lambda^2 b_1/2)x^2 + \lambda b_1 xy - (b_1/2)y^2 + \sigma x - \mu y.$$

In the region $R_{23} = \{(x,y) : y^2 - x^3 \leq 0\}$ we consider the linear center

$$\dot{x} = -b_2 x - \frac{4b_2^2 + \omega^2}{4} y + d, \quad \dot{y} = x + b_2 y + c, \ \ with \ \ \omega > 0, \tag{3.2}$$

with its first integral

$$H_2(x,y) = 4(x + b_2 y)^2 + 8(cx - dy) + y^2\omega^2.$$

We have to prove that the piecewise linear differential systems composed by system (3.1) and system (3.2) have at most three crossing limit cycles intersecting the cubic curve $y^2 - x^3 = 0$ in two points. For that we suppose that these systems have four crossing limit cycles intersecting the curve $y^2 - x^3 = 0$ in the points (x_i, y_i) and (z_i, w_i) for $i = 1, \ldots, 4$, which means that they satisfy the system

$$\begin{aligned} &H_1(x_i, y_i) - H_1(z_i, w_i) = 0, \\ &H_2(x_i, y_i) - H_2(z_i, w_i) = 0, \\ &y_i^2 - x_i^3 = 0, \quad w_i^2 - z_i^3 = 0. \end{aligned} \tag{3.3}$$

Such points (x_i, y_i) and (z_i, w_i) can take form $A_i = (r_i^2, r_i^3)$ and $B_i = (s_i^2, s_i^3)$, respectively. Taking in consideration that these points verify the first two equations of system (3.3), then by solving the first and the second equations when $i=1$, we get

$$\begin{aligned}\sigma = & \frac{1}{2(r_1+s_1)}(b_1r_1^5+b_1r_1^4s_1+b_1r_1^3s_1^2+b_1r_1^2s_1^3+b_1r_1s_1^4+b_1s_1^5-2b_1r_1^4\lambda-2b_1r_1^3s_1\lambda\\ & -2b_1r_1^2s_1^2\lambda-2b_1r_1s_1^3\lambda-2b_1s_1^4\lambda+b_1r_1^3\lambda^2+b_1r_1^2s_1\lambda^2+b_1r_1s_1^2\lambda^2+b_1s_1^3\lambda^2+2r_1^2\mu\\ & +2r_1s_1\mu+2s_1^2\mu),\\ d = & \frac{1}{8(r_1^2+r_1s_1+s_1^2)}(8c(r_1+s_1)+4(r_1+b_2r_1^2+s_1+b_2r_1s_1+b_2s_1^2)(r_1^2+b_2r_1^3+s_1^2\\ & +b_2s_1^3)+(r_1+s_1)(r_1^2-r_1s_1+s_1^2)(r_1^2+r_1s_1+s_1^2)\omega^2).\end{aligned}$$

Since the points $A_2=(r_2^2,r_2^3)$ and $B_2=(s_2^2,s_2^3)$ also verify system (3.3), we obtain the following expressions of the parameters $\mu=F_1/G_1$ and $c=F_2/G_2$, where

$$\begin{aligned}F_1 = & -(b_1(r_2-s_2)((r_1+s_1)(r_2+s_2)(r_1^4-r_2^4+r_1^2s_1^2+s_1^4-r_2^2s_2^2-s_2^4)-2(r_2(r_1^4+r_1^3s_1\\ & -r_2^3s_1+r_1^2s_1^2+s_1^4+r_1(-r_2^3+s_1^3))+(r_1^4+r_1^3s_1-r_2^3s_1+r_1^2s_1^2+s_1^4+r_1(-r_2^3+s_1^3))s_2\\ & -r_2^2(r_1+s_1)s_2^2-r_2(r_1+s_1)s_2^3-(r_1+s_1)s_2^4)\lambda+(r_1+s_1)(r_2+s_2)(r_1^2-r_2^2+s_1^2-s_2^2)\lambda^2),\\ G_1 = & 2(r_2^2(r_1^2+r_1(-r_2+s_1)+s_1(-r_2+s_1))-(r_1^2+r_1s_1+s_1^2)s_2^2+(r_1+s_1)s_2^3),\\ F_2 = & r_1^5(r_2^3-s_2^3)(4b_2^2+\omega^2)+r_1^4(r_2^3-s_2^3)(4b_2(2+b_2s_1)+s_1\omega^2)+r_1^3(r_2^3-s_2^3)(4(1+b_2s_1)^2\\ & +s_1^2\omega^2)+r_1^2(-4r_2^4-8b_2r_2^5-r_2^6(4b_2^2+\omega^2)+r_2^3s_1(4(1+b_2s_1)^2+s_1^2\omega^2)+s_2^3(-4s_1\\ & -8b_2s_1^2+4s_2(1+b_2s_2)^2+s_2^3\omega^2-s_1^3(4b_2^2+\omega^2)))+r_1s_1(-4r_2^4-8b_2r_2^5-r_2^6(4b_2^2+\omega^2)\\ & +r_2^3s_1(4(1+b_2s_1)^2+s_1^2\omega^2)+s_2^3(-4s_1-8b_2s_1^2+4s_2(1+b_2s_2)^2+s_2^3\omega^2-s_1^3(4b_2^2+\omega^2)))\\ & +s_1^2(-4r_2^4-8b_2r_2^5-r_2^6(4b_2^2+\omega^2)+r_2^3s_1(4(1+b_2s_1)^2+s_1^2\omega^2)+s_2^3(-4s_1-8b_2s_1^2\\ & +4s_2(1+b_2s_2)^2+s_2^3\omega^2-s_1^3(4b_2^2+\omega^2))),\\ G_2 = & 8(r_2^2(r_1^2+r_1(-r_2+s_1)+s_1(-r_2+s_1))-(r_1^2+r_1s_1+s_1^2)s_2^2+(r_1+s_1)s_2^3).\end{aligned}$$

Likewise the points $A_3=(r_3^2,r_3^3)$ and $B_3=(s_3^2,s_3^3)$ satisfy system (3.3), then from the first equation we obtain two values of λ named $\lambda^{(1)}$ and $\lambda^{(2)}$ where $\lambda^{(1,2)}=(L_1\pm(1/2)\sqrt{L_2})/L_3$, and from the second equation we obtain $\omega=-2\sqrt{D_1/D_2}$. The values of L_1, L_2, L_3, D_1 and D_2 are given in the appendix of chapter 3.

Since the points $A_4=(r_4^2,r_4^3)$ and $B_4=(s_4^2,s_4^3)$ satisfy system (3.3), then we obtain $b_1=0$ from the first equation. This is a contradiction to the assumption.

Finally, we proved that the maximum number of limit cycles for systems of family $\mathcal{F}_1$ separated by the irreducible cubic curve c_3 is at most three.

Now we shall provide differential systems of family $\mathcal{F}_1$ separated by the cubic c_k with three limit cycles for $k = 1, \dots, 5$.

We will explain the method for constructing an example of three crossing limit cycles intersecting the curve c_1 in two points, and by a similar way we build examples of three crossing limit cycles intersecting the curve $c_k = 0$ in two points, when $k \in \{2, 3, 4, 5\}$.

Proof. **(Proof of statement (i) of Theorem 3.1.)**

***Example* 1. *Three limit cycles when the cubic of separation is* c_1.** *Here we define the region:*

$$R_{11} = \{(x, y) : y^2 - x(x^2 + x + 1) \geq 0\},\\ R_{12} = \{(x, y) : y^2 - x(x^2 + x + 1) \leq 0\}. \tag{3.4}$$

To construct three crossing limit cycles intersecting the curve c_1 in two points (α, β) and (γ, δ), these points must satisfy the system of equations

$$\begin{aligned} &e_1 = H_1(\alpha, \beta) - H_1(\gamma, \delta) = 0,\\ &e_2 = H_2(\alpha, \beta) - H_2(\gamma, \delta) = 0,\\ &c_i(\alpha, \beta) = 0, \quad c_i(\gamma, \delta) = 0, \, when \, i = 1. \end{aligned} \tag{3.5}$$

Then we suppose the existence of three real solutions of (3.5) *given by*

$$\begin{aligned} (\alpha_1, \beta_1, \gamma_1, \delta_1) &= (0.196374.., -0.492452.., 0.16032.., 0.436053..),\\ (\alpha_2, \beta_2, \gamma_2, \delta_2) &= (0.493859.., -0.926394.., 0.509449.., 0.94932..),\\ (\alpha_3, \beta_3, \gamma_3, \delta_3) &= (0.671593.., -1.19396.., 0.7124.., 1.25756..). \end{aligned}$$

So, by using equation e_1 we obtain the values of the three parameters λ, γ and σ in function of b_1, after that we fixe $b_1 = 5$ we obtain in the region R_{11}, the linear Hamiltonian system

$$(\dot{x}, \dot{y}) = \Big(-\frac{x}{2} + 5y + \frac{1}{5}, -\frac{x}{20} + \frac{y}{2} + \frac{4}{5} \Big), \tag{3.6}$$

with its Hamiltonian function $H_1(x, y) = -\dfrac{x^2}{40} + \dfrac{xy}{2} + \dfrac{4x}{5} - \dfrac{5y^2}{2} - \dfrac{y}{5}$.

Now by using equation e_2 we obtain the values of the three parameters b_2, c and d in function of ω, after that we fixe $\omega = 3$. Therefore we obtain in the region R_{12} the linear differential center

$$(\dot{x}, \dot{y}) = (0.271713..x - 2.32383..y - 0.10829.., x - 0.271713..y - 0.332792..). \tag{3.7}$$

This differential center has the first integral

$$H_2(x,y) = 4(x - 0.271713..y)^2 + 8(0.10829..y - 0.332792..x) + 9y^2.$$

Then the discontinuous piecewise differential system (3.6)–(3.7) *has exactly three limit cycles, see* (C_1) *of Figure 3.1.*

Example 2. Three limit cycles when the cubic of separation is c_3***.***

In the region R_{31} *we consider the Hamiltonian system*

$$(\dot{x},\dot{y}) = (-\frac{x}{2} + 5y + \frac{1}{5}, -\frac{x}{20} + \frac{y}{2} + \frac{4}{5}). \tag{3.8}$$

It has the Hamiltonian function $H_1(x,y) = -\frac{x^2}{40} + \frac{xy}{2} + \frac{4x}{5} - \frac{5y^2}{2} - \frac{y}{5}$.

In the region R_{32} *we consider the linear differential center*

$$(\dot{x},\dot{y}) = (-0.0813117..x - 1.00661..y + 0.56076.., x + 0.0813117..y + 8.22904..). \tag{3.9}$$

This differential center has the first integral

$$H_2(x,y) = 4(x + 0.0813117..y)^2 + 8(8.22904..x - 0.56076..y) + 4y^2.$$

The discontinuous piecewise differential system (3.8)–(3.9) *has exactly three limit cycles, because the system of equations* (3.5) *when* $i = 3$ *has only the three real solutions*

$$(\alpha_1,\beta_1,\gamma_1,\delta_1) = (0.700707.., -0.58655.., 0.765078.., 0.669204..),$$
$$(\alpha_2,\beta_2,\gamma_2,\delta_2) = (0.969795.., -0.955036.., 1.06038.., 1.09192..),$$
$$(\alpha_3,\beta_3,\gamma_3,\delta_3) = (1.13263.., -1.2054.., 1.23647.., 1.37492..).$$

see (C_3) *of Figure 3.1.*

Example 3. Three limit cycles when the cubic of separation is c_4***.*** *We define the following regions associated to the curve* c_4*.*

$$\begin{aligned} R_{41} &= \{(x,y) : y^2 - x^2(x-1) \le 0\}, \\ R_{42} &= \{(x,y) : y^2 - x^2(x-1) \ge 0\}. \end{aligned} \tag{3.10}$$

We consider the Hamiltonian system

$$(\dot{x},\dot{y}) = \left(\frac{2x}{5} - 4y + \frac{2}{5}, \frac{x}{25} - \frac{2y}{5} + \frac{1}{2}\right), \tag{3.11}$$

in the region R_{41}. *This Hamiltonian system has the Hamiltonian function* $H_1(x,y) = \dfrac{x^2}{50} - \dfrac{2xy}{5} + \dfrac{x}{2} + 2y^2 - \dfrac{2y}{5}$.

In the region R_{42} *we consider the linear center*

$$(\dot{x},\dot{y}) = (0.206345..x - 2.29258..y + 0.369992.., x - 0.206345..y - 0.332766..), \tag{3.12}$$

which has the first integral

$$H_2(x,y) = 4(x - 0.206345..y)^2 + 8(-0.332766..x - 0.369992..y) + 9y^2.$$

The discontinuous piecewise differential system (3.11)–(3.12) *has exactly three limit cycles, because the system of equations* (3.5) *when* $i = 4$ *has only three real solutions*

$$(\alpha_1,\beta_1,\gamma_1,\delta_1) = (1.29278.., -0.699517.., 1.55103.., 1.15135..),$$
$$(\alpha_2,\beta_2,\gamma_2,\delta_2) = (1.18739.., -0.513997.., 1.42939.., 0.936654..),$$
$$(\alpha_3,\beta_3,\gamma_3,\delta_3) = (1.05976.., -0.259058.., 1.25796.., 0.638917..),$$

see (C_4) *of Figure* 3.1. *This completes the proof of statement* (i) *for Theorem* 3.1. ■

Proof. (Proof of statement (ii) of Theorem 3.1.) ***Seven examples with three limit cycles when the cubic of separation is*** c_2. *We define the following regions associated to the curve* c_2.

$$\begin{aligned} R_{21} &= \{(x,y) : y^2 - x(x-1)(x-3) \geq 0\}, \\ R_{22} &= \{(x,y) : y^2 - x(x-1)(x-3) \leq 0, x \geq 3\}, \\ R_{23} &= \{(x,y) : y^2 - x(x-1)(x-3) \leq 0, 0 \leq x \leq 1\}. \end{aligned} \tag{3.13}$$

Example 4.1. *For the first configuration of three limit cycles separated by the curve* c_2, *we consider the Hamiltonian system*

$$(\dot{x},\dot{y}) = \left(\frac{39}{10}x - 3y - \frac{3}{5}, 0.0507..x - \frac{39}{10}y + 1\right), \tag{3.14}$$

in the region R_{22}, which has the Hamiltonian function

$$H_1(x,y) = 0.02535..x^2 - \frac{39}{10}xy + x + \frac{3y^2}{2} + \frac{3y}{5}.$$

In the region R_{21} we consider the linear differential center

$$(\dot{x},\dot{y}) = (0.138717..x - 1.01924..y - 0.161038.., x - 0.138717..y - 1.51992..), \tag{3.15}$$

which has the first integral

$$H_2(x,y) = 4(x - 0.138717..y)^2 + 8(0.161038..y - 1.51992..x) + 4y^2.$$

The discontinuous piecewise differential system (3.14)–(3.15) has exactly three limit cycles, because the system of equations (3.5) when $i = 2$, has only the three real solutions

$$(\alpha_1,\beta_1,\gamma_1,\delta_1) = (3.04422.., -0.524589.., 3.11914.., 0.887429..),$$
$$(\alpha_2,\beta_2,\gamma_2,\delta_2) = (3.1128.., -0.861296.., 3.21824.., 1.24817..),$$
$$(\alpha_3,\beta_3,\gamma_3,\delta_3) = (3.17852.., -1.11182.., 3.30364.., 1.52013..),$$

see (C_2^1) of Figure 3.2.

Example 4.2. *For the second configuration we consider the Hamiltonian system*

$$(\dot{x},\dot{y}) = \left(1.x - 5y - \frac{3}{10}, \frac{1}{5}x - y - 2\right), \tag{3.16}$$

in the region R_{21}. It has the Hamiltonian function $H_1(x,y) = \frac{1}{10}x^2 - xy - 2x + \frac{5y^2}{2} + \frac{3}{10}y$. Now we consider the second Hamiltonian system

$$(\dot{x},\dot{y}) = (0.214743..x - 1.04611y + 0.0404675.., x - 0.214743..y - 2.71521..), \tag{3.17}$$

in the region R_{22}. This system has the first integral

$$H_2(x,y) = 4(x - 0.311552..y)^2 + 8(0.286927..y - 3.12887..x) + 9y^2.$$

The discontinuous piecewise differential system (3.16)–(3.17) has exactly three limit cycles, *because the system of equations (3.5) when $i = 2$, has only the three real solutions*

$$(\alpha_1, \beta_1, \gamma_1, \delta_1) = (3.00992.., -0.244971.., 3.30407.., 1.52144..),$$
$$(\alpha_2, \beta_2, \gamma_2, \delta_2) = (3.02738.., -0.409955.., 3.36189.., 1.69517..),$$
$$(\alpha_3, \beta_3, \gamma_3, \delta_3) = (3.04843.., -0.549942.., 3.41294.., 1.84408..),$$

see (C_2^2) of Figure 3.2.

Example 4.3. *To obtain the third configuration we consider in the region R_{21} the Hamiltonian system*

$$(\dot{x}, \dot{y}) = \left(0.63x - 3y - \frac{1}{5}, -\frac{23x}{1000} - \frac{23y}{100} + 1\right), \tag{3.18}$$

which has the Hamiltonian function $H_1(x, y) = 0.06615x^2 - 0.63xy - 2.4x + \dfrac{3y^2}{2} + 0.2y$. In the region R_{22} we consider the linear center

$$(\dot{x}, \dot{y}) = (0.222578x - 1.04954y + 0.0489144.., x - 0.222578..y - 3.2101..). \tag{3.19}$$

This differential system has the first integral

$$H_2(x, y) = 4(x - 0.222578..y)^2 + 8(-3.2101..x - 0.0489144..y) + 4y^2.$$

The discontinuous piecewise differential system (3.18)–(3.19) has exactly three limit cycles, due to the fact that the system (3.5) has three real solutions when $i = 2$. These solutions are

$$(\alpha_1, \beta_1, \gamma_1, \delta_1) = (3.20789.., -1.21343.., 3.74104.., 2.7566..),$$
$$(\alpha_2, \beta_2, \gamma_2, \delta_2) = (3.3699.., -1.71878.., 3.9416.., 3.30416..),$$
$$(\alpha_3, \beta_3, \gamma_3, \delta_3) = (3.51026.., -2.12043.., 4.10331.., 3.74825..),$$

see (C_2^3) of Figure 3.3.

Example 4.4. *To obtain the fourth configuration we consider in the region R_{21} the Hamiltonian system*

$$(\dot{x}, \dot{y}) = \left(\frac{1}{5}x - y - \frac{1}{5}, 0.04x - \frac{1}{5}y - 2\right), \tag{3.20}$$

which has the Hamiltonian function $H_1(x, y) = 0.02..x^2 - \dfrac{1}{5}xy - 2x + \dfrac{y^2}{2} + \dfrac{1}{5}y$.

In the region R_{22} we consider the linear center

$$(\dot{x},\dot{y}) = \begin{aligned}&(0.0193278..x - 0.0628736..y + 0.0594448.., x - 0.0193278..y \\ &-2.86499..).\end{aligned} \tag{3.21}$$

This differential system has the first integral

$$H_2(x,y) = 4(x - 0.0193278..y)^2 + 8(-2.86499..x - 0.0594448..y) + \frac{y^2}{4}.$$

The discontinuous piecewise differential system (3.20)–(3.21) *has exactly three limit cycles, because the system* (3.5) *has three real solutions when $i = 2$, which are*

$$\begin{aligned}(\alpha_1,\beta_1,\gamma_1,\delta_1) &= (3.33264.., -1.60807.., 3.86848.., 3.1044..),\\ (\alpha_2,\beta_2,\gamma_2,\delta_2) &= (3.71366.., -2.68179.., 4.26489.., 4.19675..),\\ (\alpha_3,\beta_3,\gamma_3,\delta_3) &= (3.97997.., -3.40919.., 4.54158.., 4.9795..),\end{aligned}$$

see (C_2^4) of Figure 3.3.

Example 4.5. *For the fifth configuration we consider in the region R_{21} the Hamiltonian system*

$$(\dot{x},\dot{y}) = \Big(-\frac{14x}{5} - 7y + \frac{163}{100}, \frac{28x}{25} + \frac{14y}{5} + \frac{2}{5}\Big), \tag{3.22}$$

which has the Hamiltonian function $H_1(x,y) = \dfrac{14x^2}{25} + \dfrac{14xy}{5} + \dfrac{2x}{5} + \dfrac{7y^2}{2} - \dfrac{163y}{100}$.
In the region R_{23} we consider the linear center

$$(\dot{x},\dot{y}) = \begin{aligned}&(0.338865..x - 0.142607..y - 0.170936.., 0.338865..x \\ &-0.142607..y - 0.170936..).\end{aligned} \tag{3.23}$$

This differential system has the first integral

$$H_2(x,y) = 4(x - 0.338865..y)^2 + 8(0.170936..y - 0.0452618..x) + \frac{y^2}{9}.$$

When $i = 2$ in the system of equations (3.5) *the discontinuous piecewise differential system*

(3.22)–(3.23) *has exactly three limit cycles intersecting the cubic curve* c_2 *in the points*

$$(\alpha_1, \beta_1, \gamma_1, \delta_1) = (0.748414.., 0.651116.., 0.605489.., -0.756295..),$$
$$(\alpha_2, \beta_2, \gamma_2, \delta_2) = (0.911161.., 0.4112.., 0.747078.., -0.652453..),$$
$$(\alpha_3, \beta_3, \gamma_3, \delta_3) = (0.988069.., 0.154006.., 0.87431.., -0.483319..),$$

see (C_2^5) *of Figure 3.4.*

Example 4.6. *Now we give un example for the sixth configuration. We consider in the region* R_{21} *the Hamiltonian system*

$$(\dot{x}, \dot{y}) = (62.8408..x + 3y + 11.8867.., -1316.32..x - 62.8408..y + 2292.58..), \tag{3.24}$$

with its Hamiltonian function

$$H_1(x, y) = -658.162..x^2 - 62.8408..xy + 2292.58..x - \frac{3y^2}{2} - 11.8867..y.$$

In the region $R_{22} \cup R_{23}$ *we consider the linear center*

$$(\dot{x}, \dot{y}) = (-0.4x - 2.12..y + 0.2.., x + 0.4y - 1.58..). \tag{3.25}$$

This differential system has the first integral

$$H_2(x, y) = 4x^2 + 3.2..xy - 12.64x + 8.48..y^2 - 1.6..y.$$

For this configuration the system (3.5) *has three real solutions when* $i = 2$*. Hence, the discontinuous piecewise differential system* (3.24)–(3.25) *has exactly three limit cycles given by*

$$(\alpha_1, \beta_1, \gamma_1, \delta_1) = (0.953136.., 0.302372.., 0.905745.., -0.422833..),$$
$$(\alpha_2, \beta_2, \gamma_2, \delta_2) = (0.788975.., 0.60673.., 0.73033.., -0.668586..),$$
$$(\alpha_3, \beta_3, \gamma_3, \delta_3) = (3.02633.., 0.401853.., 3.21618.., -1.24131..),$$

see (C_2^6) *of Figure 3.4.*

Example 4.7. *Finally for the seventh configuration we consider in the region* R_{21} *the Hamil-*

tonian system

$$(\dot{x},\dot{y}) = (98.4926..x + 3y + 225.612.., -3233.6..x - 98.4926..y + 7255.43..), \tag{3.26}$$

with its Hamiltonian function

$$H_1(x,y) = -1616.8..x^2 - 98.4926..xy + 7255.43..x - \frac{3y^2}{2} - 225.612..y.$$

In the region $R_{22} \cup R_{23}$ we consider the linear center

$$(\dot{x},\dot{y}) = \Big(-\frac{2x}{5} - \frac{29y}{25} + \frac{1}{5}, x + \frac{2y}{5} - \frac{79}{50}\Big). \tag{3.27}$$

This differential system has the first integral

$$H_2(x,y) = 4x^2 + 3.2..xy - 12.64..x + 4.64..y^2 - 1.6..y.$$

The system of equations (3.5) *has three real solutions when $i = 2$. So, the discontinuous piecewise differential system* (3.26)–(3.27) *has exactly three limit cycles intersecting the cubic curve c_2 in the points*

$$\begin{aligned}(\alpha_1,\beta_1,\gamma_1,\delta_1) &= (0.78786.., 0.608054.., 0.707741.., -0.688578..),\\ (\alpha_2,\beta_2,\gamma_2,\delta_2) &= (3.02845.., 0.418065.., 3.43049.., -1.89457..),\\ (\alpha_3,\beta_3,\gamma_3,\delta_3) &= (3.07065.., 0.670216.., 3.54067.., -2.20536..),\end{aligned}$$

see (C_2^7) of Figure 3.5. This completes the proof of statement (ii) for Theorem 3.1. ■

Proof. **(Proof of statement (iii) of Theorem 2.3.)** ***Seven examples with three limit cycles when the cubic of separation is c_5.***

We define the following three regions associated to the curve c_5

$$\begin{aligned}R_{51} &= \{(x,y) : y^2 - x^2(x+1) \le 0, x \ge 0\},\\ R_{52} &= \{(x,y) : y^2 - x^2(x+1) \ge 0\},\\ R_{53} &= \{(x,y) : y^2 - x^2(x+1) \le 0, -1 \le x \le 0\}.\end{aligned} \tag{3.28}$$

Example 5.1. *For the first configuration of the class C_5 and in the region R_{51} we consider the*

Hamiltonian system

$$(\dot{x},\dot{y}) = \Big(\frac{9x}{10} - 3y + \frac{1}{5}, \frac{27x}{100} - \frac{9y}{10} + \frac{6}{5}\Big), \tag{3.29}$$

which has the Hamiltonian function $H_1(x,y) = \dfrac{27x^2}{200} - \dfrac{9xy}{10} + \dfrac{6x}{5} + \dfrac{3y^2}{2} - \dfrac{y}{5}$.
In the region R_{52} *we consider the linear differential center*

$$(\dot{x},\dot{y}) = (0.148466..x - 0.744542..y + 0.46566.., x - 0.148466..y + 2.05075..). \tag{3.30}$$

Its corresponding first integral is

$$H_2(x,y) = 4(x - 0.148466..y)^2 + 8(2.05075..x - 0.46566..y) + \frac{289y^2}{100}.$$

The discontinuous piecewise differential system (3.29)–(3.30) *has exactly three limit cycles, because the system of equations* (3.5) *when* $i = 5$, *has the three real solutions*

$$(\alpha_1,\beta_1,\gamma_1,\delta_1) = (0.482359.., -0.587282.., 0.727878.., 0.956787..),$$
$$(\alpha_2,\beta_2,\gamma_2,\delta_2) = (0.644665.., -0.826748.., 0.94961.., 1.32593..),$$
$$(\alpha_3,\beta_3,\gamma_3,\delta_3) = (0.770799.., -1.02571.., 1.11523.., 1.62197..),$$

see (C_5^1) *of Figure* 3.6.

Example 5.2. *For the second configuration of the class* C_5 *and in the region* R_{51} *we consider the Hamiltonian system*

$$(\dot{x},\dot{y}) = \Big(-\frac{42x}{25} - 8y + 1, \frac{441x}{1250} + \frac{42y}{25} + \frac{12}{5}\Big), \tag{3.31}$$

with its Hamiltonian function $H_1(x,y) = \dfrac{441x^2}{2500} + \dfrac{42xy}{25} + \dfrac{12x}{5} + 4y^2 - y$.
Now we consider the linear center

$$(\dot{x},\dot{y}) = (-0.680215..x - 2.02519..y - 0.290646.., x + 0.680215..y + 0.771302..), \tag{3.32}$$

in the region R_{52}. *This differential system has the first integral*

$$H_2(x,y) = 4(x + 0.680215..y)^2 + 8(0.771302..x + 0.290646..y) + 6.25..y^2.$$

The discontinuous piecewise differential system (3.31)–(3.32) has exactly three limit cycles, because the system of equations (3.5) when $i = 5$, has the three real solutions

$$(\alpha_1, \beta_1, \gamma_1, \delta_1) = (-0.999059.., 0.0306461.., -0.857795.., 0.323475..),$$
$$(\alpha_2, \beta_2, \gamma_2, \delta_2) = (-0.969398.., -0.16958.., -0.614006.., 0.381472..),$$
$$(\alpha_3, \beta_3, \gamma_3, \delta_3) = (-0.888264.., -0.296919.., -0.329658.., 0.269906..),$$

see (C_5^2) of Figure 3.6.

Example 5.3. *To obtain the third configuration of the class C_5 and in the region R_{51} we consider the linear center*

$$(\dot{x}, \dot{y}) = (0.371399..x - 1.13794..y - 2.0777.., x - 0.371399..y - 0.612049..), \tag{3.33}$$

with its first integral $H_2(x, y) = 4(x - 0.371399y)^2 + 8(2.0777y - 0.612049x) + 4y^2$.
In the region R_{52} we consider the Hamiltonian system

$$(\dot{x}, \dot{y}) = (2(x - 2y - 3), -4 + x - 2y), \tag{3.34}$$

which has the Hamiltonian function $H_1(x, y) = \dfrac{x^2}{2} - 2xy - 4x + 2y^2 + 6y$.
In this case we have also three limit cycles, due to the fact that the system of equations (3.5) when $i = 5$ has only three real solutions. Hence, the discontinuous piecewise differential system (3.33)–(3.34) has exactly three limit cycles intersecting the curve c_5 in the points

$$(\alpha_1, \beta_1, \gamma_1, \delta_1) = (1.26688.., -1.90744.., 0.409461.., -0.486115..),$$
$$(\alpha_2, \beta_2, \gamma_2, \delta_2) = (1.19339.., -1.76742.., 0.502225.., -0.615553..),$$
$$(\alpha_3, \beta_3, \gamma_3, \delta_3) = (1.09399.., -1.58308.., 0.620329.., -0.78963..),$$

see (C_5^3) of Figure 3.7.

Example 5.4. *To get the fourth configuration of the class C_5 and in the region $R_{51} \cup R_{53}$ we consider the Hamiltonian system*

$$(\dot{x}, \dot{y}) = \left(\frac{9x}{10} - 3y + \frac{1}{5}, \frac{27x}{100} - \frac{9y}{10} + \frac{6}{5}\right), \tag{3.35}$$

which has the Hamiltonian function $H_1(x,y) = \dfrac{27x^2}{200} - \dfrac{9xy}{10} + \dfrac{6x}{5} + \dfrac{3y^2}{2} - \dfrac{y}{5}$.
Now we consider the linear center

$$(\dot{x},\dot{y}) = (0.685337..x - 2.15969..y + 0.24115..,x - 0.685337..y + 1.13465..), \qquad (3.36)$$

in the region R_{52}. *This differential system has the first integral*

$$H_2(x,y) = 4(x - 0.685337..y)^2 + 8(1.13465..x - 0.24115..y) + \frac{169y^2}{25}.$$

The discontinuous piecewise differential system (3.35)–(3.36) *has exactly three limit cycles, because the system of equations* (3.5) *when* $i = 5$, *has the three real solutions*

$$(\alpha_1,\beta_1,\gamma_1,\delta_1) = (-0.436282..,-0.327565..,-0.713688..,0.381881..),$$
$$(\alpha_2,\beta_2,\gamma_2,\delta_2) = (0.501048..,-0.613871..,0.754011..,0.998606..),$$
$$(\alpha_3,\beta_3,\gamma_3,\delta_3) = (0.782108..,-1.04408..,1.12985..,1.64891..),$$

see (C_5^4) *of Figure 3.7.*

Example 5.5. *For the fifth configuration of the class* C_5 *and in the region* $R_{51} \cup R_{53}$ *we consider the Hamiltonian system*

$$(\dot{x},\dot{y}) = \left(\frac{1}{200}(367x - 1468y + 40), \frac{367x}{800} - \frac{367y}{200} + \frac{6}{5}\right), \qquad (3.37)$$

which has the Hamiltonian function $H_1(x,y) = 0.229375..x^2 - 1.835..xy + \dfrac{6x}{5} + 3.67..y^2 - \dfrac{y}{5}$.
Now we consider the linear center

$$(\dot{x},\dot{y}) = (0.591399..x - 2.59975..y + 0.172769..,x - 0.591399..y + 0.826843..), \qquad (3.38)$$

in the region R_{52}. *This differential system has the first integral*

$$H_2(x,y) = 4(x - 0.591399..y)^2 + 8(0.826843..x - 0.172769..y) + 9y^2.$$

The system of equations (3.5) *when* $i = 5$ *has three real solutions, which means that the discontinuous piecewise differential system* (3.37)–(3.38) *has exactly three limit cycles intesecting* *the curve c_5 in the following points*

$$(\alpha_1,\beta_1,\gamma_1,\delta_1) = (-0.801385..,-0.357147..,-0.992606..,0.0853535..),$$
$$(\alpha_2,\beta_2,\gamma_2,\delta_2) = (-0.457786..,-0.337092..,-0.925215..,0.253018..),$$
$$(\alpha_3,\beta_3,\gamma_3,\delta_3) = (0.0989294..,-0.103708..,0.147927..,0.158491..),$$

see (C_5^5) of Figure 3.8.

Example 5.6. *Now we give the limit cycles of the sixth configuration of the class C_5. In the region R_{51} we consider the Hamiltonian system*

$$(\dot{x},\dot{y}) = (-5-3.2x-8y,-3+1.28x+3.2y), \tag{3.39}$$

which has the Hamiltonian function $H_1(x,y) = 0.64x^2+3.2xy-3x+4y^2+5y$.

Now we consider in the region R_{52} the linear center

$$(\dot{x},\dot{y}) = (-0.213131..x-0.535425..y-0.615264..,x+0.213131..y-0.568363..), \tag{3.40}$$

with its first integral $H_2(x,y) = 4(x+0.213131..y)^2+8(0.615264..y-0.568363..x)+1.96..y^2$. The discontinuous piecewise differential system (3.39)–(3.40) has exactly three limit cycles, because the system of equations (3.5) when $i=5$, has the three real solutions

$$(\alpha_1,\beta_1,\gamma_1,\delta_1) = (2.19804..,-3.93077..,0.809528..,1.08897..),$$
$$(\alpha_2,\beta_2,\gamma_2,\delta_2) = (1.99735..,-3.45798..,0.547303..,0.680793..),$$
$$(\alpha_3,\beta_3,\gamma_3,\delta_3) = (1.71246..,-2.82035..,0.123691..,-0.131118..),$$

see (C_5^6) of Figure 3.8.

Example 5.7. *Finally and for the seventh configuration of the class C_5 and in the region R_{51} we consider the Hamiltonian system*

$$(\dot{x},\dot{y}) = \left(-\frac{7x}{5}-7y-9,\frac{7x}{25}+\frac{7y}{5}-5\right), \tag{3.41}$$

which has the Hamiltonian function $H_1(x,y) = \dfrac{7x^2}{50}+\dfrac{7xy}{5}-5x+\dfrac{7y^2}{2}+9y$.

Now we consider the linear center

$$(\dot{x}, \dot{y}) = \begin{array}{l}(-0.801265..x - 0.944526..y - 0.242753.., x + 0.801265..y.. \\ -0.973031..),\end{array} \tag{3.42}$$

in the region R_{52}. *This differential system has the first integral*

$$H_2(x,y) = 4(x + 0.801265..y)^2 + 8(0.242753..y - 0.973031..x) + 1.21..y^2.$$

Due to the fact that the system of equations (3.5) *when* $i = 5$, *has three real solutions, we know that the discontinuous piecewise differential system* (3.41)–(3.42) *has exactly three limit cycles intersecting the curve* c_5 *in the points*

$$(\alpha_1, \beta_1, \gamma_1, \delta_1) = (2.55092.., -4.80692.., 0.700327.., 0.913203..),$$
$$(\alpha_2, \beta_2, \gamma_2, \delta_2) = (2.29096.., -4.15603.., 0.0707197.., -0.0731776..),$$
$$(\alpha_3, \beta_3, \gamma_3, \delta_3) = (1.8659.., -3.15879.., 0.786808.., -1.05174..),$$

see (C_5^7) *of Figure 3.9. This completes the proof of statement* (iii) *for Theorem 3.1.* ■

Section 3.2 LC of Discontinuous PWLS Intersecting c_2 or c_5 in four or two points

The main goal of this part is to provide the maximum number of crossing limit cycles of the family $\mathcal{F}_2$ which intersect the irreducible cubic curve c_2 or c_5 in four points or in four and two points simultaneously.

3.2.1 Statement of the second main result

Now we denote by $\mathcal{F}_2$ the family of discontinuous piecewise differential systems separated by the cubic curve c_2 or c_5 contained in three regions. When the cubic of separation is c_2, we define in regions R_{22} and R_{23} a Hamiltonian system without equilibrium points, and we define in the region R_{21} a linear center. When the cubic of separation is c_5, we define in regions R_{51} and R_{53} a Hamiltonian system without equilibrium points, and in the region R_{52} we define a linear center. Then we have the following result.

Theorem 3.2 *The following statements hold.*

(a) The maximum number of crossing limit cycles of systems of the family $\mathcal{F}_2$ which intersect the cubic curve c_2 or c_5 in four points is three.

This maximum is reached, see Example 6 of C_2 and Example 7 of C_5.

(b) The maximum number of crossing limit cycles of discontinuous piecewise differential systems of the family $\mathcal{F}_2$ intersecting simultaneously in four points and two points the cubic c_2 or c_5 is six.

This maximum is reached, see Example 8.1 of C_2 and Example 8.2 of C_5.

(C_2^1) (C_5^1)

Figure 3.10: The three limit cycles of the discontinuous piecewise differential system (C_2^1) for (3.48)–(3.49), and (C_5^1) for (3.50)–(3.51).

(C_2^2) (C_5^2)

Figure 3.11: The six limit cycles of the discontinuous piecewise differential system (C_2^2) for (3.52)–(3.53), and (C_5^2) for (3.54)–(3.55).

3.2.2 Proof of the second main result

Proof. **(Proof of statement (a) of Theorem 3.2.)** *We are going to prove it for the class C_5, and by a similar way we get the proof of the statement for the class C_2.*

We consider the three regions defined in (3.28).

In the region R_{51} we consider the linear differential center

$$\dot{x} = -\frac{1}{4}y(4b_2^2 + \omega^2) - b_2 x + d, \quad \dot{y} = b_2 y + c + x, \tag{3.43}$$

its first integral is $H_2(x,y) = 4(b_2 y + x)^2 + 8(cx - dy) + y^2\omega^2$.

Now we consider the two Hamiltonian systems

$$\begin{aligned} &\dot{x} = -\lambda_1 b_1 x + b_1 y + \mu_1, \quad \dot{y} = -\lambda_1^2 b_1 x + \lambda_1 b_1 y + \sigma_1, \text{ in the region } R_{52}, \\ &\dot{x} = -\lambda_3 b_3 x + b_3 y + \mu_3, \quad \dot{y} = -\lambda_3^2 b_3 x + \lambda_3 b_3 y + \sigma_3, \text{ in the region } R_{53}, \end{aligned} \tag{3.44}$$

with $b_i \neq 0$ and $\sigma_i \neq \lambda_i \mu_i$, when $i = 1, 3$. Their corresponding Hamiltonian first integrals are

$$\begin{aligned} H_1(x,y) &= -(\lambda_1^2 b_1/2)x^2 + \lambda_1 b_1 xy - (b_1/2)y^2 + \sigma_1 x - \mu_1 y, \\ H_3(x,y) &= -(\lambda_3^2 b_3/2)x^2 + \lambda_3 b_3 xy - (b_3/2)y^2 + \sigma_3 x - \mu_3 y. \end{aligned} \tag{3.45}$$

If we suppose that the discontinuous piecewise differential system (3.43)–(3.44) has three limit cycles which intersect the cubic c_5 in the points $p_1^{(i)} = (\alpha_i^2 - 1, \alpha_i(\alpha_i^2 - 1))$, $p_2^{(i)} = (\beta_i^2 - 1, \beta_i(\beta_i^2 - 1))$, $p_3^{(i)} = (\gamma_i^2 - 1, \gamma_i(\gamma_i^2 - 1))$ and $p_4^{(i)} = (\delta_i^2 - 1, \delta_i(\delta_i^2 - 1))$ they must satisfy the following system

$$\begin{aligned} &H_1(\alpha_i^2 - 1, \alpha_i(\alpha_i^2 - 1)) - H_1(\beta_i^2 - 1, \beta_i(\beta_i^2 - 1)) = 0, \\ &H_2(\beta_i^2 - 1, \beta_i(\beta_i^2 - 1)) - H_2(\delta_i^2 - 1, \delta_i(\delta_i^2 - 1)) = 0, \\ &H_2(\alpha_i^2 - 1, \alpha_i(\alpha_i^2 - 1)) - H_2(\gamma_i^2 - 1, \gamma_i(\gamma_i^2 - 1)) = 0, \\ &H_3(\gamma_i^2 - 1, \gamma_i(\gamma_i^2 - 1)) - H_3(\delta_i^2 - 1, \delta_i(\delta_i^2 - 1)) = 0. \end{aligned} \tag{3.46}$$

We deal with the first and the last equations. By solving the first one for $i = 1, 2, 3$ we obtain the expression of λ_1, μ_1 and σ_1, and by solving the last equation we get the expression of λ_3, μ_3 and σ_3. Now we suppose that system (3.43)–(3.44) has a fourth limit cycle, so system (3.46) satisfied for $i = 4$. We solve the first and the last equations we get $b_1 = 0$ and $b_3 = 0$, respectively. This is a contradiction of the assumption. Thus we prove that the maximum *number of limit cycles intersected with the cubic curve c_5 in four points is three.*

Example 6. Three limit cycles intersecting the curve c_2 in four points. *We consider the regions defined in (3.13).*

We will explain the method for constructing an example of three limit cycles intersecting the curve c_2 in four points. These points must satisfy the system of equations

$$\begin{aligned}
&e_1 = H_1(\alpha_s,\beta_s)) - H_1(\gamma_s,\delta_s) = 0,\\
&e_2 = H_2(\alpha_s,\beta_s) - H_2(f_s,g_s) = 0,\\
&e_3 = H_2(\gamma_s,\delta_s) - H_2(h_s,k_s) = 0,\\
&e_4 = H_3(f_s,g_s) - H_3(h_s,k_s) = 0,\\
&c_i(\alpha_s,\beta_s) = c_i(\gamma_s,\delta_s) = 0,\ c_i(f_s,g_s) = c_i(h_s,k_s) = 0,
\end{aligned} \tag{3.47}$$

with $s = 1,2,3$ and $i = 2$.

In the region R_{21} and by using the software Mathematica we construct the curves

$$H_2(x,y) = 4x^2 + 8\left(\frac{7y}{5} - 2x\right) + 16y^2 = 3k - 8, ...(I)\ \ k \in \{1,2,3\}.$$

Where the level curve corresponding to $k = 1$ in (I), intersects the cubic c_2, in $(\alpha_1,\beta_1) =$ $(3.02482..,0.389889..)$, $(f_1,g_1) = (0.927682..,0.372865..)$, $(\gamma_1,\delta_1) = (3.15833..,-1.03889..)$ and $(h_1,k_1) = (0.434221..,-0.793941..)$. These points satisfy equation e_2 and e_3 of system (3.47) for $s = 1$ and $i = 2$.

The level curve corresponding to $k = 2$ in (I), intersects the cubic c_2 in $(\alpha_2,\beta_2) = (3.04069..,$ $0.502491..)$, $(f_2,g_2) = (0.878213..,0.47637..)$, $(\gamma_2,\delta_2) = (3.19002..,-1.15216..)$ and $(h_2,k_2) =$ $(0.017634..,-0.227296..)$. These points satisfy the equation e_2 and e_3 of system (3.47) for $s = 2$ and $i = 2$.

The level curve corresponding to $k = 3$ in (I), intersects c_2 in $(\alpha_3,\beta_3) = (3.05752..,$ $0.601532..)$, $(f_3,g_3) = (0.821437..,0.565285..)$, $(\gamma_3,\delta_3) = (3.21933..,-1.25183..)$ and (h_2,k_2) $= (0.00229058..,0.0827693..)$. These points satisfy the equation e_2 and e_3 of system (3.47) for $s = 3$ and $i = 2$.

In short, in the region R_{21} we consider the linear differential center

$$\dot{x} = -1.4 - 4y, \quad \dot{y} = -2 + x, \tag{3.48}$$

with its first integral $H_2(x,y) = 4x^2 + 8\left(\frac{7y}{5} - 2x\right) + 16y^2$.

Now we have to give the expression of $H_1(x,y)$ *in the region* R_{22} *and* $H_3(x,y)$ *in the region* R_{23}. *Then we deal with the equations* e_1 *and* e_4 *in* (3.47). *By solving these equations for* $s = 1,2,3$, *then fixing* $b_1 = -2/25$ *in equation* e_1 *and* $\sigma_1 = -2$ *in equation* e_4, *we obtain the Hamiltonian systems*

$$
\begin{aligned}
\dot{x} = \;& -0.00039097..x - 0.08..y - 0.0252659.., \dot{y} = \frac{1.91072..}{10^6}x \\
& +0.00039097..y + 0.00562837.., \text{ in } R_{22}, \\
\dot{x} = \;& -0.0474089..x - 0.000365066..y + 0.955189.., \dot{y} = 6.15669..x \\
& +0.0474089..y - 2, \text{ in } R_{23}.
\end{aligned} \tag{3.49}
$$

The Hamiltonian first integrals of the Hamiltonian systems (3.49) *are*

$$
\begin{aligned}
H_1(x,y) &= \frac{9.55359x^2}{10^7} + 0.00039097..xy + 0.00562837..x + 0.04..y^2 + 0.0252659..y, \\
H_3(x,y) &= 3.07835..x^2 + 0.0474089..xy - 2x + 0.000182533..y^2 - 0.955189..y.
\end{aligned}
$$

For the discontinuous piecewise differential system (3.48)–(3.49) *all the real solutions of the system of equations* (3.47) *with* $s = 1,2,3$ *and* $i = 2$ *are*

$$
\begin{aligned}
(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = \;& (3.02482..,0.389889..,3.15833..,-1.03889.. \\
& 0.927682..,0.372865..,0.434221..,-0.793941..), \\
(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = \;& (3.04069..,0.502491..,3.19002..,-1.15216.. \\
& 0.878213..,0.476377..,0.017634..,-0.227296..), \\
(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = \;& (3.05752..,0.601532..,3.21933..,-1.25183.. \\
& 0.821437..,0.565285..,0.00229058..,0.0827693..).
\end{aligned}
$$

Then the discontinuous piecewise linear differential system (3.48)–(3.49) *has exactly three limit cycles, see* (C_2^1) *of Figure 3.10.*

Example 7. Three limit cycles intersecting the curve c_5 ***in four points.*** *We consider the regions defined in* (3.28). *In the region* R_{52} *we consider the linear differential center*

$$
\dot{x} = -\frac{x}{5} - \frac{629y}{100} + \frac{2}{5}, \dot{y} = x + \frac{y}{5} - \frac{4}{5}, \tag{3.50}
$$

its first integral is $H_2(x,y) = 4(x+\frac{y}{5})^2 + 8(-\frac{4x}{5} - \frac{2y}{5}) + 25y^2$.

Now we consider the following Hamiltonian systems

$$\begin{aligned} \dot{x} = &\ \ 29.8246..x + y - 34.8532.., \quad \dot{y} = -889.505..x - 29.8246..y \\ & \qquad\qquad\qquad\qquad\qquad\qquad +97.1847.., \ in\ R_{51}, \\ \dot{x} = &\ \ 0.00341157..x + \frac{3y}{2} - 0.095202.., \dot{y} = -\frac{7.7591..}{10^6}x \\ & \qquad\qquad\qquad -0.00341157..y + 0.198512.., \ in\ R_{53}. \end{aligned} \tag{3.51}$$

The Hamiltonian first integrals of the Hamiltonian systems (3.51) *are*

$$\begin{aligned} H_1(x,y) &= -444.753..x^2 - 29.8246..xy + 97.1847..x - \frac{y^2}{2} + 34.8532..y, \\ H_3(x,y) &= -\frac{3.8796}{10^6}x^2 - 0.00341157..xy + 0.198512..x - \frac{3y^2}{4} + 0.095202..y, \end{aligned}$$

respectively. For the discontinuos piecewise linear differential system (3.50)–(3.51) *the real solutions of the system of equations* (3.46) *are*

$$\begin{aligned} (\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = &\ \ (0.343158..,0.397701..,0.23703..,-0.263629.. \\ & -0.163657..,0.149667..,-0.0982745..,-0.0933207..), \\ (\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = &\ \ (0.396..,0.467884..,0.304088..,-0.347259.. \\ & -0.261162..,0.224484..,-0.173194..,-0.157483..), \\ (\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = &\ \ (0.438244..,0.525572..,0.354392..,-0.412436.. \\ & -0.345135..,0.279296..,-0.23907..,-0.208544..), \end{aligned}$$

where $\alpha_i = r_i^2 - 1$, $\beta_i = r_i(r_i^2 - 1)$, $\gamma_i = s_i^2 - 1$, $\delta_i = s_i(s_i^2 - 1)$, $a_i = f_i^2 - 1$, $b_i = f_i(f_i^2 - 1)$, $c_i = h_i^2 - 1$, $d_i = h_i(h_i^2 - 1)$.

Then the discontinuous piecewise linear differential system (3.50)–(3.51) *has exactly three limit cycles, see* (C_5^1) *of Figure 3.10.*

This completes the proof of statement (a) *of Theorem 3.2.* ∎

Proof. **(Proof of statement (b) of Theorem 3.2.)** *In order to have limit cycles with two and four intersection points, simultaneously, with the cubic* c_2*, the intersection points of the limit cycles in two points with* c_2 *must satisfy system* (3.5) *for* $i = 2$*, and the intersection points of the limit cycles in four points with* c_2 *must satisfy system* (3.46)*. In statement (ii)* *of Theorem 3.1 we proved that the maximum number of limit cycles with two intersection points with c_2 is three, and we also proved in the first statement of this Theorem that the maximum number of limit cycles with four intersection points with c_2 is three, then we have that the upper bound of maximum number of limit cycles with two and four intersection points, simultaneously, is six. This upper bound is reached.*

Examples 8.1. *Three limit cycles with four intersection points on c_2 and three limit cycles with two intersection points on c_2.*
Habitually we consider the regions defined in (3.13). In the region R_{21} we consider the linear differential center

$$\dot{x} = 0.0408657..x - 0.18657..y - 0.161702.., \dot{y} = x - 0.0408657..y - 2.22076, \tag{3.52}$$

its first integral is $H_2(x,y) = 4(x - 0.0413008..y)^2 + 8(0.163443..y - 2.21892..x) + 0.7569..y^2$. Now we consider the Hamiltonian systems

$$\begin{aligned} \dot{x} = &\quad 0.5x - 5y - 2, \quad \dot{y} = 0.05x - 0.5y + 2, \ in \qquad R_{22}, \\ \dot{x} = &\quad 0.275117..x + 4y - 0.895428.., \quad \dot{y} = -0.0189223..x \\ & \qquad\qquad -0.275117..y - 10.3678.., \ in \ R_{23}. \end{aligned} \tag{3.53}$$

The Hamiltonian first integrals of the systems (3.53) are

$$\begin{aligned} H_1(x,y) = &\quad 0.025x^2 - 0.5xy + 2x + \frac{5y^2}{2} + 2y, \\ H_3(x,y) = &\quad -0.00838913..x^2 - 0.28964..xy - 12.0087..x - \frac{5y^2}{2} + 0.897776..y. \end{aligned}$$

The discontinuos piecewise differential system (3.52)–(3.53) has three limit cycles intersecting the cubic c_2 in four points satisfying system (3.47) and three limit cycles intersecting the cubic c_2 in two points satisfying system (3.5) for $i = 2$, because all the real solutions of these two *systems are*

$$\begin{aligned}(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = & \ (3.29854..,-1.50447..,3.25648..,1.37282..,\\ & \ 0.996543..,0.0830817..,0.967473..,-0.252974..),\\ (\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = & \ (3.33756..,-1.62283..,3.29637..,1.49782..,0.959503..,\\ & \ 0.281699..,0.897792..,-0.44042..),\\ (\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = & \ (3.37472..,-1.73293..,3.33474..,1.61438..,0.916959..,\\ & \ 0.398981..,0.836989..,-0.547219..),\end{aligned}$$

$$\begin{aligned}(\alpha_4,\beta_4,\gamma_4,\delta_4) = & \ (3.1668..,-1.06984..,3.12648..,0.916988..),\\ (\alpha_5,\beta_5,\gamma_5,\delta_5) = & \ (3.21363..,-1.23276..,3.17161..,1.0872..),\\ (\alpha_6,\beta_6,\gamma_6,\delta_6) = & \ (3.25735..,-1.37562..,3.21493..,1.23712..).\end{aligned}$$

Then the discontinuous piecewise differential system (3.52)–(3.53) *has exactly six limit cycles, see* (C_2^2) *of Figure 3.11.*

Example 8.2. *Three limit cycles with four intersection points on c_5 and three limit cycles with two intersection points on c_5.*
In order to have limit cycles with two and four intersection points with the cubic c_5 simultaneously, the points of intersection of the limit cycles in two points with c_5 must satisfy system (3.5) *for $i = 5$, and the intersection points of the limit cycles in four points with c_5 must satisfy system* (3.46)*. In statement (iii) of Theorem 3.1 we proved that the maximum number of limit cycles with two intersection points with c_5 is three, and we also proved in statement (i) of Theorem 3.2 that the maximum number of limit cycles with four intersection points with c_5 is three, then we have that the upper bound of the maximum number of limit cycles with two and four intersection points, simultaneously, is six. The result is reached in the following example.*
Habitually we consider the regions defined in (3.28)*.*
In the region R_{52} we consider the linear differential center

$$\dot{x} = 0.208169..x - 0.0463593..y + 0.0028092.., \quad \dot{y} = x - 0.208169..y - 0.217992. \quad (3.54)$$

Its first integral is $H_2(x,y) = 4(x-0.208169..y)^2+8(-0.217992..x-0.0028092..y)+0.0121y^2$.

Now we consider the Hamiltonian systems

$$
\begin{aligned}
\dot{x} = &\quad 40x - 8y + 9, \quad \dot{y} = -33 + 200x - 40y, \qquad\qquad in\ R_{51},\\
\dot{x} = &\quad 54.173..x - 6y + 31.215.., \quad \dot{y} = 489.119..x - 54.173..y\\
&\qquad\qquad\qquad\qquad\qquad -0.0147783.., \ in\ R_{53},
\end{aligned}
\tag{3.55}
$$

and their Hamiltonian first integrals

$$
\begin{aligned}
H_1(x,y) &= 100x^2 - 40xy - 33x + 4y^2 - 9y,\\
H_3(x,y) &= 244.56..x^2 - 54.173..xy - 0.0147783..x + 3y^2 - 31.215..y,
\end{aligned}
$$

respectively.

Due to the fact that system (3.46) has three real solutions and system (3.5) for $i = 5$ has three real solutions, it results that the discontinuos piecewise differential systems (3.54)–(3.55) have three limit cycles intersecting the cubic c_5 in four points and three limit cycles intersecting the cubic c_5 in two points and we have H_3 instead of H_1. The real solutions of the two systems (3.46) and (3.5) for $i = 5$ are

$$
\begin{aligned}
(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) = &\ (0.294986..,-0.335687..,1.01573..,1.44209..,\\
&\ -0.113479..,0.106846..,-0.0149546..,-0.0148423..),\\
(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) = &\ (0.34172..,-0.395823..,1.09233..,1.58004..,\\
&\ -0.159928..,0.146583..,-0.0691478..,-0.0667143..),\\
(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) = &\ (0.34172..,-0.395823..,1.09233..,1.58004..,\\
&\ -0.200605..,0.179359..,-0.119305..,-0.111962..),\\
(\alpha_4,\beta_4,\gamma_4,\delta_4) = &\ (0.203429..,0.223163..,0.620097..,0.789278..),\\
(\alpha_5,\beta_5,\gamma_5,\delta_5) = &\ (0.126866..,0.134673..,0.705726..,0.921702..),\\
(\alpha_6,\beta_6,\gamma_6,\delta_6) = &\ (0.0684267..,0.070729..,0.773252..,1.02969..).
\end{aligned}
$$

Then the discontinuous piecewise differential systems (3.54)–(3.55) have exactly six limit cycles, see (C_5^2) of Figure 3.11.

This completes the proof of statement (b) of Theorem 3.2. ∎

Chapter 4

Limit Cycles of Planar Piecewise Linear Hamiltonian Systems Without Equilibrium Points Separated by two Circles

The solution of the 16th Hilbert problem of discontinuous piecewise linear differential Hamiltonian systems without equilibrium points separated by either reducible, or irreducible cubics is solved by Benterki and Llibre, see [4, 7, 11, 22]. In this chapter we give the solution of the extended 16th Hilbert problem for discontinuous piecewise differential systems formed by three linear Hamiltonian systems without equilibrium points separated by two concentric circles $\mathbb{S}_1$ and $\mathbb{S}_2$ such that $\mathbb{S}_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$; and $\mathbb{S}_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = a^2\}$; where $a > 1$.

We denote by $\mathcal{F}$ the family of discontinuous piecewise linear Hamiltonian systems without equilibrium points separated by the two circles $\mathbb{S}_1$ and $\mathbb{S}_2$.

The upper bound of crossing limit cycles of linear Hamiltonian systems without equilibria intersecting only in two points the circle $\mathbb{S}_1$ or the circle $\mathbb{S}_2$ was studied in [4], where the authors proved that such systems can have at most three crossing limit cycles intersecting a circle in two points. For that, in our chapter we are intersting in studying the upper bound of crossing limit cycles for systems in the class $\mathcal{F}$ intersecting the circle $\mathbb{S}_1$ in two points and intersecting the circle $\mathbb{S}_2$ in two points. For this class we get the following three zones

$$\begin{aligned} Z^1 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}, \\ Z^2 &= \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < a\}, \\ Z^3 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > a\}. \end{aligned} \tag{4.1}$$

Section 4.1 Statement of the main result

Our main result is the following.

Theorem 4.1 *The following statements hold.*

(i) The maximum number of crossing limit cycles of systems in $\mathcal{F}$ is three.

(ii) There are systems in $\mathcal{F}$ exhibiting exactly three limit cycles, see Figure 4.1.

(iii) There are systems in $\mathcal{F}$ exhibiting exactly two limit cycles, see Figure 4.2.

(iv) There are systems in $\mathcal{F}$ exhibiting exactly one limit cycle, see Figure 4.3.

(v) There are systems in $\mathcal{F}$ without limit cycles.

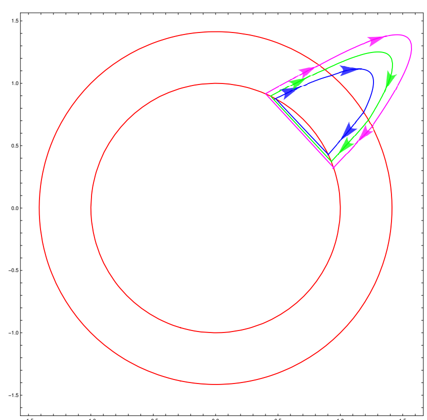


Figure 4.1: The three limit cycles of the discontinuous piecewise differential system (4.6).

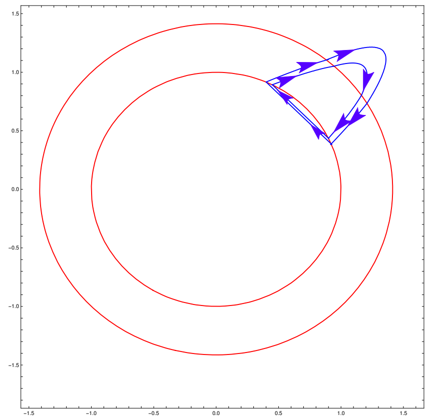


Figure 4.2: The two limit cycles of the discontinuous piecewise differential system (4.7).

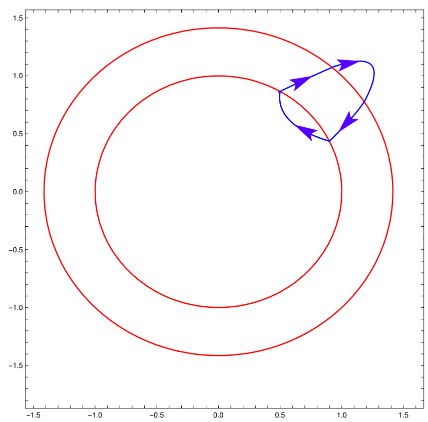


Figure 4.3: The one limit cycle of the discontinuous piecewise differential system (4.8).

4.1.1 Proof of the main result

To prove Theorem 4.1, we use lemma 1 in chapter 3 which provides the normal form for an arbitrary linear Hamiltonian differential system without equilibrium points.

Proof. **(Proof of statement (i) of Theorem 4.1)** *In the zone Z^1 we consider the arbitrary linear Hamiltonian system*

$$\dot{x} = -\lambda_1 b_1 x + b_1 y + \mu_1, \quad \dot{y} = -\lambda_1^2 b_1 x + \lambda_1 b_1 y + \sigma_1, \tag{4.2}$$

with $b_1 \neq 0$ and $\sigma_1 \neq \lambda_1\mu_1$. This system has the first integral

$$H_1(x,y) = -\Big(\lambda_1^2\frac{b_1}{2}\Big)x^2 + \lambda_1 b_1 xy - \Big(\frac{b_1}{2}\Big)y^2 + \sigma_1 x - \mu_1 y.$$

In the zone Z^2 we consider the discontinuous piecewise linear Hamiltonian system

$$\dot{x} = -\lambda_2 b_2 x + b_2 y + \mu_2, \quad \dot{y} = -\lambda_2^2 b_2 x + \lambda_2 b_2 y + \sigma_2, \tag{4.3}$$

with $b_2 \neq 0$ and $\sigma_2 \neq \lambda_2\mu_2$, with its first integral

$$H_2(x,y) = -\Big(\lambda_2^2\frac{b_2}{2}\Big)x^2 + \lambda_2 b_2 xy - \Big(\frac{b_2}{2}\Big)y^2 + \sigma_2 x - \mu_2 y.$$

In the zone Z^3 we consider the discontinuous piecewise linear Hamiltonian system

$$\dot{x} = -\lambda_3 b_3 x + b_3 y + \mu_3, \quad \dot{y} = -\lambda_3^2 b_3 x + \lambda_3 b_3 y + \sigma_3, \tag{4.4}$$

with $b_3 \neq 0$ and $\sigma_3 \neq \lambda_3\mu_3$, which has the first integral

$$H_3(x,y) = -\Big(\lambda_3^2\frac{b_3}{2}\Big)x^2 + \lambda_3 b_3 xy - \Big(\frac{b_3}{2}\Big)y^2 + \sigma_3 x - \mu_3 y.$$

We are going to prove that the piecewise linear differential Hamiltonian systems formed by system (4.2), (4.3) and (4.4) have at most three crossing limit cycles intersecting the two circles $\mathbb{S}_1$ and $\mathbb{S}_2$ in four points.

In order to have a crossing limit cycle which intersects $\mathbb{S}_1$ at the real points $A_i = (x_i, y_i)$ and $B_i = (z_i, w_i)$ and intersects $\mathbb{S}_2$ at the real points $C_i = (f_i, g_i)$ and $D_i = (h_i, k_i)$, with $A_i \neq B_i$ and $D_i \neq C_i$, these points must satisfy the following system

$$\begin{aligned}
&e_1 = H_1(x_i,y_i) - H_1(z_i,w_i) = 0,\\
&e_2 = H_2(f_i,g_i) - H_2(x_i,y_i) = 0,\\
&e_3 = H_2(z_i,w_i) - H_2(h_i,k_i) = 0,\\
&e_4 = H_3(f_i,g_i) - H_3(h_i,k_i) = 0,\\
&x_i^2 + y_i^2 = 1,\ z_i^2 + w_i^2 = 1,\\
&f_i^2 + g_i^2 = a^2,\ h_i^2 + k_i^2 = a^2.
\end{aligned} \tag{4.5}$$

The points A_i, B_i, C_i and D_i can take the form $A_i = \left(\frac{r_i^2-1}{r_i^2+1}, \frac{2r_i}{r_i^2+1}\right)$, $B_i = \left(\frac{s_i^2-1}{s_i^2+1}, \frac{2s_i}{s_i^2+1}\right)$, $C_i = \left(a\frac{n_i^2-1}{n_i^2+1}, a\frac{2n_i}{n_i^2+1}\right)$, *and* $D_i = \left(a\frac{m_i^2-1}{m_i^2+1}, a\frac{2m_i}{m_i^2+1}\right)$.

Assume that the discontinuous piecewise linear differential systems formed by (4.2), (4.3) and (4.4) have four limit cycles. For that we must suppose that system (4.5) has the four real solutions, A_i, B_i, C_i and D_i with $i = 1, \ldots, 4$. Firstly, we fixed the points A_1 and B_1, then by the equation $e_2 = 0$ we obtain the value of $n_1 = \phi_1(a, b_2, r_1, \lambda_2, \gamma_2, \sigma_2)$ (or simply $n_1 = \phi_1$), where ϕ_1 is a function which depends on parameters a, b_2, r_1, λ_2, γ_2 and σ_2. Due to the huge expression of the function ϕ_1 we will omit it. So we get the value of the point C_1.

From equation $e_3 = 0$ we obtain the value of the parameter $m_1 = \phi_2(a, b_2, s_1, \lambda_2, \gamma_2, \sigma_2)$ (or simply $m_1 = \phi_2$), where ϕ_2 is a function which depends on parameters a, b_2, s_1, λ_2, γ_2 and σ_2, and for the same reason as the function ϕ_1, we will not give its expression. In this case we get the value of the point D_1.

From equation $e_1 = 0$, and by assuming that $r_1 + s_1 \neq 0$ we obtain the expression of the parameter

$$\begin{aligned}\sigma_1 = \; & \frac{1}{(1+r_1^2)(r_1+s_1)(1+s_1^2)}\Big(-2(1+r_1^2)(r_1-s_1)(-1+r_1s_1)(1+s_1^2)\gamma_1 + 2b_1(r_1-s_1)(1 \\ & +r_1s_1)(-1+s_1\lambda_1+r_1(s_1+\lambda_1))(-s_1-\lambda_1+r_1(-1+s_1\lambda_1)\Big).\end{aligned}$$

Now assuming $\phi_1 + \phi_2 \neq 0$, then from equation $e_4 = 0$ we obtain

$$\begin{aligned}\sigma_3 = \; & \frac{-1}{2a(1+\phi_2^2)(\phi_2-\phi_1)(\phi_1+\phi_2)(1+\phi_1^2)}\Big(2a(1+\phi_2^2)(\phi_2-\phi_1)(-1+\phi_2\phi_1)(1+\phi_1^2)\gamma_3 \\ & -2a^2b_3(\phi_2-\phi_1)(1+\phi_2\phi_1)(-1+\phi_1\lambda_3+\phi_2(\phi_1+\lambda_3))(-\phi_1-\lambda_3+\phi_2(-1+\phi_1\lambda_3))\Big).\end{aligned}$$

We suppose the second solution of system (3.3), we fixed the points A_2, B_2, C_2 , then from equation $e_3 = 0$ we obtain the value of the parameter $m_2 = \phi_3(a, b_2, s_2, \lambda_2, \gamma_2, \sigma_2)$ (or simply $m_2 = \phi_3$), which depends on parameters a, b_2, s_2, λ_2, γ_2 and σ_2, and for the same reason as the function ϕ_1, we will not give its expression. In this case we get the value of the point D_2.

From equation $e_2 = 0$, and by assuming $B \neq 0$ we obtain the parameter $\sigma_2 = A/B$, where

$$\begin{aligned}A = \; & 4an_2(1+n_2^2)(1+r_2^2)^2\gamma_2 - 4(1+n_2^2)^2(r_2+r_2^3)\gamma_2 + a^2b_2(1+r_2^2)^2(\lambda_2+n_2(2-n_2\lambda_2))^2 \\ & -b_2(1+n_2^2)^2(\lambda_2+r_2(2-r_2\lambda_2))^2, \\ B = \; & 2(1+n_2^2)(1+r_2^2)(1-a+n_2^2+an_2^2-r_2^2-ar_2^2-n_2^2r_2^2+an_2^2r_2^2).\end{aligned}$$

By solving equation $e_1 = 0$, and by assuming that $D \neq 0$ we have the expression of $\gamma_1 = C/D$, where

$$
\begin{aligned}
C = \ & (b_1(-s_1(r_2+s_2)(-s_1^2+s_2^2+r_2^2(1+(2+s_1^2)s_2^2))+(r_2(-1+s_1^2-3(s_1+s_1^3)s_2+(-1+s_1^2) \\
& s_2^2-(s_1+s_1^3)s_2^3)-(s_1-s_2)(-1+(s_1(-1+s_2)-s_2)(s_1+s_2+s_1s_2))+r_2^3(-1+s_1s_2)(1 \\
& +2s_1s_2+s_2^2+s_1^2(-1+s_2^2))-r_2^2(s_1+s_2+3s_1s_2^2+s_2^3+s_1^3(1+3s_2^2)-s_1^2(s_2+s_2^3)))\lambda_1 \\
& +s_1(r_2+s_2)(-s_1^2+s_2^2+r_2^2(1+(2+s_1^2)s_2^2))\lambda_1^2+r_1^3((r_2+s_2)(1-r_2^2s_2^2+s_1^2(2+r_2^2+s_2^2)) \\
& -((-1+s_1s_2)(1+s_1^2+2s_1s_2+(-1+s_1^2)s_2^2)+r_2^2(1+s_1^2+s_1(-1+s_1^2)s_2+3(1+s_1^2)s_2^2 \\
& +s_1(-1+s_1^2)s_2^3)+r_2^3(s_1-s_2)(-1+2s_1s_2+s_2^2+s_1^2(1+s_2^2))+r_2(3s_2+s_2^3-s_1(1+s_2^2) \\
& +s_1^3(1+s_2^2)+s_1^2s_2(3+s_2^2)))\lambda_1-(r_2+s_2)(1-r_2^2s_2^2+s_1^2(2+r_2^2+s_2^2))\lambda_1^2)+r_1^2(s_1(r_2+s_2) \\
& (1-r_2^2s_2^2+s_1^2(2+r_2^2+s_2^2))+(s_1+s_1^3+s_2+3s_1^2s_2-s_1(1+s_1^2)s_2^2+(1+3s_1^2)s_2^3 \\
& +r_2^3(1+s_2^2+s_1s_2(-1+s_2^2)+s_1^3s_2(-1+s_2^2)+3s_1^2(1+s_2^2))-r_2(-1+s_1s_2)(1-2s_1s_2 \\
& +s_2^2+s_1^2(3+s_2^2))-r_2^2(s_1-s_2)(1-2s_1s_2+s_2^2+s_1^2(1+3s_2^2)))\lambda_1-s_1(r_2+s_2)(1-r_2^2s_2^2 \\
& +s_1^2(2+r_2^2+s_2^2))\lambda_1^2)+r_1(-(r_2+s_2)(-s_1^2+s_2^2+r_2^2(1+(2+s_1^2)s_2^2))+(1+s_1^2+s_1(3 \\
& +s_1^2)s_2-(1+s_1^2)s_2^2+s_1(3+s_1^2)s_2^3+r_2(s_1-s_2)(3+s_1^2-2s_1s_2+(1+s_1^2)s_2^2)+r_2^2(-1 \\
& +s_1s_2)(1+s_1^2-2s_1s_2+(3+s_1^2)s_2^2)+r_2^3(s_1(3+s_1^2)-(1+s_1^2)s_2+s_1(3+s_1^2)s_2^2+(1 \\
& +s_1^2)s_2^3))\lambda_1+(r_2+s_2)(-s_1^2+s_2^2+r_2^2(1+(2+s_1^2)s_2^2))\lambda_1^2))), \\
D = \ & (1+r_1^2)(1+r_2^2)(1+s_1^2)(r_1-r_2+s_1+r_1r_2s_1-s_2-r_1r_2s_2+r_1s_1s_2-r_2s_1s_2)(1+s_2^2).
\end{aligned}
$$

From equation $e_4 = 0$, and by assuming $F \neq 0$ we have the expression of $\gamma_3 = E/F$, where E and F are given in the appendix 4.

Likewise, we consider the third solution, and we fixed the point A_3, B_3, C_3. Then from equation $e_3 = 0$ we obtain the value of the parameter $m_3 = \phi_4(a, b_2, n_2, r_2, s_3, \lambda_2, \gamma_2)$ (or simply $m_3 = \phi_4$), which depends on parameters a, b_2, s_2, λ_2 and γ_2. In this case we get the value of the point D_3.

By solving equation $e_1 = 0$, we obtain the parameter $\lambda_1 = (G_1 \pm \sqrt{H_1})/R_1$, with $R_1 \neq 0$, and G_1, H_1 and R_1 are given in the appendix of chapter 4.

Now solving the equation $e_2 = 0$, we obtain the parameter $\gamma_2 = G_2/R_2$, with $R_2 \neq 0$, and we give the expressions of G_2 and R_2 in the appendix of chapter 4.

Finally, if we fix the four points A_4, B_4, C_4 and D_4, then from the equation $e_1 = 0$ and $e_4 = 0$ we have that $b_1 = 0$ and $b_3 = 0$ which is a contradiction to the assumptions. Therefore we have proved that the maximum number of crossing limit cycles for the family $\mathcal{F}$ intersecting the two circles $\mathbb{S}_1$ and $\mathbb{S}_2$ in four points is three.

Now we shall provide differential systems of the family $\mathcal{F}$ intersecting the two circles $\mathbb{S}_1$ and $\mathbb{S}_1$ in four points, and exhibiting three or two orone limit cycle. ∎

Section 4.2 Numericals Examples

Example 1: Three crossing limit cycles for systems in the family $\mathcal{F}$.

Here we consider the three zones defined in (4.1), and we consider the Hamiltonian systems

$$\begin{aligned}
\dot{x} = & \ \frac{9}{10} - \frac{63}{50}x + 14y, \ \ \dot{y} = -1 - \frac{567}{50}x + \frac{63}{50}y, \ \text{ in } Z^3, \\
\dot{x} = & \ -0.072.. - 0.7313..x + 0.9..y, \ \ \dot{y} = -0.1726.. - 0.5943..x \\
& +0.731367y, \ \text{ in } Z^2, \\
\dot{x} = & \ -11.7461.. + 8.864..x + 8y, \ \ \dot{y} = 12.996.. - 9.8228..x - 8.8646..y, \ \text{ in } Z^1.
\end{aligned} \tag{4.6}$$

The first integrals of the linear Hamiltonian systems (4.6) are

$$\begin{aligned}
H_1(x,y) = & \ -x - \frac{567}{100}x^2 - \frac{9}{10}y + \frac{63}{50}xy - 7y^2, \\
H_2(x,y) = & \ -0.172..x - 0.297..x^2 + 0.0720..y + 0.731..xy - 0.45y^2, \\
H_3(x,y) = & \ 12.996..x - 4.911..x^2 + 11.746..y - 8.864..xy - 4y^2,
\end{aligned}$$

respectively. The discontinuous piecewise linear differential system formed by the linear Hamiltonian systems (4.6) has exactly three crossing limit cycles, because the system of equations (4.5) has the three real solutions

$$\begin{aligned}
S_1 &= (0.932.., 1.063.., 1.187.., 0.768.., 0.48, 0.877.., 0.646.., 0.763..), \\
S_2 &= (0.875.., 1.110.., 1.225.., 0.705.., 0.44, 0.897.., 0.681.., 0.731..), \\
S_3 &= (0.830.., 1.145.., 1.253.., 0.654.., 0.4.., 0.916.., 0.710.., 0.703..).
\end{aligned}$$

Then these three limit cycles are drawn in Figure 4.1.

Now we give an example of systems of the family $\mathcal{F}$ intersecting the two circles $\mathbb{S}_1$ and $\mathbb{S}_2$ in four points, and exhibiting two limit cycles.

Example 2: Two crossing limit cycles for systems in the family $\mathcal{F}$.

We consider the Hamiltonian systems

$$
\begin{aligned}
\dot{x} = & \ \frac{8}{5}-\frac{63}{50}x+14y, \quad \dot{y}=-\frac{6}{5}-\frac{567}{50}x+\frac{63}{50}y, \text{ in } Z^3,\\
\dot{x} = & \ -0.19731..-1.4359..x+2y, \quad \dot{y}=-0.43806..-1.03101..x+1.43597..y, \text{ in } Z^2,\\
\dot{x} = & \ -31.321..+5.69824..x+8y, \quad \dot{y}=29.0365..-4.05874..x-5.6982..y, \text{ in } Z^1.
\end{aligned}
\tag{4.7}
$$

The first integrals of the linear Hamiltonian systems (4.7) are

$$
\begin{aligned}
H_1(x,y) = & \ -\frac{6}{5}x-\frac{567}{100}x^2-\frac{8}{5}y+\frac{63}{50}xy-7y^2,\\
H_2(x,y) = & \ -0.438..x-0.515..x^2+0.197..y+1.435..xy-y^2,\\
H_3(x,y) = & \ 29.036..x-2.029..x^2+31.321..y-5.698..xy-4y^2,
\end{aligned}
$$

respectively. The discontinuous piecewise linear differential system formed by the linear Hamiltonian systems (4.7) has exactly two crossing limit cycles, because the system of equations (4.5) has the two real solutions

$$
\begin{aligned}
S_1 &= (0.945..,1.051..,1.163..,0.803..,0.450,0.893..,0.9,0.435..),\\
S_2 &= (0.882..,1.104..,1.208..,0.734..,0.4,0.916..,0.921..,0.387..).
\end{aligned}
$$

Then these two limit cycles are drawn in Figure 4.2.

In what follows we give an example of systems of the family $\mathcal{F}$ intersecting the two circles $\mathbb{S}_1$ and $\mathbb{S}_2$ in four points, and exhibiting one limit cycle.

Example 3: One crossing limit cycle for systems in the family $\mathcal{F}$. We consider the Hamiltonian systems

$$
\begin{aligned}
\dot{x} = & \ \frac{8}{5}-\frac{63}{50}x+14y, \ \dot{y}=-\frac{6}{5}-\frac{567}{50}x+\frac{63}{50}y, \text{ in } Z^3\\
\dot{x} = & \ -0.103413..-1.38..x+2y, \ \dot{y}=-0.240661..-0.9522..x+1.38..y, \text{ in } Z^2\\
\dot{x} = & \ -2-0.9x+3y, \ \dot{y}=0.331279-0.27x+0.9y, \text{ in } Z^1.
\end{aligned}
\tag{4.8}
$$

The first integrals of the linear Hamiltonian systems (4.8) are

$$
\begin{aligned}
H_1(x,y) = &\ -\frac{6}{5}x - \frac{567}{100}x^2 - \frac{8}{5}y + \frac{63}{50}xy - 7y^2,\\
H_2(x,y) = &\ -0.240..x - 0.476..x^2 + 0.103..y + 1.38..xy - y^2,\\
H_3(x,y) = &\ 0.331..x - 0.135..x^2 + 2y + 0.899..xy - \frac{3y^2}{2},
\end{aligned}
$$

respectively. The discontinuous piecewise linear differential system formed by the linear Hamiltonian systems (4.8) has exactly one crossing limit cycle, because the system of equations (4.5) has the unique real solution

$$S_1 = (0.921.., 1.073.., 1.182.., 0.776.., 0.5, 0.866.., 0.9, 0.435..).$$

This limit cycle is drawn in Figure 4.3.

Chapter 5

Centers and Limit Cycles for a Kukles Differential Systems of degree eight

We consider the *Kukles* homogeneous differential systems, see Giné [34]

$$\dot{x} = -y, \quad \dot{y} = x + Q_n(x,y), \tag{5.1}$$

where Q_n is a real homogeneous polynomial of degree n, with the variables x and y over $\mathbb{R}$.

For the global phase portraits of Kukles differential systems (5.1) we can site the following works. Vulpe [73] studied the global phase portraits for all center quadratic differential systems and since systems (5.1) for $n = 2$ is a particular case of these systems, we know that their phase portraits are studied. Buzzi et al. [20], Malkin [60], Vulpe and Sibirskii [74] and Żołądek [75, 76] classifed the global phase portraits of cubic polynomial differential systems with a symmetry with respect to a straight line. Benterki and Llibre [5] provided the global phase portraits of systems (5.1) for $n = 4$.

Llibre and Silva [51, 52] classified the phase portraits of systems (5.1) with $n = 5,6$, and the global phase portraits for the case $n = 7$ was studied by Benterki and Llibre [12].

Section 5.1 The first Main Result

The first objective of this chapter is to classify all the global phase portraits of the generalized Kukles differential systems of degree 8 symmetric with respect to the x–axis

$$\dot{x} = -y, \quad \dot{y} = x + ax^8 + bx^6y^2 + cx^4y^4 + dx^2y^6 + ey^8. \tag{5.2}$$

In the following remark we will show how can we restruct the study of systems (5.2) according to the sign of their parameters.

Remark 3

The Kukles differential systems (5.2) are invariant under the transformation $(x, y, t, a, b, c, d, e) \longrightarrow (-x, y, -t, -a, -b, -c, -d, -e)$, then we only need to study the system for $e > 0$, or $e = 0$ and $d > 0$, or $e = d = 0$ and $c > 0$, or $e = d = c = 0$ and $b > 0$, or $e = d = c = b = 0$ and $a \neq 0$.

We know that to study the global phase portraits of such differential systems, we have to compute its equilibrium points, for this reason we need to know the real roots of the polynomial $P(x, y) = ax^8 + bx^6y^2 + cx^4y^4 + dx^2y^6 + ey^8$.

By doing the following change of variables $X = x^2$ and $Y = y^2$ the polynomial $P(x, y)$ becomes

$$P(X, Y) = aX^4 + bX^3Y + cX^2Y^2 + dXY^3 + eY^4,$$

which is a polynomial of degree four, so it has many different kind of roots which are summarized in the following 69 cases.

If the polynomial $P(x, y)$ has four simple real roots, so

$P(x, y) = e(y^2 - r_1x^2)(y^2 - r_2x^2)(y^2 - r_3x^2)(y^2 - r_4x^2)$ and we distinguish the 9 subcases:

(1) $0 < r_1 < r_2 < r_3 < r_4$,

(2) $r_1 = 0 < r_2 < r_3 < r_4$,

(3) $r_1 < 0 < r_2 < r_3 < r_4$,

(4) $r_1 < r_2 = 0 < r_3 < r_4$,

(5) $r_1 < r_2 < 0 < r_3 < r_4$,

(6) $r_1 < r_2 < r_3 = 0 < r_4$,

(7) $r_1 < r_2 < r_3 < 0 < r_4$,

(8) $0 < r_1 < r_2 < r_3 < r_4 = 0$,

(9) $r_1 < r_2 < r_3 < r_4 < 0$.

If the polynomial $P(x,y)$ has three real roots: two simples and one double, so $P(x,y) = e(y^2 - r_1x^2)(y^2 - r_2x^2)(y^2 - r_3x^2)^2$ and we distinguish the 7 subcases:

(10) $0 < r_1 < r_2 < r_3$,

(11) $r_1 = 0 < r_2 < r_3$,

(12) $r_1 < 0 < r_2 < r_3$,

(13) $r_1 < r_2 = 0 < r_3$,

(14) $r_1 < r_2 < 0 < r_3$,

(15) $r_1 < r_2 < r_3 = 0$,

(16) $r_1 < r_2 < r_3 < 0$.

If the polynomial $P(x,y)$ has two double real roots, so $P(x,y) = e(y^2 - r_1x^2)^2(y^2 - r_2x^2)^2$ and we distinguish the 5 subcases:

(17) $0 < r_1 < r_2$,

(18) $r_1 = 0 < r_2$,

(19) $r_1 < 0 < r_2$,

(20) $r_1 < r_2 = 0$,

(21) $r_1 < r_2 < 0$.

If the polynomial $P(x,y)$ has one simple and one triple real roots, so $P(x,y) = e(y^2 - r_1x^2)(y^2 - r_2x^2)^3$ and we distinguish the 5 subcases:

(22) $0 < r_1 < r_2$,

(23) $r_1 = 0 < r_2$,

(24) $r_1 < 0 < r_2$,

(25) $r_1 < r_2 = 0$,

(26) $r_1 < r_2 < 0$.

If the polynomial $P(x,y)$ has one quarter real root, so $P(x,y) = e(y^2 - r_1x^2)^4$ and we distinguish the 3 subcases:

(27) $0 < r_1$,

(28) $r_1 = 0$,

(29) $r_1 < 0$.

If the polynomial $P(x,y)$ has two simple real roots and two complexes, so $P(x,y) = e(y^2 - r_1x^2)(y^2 - r_1x^2)(y^4 - 2\alpha x^2y^2 + (\alpha^2 + \beta^2)x^4)$ and we distinguish the 5 subcases:

(30) $0 < r_1 < r_2$,

(31) $r_1 = 0 < r_2$,

(32) $r_1 < 0 < r_2$,

(33) $r_1 < r_2 = 0$,

(34) $r_1 < r_2 < 0$.

If the polynomial $P(x,y)$ has one double real root and two complexes, so $P(x,y) = e(y^2 - r_1x^2)^2(y^4 - 2\alpha x^2y^2 + (\alpha^2 + \beta^2)x^4)$ and we distinguish the 3 subcases:

(35) $0 < r_1$,

(36) $r_1 = 0$,

(37) $r_1 < 0$.

If the polynomial $P(x,y)$ has no real root, so $P(x,y) = e(y^4 - 2\alpha x^2y^2 + (\alpha^2 + \beta^2)x^4)^2$ and we have to study the case:

(38) $e > 0$.

If $e = 0$ and the polynomial $P(x,y)$ has three simple real roots, so $P(x,y) = dx^2(y^2 - r_1x^2)(y^2 - r_2x^2)(y^2 - r_3x^2)$ and we distinguish the 7 subcases:

(39) $0 < r_1 < r_2 < r_3$,

(40) $r_1 = 0 < r_2 < r_3$,

(41) $r_1 < 0 < r_2 < r_3$,

(42) $r_1 < r_2 = 0 < r_3$,

(43) $r_1 < r_2 < 0 < r_3$,

(44) $r_1 < r_2 < r_3 = 0$,

(45) $r_1 < r_2 < r_3 < 0$.

If $e = 0$ and the polynomial $P(x,y)$ has two real roots one simple and one double, so $P(x,y) = dx^2(y^2 - r_1x^2)(y^2 - r_2x^2)^2$ and we distinguish the 5 subcases:

(46) $0 < r_1 < r_2$,

(47) $r_1 = 0 < r_2$,

(48) $r_1 < 0 < r_2$,

(49) $r_1 < r_2 = 0$,

(50) $r_1 < r_2 < 0$.

If $e = 0$ and the polynomial $P(x,y)$ has one triple real root, so $P(x,y) = dx^2(y^2 - r_1x^2)^3$ and we distinguish the 3 subcases:

(51) $0 < r_1$,

(52) $r_1 = 0$,

(53) $r_1 < 0$.

If $e = 0$ and the polynomial $P(x,y)$ has one simple real root and two complexes, so $P(x,y) = dx^2(y^2 - r_1x^2)(y^4 - 2\alpha x^2y^2 + (\alpha^2 + \beta^2)x^4)$ and we distinguish the 3 subcases:

(54) $0 < r_1$,

(55) $r_1 = 0$,

(56) $r_1 < 0$.

If $e = 0$, $d = 0$ and the polynomial $P(x,y)$ has two simple real roots, so $P(x,y) = cx^4(y^2 - r_1x^2)(y^2 - r_2x^2)$ we have distinguish the 5 subcases:

(57) $0 < r_1 < r_2$,

(58) $r_1 = 0 < r_2$,

(59) $r_1 < 0 < r_2$,

(60) $r_1 < r_2 = 0$,

(61) $r_1 < r_2 < 0$.

If $e = 0$, $d = 0$ and the polynomial $P(x,y)$ has one double real root, so $P(x,y) = cx^4(y^2 - r_1x^2)^2$ and we distinguish the 3 subcases:

(62) $0 < r_1$,

(63) $r_1 = 0$,

(64) $r_1 < 0$.

If $e = 0$, $d = 0$ and the polynomial $P(x,y)$ has two complexes roots, so $P(x,y) = cx^4(y^4 - 2\alpha x^2y^2 + (\alpha^2 + \beta^2)x^4)$ then we have the following subcase:

(65) $c > 0$,

If $e = 0$, $d = 0$ and $c = 0$, the polynomial $P(x,y)$ has one simple real root, so $P(x,y) = bx^6(y^2 - r_1x^2)$ then we have the following 3 subcases:

(66) $0 < r_1$,

(67) $r_1 = 0$,

(68) $r_1 < 0$.

If $e = 0$, $d = 0$, $c = 0$ and $b = 0$, the polynomial $P(x,y)$ has one simple real root, so $P(x,y) = ax^8$ we have the following subcase:

(69) $a \neq 0$.

Remark 4 *Some subcases for the previous ones have more then one configuration. According to the position of their separatrices:*

- *the subcase* (7) *has two different configurations* (7.1) *and* (7.2)*,*
- *the subcase* (11) *has two different configurations* (11.1) *and* (11.2)*,*
- *the subcase* (12) *has two different configurations* (12.1) *and* (12.2)*,*
- *the subcase* (13) *has two different configurations* (13.1) *and* (13.2)*,*
- *the subcase* (17) *has two different configurations* (17.1) *and* (17.2)*,*
- *the subcase* (22) *has two different configurations* (22.1) *and* (22.2)*,*
- *the subcase* (30) *has two different configurations* (30.1) *and* (30.2)*,*
- *the subcase* (31) *has two different configurations* (31.1) *and* (31.2)*,*
- *the subcase* (32) *has two different configurations* (32.1) *and* (32.2)*,*
- *the subcase* (39) *has two different configurations* (39.1) *and* (39.2)*,*
- *the subcase* (40) *has two different configurations* (40.1) *and* (40.2)*,*
- *the subcase* (41) *has two different configurations* (41.1) *and* (41.2)*,*
- *the subcase* (43) *has three different configurations* (43.1)*,* (43.2) *and* (43.3)*,*
- *the subcase* (51) *has four different configurations* (51.1)*,* (51.2)*,* (51.3) *and* (51.4)*,*
- *the subcase* (57) *has two different configurations* (57.1) *and* (57.2)*,*
- *the subcase* (58) *has two different configurations* (58.1) *and* (58.2)*,*
- *the subcase* (59) *has two different configurations* (59.1) *and* (59.2)*,*
- *the subcase* (60) *has two different configurations* (60.1) *and* (60.2)*,*
- *the subcase* (62) *has two different configurations* (62.1) *and* (62.2).

5.1.1 Finite and infinite singularities

To study the phase portraits of systems (5.2) we identify all the finite singular points and their local phase portrait. We go through the same steps to study the local phase portrait for the infinite ones.

Finite singular points. We identify the finite singular points of the generalized kukels polynomial differential systems (5.2) in the following Proposition.

Proposition 5.1
The differential systems (5.2) have a center at the origin of coordinates and an hyperbolic saddle at $(-a^{-1/7}, 0)$*, if* $a \neq 0$*.*

Proof. *The eigenvalues of the linear part of systems (5.2) at the origin are* $\pm i$ *so the origin is either a focus or a center, but due to the fact that these systems are symmetric with respect to* x*-axis, we know that the origin is a center. For* $a < 0$*, in addition to the origin systems (5.2) have the second equilibrium point* $(-a^{-1/7}, 0)$ *with eginvalues* $\pm\sqrt{7}$*. By using Theorem 2.15 of [24] we conclude that this equilibria is a saddle.* ∎

Infinite singular points. By using the preliminaries given in chapter 1 we study the infinite singular points and their nature in the Poincaré disc.

Proposition 5.2
The local phase portraits at the infinite equilibrium points $(\sqrt{r_j}, 0)$ *with* $r_j \geq 0$ *of the local chart* U_1 *are*

(a) four semi–hyperbolic saddle–nodes for the subcase (1)*,*

(b) a linearly zero singularity at the origin of coordinates with four hyperbolic sectors, and three semi–hyperbolic saddle–nodes for the subcase (2)*,*

(c) Three semi–hyperbolic saddle–nodes for the subcases (3) *and* (39)*,*

(d) a linearly zero singularity at the origin of coordinates with four hyperbolic sectors,

and tow semi–hyperbolic saddle–nodes for the subcases (4) *and* (40),

(e) *tow semi–hyperbolic saddle–nodes for the subcases* (5), (41) *and* (57),

(f) *linearly zero singularity at the origin of coordinates with four hyperbolic sectors, and one semi–hyperbolic saddle–node for the subcases* (6), (42) *and* (58),

(g) *tow semi–hyperbolic saddle–nodes for the subcases* (7), (43), (54), (59) *and* (66),

(h) *a linearly zero singularity at the origin of coordinates with one hyperbolic, one elliptic and two parabolic sectors for the subcases* (8), (15), (20), (44) *and* (60),

(i) *no infinite equilibria in the local chart* U_1 *for the subcases* (9), (16), (21), (26), (29), (37), (38), (45), (50), (53), (55), (61), (64), (65), (68) *and* (69),

(j) *two semi–hyperbolic saddle–nodes and a linearly zero singularity with one parabolic and two hyperbolic sectors for the subcase* (10),

(k) *a semi–hyperbolic saddle–node, a linearly zero singularity with one parabolic and two hyperbolic sectors and another linearly zero singularity at the origin of coordinates where its local phase portrait consists of four hyperbolic sectors, for the subcase* (11),

(l) *a semi–hyperbolic saddle–node and a linearly zero singularity with one parabolic and two hyperbolic sectors, for the subcases* (12), (22) *and* (46),

(m) *two linearly zero singularities, where the local phase portrait at the first one consists of one parabolic and two hyperbolic sectors, and the local phase portrait of the second one which is located at the origin of coordinates consists of four hyperbolic sectors, for the subcases* (13), (18), (23) *and* (47),

(n) *a linearly zero singularity where its local phase portrait consists of one parabolic and two hyperbolic sectors, for the subcases* (14), (19), (24), (48) *and* (51),

(o) *two linearly zero singularities where their local phase portraits consist of one parabolic and two hyperbolic sectors, for the subcases* (17), (27) *and* (35),

(p) *a linearly zero singularity at the origin of coordinates with two hyperbolic sectors and one parabolic sector, for the subcases* (25) *and* (47)*,*

(q) *linearly zero singularity at the origin of coordinates with four hyperbolic sectors for the subcases* (28),(36), (52), (55), (63) *and* (67)*,*

(r) *a linearly zero singularity (not at the origin) with four hyperbolic sectors for the subcase* (62)*.*

Proof. *As we have mentioned at the beginning of this section the phase portraits of the semi–hyperbolic equilibrium can be determined by using Theorem 2.19 of [24], and the phase portraits of the linearly zero equilibrium points doing the blow–up changes of variables. Here we shall prove with all details the statements (a) and (b), the other statements are proved in a similar way. In statements (a) and (b) system* (5.2) *write as*

$$\dot{x} = -y; \qquad \dot{y} = x + e(y^2 - r_1x^2)(y^2 - r_2x^2)(y^2 - r_3x^2)(y^2 - r_4x^2), \tag{5.3}$$

with $e > 0$*. This system in the local chart* U_1 *becomes*

$$\begin{aligned} \dot{u} = {} & er_1r_2r_3r_4 - e(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)u^2 + e(r_1r_2 + r_1r_3 \\ & + r_2r_3 + r_1r_4 + r_2r_4 + r_3r_4)u^4 - e(r_1 + r_2 + r_3 + r_4)u^6 + eu^8 \\ & + v^7 + u^2v^7, \\ \dot{v} = {} & uv^8. \end{aligned} \tag{5.4}$$

Assume $0 < r_1 < r_2 < r_3 < r_4$*, then the eigenvalues at the infinite equilibrium point* $(\sqrt{r_j}, 0)$ *for* $j = 1, 2, 3, 4$ *are* 0*, and* $h'(\sqrt{r_j}) = e\sqrt{r_j}(-6r_j^2(r_1+r_2+r_3+r_4)+4r_j(r_1(r_2+r_3+r_4)+r_2(r_3+r_4)+r_3r_4)- 2r_1r_2r_3-2r_1r_2r_4-2r_1r_3r_4-2r_2r_3r_4+8r_j^3) \neq 0$*, because the four roots* $\sqrt{r_j}$ *for* $j = 1, 2, 3, 4$ *of the polynomial* $h(u) = \dot{u}|_{v=0} = er_1r_2r_3r_4-e(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)u^2+e(r_1r_2+r_1r_3+r_2r_3+ r_1r_4+r_2r_4+r_3r_4)u^4-e(r_1+r_2+r_3+r_4)u^6+eu^8$ *are simples. The point* $(\sqrt{r_j}, 0)$ *for* $j = 1, \ldots, 4$ *is a semi–hyperbolic equilibria. In order to know the local phase portraits of the infinite equilibria* $(\sqrt{r_j}, 0)$ *for* $j = 1, 2, 3, 4$ *we apply Theorem 2.19 of [24]. First we translate the equilibrium point* $(\sqrt{r_j}, 0)$ *to the origin of coordinates by doing the change* $(u, v) = (Y + \sqrt{r_j}, X)$*. Thus we* *obtain the differential system*

$$\begin{aligned}\dot{X} &= A(X,Y) = \text{sign}(h'(\sqrt{r_j}))\Big(\sqrt{r_j}X^8 + X^8 Y\Big),\\ \dot{Y} &= \text{sign}(h'(\sqrt{r_j}))Y + B(X,Y)\Big).\end{aligned} \tag{5.5}$$

Where

$$\begin{aligned}B(X,Y) = \; & +8e\sqrt{r_j}Y^7 + eY^8 - e(r_1+r_2+r_3+r_4-28r_j)Y^6 - 2e(3r_1+3r_2+3r_3\\ & +3r_4-28r_j)\sqrt{r_j}Y^5 + (-e(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4-6r_1r_2r_j\\ & -6r_1r_3r_j-6r_2r_3r_j-6r_1r_4r_j-6r_2r_4r_j-6r_3r_4r_j+15r_1r_j^2+15r_2r_j^2\\ & +15r_3r_j^2+15r_4r_j^2-28r_j^3)+X^7)Y^2+4e\sqrt{r_j}(r_1r_2+r_1r_3+r_2r_3+r_1r_4\\ & +r_2r_4+r_3r_4-5r_1r_j-5r_2r_j-5r_3r_j-5r_4r_j+14r_j^2)Y^3+e(r_1r_2+r_1r_3\\ & +r_2r_3+r_1r_4+r_2r_4+r_3r_4-15r_1r_j-15r_2r_j-15r_3r_j-15r_4r_j+70r_j^2)Y^4\\ & +(1+r_j)X^7+2\sqrt{r_j}X^7Y.\end{aligned}$$

Applying Theorem 2.19 we get the following expressions of the functions $f(X)$ *and* $g(X)$

$$\begin{aligned}f(X) &= -\frac{1+r_j}{h'(\sqrt{r_j})}X^7 + \text{h.o.t.},\\ g(X) &= \text{sign}(h'(\sqrt{r_j}))\sqrt{r_j}X^8 + \text{h.o.t.},\end{aligned}$$

where h.o.t. denotes higher order terms. So using the notation of Theorem 2.19 we have that $a_m = \text{sign}(h'(\sqrt{r_j}))\sqrt{r_j}$ *and* $m = 8$ *is even. Then we know that the equilibria* $(\sqrt{r_j}, 0)$ *for* $j = 1, \ldots, 4$ *is a saddle–node. This completes the proof of statement (a).*

Assume $r_1 = 0 < r_2 < r_3 < r_4$, *then to prove that the equilibria* $(\sqrt{r_2}, 0)$,$(\sqrt{r_3}, 0)$ *and* $(\sqrt{r_4}, 0)$ *are semi–hyperbolic saddle–nodes, we use the same steps as the proof of statement (a). Then we shall show that the local phase portrait of the equilibrium* $(\sqrt{r_1}, 0) = (0,0)$ *is constituted by four hyperbolic sectors. After doing the blow–up change of variables* $v = wu$ *to system* (5.4) *and by eliminating the common factor* u *of* $\dot{u}$ *and* $\dot{w}$ *by doing the rescaling* $d\tau = udt$ *in the* *independent variable, we get the differential system*

$$\begin{aligned} u' = & \; -er_2r_3r_4u + er_2r_3u^3 + er_2r_4u^3 + er_3r4u^3 - er_2u^5 \\ & -er_3u^5 - er_4u^5 + eu^7 + u^6w^7 + u^8w^7, \\ w' = & \; er_2r_3r_4w - er_2r_3u^2w - er_2r_4u^2w - er_3r_4u^2w + er_2u^4w \\ & +er_3u^4w + er_4u^4w - eu^6w - u^5w^8. \end{aligned} \tag{5.6}$$

The unique equilibrium on the w–axis of system (5.6) is the origin of coordinates, which is a saddle whose four separatrices are contained in the two axes. Going back through the two changes of variables, the first change is $d\tau = udt$ and the second change is $v = wu$, and taking into account that in system (5.4) we have that $\dot{u}|_{u=0} = v^7$, we obtain that the local phase portrait at the origin of system (5.4) is formed by two hyperbolic sectors. This completes the proof of statement (b). ■

PROPOSITION 5.3 *The local phase portrait at the origin of the local chart U_2 is a hyperbolic stable node in all the subcases.*

Proof. *We prove the Proposition 5.3 for the statement (a) and the other statements are proved in a similar way.*

In the subcase (1) system (5.2) becomes system (5.3). In the local chart U_2 it becomes

$$\begin{aligned} \dot{u} = & \; -eu + (er_1 + er_2 + er_3 + er_4)u^3 + (-er_1r_2 - er_1r_3 \\ & -er_2r_3 - er_1r_4 - er_2r_4 - er_3r_4)u^5 + (er_1r_2r_3 \\ & +er_1r_2r_4 + er_1r_3r_4 + er_2r_3r_4)u^7 - er_1r_2r_3r_4u^9 - v^7 \\ & -u^2v^7, \\ \dot{v} = & \; -ev + (er_1 + er_2 + er_3 + er_4)u^2v + (-er_1r_2 - er_1r_3 \\ & -er_2r_3 - er_1r_4 - er_2r_4 - er_3r_4)u^4v + (er_1r_2r_3 \\ & +er_1r_2r_4 + er_1r_3r_4 + er_2r_3r_4)u^6v - er_1r_2r_3r_4u^8v \\ & -uv. \end{aligned} \tag{5.7}$$

The origin is a singular point of system (5.7), and the eigenvalues of its associated Jacobian *matrix are* $(-e)$ *and* $(-e)$*. Then the origin is a stable node.* ■

Theorem 5.1 *Suppose that* $a^2 + b^2 + c^2 + d^2 + e^2 \neq 0$*. Then the polynomial differential system* (5.2) *has* 38 *topologically non–equivalent phase portraits in the Poincaré disc. More precisely, the phase portrait in the Poincaré disc of*

- *(1) of Figure 5.1 is realizable by the case* (1),
- *(2) of Figure 5.1 is realizable by the case* (2),
- *(3) of Figure 5.1is realizable by the cases* (3) *and* (39.1),
- *(4) of Figure 5.1 is realizable by the case* (4) *and* (40.1),
- *(5) of Figure 5.1 is realizable by the cases* (5), (30.1), (41.1) *and* (57.1),
- *(6) of Figure 5.1 is realizable by the cases* (6), (31.1) *and* (58.2),
- *(7) of Figure 5.2 is realizable by the cases* (7.1), (32.2), (43.2), (51.2) *and* (59.1),
- *(8) of Figure 5.2 is realizable by the cases* (7.2), (32.1), (43.1), (54) *and* (59.2),
- *(9) of Figure 5.2 is realizable by the case* (8), (15), (20), (44) *and* (60.1),
- *(10) of Figure 5.2 is realizable by the case* (9), (16), (21), (26), (29), (34), (37), (38), (45), (50), (53), (56), (61), (64), (65), (68) *and* (69),
- *(11) of Figure 5.2 is realizable by the cases* (10),
- *(12) of Figure 5.2 is realizable by the cases* (11.1),
- *(13) of Figure 5.2 is realizable by the cases* (11.2),
- *(14) of Figure 5.2 is realizable by the cases* (12.1),
- *(15) of Figure 5.2 is realizable by the cases* (12.2) *and* (46),
- *(16) of Figure 5.2 is realizable by the cases* (13.1), (18) *and* (47),

- *(17) of Figure 5.2 is realizable by the cases* (13.2)*,*
- *(18) of Figure 5.2 is realizable by the cases* (14)*,* (19)*,* (27)*,* (35)*,* (48) *and* (62.2)*,*
- *(19) of Figure 5.3 is realizable by the case* (17.1)*,*
- *(20) of Figure 5.3 is realizable by the case* (17.2)*,*
- *(21) of Figure 5.3 is realizable by the case* (22.1)*,*
- *(22) of Figure 5.3 is realizable by the case* (22.2)*,*
- *(23) of Figure 5.3 is realizable by the case* (23)*,*
- *(24) of Figure 5.3 is realizable by the case* (24)*, and* (51.1)*,*
- *(25) of Figure 5.3 is realizable by the case* (25) *and* (49)*,*
- *(26) of Figure 5.3 is realizable by the cases* (28)*,* (33)*,* (36)*,* (52)*,* (55)*,* (60.2)*,* (63) *and* (67)*,*
- *(27) of Figure 5.3 is realizable by the case* (30.2)*,*
- *(28) of Figure 5.3 is realizable by the case* (31.2)*,*
- *(29) of Figure 5.3 is realizable by the cases* (39.2)*,*
- *(30) of Figure 5.3 is realizable by the cases* (40.2)*,*
- *(31) of Figure 5.4 is realizable by the cases* (40.3)*,*
- *(32) of Figure 5.4 is realizable by the case* (41.2)*,*
- *(33) of Figure 5.4 is realizable by the cases* (42) *and* (58.1)*,*
- *(34) of Figure 5.4 is realizable by the cases* (51.3)*,*
- *(35) of Figure 5.4 is realizable by the cases* (51.4) *and* (43.3)*,*
- *(36) of Figure 5.4 is realizable by the cases* (57.2)*,*

- *(37) of Figure 5.4 is realizable by the cases* (62.1)*,*
- *(38) of Figure 5.4 is realizable by the cases* (66)*.*

5.1.2 Global phase portraits of Kukles differential system

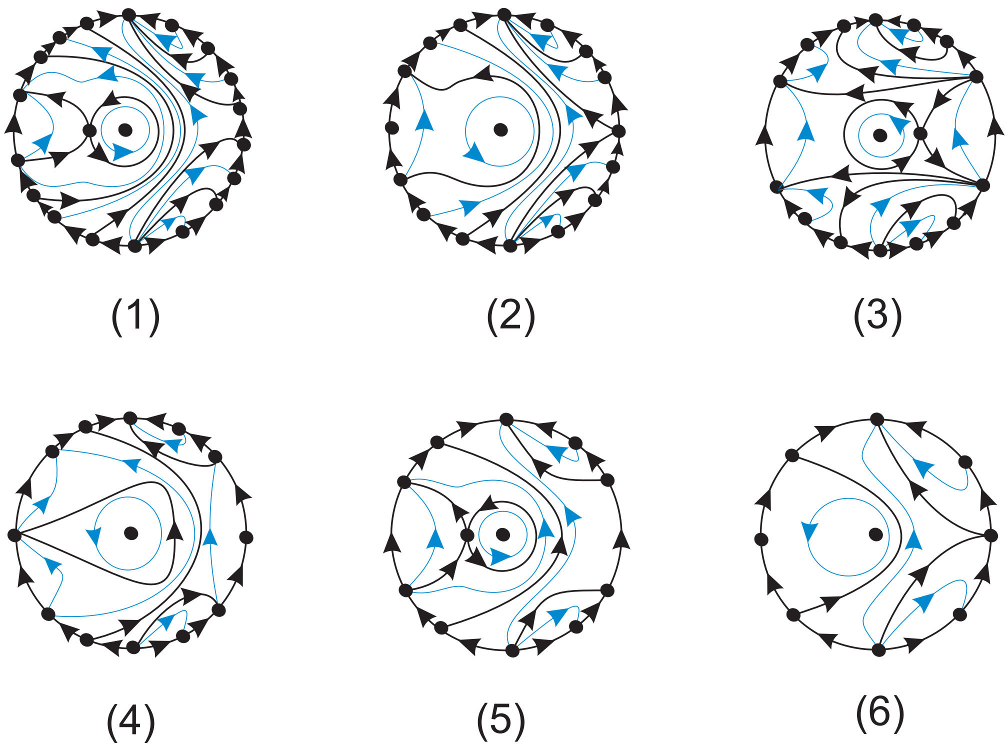


Figure 5.1: Global phase portraits of Kukles differential systems (5.2).

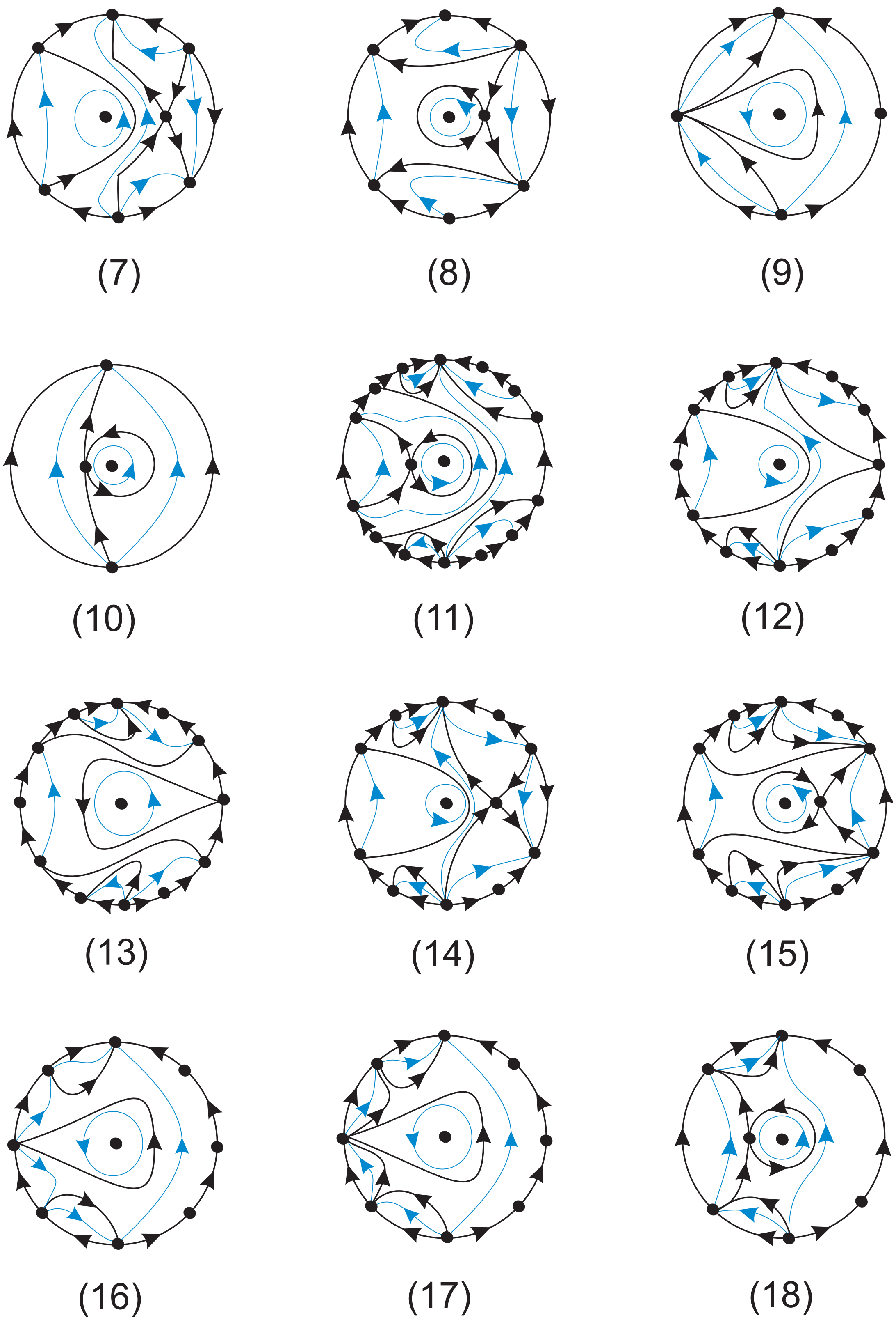


Figure 5.2: Continuation of Figure 5.1.

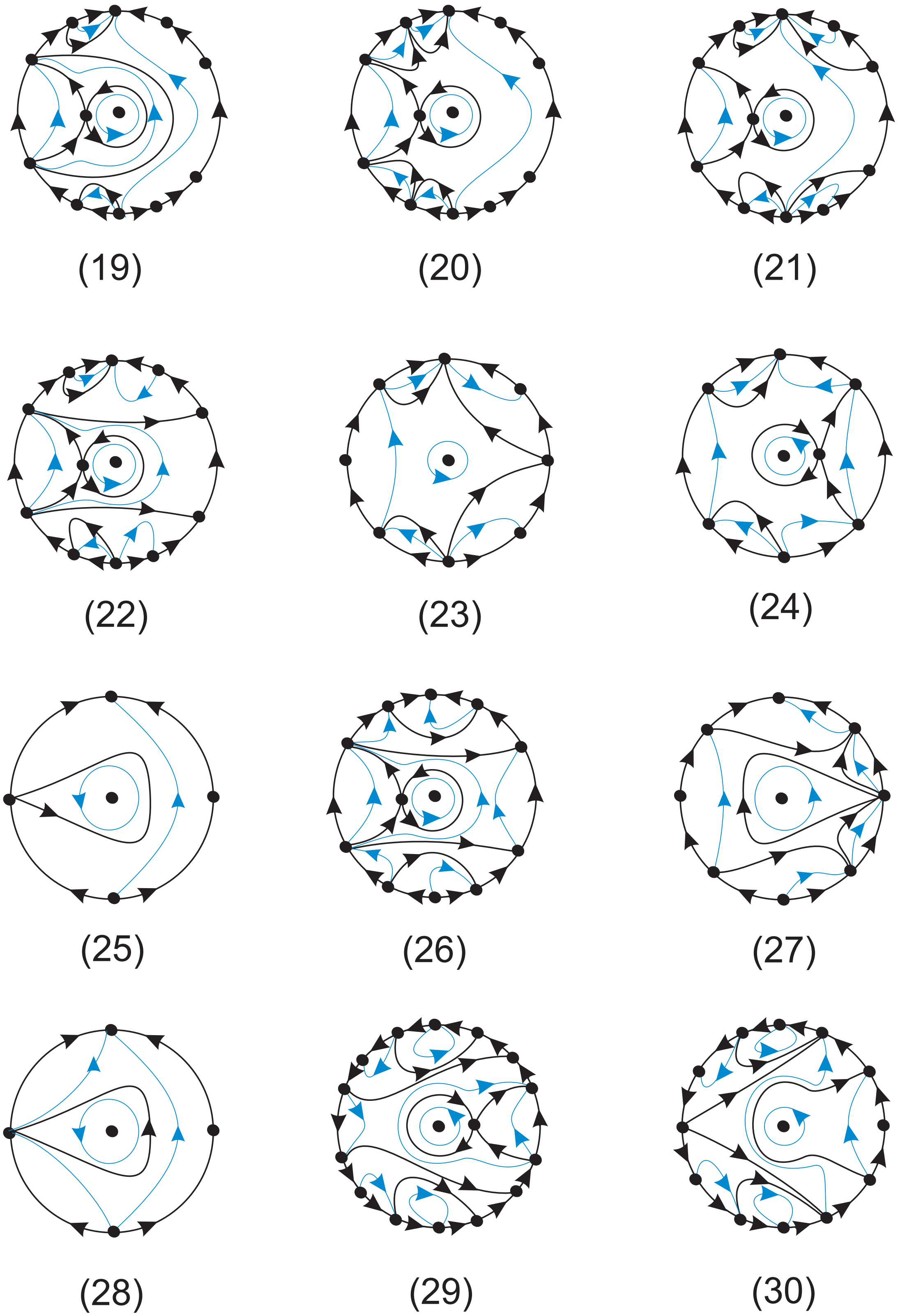


Figure 5.3: Continuation of Figure 5.1.

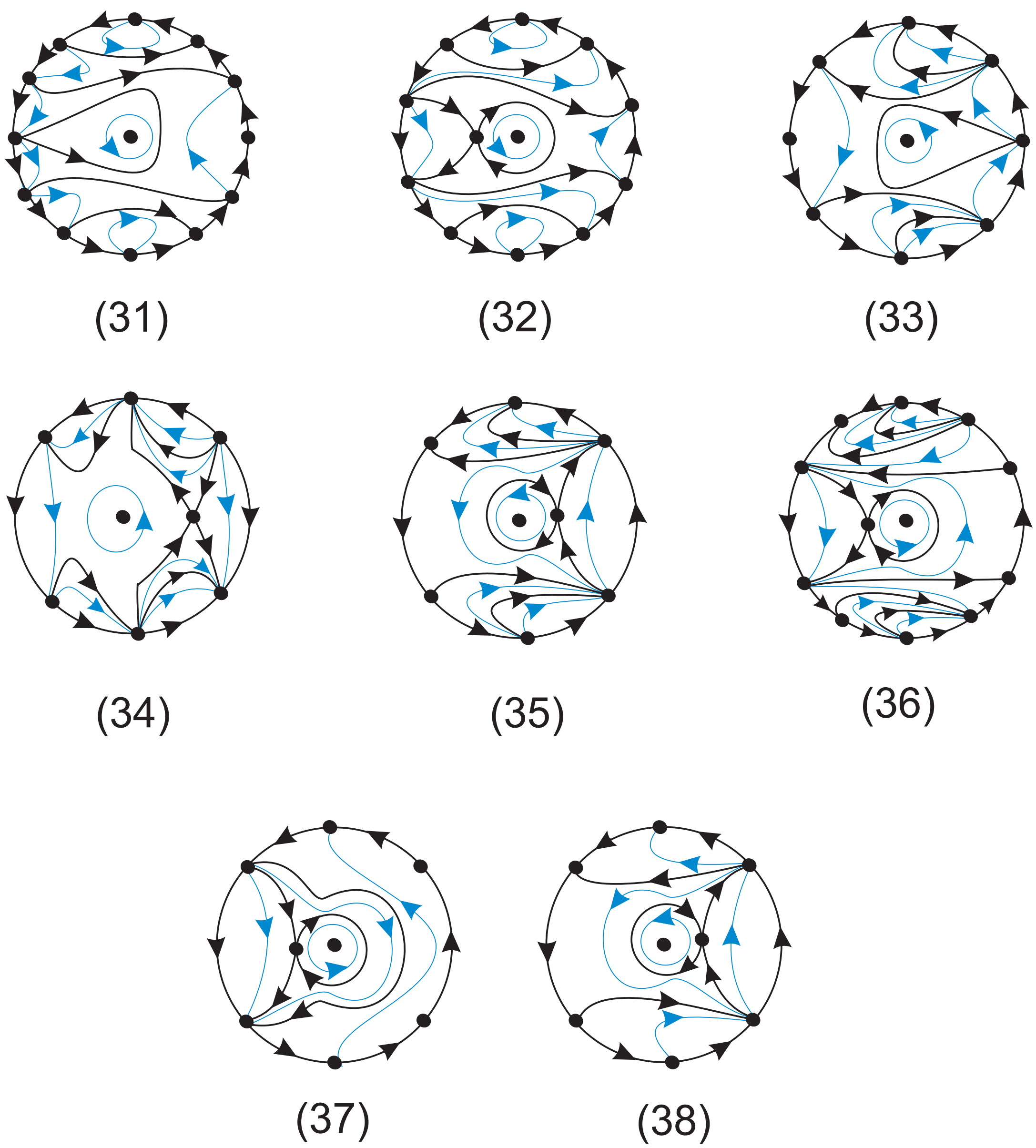


Figure 5.4: Continuation of Figure 5.1.

5.1.3 Limit cycles of Kukles differential systems via averaging theory

Now we consider the application of the developed averaging methods. As the first application we perturbed the polynomial differential systems (5.2) with polynomials of degree eight, we get

$$\begin{aligned} \dot{x} &= -y + \sum_{s=1}^{7} \varepsilon^s \sum_{0\le i+j\le 8} a_{ij}^s x^i y^j, \\ \dot{y} &= x + ax^8 + bx^6y^2 + cx^4y^4 + dx^2y^6 + ey^8 + \sum_{s=1}^{7} \varepsilon^s \sum_{0\le i+j\le 8} b_{ij}^s x^i y^j, \end{aligned} \tag{5.8}$$

where a_{ij}^s and b_{ij}^s are real parameters, for $0 \le i, j \le 8$ and $1 \le s \le 7$. For more information about the averaging theory of higher order see chapter 1.

Section 5.2 The second Main Result

The second objective of this chapter is to solve the second part of the Hilbert 16th problem for system (5.2), when we perturb it inside the class of all polynomial differential system of degree 8.

Our second main result is given in the following Theorem.

Theorem 5.2 *Assume that $f_j = 0$ for $j = 1, \ldots, k-1$ and $f_k \neq 0$. Then if $\bar{r}$ is a simple zero of f_k, the small amplitude limit cycle $(x(t,\varepsilon), y(t,\varepsilon))$ associated to this zero is of the form $(x(t,\varepsilon), y(t,\varepsilon)) = \varepsilon(\bar{r}\cos t, \bar{r}\sin t) + O(\varepsilon^2)$. Moreover, for $|\varepsilon| \neq 0$ sufficiently small the maximum number of small amplitude limit cycles of the differential system (5.8) bifurcating from the periodic solutions of the center (5.2), and which are in correspondence with the zeros $\bar{r}_i$ of f_j, is*

(a) 0 *if the first order average function f_1 is non-zero,*

(b) 0 *if $f_1 \equiv 0$ and the second order average function f_2 is non-zero,*

(c) 1 *if $f_1 \equiv f_2 \equiv 0$ and the third order average function f_3 is non-zero, where $\bar{r}_1 =$*

$\sqrt{-M_0/M_2}$ *whith* $M_0M_2 < 0$,

(d) 1 *if* $f_1 \equiv f_2 \equiv f_3 \equiv 0$ *and the fourth order average function* f_4 *is non-zero, where* $\overline{r}_1 = \sqrt{-N_0/N_2}$ *with* $N_0N_2 < 0$,

(e) 2 *if* $f_1 \equiv f_2 \equiv f_3 \equiv f_4 \equiv 0$ *and the fifth order average function* f_5 *is non-zero, when the equation* $H_4r^4 + H_2r^2 + H_0 = 0$ *has four simple solutions,*

(f) 2 *if* $f_1 \equiv f_2 \equiv f_3 \equiv f_4 \equiv f_5 \equiv 0$ *and the sixth order average function* f_6 *is non-zero, when the equation* $T_4r^4 + T_2r^2 + T_0 = 0$ *has four simple solutions,*

(g) 3 *if* $f_1 \equiv f_2 \equiv f_3 \equiv f_4 \equiv f_5 \equiv f_6 \equiv 0$ *and the seventh order average function* f_7 *is non-zero, when the equation* $G_6r^6 + G_4r^4 + G_2r^2 + G_0 = 0$ *has six simple solutions.*

Section 5.3 Proof of Theorem 5.2

For studying the limit cycles which bifurcate in a Hopf bifurcation from the center of the differential systems (5.2) when it is perturbed inside the class of all polynomial differential systems of degree 8, see (5.8), we proceed as follows.

First, doing the scaling $x = \varepsilon X$, $y = \varepsilon Y$ we introduce a small parameter ε. Thus we obtain the differential system $(\dot{X}, \dot{Y})$. Now performing the polar change of coordinates $X = r\cos\theta$, $Y = r\sin\theta$, the differential system $(\dot{X}, \dot{Y})$ written in polar coordinates becomes a differential system $(\dot{r}, \dot{\theta})$. Taking as independent variable the angle θ the differential system $(\dot{r}, \dot{\theta})$ produces the differential equation $dr/d\theta$. Finally, doing a Taylor expansion in the variable r at $r = 0$ and truncating at 7–th order in ε we obtain the differential equation

$$r' = \frac{dr}{d\theta} = \sum_{i=0}^{7} \varepsilon^i F_i(\theta, r) + O(\varepsilon^8). \tag{5.9}$$

The functions $F_i(\theta, r)$ for $i = 1, \ldots, 7$, of the differential system (5.9) are analytic, and since the independent variable θ appears through the sinus and cosinus of θ, they are 2π–periodic. Hence the assumptions for applying the averaging theory described in subsection 1.6 chapter 1 are satisfied. Now we shall study the limit cycles bifurcating from the center of systems (5.2) when it is perturbed as in (5.8). We give only the expressions of functions $F_1(r,\theta)$ and $F_2(r,\theta)$. The explicit expressions of $F_i(r,\theta)$ for $i = 3, \ldots, 7$

are very long, therefore we shall omit them here. Thus we have

$$F_1(r,\theta) = a_{00}^2 \cos\theta + b_{00}^2 sin\theta + \frac{1}{2} r(b_{01}^1 + a_{10}^1 - b_{01}^1 \cos 2\theta + a_{10}^1 \cos 2\theta + b_{10}^1 \sin 2\theta + a_{01}^1 \sin 2\theta),$$

and

$$\begin{aligned} F_2(r,\theta) = \quad & \frac{1}{r}(a_{00}^2 \cos\theta + a_{10}^1 r \cos^2\theta + b_{00}^2 \sin\theta + b_{10}^1 r \cos\theta \sin\theta + a_{01}^1 r \cos\theta \sin\theta + b_{01}^1 \\ & r \sin^2\theta)(-b_{00}^2 \cos\theta - b_{10}^1 r \cos^2\theta + a_{00}^2 \sin\theta - b_{01}^1 r \cos\theta \sin\theta + a_{10}^1 r \cos\theta \\ & \sin\theta + a_{01}^1 r \sin^2\theta) + (a_{00}^3 \cos\theta + a_{10}^2 r \cos^2\theta + a_{20}^1 r^2 \cos^3\theta + b_{00}^3 \sin\theta \\ & + b_{10}^2 r \cos\theta \sin\theta + a_{01}^2 r \cos\theta \sin\theta + b_{20}^1 r^2 \cos^2\theta \sin\theta + a_{11}^1 r^2 \cos^2\theta \\ & \sin\theta + b_{00}^2 1 r \sin^2\theta + b_{11}^1 r^2 \cos\theta \sin^2\theta + a_{02}^1 r^2 \cos\theta \sin^2\theta + b_{02}^1 r^2 \sin^3\theta). \end{aligned}$$

Using the formulas given in subsection 1.6 chapter 1, the averaged function of first order is

$$f_1(r) = (b_{01}^1 + a_{10}^1) r.$$

Clearly the polynomial $f_1(r)$ has no positive root, so the first average function does not provide any information on the limit cycles that bifurcate from the center when we perturb it as in system 5.8. So, the proof of the theorem follows for $k = 1$.

Taking $b_{01}^1 = -a_{10}^1$, we obtain $f_1(r) \equiv 0$. We apply the averaging theory of second order and we get the averaged function of second order $f_2(r) = (b_{01}^2 + a_{10}^2) r$.

Again the second averaged function does not provide any limit cycles. The proof of statement (b) holds.

Now taking $b_{01}^2 = -a_{10}^2$, we get $f_2(r) \equiv 0$, and applying the averaging theory of third order, we obtain the third averaged function $f_3(r) = r(M_2 r^2 + M_0)$ where

$$\begin{aligned} M_0 = \quad & -(b_{11}^1 b_{00}^2 - b_{01}^3 + 2 b_{00}^2 a_{20}^1 - 2 b_{02}^1 a_{00}^2 - a_{11}^1 a_{00}^2 - a_{10}^3), \\ M_2 = \quad & \frac{1}{4}(3 b_{03}^1 + b_{21}^1 + a_{12}^1 + 3 a_{30}^1). \end{aligned}$$

Clearly the coefficients M_0 and M_2 are independent, so the polynomial $f_3(r)$ can have at most one positive real root, so statement (c) of the theorem is proved.

In order to apply the averaging theory of fourth order we need that $f_3(r) \equiv 0$. So we take

$$a_{10}^3 = b_{11}^1 b_{00}^2 - b_{01}^3 + 2b_{00}^2 a_{20}^1 - 2b_{02}^1 a_{00}^2 - a_{11}^1 a_{00}^2, \qquad a_{12}^1 = -3b_{03}^1 - b_{21}^1 - 3a_{30}^1.$$

Computing the function $f_4(r)$, we obtain $f_4(r) = r(N_2 r^2 + N_0)$, where

$$\begin{aligned} N_2 = \;& -\frac{1}{4}(b_{02}^1 b_{11}^1 + b_{11}^1 b_{20}^1 + b_{10}^1 b_{21}^1 - 3b_{03}^2 - b_{21}^2 + b_{21}^1 a_{01}^1 - a_{12}^2 - 3a_{30}^2 + 3b_{10}^1 a_{30}^1 \\ & -2b_{02}^1 a_{02}^1 - 2b_{12}^1 a_{10}^1 - a_{02}^1 a_{11}^1 + 2b_{20}^1 a_{20}^1 - a_{11}^1 a_{20}^1 - 2a_{10}^1 a_{21}^1 + 3a_{01}^1 a_{30}^1) \end{aligned}$$

$$\begin{aligned} N_0 = \;& -(-b_{10}^1 b_{11}^1 b_{00}^2 + b_{00}^2 b_{11}^2 + b_{11}^1 b_{00}^3 - b_{01}^4 + 2b_{02}^1 b_{00}^2 a_{10}^1 - a_{11}^1 a_{00}^3 - a_{10}^4 + 2b_{00}^2 a_{20}^2 \\ & +b_{00}^2 a_{10}^1 a_{11}^1 - 2b_{10}^1 b_{00}^2 a_{20}^1 + 2b_{00}^3 a_{20}^1 - 2b_{02}^2 a_{00}^2 - 2b_{02}^1 a_{01}^1 a_{00}^2 - 2b_{02}^1 a_{00}^3 \\ & -b_{11}^1 a_{10}^1 a_{00}^2 - a_{01}^1 a_{11}^1 a_{00}^2 - 2a_{10}^1 a_{20}^1 a_{00}^2 - a_{00}^2 a_{11}^2). \end{aligned}$$

From the expression of the polynomial $f_4(r)$ we know that it has at most one positive real root. Hence statement (d) of the theorem is proved.

Doing $N_2 = 0$ and $N_0 = 0$ and we extract the value of a_{12}^2 from $N_2 = 0$ and the value of a_{10}^4 from $N_0 = 0$, we have that $f_4(r) \equiv 0$, which allows us to apply the averaging theory of order five, and we obtain the fifth averaged function $f_5(r) = r(H_4 r^4 + H_2 r^2 + H_0)$, where

$$\begin{aligned} H_0 = \;& -((b_{10}^1)^2 b_{11}^1 b_{00}^2 - b_{21}^1 (b_{00}^2)^2 - b_{11}^1 b_{00}^2 b_{10}^2 - b_{10}^1 b_{00}^2 b_{11}^2 - b_{10}^1 b_{11}^1 b_{00}^3 + b_{11}^2 b_{00}^3 + b_{00}^2 b_{11}^3 \\ & +2b_{00}^2 b_{02}^2 a_{10}^1 + 2b_{02}^1 b_{00}^3 a_{10}^1 + 2b_{02}^1 b_{00}^2 a_{01}^1 a_{10}^1 + b_{11}^1 b_{00}^2 (a_{10}^1)^2 - 2b_{02}^1 b_{10}^1 b_{00}^2 a_{10}^1 - b_{10}^1 \\ & b_{00}^2 a_{10}^1 a_{11}^1 + b_{00}^3 a_{10}^1 a_{11}^1 + b_{00}^2 a_{01}^1 a_{10}^1 a_{11}^1 + 2(b_{10}^1)^2 b_{00}^2 a_{20}^1 - 2b_{00}^2 b_{10}^2 a_{20}^1 - a_{10}^5 - a_{11}^2 a_{00}^3 \\ & +2b_{00}^4 a_{20}^1 + 2b_{00}^2 (a_{10}^1)^2 a_{20}^1 - 3(b_{00}^2)^2 a_{30}^1 + b_{11}^1 b_{00}^4 - 2b_{10}^1 b_{00}^3 a_{20}^1 + 2b_{12}^1 b_{00}^2 a_{00}^2 - b_{01}^5 \\ & -2b_{00}^3 2a_{00}^2 - 2b_{02}^2 a_{01}^1 a_{00}^2 - 2b_{02}^1 (a_{01}^1)^2 a_{00}^2 + b_{10}^1 b_{11}^1 a_{10}^1 a_{00}^2 - b_{11}^2 a_{10}^1 a_{00}^2 + 2b_{00}^2 a_{20}^3 - b_{11}^1 \\ & a_{01}^1 a_{10}^1 a_{00}^2 - 2b_{02}^1 (a_{10}^1)^2 a_{00}^2 + b_{21}^1 (a_{00}^2)^2 + 3a_{30}^1 (a_{00}^2)^2 - 2b_{02}^1 a_{00}^2 a_{01}^2 - a_{00}^2 a_{11}^3 - a_{11}^1 a_{00}^4 \\ & -(a_{01}^1)^2 a_{11}^1 a_{00}^2 - (a_{10}^1)^2 a_{11}^1 a_{00}^2 + 2b_{10}^1 a_{10}^1 a_{20}^1 a_{00}^2 - 2a_{01}^1 a_{10}^1 a_{20}^1 a_{00}^2 + 2b_{00}^2 a_{21}^1 a_{00}^2 + b_{00}^2 \\ & a_{11}^1 a_{10}^2 - b_{11}^1 a_{00}^2 a_{10}^2 - 2a_{20}^1 a_{00}^2 a_{10}^2 + b_{00}^2 a_{10}^1 a_{11}^2 - a_{01}^1 a_{00}^2 a_{11}^2 - a_{11}^1 a_{00}^2 a_{01}^2 - 2b_{02}^1 a_{00}^4 \\ & -2b_{10}^1 b_{00}^2 a_{20}^2 + 2b_{00}^3 a_{20}^2 - 2a_{10}^1 a_{00}^2 a_{20}^2 - 2b_{02}^2 a_{00}^3 - 2b_{02}^1 a_{01}^1 a_{00}^3 + 2b_{02}^1 b_{00}^2 a_{10}^2 - b_{11}^1 \\ & a_{10}^1 a_{00}^3 - a_{01}^1 a_{11}^1 a_{00}^3 - 2a_{10}^1 a_{20}^1 a_{00}^3), \end{aligned}$$

$$
\begin{aligned}
H_2 = & -\frac{1}{4}(-b_{02}^1 b_{10}^1 b_{11}^1 - 2b_{10}^1 b_{11}^1 b_{20}^1 - (b_{10}^1)^2 b_{21}^1 + 3b_{13}^1 b_{00}^2 + 3b_{31}^1 b_{00}^2 + b_{11}^1 b_{02}^2 + b_{21}^1 b_{10}^2 - 3a_{30}^3 \\
& + b_{02}^1 b_{11}^2 + b_{20}^1 b_{11}^2 + b_{11}^1 b_{20}^2 + b_{10}^1 b_{21}^2 - 3b_{03}^3 - b_{11}^3 - b_{11}^1 b_{20}^1 a_{01}^1 - b_{10}^1 b_{21}^1 a_{01}^1 - 2b_{02}^2 a_{02}^1 \\
& - 2b_{02}^1 a_{01}^1 a_{02}^1 + 2(b_{02}^1)^2 a_{10}^1 + (b_{11}^1)^2 a_{10}^1 + 2b_{10}^1 b_{12}^1 a_{10}^1 + b_{21}^2 a_{01}^1 + 2b_{02}^1 b_{20}^1 a_{10}^1 - 2b_{12}^2 a_{10}^1 \\
& - b_{11}^1 a_{02}^1 a_{10}^1 - a_{01}^1 a_{02}^1 a_{11}^1 - b_{02}^1 a_{10}^1 a_{11}^1 + 3b_{10}^2 a_{30}^1 + 3b_{10}^1 a_{30}^2 + 3a_{01}^1 a_{30}^2 - p_{12}^3 - a_{11}^1 a_{20}^2 \\
& + b_{20}^1 a_{10}^1 a_{11}^1 - a_{10}^1 (a_{11}^1)^2 - 4b_{10}^1 b_{20}^1 a_{20}^1 + 2b_{20}^2 a_{20}^1 - 2b_{20}^1 a_{01}^1 a_{20}^1 - 3(b_{10}^1)^2 a_{30}^1 - a_{20}^1 a_{11}^2 \\
& + b_{11}^1 a_{10}^1 a_{20}^1 - 2a_{02}^1 a_{10}^1 a_{20}^1 + b_{10}^1 a_{11}^1 a_{20}^1 - 2a_{10}^1 (a_{20}^1)^2 + 2b_{10}^1 a_{10}^1 a_{21}^1 + 2b_{00}^2 a_{22}^1 - a_{02}^1 a_{11}^2 \\
& - 12b_{04}^1 a_{00}^2 - 2b_{22}^1 a_{00}^2 - 3a_{13}^1 a_{00}^2 - 3a_{31}^1 a_{00}^2 + b_{21}^1 a_{01}^2 - 3b_{10}^1 a_{01}^1 a_{30}^1 + 12b_{00}^2 a_{40}^1 - a_{11}^1 a_{00}^2 \\
& + 3a_{30}^1 a_{01}^2 - 2b_{02}^1 a_{02}^2 - 2b_{12}^1 a_{10}^2 - 2a_{21}^1 a_{10}^2 + 2b_{20}^1 a_{20}^2 - 2a_{10}^1 a_{21}^2), \\
H_4 = & \frac{1}{8}(5b_{05}^1 + b_{23}^1 + b_{41}^1 + a_{14}^1 + a_{32}^1 + 5a_{50}^1).
\end{aligned}
$$

The rank of the Jacobian matrix of the function $\mathcal{H} = (H_0, H_2, H_4)$ with respect to the coefficients a_{ij}^s and b_{ij}^s which appear in their expressions is maximal, i.e. it is 3. Then the coefficients H_i for $i = 0, 2, 4$ which appear in the expression of $f_5(r)$ are linearly independent. By the roots of a quadratic polynomial in the variable r^2 it follows that $f_5(r)$ can have at most two positive real roots. Therefore statement (e) of the theorem is proved. Imposing that $H_0 = 0$, $H_2 = 0$ and $H_4 = 0$ we know that $f_5(r) \equiv 0$, and applying the averaging theory of order six we obtain the sixth average function $f_6(r) = r(T_4 r^4 + T_2 r^2 + T_0)$. The values of T_0, T_2 and T_4 are given in the appendix of chapter 5.

Taking $T_0 = 0, T_2 = 0$ and $T_4 = 0$ we obtain that $f_6(r) \equiv 0$. Computing the seventh average function we obtain $f_7(r) = r(G_6 r^6 + G_4 r^4 + G_2 r^2 + G_0)$. We do not give the big expressions of the coefficients G_i for $i = 0, 2, 4, 6$. According to the expression of the polynomial $f_7(r)$ it follows easily that $f_7(r)$ can have at most three positive real roots.

Now we are going to reach our result by giving an example with exactely three limit cycles.

Example with three limit cycles. We consider the kukles differential systems of degree 8 symmetric with respect to the x–axis, with $a^2 + b^2 + c^2 + d^2 + e^2 \neq 0$, perturbed inside a class of polynomial differential systems of degree 8

$$\begin{aligned}\dot{x} = & -y+\epsilon^7\left(r_0+r_2x^2\right)+\epsilon^6\left(u_2x^2-1\right)+\epsilon^4\left(\frac{815x^3}{36}-8xy^2-5xy+\frac{17x}{2}\right)+\epsilon^5(x^3+x^2\\ & -xy)+\epsilon^3(-\frac{851x^5}{30}-x^3y+x^3-7x^2y-\frac{37xy^2}{2}-\frac{3x}{2})+\epsilon^2(-\frac{14x^5}{3}+2x^3y^2-9x^2y^3\\ & +x^2-2xy^2+xy-x-2y^3+y^2+1)+\epsilon(-2x^7-4x^4y^2-\frac{2x^4y}{3}+2x^3y^4-x^3y^3-7x^3y^2\\ & +x^3y-7x^2y^4+x^2y^2+\frac{73xy^6}{5}-3xy^5+2xy^2-2x-2y^4+4y^3+2y^2-2y),\\ \dot{y} = & x+ax^8+bx^6y^2+cx^4y^4+dx^2y^6+ey^8+\epsilon^6(g_1x+g_2x^2)+\epsilon^7(k_0+k_1x)+\epsilon^5(\frac{387x^2y}{4}\\ & -\frac{1743xy}{4}-2y^3+9y^2-32y-\frac{122}{3})+\epsilon^4\left(-5x^2y-6xy^2+xy-\frac{y^3}{4}\right)+\epsilon^2(-x^2y^2\\ & -2xy^4+4x+y^5+y-\frac{1}{2})+\epsilon^3\left(4x^4y-2x^3y-4x^2-\frac{xy^3}{2}-\frac{xy^2}{3}-3y^5\right)+\epsilon(5x^4y^2\\ & +2x^4+x^3y^3+x^3y+x^3+4x^2y^5+2x^2y^3-2x^2y+3xy+3x+y^7+y^5+2y),\end{aligned} \tag{5.10}$$

where $r_0, r_2, u_2, g_1, g_2, k_0, k_1$ are real parametres.

An exhausting computation shows that for the polynomial differential systems (5.10), we obtain $f_1(r) \equiv f_2(r) \equiv f_3(r) \equiv f_4(r) \equiv f_5(r) \equiv f_6(r) \equiv 0$, and

$$f_7(r) = r(r-3)(r-2)(r-1)(r+1)(r+2)(r+3).$$

Then for this systems we have three limit cycles bifurcating from the periodic orbits of the center $\dot{x} = -y, \dot{y} = x+ax^8+bx^6y^2+cx^4y^4+dx^2y^6+ey^8$. Moreover, in polar coordinates (r,θ) the periodic orbits that bifurcate are $r = 1,2,3$. This completes the proof of the theorem when $k = 7$.

In this part dedicated to giving appendixes to the chapters 3, 4 and 5

Section The appendix of Chapter 3

Here we provide the values of L_1, L_2, L_3, D_1 and D_2 that appear in the proof of Theorem 3.1.

$$
\begin{aligned}
L_1 \;&= r_1^4r_2^2r_3 - r_1^2r_2^4r_3 - r_1^4r_2r_3^2 + r_1r_2^4r_3^2 + r_1^2r_2r_3^4 - r_1r_2^2r_3^4 + r_1^3r_2^2r_3s_1 - r_1r_2^4r_3s_1 - r_1^3r_2r_3^2s_1 + r_2^4r_3^2\\
&s_1 + r_1r_2r_3^4s_1 - r_2^2r_3^4s_1 + r_1^2r_2^2r_3s_1^2 - r_2^4r_3s_1^2 - r_1^2r_2r_3^2s_1^2 + r_2r_3^4s_1^2 + r_1r_2^2r_3s_1^3 - r_1r_2r_3^2s_1^3 + r_2^2r_3s_1^4\\
&-r_2r_3^2s_1^4 + r_1^4r_2r_3s_2 - r_1^2r_2^3r_3s_2 - r_1^4r_3^2s_2 + r_1r_2^3r_3^2s_2 + r_1^2r_3^4s_2 - r_1r_2r_3^4s_2 + r_1^3r_2r_3s_1s_2 - r_3^4s_1s_2^2\\
&-r_1r_2^3r_3s_1s_2 - r_1^3r_3^2s_1s_2 + r_2^3r_3^2s_1s_2 + r_1r_3^4s_1s_2 - r_2r_3^4s_1s_2 + r_1^2r_2r_3s_1^2s_2 - r_2^3r_3s_1^2s_2 + r_2^2r_3^2s_1s_2^2\\
&+r_3^4s_1^2s_2 + r_1r_2r_3s_1^3s_2 - r_1r_3^2s_1^3s_2 + r_2r_3s_1^4s_2 - r_3^2s_1^4s_2 - r_1^2r_3^2s_1^2s_2 + r_1^4r_3s_2^2 - r_1r_2^2r_3s_1s_2^2 + r_1r_2r_3^3\\
&-r_1^2r_2^2r_3s_2^2 + r_1r_2^2r_3^2s_2^2 - r_1r_3^4s_2^2 + r_1^3r_3s_1s_2^2 + r_1^2r_3s_1^2s_2^2 - r_2^2r_3s_1^2s_2^2 + r_1r_3s_1^3s_2^2 + r_3s_1^4s_2^2 - r_1^2r_2r_3s_2^3\\
&+r_1r_2r_3^2s_2^3 - r_1r_2r_3s_1s_2^3 + r_2r_3^2s_1s_2^3 - r_2r_3s_1^2s_2^3 - r_1^2r_3s_2^4 + r_1r_3^2s_2^4 - r_1r_3s_1s_2^4 + r_3^2s_1s_2^4 - r_1^4r_2r_3s_3\\
&-r_3s_1^2s_2^4 + r_1^4r_2^2s_3 - r_1^2r_2^4s_3 + r_1r_2^4r_3s_3 + r_1^2r_2r_3^3s_3 - r_1r_2^2r_3^3s_3 + r_1^3r_2^2s_1s_3 - r_1r_2^4s_1s_3 - r_1^3r_2r_3s_1s_3\\
&+r_2^4r_3s_1s_3s_1s_3 - r_2^2r_3^3s_1s_3 + r_1^2r_2^2s_1^2s_3 - r_2^4s_1^2s_3 - r_1^2r_2r_3s_1^2s_3 + r_2r_3^3s_1^2s_3 + r_1r_2^2s_1^3s_3 - r_1r_2r_3s_1^3s_3\\
&+r_1^3r_2s_1s_2s_3 - r_2r_3s_1^4s_3 + r_1^4r_2s_2s_3 - r_1^2r_2^3s_2s_3 - r_1^4r_3s_2s_3 + r_1r_2^3r_3s_2s_3 + r_1^2r_3^3s_2s_3 - r_1r_2r_3^3s_2s_3\\
&-r_1r_2^3s_1s_2s_3 - r_1^3r_3s_1s_2s_3 + r_2^3r_3s_1s_2s_3 + r_1r_3^3s_1s_2s_3 - r_2r_3^3s_1s_2s_3 + r_1^2r_2s_1^2s_2s_3 - r_1^2r_3s_1^2s_2s_3\\
&+r_2^2s_1^4s_3 - r_2^3s_1^2s_2s_3 + r_3^3s_1^2s_2s_3 + r_1r_2s_1^3s_2s_3 - r_1r_3s_1^3s_2s_3 + r_2s_1^4s_2s_3 - r_3s_1^4s_2s_3 + r_2^2r_3s_1s_2^2s_3\\
&-r_1^2r_2^2s_2^2s_3 + r_1r_2^2r_3s_2^2s_3 - r_1r_3^3s_2^2s_3 + r_1^3s_1s_2^2s_3 - r_1r_2^2s_1s_2^2s_3 - r_3^3s_1s_2^2s_3 - r_1r_2s_1s_2^3s_3 + r_1^2s_1^2s_2^2s_3\\
&-r_2^2s_1^2s_2^2s_3 + r_1s_1^3s_2^2s_3 + s_1^4s_2^2s_3 - r_1^2r_2s_2^3s_3 + r_1r_2r_3s_2^3s_3 + r_2r_3s_1s_2^3s_3 + r_1^4s_2^2s_3 - r_1^4r_2s_3^2 + r_1r_2^4s_3^2
\end{aligned}
$$

$$
\begin{aligned}
&-r_2s_1^2s_2^3s_3 - r_1^2s_2^4s_3 + r_1r_3s_2^4s_3 - r_1s_1s_2^4s_3 + r_3s_1s_2^4s_3 - s_1^2s_2^4s_3 + r_1^2r_2r_3^2s_3^2 - r_1^3r_2s_1s_3^2 - r_1r_2^2r_3^2s_3^2\\
&+r_1r_2r_3^2s_1s_3^2 - r_2^2r_3^2s_1s_3^2 - r_1^2r_2s_1^2s_3^2 + r_2r_3^2s_1^2s_3^2 - r_1r_2s_1^3s_3^2 - r_2s_1^4s_3^2 - r_1^4s_2s_3^2 + r_1r_2^3s_2s_3^2 + r_2^4s_1s_3^2\\
&+r_1^2r_3^2s_2s_3^2 - r_1r_2r_3^2s_2s_3^2 - r_1^3s_1s_2s_3^2 + r_2^3s_1s_2s_3^2 + r_1r_3^2s_1s_2s_3^2 - r_2r_3^2s_1s_2s_3^2 - r_1^2s_1^2s_2s_3^2 + r_3^2s_1^2s_2s_3^2\\
&-r_1s_1^3s_2s_3^2 - s_1^4s_2s_3^2 + r_1r_2^2s_2^2s_3^2 - r_1r_3^2s_2^2s_3^2 + r_2^2s_1s_2^2s_3^2 - r_3^2s_1s_2^2s_3^2 + r_1r_2s_2^3s_3^2 + r_2s_1s_2^3s_3^2 + r_1s_2^4s_3^2\\
&+s_1s_2^4s_3^2 + r_1^2r_2r_3s_3^3 - r_1r_2^2r_3s_3^3 + r_1r_2r_3s_1s_3^3 - r_2^2r_3s_1s_3^3 + r_2r_3s_1^2s_3^3 + r_1^2r_3s_2s_3^3 - s_1s_2^2s_3^4 + r_1^2r_2s_3^4\\
&-r_1r_2^2s_3^4 + r_1r_2s_1s_3^4 - r_2^2s_1s_3^4 + r_2s_1^2s_3^4 + r_1^2s_2s_3^4 - r_1r_2s_2s_3^4 + r_1s_1s_2s_3^4 - r_2s_1s_2s_3^4 + s_1^2s_2s_3^4 - r_1s_2^2s_3^4\\
&-r_1r_2r_3s_2s_3^3 + r_1r_3s_1s_2s_3^3 - r_2r_3s_1s_2s_3^3 + r_3s_1^2s_2s_3^3 - r_1r_3s_2^2s_3^3 - r_3s_1s_2^2s_3^3,
\end{aligned}
$$

$$
\begin{aligned}
L_2 = \;& 4(r_1^4(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+s_3)))\\
&+r_1^3s_1(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+s_3)))\\
&+r_1^2(-r_2^4(r_3+s_3)-r_2^3s_2(r_3+s_3)+r_2^2(s_1^2-s_2^2)(r_3+s_3)+r_2(r_3^4+r_3^3s_3+r_3^2(-s_1^2+s_3^2)+r_3(-s_2^3\\
&+s_1^2(s_2-s_3)+s_3^3)+s_3(-s_2^3+s_1^2(s_2-s_3)+s_3^3))+s_2(r_3^4+r_3^3s_3+r_3^2(-s_1^2+s_3^2)+r_3(-s_2^3+s_1^2(s_2\\
&-s_3)+s_3^3)+s_3(-s_2^3+s_1^2(s_2-s_3)+s_3^3)))+r_1(r_2^4(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))+r_2^3s_2(r_3^2\\
&+r_3(-s_1+s_3)+s_3(-s_1+s_3))-r_2^2(r_3^4+r_3^3s_3+r_3^2(-s_2^2+s_3^2)+r_3(-s_1^3+s_1s_2^2-s_2^2s_3+s_3^3)+s_3\\
&(-s_1^3+s_1s_2^2-s_2^2s_3+s_3^3))+r_2(r_3^4(s_1-s_2)+r_3^3(s_1-s_2)s_3+r_3^2(-s_1^3+s_2^3+s_1s_3^2-s_2s_3^2)+r_3\\
&(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))+s_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3)))\\
&+s_2(r_3^4(s_1-s_2)+r_3^3(s_1-s_2)s_3+r_3^2(-s_1^3+s_2^3+s_1s_3^2-s_2s_3^2)+r_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)\\
&+s_1(-s_2^3+s_3^3))+s_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))))+s_1(r_2^4(r_3^2+r_3(-s_1+s_3)\\
&+s_3(-s_1+s_3))+r_2^3s_2(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))-r_2^2(r_3^4+r_3^3s_3+r_3^2(-s_2^2+s_3^2)+r_3\\
&(-s_1^3+s_1s_2^2-s_2^2s_3+s_3^3)+s_3(-s_1^3+s_1s_2^2-s_2^2s_3+s_3^3))+r_2(r_3^4(s_1-s_2)+r_3^3(s_1-s_2)s_3+r_3^2\\
&(-s_1^3+s_2^3+s_1s_3^2-s_2s_3^2)+r_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))+s_3(s_1^3(s_2-s_3)+s_2\\
&s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3)))+s_2(r_3^4(s_1-s_2)+r_3^3(s_1-s_2)s_3+r_3^2(-s_1^3+s_2^3+s_1s_3^2-s_2s_3^2)+r_3\\
&(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))+s_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3)))))^2\\
&-4(r_1^2(-r_2^3(r_3+s_3)+r_2^2(s_1-s_2)(r_3+s_3)+r_2(r_3^3+r_3(s_1-s_2-s_3)(s_2-s_3)+(s_1-s_2-s_3)\\
&(s_2-s_3)s_3+r_3^2(-s_1+s_3))+s_2(r_3^3+r_3(s_1-s_2-s_3)(s_2-s_3)+(s_1-s_2-s_3)(s_2-s_3)s_3+r_3^2\\
&(-s_1+s_3)))+r_1^3(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3\\
&(-s_2+s_3)))+r_1(r_2(s_1-s_2)(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)\\
&+(s_1-s_2)s_2(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)+r_2^3(r_3^2+r_3(-s_1\\
&+s_3)+s_3(-s_1+s_3))-r_2^2(r_3^3+r_3^2(-s_2+s_3)-r_3(s_1-s_3)(s_1-s_2+s_3)-(s_1-s_3)s_3(s_1-s_2\\
&+s_3)))+s_1(r_2(s_1-s_2)(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)+(s_1\\
&-s_2)s_2(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)+r_2^3(r_3^2+r_3(-s_1+s_3)\\
&+s_3(-s_1+s_3))-r_2^2(r_3^3+r_3^2(-s_2+s_3)-r_3(s_1-s_3)(s_1-s_2+s_3)-(s_1-s_3)s_3(s_1-s_2+s_3))))\\
&(r_1^5(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+s_3)))+r_1^4
\end{aligned}
$$

$$
\begin{aligned}
&s_1(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+s_3)))+r_1^3\\
&s_1^2(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+s_3)))+r_1^2\\
&(-r_2^5(r_3+s_3)-r_2^4s_2(r_3+s_3)-r_2^3s_2^2(r_3+s_3)+r_2^2(s_1^3-s_2^3)(r_3+s_3)+r_2(r_3^5+r_3^4s_3+r_3^3s_3^2+r_3^2\\
&(-s_1^3+s_3^3)+r_3(-s_2^4+s_1^3(s_2-s_3)+s_3^4)+s_3(-s_2^4+s_1^3(s_2-s_3)+s_3^4))+s_2(r_3^5+r_3^4s_3+r_3^3s_3^2+r_3^2
\end{aligned}
$$

$$
\begin{aligned}
&(-s_1^3+s_3^3)+r_3(-s_2^4+s_1^3(s_2-s_3)+s_3^4)+s_3(-s_2^4+s_1^3(s_2-s_3)+s_3^4)))+r_1(r_2^5(r_3^2+r_3(-s_1+s_3)\\
&+s_3(-s_1+s_3))+r_2^4s_2(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))+r_2^3s_2^2(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))
\end{aligned}
$$

$$+s_3^4) + s_3(-s_1^4 + s_1 s_2^3 - s_2^3 s_3 + s_3^4)) + r_2(r_3^5(s_1 - s_2) + r_3^4(s_1 - s_2)s_3 + r_3^3(s_1 - s_2)s_3^2 + r_3^2(-s_1^4 + s_2^4$$
$$+s_1 s_3^3 - s_2 s_3^3) + r_3(s_1^4(s_2 - s_3) + s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4 + s_3^4)) + s_3(s_1^4(s_2 - s_3) + s_2 s_3(s_2^3 - s_3^3) + s_1$$
$$(-s_2^4 + s_3^4))) + s_2(r_3^5(s_1 - s_2) + r_3^4(s_1 - s_2)s_3 + r_3^3(s_1 - s_2)s_3^2 + r_3^2(-s_1^4 + s_2^4 + s_1 s_3^3 - s_2 s_3^3) + r_3(s_1^4(s_2$$
$$-s_3) + s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4 + s_3^4)) + s_3(s_1^4(s_2 - s_3) + s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4 + s_3^4)))) + s_1(r_2^5(r_3^2$$
$$+r_3(-s_1 + s_3) + s_3(-s_1 + s_3)) + r_2^4 s_2(r_3^2 + r_3(-s_1 + s_3) + s_3(-s_1 + s_3)) + r_2^3 s_2^2(r_3^2 + r_3(-s_1 + s_3) + s_3$$
$$(-s_1 + s_3)) - r_2^2(r_3^5 + r_3^4 s_3 + r_3^3 s_3^2 + r_3^2(-s_2^3 + s_3^3) + r_3(-s_1^4 + s_1 s_2^3 - s_2^3 s_3 + s_3^4) + s_3(-s_1^4 + s_1 s_2^3 - s_2^3 s_3$$
$$+s_3^4)) + r_2(r_3^5(s_1 - s_2) + r_3^4(s_1 - s_2)s_3 + r_3^3(s_1 - s_2)s_3^2 + r_3^2(-s_1^4 + s_2^4 + s_1 s_3^3 - s_2 s_3^3) + r_3(s_1^4(s_2 - s_3)$$
$$+s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4 + s_3^4)) + s_3(s_1^4(s_2 - s_3) + s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4 + s_3^4))) + s_2(r_3^5(s_1 - s_2)$$
$$+r_3^4(s_1 - s_2)s_3 + r_3^3(s_1 - s_2)s_3^2 + r_3^2(-s_1^4 + s_2^4 + s_1 s_3^3 - s_2 s_3^3) + r_3(s_1^4(s_2 - s_3) + s_2 s_3(s_2^3 - s_3^3) + (-s_2^4$$
$$+s_3^4))s_1 + s_3(s_1^4(s_2 - s_3) + s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4 + s_3^4))))),$$

$$L_3 = \quad r_1^2(-r_2^3(r_3 + s_3) + r_2^2(s_1 - s_2)(r_3 + s_3) + r_2(r_3^3 + r_3(s_1 - s_2 - s_3)(s_2 - s_3) + (s_1 - s_2 - s_3)(s_2$$
$$-s_3)s_3 + r_3^2(-s_1 + s_3)) + s_2(r_3^3 + r_3(s_1 - s_2 - s_3)(s_2 - s_3) + (s_1 - s_2 - s_3)(s_2 - s_3)s_3 + r_3^2(-s_1$$
$$+s_3))) + r_1^3(r_2^2(r_3 + s_3) + s_2(-r_3^2 + r_3(s_2 - s_3) + (s_2 - s_3)s_3) - r_2(r_3^2 + r_3(-s_2 + s_3) + s_3(-s_2$$
$$+s_3))) + r_1(r_2(s_1 - s_2)(r_3^3 + r_3(s_1 - s_3)(s_2 - s_3) - r_3^2(s_1 + s_2 - s_3) + (s_1 - s_3)(s_2 - s_3)s_3) + (s_1$$
$$-s_2)s_2(r_3^3 + r_3(s_1 - s_3)(s_2 - s_3) - r_3^2(s_1 + s_2 - s_3) + (s_1 - s_3)(s_2 - s_3)s_3) + r_2^3(r_3^2 + r_3(-s_1$$
$$+s_3) + s_3(-s_1 + s_3)) - r_2^2(r_3^3 + r_3^2(-s_2 + s_3) - r_3(s_1 - s_3)(s_1 - s_2 + s_3) - (s_1 - s_3)s_3(s_1 - s_2$$
$$+s_3))) + s_1(r_2(s_1 - s_2)(r_3^3 + r_3(s_1 - s_3)(s_2 - s_3) - r_3^2(s_1 + s_2 - s_3) + (s_1 - s_3)(s_2 - s_3)s_3)$$
$$+(s_1 - s_2)s_2(r_3^3 + r_3(s_1 - s_3)(s_2 - s_3) - r_3^2(s_1 + s_2 - s_3) + (s_1 - s_3)(s_2 - s_3)s_3) + r_2^3(r_3^2 + r_3$$
$$(-s_1 + s_3) + s_3(-s_1 + s_3)) - r_2^2(r_3^3 + r_3^2(-s_2 + s_3) - r_3(s_1 - s_3)(s_1 - s_2 + s_3) - (s_1 - s_3)s_3$$
$$(s_1 - s_2 + s_3))).$$

$$D_1 = \quad \frac{1}{-(r_1 + s_1)s_2^2 + (r_1^2 + (s_1 - r_2)r_1 + s_1(s_1 - r_2))s_2 + r_2(r_1^2 + (s_1 - r_2)r_1 + s_1(s_1 - r_2))}\Big((b_2^2$$
$$(-(r_2 + s_2)s_3^3 + (r_2^2 + s_2 r_2 + s_2^2)s_3^2 - r_3^2(r_2^2 + (s_2 - r_3)r_2 + s_2(s_2 - r_3)))r_1^5 + b_2(b_2 s_1 + 2)(r_3$$
$$-s_3)((r_3^2 + (s_3 - s_2)r_3 + s_3(s_3 - s_2))r_2 - r_2^2(r_3 + s_3) + s_2(r_3^2 + (s_3 - s_2)r_3 + s_3(s_3 - s_2)))r_1^4$$

$$+(b_2 s_1 + 1)^2(r_3 - s_3)((r_3^2 + (s_3 - s_2)r_3 + s_3(s_3 - s_2))r_2 - r_2^2(r_3 + s_3) + s_2(r_3^2 + (s_3 - s_2)r_3$$
$$+s_3(s_3 - s_2)))r_1^3 + (r_3 - s_3)(b_2^2(r_3 + s_3)r_2^5 + b_2(b_2 s_2 + 2)(r_3 + s_3)r_2^4 + (b_2 s_2 + 1)^2(r_3 + s_3)$$
$$r_2^3 - (s_1 - s_2)((s_1^2 + s_2 s_1 + s_2^2)b_2^2 + 2(s_1 + s_2)b_2 + 1)(r_3 + s_3)r_2^2 + (-2b_2 s_3^4 - 2b_2 r_3 s_3^3 + (-2$$
$$b_2 r_3^2 + s_1 + b_2 s_1^2(b_2 s_1 + 2))s_3^2 + (b_2^2 s_2^4 + 2b_2 s_2^3 + s_2^2 - s_1(b_2 s_1 + 1)^2 s_2 + r_3(-2b_2 r_3^2 + s_1 + b_2$$
$$s_1^2(b_2 s_1 + 2)))s_3 + r_3(b_2^2 s_2^4 + 2b_2 s_2^3 + s_2^2 - s_1(b_2 s_1 + 1)^2 s_2 + r_3(-2b_2 r_3^2 + s_1 + b_2 s_1^2(b_2 s_1$$

$+2))))r_2 + s_2(-2b_2s_3^4 - 2b_2r_3s_3^3 + (-2b_2r_3^2 + s_1 + b_2s_1^2(b_2s_1+2))s_3^2 + (b_2^2s_2^4 + 2b_2s_2^3 + s_2^2$
$-s_1(b_2s_1+1)^2s_2 + r_3(-2b_2r_3^2 + s_1 + b_2s_1^2(b_2s_1+2)))s_3 + r_3(b_2^2s_2^4 + 2b_2s_2^3 + s_2^2 - s_1(b_2s_1+1)^2$

$s_2 + r_3(-2b_2r_3^2 + s_1 + b_2s_1^2(b_2s_1+2)))))r_1^2 + (b_2^2(-r_3^3 + s_1r_3^2 + s_3^2(s_3-s_1))r_2^5 - b_2(b_2s_2+2)(r_3-s_3)$
$(r_3^2 + (s_3-s_1)r_3 + s_3(s_3-s_1))r_2^4 - (b_2s_2+1)^2(r_3-s_3)(r_3^2 + (s_3-s_1)r_3 + s_3(s_3-s_1))r_2^3 + (-2b_2s_3^5$
$+s_2(b_2s_2+1)^2s_3^3 + s_1(b_2^2s_1^3 + 2b_2s_1^2 + s_1 - s_2(b_2s_2+1)^2)s_3^2 - r_3^2(-2b_2r_3^3 + b_2^2(r_3-s_1)s_2^3 + s_1^2 + 2b_2$
$(r_3-s_1)s_2^2 + b_2s_1^3(b_2s_1+2) + (r_3-s_1)s_2))r_2^2 + (s_1-s_2)(2b_2s_3^5 - (b_2s_1^2 + s_1 + b_2s_2^2 + s_2)(b_2(s_1+s_2)$
$+1)s_3^3 + s_1s_2((s_1^2 + s_2s_1 + s_2^2)b_2^2 + 2(s_1+s_2)b_2 + 1)s_3^2 + r_3^2((r_3(s_1+s_2)(s_1^2+s_2^2) - s_1s_2(s_1^2 + s_2s_1$
$+s_2^2))b_2^2 - 2(r_3-s_1)(r_3-s_2)(r_3+s_1+s_2)b_2 - s_1s_2 + r_3(s_1+s_2)))r_2 + s_2(s_2-s_1)(-2b_2s_3^5 + (b_2s_1^2$
$+s_1 + b_2s_2^2 + s_2)(b_2(s_1+s_2)+1)s_3^3 - s_1s_2((s_1^2 + s_2s_1 + s_2^2)b_2^2 + 2(s_1+s_2)b_2 + 1)s_3^2 + r_3^2(-r_3(s_1+s_2)$
$(s_1^2+s_2^2)b_2^2 + s_1s_2(s_1^2 + s_2s_1 + s_2^2)b_2^2 + 2(r_3-s_1)(r_3-s_2)(r_3+s_1+s_2)b_2 + s_1s_2 - r_3(s_1+s_2))))r_1 + s_1$
$(b_2^2(-r_3^3 + s_1r_3^2 + s_3^2(s_3-s_1))r_2^5 - b_2(b_2s_2+2)(r_3-s_3)(r_3^2 + (s_3-s_1)r_3 + s_3(s_3-s_1))r_2^4 - (b_2s_2+1)^2$
$(r_3-s_3)(r_3^2 + (s_3-s_1)r_3 + s_3(s_3-s_1))r_2^3 + (-2b_2s_3^5 + s_2(b_2s_2+1)^2s_3^3 + s_1(b_2^2s_1^3 + 2b_2s_1^2 + s_1 - s_2(b_2$
$s_2+1)^2)s_3^2 - r_3^2(-2b_2r_3^3 + b_2^2(r_3-s_1)s_2^3 + s_1^2 + 2b_2(r_3-s_1)s_2^2 + b_2s_1^3(b_2s_1+2) + (r_3-s_1)s_2))r_2^2 + (s_1$
$-s_2)(2b_2s_3^5 - (b_2s_1^2 + s_1 + b_2s_2^2 + s_2)(b_2(s_1+s_2)+1)s_3^3 + s_1s_2((s_1^2 + s_2s_1 + s_2^2)b_2^2 + 2(s_1+s_2)b_2 + 1)$
$s_3^2 + r_3^2((r_3(s_1+s_2)(s_1^2+s_2^2) - s_1s_2(s_1^2 + s_2s_1 + s_2^2))b_2^2 - 2(r_3-s_1)(r_3-s_2)(r_3+s_1+s_2)b_2 - s_1s_2 + r_3$
$(s_1+s_2)))r_2 + s_2(s_2-s_1)(-2b_2s_3^5 + (b_2s_1^2 + s_1 + b_2s_2^2 + s_2)(b_2(s_1+s_2)+1)s_3^3 - s_1s_2((s_1^2 + s_2s_1 + s_2^2)$
$b_2^2 + 2(s_1+s_2)b_2 + 1)s_3^2 + r_3^2(-r_3(s_1+s_2)(s_1^2+s_2^2)b_2^2 + s_1s_2(s_1^2 + s_2s_1 + s_2^2)b_2^2 + 2(r_3-s_1)(r_3-s_2)(r_3$
$+s_1+s_2)b_2 + s_1s_2 - r_3(s_1+s_2)))))\Big) + (s_3^4 - r_3^4)a^2 + b_2^2(s_3^6 - r_3^6),$

$$D_2 = \frac{1}{s_2(r_1^2 + r_1(s_1-r_2) + s_1(s_1-r_2)) + r_2(r_1^2 + r_1(s_1-r_2) + s_1(s_1-r_2)) + s_2^2(-(r_1+s_1))}\Big(-r_3^3(r_1$$
$+s_1)(r_2+s_2)(r_1^4 + r_1^2s_1^2 - r_2^4 - r_2^2s_2^2 + s_1^4 - s_2^4) + ar_3^2(r_1^2 + r_1s_1 + s_1^2)(r_2^2 + r_2s_2 + s_2^2)(r_1^3 - r_2^3 + s_1^3$
$-s_2^3) + as_3^2(s_3(r_1+s_1)(r_2+s_2)(r_1^4 + r_1^2s_1^2 - r_2^4 - r_2^2s_2^2 + s_1^4 - s_2^4) + (r_1^2 + r_1s_1 + s_1^2)(r_2^2 + r_2s_2 + s_2^2)$
$(-r_1^3 + r_2^3 - s_1^3 + s_2^3)) + s_3^6(-r_1^2(r_2+s_2) + r_1(r_2^2 + r_2(s_2-s_1) + s_2(s_2-s_1)) + s_1(r_2^2 + r_2(s_2-s_1)$
$+s_2(s_2-s_1))) + r_3^6(s_2(r_1^2 + r_1(s_1-r_2) + s_1(s_1-r_2)) + r_2(r_1^2 + r_1(s_1-r_2) + s_1(s_1-r_2)) + s_2^2(-(r_1$
$+s_1))\Big).$

Section The appendix of Chapter 4

Here we provide the values E, F, G_1, G_2, G_3, H_1, H_2, R_1, R_2 and R_3, that appear in the proof of Theorem 4.1.

$$
\begin{aligned}
E = \; & ab_3(-\lambda_3(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)+(\lambda_3(-1-\phi_1+(-1+\phi_1)\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2 \\
& +\phi_1\phi_2)-(\phi_1+\phi_2)(\phi_1^2+\phi_2^2+2\phi_1^2\phi_2^2)+\lambda_3^2(\phi_1+\phi_2)(\phi_1^2+\phi_2^2+2\phi_1^2\phi_2^2))\phi_3+\lambda_3(1+\phi_1^2) \\
& (\phi_1+\phi_2)(1+\phi_2^2)\phi_3^2+(1+\phi_1\phi_2)(-1+\phi_1\phi_2+\lambda_3(\phi_1+\phi_2))(-\phi_1-\phi_2+\lambda_3(-1+\phi_1\phi_2)) \\
& \phi_3^3+n_2^3((1+\phi_1\phi_2)(-1+\phi_1\phi_2+\lambda_3(\phi_1+\phi_2))(-\phi_1-\phi_2+\lambda_3(-1+\phi_1\phi_2))+\lambda_3(1+\phi_1^2) \\
& (\phi_1+\phi_2)(1+\phi_2^2)\phi_3-(-\lambda_3(-1-\phi_1+(-1+\phi_1)\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1\phi_2) \\
& -(\phi_1+\phi_2)(2+\phi_1^2+\phi_2^2)+\lambda_3^2(\phi_1+\phi_2)(2+\phi_1^2+\phi_2^2))\phi_3^2-\lambda_3(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2) \\
& \phi_3^3)+n_2^2(\lambda_3(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)+(1+\phi_1\phi_2)(-1+\phi_1\phi_2+\lambda_3(\phi_1+\phi_2))(-\phi_1-\phi_2 \\
& +\lambda_3(-1+\phi_1\phi_2))\phi_3+3\lambda_3(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)\phi_3^2-(-\lambda_3(-1-\phi_1+(-1+\phi_1)\phi_2)(1 \\
& +\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1\phi_2)-(\phi_1+\phi_2)(2+\phi_1^2+\phi_2^2)+\lambda_3^2(\phi_1+\phi_2)(2+\phi_1^2+\phi_2^2))\phi_3^3) \\
& +n_2(\lambda_3(1-\phi_1^2-3\phi_1\phi_2-\phi_1^3\phi_2-\phi_2^2-3\phi_1^2\phi_2^2-\phi_1\phi_2^3+\phi_1^3\phi_2^3+3(1+\phi_1^2)(\phi_1+\phi_2)(1 \\
& +\phi_2^2)\phi_3+(-1-\phi_1+(-1+\phi_1)\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1\phi_2)\phi_3^2+(1+\phi_1^2)(\phi_1+ \\
& \phi_2)(1+\phi_2^2)\phi_3^3)-(\phi_1+\phi_2)((\phi_2-\phi_3)(\phi_2+\phi_3)+\phi_1^2(1+\phi_2^2(2+\phi_3^2)))+\lambda_3^2(\phi_1+\phi_2)((\phi_2 \\
& -\phi_3)(\phi_2+\phi_3)+\phi_1^2(1+\phi_2^2(2+\phi_3^2))))), \\
F = \; & (1+n_2^2)(1+\phi_1^2)(1+\phi_2^2)(1+\phi_3^2)(\phi_1+\phi_2-\phi_3+\phi_1\phi_2\phi_3+n_2(-1+\phi_1\phi_2-(\phi_1+\phi_2)\phi_3)), \\
G_1 = \; & b_1D(r_3(r_3(1+r_3)(1+r_3^2)+2(-1+r_3^3)s_1-r_3(1+r_3)(1+r_3^2)s_1^2+2(-1+r_3^3)s_1^3)+2(1+r_3^2)^2 \\
& s_1(1+s_1^2)s_3-2(-1+r_3^3)(-1+s_1^2+r_3^2(-1+s_1^2)-2r_3(s_1+s_1^3))s_3^2-2(1+r_3^2)^2s_1(1+s_1^2)s_3^3 \\
& +(-2+r_3^2(-3+r_3-r_3^2+r_3^3)+2r_3(-1+r_3^3)s_1-(1+r_3^2)(-2+(-1+r_3)r_3^2)s_1^2+2r_3(-1+r_3^3) \\
& s_1^3)s_3^4+r1(r_3(-2+2r_3^3-3r_3(1+r_3)(1+r_3^2)s_1+2(-1+r_3^3)s_1^2-r_3(1+r_3)(1+r_3^2)s_1^3)+2(1+ \\
& r_3^2)^2(1+s_1^2)s_3-2(-1+r_3^3)(3s_1+s_1^3-2r_3(1+s_1^2)+r_3^2s_1(3+s_1^2))s_3^2-2(1+r_3^2)^2(1+s_1^2)s_3^3 \\
& +(2r_3(-1+r_3^3)-3(1+r_3^2)(-2+(-1+r_3)r_3^2)s_1+2r_3(-1+r_3^3)s_1^2-(1+r_3^2)(-2+(-1+r_3) \\
& r_3^2)s_1^3)s_3^4)+r1^3(r_3(-2+2r_3^3-r_3(1+r_3)(1+r_3^2)s_1+2(-1+r_3^3)s_1^2+r_3(1+r_3)(1+r_3^2)s_1^3) \\
& +2(1+r_3^2)^2(1+s_1^2)s_3+2(-1+r_3^3)(-s_1+s_1^3+r_3^2s_1(-1+s_1^2)+2r_3(1+s_1^2))s_3^2-2(1+r_3^2)^2 \\
& (1+s_1^2)s_3^3+(2r_3(-1+r_3^3)-(1+r_3^2)(-2+(-1+r_3)r_3^2)s_1+2r_3(-1+r_3^3)s_1^2+(1+r_3^2)(-2+ \\
& (-1+r_3)r_3^2)s_1^3)s_3^4)+r1^2(r_3(-r_3(1+r_3)(1+r_3^2)+2(-1+r_3^3)s_1-3r_3(1+r_3)(1+r_3^2)s_1^2+2 \\
& (-1+r_3^3)s_1^3)+2(1+r_3^2)^2s_1(1+s_1^2)s_3-2(-1+r_3^3)(1+3s_1^2+r_3^2(1+3s_1^2)-2r_3(s_1+s_1^3)) \\
& s_3^2-2(1+r_3^2)^2s_1(1+s_1^2)s_3^3+(2-r_3^2(-3+r_3-r_3^2+r_3^3)+2r_3(-1+r_3^3)s_1-3(1+r_3^2)(-2+ \\
& (-1+r_3)r_3^2)s_1^2+2r_3(-1+r_3^3)s_1^3)s_3^4)),
\end{aligned}
$$

$$
\begin{aligned}
H_1 = {} & b_1D(b_1D(r_3(-r_3(1+r_3)(1+r_3^2)-2(-1+r_3^3)s_1+r_3(1+r_3)(1+r_3^2)s_1^2-2(-1+r_3^3)s_1^3)-2 \\
& (1+r_3^2)^2s_1(1+s_1^2)s_3+2(-1+r_3^3)(-1+s_1^2+r_3^2(-1+s_1^2)-2r_3(s_1+s_1^3))s_3^2+2(1+r_3^2)^2s_1 \\
& (1+s_1^2)s_3^3+(2-r_3^2(-3+r_3-r_3^2+r_3^3)-2r_3(-1+r_3^3)s_1+(1+r_3^2)(-2+(-1+r_3)r_3^2)s_1^2-2 \\
& r_3(-1+r_3^3)s_1^3)s_3^4+r_1^3(r_3(2-2r_3^3+r_3(1+r_3)(1+r_3^2)s_1-2(-1+r_3^3)s_1^2-r_3(1+r_3)(1+r_3^2) \\
& s_1^3)-2(1+r_3^2)^2(1+s_1^2)s_3-2(-1+r_3^3)(-s_1+s_1^3+r_3^2s_1(-1+s_1^2)+2r_3(1+s_1^2))s_3^2+2(1+r_3^2)^2 \\
& (1+s_1^2)s_3^3+(-2r_3(-1+r_3^3)+(1+r_3^2)(-2+(-1+r_3)r_3^2)s_1-2r_3(-1+r_3^3)s_1^2-(1+r_3^2)(-2+ \\
& (-1+r_3)r_3^2)s_1^3)s_3^4)+r_1(r_3(2-2r_3^3+3r_3(1+r_3)(1+r_3^2)s_1-2(-1+r_3^3)s_1^2+r_3(1+r_3)(1+r_3^2) \\
& s_1^3)-2(1+r_3^2)^2(1+s_1^2)s_3+2(-1+r_3^3)(3s_1+s_1^3-2r_3(1+s_1^2)+r_3^2s_1(3+s_1^2))s_3^2+2(1+r_3^2)^2(1 \\
& +s_1^2)s_3^3+(-2r_3(-1+r_3^3)+3(1+r_3^2)(-2+(-1+r_3)r_3^2)s_1-2r_3(-1+r_3^3)s_1^2+(1+r_3^2)(-2+(-1 \\
& +r_3)r_3^2)s_1^3)s_3^4)+r_1^2(r_3(r_3(1+r_3)(1+r_3^2)-2(-1+r_3^3)s_1+3r_3(1+r_3)(1+r_3^2)s_1^2-2(-1+r_3^3) \\
& s_1^3)-2(1+r_3^2)^2s_1(1+s_1^2)s_3+2(-1+r_3^3)(1+3s_1^2+r_3^2(1+3s_1^2)-2r_3(s_1+s_1^3))s_3^2+2(1+r_3^2)^2s_1 \\
& (1+s_1^2)s_3^3+(-2+r_3^2(-3+r_3-r_3^2+r_3^3)-2r_3(-1+r_3^3)s_1+3(1+r_3^2)(-2+(-1+r_3)r_3^2)s_1^2-2r_3 \\
& (-1+r_3^3)s_1^3)s_3^4))^2-2(r_1+s_1)(-2s_3^2+r_3^2(1+r_3+(-1+r_3)s_3^2))(C(1+r_1^2)(1+r_3^2)(1+s_1^2) \\
& (1+s_3^2)(-r_3(r_3+r_3^2-2s_1)+r_1r_3(2+r_3(1+r_3)s_1)-2(1+r_3^2)(r_1+s_1)s_3+(2+2r_1r_3+r_3^2 \\
& -r_3^3+2r_3s_1+r_1(-2+(-1+r_3)r_3^2)s_1)s_3^2)+b_1D(r_1+s_1)(-r_3^2(-1-2r_1^2+r_3+r_3^2+r_3^3)+r_3^2 \\
& (2+r_1^2(3+r_3+r_3^2+r_3^3))s_1^2+2(-s_1^2+r_1^2(-1-r_3^4+(-2+(-1+r_3)r_3^2(1+r_3^2))s_1^2)-r_3^2(-1 \\
& +r_3+r_3^2(1+r_3+s_1^2)))s_3^2+(2+r_3^2(5+2r_1^2+r_3(-1+r_3-r_3^2))+(2r_3^2+r_1^2(-2+(-1+r_3)r_3^2(1 \\
& +r_3^2)))s_1^2)s_3^4))(-2(s_1-s_3)(s_1+s_3)+r_3^2(1+r_3-s_1^2+r_3s_1^2+(3+r_3+(1+r_3)s_1^2)s_3^2)+r_1^2(r_3^3(1 \\
& +s_1^2)(1+s_3^2)-2(1+s_1^2(2+s_3^2))+r_3^2(-1+s_3^2-s_1^2(3+s_3^2))))),
\end{aligned}
$$

$$
\begin{aligned}
R_1 = {} & b_1D(r_1+s_1)(-2s_3^2-r_3^2(-1+s_3^2)+r_3^3(1+s_3^2))(-2s_1^2+2s_3^2+r_3^3(1+s_1^2)(1+s_3^2)+r_3^2(1+3s_3^2 \\
& +s_1^2(-1+s_3^2))+r_1^2(r_3^3(1+s_1^2)(1+s_3^2)-2(1+{}^2(2+s_3^2))+r_3^2(-1+s_3^2-s_1^2(3+s_3^2)))), \\
G_2 = {} & -4b_2(1+n_2^2)(1+n_3^2)(1+r_2^2)(1+r_3^2)(a^2(n_2-n_3)(1+n_2n_3)(1+r_2^2)(1+r_3^2)+(1+n_2^2)(1 \\
& +n_3^2)(r_2+(-1+r_2^2)r_3-r_2r_3^3)+a(n_2(1+n_3^2)(1+r_2^2)(-1+r_3^3)+(n_3-r_2+r_3+n_3r_2r_3)(1 \\
& -r_2r_3+n_3(r_2+r_3))-n_2^2(1+n_3(r_2-r_3)+r_2r_3)(r_2+r_3+n_3(-1+r_2r_3))))(-2a^3(n_2-n_3) \\
& (1+r_2^2)^2(1+r_3^2)^2(-2(n_2+n_3)(1+n_2^2n_3^2)+2(-1+n_2^2)(-1+n_2n_3)(-1+n_3^2)\lambda_2+(-1 \\
& +n_2^2)(n_2+n_3)(-1+n_3^2)\lambda_2^2)-a(1+n_2^2)(1+n_3^2)(-4(1+n_2^2)(-1+n_3^2)r_2^2+4((-1+n_2^2)(1 \\
& +n_3^2)+4(n_2-n_3)(n_2+n_3)r_2^2+(-1+n_2^2)(1+n_3^2)r_2^4)r_3^2-4(1+n_2^2)(-1+n_3^2)r_2^2r_3^4-4((1 \\
& +n_2^2)(-1+(-1+n_2^2)(1+n_3^2)r_3(-1+r_3^3)+2(-1+n_2^2)(1+n_3^2)r_2^2r_3(-1+r_3^3)+(-1+n_2^2)(1 \\
& +n_3^2)r_2^4r_3(-1+r_3^3))\lambda_2+(-4r_2^2-2n_3^2(1+r_2^4)+2n_2^2(1+2n_3^2r_2^2+r_2^4)-2(1+n_2^2)(-1+n_3^2) \\
& (-1+r_2^2)^2r_3^2-2(-1+n_2^2)(1+n_3^2)(1+r_2^2)^2r_3^3-(1+n_2^2)(-1+n_3^2)(-1+r_2^2)^2r_3^4+(-1+n_2^2)
\end{aligned}
$$

$$(1+n_3^2)(1+r_2^2)^2r_3^6)\lambda_2^2)+(1+n_2^2)^2(1+n_3^2)^2(-4r_3^2+4r_2^2(1-r_3^2(-1-r_2^2+r_3+r_3^3))-4(-1 +r_2)(1+r_2)(r_2-r_3)(-1+r_3)(-1+r_2r_3)(1+r_3+r_3^2)\lambda_2+(-1+r_2^2)(-1+r_3^3)(r_3^2+r_3^3+r_2^2 (-2+(-1+r_3)r_3^2))\lambda_2^2)+a^2(1+r_2^2)(1+r_3^2)(-4n_3^2(-1+r_2^2)(1+r_3^2)+4n_3(-1+n_3^2)(-1+r_2^2)(1 +r_3^2)\lambda_2+4n_2(1+n_3^2)^2(1+r_2^2)(-1+r_3^3)\lambda_2-4n_2^3(1+n_3^2)^2(1+r_2^2)(-1+r_3^3)\lambda_2+(-4n_3^2-2(1 +n_3^4)r_2^2-(-1+n_3^2)^2(-1+r_2^2)r_3^2+(1+n_3^2)^2(1+r_2^2)r_3^3)\lambda_2^2+n_2^4(-4n_3^2(-1+r_2^2)(1+r_3^2)+4n_3(-1 +n_3^2)(-1+r_2^2)(1+r_3^2)\lambda_2+(-4n_3^2-2(1+n_3^4)r_2^2-(-1+n_3^2)^2(-1+r_2^2)r_3^2+(1+n_3^2)^2(1+r_2^2)r_3^3) \lambda_2^2)+2n_2^2(-2(1+n_3^4)-2(1+4n_3^2+n_3^4)r_2^2-4n_3^2(-1+r_2^2)r_3^2+2(1+n_3^2)^2(1+r_2^2)r_3^3+4n_3(-1 +n_3^2)(-1+r_2^2)(1+r_3^2)\lambda_2-(-2(1+n_3^4+2n_3^2r_2^2)+(-1+n_3^2)^2(-1+r_2^2)r_3^2+(1+n_3^2)^2(1+r_2^2)r_3^3) \lambda_2^2))),$$

$$R_2 = 4(1+n_2^2)(1+n_3^2)(1+r_2^2)(1+r_3^2)(a^2(n_2-n_3)(1+n_2n_3)(1+r_2^2)(1+r_3^2)+(1+n_2^2)(1+n_3^2) (r_2+(-1+r_2^2)r_3-r_2r_3^3)+a(n_2(1+n_3^2)(1+r_2^2)(-1+r_3^3)+(n_3-r_2+r_3+n_3r_2r_3)(1-r_2r_3 +n_3(r_2+r_3))-n_2^2(1+n_3(r_2-r_3)+r_2r_3)(r_2+r_3+n_3(-1+r_2r_3)))),$$

$$G_3 = ab_3((1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)-(-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1 \phi_2)\phi_4-(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)\phi_4^2-(-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2 +\phi_1\phi_2)\phi_4^3+n_3^2(-(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)-(-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1 +\phi_1+\phi_2+\phi_1\phi_2)\phi_4-3(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)\phi_4^2-(-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1 \phi_2)(-1+\phi_1+\phi_2+\phi_1\phi_2)\phi_4^3)+n_3(-1+\phi_1^2+3\phi_1\phi_2+\phi_1^3\phi_2+\phi_2^2+3\phi_1^2\phi_2^2+\phi_1\phi_2^3-\phi_1^3 \phi_2^3-3(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)\phi_4-(-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2 +\phi_1\phi_2)\phi_4^2-(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)\phi_4^3) +n_3^3(-1+\phi_1^2+3\phi_1\phi_2+\phi_1^3\phi_2+\phi_2^2+3\phi_1^2\phi_2^2+\phi_1\phi_2^3-\phi_1^3\phi_2^3-(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2) \phi_4-(-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1\phi_2)\phi_4^2+(1+\phi_1^2)(\phi_1+\phi_2)(1 +\phi_2^2)\phi_4^3)),$$

$$H_2 = ab_3(ab_3(n_3(-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1\phi_2)+n_3^3(-1+\phi_1(-1 +\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1\phi_2)-(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)+n_3^2(1+\phi_1^2)(\phi_1 +\phi_2)(1+\phi_2^2)+((-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2\phi_1\phi_2)+n_3^2(-1+\phi_1(-1 +\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1\phi_2)+3n_3(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)+n_3^3(1+\phi_1^2) (\phi_1+\phi_2)(1+\phi_2^2))\phi_4+(n_3(-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1\phi_2)+n_3^3 (-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1\phi_2)+(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2)+3 n_3^2(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2))\phi_4^2-(-(-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1 \phi_2)-n_3^2(-1+\phi_1(-1+\phi_2)-\phi_2)(1+\phi_1\phi_2)(-1+\phi_1+\phi_2+\phi_1\phi_2)-n_3(1+\phi_1^2)(\phi_1+\phi_2)(1 +\phi_2^2)+n_3^3(1+\phi_1^2)(\phi_1+\phi_2)(1+\phi_2^2))\phi_4^3)^2+(1/F)4(\phi_1+\phi_2)(n_3+\phi_4)((\phi_2-\phi_4)(\phi_2+\phi_4)$$

$$+n_3^2(-1+\phi_1^2\phi_2^2-(2+\phi_1^2+\phi_2^2)\phi_4^2)+\phi_1^2(1+\phi_2^2(2+\phi_4^2)))(E(1+n_3^2)(1+\phi_1^2)(1+\phi_2^2)(1+\phi_4^2)$$
$$(\phi_1+\phi_2-\phi_4+\phi_1\phi_2\phi_4+n_3(-1+\phi_1\phi_2-(\phi_1+\phi_2)\phi_4))+ab_3F(\phi_1+\phi_2)(n_3+\phi_4)((\phi_2$$
$$-\phi_4)(\phi_2+\phi_4)+n_3^2(-1+\phi_1^2\phi_2^2-(2+\phi_1^2+\phi_2^2)\phi_4^2)+\phi_1^2(1+\phi_2^2(2+\phi_4^2))))),$$
$$R_3 = (2ab_3(\phi_1+\phi_2)(n_3+\phi_4)(\phi_1^2+\phi_2^2+2\phi_1^2\phi_2^2+n_3^2(-1+\phi_1^2\phi_2^2)-(1-\phi_1^2\phi_2^2+n_3^2(2+\phi_1^2+\phi_2^2))$$
$$\phi_4^2)).$$

Section The appendix of Chapter 5

Here we provide the values of T_0, T_2 and T_4 that appear in the proof of Theorem 5.2.

$$
\begin{aligned}
T_0 = \; & (b^1_{10})^3 b^1_{11} b^2_{00} - 2b^1_{02} b^1_{11} (b^2_{00})^2 - b^1_{11} b^1_{20} (b^2_{00})^2 - 2b^1_{10} b^1_{21} (b^2_{00})^2 - 2b^1_{10} b^1_{11} b^2_{00} b^2_{10} - (b^1_{10})^2 b^2_{00} \\
& b^2_{11} + b^2_{00} b^2_{10} b^2_{11} + (b^2_{00})^2 b^2_{21} - (b^1_{10})^2 b^1_{11} b^3_{00} + 2b^1_{21} b^2_{00} b^3_{00} + b^1_{11} b^2_{10} b^3_{00} + b^1_{10} b^2_{11} b^3_{00} + b^2_{00} b^3_{01} \\
& 2b^1_{02} + b^1_{11} b^2_{00} b^3_{10} + b^1_{10} b^1_{11} b^4_{00} - b^2_{11} b^4_{00} - b^2_{00} b^4_{11} - 2b^1_{02} (b^1_{10})^2 b^2_{00} a^1_{10} + 2b^1_{12} (b^2_{00})^2 a^1_{10} + b^6_{01} \\
& +2b^1_{10} b^2_{00} b^2_{02} a^1_{10} + 2b^1_{02} b^2_{00} b^2_{10} a^1_{10} - 2b^2_{02} b^3_{00} a^1_{10} - 2b^2_{00} b^3_{02} a^1_{10} - 2b^1_{02} b^4_{00} a^1_{10} 2b^1_{02} b^1_{10} + 2b^1_{02} \\
& b^1_{10} b^3_{00} a^1_{10} + b^2_{00} a^1_{01} a^1_{10} - 2b^2_{00} b^2_{02} a^1_{01} a^1_{10} - 2b^1_{02} b^3_{00} a^1_{01} a^1_{10} + 2b^1_{10} b^1_{11} b^2_{00} (a^1_{10})^2 - b^2_{00} b^2_{11} (a^1_{10})^2 \\
& -b^1_{11} b^3_{00} (a^1_{10})^2 - b^1_{11} b^2_{00} a^1_{01} (a^1_{10})^2 - 2b^1_{02} b^2_{00} (a^1_{10})^3 - (b^1_{11} b^2_{00})^2 a^1_{11} + b^2_{00} b^3_{00} a^1_{11} - b^4_{00} a^1_{10} a^1_{11} \\
& +b^2_{00} b^2_{10} a^1_{10} a^1_{11} + b^1_{10} b^3_{00} a^1_{10} a^1_{11} - (b^1_{10})^2 b^2_{00} a^1_{10} a^1_{11} + b^1_{10} b^2_{00} a^1_{01} a^1_{10} a^1_{11} - b^3_{00} a^1_{01} a^1_{10} a^1_{11} - b^2_{00} \\
& (a^1_{01})^2 a^1_{10} a^1_{11} - b^2_{00} (a^1_{10})^3 a^1_{11} + 2(b^1_{10})^3 b^2_{00} a^1_{20} - 2b^1_{02} (b^2_{00})^2 a^1_{20} - 2b^1_{20} (b^2_{00})^2 a^1_{20} + 2b^2_{00} b^3_{10} a^1_{20} \\
& -4b^1_{10} b^2_{00} b^2_{10} a^1_{20} - 2(b^1_{10})^2 b^3_{00} a^1_{20} + 2b^2_{10} b^3_{00} a^1_{20} + 2b^1_{10} b^4_{00} a^1_{20} + 4b^1_{10} b^2_{00} (a^1_{10})^2 a^1_{20} - 2b^3_{00} a^1_{20} \\
& (a^1_{10})^2 - 2b^2_{00} a^1_{01} (a^1_{10})^2 a^1_{20} - (b^2_{00})^2 a^1_{11} a^1_{20} + 2(b^2_{00})^2 a^1_{10} a^1_{21} - 6b^1_{10} (b^2_{00})^2 a^1_{30} + 4(b^1_{02})^2 b^2_{00} a^2_{00} \\
& +6b^2_{00} b^3_{00} a^1_{30} + (b^1_{11})^2 b^2_{00} a^2_{00} + 2b^1_{10} b^1_{12} b^2_{00} a^2_{00} - 2b^2_{00} b^2_{12} a^2_{00} - 2b^1_{12} b^3_{00} a^2_{00} + 2b^2_{02} (a^1_{01})^2 a^2_{00} \\
& -b^1_{11} b^3_{00} a^2_{00} + 2b^4_{02} a^2_{00} - 2b^1_{12} b^2_{00} a^1_{01} a^2_{00} + 2b^3_{00} a^1_{01} a^2_{00} + 2b^1_{02} (a^1_{01})^3 a^2_{00} + +2(a^1_{01} a^1_{10})^2 a^1_{11} a^2_{00} \\
& -b^1_{11} b^2_{10} a^1_{10} a^2_{00} - b^1_{10} b^2_{11} a^1_{10} a^2_{00} + b^3_{11} a^1_{10} a^2_{00} - b^1_{10} b^1_{11} a^1_{01} a^1_{10} a^2_{00} + b^2_{11} a^1_{01} a^1_{10} a^2_{00} + a^1_{10} a^2_{00} b^1_{11} \\
& (a^1_{01})^2 - 2b^1_{02} b^1_{10} (a^1_{10})^2 a^2_{00} + 2b^2_{02} (a^1_{10})^2 a^2_{00} 4b^1_{02} a^1_{01} (a^1_{10})^2 a^2_{00} + b^1_{11} (a^1_{10})^3 a^2_{00} + 2b^1_{02} b^2_{00} a^1_{11} \\
& a^2_{00} + (a^1_{01})^3 a^1_{11} a^2_{00} - b^1_{10} (a^1_{10})^2 a^1_{11} a^2_{00} + 2b^1_{11} b^2_{00} + (b^1_{10})^2 b^1_{11} a^1_{10} a^2_{00} a^1_{20} a^2_{00} + 2(a^1_{10})^3 a^1_{20} a^2_{00} \\
& +2(b^1_{10})^2 a^1_{10} a^1_{20} a^2_{00} - 2b^2_{10} a^1_{10} a^1_{20} a^2_{00} - 2b^1_{10} a^1_{01} a^1_{10} a^1_{20} - 2b^3_{01} a^1_{20} a^2_{00} a^2_{00} + 2(a^1_{01})^2 a^1_{10} a^1_{20} a^2_{00} \\
& +2b^1_{10} b^2_{00} a^1_{21} a^2_{00} - 2b^3_{00} a^1_{21} a^2_{00} - 2b^2_{00} a^1_{01} a^1_{21} a^2_{00} + b^1_{11} b^1_{20} (a^2_{00})^2 + b^1_{10} b^1_{21} (a^2_{00})^2 - b^2_{21} (a^2_{00})^2 \\
& -b^1_{21} a^1_{01} (a^2_{00})^2 - 2b^1_{02} a^1_{20} (a^2_{00})^2 + 2b^1_{20} a^1_{20} (a^2_{00})^2 - a^1_{11} a^1_{20} (a^2_{00})^2 + 3b^1_{10} a^1_{30} (a^2_{00})^2 - 3a^1_{01} a^1_{30} \\
& (a^2_{00})^2 - 2b^1_{02} b^2_{00} a^1_{10} a^2_{01} - b^2_{00} a^1_{10} a^1_{11} a^2_{01} + 2b^2_{02} a^2_{00} a^2_{01} + 4b^1_{02} a^1_{01} a^2_{00} a^2_{01} + b^1_{11} a^1_{10} a^2_{00} a^2_{01} + a^1_{01} \\
& 2a^1_{11} a^2_{00} a^2_{01} + 2a^1_{10} a^1_{20} a^2_{00} a^2_{01} + 2b^1_{02} b^1_{10} b^2_{00} a^2_{10} - 2b^2_{00} b^2_{02} a^2_{10} - 2b^1_{02} b^3_{00} a^2_{10} - 2b^1_{02} b^2_{00} a^1_{01} a^2_{10} \\
& -2b^1_{11} b^2_{00} a^1_{10} a^2_{10} + b^1_{10} b^2_{00} a^1_{11} a^2_{10} - b^3_{00} a^1_{11} a^2_{10} - b^2_{00} a^1_{01} a^1_{11} a^2_{10} - 4b^2_{00} a^1_{10} a^1_{20} a^2_{10} - b^1_{10} b^1_{11} a^2_{00} \\
& a^2_{10} + b^2_{11} a^2_{00} a^2_{10} + b^1_{11} a^1_{01} a^2_{00} a^2_{10} + 4b^1_{02} a^1_{10} a^2_{00} a^2_{10} + 2a^1_{10} a^1_{11} a^2_{00} a^2_{10} - 2b^1_{10} a^1_{20} a^2_{00} a^2_{10}
\end{aligned}
$$

$$+3(b^2_{00})^2 a^2_{30} + 2a^1_{01}a^1_{20}a^2_{00}a^2_{10} + b^1_{10}b^2_{00}a^1_{10}a^2_{11} - b^3_{00}a^1_{10}a^2_{11} - b^2_{00}a^1_{01}a^1_{10}a^2_{11} + (a^1_{01})^2 a^2_{00}a^2_{11}$$

$$+(a^1_{10})^2 a^2_{00}a^2_{11} + a^2_{00}a^2_{01}a^2_{11} - b^2_{00}a^2_{10}a^2_{11} - 2(b^1_{10})^2 b^2_{00}a^2_{20} + 2b^2_{00}b^2_{10}a^2_{20} + 2b^1_{10}b^3_{00}a^2_{20} - 2b^2_{00}$$

$$(a^1_{10})^2 a^2_{20} - 2b^4_{00}a^2_{20} - 2b^1_{10}a^1_{10}a^2_{00}a^2_{20} + 2a^1_{01}a^1_{10}a^2_{00}a^2_{20} + 2a^2_{00}a^2_{10}a^2_{20} - 2b^2_{00}a^2_{00}a^2_{21} - 3(a^2_{00})^2$$

$$a^2_{30} - 2b^1_{12}b^2_{00}a^3_{00} + 2b^3_{00}2a^3_{00}2b^2_{02}a^1_{01}a^3_{00} + 2b^1_{02}(a^1_{01})^2 a^3_{00} - b^1_{10}b^1_{11} + a^1_{10}a^3_{00} + b^1_{11}a^1_{01}a^1_{10}a^3_{00}$$

$$+b^2_{11}a^1_{10}a^3_{00} + 2b^1_{02}(a^1_{10})^2 a^3_{00} + (a^1_{01})^2 a^1_{11}a^3_{00} + (a^1_{10})^2 a^1_{11}a^3_{00} - 2b^1_{10}a^1_{10}a^1_{20}a^3_{00} - 2b^2_{00}a^1_{21}a^3_{00}$$

$$+2a^1_{01}a^1_{10}a^1_{20}a^3_{00} - 2b^1_{21}a^2_{00}a^3_{00} - 6a^1_{30}a^2_{00}a^3_{00} + 2b^1_{02}a^2_{01}a^3_{00} + a^1_{11}a^2_{01}a^3_{00} + b^1_{11}a^2_{10}a^3_{00} + 2a^1_{20}$$

$$a^2_{10}a^3_{00} + a^1_{01}a^2_{11}a^3_{00} + 2a^1_{10}a^2_{20}a^3_{00} + 2b^1_{02}a^2_{01}a^3_{00} + a^1_{11}a^2_{01}a^3_{00} - b^2_{00}a^1_{10}a^3_{11} + a^1_{01}a^2_{00}a^3_{11} + a^3_{00}$$

$$a^3_{11} + 2b^1_{10}b^2_{00}a^3_{20} - 2b^3_{00}a^3_{20} + 2a^1_{10}a^2_{00}a^3_{20} + 2b^1_{02}a^5_{00} + a^1_{11}a^5_{00} + a^6_{10} + 2b^2_{02}a^4_{00} + 2b^1_{02}a^1_{01}a^4_{00}$$

$$+b^1_{11}a^1_{10}a^4_{00} + a^1_{01}a^1_{11}a^4_{00} + 2a^1_{10}a^1_{20}a^4_{00} + a^2_{11}a^4_{00} + a^2_{00}a^4_{11} - 2b^2_{00}a^4_{20} - b^1_{11}b^5_{00} - 2a^1_{20}b^5_{00},$$

$$T_2 = \frac{1}{4}(-b^1_{02}(b^1_{10})^2 b^1_{11} - 3(b^1_{10})^2 b^1_{11}b^1_{20} - (b^1_{10})^3 b^1_{21} + 6b^1_{02}b^1_{03}b^2_{00} + 3b^1_{11}b^1_{12}b^2_{00} + 3b^1_{10}b^1_{13}b^2_{00} + 4b^1_{02}$$

$$b^1_{21}b^2_{00} + 4b^1_{20}b^1_{21}b^2_{00} + 3b^1_{11}b^1_{30}b^2_{00} + 6b^1_{10}b^1_{31}b^2_{00} + b^1_{10}b^1_{11}b^2_{02} + b^1_{02}b^1_{11}b^2_{10} + 2b^1_{11}b^1_{20}b^2_{10} + 2b^1_{10}$$

$$b^1_{21}b^2_{10} + b^1_{02}b^1_{10}b^2_{11} + 2b^1_{10}b^1_{20}b^2_{11} - b^2_{02}b^2_{11} - 3b^2_{00}b^2_{13} + 2b^1_{10}b^1_{11}b^2_{20} - b^1_{11}b^2_{20} + (b^1_{10})^2 b^2_{21} - b^2_{10}$$

$$b^2_{21} - 3b^2_{00}b^2_{31} - 3b^1_{13}b^3_{00} - 3b^1_{31}b^3_{00} - 2b^1_{12}b^3_{01} - b^1_{11}b^3_{02} - b^1_{21}b^3_{10} - b^1_{02}b^3_{11} - b^1_{20}b^3_{11} - b^1_{11}b^3_{20}$$

$$-b^1_{10}b^3_{21} + 3b^4_{03} + b^4_{21} - 2b^1_{10}b^1_{11}b^1_{20}a^1_{01} - (b^1_{10})^2 b^1_{21}a^1_{01} + 3b^1_{31}b^2_{00}a^1_{01} + b^1_{21}b^2_{10}a^1_{01} + b^1_{20}b^2_{11}a^1_{01}$$

$$+b^1_{11}b^2_{20}a^1_{01} + b^1_{10}b^2_{21}a^1_{01} - b^3_{21}a^1_{01} - 2b^1_{12}b^2_{00}a^1_{02} + 2b^3_{00}2a^1_{02} + 2b^2_{02}a^1_{01}a^1_{02} + 2b^1_{02}(a^1_{01})^2 a^1_{02}$$

$$+2(b^1_{02})^2 b^1_{10}a^1_{10} + 2b^1_{10}(b^1_{11})^2 a^1_{10} + 2(b^1_{10})^2 b^1_{12}a^1_{10} + 4b^1_{02}b^1_{10}b^1_{20}a^1_{10} - 12b^1_{04}b^2_{00}a^1_{10} - 6b^1_{22}b^2_{00}$$

$$a^1_{10} - 4b^1_{02}b^2_{02}a^1_{10} - 2b^1_{20}b^2_{02}a^1_{10} - 2b^1_{12}b^2_{10}a^1_{10} - 2b^1_{11}b^2_{11}a^1_{10} - 2b^1_{10}b^2_{12}a^1_{10} - 2b^1_{02}b^2_{20}a^1_{10} + 2b^3_{12}$$

$$a^1_{10} - 2(b^1_{02})^2 a^1_{01}a^1_{10} - b^1_{10}b^1_{11}a^1_{02}a^1_{10} + b^2_{11}a^1_{02}a^1_{10} + b^1_{11}a^1_{01}a^1_{02}a^1_{10} - 3b^1_{02}b^1_{11}(a^1_{10})^2 - b^1_{11}b^1_{20}$$

$$(a^1_{10})^2 + 2b^1_{02}a^1_{02}(a^1_{10})^2 + 3b^1_{03}b^2_{00}a^1_{11} + 2b^1_{21}b^2_{00}a^1_{11} + (a^1_{01})^2 a^1_{02}a^1_{11} - b^1_{02}b^1_{10}a^1_{10}a^1_{11} + 2b^1_{10}b^1_{20}$$

$$a^1_{10}a^1_{11} + b^2_{02}a^1_{10}a^1_{11} - b^2_{20}a^1_{10}a^1_{11} + b^1_{02}a^1_{01}a^1_{10}a^1_{11} + a^1_{02}(a^1_{10})^2 a^1_{11} - b^1_{10}a^1_{10}(a^1_{11})^2 - 3b^2_{00}a^1_{10}a^1_{13}$$

$$+a^1_{01}a^1_{10}(a^1_{11})^2 - 6(b^1_{10})^2 b^1_{20}a^1_{20} + 6b^1_{30}b^2_{00}a^1_{20} + 4b^1_{20}b^2_{10}a^1_{20} + 4b^1_{10}b^2_{20}a^1_{20} - 2b^3_{20}a^1_{20} - 4b^1_{10}b^1_{20}$$

$$+2b^2_{20}a^1_{01}a^1_{20} + 2b^1_{10}b^1_{11}a^1_{10}a^1_{20} - b^2_{11}a^1_{10}a^1_{20} - 2b^1_{10}a^1_{02}a^1_{10}a^1_{20} + 2a^1_{01}a^1_{02}a^1_{10}a^1_{20} - 2(b^1_{20}a^1_{10})^2 a^1_{20}$$

$$+(b^1_{10})^2 a^1_{11}a^1_{20} - b^2_{10}a^1_{11}a^1_{20} + 3(a^1_{10})^2 a^1_{11}a^1_{20} - 4b^1_{10}a^1_{10}(a^1_{20})^2 + 2b^1_{11}b^2_{00}a^1_{21} - 2b^2_{00}a^1_{02}a^1_{21}2b^3_{00}$$

$$a^1_{21} + 2(b^1_{10})^2 a^1_{10}a^1_{21} - 2b^2_{10}a^1_{10}a^1_{21} - 2b^2_{00}a^1_{20}a^1_{21} + 2b^1_{10}b^2_{00}a^1_{22} - 2b^3_{00}a^1_{22} - 3(b^1_{10})^3 a^1_{30} + 6b^1_{02}b^2_{00}$$

$$a^1_{30} + 12b^1_{20}b^2_{00}a^1_{30} + 6b^1_{10}b^2_{10}a^1_{30} - 3b^3_{10}a^1_{30} - 3(b^1_{10})^2 a^1_{01}a^1_{30} + 3b^2_{10}a^1_{01}a^1_{30} + 3b^2_{00}a^1_{11}a^1_{30} - 9b^2_{00}$$

$a^1_{10}a^1_{31} + 24b^1_{10}b^2_{00}a^1_{40} - 12b^3_{00}a^1_{40} + 12b^2_{00}a^1_{01}a^1_{40} - 3b^1_{03}b^1_{11}a^2_{00} - 6b^1_{02}b^1_{12}a^2_{00} - 2b^1_{12}b^1_{20}a^2_{00} - 2b^1_{11}$
$b^1_{21}a^2_{00} - 2b^1_{10}b^1_{22}a^2_{00} + 12b^2_{00}4a^2_{00} + 2b^2_{22}a^2_{00} + 12b^1_{04}a^1_{01}a^2_{00} - 4b^1_{21}a^1_{02}a^2_{00} + 6b^1_{02}a^1_{03}a^2_{00} + 9b^1_{13}$
$a^1_{10}a^2_{00} + 3b^1_{31}a^1_{10}a^2_{00} + 3a^1_{03}a^1_{11}a^2_{00} + 3a^1_{01}a^1_{13}a^2_{00} - 6b^1_{03}a^1_{20}a^2_{00} - 4b^1_{21}a^1_{20}a^2_{00} - 4b^1_{02}a^1_{21}a^2_{00} - 2b^1_{20}$
$a^1_{21}a^2_{00} + a^1_{11}a^1_{21}a^2_{00} + 6a^1_{10}a^1_{22}a^2_{00} - 3b^1_{11}a^1_{30}a^2_{00} - 12a^1_{02}a^1_{30}a^2_{00} - 6a^1_{20}a^1_{30}a^2_{00} - 3b^1_{10}a^1_{31}a^2_{00} + b^1_{11}$
$b^1_{20}a^2_{01} + 12a^1_{10}a^1_{40}a^2_{00} + b^1_{10}b^1_{21}a^2_{01} - b^2_{21}a^2_{01} + 2b^1_{02}a^1_{02}a^2_{01} + a^1_{02}a^1_{11}a^2_{01} + 2b^1_{20}a^1_{20}a^2_{01} + 3b^1_{10}a^1_{30}a^2_{01}$
$+2b^2_{02}a^2_{02} + 2b^1_{02}a^1_{01}a^2_{02} + b^1_{11}a^1_{10}a^2_{02} + a^1_{01}a^1_{11}a^2_{02} + 2a^1_{10}a^1_{20}a^2_{02} - 2(b^1_{02})^2a^2_{10} - (b^1_{11})^2a^2_{10} - 2b^1_{10}b^1_{12}$
$a^2_{10} - 2b^1_{02}b^1_{20}a^2_{10} + 2b^2_{12}a^2_{10} + b^1_{11}a^1_{02}a^2_{10} + b^1_{02}a^1_{11}a^2_{10} - b^1_{20}a^1_{11}a^2_{10} + (a^1_{11})^2a^2_{10} - b^1_{11}a^1_{20}a^2_{10}3a^1_{01}a^3_{30}$
$2a^1_{02}a^1_{20}a^2_{10} + 2(a^1_{20})^2a^2_{10} - 2b^1_{10}a^1_{21}a^2_{10} + a^1_{01}a^1_{02}a^2_{11} + b^1_{02}a^1_{10}a^2_{11} - b^1_{20}a^1_{10}a^2_{11} + 2a^1_{10}a^1_{11}a^2_{11} + a^4_{12}$
$+a^2_{00}a^2_{11} + 3a^2_{00}a^2_{13} + 4b^1_{10}b^1_{20}a^2_{20} - 2b^2_{20}a^2_{20}2b^1_{20}a^1_{01}a^2_{20} - b^1_{11}a^1_{10} + a^2_{20} + 2a^1_{02}a^1_{10}a^2_{20} - b^1_{10}a^1_{11}a^2_{20}$
$+4a^1_{10}a^1_{20}a^2_{20} + a^2_{11}a^2_{20} - 2b^1_{10}a^1_{10}a^2_{21} + 2a^2_{10}a^2_{21} - 2b^2_{00}a^2_{22} + 3(b^1_{10})^2a^2_{30} - 3b^2_{10}a^2_{30} + 3b^1_{10}a^1_{01}a^2_{30} - 3$
$a^2_{01}a^2_{30} + 3a^2_{00}a^2_{31} - 12b^2_{00}a^2_{40} - b^1_{10}a^1_{20}a^2_{11} + 3a^4_{30} + a^1_{20}a^3_{11} + 12b^1_{04}a^3_{00} + 2b^1_{22}a^3_{00} + 3a^1_{13}a^3_{00} + 3a^1_{31}$
$a^3_{00} - b^1_{21}a^3_{01} - 3a^1_{30}a^3_{01} + 2b^1_{02}a^3_{02} + a^1_{11}a^3_{02} + a^1_{02}a^3_{11} - 2b^1_{20}a^3_{20} + a^1_{11}a^3_{20} + 2a^1_{10}a^3_{21} - 3b^1_{10}a^3_{30}),$

$T_4 = \frac{1}{24}(15b^1_{05}b^1_{10} + 5b^1_{04}b^1_{11} - 6b^1_{03}b^1_{12} - 9b^1_{02}b^1_{13} - 3b^1_{13}b^1_{20} - 3b^1_{12}b^1_{21} - b^1_{11}b^1_{22} - 3b^1_{21}b^1_{30} - 7b^1_{02}$
$b^1_{31} - 5b^1_{20}b^1_{31} - 3b^1_{11}b^1_{40} - 3b^1_{10}b^1_{41} + 15b^2_{05} + 3b^2_{23} + 3b^2_{41} + 15b^1_{05}a^1_{01} - 3b^1_{41}a^1_{01} + 20b^1_{04}a^1_{02}$
$+2b^1_{22}a^1_{02} - 3b^1_{21}a^1_{03} + 6b^1_{02}a^1_{04} + 12b^1_{14}a^1_{10} + 6b^1_{32}a^1_{10} - 2b^1_{31}a^1_{11} + 3a^1_{04}a^1_{11} + 2b^1_{11}a^1_{13} + 5a^1_{02}a^1_{13}$
$+3b^1_{10}a^1_{14} + 3a^1_{01}a^1_{14} + 22b^1_{04}a^1_{20} + 4b^1_{22}a^1_{20} - 6b^1_{40}a^1_{20} + 7a^1_{13}a^1_{20} - 6b^1_{03}a^1_{21} - 3b^1_{21}a^1_{21} - 4b^1_{02}a^1_{22}$
$-2b^1_{20}a^1_{22} + a^1_{11}a^1_{22} + 6a^1_{10}a^1_{23} - 3b^1_{12}a^1_{30} - 9b^1_{30}a^1_{30} - 9a^1_{03}a^1_{30} - 3a^1_{21}a^1_{30} + 3a^1_{02}a^1_{31} + 9a^1_{20}a^1_{31}$
$-22b^1_{02}a^1_{40} - 20b^1_{20}a^1_{40} - 5a^1_{11}a^1_{40} + 12a^1_{10}a^1_{41} - 15b^1_{10}a^1_{50} - 15a^1_{01}a^1_{50} + 3a^2_{14} + 3a^2_{32} + 15a^2_{50}).$

Conclusion

In this work, we solved the second part of the sixteenth Hilbert problem for three families of planar discontinuous piecewise differential systems, the first one formed by a linear differential centers separated by irreducible cubic algebraic curves, the second family formed by differential Hamiltonian systems without equilibrium points and linear differential centers separated by irreducible cubic curves, and the third family is formed by linear differential Hamiltonian systems without equilibrium points separated by two circles.

On the other hand, we considered Kukles differential systems of degree eight and provided all their global phase portraits in the Poincaré disk, by using the classical method. We also solved the second part of the sixteenth Hilbert problem for these systems by applying the averaging theory up to seven order, and we succeeded in showing a certain number of limit cycles.

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