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Time dependent Non-Hermitian Quantum Systems: Non-Hermitian Magnetic Effect

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DEDICATION

I dedicate this work as a mark of respect and gratitude to my very dear parents, who have worked so hard to ensure the success of my studies. I express my deep gratitude and sincere thanks for their courage. I owe what I am today to your love, patience and countless sacrifices.

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Contents

Introduction

The quantum mechanics represents an important part of studying and describing the fundamental phenomena of physical systems at the atomic and subatomic scale. The microscopic behavior of atomic objects, which make up matter, gives predictions in which these objects behave and led to the formulation of quantum theory by Schrödinger, Heisenberg, and Dirac. Quantum Mechanics is based on a set of axioms [1, 2] among them we mention: i) the inner products of state vectors have a positive norm, ii) evolution is unitary, iii) the Hamiltonian ($h = h^+$) of a system must be Hermitian operator.

The hermiticity of a Hamiltonian is a sufficient condition for the reality of the energy spectrum and needed for the unitarity of evolution. In 1998 Bender and Boettcher worked on the non-Hermitian Hamiltonians ($H \neq H^+$) and interpreted the reality of the spectrum as being due to its PT -symmetry which comes from the invariance of PT-symmetric Hamiltonians under both parity and time reversal transformation $HPT = PTH$. The more general than the PT concept, is pseudo-Hermiticity for a time-independent Hamiltonian ($H = H_0$) which has been introduced by Mostafazadeh in 2002 [24].

For the time dependent non-Hermitian systems, described by non-Hermitian Hamiltonians ($H(t) \neq H^+(t)$) adapting the invariant operator approach to this setting. Our key assumption is that the stationary theory sumarized in pseudo-Hermiticity concept remains valid for the time-dependent systems in which invariant operator $I(t)$ generalized into the pseudo-Hermitian invariant $I^{PH}(t)$. However for time dependent non-Hermitian Hamiltonians it has been found a problem for the conservation of the propability of presence ($\langle\psi(t)\,|\psi(t)\rangle = f(t)$), and the mean values of observables. To solve this problem, it was necessary to introduce the renormalization of the probability by introducing the metric operator so called η.

We start the first chapter by introducing the notions of PT-symmetry, and pseudo-Hermiticity, which are the basic and the fundamental concepts in the non-Hermitian theory. In the seconde chapter, we provide the method of pseudo-Hermitian invariants to deduce the solution of time-dependent non-Hermitian systems, and then we suggest non-Hermitian perturbation method as preliminary approach in time dependent non-Hermitian quantum systems. The third chapter is devoted to study a spin interacting with a complex external magnetic field, applying the pseudo Hermitian invariant method, and non hermitian perturbation approach.

Chapter 1

Non Hermitian Hamiltonians

In 1959 the concept of a non-Hermitian Hamiltonians was one of the most important events in the literature, where Wu Tai Tsun published an article [4] aimed to calculate the ground-state energy of "Bose spheres". According to the paper, a serious problem in this type of calculation was that the ground-state energy "diverges". In ordre to solve this problem, it has been used a non-Hermitian and non-diagonalizable Hamiltonian. What's impressive is that this Hamiltonian has real eigenvalues, although the research didn't explain this representation.

In 1967, Jack Wong published a paper on non-Hermitian Hamiltonians [5], and showed that closed-system are described by Hermitian Hamiltonians, but when the external interaction is considered, the Hamiltonian loses its Hermiticity. These Hamiltonians may have part of the discrete spectrum. Furthermore, complex eigenvalues seem acceptable but there is not an explanation for their being physically reasonable. Until 1998, Bender and Boettcher have introduced the PT-symetry theory as first step in non Hermitian Quantum Systems, and in 2002, Ali Mostafazadeh has well formulated the theory of non Hermitian Hamiltonians. And in the recent decades the theory of non-Hermitian Hamiltonians has been expanded to describe various physically observable phenomena in optical [6], photonic [7], and condensed matter systems [8, 9, 10, 11].

1.1 PT- symmetry theory

The PT-symmetry theory, which is an alternative formulation of conventional quantum theory, has been principally developed by Bender and his Collaborators. Where explicitly studied a class of non-Hermitian Hamiltonians, and demonstrated that the one-dimensional non-Hermitian Hamiltonian spectrum is real, positive, and discrete. It is also invariant under the PT-symmetry transformation. the PT-symmetric quantum mechanics has allowed physicists to study several phenomena in different areas of physics.

The notion of PT-symmetry was first introduced in 1998 by Bender et al [12], When they studied a class of Hamiltonians in quantum mechanics:

$$H = p^2 + x^2 - (ix)^{\epsilon} \tag{1.1}$$

and demonstrated that for $\epsilon \geq 0$, the non-Hermitian Hamiltonian (1.1) has a real and positive spectrum. For negative values of ϵ, the spectrum is complex. Thus the PT-symmetry is the key reason for the reality of the spectrum.

A non-Hermitian Hamiltonian H is said to be PT-symmetric, if it is invariant by the PT transformation, i.e;

$$H = H^{PT} = (PT)\, H\, (PT) \tag{1.2}$$

without violating any of the physical axioms of quantum mechanics. If a Hamiltonian H is PT-symmetric, then it commutes with the PT operator.

$$[H; PT] = 0; \tag{1.3}$$

where P and T are respectively the parity and time reversal operators. These operators are defined by their effect on the position x and the momentum p operators , as follows

$$PxP = -x \tag{1.4}$$

and

$$PpP = -p \tag{1.5}$$

We note that the effect of the linear operator P is to change the signs of the operators x and p.

$$TxT = x \tag{1.6}$$

and

$$TpT = -p \tag{1.7}$$

$$TiT = -i \tag{1.8}$$

While the antilinear operator T only effects the sign of the operator p by changing the sign of the pure imaginary complex number i. Where x and p operators verify the commutation relation;

$$[x, p] = i\hbar$$

Furthermore, since P and T are reflection operators, their squares equal the unity operator

$$P^2 = T^2 = 1 \tag{1.9}$$

but are not equal ($P \neq T$). Besides, P and T commute:

$$[P, T] = 0 \tag{1.10}$$

1.1.1 Eigenvalues of PT-symmetric Hamiltonians

In fact, to construct a quantum theory of PT-symmetric Hamiltonians, we require that the symmetry be unbroken. With this condition, we can demonstrate the reality of the eigenvalues of PT-symmetric Hamiltonians.

To demonstrate this, one first needs to define the eigenfunctions of the Hamiltonian, so we can write:

$$H\left|\psi_n\right\rangle = E_n\left|\psi_n\right\rangle \tag{1.11}$$

and the eigenvalue equation of the PT operator

$$PT\left|\psi_n\right\rangle = \lambda_n\left|\psi_n\right\rangle \tag{1.12}$$

where E_n and λ_n are the eigenvalues corresponding to H and PT , respectively. Then,

$$(PT)^2 |\psi_n\rangle = |\lambda_n|^2 |\psi_n\rangle \tag{1.13}$$

using

$$(PT)^2 = 1 \tag{1.14}$$

leads to

$$|\lambda_n|^2 = 1 \tag{1.15}$$

Since H and PT commute, we can write

$$PTH = HPT \tag{1.16}$$

By multiplying both sides of the right hand side of this equation by PT, and for $(PT)^2 = 1$ we obtain:

$$PTHPT = H \tag{1.17}$$

The relation (1.2), allows to write

$$H |\psi_n\rangle = PTHPT |\psi_n\rangle = PTH\lambda_n |\psi_n\rangle = PTE_n\lambda_n |\psi_n\rangle \tag{1.18}$$

Using the property $TE_n\lambda_n = E_n^*\lambda_n^*T$, with the equations; (1.12) and (1.15), to find

$$H |\psi_n\rangle = E_n^* |\psi_n\rangle \tag{1.19}$$

From equations (1.11) and (1.19) we obtain:

$$E_n |\psi_n\rangle = E_n^* |\psi_n\rangle \tag{1.20}$$

Indeed, this gives us a confirmation that the eigenvalues E_n of the PT-symmetric Hamiltonian H are real.

1.1.2 PT-inner product

The question that arises now is whether the eigenvector norms of the Hermitian Hamiltonian in Hilbert space are positive. Furthermore, it is also necessary to conserve the inner product of any two eigenvectors in Hilbert space, which is an essential property for quantum theory to be valid. Indeed, both these two requirements are satisfied in the conventional Quantum theory with Hermitian Hamiltonians. The first allows us to interpret the state norm as a probability that must be positive and definite. However, in non-Hermitian quantum theory, to validate the orthogonality of the eigenvectors of a PT-symmetric Hamiltonian, Carl Bender [13] first introduced an inner product called the "PT-inner product" associated with the PT-symmetric Hamiltonian, defined by two arbitrary functions $f(x), g(x)$

$$(f, g)_{PT} = \int dx \, [PTf(x)] \, g(x) \tag{1.21}$$

where

$$PTf(x) = f^*(-x) \tag{1.22}$$

The inner product has the advantage that the PT norm is the quantity that is conserved. However, this approach assumes negative norms for some eigenstates of PT-symmetric Hamiltonians.

If ψ_m and ψ_n represents the eigenfunctions of H and is orthogonal to $n \neq m$, the PT-inner product becomes

$$\langle \psi_m, \psi_n \rangle_{PT} = \int dx[PT\psi_m(x)]\psi_n(x) = \int dx\psi_m^*(-x)\psi_n(x) = (-1)^n \delta_{mn} \tag{1.23}$$

If $m = n$, the PT-norms of these eigenfunctions are not always positive [37]:

$$\langle \psi_n, \psi_n \rangle_{PT} = \int dx\psi_n^*(-x)\psi_n(x) = (-1)^n \tag{1.24}$$

The negative norm $(-1)^n$ means; the relation (1.21) defining the inner product is insufficient to formulate a valid quantum theory, this negates the Hermitian Quantum Mechanics axiom.

So, it is necessary to construct a new inner product where the norm is positive, this led Bender to construct a new inner product with a positive norm; the "CPT-inner product".

1.1.3 C operator and CPT-inner product

To solve the problem of the negative norm, Bender [13] introduced another symmetry generated by a new linear operator noted C. The properties of this operator are almost identical (mathematically similar) to those of the charge conjugation operator in quantum field theory. The linear operator C is represented in the coordinate-space by the sum of the Hamiltonian's eigenfunctions

$$C(x,y) = \sum_n \psi_n(x)\psi_n(y) \tag{1.25}$$

We can verify that the square of C is equal to unity [13]

$$\int dz C(x,y)C(y,z) = \delta(x-y) \tag{1.26}$$

The operator C verifies the following commutation relations

$$[C,H] = [C,PT] = 0 \tag{1.27}$$

but not either P or T separately

$$[C,T] \neq 0 \tag{1.28}$$

$$[C,P] \neq 0 \tag{1.29}$$

consequently

$$C^2 = 1 \tag{1.30}$$

One observes that the eigenvalues of the operator C are ± 1, the act of C on the eigenfunctions ψ_n of H is given by

$$\begin{aligned} C\psi_n(x) &= \int dy C(x,y)\psi_n(y) \\ &= \sum_m \psi_m(x) \int dy \psi_m(y)\psi_n(y) = (-1)^n \psi_n(x) \end{aligned} \tag{1.31}$$

In coordinate space, The linear operator P is defined in terms of the eigenfunctions $\psi_n(x)$ of H by [13]

$$P(x,y) = \sum_n (-1)^n \psi_n(x)\psi_n(-y) = \delta(x+y) \tag{1.32}$$

As the square of the two operators P and C is equal to unity

$$P^2 = C^2 = 1 \tag{1.33}$$

but P and T are not identical

$$P \neq C \tag{1.34}$$

Indeed the operator P is real, while C is complex. Specifically, in the position representation;

$$(CP)(x,y) = \sum_n \psi_n(x)\psi_n(-y) \tag{1.35}$$

$$(PC)(x,y) = \sum_n \psi_n(-x)\psi_n(y) \tag{1.36}$$

therefore it's clear that

$$(CP) = (PC)^* \tag{1.37}$$

After introduced the properties of the operator C, we can then define a new inner product called "CPT-inner product" [14, 15]: by

$$\langle \psi_m, \psi_n \rangle_{CPT} = \int dx \, [CPT\psi_m(x)] \, \psi_n(x) \tag{1.38}$$

where

$$CPT\psi_m(x) = \int dy CPT(x,y)\psi_m^*(-y) \tag{1.39}$$

This CPT-inner product is positive defined, and the eigenfunctions of H are orthonormal.

$$\langle \psi_m, \psi_n \rangle_{CPT} = \int dx \, [CPT\psi_m(x)] \, \psi_n(x) = \delta_{mn} \tag{1.40}$$

So this CPT-inner product satisfies all the conditions for the quantum theory defined by H to be unitary. In this case, the fermeture relation, is given by

$$\sum_n \psi_n(x)\left[CPT\psi_n(y)\right] = \delta(x-y) \tag{1.41}$$

1.1.4 Application: Harmonic oscillator interacts with complex electrical Field

A harmonic oscillator; $\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2$, placed in complex uniform electrical field, can be described by the following hamiltonian;

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + iFx, \tag{1.42}$$

where m is the masse, ω is the frequancy, and $F = qE$ is the electrical force. We note that H is non Hermitian, but $\mathcal{PT}$ symetric, i.e.

$$H^{\mathcal{PT}} = (\mathcal{PT})\,H\,(\mathcal{PT}) = (\mathcal{PT})\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + iFx\right](\mathcal{PT}) = H. \tag{1.43}$$

The eigenvalue equation associated with H is written

$$H\left|\psi_n\right\rangle = E_n\left|\psi_n\right\rangle, \tag{1.44}$$

We perform the transformation D

$$D = \exp\left(\frac{F}{m\omega^2}p\right), \tag{1.45}$$

The equation (1.44) becomes

$$DHD^{-1}\left|\varphi_n\right\rangle = E_n\left|\varphi_n\right\rangle \quad \text{with} \quad \left|\varphi_n\right\rangle = D\left|\psi_n\right\rangle,$$

Where

$$DHD^{-1} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{1}{2}\frac{F^2}{m\omega^2}. \tag{1.46}$$

The transformed hamiltonian is hermitian, than its eigenvalues are real and given by

$$E_n = \omega\left(n + \frac{1}{2}\right) + \frac{1}{2}\frac{F^2}{m\omega^2} \tag{1.47}$$

and the corresponding eigenfunctions are

$$\varphi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega^2}{2}x^2\right) H_n\left(\sqrt{m\omega}x\right), \tag{1.48}$$

Where H_n are Hermite's polynomials , then the eigenfunctions $|\psi_n\rangle$ have the form

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{F}{m\omega^2}p\right) \exp\left(-\frac{m\omega^2}{2}x^2\right) H_n\left(\sqrt{m\omega}x\right), \tag{1.49}$$

And are orthonormal in sens of $\mathcal{CPT}$ scalar product

$$\begin{aligned} \langle\psi_n(x)|\, \psi_m(x)\rangle_{\mathcal{CPT}} &= \int dx \left[\mathcal{CPT}\psi_n(x)\right] \psi_m(x) \\ &= \int dx \psi_m(x) \exp\left(\frac{2F}{m\omega^2}p\right) \psi_n^*(x) = \delta_{mn}, \end{aligned} \tag{1.50}$$

Where $\mathcal{CP} = \exp\left(\frac{2F}{m\omega^2}p\right)$.

1.2 Pseudo-Hermiticity

The concept of pseudo-Hermiticity was introduced in the 40s by Dirac and Pauli [17, 18, 19, 20], and later discussed by Lee and Sudarshan [21, 22], who were trying to solve various problems in several domains of physics that arise in quantization in electrodynamics and many quantum field theories, where the negative-norm states appear as a consequence of renormalization.

In 1992, Scholtz [23] discussed another notion related to pseudo-Hermiticity is quasi-Hermiticity. Who showed how to construct a similar transformation from Hermitian operators to the corresponding quasi-Hermitian operators.

Recently, in 2002, Mostafazadeh published three articles [24, 25, 26], where presented an alternative to conventional quantum mechanics for non-Hermitian Hamiltonians with real spectraum. This theory is called "pseudo-Hermiticity". Mostafazadeh has also demonstrated the existence of PT-symmetric Hamiltonians whose spectrum is not real. In the other hand the PT-symmetry is not sufficient or necessary to guarantee the reality of the spectrum.

1.2.1 Pseudo-Hermitian Hamiltonians

As mentioned in [24], an operator H that acts in the Hilbert space, is said to be pseudo-Hermitian, if there exists a linear, Hermitian and invertible operator η such that

$$H^{+} = \eta H \eta^{-1} \tag{1.51}$$

where η called "metric operator" and H^{+} is the adjoint operator of H.

If we make a particular choice of η, we say that H is η-pseudo-Hermitian. The condition (1.51) can be expressed in the form:

$$H^{\#} = H \tag{1.52}$$

where

$$H^{\#} = \eta^{-1} H^{+} \eta \tag{1.53}$$

is the pseudo adjoint of H [24]:

Note that the choice of $\eta = 1$ reduces equation (1.51) to ordinary Hermeticity, so pseudo-Hermeticity is a generalization of Hermeticity. This condition (1.51) reduces to PT-symmetry when $\eta = P$.

Pseudo-Hermitian conjugation ($^{\#}$) has the same properties as Hermitian conjugation ($^{+}$), we mention a certain number of them;

$$1^{\#} = 1 \tag{1.54}$$

$$\left(A^{\#}\right)^{\#} = A \tag{1.55}$$

$$(AB)^{\#} = B^{\#} A^{\#} \tag{1.56}$$

$$(\alpha A + \beta B)^{\#} = \alpha^{*} A^{\#} + \beta^{*} B^{\#} \tag{1.57}$$

where A and B are linear operators, 1 is the identity operator, α^{*} and $\beta^{*} \in \mathbb{C}$, α^{*} and β^{*} are the complex conjugates of α and β respectively.

In general, the pseudo-Hermitian conjugate of any expression is obtained by:

1.Reversing the order of terms.

2.Transforming: operators into their pseudo-adjoints, kets into bras and inversely, and numbers into their complex conjugates.

The Pseudo-hermiticity allows us to pass from a Hermitian Hamiltonian to an equivalent pseudo-Hermitian Hamiltonian. In other words, every pseudo-Hermitian Hamiltonian H has an equivalent Hermitian Hamiltonian h, and they are linked by the relation

$$h = \rho H \rho^{-1} \tag{1.58}$$

Since h is Hermitian, that is to say

$$h^{+} = (\rho H \rho^{-1})^{+} \tag{1.59}$$

with ρ : a linear bounded and invertible operator. Also, it is easy to verify that;

$$\eta = \rho^{+} \rho \tag{1.60}$$

$$\eta^{-1} = \rho^{-1} (\rho^{+})^{-1} \tag{1.61}$$

The equations (1.58),(1.59), show that the Hamiltonian h is Hermitian, so its eigenvalues E_n are real and consequently the eigenvalues of H are also real.

The corresponding eigenvalue equations are then

$$H \left|\psi_n\right\rangle = E_n \left|\psi_n\right\rangle \tag{1.62}$$

and

$$h \left|\phi_n\right\rangle = E_n \left|\phi_n\right\rangle \tag{1.63}$$

It is important to note that the transformation ρ allows to pass from the eigenvectors of h to the eigenvectors of H

$$\left|\phi_n\right\rangle = \rho \left|\psi_n\right\rangle \tag{1.64}$$

The eigenvectors $\left|\phi_n\right\rangle$ form an orthonormal basis, i.e. The Hermitian Hamiltonian h preserves the ordinary inner-product. So we can write

$$\langle \phi_m | \phi_n \rangle = \delta_{mn} \tag{1.65}$$

Since the pseudo-Hermitian Hamiltonian H does not preserve the standard inner product, Mustafazadah introduced the pseudo-inner product defined by [24, 25, 26]:

$$\langle \psi_m | \psi_n \rangle_\eta = \langle \psi_m | \eta | \psi_n \rangle \tag{1.66}$$

Using the transformation (1.64) in (1.65), we obtain

$$\langle \phi_m | \phi_n \rangle = \langle \psi_m | \rho^+ \rho | \psi_n \rangle = \langle \psi_m | \eta | \psi_n \rangle = \langle \psi_m | \psi_n \rangle_\eta = \delta_{mn} \tag{1.67}$$

The last relation defines the η-inner product, also called the pseudo-inner product.

1.2.2 Application

Let Consider the previous Hamiltonian (1.42);

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 + iFx. \tag{1.68}$$

In order to determine the eigenvalues and the eigenstates (1.62) of H using the notion of pseudo-hermiticity, we introduce the metric operator η in the form

$$\eta = \exp\left(\frac{2F}{m\omega^2} p \right), \tag{1.69}$$

it is easy to verify that η connects the hamiltonian H (1.68) to its adjoint H^+ by the pseudo-hermiticity relation (1.51) and that $\rho = \eta^{1/2} = \exp\left(\frac{F}{m\omega^2} p\right)$ gives Hermitian equivalent (1.58)

$$h = \exp\left(\frac{F}{m\omega^2} p \right) H \exp\left(-\frac{F}{m\omega^2} p \right) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} \frac{F^2}{m\omega^2}, \tag{1.70}$$

whose eigenvalues are

$$E_n = \omega \left(n + \frac{1}{2} \right) + \frac{1}{2} \frac{F^2}{m\omega^2} \tag{1.71}$$

and the corresponding eigenfunctions are

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi} \right)^{\frac{1}{4}} \exp\left(-\frac{m\omega^2}{2} x^2 \right) H_n\left(\sqrt{m\omega} x \right). \tag{1.72}$$

we obtain the eigenfunctions associated with H (1.68) by the inverse tranformation $\psi_n(x) = \rho^{-1}\varphi_n(x)$

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{F}{m\omega^2}p\right) \exp\left(-\frac{m\omega^2}{2}x^2\right) H_n\left(\sqrt{m\omega}x\right), \tag{1.73}$$

Now, let calculate the pseudo-inner product

$$\langle\psi_m|\,\eta\,|\psi_n\rangle = \int dx\psi_m^*(x)\eta\psi_n(x), \tag{1.74}$$

By changing the variable $x = \frac{1}{\sqrt{m\omega}}y$, we find that the eigenfunctions ψ_n are $\eta-$orthonormal

$$\langle\psi_m|\,\eta\,|\psi_n\rangle = \delta_{mn}. \tag{1.75}$$

Chapter 2

Time dependant non-Hermitian Quantum Systems

In quantum mechanics, the dynamics of closed systems governed by Schrodinger's equation [27]:

$$H(t)\left|\Psi(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle \tag{2.1}$$

where $H(t)$ is the Hamiltonian describes the system, and $\left|\Psi(t)\right\rangle$, represents the wave function. The resolution of this equation gives the state of the system $\left|\Psi(t)\right\rangle$,which represents the time dependent state of evolution.

$$\left|\Psi(t)\right\rangle = U(t)\left|\Psi(0)\right\rangle \tag{2.2}$$

with $U(t)$ represents the evolution operator and $\left|\Psi(0)\right\rangle$ is the initial state of the system.

In general, it is difficult to find the analytic solution of time dependent Schrodinger's equation, especially when it involves with explicitly time-dependent Hamiltonians. So only a few exact solutions of Schrodinger's equation have been found so far. In certain cases, to find the $\left|\Psi(t)\right\rangle$ state one seek to approximate approaches, which are more used in fields of applied physics, such as solid-state physics, plasma physics, and classical electromagnetic fields. The choice of a particular method generally depends on the shape of the potential and the shape of the wave function required. Some of these methods include; perturbation theory, sudden approximation and adiabatic approximation. In certain cases we can use the exact method of the invariant approach to estimate the quantum state of the system. Due to their importance, we will examine them in more detail in this chapter.

2.1 pseudo Hermitian invariants Method

To study time dependent Quantum systems described by Hermitian Hamiltonians, Lewis and Riesenfeld (1969) proposed one of the most effective methods that gives an exact solutions of the Schrodinger equation, is the invariants theory [28]. The basic idea of this theory consists to derive a simple relation between the eigenvalues of the Hermitian invariant and the solution of the Schrodinger equation. This method was first applied for a harmonic oscillator with time-dependent frequency, and a charged particle in electromagnetic field. This theory has been widely used to treat many theoretical problems.

For the non-Hermitian senario we call pseudo-invariant theory [29, 30, 31]. It is generally applied to obtaine the solution of the Schrodinger equation as the function of the pseudo Hermitian invariant eigenstates multiplied by the phase. Thus, the problem is reduced to finding the explicit form of the pseudo Hermitian invariant operator and the phases associated with the evolution.

To explain this method in a simple form, we consider a system whose Hamiltonian $H(t)$, where the wave function describing the time evolution of this system obeys the Schrodinger equation;

$$H(t)\left|\psi(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle \tag{2.3}$$

where $H(t)$ is time-dependent non-hermetian Hamiltonian.

2.1.1 Pseudo Hermitian Invariant Operator

Given a non-Hermitian time-dependent Hamiltonian operator $H(t)$, it is possible to build another auxiliry time-dependent operator $I^{PH}(t)$, called a pseudo-invariant operator, which satisfies the invariance condition;

$$\frac{dI^{PH}(t)}{dt} = \frac{\partial I^{PH}(t)}{\partial t} - \frac{i}{\hbar}\left[I^{PH}(t), H(t)\right] = 0 \tag{2.4}$$

By applying (2.4) on the state vector $|\psi(t)\rangle$ solution of the Schrodinger equation (2.3), we obtain :

$$\frac{\partial I^{PH}(t)}{\partial t}|\psi(t)\rangle - \frac{i}{\hbar}\left[I^{PH}(t), H(t)\right]|\psi(t)\rangle = 0 \tag{2.5}$$

Where

$$i\hbar\frac{\partial I^{PH}(t)}{\partial t}|\psi(t)\rangle + I^{PH}(t)(H(t)|\psi(t)\rangle) - H(t)(I^{PH}(t)|\psi(t)\rangle) = 0 \tag{2.6}$$

According to (2.3)

$$i\hbar\frac{\partial I^{PH}(t)}{\partial t}|\psi(t)\rangle + I^{PH}(t)i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle - H(t)I^{PH}(t)|\psi(t)\rangle = 0 \tag{2.7}$$

Finally we get

$$i\hbar\frac{\partial}{\partial t}(I^{PH}|\psi(t)\rangle) = H(t)(I^{PH}|\psi(t)\rangle) \tag{2.8}$$

This means that the action of the pseudo-invariant operator on the Schrodinger state vector, produces another solution of the Schrodinger equation. This result is valid for any invariant.

Now let's present the spectral properties of the pseudo-Hermitian invariant operator $I^{PH}(t)$. The beginning of this section briefly describes the pseudo Invariant operator technique [30, 31]. In complete analogy to the time-independent scenario; (Chap.1. Sec.2), the Hermitian invariant operator $I^h(t)$ and the pseudo hermitian operator $I^{PH}(t)$ are related to each other by means of the similarity transformation $\rho(t)$, and can also be linked to its Hermitian conjugate $I^{PH+}(t)$ as

$$\rho(t)I^{PH}(t)\rho^{-1} \Longleftrightarrow I^{PH+}(t) = \eta(t)I^{PH}(t)\eta^{-1}(t) \tag{2.9}$$

Now suppose that this pseudo-Hermitian operator, explicitly time-dependent $I^{PH}(t)$, obeys the eigenvalue equation

$$I^{PH}(t)|\phi_n(t)\rangle = \lambda_n|\phi_n(t)\rangle \tag{2.10}$$

where λ_n represents time-independent eigenvalues, and $|\phi_n(t)\rangle$ represent the eigenstates of the pseudo Hermitian invariant operator.

The pseudo-hermiticity condition for $I^{PH}(t)$ is expressed as :

$$I^{PH+}(t) = \eta(t) I^{PH}(t) \eta^{-1}(t) \tag{2.11}$$

Using the time-dependent metric $\eta(t)$, the inner product (1.67) of the eigenstates $|\phi_n(t)\rangle$ is written in the form

$$\langle \phi_m(t)| \, \eta(t) \, |\phi_n(t)\rangle = \delta_{mn} \tag{2.12}$$

so, the eigenstate of $I^{PH}(t)$ and $I^{PH+}(t)$ verify the bi-orthogonality relation (1.67);

$$\left\langle \phi_m^{I^{PH+}}(t) \middle| \phi_n^{I^{PH}}(t) \right\rangle = \delta_{mn} \tag{2.13}$$

The eigenvalues λ_n of the operator $I^{PH}(t)$ are time-independent, and can be deduced by differentiating equation (2.10) as follows

$$\frac{\partial}{\partial t}(I^{PH}(t) \left| \phi_n^{I^{PH}}(t) \right\rangle) = \frac{\partial}{\partial t}(\lambda_n \left| \phi_n^{I^{PH}}(t) \right\rangle) \tag{2.14}$$

$$\frac{\partial I^{PH}}{\partial t} \left| \phi_n^{I^{PH}}(t) \right\rangle + I^{PH} \frac{\partial \left| \phi_n^{I^{PH}}(t) \right\rangle}{\partial t} = \frac{\partial \lambda_n}{\partial t} \left| \phi_n^{I^{PH}}(t) \right\rangle + \lambda_n \frac{\partial \left| \phi_n^{I^{PH}}(t) \right\rangle}{\partial t} \tag{2.15}$$

When multiplying the equation (2.15) by $\left\langle \phi_n^{I^{PH}}(t) \right| \eta(t)$, and using equation (2.4), we find

$$\frac{\partial \lambda_n}{\partial t} = \left\langle \phi_n^{I^{PH}}(t) \right| \eta(t) \frac{\partial I^{PH}}{\partial t} \left| \phi_n^{I^{PH}}(t) \right\rangle = 0 \tag{2.16}$$

which explicitly shown that the λ_n eigenvalues are constant.

The reality of the eigenvalues λ_n is guaranteed, since the two invariants $I^h(t)$ and $I^{PH}(t)$, which are Hermitian and non-Hermitian respectively, are related by a similarity transformation (2.9).

2.1.2 Solution of Schrodinger's equation

In order to find the connection between eigenstates of $I^{PH}(t)$ and the solutions of Schrodinger's equation (2.3), we first start by projecting equation (2.15) into $\left\langle \phi_m^{I^{PH}}(t) \right| \eta(t)$, and using equation (2.16), it follows that

$$i\hbar \left\langle \phi_m^{I^{PH}}(t) \right| \eta(t) \frac{\partial}{\partial t} \left| \phi_n^{I^{PH}}(t) \right\rangle = \left\langle \phi_m^{I^{PH}}(t) \right| \eta(t) H(t) \left| \phi_n^{I^{PH}}(t) \right\rangle \tag{2.17}$$

with $m \neq n$.

If the equation (2.17) is valid for $m = n$, then we deduce that $\left| \phi_n^{I^{PH}}(t) \right\rangle$ represents a particular solution of Schrodinger's equation. We can multiply $\left| \phi_n^{I^{PH}}(t) \right\rangle$ by a time-dependent phase factor, the new eigenstates $\left| \phi_n^{I^{PH}}(t) \right\rangle$ of $I^{PH}(t)$ become

$$|\psi_n(t)\rangle = e^{i\delta_n(t)} \left| \phi_n^{I^{PH}}(t) \right\rangle \tag{2.18}$$

The state $|\psi_n(t)\rangle$ obeys the Schrodinger equation (2.17), which is a particular solution to the Schrodinger equation.

This requirement allows us to find the first-order differential equation satisfied by the phase $\delta_n(t)$

$$\frac{d\delta_n(t)}{dt} = \left\langle \phi_n^{I^{PH}}(t) \right| \eta(t) \left[i\hbar \frac{\partial}{\partial t} - H(t) \right] \left| \phi_n^{I^{PH}}(t) \right\rangle \tag{2.19}$$

In equation (2.19), the first term on the right-hand side describes the geometric phase, while the second term represents the dynamic phase.

The general solution of Schrodinger's equation for time-dependent non-Hermitian Hamiltonian $H(t)$ is given by

$$|\psi(t)\rangle = \sum_n C_n e^{i\delta_n(t)} \left| \phi_n^{I^{PH}}(t) \right\rangle \tag{2.20}$$

where $C_n = \left\langle \phi_n^{I^{PH}}(0) \right| \eta(0) \left| \psi(0) \right\rangle$ are time-independent coefficients.

In order to preserve the quantum postulat; (conservation of probability of presence) of the state $|\psi(t)\rangle$, once use the notion of pseudo inner product mentioned in the equation (1.67); i.e. $\left\langle \phi_m^{I^{PH}}(t) \right| \eta(t) \left| \phi_n^{I^{PH}}(t) \right\rangle = \delta_{mn}$. And the phase $\delta_n(t)$ (2.19) is ensured that be real.

We have

$$\dot{\delta}_n^*(t) = -i\hbar \left\langle \frac{\partial}{\partial t} \phi_n^{I^{PH}}(t) \right| \eta(t) \left| \phi_n^{I^{PH}}(t) \right\rangle - \left\langle \phi_n^{I^{PH}}(t) \right| H^+(t) \eta(t) \left| \phi_n^{I^{PH}}(t) \right\rangle \tag{2.21}$$

On the other hand; the derivative over time of inner product (2.12) allows us to write

$$\left\langle \frac{\partial}{\partial t}\phi_n^{IPH}(t)\right| \eta(t) \left|\phi_n^{IPH}(t)\right\rangle + \left\langle \phi_n^{IPH}(t)\right| \dot{\eta}(t) \left|\phi_n^{IPH}(t)\right\rangle + \left\langle \phi_n^{IPH}(t)\right| \eta(t) \left|\frac{\partial}{\partial t}\phi_n^{IPH}(t)\right\rangle = 0 \quad (2.22)$$

then we write

$$\left\langle \frac{\partial}{\partial t}\phi_n^{IPH}(t)\right| \eta(t) \left|\phi_n^{IPH}(t)\right\rangle = -\left\langle \phi_n^{IPH}(t)\right| \eta(t)\eta^{-1}(t)\dot{\eta}(t) \left|\phi_n^{IPH}(t)\right\rangle - \left\langle \phi_n^{IPH}(t)\right| \eta(t) \left|\frac{\partial}{\partial t}\phi_n^{IPH}(t)\right\rangle \quad (2.23)$$

by replacing $\left\langle \frac{\partial}{\partial t}\phi_n^{IPH}(t)\right| \eta(t) \left|\phi_n^{IPH}(t)\right\rangle$ in (2.21), we get

$$\begin{aligned}\dot{\delta}_n^*(t) &= \Big\{ i\hbar \left\langle \phi_n^{IPH}(t)\right| \eta(t)\eta^{-1}(t)\dot{\eta}(t) \left|\phi_n^{IPH}(t)\right\rangle + i\hbar \left\langle \phi_n^{IPH}(t)\right| \eta(t) \left|\frac{\partial}{\partial t}\phi_n^{IPH}(t)\right\rangle \\ &\quad - \left\langle \phi_n^{IPH}(t)\right| \eta(t)\eta^{-1}(t)H^{+}(t)\eta(t) \left|\phi_n^{IPH}(t)\right\rangle \Big\} \end{aligned} \quad (2.24)$$

And using the relation [30, 31]:

$$H^{+}(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\dot{\eta}(t)\eta^{-1}(t) \quad (2.25)$$

we deduce

$$\dot{\delta}_n^{\,*}(t) = \dot{\delta}_n(t) \quad (2.26)$$

which guarantees the conservation of the probability of the presence; combining the equations: (2.12), (2.20), and (2.26), gives:

$$\langle\psi(t)|\, \eta(t)\, |\psi(t)\rangle = \sum_n |C_n|^2 = Conste \quad (2.27)$$

2.2 Non Hermitian perturbation approach

When a quantum system is subjected to time-dependent perturbation [3], it means that there are external influences acting on the system, such as quantum scattering, quantum control, quantum engineering, laser-driven atomic [46, 47], molecular physics. When the Hamiltonian is Hermitian, there are well-developed mathematical tools to approximate the solution of time-dependent Schrodinger's equation [3].

In many physical problems, one often deals with systems described by a non-Hermitian Hamiltonian [32, 33]: which is an extension of the standard perturbation theory used in quantum mechanics, under the influence of time-dependent non-Hermitian perturbation. The increasing interest in non-Hermitian dynamics has motivated a new set of mathematical tools, to explore and understand these complex systems.

We proceed to investigate in detail the main idea of the method to define the principal effects that generally describes the behavior of the system, and then to detail certain quantities that included, such as the state of evolution; $|\psi(t)\rangle$, and transition probability; $P_{n\longrightarrow m}(t)$.

2.2.1 State of evolution $|\psi(t)\rangle$

The state of evolution $|\psi(t)\rangle$ can be calculated approximately from the stationary states of the system, and the various physical quantities are obtained by calculating the mean values of the corresponding operators.

Let us consider a stationary system, known by a time-independent Hamiltonian H_0, whose eigenvalues and eigenstates;

$$H_0 |n\rangle = E_n |n\rangle \qquad n = 0, 1, 2.... \tag{2.28}$$

The perturbation denoted as $W(t)$ will be time dependent non-Hermitian ($W(t) \neq W(t)^{+}$). Thus, time-dependent non-Hermitian Hamiltonian is given by;

$$H(t) = H_0 + W(t) \tag{2.29}$$

Where $W(t)$ is small compared to H_0; $W(t) \lll H_0$.

If the perturbation acts at $t = t_0$, the initial state is given by

$$|\psi(t_0)\rangle = U(t_0) \sum_n C_n(0) |n\rangle \tag{2.30}$$

where

$$U(t_0) = e^{-\frac{iH_0}{\hbar}t} \tag{2.31}$$

The solutions $|\psi(t)\rangle$ of the Schrodinger equation for the Hamiltonian $H(t)$ verifies;

$$H(t)\,|\psi(t)\rangle = (H_0 + W(t))\,|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle \tag{2.32}$$

we define the auxiliary ket $\left|\tilde{\psi}(t)\right\rangle$ related with $|\psi(t)\rangle$ as:

$$|\psi(t)\rangle = e^{-\frac{iH_0 t}{\hbar}}\left|\tilde{\psi}(t)\right\rangle \tag{2.33}$$

Taking the derivative of (2.33) and using (2.32), thus we find The Schrodinger equation for $\left|\tilde{\psi}(t)\right\rangle$;

$$\tilde{W}(t)\left|\tilde{\psi}(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\tilde{\psi}(t)\right\rangle \tag{2.34}$$

where

$$\tilde{W}(t) = e^{\frac{iH_0 t}{\hbar}}W(t)e^{-\frac{iH_0 t}{\hbar}} \tag{2.35}$$

In the perturbation theory [3], the state of evolution $\left|\tilde{\psi}(t)\right\rangle$ can be written as a linear combination of the eigenstates $|n\rangle$ of the Hamiltonian H_0 as;

$$\left|\tilde{\psi}(t)\right\rangle = \sum_n C_n(t)\,|n\rangle \tag{2.36}$$

If the perturbation is sufficiently low, we can express $\left|\tilde{\psi}(t)\right\rangle$ as:

$$\left|\tilde{\psi}(t)\right\rangle = \left|\tilde{\psi}^{(0)}(t)\right\rangle + \left|\tilde{\psi}^{(1)}(t)\right\rangle + \left|\tilde{\psi}^{(2)}(t)\right\rangle + \ldots\ldots \tag{2.37}$$

Now we apply $\left|\tilde{\psi}(t)\right\rangle$ to both sides of the Schrodinger equation (2.34), and using ∂t for the time derivative, we find

$$i\hbar\partial t\left|\tilde{\psi}^{(0)}(t)\right\rangle + i\hbar\partial t\left|\tilde{\psi}^{(1)}(t)\right\rangle + i\hbar\partial t\left|\tilde{\psi}^{(2)}(t)\right\rangle = \tilde{W}(t)\left|\tilde{\psi}^{(0)}(t)\right\rangle + \tilde{W}(t)\left|\tilde{\psi}^{(1)}(t)\right\rangle + \tilde{W}(t)\left|\tilde{\psi}^{(2)}(t)\right\rangle \tag{2.38}$$

The coefficient of each term of perturbation order must vanish, giving us

$$i\hbar\partial t\left|\tilde{\psi}^{(0)}(t)\right\rangle = 0 \tag{2.39}$$

$$i\hbar\partial t\left|\tilde{\psi}^{(1)}(t)\right\rangle = \tilde{W}(t)\left|\tilde{\psi}^{(0)}(t)\right\rangle \tag{2.40}$$

$$i\hbar\partial t\left|\tilde{\psi}^{(2)}(t)\right\rangle = \tilde{W}(t)\left|\tilde{\psi}^{(1)}(t)\right\rangle \tag{2.41}$$

by regression relation we find

$$i\hbar\partial t\left|\tilde{\psi}^{(n+1)}(t)\right\rangle = \tilde{W}(t)\left|\tilde{\psi}^{(n)}(t)\right\rangle \tag{2.42}$$

The first equation (2.39) is completely solved:

$$\left|\widetilde{\psi}(0)\right\rangle = |\psi(0)\rangle = \sum_n C_n(0)\,|n\rangle \tag{2.43}$$

The expansion (2.37) evaluated at $t = 0$ implies that:

$$\left|\widetilde{\psi}(0)\right\rangle = |\psi_n(0)\rangle = \left|\widetilde{\psi}^{(0)}(0)\right\rangle + \left|\widetilde{\psi}^{(1)}(0)\right\rangle + \left|\widetilde{\psi}^{(2)}(0)\right\rangle + ... \tag{2.44}$$

Using the zeroth order result, the first order equation (2.40) is:

$$i\hbar\partial t\left|\widetilde{\psi}^{(1)}(t)\right\rangle = \widetilde{W}\left|\widetilde{\psi}^{(0)}(0)\right\rangle = \widetilde{W}(t)\,|\psi(0)\rangle \tag{2.45}$$

The solution can be written as an integral equation:

$$\left|\widetilde{\psi}^{(1)}(t)\right\rangle = \int_0^t \frac{\widetilde{W}(t')}{i\hbar}\,|\psi(0)\rangle\,dt' \tag{2.46}$$

The next equation, of second order (2.41) gives:

$$i\hbar\partial t\left|\widetilde{\psi}^{(2)}(t)\right\rangle = \widetilde{W}\left|\widetilde{\psi}^{(1)}(t)\right\rangle \tag{2.47}$$

By integration we deduce

$$\left|\widetilde{\psi}^{(2)}(t)\right\rangle = \int_0^t \frac{\widetilde{W}(t')}{i\hbar}\left|\widetilde{\psi}^{(1)}(t')\right\rangle dt' \tag{2.48}$$

Using the previous result (2.46), then $\left|\widetilde{\psi}^{(2)}(t)\right\rangle$ becomes as follows;

$$\left|\overset{\sim(2)}{\psi}(t)\right\rangle = \int_0^t \frac{\widetilde{W}(t')}{i\hbar} dt' \int_0^{t'} \frac{\widetilde{W}(t'')}{i\hbar} |\psi(0)\rangle \, dt'' \tag{2.49}$$

Finaly, the solution $|\psi(t)\rangle$ can be determined using (2.33) and (2.37);

$$|\psi(t)\rangle = e^{-\frac{iH^{(0)}t}{\hbar}} \left(|\psi(0)\rangle + \left|\overset{\sim(1)}{\psi}(t)\right\rangle + \left|\overset{\sim(2)}{\psi}(t)\right\rangle + ... \right) \tag{2.50}$$

2.2.2 Transition Probability $P_{n \longrightarrow m}(t)$

In various Quantum systems, the transition probability plays an important role to describe the ability of the system to gain or loss the quantum of the energy, by the influence of the time dependent perturbation $W(t)$. If the system is located in the state $|n\rangle$, what is the probability that it will be found in state $|m\rangle$ at time t ? We procced to calculate the probability of transition for the system from state $|n\rangle$ at $t = 0$ to state $|m\rangle$ at t, with $m \neq n$.

The transition probability $P_{n \longrightarrow m}(t)$ is defined as

$$P_{n \longrightarrow m}(t) = |\langle m \, |\psi(t)\rangle|^2 \tag{2.51}$$

where $|m\rangle$ is the state corresponding to the energy level E_m, and $|\psi(t)\rangle$ is the state of evolution given previously in (2.50).

Once the state $|\psi(t)\rangle$ in (2.51) is replaced, one gets:

$$P_{n \longrightarrow m}(t) = \left| \langle m| \, e^{-\frac{iH_0 t}{\hbar}} \left|\widetilde{\psi}(t)\right\rangle \right|^2 = \left| \langle m \left|\widetilde{\psi}(t)\right\rangle \right|^2 \tag{2.52}$$

Now by applying the perturbation expression for $\left|\widetilde{\psi}(t)\right\rangle$ in (2.37), we have

$$P_{n \longrightarrow m}(t) = \left| \langle m| \left(|\psi(0)\rangle + \left|\overset{\sim(1)}{\psi}(t)\right\rangle + \left|\overset{\sim(2)}{\psi}(t)\right\rangle + ... \right) \right|^2 \tag{2.53}$$

From $|\psi(0)\rangle = |n\rangle$ and $\langle m \, |n\rangle = 0$, we find

$$P_{n \longrightarrow m}(t) = \left| \langle m \left|\overset{\sim(1)}{\psi}(t)\right\rangle + \langle m \left|\overset{\sim(2)}{\psi}(t)\right\rangle + ... \right|^2 \tag{2.54}$$

A simple expression for $P^{(1)}_{n \longrightarrow m}(t)$ can be found by first-order perturbation theory, by keeping only the first term in the sum and using our result for $\left|\overset{\sim(1)}{\psi}(t)\right\rangle$ shown in (2.46);

$$P^{(1)}_{n\longrightarrow m}(t) = \left| \langle m| \int_0^t \frac{\tilde{W}(t')}{i\hbar} |n\rangle \, dt' \right|^2 = \left| \int_0^t \frac{\langle m| \tilde{W}(t') |n\rangle}{i\hbar} dt' \right|^2 \tag{2.55}$$

Finally, the transition probability to first order can be written as

$$P^{(1)}_{n\longrightarrow m}(t) = \left| \int_0^t e^{i\omega_{mn}t'} \frac{W_{mn}(t')}{i\hbar} dt' \right|^2 \tag{2.56}$$

where

$$\langle m| \tilde{W}(t') |n\rangle = \langle m| e^{\frac{iH_0 t}{\hbar}} W(t) e^{-\frac{iH_0 t}{\hbar}} |n\rangle = e^{i\omega_{mn}t'} W_{mn}(t') \tag{2.57}$$

The second order in perturbation, we keep the first and second term in the sum (2.54), and using (2.46), and (2.49) we find

$$\begin{aligned} P^{(2)}_{n\longrightarrow m}(t) &= \left| \langle m \left| \overset{\sim}{\psi}^{(1)}(t) \right\rangle + \langle m \left| \overset{\sim}{\psi}^{(2)}(t) \right\rangle \right|^2 \\ &= \left| \langle m| \int_0^t \frac{\tilde{W}(t')}{i\hbar} |n\rangle \, dt' + \langle m| \int_0^t \int_0^{t'} \frac{\tilde{W}(t')}{i\hbar} \frac{\tilde{W}(t'')}{i\hbar} |n\rangle \, dt' dt'' \right|^2 \end{aligned} \tag{2.58}$$

Thus

$$P^{(2)}_{n\longrightarrow m}(t) = \left| \int_0^t \frac{\langle m| \tilde{W}(t') |n\rangle}{i\hbar} dt' + \int_0^t \int_0^{t'} \frac{\langle m| \tilde{W}(t')\tilde{W}(t'') |n\rangle}{(i\hbar)^2} dt' dt'' \right|^2 \tag{2.59}$$

So, the probability to the second order in time-dependent perturbation is written as

$$P^{(2)}_{n\longrightarrow m_-}(t) = \left| \int_0^t e^{i\omega_{mn}t'} \frac{W_{mn}(t')}{i\hbar} dt' + \int_0^t \int_0^{t'} e^{i\omega_{mn}t'} \frac{\langle m| W(t')W(t'') |n\rangle}{(i\hbar)^2} e^{i\omega_{mn}t''} dt' dt'' \right|^2 \tag{2.60}$$

Chapter 3

Application: Non Hermitian Magnetic Effect

The main of this chapter is to solve time-dependent Non-Hermetian Schrodinger equation for Spin interacting with Complex external Magnetic Field, using the pseudo-Hermitian invariant operator $I^{PH}(t)$ method mensioned in the second chapter (Sec.1) as well, where we seek to find the phase of Lewis Riesenfeld, and then the state of evolution $|\psi(t)\rangle$. The same system, will be studied using non-Hermitian perturbation (Chap2. Sec 2).

3.1 Spin interacting with Complex external Magnetic Field: Pseudo Hermitian Invariant method

The interaction of spin with a complex external magnetic field is described by the Hamiltonian;

$$H(t) = -\overrightarrow{\mu_s}.\overrightarrow{B(t)} = \frac{e}{m}\overrightarrow{S}.\overrightarrow{B(t)} \tag{3.1}$$

where $\overrightarrow{\mu_S}$ is magnetic momentum, $\overrightarrow{B(t)} \in \mathbb{C}$; is time dependent complex external magnetic field and $\overrightarrow{S}$ is the spin operator; $\overrightarrow{S} = \frac{\hbar}{2}\overrightarrow{\sigma}$. Once inserting the spin $\overrightarrow{S}$ into the hamiltonian $H(t)$ one gets:

$$H(t) = \mu_B\left[\sigma_x B_x(t) + \sigma_y B_y(t) + \sigma_z B_z(t)\right] \tag{3.2}$$

where $\mu_B = \frac{\hbar e}{2m}$is bohr magneton, and $\sigma_x, \sigma_y, \sigma_z$ are the standard Pauli matrices given by (2×2) matrices as follows**:**

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.3}$$

Pauli matrices obey the following commutation relation:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{3.4}$$

where i is the imaginary unit of complex numbers, and ϵ_{ijk} is levi-civita symbol.

Once replacing σ_x, σ_y, σ_z in the equation (3.2) one obtains;

$$H(t) = \mu_B \begin{pmatrix} B_z(t) & B_-(t) \\ B_+(t) & -B_z(t) \end{pmatrix} \tag{3.5}$$

where

$$B_\pm(t) = B_x(t) \pm iB_y(t) \tag{3.6}$$

Let us now use the pseudo-hermitian invariant method presented previously using the same steps :

1. We choose the pseudo-Hermitian invariant operator $I^{PH}(t)$ in the form:

$$\begin{aligned} I^{PH}(t) &= \alpha(t)\sigma_x + \beta(t)\sigma_y + \gamma(t)\sigma_z \\ &= \begin{pmatrix} \gamma(t) & \alpha(t) - i\beta(t) \\ \alpha(t) + i\beta(t) & -\gamma(t) \end{pmatrix} \end{aligned} \tag{3.7}$$

where $\gamma(t)$ and $\beta(t)$ are real time-dependent functions, while $\alpha(t)$ is complex time-dependent function.

2. We impose the invariance condition (2.4) for $I^{PH}(t)$ to find;

$$\begin{aligned} \frac{i}{\hbar}\left[I^{PH}(t), H(t)\right] &= \frac{i}{\hbar}\left[\left(\sigma_x\alpha(t) + \sigma_y\beta(t) + \sigma_z\gamma(t)\right), \mu_B\left(\sigma_x B_x(t) + \sigma_y B_y(t) + \sigma_z B_z(t)\right)\right] \\ &= -\frac{2\mu_B}{\hbar}\left\{\left[\beta(t)B_z - \gamma(t)B_y(t)\right]\sigma_x + \left[\gamma(t)B_x(t) - \alpha(t)B_z(t)\right]\sigma_y \right. \\ &\quad \left. + \left[\alpha(t)B_y - \beta(t)B_x(t)\right]\sigma_z\right\} \end{aligned} \tag{3.8}$$

and

$$\frac{\partial I(t)}{\partial t} = \dot{\alpha}(t)\sigma_x + \dot{\beta}(t)\sigma_y + \dot{\gamma}(t)\sigma_z \tag{3.9}$$

By inserting the two equations (3.8) and (3.9) into (2.4) we get;

$$\dot{\alpha}(t) = -\frac{2\mu_B}{\hbar}(\beta(t)B_z(t) - \gamma(t)B_y(t)) \tag{3.10}$$

$$\dot{\beta}(t) = -\frac{2\mu_B}{\hbar}(\gamma(t)B_x(t) - \alpha(t)B_z(t)) \tag{3.11}$$

$$\dot{\gamma}(t) = -\frac{2\mu_B}{\hbar}(\alpha(t)B_y(t) - \beta(t)B_x(t)) \tag{3.12}$$

3. Let us search for the eigenstates; $\left|\phi_n^{I^{PH}}(t)\right\rangle$, $\left|\phi_n^{I^{PH+}}(t)\right\rangle$, of $I^{PH}(t)$, $I^{PH+}(t)$ respectively;

$$I^{PH}(t)\left|\phi_n^{I^{PH}}(t)\right\rangle = \lambda_n \left|\phi_n^{I^{PH}}(t)\right\rangle \tag{3.13}$$

the eigenvalues λ_n verify

$$\det\left|I^{PH}(t) - \lambda_n 1\right| = 0 \tag{3.14}$$

$$\begin{vmatrix} \gamma(t) - \lambda & \alpha(t) - i\beta(t) \\ \alpha(t) + i\beta(t) & -\gamma(t) - \lambda \end{vmatrix} = 0 \tag{3.15}$$

which gives

$$\lambda_\pm = \pm\sqrt{\alpha^2(t) + \beta^2(t) + \gamma^2(t)} \tag{3.16}$$

let's consider $\left|\phi_+^{I^{PH}}(t)\right\rangle = \begin{pmatrix} A \\ B \end{pmatrix}$ and $\left|\phi_-^{I^{PH}}(t)\right\rangle = \begin{pmatrix} A' \\ B' \end{pmatrix}$ are the eigenvectors of $I^{PH}(t)$.Then, we substitute $\left|\phi_+^{I^{PH}}(t)\right\rangle$ and $\left|\phi_-^{I^{PH}}(t)\right\rangle$ in equation (3.13) respectively to obtain

$$I^{PH}(t)\left|\phi_{+}^{I^{PH}}(t)\right\rangle = \lambda_{+}\left|\phi_{+}^{I^{PH}}(t)\right\rangle \tag{3.17}$$

$$\begin{pmatrix} \gamma(t) & \alpha(t)-i\beta(t) \\ \alpha(t)+i\beta(t) & -\gamma(t) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \lambda_{+}\begin{pmatrix} A \\ B \end{pmatrix}$$

So that the eigenstates of $I^{PH}(t)$ are given by

$$\left|\phi_{+}^{I^{PH}}(t)\right\rangle = \begin{pmatrix} 1 \\ \frac{\lambda_{+}-\gamma(t)}{\alpha(t)-i\beta(t)} \end{pmatrix} \tag{3.18}$$

$$\left|\phi_{-}^{I^{PH}}(t)\right\rangle = \begin{pmatrix} 1 \\ \frac{\lambda_{-}-\gamma(t)}{\alpha(t)-i\beta(t)} \end{pmatrix} \tag{3.19}$$

The eigenvalue equation for the adjoint $I^{PH+}(t)$ is

$$I^{PH+}(t)\left|\phi_{n}^{I^{PH+}}(t)\right\rangle = \lambda'_{n}\left|\phi_{n}^{I^{PH+}}(t)\right\rangle \tag{3.20}$$

where

$$I^{PH+}(t) = \left(I^{PH}(t)\right)^{+} = \begin{pmatrix} \gamma^{*}(t) & (\alpha^{*}(t)-i\beta^{*}(t)) \\ (\alpha^{*}(t)+i\beta^{*}(t)) & -\gamma^{*}(t) \end{pmatrix} \tag{3.21}$$

So, the eigenvalues of $I^{PH+}(t)$ are given by :

$$\lambda'_{\pm} = \pm\sqrt{\alpha^{*2}(t)+\beta^{*2}(t)+\gamma^{*2}(t)} \tag{3.22}$$

We consider $\left|\phi_{+}^{I^{PH+}}(t)\right\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\left|\phi_{-}^{I^{PH+}}(t)\right\rangle = \begin{pmatrix} a' \\ b' \end{pmatrix}$ are the eigenvectors of $I^{PH+}(t)$. after that, we substitute those vectors in equation (3.20) respectively to find

$$\left|\phi_{+}^{I^{PH+}}(t)\right\rangle = \begin{pmatrix} 1 \\ \frac{\lambda'_{+}-\gamma(t)}{\alpha^{*}(t)-i\beta(t)} \end{pmatrix} \tag{3.23}$$

$$\left|\phi_{-}^{IPH+}(t)\right\rangle = \begin{pmatrix} 1 \\ \frac{\lambda'_{-}-\gamma(t)}{\alpha^{*}(t)-i\beta(t)} \end{pmatrix} \tag{3.24}$$

4. Determination of the solution of Schrodinger's equation; $H(t)\left|\psi_n(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\psi_n(t)\right\rangle$,which given in (2.18);

$$\left|\psi_n(t)\right\rangle = e^{i\delta_n(t)}\left|\phi_n^{IPH}(t)\right\rangle \tag{3.25}$$

Where $\delta_n(t)$ is the phase function given as

$$\dot{\delta}_n(t) = \left\langle \phi_n^{IPH+}(t)\right| i\frac{\partial}{\partial t} - \frac{1}{\hbar}H(t)\left|\phi_n^{IPH}(t)\right\rangle \tag{3.26}$$

The final step consists in determining the Schrödinger solution (3.25), which is an eigenstates of the pseudo-Hermitian invariant(3.7) multiplied by a time-dependent factor (3.26).

We need to calculate the diagonal matrix element of the operator $i\frac{\partial}{\partial t} - \frac{1}{\hbar}H(t)$;

First we have

$$\left\langle \phi_n^{IPH+}(t)\right| i\frac{\partial}{\partial t}\left|\phi_n^{IPH}(t)\right\rangle = \begin{pmatrix} 1 & \frac{\lambda_n-\gamma(t)}{\alpha^{*}(t)-i\beta(t)} \end{pmatrix} i\frac{\partial}{\partial t}\begin{pmatrix} 1 \\ \frac{\lambda_n-\gamma(t)}{\alpha(t)-i\beta(t)} \end{pmatrix} \tag{3.27}$$

which gives

$$i\left(\frac{\lambda_n-\gamma(t)}{\alpha(t)-i\beta(t)}\right)\left(\frac{(-\dot{\gamma}(t))(\alpha(t)-i\beta(t)) - \left(\dot{\alpha}(t)-i\dot{\beta}(t)\right)(\lambda_n-\gamma(t))}{(\alpha(t)-i\beta(t))^2}\right) \tag{3.28}$$

Second, the term $-\frac{\mu_B}{\hbar}\left\langle \phi_n^{IPH+}(t)\right| H(t)\left|\phi_n^{IPH}(t)\right\rangle$ gives:

$$-\frac{\mu_B}{\hbar}\begin{pmatrix} 1 & \frac{\lambda_n-\gamma(t)}{\alpha(t)-i\beta(t)} \end{pmatrix}\begin{pmatrix} B_z(t) & B_-(t) \\ B_+(t) & -B_z(t) \end{pmatrix}\begin{pmatrix} 1 \\ \frac{\lambda_n-\gamma(t)}{\alpha(t)-i\beta(t)} \end{pmatrix} \tag{3.29}$$

By simplification we deduce

$$-\frac{\mu_B}{\hbar}\left(\frac{\lambda_{n+}-\gamma(t)}{\alpha(t)-i\beta(t)}\right)\left[B_-(t)+B_+(t)+(B_z(t))\left(\frac{(\alpha(t)-i\beta(t))^2-(\lambda_{n+}-\gamma(t))^2}{(\lambda_{n+}-\gamma(t))(\alpha(t)-i\beta(t))}\right)\right] \tag{3.30}$$

Combining the equations: (3.28) and (3.30) gives

$$\begin{aligned}
&\left\langle \phi_n^{I^{PH+}}(t) \right| i\frac{\partial}{\partial t} - \frac{1}{\hbar}H(t) \left| \phi_n^{I^{PH}}(t) \right\rangle \\
= &\left\{ i\left(\frac{\lambda_n - \gamma(t)}{\alpha(t) - i\beta(t)}\right)\left(\frac{(-\dot{\gamma}(t))(\alpha(t) - i\beta(t)) - \left(\dot{\alpha}(t) - i\dot{\beta}(t)\right)(\lambda_n - \gamma(t))}{(\alpha(t) - i\beta(t))^2}\right) \right. \\
&\left. -\frac{\mu_B}{\hbar}\left(\frac{\lambda_n - \gamma(t)}{\alpha(t) - i\beta(t)}\right)\left[B_-(t) + B_+(t) + B_z(t)\left(\frac{(\alpha(t) - i\beta(t))^2 - (\lambda_n - \gamma(t))^2}{(\lambda_n - \gamma(t))(\alpha(t) - i\beta(t))}\right)\right]\right\}
\end{aligned}$$

$$\begin{aligned}
\dot{\delta}_n(t) &= \left\langle \phi_n^{I^{PH+}}(t) \right| i\frac{\partial}{\partial t} - \frac{1}{\hbar}H(t) \left| \phi_n^{I^{PH}}(t) \right\rangle \\
&= \left(\frac{\lambda_n - \gamma}{\alpha(t) - i\beta(t)}\right)\frac{\mu_B}{\hbar}\left[\begin{array}{c} B_z\left[\frac{(\beta(t) - i\alpha(t))(\lambda_n - \gamma(t))}{(\alpha(t) - i\beta(t))^2} - \frac{(\alpha(t) - i\beta(t))^2 - (\lambda_n - \gamma(t))^2}{(\lambda_n - \gamma(t))(\alpha(t) - i\beta(t))}\right] \\ +B_+\left[\frac{i\gamma(t)(\lambda_n - \gamma(t))}{(\alpha(t) - i\beta(t))^2} - 1\right] - B_- + \frac{(\alpha(t)B_y(t) - \beta(t)B_x(t))}{(\alpha(t) - i\beta(t))}\end{array}\right]
\end{aligned} \tag{3.31}$$

Then, we obtain the exact phase which is given by

$$\delta_\pm(t) = \pm\frac{\mu_B}{\hbar}\int_0^t \sqrt{|B_z(t)|^2\cos^2(\varphi_{B_z}) + |B_+(t)|\,|B_-(t)|\cos(\varphi_+)\cos(\varphi_-)}dt' = \pm\delta \tag{3.32}$$

Finaly the solution of the Schrodinger equation for the evolved spin is given by

$$|\psi(t)\rangle = \sum_n C_n e^{i\delta_n(t)}\left|\phi_n^{I^{PH}}(t)\right\rangle = C_+e^{i\delta_+(t)}\left|\phi_+^{I^{PH}}(t)\right\rangle + C_-e^{i\delta_-(t)}\left|\phi_-^{I^{PH}}(t)\right\rangle \tag{3.33}$$

Using the equations (3.18), (3.19) gives;

$$|\psi(t)\rangle = C_+\begin{pmatrix} e^{i\delta(t)} \\ \frac{\lambda - \gamma(t)}{\alpha(t) - i\beta(t)}e^{i\delta(t)} \end{pmatrix} + C_-\begin{pmatrix} e^{-i\delta(t)} \\ -\frac{\lambda + \gamma(t)}{\alpha(t) - i\beta(t)}e^{-i\delta(t)} \end{pmatrix} \tag{3.34}$$

and by chosing the constants $C_+ = C_- = \frac{1}{2}$, the general solution $|\psi(t)\rangle$ can be written in the following simplified matricial form;

$$|\psi(t)\rangle = \begin{pmatrix} \cos(\delta) \\ \frac{\lambda i\sin(\delta) - \gamma(t)\cos(\delta)}{\alpha(t) - i\beta(t)} \end{pmatrix} \tag{3.35}$$

3.2 Non Hermitian Magnetic perturbation

Let us take the previous interaction (3.1), but instead of using pseudo invariant method, we adapt Non Hermitian perturbation approach for a non Hermitian Magnetic perturbation described by the following Hamiltonian:

$$H(t) = H_0 + W(t) \tag{3.36}$$

with H_0 is time independant Hermitian;

$$H_0 = \frac{\hbar e}{2m_e}\begin{pmatrix} B_0 & 0 \\ 0 & -B_0 \end{pmatrix}, \text{ where; } B_0 \in \mathbb{R} \tag{3.37}$$

and the non hermitian Magnetic Perturbation potential $W(t)$ is given by

$$W(t) = \frac{\hbar e}{2m_e}\begin{pmatrix} Be^{i\omega t} & Be^{i\omega t} \\ Be^{-i\omega t} & -Be^{i\omega t} \end{pmatrix} \tag{3.38}$$

where: $\frac{B}{B_0} \ll 1$, and $Be^{i\omega t}$ is time dependent complexe external magnetic field. The eigenvectors and eigenvalues of H_0 can be obtained by the equation

$$\det|H_0 - E1| = 0 \tag{3.39}$$

$$\Longrightarrow \frac{\hbar e}{2m_e}\begin{vmatrix} B_0 - E & 0 \\ 0 & -B_0 - E \end{vmatrix} = 0 \tag{3.40}$$

So the eigenvalues of H_0 can be written in the form

$$E\pm = \pm\mu_B B_0 = \pm\hbar\omega_0 \tag{3.41}$$

with

$$\omega_0 = \frac{eB_0}{2m_e} \tag{3.42}$$

and for eigenvectors we take the eigenvalue equation

$$H_0 \left| E\pm \right\rangle = E \pm \left| E\pm \right\rangle \tag{3.43}$$

we obtain

$$\left| E_+ \right\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{3.44}$$

$$\left| E_- \right\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.45}$$

The initial state of the system is given by

$$\begin{aligned} \left| \psi(0) \right\rangle &= c_+(0) \left| E_+ \right\rangle + c_-(0) \left| E_- \right\rangle \\ &= \begin{pmatrix} \frac{c_+(0)}{\sqrt{2}} \\ \frac{c_-(0)}{\sqrt{2}} \end{pmatrix} \end{aligned} \tag{3.46}$$

Once Taking the normalisation condition; $\left\langle \psi(0) \right| \psi(0) \rangle$ for initial state of evolution one gets;

$$\left| \psi(0) \right\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.47}$$

As mensioned in the second chapter, in the equations: (2.46), (2.35), and (2.33), the state of first order in perturbation is given by;

$$\begin{aligned} \left| \psi^{(1)}(t) \right\rangle &= \frac{e^{-\frac{iH_0 t}{\hbar}}}{i\hbar} \int_0^t \widetilde{W}(t') \left| \psi(0) \right\rangle dt' \\ &= \frac{e^{-\frac{iH_0 t}{\hbar}}}{i\hbar} \int_0^t e^{\frac{iH_0 t'}{\hbar}} W(t') e^{-\frac{iH_0 t'}{\hbar}} \left| \psi(0) \right\rangle dt' \\ &= \frac{\mu_B e^{-\frac{iH_0 t}{\hbar}}}{i\hbar} \int_0^t e^{\frac{iH_0 t'}{\hbar}} \begin{pmatrix} Be^{i\omega t'} & Be^{i\omega t'} \\ Be^{-i\omega t'} & -Be^{i\omega t'} \end{pmatrix} e^{-\frac{iH_0 t'}{\hbar}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} dt' \\ &= \frac{\mu_B B}{\hbar(\omega - 2\omega_0)} \begin{pmatrix} e^{i\left(\frac{\omega}{2} - 2\omega_0\right)t} \sin\left(\frac{\omega - 2\omega_0}{2}\right) t \\ \frac{e^{-i2\omega_0 t} i(2\omega_0 \cos(\omega t) - i\omega \sin(\omega t))}{(\omega + 2\omega_0)} \end{pmatrix} \end{aligned} \tag{3.48}$$

Using the equation (2.49), as mensioned in the second chapter, the state of second order is written as

$$\begin{aligned}
\left|\overset{\sim}{\psi}^{(2)}(t)\right\rangle &= \frac{-1}{\hbar^2}\int_0^t \widetilde{W}(t')dt'\int_0^{t'}\widetilde{W}(t'')\left|\psi(0)\right\rangle dt'' \\
&= \frac{-1}{\hbar^2}\int_0^t \widetilde{W}(t')dt'\left|\overset{\sim}{\psi}^{(1)}(t')\right\rangle \\
&= \frac{-1}{\sqrt{2}\hbar^2}\frac{\mu_B B}{\hbar\left(\omega-2\omega_0\right)}\int_0^t \mu_B \begin{pmatrix} Be^{i\omega t'} & Be^{i\omega t'} \\ Be^{-i\omega t'} & -Be^{i\omega t'} \end{pmatrix}\begin{pmatrix} e^{i\left(\frac{\omega}{2}-2\omega_0\right)t'}\sin\left(\frac{\omega-2\omega_0}{2}\right)t' \\ \frac{e^{-i2\omega_0 t'}i(2\omega_0\cos(\omega t')-i\omega\sin(\omega t'))}{(\omega+2\omega_0)} \end{pmatrix}dt' \\
&= \frac{\mu_B^2 B^2}{2\sqrt{2}\hbar^2}\begin{pmatrix} \frac{\left[ie^{i2\omega_0 t}-i\cos(\omega t)+2\sin(\omega_0 t)\right]}{\left(\omega^2-4\omega_0^2\right)} \\ \frac{\left[ie^{i2\omega_0 t}+i\cos(\omega t)-2\sin(\omega_0 t)\right]}{\left(\omega^2-4\omega_0^2\right)} \end{pmatrix}
\end{aligned} \tag{3.49}$$

Now we determine the transition probability in first order, from the equation (2.55) and (3.38), we get

$$\begin{aligned}
P^{(1)}_{E_+\longleftarrow E_-}(t) &= \left|\frac{1}{i\hbar}\int_0^t \left\langle E_+\right|\widetilde{W}(t')\left|E_-\right\rangle dt'\right|^2 \\
&= \left|\frac{1}{i\hbar}\int_0^t e^{i2\omega_0 t'}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\mu_B \begin{pmatrix} Be^{i\omega t'} & Be^{i\omega t'} \\ Be^{-i\omega t'} & -Be^{i\omega t'} \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}dt'\right|^2 \\
&= \left|\frac{\mu_B B}{2i\hbar}\int_0^t e^{i2\omega_0 t'}e^{i\omega t'}dt'\right|^2 \\
&= \left(\frac{\mu_B B}{2\hbar(\omega+2\omega_0)}\right)^2\sin^2\left(\frac{\omega+2\omega_0}{2}t\right)
\end{aligned} \tag{3.50}$$

and the probability of second ordre in perturbation is obtained from (2.46), and the equation (3.38)

$$
\begin{aligned}
P^{(2)}_{E_+\longleftarrow E_-}(t) &= \left| \frac{1}{i\hbar}\int_0^t \langle E_+|\,\widetilde{W}(t')\,|E_-\rangle\, dt' + \frac{1}{(i\hbar)^2}\int_0^t\int_0^{t'} \langle E_+|\,\widetilde{W}(t')\widetilde{W}(t'')\,|E_-\rangle\, dt'dt'' \right|^2 \\
&= \left| \begin{array}{c} \Big\{ \frac{\mu_B B}{2i\hbar}\int_0^t e^{i(\omega+2\omega_0)t'}dt' \\ +\frac{\mu_B^2}{(i\hbar)^2}\int_0^t \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} Be^{i\omega t'} & Be^{i\omega t'} \\ Be^{-i\omega t'} & -Be^{i\omega t'} \end{pmatrix} e^{i2\omega_0 t'}dt' \\ \int_0^{t\prime} \begin{pmatrix} Be^{i\omega t''} & Be^{i\omega t''} \\ Be^{-i\omega t''} & -Be^{i\omega t''} \end{pmatrix} e^{i2\omega_0 t''} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} dt'' \Big\} \end{array} \right|^2 \\
&= \left| \begin{array}{c} \Big\{ \frac{\mu_B B e^{i\left(\frac{\omega+2\omega_0}{2}\right)t}\sin(\frac{\omega+2\omega_0}{2}t)}{2i\hbar(\omega+2\omega_0)} \\ -\frac{(\mu_B B)^2\left[ie^{i2\omega_0 t}-i\cos(\omega t)+2\sin(\omega_0 t)\right]}{\hbar^2\left(\omega^2-4\omega_0^2\right)} \Big\} \end{array} \right|^2
\end{aligned}
\tag{3.51}
$$

In conclusion, non-Hermitian magnetic perturbation provides an important theoretical framework for the study of quantum systems under the influence of complex perturbation (3.38). This offer provides effective prediction into a wide range of physical phenomena, such as: complex Zeeman field drives quantum phase transitions in the Ising model [46, 47].

Conclusion

During this thesis :

• We have recalled in historical brief, how non-Hermitian Hamiltonians have been introduced in the literature.

• It has been presented in detail the concepts of PT-symmetry, the pseudo-hermiticity, PT and CPT inner-products.

• The pseudo-hermiticity has been generalized for time dependent Quantum systems to develop the pseudo-invariant method.

• The pseudo-invariant method has been examinated to study non-Hermitian Quantum systems, and we have found that this method is effective and preserves the postulates of Quantum Mechanics.

• We have deduced with a concrete example the solution of the time-dependent Schrodinger equation for a spin interacting with complex external magnetic field.

• We have discussed the non-Hermitian magnetic perturbation.

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Abstract:

In this work, we recall the concepts of PT-symmetry, the pseudo-Hermiticity, PT and CPT inner products. We study the non-Hermitian quantum systems described by the non-Hermitian Hamiltonian with an examination of the pseudo-invariant theory. We use a pseudo Hermitian invariant operator to solve analytically time-dependent Schrodinger equation for a spin interacting with complex external magnetic field with unitary evolution. Finally, perturbation theory for the non-Hermitian case is discussed as well.

Résume :

Dans ce travail, nous rappelons les concepts de PT-symétrie, pseudo-Herméticité, et de produits scalaires PT et CPT. Nous étudions les systèmes quantiques non Hermétiques décrits par l'hamiltonien non Hermétique en examinant la méthode des invariants pseudo-Hermétique. Nous utilisons un opérateur invariant pseudo Hermétique pour résoudre analytiquement l'équation de Schrödinger dépendante du temps pour un spin en interaction avec un champ magnétique externe complexe avec une évolution unitaire. Enfin, la théorie des perturbations pour le cas non-Hermitien est également discutée.

الملخص:

في هذا العمل، نذكر بمفاهيم التناظر PT، الشبه هرميتية، والجداء السلمي PT وCPT. ندرس الأنظمة الكمومية غير الهرميتية التي تعتمد في وصفها على هاملتون غير هرميتي مع التركيز على نظرية اللا متغير الشبه هرميتي. نستعمل اللا متغير الشبه هرميتي لإيجاد الحل تحليلي لمعادلة شرودنغر المتعلقة بالزمن في حالة سبين يخضع لحقل مغناطيسي مركب مع تطور زمني محفوظ. في النهاية، نناقش نظرية الاضطراب في الحالة غير الهرميتية.