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Numerical Solution of Integro-Delay Differential Equations on a Half Line.

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Abstract

The main concern of this dissertation is concentrated on numerically solving the integro-delay differential equations with variable coefficients on a half-line, proposing a matrix-collocation method based on ortho-exponential polynomials. The method used the collocation points and the hybridized matrix relations between the ortho-exponential and Taylor polynomials, which permit to convert the integral form into a matrix form.

The approximation of a given function by ortho-exponential polynomials is effective, accurate and fast. Also it is remarkable and dependable.

Key words: Integro-delay differential equation, collocation method, ortho-exponential polynomial, approximate solution.

Résumé

L'objectif principal de cette mémoire est concentré sur la résolution numérique des équations intégro-différentielles à retard à coefficients variables sur une demi-droite, en proposant une méthode de collocation basée sur les polynômes ortho-exponentiels. La méthode utilise les points de collocation et de relations matricielles hybrides entre les polynômes orthoexponentiels et de Taylor, qui nous permettent de convertir la forme intégrale en une forme matricielle.

L'approximation d'une fonction donnée par les polynômes ortho-exponentiels est efficace, précise, rapide. Elle est également remarquable et fiable.

Mots clés: Les équations intégro différentielles à retard, méthode de collocation, les polynômes ortho-exponentiels, solution approximative.

ملخص

الهدف الرئيسي من هذه الأطروحة هو دراسة الحل العددي للمعادلات التفاضلية التكاملية للتأخير ذات معاملات متغيرة على نصف المستقيم الحقيقي، مع اقتراح طريقة تجميع المصفوفة على أساس كثيرات الحدود الأسية. استخدمت الطريقة نقاط التجميع و علاقات المصفوفة المهجنة بين كثيرات الحدود الأسية و تايلور، و التي تمكننا من تحويل شكل تكاملي إلى شكل مصفوفي.

تقريب دالة معينة بواسطة كثيرات الحدود المتعامدة الأسية دقيق و سريع. كما أنها فعالة و يمكن الاعتماد عليها.

الكلمات المفتاحية: معادلات تفاضلية تكاملية ذات تأخير، طريقة التجميع، كثيرات حدود أسية، حل تقريبي.

Introduction

The subject of this dissertation is to solve numerically the integro-delay differential equations with variable coefficients and infinite boundary on a half-line using a collocation method based on the ortho-exponential polynomials. The method also directly establishes on explicit formula of the integral form appeared in the original equation.

Delay differential equations (DDEs) are a type of differential equations, where some times called delays appear in these equations. Delay equations have been introduced to model phenomena in which there is a temporal mixture between the action on the system and the system's response to that action. For example, in the process of birth biological populations (cells, bacteria, etc.). In [8], the authors, introduce an epidemic model based on delay equations. Two mathematical models describing COVID19 are also modeled by delay equations [19]. Many phenomena encountered in physics, biology, chemistry, etc., have found in the theory of delay equations a good means of modeling (a more realistic means than in the case of ordinary differential equations). Since the 1940 s , the theory of delay equations have known a great development, in particular we find Belman and Cooke (1963), Hale (1977).

DDEs can be classified into two main types: retarded and neutral. In a retarded DDE, the delay term appears only in the function, while in a neutral DDE, the delay term appears both in the derivative and in the function itself. Neutral DDEs often arise in control theory, while retarded DDEs are more common in the modeling of physical systems.

Delay differential equations can be linear or non-linear, autonomous or non-autonomous. The delay is usually a constant positive, a variable continuously dependent on time or state or distributed and translates as a time needed for which system responds to a certain evolution, or because a certain tool must be reached before which system is activated.

At this notion, the delay can be given as an integral and therefore it depends, on the unknown functions, which are the solutions of the problem, it is called in this case, delay distributed or state-dependent delay. Numerical solutions are not available in general so numerical methodologies are thus needed under this situation.

The collocation method is a numerical method used to approximate solutions of differential equations, including delay differential equations (DDEs). The method based on the idea of "collocating" the differential equation at a finite number of points in the domain of interest and then solving the resulting system of algebraic equations.
The advantages of the proposed method are given as follows:
First, it solves problems on half-line or in $[0, T]$, where T can be sufficiently large and secondly, it gives an explicit formulation of the integral of Eq. (2.1) and finaly, one of the best avantages is that the original problem is converted to a linear algebraic system. The dissertation is divided roughly into three chapters. In the first chapter, we present some preliminaries to some mathematical aspects of DDEs, some classification of DDEs, after, we propose some mathematical models based on a set of delay differential equations that describe some phenomena in biology, life and economics, also, existence and uniqueness of solutions is investigated. Finaly, some techniques to solve DDEs are presented in the last part of this chapter. An introduction to the concept of ortho-exponential polynomials and its application for solving high-integro-delay differential equations (IDDEs) in a half line is presented in chapter 2. A numerical analysis is given in the third chapter of this dissertation to show the simple applicability and a good accuracy of this collocation technique in large interval for solving IDDEs. Finaly, a conclusion is given in the last section of this dissertation.

Chapter 1

Introduction to integro-delay differential equations

1.1 Historical context

Delay functional equations (DDE) are very important field of study for modeling heredity phenomena encountered in physics, biology, chemistry, economics, ecology, $\cdots$ etc. It has been proven that in many cases delay plays a dominant role in several domains and lagging models provide more accurate and realistic results than their lag-free counterparts.

The appearance of these equations dates back to the $18th$ century, it is due to *Bernoulli, Euler, Lagrange, Laplace, Poisson*, and others. Bernoulli in his experiments in 1728, on the vibrating wire and starting from a partial differential equation of hyperbolic type, he found the following delay equation [3]:

$$y^{'} = y(t-1),$$

After, he decided to hold it false and it is said that there were several errors to deduce the equation.

Until the beginning of the $20th$ century and the pioneering work that established the beginning of the theory were in geometry and number theory and the first papers dealing with linear delayed functional equations are due to *Polossuchin* (1910) [25] and *Schmidt* (1911) [26], and after the works of *V.Volterra* [28, 29], on predator-prey models and viscoelasticity models. He used the energy method to study a general class of nonlinear delay equations and wrote in his major work on the role of hereditary effects on models of the dynamics of several interacting species.

There has been intensive research on the subject since 1940. Regulation based on linear and stationary models with delay was addressed in 1941 by *Y.Zypkin*. In 1949, *A.D.Myshkis* laid the foundation for modern DDE theory. In particular, he was the first to formulate the statement of *Cauchy's* problem for equations with arbitrary delay.

The fifties saw an explosion of theory that was widely developed and DDE are part of the vocabulary of researchers working on viscoelasticity, mechanical problems, nuclear reactors, heat flow, neural networks, interaction, microbiological, epidemiological or physiological models, as well as many others (see [17]).

1.2 Preliminaries

In many applications, it is assumed that the future state of a process does not depend on past states and is determined by the present. For example phenomena modeled by ordinary or partial differential equations are usually considered in this case. However, in other cases, more realistic model would include some of the past states of the system. Similarly, in some situations where dependence on past system states is not imagined.

Let r be a positive number $(r>0)$ and $C([a,b],\mathbb{R}^n)$ the Banach space of continuous functions defined on $[a,b]$ with values in $\mathbb{R}^n$ with the uniform convergence norm.
For $[a,b]=[-r,0]$, let $C=C([-r,0],\mathbb{R}^n)$ and denote the norm of an element $f\in C$ by

$$\|f\| = sup\{|f(t)|, -r\leq t\leq 0\},$$

Let $t_0\in\mathbb{R}$, $\alpha\geq 0$ and $y\in C([t_0-r,t_0+\alpha],\mathbb{R}^n)$, then for $t\in[t_0,t_0+\alpha]$, we define the function $y_t\in C$ by

$$y_t(s)=y(t+s),\quad s\in[-r,0].$$

Definition 1.1 *[11] Let O be an open set of $\mathbb{R}\times C$ and $f:O\to\mathbb{R}^n$ is a continuous function. The equation*

$$y'(t)=f(t,y_t) \tag{1.1}$$

where

$$y_t(s)=y(t+s),\quad s\in[-r,0], \tag{1.2}$$

is a functional equation with delay r, called delay differential equations.

Remark 1.1 *For $r=0$, we get the case of ordinary differential equations.*

Definition 1.2 *An integro-differential equation is an equation which contain both integrals and derivatives of a function.*
This type of equations can be found in epidemiology, particularly when the models contain age-structure [6, 20].
Integro delay differential equations is a combination between integro and delay equations.

Remark 1.2

- *Equation (1.1) is said to be autonomous if the function f does not depend on t. In this case we write $f(y)$ instead of $f(t,y)$.*
- *If $f(t,y) = L(t)y$, equation (1.1) is said to be linear.*
- *Equation (1.1) is said to be non-homogeneous if $f(t,y) = L(t)y + h(t)$, where $h(t)$ is a function and $h(t) \neq 0$.*

Example 1.1 *Let given the following delay differential equations*

$$y^{'}(t) = 2y(t) + 5y(t-1) \tag{1.3}$$

$$y^{'}(t) = \alpha(t)y(t) + \beta(t)y^{'}(t - r(t)) + h(t) \tag{1.4}$$

$$y^{'}(t) = \int_{-r}^{0} y(t+s)\mathrm{d}s, \tag{1.5}$$

where $\alpha(t), \beta(t)$ and $r(t)$ are continuous functions. Equation (1.3) is an linear autonomous differential equation with $r = 1$, equation (1.4) represent a linear non-homogeneous and non-autonomous delayed equation, and equation (1.5) is a linear integro-differential equation with delay.

The initial condition at time $t = t_0$ requires the determination of the function y over the whole interval $[t_0 - r, t_0]$, i.e,

$$y(t) = \psi(t), \quad t \in [t_0 - r, t_0],$$

such that $\psi : [t_0 - r, t_0] \to \mathbb{R}^n$ is a given function assumed to be continuous called the initial condition of the delay equation (1.1). Thus, equation (1.1) can be written in the form

$$\begin{cases} y^{'}(t) = f(t, y_t), & t \geq t_0 \\ y(t) = \psi(t), & t \in [t_0 - r, t_0], \end{cases} \tag{1.6}$$

where ψ is a given continuous function on the interval $[t_0 - r, t_0]$.

Remark 1.3

- *To solve the delay differential equation:*

  $$y^{'}(t) = f(t, y(t), y(t-r)),$$

  it is necessary to know $y(t)$ on $[t_0 - r, t_0]$, of length r. On the other hand, to solve an ordinary differential equation it is enough to know $y(t)$ in a single point.

- *A linear and homogeneous delay differential equation can have nontrivial oscillating solutions, i.e. solutions that cancel several times, but they are not identically zero, and if the solution of an ordinary linear and homogeneous differential equation, cancels at one point, it is zero everywhere (uniqueness of the solution). In a general way, if two solutions of an ordinary differential equation meet at a point, and if the uniqueness condition is satisfied, then they are equal, on the whole domain of definition.*

  On the other hand, two solutions of a differential equation, with delay, can meet at several points, without being equal.

Example 1.2 *Let be the following equation $y^{'}(t) = -y(t - \frac{\pi}{2})$ which admits as solutions: $y_1(t) = cos\ t$ and $y_2(t) = sin\ t$. We notice that $y_1(\frac{\pi}{4}) = y_2(\frac{\pi}{4})$ and $y_1 \neq y_2$.*

1.3 Classification of delay equations

The aim of this part, is to define the different types of delay differential equations. Generally, the practice of modelling proves that only delayed or neutral equations are used to represent real phenomena but there are a considerable number of phenomena modelled by advanced equations.

First, we will present the class of differential equations with deviated argument.

1.3.1 Differential equations with deviated argument

Deviated argument differential equations belong to the class of functional differential equations. They describe the evolution of variables depending on the values taken in the past or in the future.

There are three main categories, advanced, delayed and mixed differential equations.

Advanced differential equations

The value of the derivative at a time t of the related variable y, does not depend only on the value of y at time t, but also on the future, and the form of these equations are written as follows

$$y^{'}(t) = f(t, y(t), y(t + \mu)), \quad \mu > 0,$$

where f is a given function.

Delayed differential equations

The value of the derivative at an instant t of the related variable y, depends not only on the value of y at the instant t, but also on the values taken before the instant t, and the form of these equations are written as follows

$$y^{'}(t) = f(t, y(t), y(t - r)), \quad r > 0,$$

where f is a given function.

Mixed differential equations

The value of the derivative at an instant t of the related variable y, depends on both the past and the future. The form of these equations are written as follows

$$y^{'}(t) = f(t, y(t), y(t+\mu), y(t-r)), \quad \mu, r > 0.$$

1.3.2 Some kind of delay differential equations

Delayed functional equations can be categorized as autonomous or non-autonomous, periodic or non-periodic, linear or non-linear, or according to the types of delay. So, we are interested in the classification of delay functional equations according to the types of delay. There exist two main classes, the first one is called delayed differential equations and the other one is called neutral type differential equations. In the following, we introduce various kind of DDE.

Differential equations with constant delay

The differential equations with constant, discrete or point delay can be written in their simplest form as follows

$$y^{'}(t) = f(t, y(t), y(t-r)),$$

where $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, a continuous function, and r a strictly positive real number called the delay. We find this kind of equations in the *Nicholson* fly model [17].

Variable delay differential equation

The delay in this case varying in the time variable or depending on the state.

Time-varying delay equation

This time-dependent equations can be raised in the transport models [17], and written in their simplest form as follows

$$y^{'}(t) = f(t, y(t), y(t-r(t))),$$

where f is a given function.

State-dependent variable delay equations

We find the state-dependent variable delay equations in the model describing the evolution of a fish population whose larvae consume a food [1]. These equations can be written as follows

$$y^{'}(t) = f(t, y(t), y(t-r(y(t)))),$$

where f is a given function.

Sometimes this classification is not sufficient in the study, which leads to adding additional constraints relating to the delay or its derivative, this leads to the identification of new sub-categories of variable delay equations such as

Arbitrary variable delay equations

For this class of equations, the delay and its derivative are not limited.

Equations with increased delay

This sub-category assumes the knowledge of a maximum value on the delay

$$0 \leq r(t) \leq r_{max},$$

if $r(t) = r$ is constant, it remains in practice uncertain and the above constraint ensures a bounded interval.

Delay equations (bi-bounded)

This sub-category is less discussed than the previous case where it is assumed that the delay verifies the constraint

$$r_{min} \leq r(t) \leq r_{max}.$$

Slowly time-varying delay equations

Suppose that $r(t)$ is a differentiable function almost every where such that

$$r'(t) \leq \lambda < 1,$$

which then indicates a limitation on the speed of variation of the delay and that the latter varies slowly in time, in other words that the delayed information arrives in chronological order.

Moderately time-varying delay equations

Suppose that $r(t)$ is a differentiable function almost every where such that

$$r'(t) \leq \lambda \quad with \quad \lambda \geq 1.$$

Fast time-varying delay equations

In this sub-category, there are no constraints on the delay and its derivative.

Differential equations with distributed delay

These equations are written in their simplest form as follows:

$$y^{'}(t) = -\alpha y(t) - \beta \int y(t-a) d\eta(a).$$

These type of equations are found in the AIDS model [21] or population dynamics model introduced by *Volterra* in 1934 where the term distributed delay was used to study the cumulative effect on the mortality rate of species.

Differential equations with unknown delay

In this case, no assumptions about the delay are taken into account, whether constant, variable or distributed.

Differential equations with neutral delay

These equations are written as follows

$$\frac{d}{dt}[Dy(t-r(t))] = f(t,y(t),y(t-r(t))),$$

where f is given and D is an operator.
This type of equation can be found in the distributed network model [17].

1.4 Relation between delay differential equation and integral equation

Let $G = C([0,r],\mathbb{R}^n)$ and $y_t \in G$ defined by

$$y_t(\tau) = y(t-\tau), \quad t \in [t_0, t_0+\alpha], \quad \alpha > 0$$

and $y \in C([t_0-r, t_0+\alpha],\mathbb{R}^n)$.
Consider the delay equation:

$$y' = f(t,y_t). \tag{1.7}$$

Lemma 1.1 *Let be a function $\psi \in G$, $t_0 \in \mathbb{R}$ and f a continuous function. To find a solution of equation (1.1) through (t_0,ψ) we will solve the following integral equation*

$$\begin{cases} y(t) = \psi(0) + \displaystyle\int_{t_0}^{t} f(v,y_v)\mathrm{d}v, \quad t \geq t_0 \\ y_{t_0} = \psi. \end{cases}$$

Proof 1 *Let's prove the equivalence.*
$\Rightarrow$) Let $y(t_0,\psi)$ be the solution of the initial value problem, i.e.

$$\begin{cases} \dfrac{\partial y(t_0,\psi)}{\partial t}(t) = f(t,y_t(t_0,\psi)) \\ y_{t_0}(t_0,\psi) \equiv \psi. \end{cases}$$

Then, we obtain,

$$\begin{aligned} y(t_0,\psi)(t) - \psi(0) &= y(t_0,\psi)(t) - y_{t_0}(t_0,\psi)(0) = y(t_0,\psi)(t) - y(t_0,\psi)(t_0) \\ &= \int_{t_0}^{t} y'(t_0,\psi)(s)\mathrm{d}s = \int_{t_0}^{t} f(v, y_v(t_0,\psi))\mathrm{d}v \end{aligned}$$

$\Leftarrow$) *we have,*

$$y'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\psi(0) + \int_{t_0}^{t} f(v, y_v)\mathrm{d}v) = f(t, y_t(t_0,\psi)),$$

hence, the delay equation is obtained.

1.5 Some models of delay differential equations

1.5.1 Nicholson's equation (sheep flies)

In 1950, the famous Australian entomologist *Alexander J. Nicholson* conducted a long series of experiments to learn more about meat-eating fly populations responsible for 90% of sheep myiasis that threaten farms in several countries such as *Australia*, *New Zealand* and *South Africa* [22, 23, 24].

This diptera fly has two pairs of wings, the first pair being membranous wings and the second pair being reduced and modified hind wings known as *"dumbbbells"* which are used for stabilization of flight. In addition this type of diptera fly having a round to oval body of length varying from 4.5 to 10 millimeters with reddish eyes and a greenish or bluish-green body with coppery reflections is part of the family *" Calliphoridae"* and is known as *"Australian copper fly"* or simply *"Australian sheep fly"* or in Latin *"Lucilia cuprina"* or *"Phaenicia cuprina"*. This type of the copper colored lucilia is presented in figure 1.1.

Figure 1.1: The copper-colored lucilia.

The development cycle of this fly (see figure 1.2.)includes four stages of growth: egg, larva, pupa and adult. The pregnant female firefly attracted by the smelly and sheep lays an average of 250 eggs on the skin of the animal and which will hatch and turn into carnivorous larvae after an incubation period does not exceed $24h$. These maggots feed on secretions from the wounds and underlying tissues of the sheep for three larval stages lasting 4 to 5 days.

After the larval phase, the fully developed larvae drop and sink into the ground to turn into pupae giving new young flies.

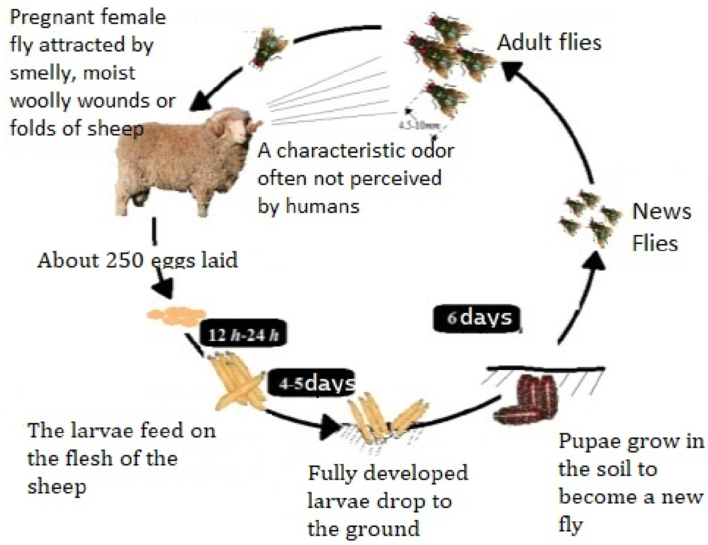

Figure 1.2: Development cycle of the copper firefly.

The following delayed equation (1.8) proposed in 1980 by *Nicholson* et al describes the evolution of the dynamics of a population over time [9]

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \beta N(t-r)exp\left(\frac{-N(t-r)}{k}\right) - \delta N(t), \tag{1.8}$$

where

- $N(t)$: is the population size at time t (adult copper fireflies at time t).
- $\frac{\mathrm{d}N(t)}{\mathrm{d}t}$: represents the rate of change in the number of people.
- β: is the maximum daily egg growth per individual.
- k: is the maximum number of individuals that the environment can support.
- δ: is the mortality rate individual (day^{-1}).
- r: is the duration of the maturation phase (the development cycle).

Several generalizations and a very active research has recently started on this equation, for example in 2004, *Liu* and *Ge* [18], studied the following neutral type equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t) - cy(t-r(t))] = -a(t)y(t) + b(t)y(t-r(t))e^{-\beta(t)y(t-r(t))}.$$

By keeping the same name, *"Nicholson equation"*, new modifications of the model have been widely considered in the study of the growths of other species.

1.5.2 Car chase model

Figure 1.3: Car chase

An example of a well-documented human activity where a delay appears are car tracking models that are used to predict traffic or to improve the safety of our vehicles [5, 10]. This model is described in figure 1.3. The following equation

$$y''_{n+1}(t+r) = \alpha(y'_n - y'_{n+1}), \tag{1.9}$$

can be used to determine the position and speed of the next car. This equation means that the acceleration of the vehicle to $y = y_{n+1}$ depends only on the difference between the y'_n and y'_{n+1} speeds. The coefficient α is positive (if $y'_n < y'_{n+1}$, the acceleration y''_{n+1} will be negative and the vehicle at y_{n+1} will brake).

If the driver reacts too abruptly (large value of α representing excessive braking) or too late (large r), the space between vehicles can become unstable (oscillations between vehicles). A typical solution for two cars is shown in figure 1.4.

The distance between the two cars decreases dangerously from 10 m to 5 m after the first car reduces its speed. A sober driver needs 1 s to start braking in front of an obstacle. With 0.5 g/l of alcohol in the blood (2 glasses of wine), the reaction is estimated at 1.5 s (http ://www.soifdevivre.com/ep/alcoolsante). Figure 1.5 shows that the oscillations are increasing.

It is clear that acceleration or braking also depends on the distance between the cars. If the driver is near the car in front of him, he will be more attentive to the changes. This

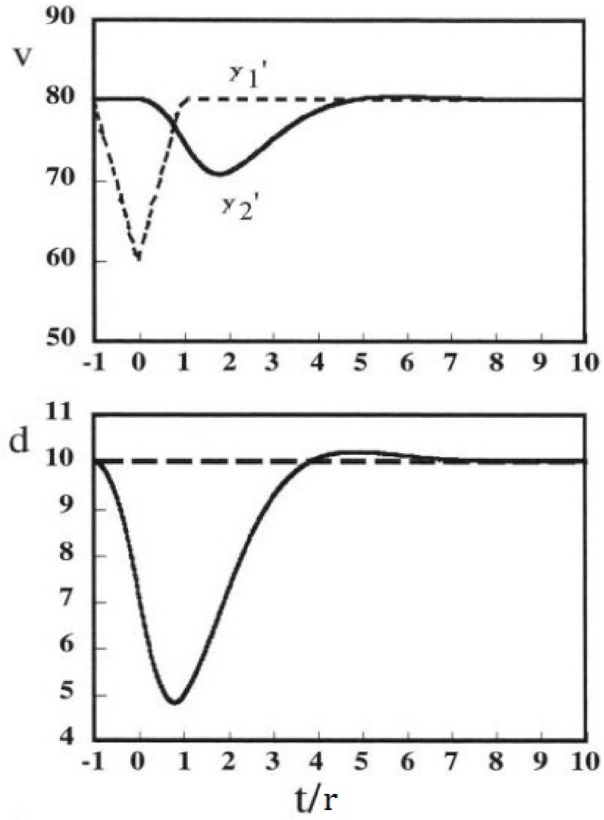

Figure 1.4: Car chase models. Top: speed of the two cars. The leading car reduces its speed from 80km/h to 60 km/h and then accelerates back to its original speed. Braking and acceleration are constant. Bottom: the distance $d = y_2 - y_1$ between the two cars. This distance is initially 10 m. $\alpha = 0.5s^{-1}$ and $r = 1s$.

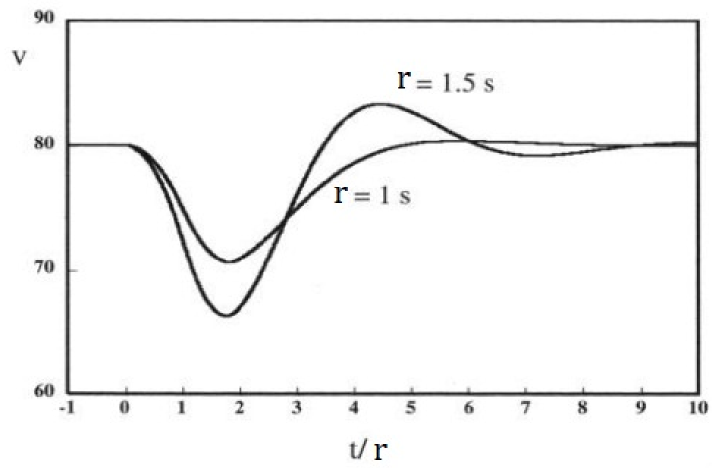

Figure 1.5: Drink or drive. Alcohol decreases the reaction time of the second driver allowing more oscillations between vehicles. $\alpha = 0.5s^{-1}$. The value of the delay r is shown in the figure.

increased sensitivity can be described by assuming that the coefficient α is inversely proportional to the distance $y_{n+1} - y_n$.

1.5.3 Economic model

Let given the following economic model

$$\begin{aligned} y(t) &= \int_0^l \rho(s) f(y(t-s)) \mathrm{d}s + c \\ &= \int_{t-l}^t \rho(t-u) f(y(u)) \mathrm{d}u + c. \end{aligned} \tag{1.10}$$

Where

- $y(t)$ is the value of capital stock.
- $f(y(t))$ is the production rate.
- l is the equipment lifetime.
- $\rho(s)$ is the value of capital equipment at time s, with $\rho(0) = 1$, $\rho(l) = 0$.
- c represents the value of non-depreciating assets.

This model is proposed by *Cooke* and *Yorke* 1973. They suppose that

- The production of new capital depends solely on $y(t)$.
- The depreciation is independent of equipment type.

Equation (1.10) indicate that at each time t, $y(t)$ is equal to the sum of capital produced over the period $[t-l, t]$.

1.6 Existence and uniqueness of solutions

1.6.1 Case of delay functional equations

In this section, we give some definitions and theorems of existence and uniqueness of solutions of DDE.

Theorem 1.1 ***(Existence)*** *[11] Let Ω be an open subset of $\mathbb{R} \times C$, where $C = \mathcal{C}(] - r, 0], \mathbb{R}^n)$ and $f \in \mathcal{C}(\Omega, \mathbb{R}^n)$ is a continuous application. For $(t_0, \psi) \in \Omega$, there exists a solution of equation (1.1) passing through (t_0, ψ).*

Definition 1.3 *[11] The function $f(t,y)$ is Lipschitzian with respect to y on a compact U of $\mathbb{R} \times C$ if there exists a constant $L > 0$ such that, for all $(t,y_i) \in U, \quad i = 1,2$, we have*

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|. \tag{1.11}$$

Theorem 1.2 ***(Uniqueness)*** *[11] Suppose that Ω is an open subset of $\mathbb{R} \times C$, $f : \Omega \longrightarrow \mathbb{R}^n$ is continuous and Lipschitzian with respect to y on any compact subset of Ω. If $(t_0, \psi) \in \Omega$, hence, there exists a unique solution of equation (1.1) passing through (t_0, ψ).*

1.6.2 Case of functional equations of neutral type

The Fréchet derivative is a derivative defined on normed spaces. It generalize the derivative of a real-valued function of a single real variable to the case of a vector-valued function of multiple real variables.

Definition 1.4 *Let S_1 and S_2 be normed vector spaces, and $O \subseteq S_1$ be an open sub-set of S_1. A function $f : O \to S_2$ is said to be Fréchet differentiable at $y \in O$ if there exists a bounded linear operator $D : S_1 \to S_2$ such that*

$$\lim_{\|h\| \to 0} \frac{\|f(y+h) - f(y) - Dh\|_{S_2}}{\|h\|_{S_1}} = 0$$

Example 1.3

1- ***Polynomial functions:*** *For example, the polynomial $f(y) = y^3 + 2y^2 - 5y + 1$ is Frechet differentiable everywhere. Its derivative is the function $f'(y) = 3y^2 + 4y - 5$.*

2- *Let consider $\|.\| : H \to \mathbb{R}^+$.*
It is stated that the Fréchet derivative of $\|.\|$ at y is defined by $Du := \left\langle u, \frac{y}{\|y\|} \right\rangle$. In effect,

$$\begin{aligned}
\frac{|\|y+h\| - \|y\| - Dh|}{\|h\|} &= \frac{|\|y\|\|y+h\| - \langle y,y \rangle - \langle y,h \rangle|}{\|y\|\|h\|} \\
&= \frac{|\|y\|\|y+h\| - \langle y,y+h \rangle|}{\|y\|\|h\|} \\
&= \frac{|\langle y,y \rangle \langle y+h,y+h \rangle - \langle y,y+h \rangle^2|}{\|y\|\|h\|(|\|y\|\|y+h\| + \langle y,y+h \rangle|)} \\
&= \frac{\langle y,y \rangle \langle h,h \rangle - \langle y,h \rangle^2}{\|y\|\|h\|(|\|y\|\|y+h\| + \langle y,y+h \rangle\||)}.
\end{aligned}$$

According to continuity of the norm and inner product, we get

$$
\begin{aligned}
\lim_{h\to 0}\frac{|\|y+h\|-\|y\|-Dh|}{\|h\|} &= \lim_{h\to 0}\frac{\langle y,y\rangle\langle h,h\rangle-\langle y,h\rangle^2}{\|y\|\|h\|(|\|y\|\|y+h\|+\langle y,y+h\rangle|)}, \quad y\neq 0\\
&= \frac{1}{2\|y\|^3}\lim_{h\to 0}\frac{\langle y,y\rangle\langle h,h\rangle-\langle y,h\rangle^2}{\|h\|}\\
&= \frac{1}{2\|y\|^3}\lim_{h\to 0}(\langle y,y\rangle\|h\|-\langle y,h\rangle\left\langle y,\frac{h}{\|h\|}\right\rangle)\\
&= \frac{1}{2\|y\|^3}(\lim_{h\to 0}\langle y,y\rangle\|h\|-\lim_{h\to 0}\langle y,h\rangle\left\langle y,\frac{h}{\|h\|}\right\rangle)\\
&= \frac{1}{2\|y\|^3}(0-\lim_{h\to 0}\langle y,h\rangle\left\langle y,\frac{h}{\|h\|}\right\rangle)\\
&= -\frac{1}{2\|y\|^3}(\lim_{h\to 0}\langle y,h\rangle\left\langle y,\frac{h}{\|h\|}\right\rangle), \qquad (1.12)
\end{aligned}
$$

when $h\to 0$, $\langle y,h\rangle\to 0$ and due to the Cauchy-Schwarz inequality $\left\langle y,\frac{h}{\|h\|}\right\rangle$ is bounded by $\|y\|$ hence, we get the result.

The definition of neutral delay differential equations NDDEs is based on the definition of atomic functions.

Definition 1.5 *[4] Let Ω be an open subset of $\mathbb{R}\times C$ with elements (t,ψ). A function $D:\Omega\longrightarrow\mathbb{R}^n$ is called atomic at the point β of Ω if D is continuous as well as its first and second derivatives in the Fréchet sense with respect to ψ and D_ψ, its derivative with respect to ψ, is atomic at β of Ω.*

Definition 1.6 *[4] Suppose that Ω is an open subset of $\mathbb{R}\times C$, $D:\Omega\longrightarrow\mathbb{R}^n$ and $f:\Omega\longrightarrow\mathbb{R}^n$ are two continuous given function with D atomic in zero. The relation*

$$\frac{d}{dt}D(t,y_t)=f(t,y_t), \qquad (1.13)$$

is said to be a differential equation of neutral type noted in abbreviation NDDE.

Definition 1.7 *A function y is a solution of (1.13) on $[t_0-r,t_0+\alpha]$ if $\exists t_0\in\mathbb{R}$, $\alpha>0$ such that $y\in C([t_0-r,t_0+\alpha],\mathbb{R}^n)$, $(t,y_t)\in\Omega$, $t\in[t_0,t_0+\alpha]$, $D(t,y_t)$ is continuously differentiable and verifies (1.13) on $[t_0,t_0+\alpha]$.*
For $t_0\in\mathbb{R},\psi\in C$, and $(t_0,\psi)\in\Omega$, $y(t,t_0,\psi)$ is said to be a solution of (1.13) with initial value ψ at t_0, if there exist an $\alpha>0$ such that $y(t,t_0,\psi)$ is a solution of (1.13) on $[t_0-r,t_0+\alpha]$ and $y_{t_0}(t_0,\psi)=\psi$. The function $y(t,t_0,\psi)$ is a solution of equation (1.13) on $[t_0-r,\infty)$ if for each $\alpha>0$, $y(t,t_0,\psi)$ is a solution of (1.13) on $[t_0-r,t_0+\alpha]$ and $y_{t_0}(t_0,\psi)=\psi$.

Theorem 1.3 ***(Existence)*** *[12] If Ω is an open subset of $\mathbb{R}\times C$ and $(t_0,\psi)\in\Omega$, then equation (1.13) has a solution passing through (t_0,ψ).*

Theorem 1.4 ***(Existence and uniqueness)*** *[12] If Ω is an open subset of $\mathbb{R} \times C$ and $f(t,y)$ is Lipschitzian with respect to y on any compact subset of Ω, then equation (1.13) has unique solution for all $(t_0, \psi) \in \Omega$, passing through (t_0, ψ).*

Example 1.4 *If $r > 0$, β is an matrix, $D(\psi) = \psi(0) - \beta\psi(-r)$, $\quad \psi \in C([-r,0], \mathbb{R}^n)$ and $f : \Omega \to \mathbb{R}^n$ is continuous, the pair (D, f) defines an NDDE*

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t) - \beta y(t-r)] = f(t, y_t).$$

Example 1.5 *If $r > 0$, y is a scalar, $D(\psi) = \psi(0) - sin\psi(-r)$, and $f : \Omega \to \mathbb{R}$ is continuous, then the pair (D, f) defined by:*

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t) - sin y(t-r)] = f(t, y_t),$$

is a NDDE.

Example 1.6 *The following equations*

$$y^{'}(t) = 3y^{'}(t-5),$$

$$y^{'}(t) = -y(t) + [y^{'}(t-4) + 1]^2,$$

$$y^{'}(t) = y(t) + y^{'}(t-2) - y^{'}(t-7),$$

are neutral equations.

1.7 Some methods for solving DDE

There are many methods that allow us to solve DDE, including: step-by-step method, *Runge Kutta's* method, *Laplace's* method.

1.7.1 Step-by-step method

The method of steps (also called the step-by-step method or the method of successive integrations) allows to solve numerically the DDE and DEN and allows at the same time to establish the existence and the uniqueness of the solution. It was presented in 1965, by R.Bellman for constant delays. Others like *El'sgl'ts and Norkin* (1973) have shown that it also remains valid for variable delays, provided that the delay never cancels.

To find the ideas, let us consider the following delayed linear functional equation with variable coefficients

$$\begin{cases} y'(t) = a(t)y(t) + b(t)y(t-r), & \text{for all} \quad t \in [0,r] \\ y(t) = \psi(t), & \text{for all} \quad t \in [-r,0]. \end{cases} \tag{1.14}$$

In the case where a and b are two real constants, then the equation is said to be of *"Frisch Holme"* type [7]. Now, we will solve this equation by the step method. The principle of this method is to look for solutions on interval, by using the following steps:

- **1st step:** In the interval $[-r,0]$ the function $y(t)$ is the given function $\psi(t)$, so the equation is solved in the interval $[-r,0]$ and we denote this solution by $y_0(t)$. Note here that if $t \in [0,r]$, then $t-r$ will reside in $[-r,0]$.

- **2nd step:** In the interval $[0,r]$, if $t \in [0,r]$, then $t-r$ will reside in $[-r,0]$. So $y(t-r) = y_0(t-r)$ in the interval $[0,r]$ and system (1.14) becomes:

$$\begin{cases} y'(t) = a(t)y(t) + b(t)y_0(t-r), & \text{for all} \quad t \in [0,r] \\ y(0) = \psi(0), \end{cases} \tag{1.15}$$

  which is an initial-valued problem for an ordinary differential equation (ODE) where $y_0(t-r) = \psi(t-r)$ is continuous. Thus, we solve this ODE in $[0,r]$ using the initial condition $y(0) = \psi(0)$ and denote by $y_1(t)$ this solution in $[0,r]$.

- **3rd step:** In interval $[r,2r]$, the system becomes:

$$\begin{cases} y'(t) = a(t)y(t) + b(t)y_1(t-r), & \text{for all} \quad t \in [r,2r] \\ y(r) = y_1(r). \end{cases} \tag{1.16}$$

  Equation (1.16) is a ordinary differential equation defined on $[r,2r]$, with exact solution $y_2(t) \in [r,2r]$ and so on.

Example 1.7 *Consider the following equation*

$$\begin{cases} y'(t) = y(t-7), & t \in [0,14] \\ y(t) = 4, & -7 \leq t \leq 0. \end{cases}$$

In $[-7,0]$, $y(t) = 4$. *Then, we solve the previous equation in the interval* $[0,7]$, *and obtain*

$$y(t) = \int_0^t y(s-7)\mathrm{d}s + y(0) = 4t + 4.$$

The solution in the interval $[7,14]$ *is given by*

$$y(t) = \int_7^t y(s-7)\mathrm{d}s + y(7) = 2t^2 - 24t + 102.$$

Example 1.8 *Let us consider the following particular case*

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = \alpha y(t-r), \quad \textit{for} \quad t \in [0,2r] \\ y(t) = \psi(t) = 1, \qquad -r \leq t \leq 0. \end{cases} \tag{1.17}$$

- ***1st step:*** *In* $[-r,0]$

$$y(t) = 1.$$

- ***2nd step:*** *Integration in* $[0,r]$
  The integration of the two members of the DDE from 0 *to* t*, gives*

$$y(t) = \alpha \int_0^t y(s-r)ds + y(0).$$

As $0 \leq s \leq r$*, then* $-r \leq s - r \leq 0$*. Knowing that* $y(t) = 1$ *for* $t \in [-r,0]$*, then*

$$y(s-r) = 1,$$

for $t \in [0,r]$ *, this leads to*

$$y(t) = \alpha t + 1,$$

- ***3rd step:*** *Integration in* $[r,2r]$
  The integration of the two members from r *to* t*, gives*

$$y(t) = \alpha \int_r^t y(s-r)ds + y(r).$$

As $r \leq s \leq 2r$*, then* $0 \leq s - r \leq r$*.*
Knowing that $y(t) = \alpha t + 1$ *for* $t \in [0,r]$*, then*

$$y(s-r) = \alpha(s-r) + 1,$$

for $s \in [r,2r]$ *, this leads to*

$$y(t) = \alpha^2 \frac{t^2}{2} + (\alpha - \alpha^2 r)t + \alpha^2 \frac{r^2}{2} + 1,$$

in the interval $[r, 2r]$.

Finally, we obtained equations given in table 1.1.

t	ODE with initial condition
$[0,r]$	$\begin{cases} y'(t) = \alpha \\ y(0) = 1 \end{cases}$
$[r,2r]$	$\begin{cases} y'(t) = \alpha^2(t-r) + \alpha \\ y(r) = \alpha r + 1. \end{cases}$

Table 1.1: Exact solution of equation (1.17).

and the solution is given by:

$$y(t) = \begin{cases} \alpha t + 1, & 0 \leq t \leq r \\ y(t) = \alpha^2 \frac{t^2}{2} + (\alpha - \alpha^2 r)t + \alpha^2 \frac{r^2}{2} + 1, & r \leq t \leq 2r. \end{cases}$$

If we take $\alpha = -1, r = 1$ *with the same initial condition* $\psi(t) = 1$*, we will have*

$$y(t) = \begin{cases} 1 - t, & 0 \leq t \leq 1 \\ \frac{t^2}{2} - 2t + \frac{3}{2}, & 1 \leq t \leq 2. \end{cases}$$

Figure 1.6 represents the graphical representation of solutions of equation (1.17).

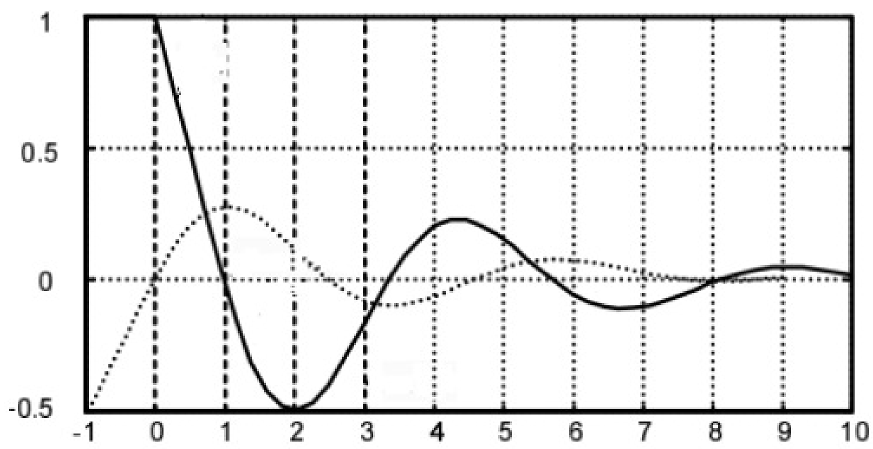

Figure 1.6: Graphical representation of the equation $y'(t) = -y(t-1)$ with $\psi(t) = 1$ (full) and $\psi(t) = -1$ (dashed line).

Chapter 2

Pseudo-spectral Method for solving integro-delay differential equations on a half line

2.1 Definition of the problem

The subject of this chapter is to approximate solution of integro-delay differential equations (IDDE). This work is inspired from the work of Ömür Kıvanç Kürkçü [16] in which the proposed algorithm has been programmed and applied to many examples with some changes.

Let given the following (IDDE)

$$\sum_{k=0}^{3}\sum_{l=0}^{m} P_{kl}(t)y^{(k)}(\alpha_{kl}t-\beta_{kl}) = f(t) + \int_{0}^{\infty}\mathcal{N}(t,s)y(\lambda s-\mu)ds, \quad t\in[0,T], \quad s\in[0,\infty], \tag{2.1}$$

with the following initial and boundary conditions:

$$y(0)=c_1, y^{'}(0)=c_2, y(T)=b,$$

where

- m is a positive integer.
- c_1, c_2, b are constants and $T>0$.
- $y(t), g(t)$ and $P_{kl}(t)$ are continuous functions on $[0,\infty]$.
- The sets $\{\alpha_{kl},\lambda\}$ and $\{\beta_{kl},\mu\}$ are the proportional $(\alpha_{kl},\lambda\in[0,1])$ and constant $(\beta_{kl},\mu\in[0,1])$ real delays.

- The kernel $\mathcal{N}(s,t)$ is supposed a Riemann integrable function on $\Omega = \{(t,s) : 0 \leqslant t \leqslant T, 0 \leqslant s < \infty\}$, such that

$$\int_0^\infty \mathcal{N}(t,s) y(\lambda s - \mu) ds < \infty, \tag{2.2}$$

  where T can be large $(T \longrightarrow \infty)$.

To find the approximate solution of equation (2.1), we propose a convergent numerical method based on hybrid ortho-exponential and Taylor polynomials.

2.2 Ortho-exponential polynomials

Jaroch [13] presented orthogonal exponential polynomials to create the technique of approximation across exponential functions, as it was applied in different kind of problems such that electrical circuit theory, hydro-meteorology and automatic control.

Definition 2.1 *[14] Let t be a real, the functions*

$$E_n^\perp(t) = \sum_{k=1}^{n} b_{nk} e^{-kt}, \quad n = 1,2,3\ldots, \tag{2.3}$$

where

$$b_{nk} = (-1)^{n+k} \binom{n}{k} \binom{n+k-1}{k-1},$$

are called orthogonal exponential polynomials.
The first four polynomials are represented as follows

- $E_1^\perp = e^{-t}$.
- $E_2^\perp = -2e^{-t} + 3e^{-2t}$.
- $E_3^\perp = 3e^{-t} - 12e^{-2t} + 10e^{-3t}$.
- $E_4^\perp = -4e^{-t} + 30e^{-2t} - 60e^{-3t} + 35e^{-4t}$.

Remark 2.1 *The ortho-exponential polynomials are obtained by orthogonalization of the system $(e^{-t}, e^{-2t}, e^{-3t}, \ldots)$ on $[0, +\infty)$ with the weight function $w(t) = 1$ that is to say in $L^2(0, +\infty)$.*

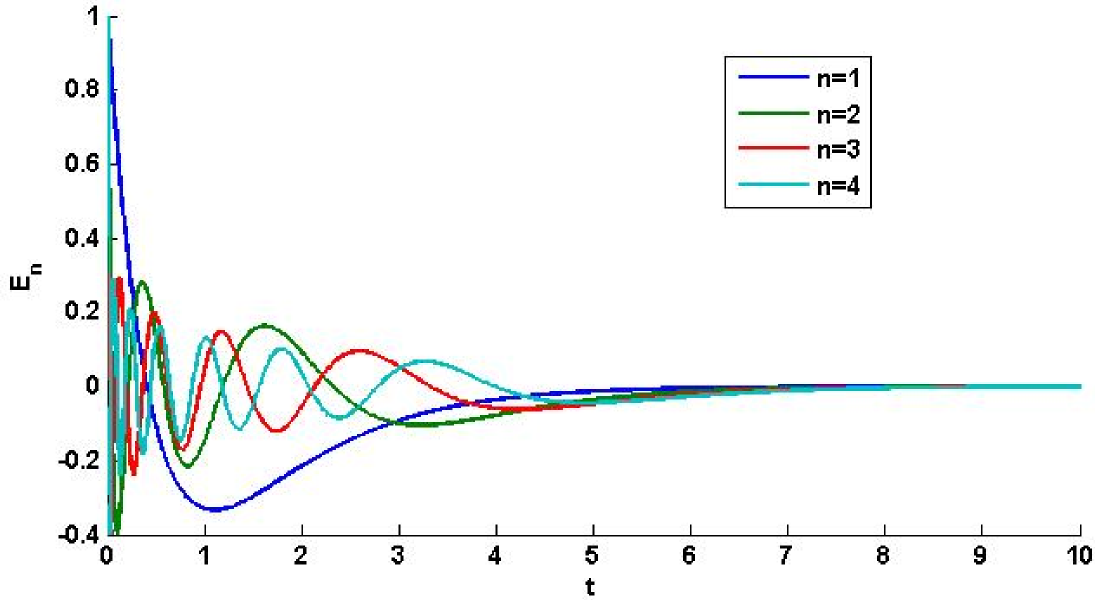

Figure 2.1: Some ortho-exponential polynomials $E_n^{\perp}(t)$.

2.2.1 Relation between ortho-exponential and Legendre polynomials

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomials which are defined on $[-1,1]$ and orthogonal with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}, \alpha > -1, \beta > -1$, and normalized with $P_n^{(\alpha,\beta)}(1) = \begin{pmatrix} n+\alpha \\ n \end{pmatrix}$ [15]. Legendre polynomials are a special case of Jacobi polynomials with $\alpha = \beta = 0$.

Definition 2.2 *[15] The polynomials*

$$P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} a_{nk} x^{n-2k}, \quad n = 0, 1, 2, \cdots,$$

where $a_{nk} = (-1)^k 2^{-n} \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} 2n-2k \\ n \end{pmatrix}$ *are called Legendre polynomials if the coefficients* a_{nk} *are chosen such that for any* m, n *the following conditions are satisfied*

$$P_n(1) = 1 \text{ and } \int_{-1}^{1} P_m(x) P_n(x) dx = \delta_{mn} \|P_n\|^2.$$

Some properties of Legendre polynomials [15] are given as follows

1.
$$P_n^{(0,0)}(1) = \begin{pmatrix} n \\ n \end{pmatrix} = 1. \tag{2.4}$$

2.
$$P_n^{(0,0)}(-1) = (-1)^n \binom{n}{n} = (-1)^n. \tag{2.5}$$

3.
$$\|P_n\| = \frac{1}{\sqrt{n+\frac{1}{2}}}, |P_n(x)| \leq 1, \quad -1 \leq x \leq 1. \tag{2.6}$$

4.
$$(1+x)P_{n-1}^{(0,1)}(x) = P_n^{(0,0)}(x) + P_{n-1}^{(0,0)}(x), \quad n \geq 1. \tag{2.7}$$

5.
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x). \tag{2.8}$$

The following theorem gives the relation between ortho-exponential and Legendre polynomials.

Theorem 2.1 *Let $E_n(t)$ be the ortho-exponential polynomials and $P_n(x)$ the Legendre polynomials. Then, for arbitrary t we have*

$$E_n^{\perp}(t) = \frac{1}{2}[P_n(2e^{-t}-1) + P_{n-1}(2e^{-t}-1)]. \tag{2.9}$$

Proof 2 *From definitions 2.1-2.2, we get equation (2.9).*

2.2.2 Properties of ortho-exponential polynomials

(i) $E_n^{\perp}(0) = 1, \quad E_n^{\perp}(+\infty) = \lim_{t\to\infty} E_n^{\perp}(t) = 0$, for all $n = 1, 2, \ldots$

(ii) We have, $\quad |E_n^{\perp}(t)| \leq 1, \quad \forall t \geq 0.$

(iii) $\int_0^{\infty} E_m^{\perp}(t)E_n^{\perp}(t)\mathrm{d}t = \delta_{mn}\|E_n^{\perp}\|^2,$
where $\|E_n^{\perp}\| = \frac{1}{\sqrt{2n}}$ and δ_{mn} is the kronecher delta function.

(iv) The ortho-exponential polynomials satisfy the relation such that, for all $n = 1, 2, 3, \cdots$, we have
$(n+1)(2n-1)E_{n+1}^{\perp}(t) = 2[(4n^2-1)e^{-t} - 2n^2]E_n^{\perp} - (n-1)(2n+1)E_{n-1}^{\perp},$
where $E_1^{\perp} = e^{-t}$ and $E_0^{\perp} = 0$.

Proof 3

(i) *According to equations (2.9) and (2.4), we have*

$$\begin{aligned} E_n^{\perp}(0) = \lim_{t\longrightarrow 0} E_n^{\perp}(t) &= \lim_{t\longrightarrow 0} \frac{1}{2}[P_n(2e^{-t}-1) + P_{n-1}(2e^{-t}-1)] \\ &= \frac{1}{2}[P_n(1) + P_{n-1}(1)] \\ &= \frac{1}{2} \times 2 \\ &= 1, \end{aligned}$$

and from the equations (2.9) and (2.5), we have

$$\begin{aligned} E_n^{\perp}(\infty) &= \lim_{t\longrightarrow\infty} E_n^{\perp}(t) = lim_{t\longrightarrow\infty}\frac{1}{2}[P_n(2e^{-t}-1) + P_{n-1}(2e^{-t}-1)] \\ &= \frac{1}{2}[P_n(-1) + P_{n-1}(-1)] \\ &= 0. \end{aligned}$$

(ii) *We have,*
$|E_n^{\perp}(t)| = |\frac{1}{2}[P_n(2e^{-t}-1) + P_{n-1}(2e^{-t}-1)]| \leq 1.$

(iii) *We prove that* $\int_0^{\infty} E_m^{\perp}(t)E_n^{\perp}(t)\mathrm{d}t = \delta_{mn}\|E_n^{\perp}\|^2$*, by replacing* $x = 2e^{-t}-1$ *in equation (2.7), we obtain*

$$2e^{-t}P_{n-1}^{(0,1)}(2e^{-t}-1) = P_n^{(0,0)}(2e^{-t}-1) + P_{n-1}^{(0,0)}(2e^{-t}-1) = 2E_n^{\perp}(t),$$

the norm of the Jacobi polynomials for the considered case is $\|P_n^{(0,1)}\| = \dfrac{\sqrt{2}}{\sqrt{n+1}}$ *[27]. As consequence,*

$$\int_0^{\infty} E_m^{\perp}(t)E_n^{\perp}(t)\mathrm{d}t = \int_0^{\infty} e^{-2t}P_{m-1}^{(0,1)}(2e^{-t}-1)P_{n-1}^{(0,1)}(2e^{-t}-1)\mathrm{d}t,$$

and, if we use the substitution $x = 2e^{-t}-1$ *we get,*

$$\int_0^{\infty} E_m^{\perp}(t)E_n^{\perp}(t)\mathrm{d}t = \frac{1}{4}\int_{-1}^{1}(1+x)P_{m-1}^{(0,1)}(x)P_{n-1}^{(0,1)}(x)\mathrm{d}x = \delta_{mn}(\frac{1}{2n}).$$

(iv) *From the relation (2.9) and (2.8), we get*

$$(n+1)(2n-1)E_{n+1}^{\perp}(t) = 2[(4n^2-1)e^{-t} - 2n^2]E_n^{\perp} - (n-1)(2n+1)E_{n-1}^{\perp}.$$

2.3 Numerical method for solving integro-delay differential equations

In this section, we will use the collocation method to approximate solution of IDDE and which performs a fast and directly convergent computation.

2.3.1 Function approximation

Let $w(t)$ denotes a non negative integral real valued function over the interval $\Lambda = [0, \infty[$, we define

$$L^2_w(\Lambda) = \{u : \Lambda \to \mathbb{R}/u \text{ is mesurable and } \|u\|_w < \infty\},$$

where

$$\|u\|^2_w = \int_0^\infty u^2(t)w(t)du,$$

is the norm induced by the inner product of the space $L^2_w(\Lambda)$.

$$(u, v)_w = \int_0^\infty u(t)v(t)w(t)dt. \tag{2.10}$$

Let $\{\varphi_j(t)\}_{j\geq 1}$ denotes a system which is mutually orthogonal under (2.10).
Let consider $\{\varphi_j(t)\}_{j\geq 1}$ is $\left\{E^\perp_j(t)\right\}_{j\geq 1}$, from the classical Weierstrass theorem implies that such a system is complete in the space $L^2_w(\Lambda)$, so for each function $y \in L^2_w(\Lambda)$, we have

$$y(t) = \sum_{i=1}^{\infty} a_i \varphi_i(t) = \sum_{i=1}^{\infty} a_i E^\perp_i(t), \tag{2.11}$$

where a_i can be determined by using equation (2.10).
If the serie (2.11) is truncated up to the $N + 1$ terms, we can write

$$y(t) \simeq y_N(t) = \sum_{i=1}^{N+1} a_i E^\perp_i(t).$$

The idea of the numerical method is to choose a finite dimensional space of candidate and a number of point called collocation points.

2.3.2 Description of the method

The ortho-exponential polynomial solution of equation (2.1) is written as

$$y(t) \simeq y_N(t) = \sum_{i=1}^{N+1} a_i E^\perp_i(t), \tag{2.12}$$

where, $a_i, \quad i = 1 : N + 1$ are unknown coefficients that are determined through the method and $E_i^{\perp}(t)$ are the ortho-exponential polynomials. Then, the functions in equation (2.1) are transformed into matrix form at the collocation points. Finally a unique matrix equation is used to obtain the unknown coefficients of equation (2.12). The numerical technique for solving (2.1) is described as follows: The equation (2.12) can be written as

$$y(t) = E_n(t)A = E(t)CA, \tag{2.13}$$

where, $E_n(t) = [E_1^{\perp}(t) \quad E_2^{\perp}(t) \quad \cdots \quad E_{N+1}^{\perp}(t)]$, $E(t) = [e^{-t} \quad e^{-2t} \quad \cdots \quad e^{-(N+1)t}]$, $A = [a_1 \quad a_2 \quad \cdots \quad a_{N+1}]^T$, and C is the matrix defined by

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1,N+1} \\ 0 & c_{22} & c_{23} & \cdots & c_{2,N+1} \\ 0 & 0 & c_{33} & \cdots & c_{3,N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{N+1,N+1} \end{bmatrix},$$

such that

$$c_{nk} = \begin{cases} 0, & n > k \\ (-1)^{k+n} \binom{k}{n} \binom{k+n-1}{n-1}, & n \leq k \end{cases}$$

with $n, k = 1, 2, \cdots N + 1$.

Hence, by equation (2.13) we get

$$y^{(k)}(t) = E_n^{(k)}(t)A = E(t)B^kCA, \tag{2.14}$$

where

$$E_n^{(k)}(t) = [E_1^{(k)\perp}(t) \quad E_2^{(k)\perp}(t) \quad \cdots \quad E_{N+1}^{(k)\perp}(t)],$$

and

$$B = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -2 & 0 & \cdots & 0 \\ 0 & 0 & -3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -(N+1) \end{bmatrix}.$$

Replacing t by $\alpha_{kl}t - \beta_{kl}$ in equation (2.14), we obtain

$$y^{(k)}(\alpha_{kl}t - \beta_{kl}) = E(\alpha_{kl}t - \beta_{kl})B^kCA, \tag{2.15}$$

then substituting (2.15) in equation (2.1), the matrix form of the differential part $H(t)$ of equation (2.1) is equal to

$$H(t_i) = \sum_{k=0}^{3}\sum_{l=0}^{m} \mathbf{P}_{kl}(t_i)E(\alpha_{kl}t_i - \beta_{kl})B^k CA, \tag{2.16}$$

where, $\mathbf{P}_{kl}(t_i) = diag[P_{kl}(t_i)]_{(N+1)\times(N+1)}$ and

$$\begin{aligned} E(\alpha_{kl}t_i - \beta_{kl}) &= \begin{bmatrix} E(\alpha_{kl}t_1 - \beta_{kl}) \\ E(\alpha_{kl}t_2 - \beta_{kl}) \\ \vdots \\ E(\alpha_{kl}t_{N+1} - \beta_{kl}) \end{bmatrix} \\ &= \begin{bmatrix} e^{-(\alpha_{kl}t_1-\beta_{kl})} & e^{-2(\alpha_{kl}t_1-\beta_{kl})} & \dots & e^{-(N+1)(\alpha_{kl}t_1-\beta_{kl})} \\ e^{-(\alpha_{kl}t_2-\beta_{kl})} & e^{-2(\alpha_{kl}t_2-\beta_{kl})} & \dots & e^{-(N+1)(\alpha_{kl}t_2-\beta_{kl})} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-(\alpha_{kl}t_{N+1}-\beta_{kl})} & e^{-2(\alpha_{kl}t_{N+1}-\beta_{kl})} & \dots & e^{-(N+1)(\alpha_{kl}t_{N+1}-\beta_{kl})} \end{bmatrix}. \end{aligned}$$

Let t_i be the collocation points on $[0, T]$, given by

$$t_i = \frac{T}{N+1}i, \quad \text{with} \quad i = 1, 2, 3, \cdots, N+1. \tag{2.17}$$

The matrix (2.16) can also be written briefly as follows

$$H = \sum_{k=0}^{3}\sum_{l=0}^{m} \mathbf{P}_{kl}E(\alpha_{kl}, \beta_{kl})B^k CA. \tag{2.18}$$

Let assumed that

$$E(t) = X(t)T, \tag{2.19}$$

where

$$X^T(t) = \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^N \end{bmatrix},$$

and

$$T^T = \begin{bmatrix} 1 & \frac{(-1)^1}{1} & \frac{(-1)^2}{2!} & \cdots & \frac{(-1)^N}{N!} \\ 1 & \frac{(-2)^1}{1} & \frac{(-2)^2}{2!} & \cdots & \frac{(-2)^N}{N!} \\ 1 & \frac{(-3)^1}{1} & \frac{(-3)^2}{2!} & \cdots & \frac{(-3)^N}{N!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{(-N-1)^1}{1} & \frac{(-N-1)^2}{2!} & \cdots & \frac{(-N-1)^N}{N!} \end{bmatrix}.$$

Since T is invertible then

$$X(t) = E(t)T^{-1}.$$

We have

$$E^T(t) = (X(t)T)^T = T^T X^T(t),$$

then it's equivalent to

$$\begin{bmatrix} e^{-t} \\ e^{-2t} \\ e^{-3t} \\ \vdots \\ e^{-(N+1)t} \end{bmatrix} = \begin{bmatrix} 1 & \frac{(-1)^1}{1} & \frac{(-1)^2}{2!} & \cdots & \frac{(-1)^N}{N!} \\ 1 & \frac{(-2)^1}{1} & \frac{(-2)^2}{2!} & \cdots & \frac{(-2)^N}{N!} \\ 1 & \frac{(-3)^1}{1} & \frac{(-3)^2}{2!} & \cdots & \frac{(-3)^N}{N!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{(-N-1)^1}{1} & \frac{(-N-1)^2}{2!} & \cdots & \frac{(-N-1)^N}{N!} \end{bmatrix} \begin{bmatrix} 1 \\ t^1 \\ t^2 \\ \vdots \\ t^N \end{bmatrix}. \tag{2.20}$$

So, $H(t)$ can be written as follows

$$H = \sum_{k=0}^{3} \sum_{l=0}^{m} \mathbf{P}_{kl} X(\alpha_{kl}, \beta_{kl}) T B^k C A, \tag{2.21}$$

where

$$X(\alpha_{kl}, \beta_{kl}) = \begin{bmatrix} X(\alpha_{kl}t_1 - \beta_{kl}) \\ X(\alpha_{kl}t_2 - \beta_{kl}) \\ \vdots \\ X(\alpha_{kl}t_{N+1} - \beta_{kl}) \end{bmatrix} = \begin{bmatrix} 1 & (\alpha_{kl}t_1 - \beta_{kl}) & \cdots & (\alpha_{kl}t_1 - \beta_{kl})^N \\ 1 & (\alpha_{kl}t_2 - \beta_{kl}) & \cdots & (\alpha_{kl}t_2 - \beta_{kl})^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (\alpha_{kl}t_{N+1} - \beta_{kl}) & \cdots & (\alpha_{kl}t_{N+1} - \beta_{kl})^N \end{bmatrix}, \tag{2.22}$$

for $\alpha_{kl} = 1$, $\beta_{kl} = 0$ we have

$$X(1,0) = X = \begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_{N+1}) \end{bmatrix} = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^N \\ 1 & t_2 & t_2^2 & \cdots & t_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{N+1} & t_{N+1}^2 & \cdots & t_{N+1}^N \end{bmatrix}. \tag{2.23}$$

Next, we write the integral part noted $I(t)$ for equation (2.1) in matrix form.

Theorem 2.2 *The integral part $I(t)$ of equation (2.1) is determined by*

$$I = XK(T^{-1})^T RCA, \tag{2.24}$$

where $K = \frac{1}{i!j!}\frac{\partial^{i+j}\mathcal{N}(0,0)}{\partial t^i \partial s^i}$, $\quad i,j = 0,1,\cdots,N$ *and* $R = (R_{ij}) = \left[\frac{e^{\mu j}}{i+j\lambda}\right]$, $\quad i,j = 1,2,\cdots,N+1$.

Proof 4 *The Kernel function can be written as,*

$$[\mathcal{N}(t,s)] = X(t)KX^T(s), \tag{2.25}$$

where

$$K = [k_{ij}] = \frac{1}{i!j!}\frac{\partial^{i+j}\mathcal{N}(0,0)}{\partial t^i \partial s^i}, \quad i,j = 0,1,\cdots,N. \tag{2.26}$$

Substituting relations (2.13) and (2.25) in $I(t)$, we obtain

$$\begin{aligned} I(t) &= \int_0^\infty X(t)KX^T(s)E(\lambda s-\mu)\mathrm{d}s \;\; CA \\ &= X(t)K\int_0^\infty X^T(s)E(\lambda s-\mu)\mathrm{d}s \;\; CA, \end{aligned} \tag{2.27}$$

and we have $X^T(s) = (T^{-1})^T E^T(s)$, substituting it in equation (2.27), we get

$$I(t) = X(t)K(T^{-1})^T RCA, \tag{2.28}$$

where

$$\begin{aligned} R &= \int_0^\infty E^T(s)E(\lambda s-\mu)\mathrm{d}s \\ &= \int_0^\infty \begin{bmatrix} e^{-s} \\ e^{-2s} \\ \vdots \\ e^{-(N+1)s} \end{bmatrix} \begin{bmatrix} e^{-(\lambda s-\mu)} & e^{-2(\lambda s-\mu)} & \dots & e^{-(N+1)(\lambda s-\mu)} \end{bmatrix} \mathrm{d}s \\ &= \int_0^\infty \left[e^{-s(i+j\lambda)+\mu j}\right] \mathrm{d}s, \quad i,j = 1:N+1. \end{aligned}$$

The matrix R can be written as

$$R = \left[\lim_{\epsilon\to\infty}\int_0^\epsilon e^{-s(i+j\lambda)+\mu j} ds\right] = \left[\lim_{\epsilon\to\infty}\left(\frac{e^{-s(i+j\lambda)+\mu j}}{-i-j\lambda}\right)\Big|_0^\epsilon\right] = \left[\frac{e^{\mu j}}{i+j\lambda}\right]. \tag{2.29}$$

Finally, by inserting collocation points (2.17) into (2.28), we obtain the following final form

$$I = XK(T^{-1})^T RCA. \tag{2.30}$$

Now, we add the matrix (2.18) (or (2.21)) and (2.24), we obtain

$$\left[\sum_{k=0}^{3}\sum_{l=0}^{m}\mathbf{P}_{kl}E(\alpha_{kl},\beta_{kl})B^{k}+XK(T^{-1})^{T}R\right]CA=G, \tag{2.31}$$

or

$$\left[\sum_{k=0}^{3}\sum_{l=0}^{m}\mathbf{P}_{kl}X(\alpha_{kl},\beta_{kl})TB^{k}+XK(T^{-1})^{T}R\right]CA=G, \tag{2.32}$$

where

$$G=\begin{bmatrix}g(t_1) & g(t_2) & \cdots & g(t_{N+1})\end{bmatrix}^{T}.$$

Through the matrix formulas (2.31) and (2.32), the matrix form of equation (2.1) can be expressed as follows

$WA=G,$

where

$$W=\left[\sum_{k=0}^{3}\sum_{l=0}^{m}\mathbf{P}_{kl}E(\alpha_{kl},\beta_{kl})B^{k}+XK(T^{-1})^{T}R\right]C, \tag{2.33}$$

or

$$W=\left[\sum_{k=0}^{3}\sum_{l=0}^{m}\mathbf{P}_{kl}X(\alpha_{kl},\beta_{kl})TB^{k}+XK(T^{-1})^{T}R\right]C. \tag{2.34}$$

Using the matrix relations (2.13) and (2.14) we can determine the matrix associated to boundary and initial conditions as follows

$$\begin{cases} y(0)=E(0)CA=c_1=[u_{1j}:c_1] \\ y^{'}(0)=E(0)B^{1}CA=c_2=[u_{2j}:c_2] \\ y(T)=E(T)CA=b=[u_{3j}:b] \end{cases} \tag{2.35}$$

where $j=1,2,\cdots,N+1$ and u_{1j},u_{2j} and u_{3j} are the elements corresponding to initial and boundary conditions.

After substituting matrix (2.35) into any row(s) of the system $[W:G]$, we get the augmented matrix system $\left[\tilde{W}:\tilde{G}\right]$, where

$$[\tilde{W} : \tilde{G}] = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1,N+1} & : & g(t_1) \\ w_{21} & w_{22} & \cdots & w_{2,N+1} & : & g(t_2) \\ \vdots & \vdots & \vdots & \vdots & : & \vdots \\ w_{N-2,1} & w_{N-2,2} & \cdots & w_{N-2,N+1} & : & g(t_{N-2}) \\ u_{11} & u_{12} & \cdots & u_{1,N+1} & : & c_1 \\ u_{21} & u_{22} & \cdots & u_{2,N+1} & : & c_2 \\ u_{31} & u_{32} & \cdots & u_{3,N+1} & : & b \end{bmatrix}. \tag{2.36}$$

Solving this system we get the unknown coefficients a_i, $\quad i = 1 : N+1$, substituting a_i in equation (2.12) we get the approximate solution of equation (2.1).

There are some proposals for the convergence of the proposed technique which does not give the error estimate.

Chapter 3

Numerical experiments and analysis

To show the applicability and effectiveness of the proposed numerical algorithm for solving integro-delay differential equations in half line, some numerical experiments are presented. The root mean square L^2-error is computed. The obtaining numerical results are summarized in tables and figures.

3.1 Numerical experiments

Example 3.1 *[16] Consider the following integro-delay differential equation*

$$y^{''}(t) + ty(0.1t - 0.5) - ty(t) = f(t) + \int_0^{\infty} exp(-t-s)y(0.2s - 0.3)\mathrm{d}s,$$

where $t \in [0,1]$, $s \in [0,\infty)$ *and* $f(t) = 0.100435\, exp(-t) - t\, sin(0.5 - 0.1t) - t\, sin(t) - sin(t)$, *such that the exact solution is given by*

$$y(t) = sin(t).$$

The numerical results are given in table 3.1 and figures 3.1-3.2, where

$$t(i) = (i-1) \times \frac{T}{N}, \quad i = 1 : N+1.$$

N	$N=5$	$N=8$	$N=9$	$N=12$
T	$T=1$	$T=1$	$T=1$	$T=1$
t_1	3.19744×10^{-14}	6.82121×10^{-13}	9.67759×10^{-12}	1.32243×10^{-10}
t_2	0.00319505	0.00352549	0.00353467	0.00280308
t_3	0.0176328	0.00100804	0.00204201	0.00248273
t_4	0.0297426	0.00472524	0.00248313	0.000101542
t_5	0.0259093	0.010999	0.00813736	0.00411745
t_6	1.42109×10^{-14}	0.0153523	0.013133	0.00876043
t_7		0.0157399	0.0158691	0.0132537
t_8		0.0107903	0.0150571	0.0168514
t_9		6.62314×10^{-13}	0.00986136	0.0188507
t_{10}			2.01617×10^{-12}	0.0186165
t_{11}				0.015619
t_{12}				0.00947733
t_{13}				3.5655×10^{-11}
L^2-error	0.02893635301	0.04003817508	0.02750674145	0.04332475047

Table 3.1: Computed errors for Example 3.1 for different values of N.

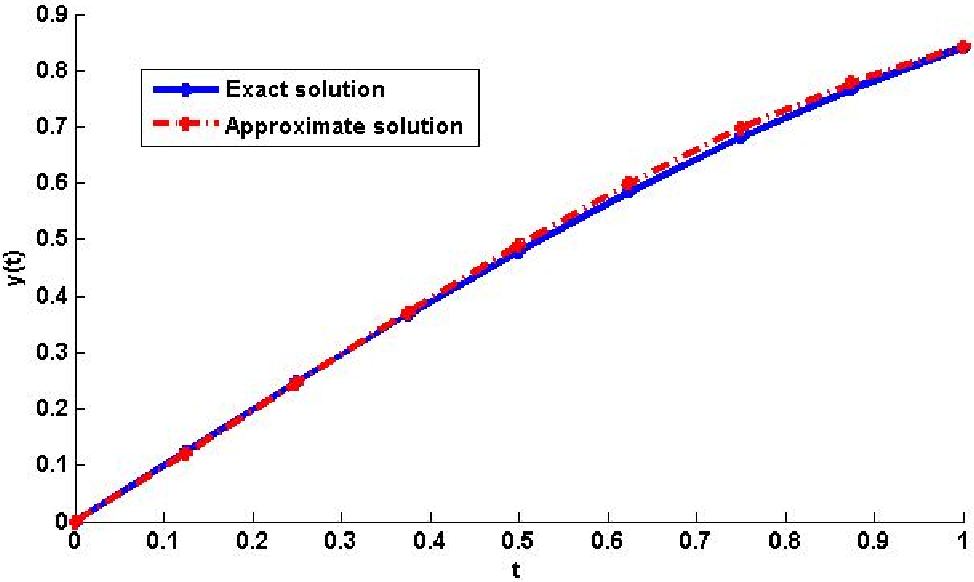

Figure 3.1: Exact and approximate solutions for Example 3.1 for $N=8$.

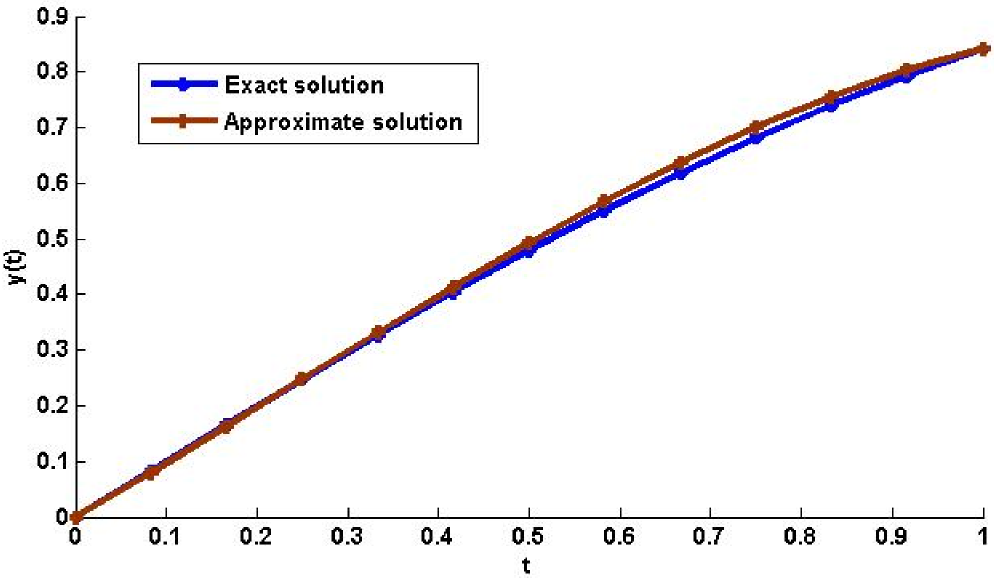

Figure 3.2: Exact and approximate solutions for Example 3.1 for $N = 12$.

Example 3.2 *[2] Let given the following equation*

$$y^{'''}(t) = ty^{''}(2t) - y'(t) - y\left(\frac{t}{2}\right) + tcos(2t) + cos\left(\frac{t}{2}\right),$$

with $y(0) = 1, y'(0) = 0, y''(0) = -1$, *such that the analytical solution of this problem is* $y(t) = cos(t)$.

The numerical experiments are presented in table 3.2 and figures 3.3-3.4, where $t(i) = (i-1) \times \dfrac{T}{N-1}, \quad i = 1 : N.$

Example 3.3 *[16] Consider the following integro-differential equation*

$$y^{''}(t) - 2y^{'}(t) - 8y(t) = f(t) + \int_0^{\infty} (ts^2 + t)y(s)\mathrm{d}s, \quad t \in [0, T],\ s \in [0, \infty),$$

where $f(t) = \dfrac{-3t}{4}$, *and* $y(0) = 1, y(1) = exp(-2)$. *The exact solution is written as*

$$y(t) = exp(-2t).$$

The augmented matrix formed by equation (2.36) is given for $N = 2$ *as follows*

$$[\tilde{W} : \tilde{G}] = \begin{bmatrix} -5 & 10 & 55 & : & 0 \\ 1 & 1 & 1 & : & 1 \\ \frac{1}{e} & \frac{3}{e^2} - \frac{2}{e} & \frac{10}{e^3} - \frac{12}{e^2} + \frac{3}{e} & : & 0.135335 \end{bmatrix},$$

N	$N=8$	$N=10$	$N=12$	$N=15$
T	$T=1$	$T=1$	$T=1$	$T=1$
t_1	2.84217×10^{-14}	1.64846×10^{-12}	1.13687×10^{-11}	1.0652×10^{-8}
t_2	0.000986862	0.00307169	0.000126917	0.0000804563
t_3	0.00383845	0.00123617	0.000507379	0.000321822
t_4	0.0084656	0.00278155	0.00114108	0.000724051
t_5	0.0137757	0.00489203	0.00202385	0.00128673
t_6	0.016905	0.00731151	0.00313101	0.00200697
t_7	0.0137282	0.00937482	0.00437843	0.00287304
t_8	5.4956×10^{-15}	0.00992113	0.00556875	0.00385224
t_9		0.00738189	0.00634372	0.00487419
t_{10}		3.24629×10^{-13}	0.00616341	0.000581346
t_{11}			0.0043223	0.00647586
t_{12}			4.14824×10^{-12}	0.00659243
t_{13}				0.00582314
t_{14}				0.00377032
t_{15}				2.46814×10^{-10}
L^2-error	0.02741141704	0.01809856395	0.01275026154	0.01484880703

Table 3.2: Computed errors for Example 3.2 for different values of N.

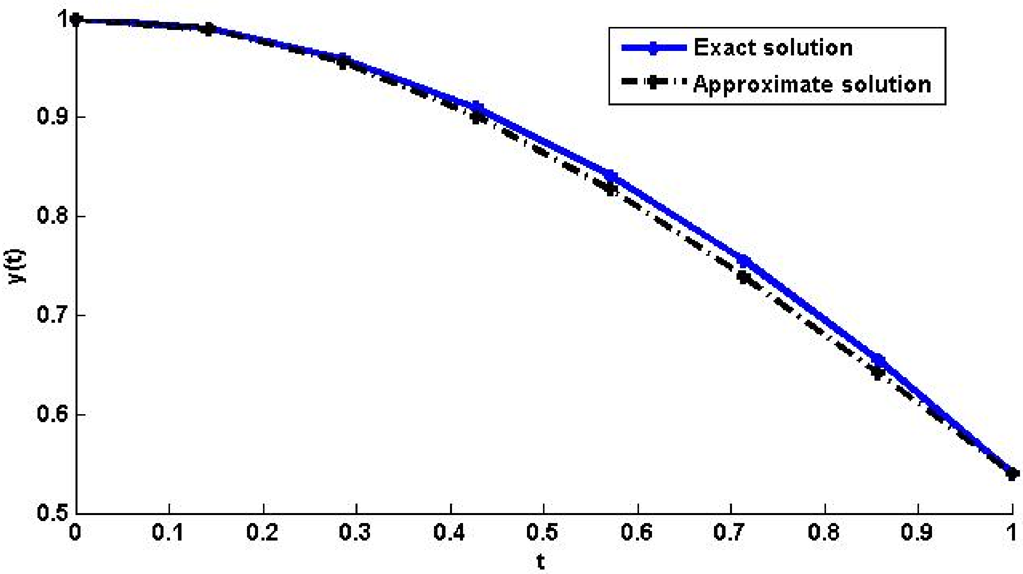

Figure 3.3: Exact and approximate solutions for Example 3.2 for $N=8$.

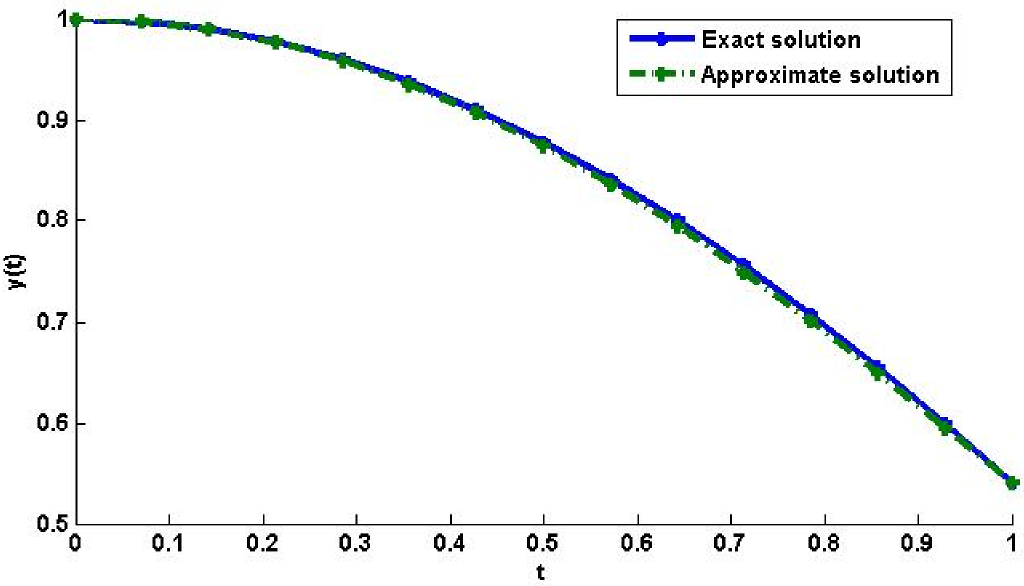

Figure 3.4: Exact and approximate solutions for Example 3.2 for $N = 15$.

after solving the system, the coefficients matrix are produced as follows

$$A = \begin{bmatrix} 0.\bar{6} & 0.\bar{3} & 0 \end{bmatrix}^T.$$

Now replacing A into the solution form (2.13), the approximate solution is directly given by

$$\begin{aligned} y_2(t) &= E_2(t)A \\ &= \begin{bmatrix} e^{-t} & 3e^{-2t} - 2e^{-t} & 10e^{-3t} - 12e^{-2t} + 3e^{-t} \end{bmatrix} \begin{bmatrix} 0.\bar{6} \\ 0.\bar{3} \\ 0 \end{bmatrix} = e^{-2t}, \end{aligned}$$

and this is the exact solution.

The numerical results are given in table 3.3 and figures 3.5-3.6, where $t(i) = (i-1) \times \frac{T}{N}, i = 1 : N+1.$

Example 3.4 *[16] Let given the following integro-differential equation*

$$y(t) = f(t) + \int_0^\infty e^{(-s-ts)} y(s) \mathrm{d}s, \quad t \in [0, T],\ s \in [0, \infty),$$

with $f(t) = e^{-t} sin(\omega t) - \frac{\omega}{(2+t)^2 + \omega^2}$, *such that the exact solution is given by*

$$y(t) = e^{-t} sin(\omega t).$$

N	$N=2$	$N=5$	$N=8$	$N=12$
T	$T=1$	$T=1$	$T=1$	$T=1$
t_1	0	6.25278×10^{-13}	3.21165×10^{-12}	7.98013×10^{-9}
t_2	2.37355×10^{-41}	0.00136043	0.00105764	0.000247277
t_3	1.78545×10^{-41}	0.00247307	0.002205589	0.000519893
t_4		0.00253664	0.0032624	0.00080401
t_5		0.00155558	0.00400137	0.00108397
t_6		1.03029×10^{-13}	0.0041728	0.00134106
t_7			0.00358345	0.00155212
t_8			0.00216671	0.00168839
t_9			6.39488×10^{-14}	0.00171548
t_{10}				0.00159525
t_{11}				0.00128959
t_{12}				0.00076546
t_{13}				5.37884×10^{-10}
L^2-error	$2.970108108\times10^{-41}$	0.004101368085	0.008221116067	0.004116760354

Table 3.3: Computed errors for Example 3.3 for different values of N.

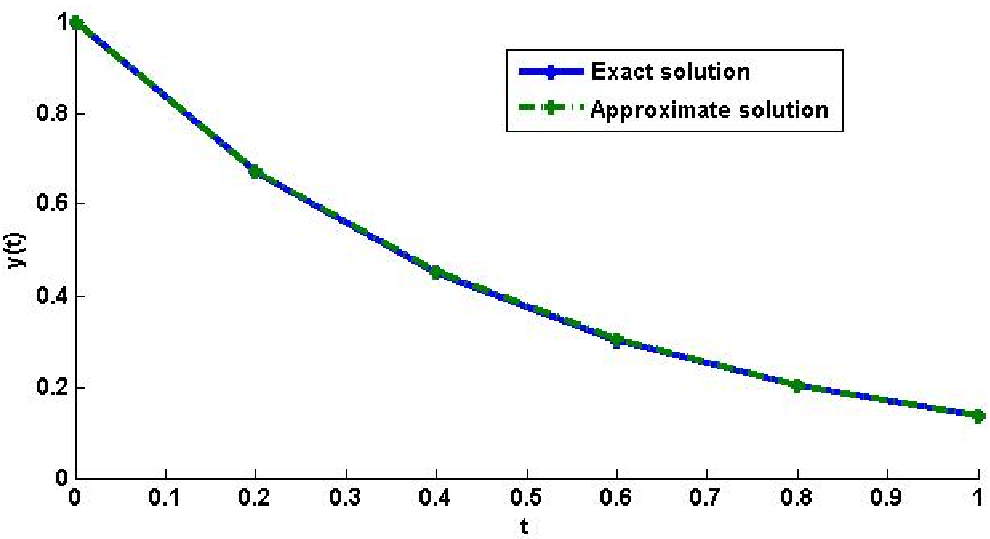

Figure 3.5: Exact and approximate solutions for Example 3.3 for $N=5$.

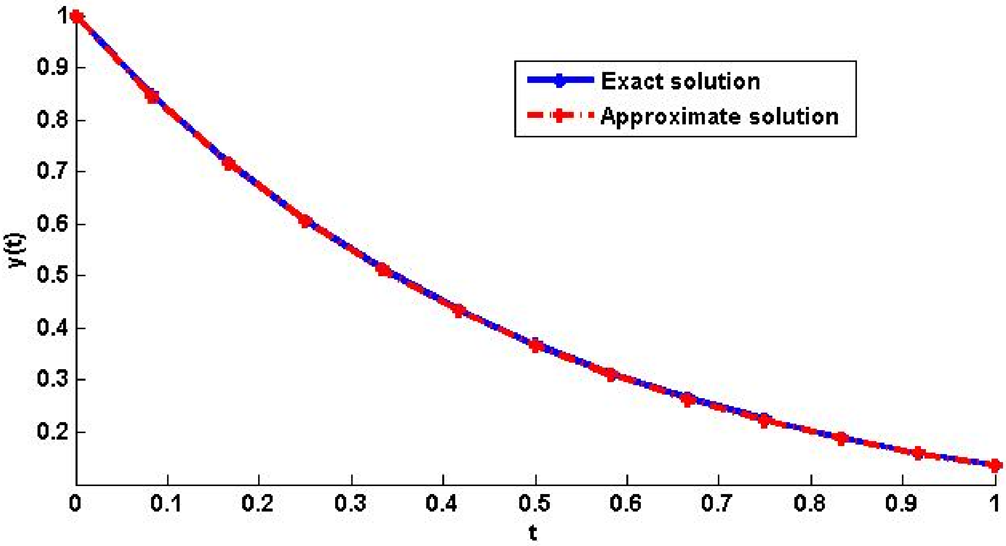

Figure 3.6: Exact and approximate solutions for Example 3.3 for $N = 12$.

The numerical results are presented in tables 3.4-3.5 and figures 3.7-3.8-3.9 and 3.10, where $t(i) = (i-1) \times \frac{T}{N}, \quad i = 1 : N+1.$

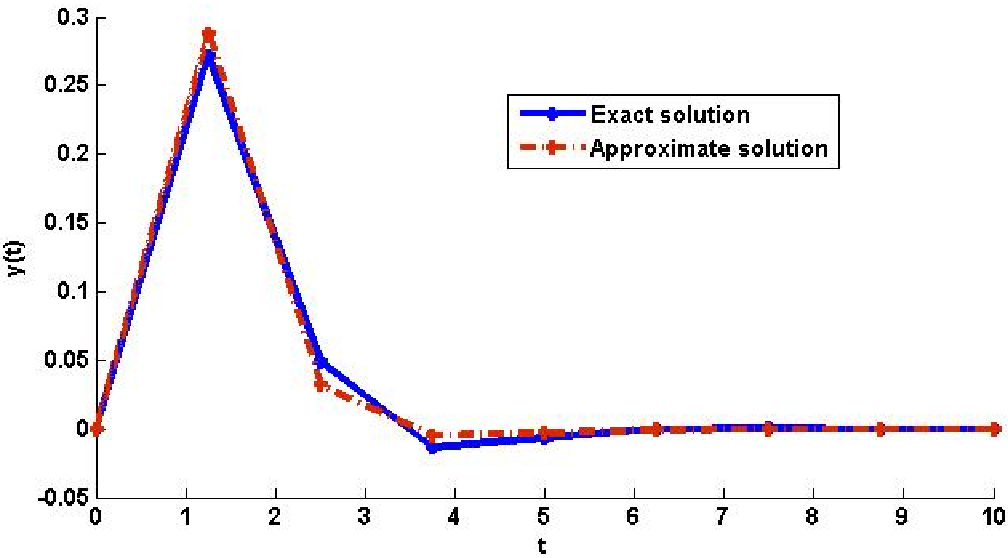

Figure 3.7: Exact and approximate solutions for Example 3.4 for $N = 8$ and $\omega = 1$.

Example 3.5 *[2] Consider the following delay equation*

$$y'(t) = -y(t) + \mu_1(t)y(\frac{t}{2}) + \mu_2(t)y(\frac{t}{4}),$$

N	$N=8$	$N=12$	$N=8$	$N=12$
$\omega\ /T$	ω=1, $T=10$	ω=1, $T=10$	ω=0.7, $T=10$	ω=0.7, $T=10$
t_1	1.30104×10^{-18}	5.42101×10^{-19}	2.1684×10^{-18}	4.33681×10^{-19}
t_2	0.0155812	0.000226825	0.00514277	0.000106836
t_3	0.0173223	0.00171303	0.0120525	0.000314909
t_4	0.00838099	0.0022631	0.00549275	0.00486416
t_5	0.00346631	0.00908404	0.00693767	0.00338741
t_6	0.000931096	0.00933951	0.00310208	0.00796914
t_7	0.000815413	0.00368451	0.000839378	0.00703784
t_8	0.000184928	0.000150121	0.000129129	0.00433915
t_9	5.18808×10^{-21}	0.00113686	3.91753×10^{-21}	0.0021163
t_{10}		0.000813973		0.000840038
t_{11}		0.00034309		0.000262708
t_{12}		0.0000833699		0.0000547643
t_{13}		5.39984×10^{-21}		3.59989×10^{-21}
L^2-error	0.02503314103	0.01391156115	0.01613553221	0.0131289066

Table 3.4: Computed errors for Example 3.4 for different values of N.

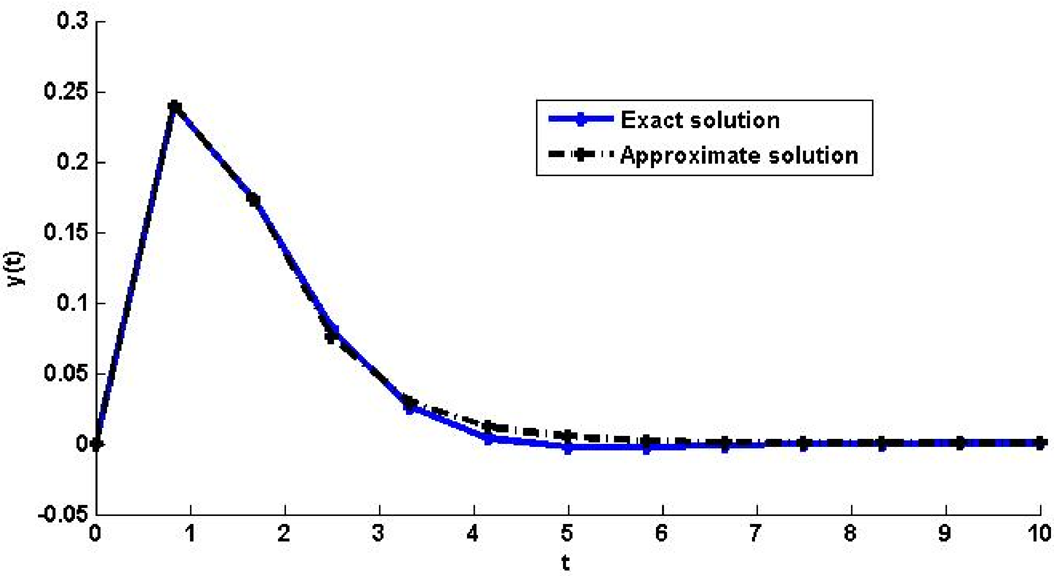

Figure 3.8: Exact and approximate solutions for Example 3.4 for $N=12$ and $\omega=0.7$.

N	$N=8$	$N=12$	$N=8$	$N=12$
ω/T	ω=0.7, $T=20$	ω=0.5, $T=20$	ω=0.01, $T=20$	ω=0.01, $T=20$
t_1	8.94467×10^{-19}	4.33681×10^{-19}	1.10114×10^{-20}	1.01644×10^{-20}
t_2	0.00417482	0.00292087	0.00856975	0.000313783
t_3	0.00901411	0.0369009	0.000957104	0.00365079
t_4	0.0010228	0.00695779	0.0000681329	0.0000911203
t_5	0.0000151453	0.00042269	0.00000448508	0.000164743
t_6	0.00000136351	0.000075663	2.75738×10^{-7}	0.0000276278
t_7	5.7213×10^{-7}	0.0000188713	1.50598×10^{-8}	0.00000448291
t_8	3.26863×10^{-8}	9.35049×10^{-7}	6.167331×10^{-10}	7.05285×10^{-7}
t_9	3.16655×10^{-25}	0.00000148702	2.01948×10^{-26}	1.06452×10^{-7}
t_{10}		4.53352×10^{-7}		1.50597×10^{-8}
t_{11}		8.26977×10^{-8}		1.89355×10^{-9}
t_{12}		$8.7223\times10-9$		1.78546×10^{-10}
t_{13}		2.58494×10^{-26}		1.85793×10^{-26}
L^2-error	0.00998647485	0.03766704855	0.008623297953	0.003779545791

Table 3.5: Computed errors for Example 3.4 for different values of N.

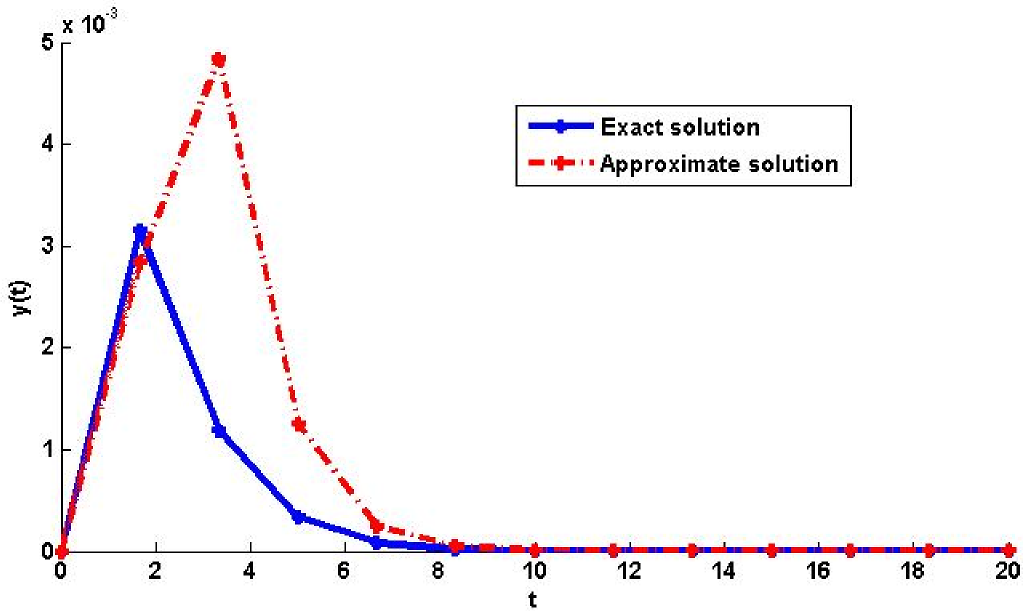

Figure 3.9: Exact and approximate solutions for Example 3.4 for $N=12$ and $\omega=0.01$.

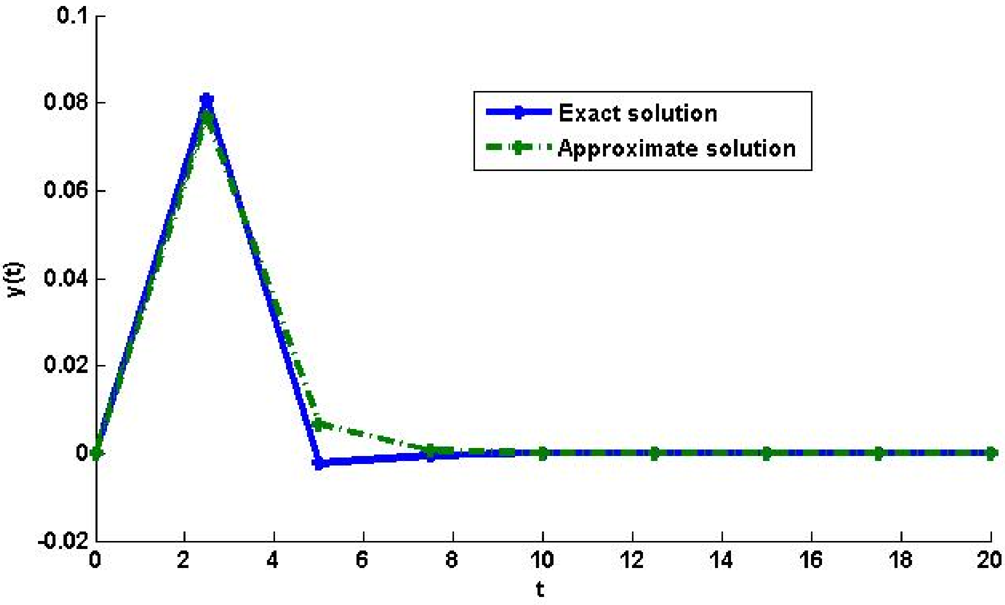

Figure 3.10: Exact and approximate solutions for Example 3.4 for $N = 8$ and $\omega = 0.7$.

with $\mu_1(t) = -e^{-0.5t} sin(0.5t)$, $\mu_2(t) = -2e^{-0.75t} cos(0.5t) sin(0.25t)$. *Such that the exact solution of this problem is given by*

$$y(t) = e^{(-t)} cos(t).$$

The numerical results are given in table 3.6 and figures 3.11-3.12, where $t(i) = (i-1) \times \frac{T}{N}, \quad i = 1 : N+1.$

The numerical errors by taking as collocation points Chebyshev nodes $t^{'}(i) = \frac{T(1+x_i)}{2}$, $i = 1 : N+1$, where $x_i = -cos\left(\frac{(i-1)\pi}{N}\right), \quad i = 1 : N+1$, are summarized in tables 3.7-3.8. Tables 3.9-3.10 presents, the numerical errors at equidistant points and using Chebyshev nodes.

3.2 Analysis of the numerical tests

1- This algorithm concerned both decaying and a non-decaying behavior of solutions.

2- The numerical and exact solutions matches when T is sufficiently large.

3- The algorithm is simple to implement and confirm the validity of the proposed technique.

4- The results of the error computations is highly effective as N increases.

N	$N=7$	$N=10$	$N=12$	$N=15$
T	$T=1$	$T=1$	$T=1$	$T=1$
t_1	1.46937×10^{-39}	1.76324×10^{-38}	8.22846×10^{-38}	1.88079×10^{-37}
t_2	0.0000490834	4.32524×10^{-7}	1.39862×10^{-8}	5.36602×10^{-11}
t_3	0.0000289848	2.93105×10^{-7}	1.12086×10^{-8}	4.91127×10^{-11}
t_4	0.000111252	3.43555×10^{-7}	1.05938×10^{-8}	4.09273×10^{-11}
t_5	0.000406026	8.83404×10^{-7}	1.37993×10^{-8}	4.00178×10^{-11}
t_6	0.000719877	0.00000409216	5.16302×10^{-8}	5.18412×10^{-11}
t_7	0.000690629	0.0000136831	2.42293×10^{-7}	1.80762×10^{-10}
t_8	2.84217×10^{-14}	0.0000301406	8.4269×10^{-7}	8.83119×10^{-10}
t_9		0.0000458162	0.0000021348	3.35331×10^{-9}
t_{10}		0.000043598	0.00000407024	9.42805×10^{-9}
t_{11}		4.26326×10^{-14}	0.00000581206	1.97776×10^{-8}
t_{12}			0.00000543478	3.03995×10^{-8}
t_{13}			2.84217×10^{-14}	3.04403×10^{-8}
t_{14}				7.97962×10^{-9}
t_{15}				2.76841×10^{-8}
t_{16}				6.75016×10^{-14}
L^2-error	0.00108428504	0.00007150883173	0.000009231096198	0.00000005632875356

Table 3.6: Computed errors for Example 3.5 for different values of N.

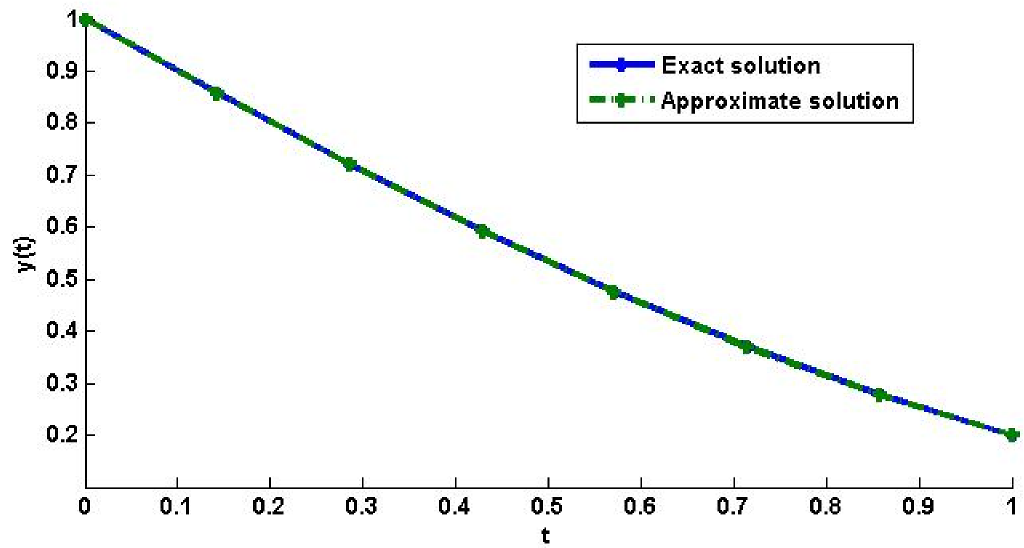

Figure 3.11: Exact and approximate solutions for Example 3.5 for $N=7$.

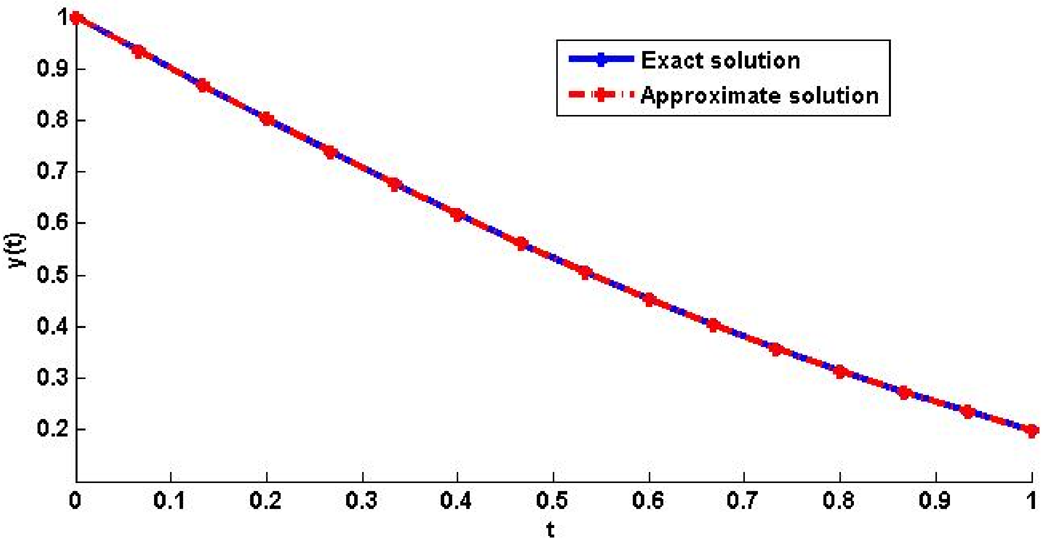

Figure 3.12: Exact and approximate solutions for Example 3.5 for $N = 15$.

nbr of Exp	Exp 3.1	Exp 3.2	Exp 3.3
N	$N = 12$	$N = 15$	$N = 12$
T	$T = 1$	$T = 1$	$T = 1$
t_1'	1.16233×10^{-9}	3.07336×10^{-8}	4.55504×10^{-38}
t_2'	0.000871705	0.000003335	0.0000497081
t_3'	0.00253447	0.0000517446	0.000203965
t_4'	0.00281722	0.000252383	0.000469649
t_5'	0.0000734662	0.000757421	0.000835102
t_6'	0.00612188	0.00172937	0.00124937
t_7'	0.0132011	0.00329748	0.00161157
t_8'	0.0181363	0.00548512	0.00178425
t_9'	0.0185545	0.00806245	0.00165424
t_{10}'	0.0143765	0.0103878	0.00122613
t_{11}'	0.00785273	0.0115101	0.000657067
t_{12}'	0.00220168	0.0106811	0.000182851
t_{13}'	2.59543×10^{-11}	0.00794332	2.08278×10^{-13}
t_{14}'		0.00428164	
t_{15}'		0.00119825	
t_{16}'			
L^2-error	0.03425294186	0.02338661595	0.003606768627

Table 3.7: Computed errors at Chebyshev nodes and by using Chebyshev collocation points.

nbr of Exp	Exp 3.3	Exp 3.4	Exp 3.5
N	$N = 12$	$N = 12$	$N = 15$
T	$T = 10, \omega = 1$	$T = 20, \omega = 0.01$	$T = 1$
$t_1^{'}$	6.93889×10^{-18}	1.69195×10^{-19}	$1.88079096 \times 10^{-37}$
$t_2^{'}$	0.0000478324	0.00012126	$1.45519152 \times 10^{-11}$
$t_3^{'}$	0.0000572711	0.00185952	$3.6379788 \times 10^{-12}$
$t_4^{'}$	0.00133589	0.00346272	$7.27595761 \times 10^{-12}$
$t_5^{'}$	0.00563831	0.000886733	0
$t_6^{'}$	0.0101551	0.0000743214	0
$t_7^{'}$	0.003642	0.00000448171	$2.36468622 \times 10^{-11}$
$t_8^{'}$	0.000961084	2.49459×10^{-7}	$6.32326191 \times 10^{-10}$
$t_9^{'}$	0.000814251	1.50597×10^{-8}	5.6634235×10^{-9}
$t_{10}^{'}$	0.000258749	1.10999×10^{-9}	$2.38007942 \times 10^{-8}$
$t_{11}^{'}$	0.0000565702	1.03469×10^{-10}	$5.04429635 \times 10^{-8}$
$t_{12}^{'}$	0.0000080736	9.68105×10^{-12}	$5.47944694 \times 10^{-8}$
$t_{13}^{'}$	5.18808×10^{-12}	$1.53481 \times^{-26}$	$2.38859456 \times 10^{-8}$
$t_{14}^{'}$			$6.48554987 \times 10^{-9}$
$t_{15}^{'}$			$7.25560767 \times 10^{-9}$
$t_{16}^{'}$			$4.26325641 \times 10^{-14}$
L^2-error	0.01231371843	0.004031722666	$8.252934899 \times 10^{-8}$

Table 3.8: Computed errors at Chebyshev nodes and by using Chebyshev collocation points.

nbr of Exp	Exp 3.1	Exp 3.2	Exp 3.3
N	$N = 12$	$N = 15$	$N = 12$
T	$T = 1$	$T = 1$	$T = 1$
t_1	1.16233×10^{-19}	3.07336×10^{-8}	4.55504×10^{-38}
t_2	0.00281242	0.000141228	0.000256854
t_3	0.00250149	0.000564805	0.000540022
t_4	0.0000734669	0.0012707	0.000835102
t_5	0.00408042	0.00225815	0.0011258
t_6	0.00871511	0.00352191	0.00139267
t_7	0.0132011	0.0050406	0.00161157
t_8	0.0167931	0.00675465	0.00175258
t_9	0.0187887	0.00853633	0.00177997
t_{10}	0.0185545	0.0101598	0.00165424
t_{11}	0.0155639	0.0112802	0.00133625
t_{12}	0.00944083	0.01143	0.000792425
t_{13}	2.59543×10^{-11}	0.010035	2.08278×10^{-13}
t_{14}		0.00644882	
t_{15}		1.51623×10^{-9}	
L^2-error	0.0398961124	0.02582159327	0.004271883455

Table 3.9: Computed errors at equidistant nodes and by using Chebyshev collocation points.

nbr of Exp	Exp 3.3	Exp 3.4	Exp 3.5
N	$N = 12$	$N = 12$	$N = 15$
T	$T = 20, \omega = 0.01$	$T = 10, \omega = 1$	$T = 1$
t_1	1.69195×10^{-19}	6.93889×10^{-18}	$1.88079096 \times 10^{-37}$
t_2	0.0011137	0.0050486	0
t_3	0.00315307	0.00420489	0
t_4	0.000886733	0.00563831	$3.6379788 \times 10^{-12}$
t_5	0.000163818	0.00813501	$3.18323145 \times 10^{-12}$
t_6	0.0000275944	0.00912875	$3.72892827 \times 10^{-11}$
t_7	0.00000448171	0.003642	$3.83465703 \times 10^{-10}$
t_8	7.05242×10^{-7}	0.000158321	$2.08956407 \times 10^{-9}$
t_9	1.0645×10^{-7}	0.00113839	$7.79607489 \times 10^{-9}$
t_{10}	1.50597×10^{-8}	0.000814251	$2.08819983 \times 10^{-8}$
t_{11}	1.89354×10^{-9}	0.000343137	$4.08145339 \times 10^{-8}$
t_{12}	1.78546×10^{-10}	0.0000833761	$5.67365532 \times 10^{-8}$
t_{13}	1.53481×10^{-26}	5.18808×10^{-21}	$4.83798388 \times 10^{-8}$
t_{14}			$8.44158876 \times 10^{-9}$
t_{15}			$4.26325641 \times 10^{-14}$
t_{16}			$2.55560024 \times 10^{-7}$
L^2-error	0.003463533665	0.0154869415	$2.703867023 \times 10^{-7}$

Table 3.10: Computed errors at equidistant nodes by using Chebyshev collocation points.

5- In all the examples, it is confirmed that the ortho-exponential polynomial solution method yields quite acceptable results and the accuracy of the solution can significantly be increased by error correction (changing collocation points).

6- In general, using Chebyshev nodes as collocation points gives a better accuracy then that obtained by equidistant nodes.

Conclusion

The aim of this dissertation, is to introduce a new matrix method based on ortho-exponential polynomials and collocation method for solving linear integro-delay equations of high order.

The advantages of using these polynomials is that they are effective in approximating the given function in the half line, and it is fast and simple to implement.

The obtained errors are given by using residual function. The proposed technique is applied to five different tests, in order to see the applicability and validity of the technique.

In all proposed experiments, it is affirmed that the ortho-exponential polynomials solution yields to acceptable and accurate results.

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