الجمهورية الجزائرية الديمقراطية الشعبية

وزارة التعليم العالي والبحث العلمي

جامعة محمد البشير الإبراهيمي برج بوعريريج

كلية العلوم الاقتصادية والتجارية وعلوم التسيير

UNIVERSITE MOHAMED EL BACHIR EL IBRAHIMI
BORDJ BOU ARRERIDJ

Statistics 2 lectures

Supported by series of solved exercises

For first year students

Specialty: Common trunk

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سلسلة المطبوعات الجامعية المحكمة

Contents

Introduction:

This course presents a set of Statistics 2 lessons for first-year students of economics, business and management sciences, which help them to understand and acquire the most important knowledge and concepts related to this subject included in the official academic program, as this course mainly aims to learn about probability theory and study the laws of probability used in the study of random variables and on which we rely to make decisions, especially those related to economics, commerce and management, and how to apply them in the case of real data.

General objectives of the course:

After studying the Statistics 2 standard, the student will be able to understand the different relationships and properties that characterize random events and be able to calculate the probability that corresponds to them, as well as learn about the different types of random variables and their properties and numerical values. features such as moments, expected value and variance, as well as learn about the most important laws and probability distributions and how to apply them.

These lessons will allow the student to build a real theoretical and practical base that will allow him to delve deeply into the subjects that he will study in the rest of his university course, in which he will deal with many random variables such as inferential statistics, econometrics, feasibility study, project evaluation, risk study, questionnaire analysis...

Prior knowledge:

Probability and statistics have been studied extensively in high school, but in order to make good use of the scientific concepts present in the course, the student must be proficient in the technical knowledge acquired in mathematics, especially the methods of dealing with various mathematical functions and the methods of their derivation and integrations, for which these concepts have been reviewed. In the first hexagram in mathematics lectures, this is in addition to the properties of logarithmic and exponential functions, which are important and necessary prerequisites.

The following table gives us the most important information related to this teaching subject.

Course name: Statistics 2			
Domain	Economics, commercial sciences and management sciences	**Branch/ Division:**	all branches
Specialization:	Common trunk	**level:**	The first is a bachelor's degree
semester	Second	**University year**	**../..**

Course name:	Statistics 2	**teaching unit**	methodology
Number of credits	**3**	**coefficient**	**3**
Weekly hourly volume	**4.5 h**	**Lecture (number of hours per week)**	**3 h**
		Directed/practical work (number of hours per week)	**1.5 h**

Chapter One:

General concepts about events

1. Experiment:

It is every process of intentional and artificial repetition of a specific phenomenon that results in a set of possible results, with the aim of observation or measurement.

We distinguish between a **systematic experiment** that is conducted under specific conditions using special equipment and specific tools, and its results are linked to scientific laws and universal laws. Here we obtain the same results no matter how the experiment is repeated. When water is placed at zero degrees Celsius, it will freeze, and when it is placed at 100 degrees, it will It evaporates,

On other hand there is a **random experiment** that is Any activity or situation in which there is uncertainty about which of two or more possible outcomes will result, its outcome cannot be determined in advance before conducting the experiment alone because it depends on random chance, and in our course, we are interested in a random experiment or a probabilistic experiment.

2. Sample spaces

Sample spaces are simply set which its elements describe the outcomes of the experiment in which we are interested, so that each possible result of this experiment corresponds to an element of the sample space and represents an event in itself.

We start with the most basic experiment: the tossing of a coin, assuming that we will never see the coin land on its rim, there are two possible outcomes: heads and tails. We therefore take as the sample space associated with this experiment the set $\Omega = \{H, T\}$.

Also, the tossing of balanced die; there are 6 possibles outcomes, therefore the sample space associated with this experiment the set $\Omega = \{1, 2, 3, 4, 5, 6\}$.

3. The event:

The word event is considered an important term in probability theory. An event is defined as the fact or phenomenon that may or may not occur during conducting the experiment, and we must always remember that both cases of verification and non-verification can include a large number of possible cases for each of them, so the event is associated to set of outcomes from all possible outcomes of the experiment, it is subset of elements of the sample space Ω.

4. Types of events:

The events differ from each other according to the number of elements including the set that correspond to the occurring of that event and the relationship that links it to the other elements of the sample space Ω We distinguish between the following type of incidents:

4.1 Elementary event:

It is the event whose realization corresponds to obtaining one and only one result from the results of the probabilistic experiment. It corresponds to one element of the set of primary events, and we symbolize it with the symbol (w_i).

When we throw a balanced die, the event corresponding to obtain the number **5** which is named **A** is elementary event, because the set corresponding to this event includes only one element and we note **A = {5}**.

4.2 Compound event:

It is the event whose realization corresponds to obtaining more than one result from the probability experiment, that is, it consists of more than one element of the set of initial accident spaces (Ω).

When we throw a balanced die, the event corresponding to obtain an even number which is named **B** is compound event, because the set corresponding to this event includes more than one element and we note **B = {2, 4, 6}**.

4.3 The sample set

It is the event whose realization corresponds to obtaining all possible results of the experiment. It corresponds to a set that contains all the elements of the set of events, and we symbolize it with the symbol **(Ω).**

When we throw a balanced die, the event corresponding to obtain number less than 7 which is named **C** is the sample set, because the set corresponding to this event includes all the elements of the set of events and we note **C = Ω = {1, 2, 3, 4, 5, 6}.**

4.4 The empty set

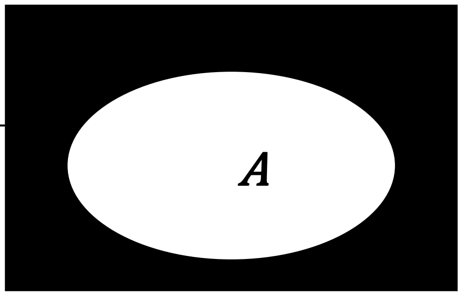
It is the event whose realization does not correspond to any of the possible results of the experiment, meaning that none of the elements associated with its realization belong to the set of primary events, and we usually symbolize it as the empty set **Ø.**

When we throw a balanced die, the event corresponding to obtain number greater than 6 which is named **D** is the empty set, because the set corresponding to this event does not include any element of the set of events and we note **D = { } = Ø.**

4.5 The complement event:

The event $\mathbf{A^c}$ or $\bar{\mathbf{A}}$ is called the complement of **A** it occurs if and only if **A** does *not* occur, it consists of all experimental outcomes that are not in event **A**, the complement of an event A is the set of all outcomes from the sample space that don’t reside in **A**, and we denote:

$$\bar{A} = \{w_i \in \Omega : w_i \notin A\}$$



Let us take the experiment of tossing a balanced die, and let us take event A, which is obtaining one of the numbers **(1, 2, 5)**, then its complementary event is obtaining one of the following numbers **(3, 4, 6)**. We note here that the realization of event **A** is obtaining a partial set from the space of primary events **(Ω)** and we write **A = {1, 2, 5}**.

If the complement event **occurs**, it also agrees with a subset of **(Ω)**, but none of its elements belong to the subset **A**, since $\mathbf{A^c}$ or $\bar{A} = \{3, 4, 6\}$.

These two events cannot occur at the same time. Rather, they are completely opposite events. We note that if the basic event does not occur, the complementary or opposite event must occur, meaning that the event and its complement constitute a definite event, and therefore:

$$A \cap \bar{A} = \emptyset$$

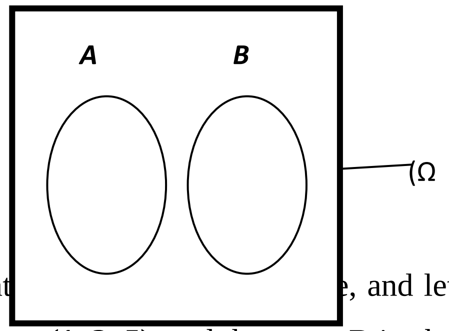
$$A \cup \bar{A} = \Omega$$

5. The relationship between events:

5.1 Mutually exclusive:

We say that the two events **A** and **B** associated with the random experiment are mutually exclusive events if obtaining one of the results that agree with the occurring of one of the two events does not agree with any of the results that agree with the occurring of the other event, the two events are mutually exclusive if they have no outcomes in common. The term disjoint is also sometimes used to describe events that have no outcomes in common.

$$A \cap B = \emptyset \Leftrightarrow \begin{cases} \forall w_i \in A : w_i \notin B \\ \forall w_i \in B : w_i \notin A \end{cases}$$



Let us take the experiment e, and let us take event **A**, which is obtaining one of the numbers **(1, 3, 5)**, and the event B is obtaining one of the following numbers **(4, 6)**, we note that these two events are mutually exclusive if they have no outcomes in common, and we write:

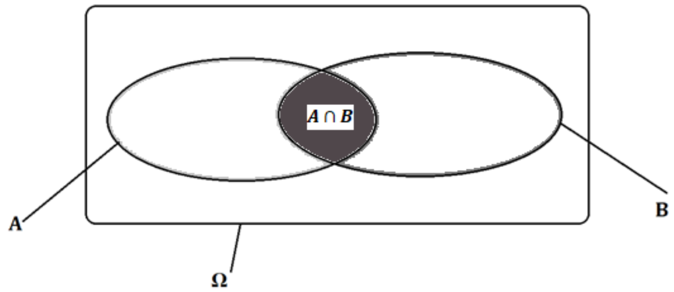
$$\mathbf{A = \{1, 3, 5\}}$$

$$\mathbf{B = \{4, 6\}}$$

$$\boldsymbol{A \cap B = \emptyset}$$

5.2 The intersection of events:

The intersection of the two events **A** and **B** associated with the random experiment represents an event that expresses the occurring of both events at the same time, it is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as $\boldsymbol{A \cap B}$



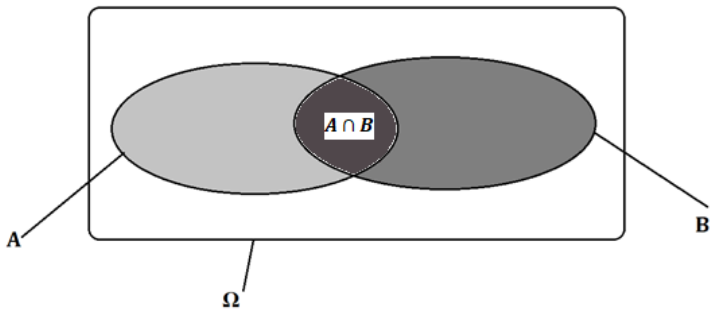
$$\boldsymbol{A = \{w_i \in \Omega: w_i \in A\}}$$

$$\boldsymbol{B = \{w_i \in \Omega: w_i \in B\}}$$

$$\mathbf{A \cap B = B \cap A = \{w_i \in \Omega: w_i \in A \wedge w_i \in B\}}$$

5.3 The union of events:

The union of two events **A** and **B** associated with the random experiment represents an event that expresses the occurring of at least one of the two events, that is, the occurring of event **A**, the occurring of event **B**, or the occurring of both events **A** and **B** together, it consists of all experimental outcomes that are in at least one of the two events, that is, in *A* or in *B* or in both of them, **A** or **B** is called the union of the two events and is denoted by $\boldsymbol{A \cup B}$**.**



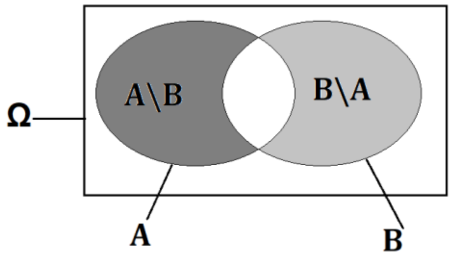
$$\boldsymbol{A = \{w_i \in \Omega: w_i \in A\}}$$

$$\boldsymbol{B = \{w_i \in \Omega: w_i \in B\}}$$

$$\mathbf{A \cup B = B \cup A = \{w_i \in \Omega : w_i \in A \vee w_i \in B\}}$$

5.4 The difference between events:

The difference between events **A** and **B** associated with the random experiment represents an event that expresses the fulfillment of one of the two events without the fulfillment of the other event



$$\boldsymbol{A = \{w_i \in \Omega : w_i \in A\}}$$

$$\boldsymbol{B = \{w_i \in \Omega : w_i \in B\}}$$

$$\boldsymbol{A \backslash B = \{w_i \in \Omega : w_i \in A \wedge w_i \notin B\} = A \cap \bar{B}}$$

$$\boldsymbol{B \backslash A = \{w_i \in \Omega : w_i \in B \wedge w_i \notin A\} = B \cap \bar{A}}$$

$$\boldsymbol{A \backslash B \neq B \backslash A}$$

Solved exercises related to the chapter

<u>Exercise 01:</u>

We throw a randomly balanced dice whose sides are numbered from 1 to 6. Let event A get an even number and event B get a number greater than 3.

Find the following events:

1- $\mathbf{A \cup B}$

2- $\mathbf{A \cap B}$

3- $\mathbf{\bar{A}, \bar{B}}$

4- $\mathbf{\bar{A} \cap B}$

5- $\mathbf{\bar{A} \cup \bar{B}}$

6- $\mathbf{\overline{A \cup B}}$

7- $\mathbf{\overline{A \cap B}}$

8- $\mathbf{\cap \bar{B}\bar{A}}$

9- $\mathbf{B \backslash A}$

Solution:

This exercise aims to find the relationship between two events of the same type, as the results of the experiment, where we define events as groups consisting of elements corresponding to the fulfillment of each event, are as follows:

Ω = {1, 2, 3, 4, 5, 6} **A = {2, 4, 6}** **B = {4, 5, 6]**

1- **<u>A ∪ B</u>**:

The event **A ∪ B** corresponds to the realization of **A** or **B**, meaning that the number obtained is an even number or greater than **3**. In other words:

$$\boldsymbol{A \cup B = B \cup A = \{w_i \in \Omega : w_i \in A \vee w_i \in B\}}$$

$$A \cup B = B \cup A = \{2, 4, 5, 6\}$$

2- **$\underline{A \cap B}$:**

The event **A ∩ B** agrees with the realization of **A** and **B**, meaning that the resulting number is an even number and greater than **3**. In other words:

$$A \cap B = B \cap A = \{w_i \in \Omega : w_i \in A \land w_i \in B\}$$
$$A \cap B = B \cap A = \{4, 6\}$$

3- **$\underline{\bar{B}\ , \bar{A}}$:**

The event $\bar{A}$ corresponds to the non-occurrence of A, meaning that the number obtained is not an even number, and they are the elements that do not belong to A. In other words:

$$\bar{A} = \{w_i \in \Omega : w_i \notin A\}$$
$$\bar{A} = \{1, 3, 5\}$$

The event $\bar{B}$ corresponds to the non-occurrence of **B**, meaning that the number obtained is not an even number, and they are the elements that do not belong to **B**. In other words:

$$\bar{B} = \{w_i \in \Omega : w_i \notin B\}$$
$$\bar{A} = \{1, 2, 3\}$$

4- **$\underline{\bar{A} \cap B}$:**

The event $\underline{\bar{A} \cap B}$ corresponds to the non-occurrence of **A** and the occurrence of **B**, meaning that the resulting number is not an even number and is a number greater than 3. In other words:

$$\bar{A} \cap B = B \cap \bar{A} = \{w_i \in \Omega : w_i \in \bar{A} \land w_i \in B\}$$
$$\bar{A} \cap B = B \cap \bar{A} = \{w_i \in \Omega : w_i \notin A \land w_i \in B\}$$
$$\bar{A} \cap B = B \cap \bar{A} = \{5\}$$

5- **$\underline{\bar{B} \cup \bar{A}}$**

The event $\bar{B} \cup \bar{A}$ corresponds to the non-occurrence of **A** or the non-occurrence of **B**, i.e. not obtaining an even number or not obtaining greater than **3**, i.e. the fulfillment of only one of the two events. In other words:

$$\bar{A} \cup \bar{B} = \bar{B} \cup \bar{A} = \{w_i \in \Omega : w_i \in \bar{A} \lor w_i \in \bar{B}\}$$
$$\bar{A} \cup \bar{B} = \bar{B} \cup \bar{A} = \{1, 2, 3, 5\}$$

6- **$\underline{\overline{A \cup B}}$:**

The event $\underline{\overline{A \cup B}}$ agrees that A or B does not occur, meaning that the number obtained is not even and is not greater than 3 at the same time, i.e.:

$$\overline{A \cup B} = \overline{B \cup A} = \{w_i \in \Omega : w_i \notin A \cup B\}$$
$$\overline{A \cup B} = \overline{B \cup A} = \{1\}$$

7- $\overline{\mathbf{A} \cap \mathbf{B}}$:

The event $\overline{\mathbf{A} \cap \mathbf{B}}$ corresponds to the non-occurrence of A and B together, meaning that the number obtained is not even and is not greater than 3 at the same time, meaning that only one of the two events occurred. In other words, it is:

$$\overline{A \cap B} = \overline{B \cap A} = \{w_i \in \Omega : w_i \notin A \cap B\}$$

$$\overline{A \cap B} = \overline{B \cap A} = \{1, 2, 3, 5\} = \bar{A} \cup \bar{B}$$

8- $\bar{\mathbf{A}} \cap \bar{\mathbf{B}}$:

The event $\bar{\mathbf{A}} \cap \bar{\mathbf{B}}$ corresponds to both the non-fulfillment of A and the non-fulfillment of B, meaning that the number obtained must not be even and must not be greater than 3. In other words:

$\bar{A} \cap \bar{B} = \bar{B} \cap \bar{A} = \{w_i \in \Omega : w_i \in \bar{A} \wedge w_i \in \bar{B}\}$

$\bar{A} \cap \bar{B} = \bar{B} \cap \bar{A} = \{w_i \in \Omega : w_i \notin \bar{A} \wedge w_i \notin \bar{B}\}$

$\bar{A} \cap \bar{B} = \bar{B} \cap \bar{A} = \{1\} = \overline{A \cup B}$

9- **B\A**:

The occurrence of event B\A corresponds to the occurrence of event B and the failure of event A, meaning that the number obtained is greater than 3 and is not even. In other words:

$B \backslash A = \{w_i \in \Omega : w_i \in B \wedge w_i \notin A\}$

$B \backslash A = \{w_i \in \Omega : w_i \in B \wedge w_i \in \bar{A}\}$

$B \backslash A = \{w_i \in \Omega : w_i \in B \cap \bar{A}\} = \{5, 3\}$

Exercise 02:

An experiment consists of throwing a coin and a dice at the same time. Let the following events occur:

A = Appearance of writing with an odd number.

B = head appears with an even number.

C = appearance of a prime number.

1. Find the space sample corresponding to this experiment.
2. Identify incidents C, B, A.
3. Express the incidents: the occurrence of A or B, the occurrence of B and C, the occurrence of event B only, the occurrence of all incidents, the occurrence of at most one event.
4. Which of the events C, B, A are incompatible with each other?

Exercise 03:

Three consecutive shots are fired at a target, and let the event $\mathbf{A}_k$ hit the target with the k^{th} shot, and the event $\overline{\mathbf{A}}_k$ not hit the target with the k^{th} shot, so that k = 1, 2, 3. Using operations on the events $\mathbf{A}_k$, $\overline{\mathbf{A}}_k$, write the following events:

B: Hitting the target in all three shots.

C: The target was not hit in all three shots.

D: Hitting the target in one shot.

E: Hitting the target in two shots.

F: Hitting the target with at most two shots.

G: Hit the target at least once.

Solution :

1. **Writing the event B:**

Event **B** is the target being hit in three shots, i.e. hitting the target with the first shot (i.e. achieving $\mathbf{A}_1$), hitting it with the second shot (i.e. achieving $\mathbf{A}_2$) and hitting it with the third shot (i.e. achieving $\mathbf{A}_3$). We write:

$$\mathbf{B} = \mathbf{A}_1 \cap \mathbf{A}_2 \cap \mathbf{A}_3$$

2. **Writing the event C:**

Event **C** is that the target was not hit in the three shots, that is, it was not hit in the first shot (i.e., $\overline{\mathbf{A}}_1$ was achieved), it was not hit in the second shot (i.e., $\overline{\mathbf{A}}_2$ was achieved), and it was not hit in the third shot (i.e., $\overline{\mathbf{A}}_3$ was achieved), and we write:

$$\mathbf{C} = \overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2 \cap \overline{\mathbf{A}}_3$$

3. **Writing the eventD:**

Event D is the target being hit in one shot, i.e. hitting the target with the first shot (i.e., achieving $\mathbf{A}_1$), not hitting it with the second shot (i.e., achieving $\overline{\mathbf{A}}_2$), not hitting it with the third shot (i.e., achieving $\overline{\mathbf{A}}_3$), or not hitting it with the first shot (i.e., achieving $\overline{\mathbf{A}}_1$) and he was injured in the second shot (i.e., $\mathbf{A}_2$ was achieved) and he was not injured in the third shot (i.e., $\overline{\mathbf{A}}_3$ was achieved), or he was not injured in the first shot (i.e., $\overline{\mathbf{A}}_1$was achieved) and he was not injured in the second shot (i.e., $\overline{\mathbf{A}}_2$ was achieved) and he was injured in the third shot (i.e., $\mathbf{A}_3$ was achieved).) and we write:

$$\mathbf{D} = (\mathbf{A}_1 \cap \overline{\mathbf{A}}_2 \cap \overline{\mathbf{A}}_3) \cup (\overline{\mathbf{A}}_1 \cap \mathbf{A}_2 \cap \overline{\mathbf{A}}_3) \cup (\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2 \cap \mathbf{A}_3)$$

4. **Writing the event E:**

Event **E** is the target being hit in two shots, i.e. hitting the target with the first shot (i.e. achieving $\mathbf{A}_1$) and hitting it with the second shot (i.e. achieving $\mathbf{A}_2$) and not hitting it with the third shot (i.e. achieving $\overline{\mathbf{A}}_3$), or hitting it with the first shot (i.e. achieving $\mathbf{A}_1$) and not hitting it In the second shot (i.e. $\overline{\mathbf{A}}_2$ was achieved) and he was injured in the third shot (i.e. A_3 was achieved), or he was not injured in the first shot (i.e. $\bar{A}_1$ was achieved) and he was injured in the second shot (i.e. A_2 was achieved) and he was injured in the third shot (i.e. A_3 was achieved) and we write:

$$E = (A_1 \cap A_2 \cap \bar{A}_3) \cup (A_1 \cap \bar{A}_2 \cap A_3) \cup (\bar{A}_1 \cap A_2 \cap A_3)$$

5. <u>Writing the event F:</u>

The event **F** is when the target was hit with two shots at most, that is, it was not hit by all three shots (event **C** occurred), or it was hit by one shot (event **D** occurred), or it was hit by two shots (event **E** occurred). We write:

F = C U D U E

$$F = (\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)$$
$$\cup ((A_1 \cap \bar{A}_2 \cap \bar{A}_3) \cup (\bar{A}_1 \cap A_2 \cap \bar{A}_3) \cup (\bar{A}_1 \cap \bar{A}_2 \cap A_3))$$
$$\cup ((A_1 \cap A_2 \cap \bar{A}_3) \cup (A_1 \cap \bar{A}_2 \cap A_3) \cup (\bar{A}_1 \cap A_2 \cap A_3))$$

6. <u>Writing the event G:</u>

The event **G** is when the target was hit by at least one shot, that is, the target was hit by one shot (event **D** occurred), or it was hit by two shots (event **E** occurred), or it was hit by three shots (event **B** occurred). We write:

G = D U E U B

$$G = ((A_1 \cap \bar{A}_2 \cap \bar{A}_3) \cup (\bar{A}_1 \cap A_2 \cap \bar{A}_3) \cup (\bar{A}_1 \cap \bar{A}_2 \cap A_3))$$
$$\cup ((A_1 \cap A_2 \cap \bar{A}_3) \cup (A_1 \cap \bar{A}_2 \cap A_3) \cup (\bar{A}_1 \cap A_2 \cap A_3))$$
$$\cup (\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)$$

Chapter two:

Somebasic laws of probability

1. Definition of probability:

It is used to quantify the chance or expresses the quantitative estimate of the possibility that an outcome of a random experiment will occur by assigning a positive value ranged between zero and one (or a percentage from 0 to 100%).

1.1 Classical Approach

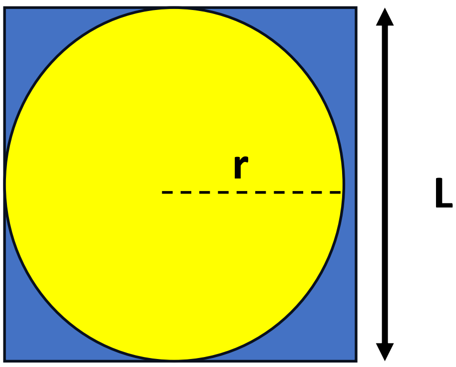
When the outcomes in the sample space of a chance experiment are equally likely, the probability of an event **A,** denoted by **P(A),** is the ratio of the number of outcomes favorable to **A** to the total number of outcomes in the sample space

Using the principle of insufficient cause, i.e. the principle of equality of possibility of appearance, or in general, if we consider **N** simple events $\mathbf{w_i \in \Omega}$ of equal possibility, then each simple event can be associated to the same probability, which is the reciprocal of the number of events composing $\mathbf{\Omega}$, that is, it is equal to $\frac{1}{N}$

$$P(A) = \frac{\textbf{Number of Favorable Outcomes for Event A}}{\textbf{Total Number of Possible Outcomes in the Sample Space}}$$

1.2 Geometric Approach

Let **(Ω)** be the space of elementary events corresponding to the experiment **(E)** that includes an infinite number of elementary events $\mathbf{(w_i)}$ that can be interpreted as the coordinates of a point in the space $\mathfrak{R}^n$ (n=1,2,3), and the events as some region of this space Ω, and thus the events have a measure ($\mathfrak{R}$) is length, ($\mathfrak{R}^2$) is area, and ($\mathfrak{R}^3$) is volume.



$$P(A) = \frac{area\ of\ circle}{area\ of\ square}$$

$$P(A) = \frac{\pi r^2}{L^2}$$

1.3 Relative Frequency Approach

The probability of an event **A**, denoted by **P(A)**, is defined to be the value approached by the relative frequency of occurrence of **A** when a chance experiment is performed many times. If the number of times the chance experiment is performed is quite large

$$P(A) \approx \frac{number\ of\ times\ A\ occurs}{number\ of\ times\ the\ experiment\ is\ performed}$$

$$= \lim \frac{n}{}$$

2. Fundamental Properties of Probability

In probability theory, there are three basic axioms on which probability calculations are based, called Kolmogorov's axioms, which are:

1. If **A** is a specific event in the sample space, then the probability is a positive numerical value between zero and one, and we denote:

$$0 \leq P(A_i) \leq 1$$

2. The probability of the sure event **Ω** being achieved is equal to one, and therefore the probability of an impossible event being achieved **∅** is equal to zero, we denote it:

$$P(\Omega) = 1 \Longrightarrow P(\emptyset) = 0$$

3. If we have n mutually exclusive events, A_i, two by two, then the probability of the union of these events is equal to the sum of the probabilities of these events.

$$P(A_1 \cup A_2 \cup \dots A_n) = P(A_1) + P(A_2) + \cdots P(A_n) \quad \forall A_i \neq A_j$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Based on these axioms, it can be concluded that the probability of the complementary event is equal to one minus the probability of the original event, as we have:

$$A \cup \bar{A} = \Omega \Longrightarrow P(A \cup \bar{A}) = P(\Omega) = 1$$

Since the:

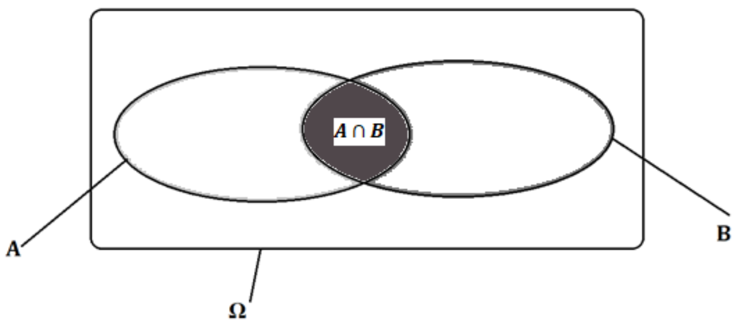
$$A \cap \bar{A} = \emptyset \Longrightarrow P(A \cup \bar{A}) = P(A) + P(\bar{A}) = 1$$

Hence:

$$P(\bar{A}) = 1 - P(A)$$

3. The rule of calculating the probability of two events union:

If we have two events related to the experiment, and let us assume that they intersect and are not mutually exclusive, as shown in the figure.



The union of these two events is the event associated with the fulfillment of at least one of the two events, that is, the first event achieved obtaining one of the results corresponding to it without the second event being achieved, and it is written with $(A \cap \bar{B})$, or the second event was achieved by obtaining one of the results corresponding to it without the first event being achieved, and it is written as $(B \cap \bar{A})$, or they were achieved simultaneously, meaning obtaining one of the results corresponding to the first event and the second event at the same time, written as (A∩B).

$$\boldsymbol{A \cup B = \left\{w_i \in \Omega, w_i \in \mathbf{A} \text{ أو } w_i \in \mathbf{B}\right\}}$$

We can write event **A** as the union of two mutually exclusive events as follows:

$\boldsymbol{A = (A \cap \bar{B}) \cup (A \cap B)}$ whereas $\boldsymbol{(A \cap \bar{B}) \cap (A \cap B) = \emptyset}$

Relying on the third axioms we find:

$$\boldsymbol{P(A) = P\big((A \cap \bar{B}) \cup (A \cap B)\big) = P(A \cap \bar{B}) + P(A \cap B)}$$

Hence:

$$\boldsymbol{P(A \cap \bar{B}) = P(A) - P(A \cap B)}$$

Also, we can write event **B** as the union of two mutually exclusive events as follows:

$\boldsymbol{B = (B \cap \bar{A}) \cup (A \cap B)}$ whereas $\boldsymbol{(B \cap \bar{A}) \cap (A \cap B) = \emptyset}$

Relying on the third axioms we find:

$$\boldsymbol{P(B) = P\big((B \cap \bar{A}) \cup (A \cap B)\big) = P(B \cap \bar{A}) + P(A \cap B)}$$

Hence:

$$\boldsymbol{P(B \cap \bar{A}) = P(B) - P(A \cap B)}$$

So, the probability of the union of A and B is equal to the probability of union of the three mutually exclusive events: $(A \cap \bar{B}), (B \cap \bar{A})$ and (A∩B).

Relying on the third axioms, the axioms of probability then the probability of the union of these events is equal to the sum of the probabilities of these events, and we denote:

$$\boldsymbol{P(A \cup B) = P\big((A \cap \bar{B}) \cup P(A \cap B) \cup (B \cap \bar{A})\big)}$$

$$\boldsymbol{P(A \cup B) = P(A \cap \bar{B}) + P(A \cap B) + P(B \cap \bar{A})}$$

$$\boldsymbol{P(A \cup B) = P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B)}$$

Hence, the probability of the combination of two events is equal to the sum of their probability minus the probability of their intersection.

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

4.

Conditional Probability:

Sometimes the knowledge that one event has occurred changes our assessment of the likelihood that another event occurs, and probabilities need to be reevaluated as additional information becomes available.

To understand this concept let us these two events:

L= born in a long month = {Jan, Mar, May, Jul, Aug, Oct, Dec}

R = born in a month with the letter r = {Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec}

$$P(L) = \frac{7}{12} \qquad\qquad P(R) = \frac{8}{12}$$

R ∩ L = {Jan, Mar, Oct, Dec} ⇒ P(R ∩ L) = 4/12 =1/3

Now suppose that it is known about the person we meet in the street that he was born in a "long month," and we wonder whether he was born in a "month with the letter r." We call this the conditional probability of **R** given **L** and we denote it:

$$\boldsymbol{P(R|L) = \frac{4}{7}}$$

To generalize this concept, let us have two events A and B. The conditional probability is the association of event A with event B. The probability of A is calculated after event B occurs, and we symbolize it with the symbol P(A/B) and it is equal to:

$$\boldsymbol{P(A/B) = \frac{P(A \cap B)}{P(B)} \quad .P(B) > 0}$$

$$\boldsymbol{P(B/A) = \frac{P(A \cap B)}{P(A)} \quad .P(A) > 0}$$

Hence, the probability of the intersection of two events **A** and **B** is the product of the probability of one of them occurring multiplied by the probability of the other event occurring, knowing that the first has occurred.

$$\boldsymbol{P(A \cap B) = P(B).P(A/B)}$$

Or

$$\boldsymbol{P(A \cap B)P(A).P(B/A)}$$

Notes:

1. We say that two events (A and B) are independent of each other if:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = P(B)$$

Hence, the probability of intersection of two independent events becomes as follows:

$$P(A \cap B) = P(B \cap A) = P(A).P(B)$$

Here the general case is written in the form for **n** events independent of each other:

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i) = P(A_1).P(A_2)\ldots P(A_n)$$

2. We say that two events (**A** and **B**) are related if:

$$P(A/B) \neq P(B)$$

$$P(B/A) \neq P(A)$$

5. Total probability:

If **A** is the result of one of the mutually exclusive and complementary events $\mathbf{H_1}$, $\mathbf{H_2}$…$\mathbf{H_n}$ as in the following figure:

$$\begin{cases} \xrightarrow{P(H_1)} H_1 \begin{cases} \rightarrow \bar{A} \\ \rightarrow A \end{cases} \\ \xrightarrow{P(H_2)} H_2 \begin{cases} \rightarrow \bar{A} \\ \rightarrow A \end{cases} \\ \quad\quad\quad \vdots \\ \xrightarrow{P(H_2)} H_n \begin{cases} \rightarrow \bar{A} \\ \rightarrow A \end{cases} \end{cases}$$

The occurrence of event **A** is the result of the occurrence of one of the incidents $\mathbf{H_i}$, since **A** cannot be achieved by more than one event, $\mathbf{H_i}$. In other words, the occurrence of event **A** as a result of more than one event $\mathbf{H_i}$ is an impossible event, which that indicates that events $\mathbf{H_i}$ are mutually exclusive events, and their intersection is an impossible event, and this can be written using the group expression as follows:

$$H_i \cap H_j = \emptyset \quad \forall i \neq j$$

On the other hand, the realization of event **A** cannot occur except by the realization of one of the events $\mathbf{H_i}$, and therefore the realization of event **A** is the result of the realization of all the events $\mathbf{H_i}$, which that indicates that the events $\mathbf{H_i}$ are integrated events and their union is equal to the empty space of the primary events **Ω**. This can be written using the group expression as follows:

$$\bigcup_{i=1}^{n} H_i = H_1 \cup H_2 \cup \dots \cup H_n = \Omega$$

The realization of incident **A** is directly linked to the realization of one of the incidents **H_i**, meaning that incident **A** is achieved by the occurrence of incident **H_1**, or by the occurrence of incident **H_2**, or...or by the realization of incident **H_n**, that is, it can be written:

$$A = A \cap (H_1 \cup H_2 \cup \dots . H_n)$$

Since the intersection process **(∩)** is distributive to the union process **(∪)**, the previous relationship becomes as follows:

$$A = (A \cap H_1) \cup (A \cap H_2) \cup (A \cap H_3) \cup \dots \cup (A \cap H_n)$$

Then calculating the probability of event, **A** takes us back to calculating the probability of the second side of the equation, which is the union of a group of events, i.e. writing the following:

$$P(A) = P\big((A \cap H_1) \cup (A \cap H_2) \cup (A \cap H_3) \cup \dots \cup (A \cap H_n)\big)$$

By applying the probability rule for the combination of mutually exclusive events, which stipulates that the probability of a group of mutually exclusive events coming together twofold is equal to the sum of the probabilities of these events, i.e.:

$$P(A) = P(A \cap H_1) + P(A \cap H_2) + P(A \cap H_3) + \dots + P(A \cap H_n)$$

We note that the probability of event **A** occurring is equal to the sum of a set of terms, each term representing the probability of the intersection of two events, which is equal to the probability of one of these events multiplied by the probability of the other event, knowing that the first event has occurred, according to the constitution of the probability of the intersection of two events, meaning that:

$$\begin{cases} P(A \cap H_1) = P(H_1).P(A/H_1) \\ P(A \cap H_2) = P(H_2).P(A/H_2) \\ \vdots \\ P(A \cap H_n) = P(H_n).P(A/H_n) \end{cases}$$

From it we have:

$$P(A) = P(H_1).P(A/H_1) + P(H_2).P(A/H_2) + \dots + P(H_n).P(A/H_n)$$

In the end, the relationship becomes:

$$\boxed{P(A) = \sum_{i=1}^{n} P(H_i).P(A/H_i)}$$

6. Bayes' conditional probability rule:

If **(Ω)** is the space of the initial events of an experiment, and **A** is an event related to one of the events H_i constituting **(Ω)** and H_k is any event from **(Ω)**, then:

$$P(A) = \sum_{i=1}^{n} P(H_i).P(A/H_i)$$

$$P(A \cap H_k) = P(A).P(H_k/A)$$

$$P(A \cap H_k) = P(H_k).P(A/H_k)$$

Hence:

$$P(A).P(H_k/A) = P(H_k).P(A/H_k)$$

Therefore:

$$P(H_k/A) = \frac{P(H_k).P(A/H_k)}{P(A)}$$

The Bayes' conditional probability rule is given as follows:

$$P(H_k/A) = \frac{P(H_k).P(A/H_k)}{\sum_{i=1}^{n} P(H_i).P(A/H_i)}$$

Example:

A factory for producing electronic parts uses three machines, M_1, M_2, and M_3. The first machine, M_1, is used daily to produce **40%** of the parts, the second machine, M_2, is used to produce **35%** of the parts, while the third machine, M_3, is used to produce **25%** of the parts daily.

According to the Quality Control Department, **10%** of the pieces produced using the first machine, M_1, do not meet quality standards. In contrast, **5%** of the pieces produced using the second machine, M_2, do not meet quality standards, while **1%** of the pieces produced using the third machine, M_3, do not meet quality standards.

If a piece is chosen randomly, what is the probability that this piece does not meet the quality standards?

Solution:

To answer this question, we use the principle of total probability, but before that, let us know the following events:

1. **A**: The chosen piece does not meet quality standards.
2. H_i: the piece must be produced using the machine M_i, where **i=1,2,3**, meaning:

- H_1 that the part is produced using machine M_1;
- H_2 that the part is produced using machine M_2;
- H_3 that the part is produced using the M_3 machine.

Conversely, let us know the following possibilities:

1. **P(A)** The probability that the chosen piece does not meet quality

2. **$P(H_i)$** is the probability that the chosen piece will be produced using the machine **M_i**, where **i = 1, 2, 3**, meaning that:

- **$P(H_1)$** is the probability that the drawn piece was produced using machine **M_1** and is equal to the percentage of pieces produced using machine **M_1** out of the total pieces produced daily, where **$P(H_1) = 0.4$**.
- **$P(H_2)$** is the probability that the drawn piece was produced using the **M_2** machine and is equal to the percentage of the pieces produced using the **M_2** machine out of the total pieces produced daily, where **$P(H_2) = 0.35$**.
- **$P(H_3)$** is the probability that the drawn piece was produced using the **M_3** machine and is equal to the percentage of pieces produced using the M3 machine out of the total pieces produced daily, where **$P(H_3) = 0.35$.**

3. **$P(A/H_i)$**, which is the probability that a piece that does not meet quality standards was produced using the machine **M_i**, where **i = 1, 2, 3**, meaning that:

- **$P(A/H_1)$** is the probability that a part that does not meet quality standards was produced using machine **M_1**, since **$P(A/H_1) = 0.10$**.
- **$P(A/H_2)$** is the probability that a part that does not meet quality standards was produced using machine **M_2**, since **$P(A/H_2) = 0.05$**.
- **$P(A/H_3)$** is the probability that a part that does not meet quality standards was produced using machine **M_3**, where **$P(A/H_3)=0.01$**.

$$\Omega: \begin{cases} \xrightarrow{P(H_1)=0.4} M_1 \begin{cases} \xrightarrow{0.90} \bar{A} \\ \xrightarrow{0.10} A \end{cases} \\ \xrightarrow{P(H_2)=0.35} M_2 \begin{cases} \xrightarrow{0.95} \bar{A} \\ \xrightarrow{0.05} A \end{cases} \\ \xrightarrow{P(H_3)=0.25} M_3 \begin{cases} \xrightarrow{0.99} \bar{A} \\ \xrightarrow{0.01} A \end{cases} \end{cases}$$

The occurring of the event **A** is directly linked to the occurring of one of the events **H_i**, since for this piece to be non-compliant with quality standards, it must be produced using one of the machines **M_i** and not conform to quality standards. In more detailed words, the part must be produced using machine **M_1** (i.e., verification **H_1**) and that it is simultaneously non-compliant with quality standards (i.e., incident **A** is achieved), or the part is produced using machine **M_2** (i.e., **H_2** is verified) and it is simultaneously non-compliant with quality standards (i.e., incident **A** is achieved), or the part is produced Using machine **M_3** (i.e., **H_3** is achieved) and at the same time not meeting the quality standards (i.e., incident **A** is achieved). The previous paragraph can be rewritten using the group expression as follows:

$$A = (A \cap H_1) \cup (A \cap H_2) \cup (A \cap H_3)$$

Then calculating the probability of the event **A** takes us back to calculating the probability of the second side of the equation, which is the union of a group of events, i.e. writing the following:

$$P(A) = P\big((A \cap H_1) \cup (A \cap H_2) \cup (A \cap H_3)\big)$$

Also, we have to indicate that the process of producing of any specific piece is done using one machine and only one machine, as it is not possible to find an electronic piece that was produced using two or more machines. In other words, producing a piece using more than one machine is an impossible event, and this, if this indicates anything, indicates that accidents **H_i**, mutually exclusive events, and their intersection is an impossible event, and this can be written using the set expression as follows:

$$H_i \cap H_j = \emptyset \quad \forall i \neq j \quad \text{أي} \quad \begin{cases} H_1 \cap H_2 = \emptyset \\ H_1 \cap H_3 = \emptyset \\ H_2 \cap H_3 = \emptyset \end{cases}$$

On the other hand, any electronic piece of the total daily production is produced using one of the **M_i** machines, and therefore the total daily production of electronic parts is equal to the sum of the total production of machine **M_1**, the total production of machine **M_2**, and the total production of machine **M_3**. Which that means that the events **H_i**, are integrated incidents and their union is equal to the sample space of the primary events **Ω**, and this can be written using the group expression as follows:

$$\bigcup_{i=1}^{3} H_i = H_1 \cup H_2 \cup H_3 = \Omega$$

By applying the rule of calculating the probability of mutually exclusive events union, which stipulates that is equal to the sum of the probabilities of these events, i.e.:

$$P(A) = P(A \cap H_1) + P(A \cap H_2) + P(A \cap H_3)$$

We note that the probability of event A occurring is equal to the sum of a set of terms, each term representing the probability of the intersection of two events, which is equal to the probability of one of these events multiplied by the probability of the other event, knowing that the first event has occurred, according to the constitution of the probability of the intersection of two events, meaning that:

$$\begin{cases} P(A \cap H_1) = P(H_1).P(A/H_1) \\ P(A \cap H_2) = P(H_2).P(A/H_2) \\ P(A \cap H_3) = P(H_3).P(A/H_3) \end{cases}$$

From it we have:

$$P(A) = P(H_1).P(A/H_1) + P(H_2).P(A/H_2) + P(H_3).P(A/H_3)$$

Substituting the value of the probabilities with their equal value, we find:

$$P(A) = \left((0.4) \times (0.1)\right) + \left((0.35) \times (0.05)\right) + \left((0.25) \times (0.01)\right)$$

$$P(A) = (0.04) + (0.0175) + (0.0025) = 0.06$$

The probability that any randomly chosen piece from the daily production of this factory does not meet quality standards is equal to **0.06**, meaning that **6%** of this institution's daily production does not meet quality standards.

To answer the second question, we are going to use the Bays' Conditional Probability rule; but before that, let us know the events **P(H$_i$/A)**, which is the probability that the drawn piece was produced using the machine **M$_i$** even though it does not meet the quality standards, since **i = 1, 2, 3**, that is:

Probability **P(H$_1$/A)**, which is the probability that the drawn piece was produced using machine **M$_1$** even though it did not meet the quality standards.

Probability **P(H$_2$/A)**, which is the probability that the drawn piece was produced using machine **M$_2$** even though it did not meet the quality standards.

Probability **P(H$_3$/A)**, which is the probability that the drawn piece was produced using the **M$_3$** machine even though it did not meet the quality standards. Applying Bayes' rule to conditional probability, we find:

$$P(H_1/A) = \frac{P(H_1)P(A/H_1)}{P(A)} = \frac{(0.4) \times (0.1)}{0.06} = 0.67$$

$$P(H_2/A) = \frac{P(H_2)P(A/H_2)}{P(A)} = \frac{(0.35) \times (0.05)}{0.06} = 0.29$$

$$P(H_3/A) = \frac{P(H_3)P(A/H_3)}{P(A)} = \frac{(0.25) \times (0.01)}{0.06} = 0.04$$

Solved exercises related to the chapter

Exercise 01:

Let we have two events **A** and **B**, prove that:

P (A\ B) = P (A) – P (A ∩ B)

P (A ∪ B) = 1 – P($\bar{A}$ ∩ $\bar{B}$)

Let us now assume the two events A and B such that:

$$P(A) = \frac{3}{8} \qquad P(B) = \frac{1}{2} \qquad P(A \cap B) = \frac{1}{4}$$

Calculate the following probabilities:

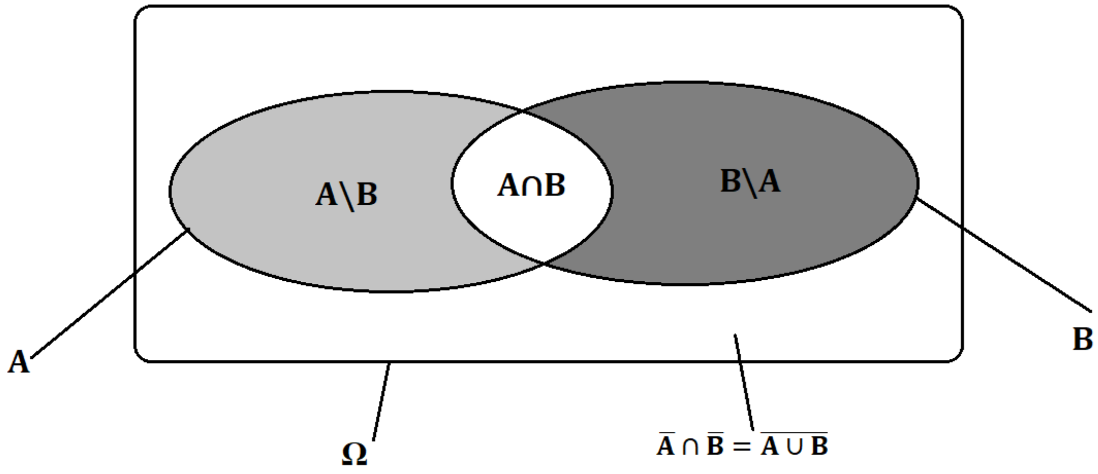
1- **P(A ∪ B)** **4 -** **P($\bar{A}$ ∪ $\bar{B}$)**

2- $P(\bar{A})$, $P(\bar{B})$ 5 - $P(A \cap \bar{B})$

3- $P(\bar{A} \cap \bar{B})$ 6 - $P(\bar{A} \cap B)$

Solution:

Let us define the two events A and B associated with the random experiment and which belong to the sample set of simple events Ω, as shown in the following figure:



Let us know the following events:

$\bar{A} = \{w_i \in \Omega : w_i \notin A\}$

$\bar{B} = \{w_i \in \Omega : w_i \notin B\}$

$A \backslash B = \{w_i \in \Omega : w_i \in A \wedge w_i \notin B\} = A \cap \bar{B}$

$B \backslash A = \{w_i \in \Omega : w_i \in B \wedge w_i \notin A\} = \bar{A} \cap B$

1- **We can write the event A as following:**

$$A = (A \backslash B) \cup (A \cap B) = (A \cap \bar{B}) \cup (A \cap B)$$

Hence, the probability of event A is equal to the probability of the two events $(A \cap \bar{B})$ **and** $(A \cap B)$ **that is:**

$$P(A) = P\big((A \cap \bar{B}) \cup (A \cap B)\big)$$

Since $(A \cap \bar{B})$and $(A \cap B)$ are two mutually exclusive events:

$$(A \cap \bar{B}) \cap (A \cap B) = (A \cap A) \cap (\bar{B} \cap B)$$

$$A \cap \emptyset = \emptyset$$

The probability of their union is equal to the sum of the two probabilities as:

$$P(A) = P(A \cap \bar{B}) + P(A \cap B)$$

Hence:

$$P(A \cap \bar{B}) = P(A) + P(A \cap B)$$

And we have:

$$P(A\backslash B) = P(A) - P(A \cap B)$$

2- The events $\bar{A}$ and $\bar{B}$ are defined as:

$\bar{A} = \{w_i \in \Omega, w_i \notin A\}$

$\bar{B} = \{w_i \in \Omega, w_i \notin B\}$

$\bar{A} \cap \bar{B} = \{w_i \in \Omega, w_i \in \bar{A} \wedge w_i \in \bar{B}\}$

$\bar{A} \cap \bar{B} = \{w_i \in \Omega, w_i \notin A \wedge w_i \notin B\}$

$\bar{A} \cap \bar{B} = \{w_i \in \Omega, w_i \notin A \cup B\}$

$\bar{A} \cap \bar{B} = \{w_i \in \Omega, w_i \in \overline{A \cup B}\}$

$\bar{A} \cap \bar{B} = \overline{A \cup B}$

$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$

<u>Calculate P(A ∪ B):</u>

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = \frac{3}{8} + \frac{1}{2} - \frac{1}{4} = \frac{5}{8}$

<u>Calculate P(Ā) and P(B̄):</u>

$P(\bar{A}) = 1 - P(A) = 1 - \frac{3}{8} = \frac{5}{8}$

$P(\bar{B}) = 1 - P(B) = 1 - \frac{1}{2} = \frac{1}{2}$

<u>Calculate $P(\bar{A} \cap \bar{B})$:</u>

$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - \frac{5}{8} = \frac{3}{8}$

<u>Calculate $P(\bar{A} \cup \bar{B})$:</u>

$P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B})$

$P(\bar{A} \cup \bar{B}) = \frac{5}{8} + \frac{1}{2} - \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$

<u>Calculate $P(A \cap \bar{B})$:</u>

$P(A \cap \bar{B}) = P(A\backslash B) = P(A) - P(A \cap B)$

$P(A \cap \bar{B}) = \frac{3}{8} - \frac{1}{4} = \frac{1}{8}$

<u>Calculate $P(\bar{A} \cap B)$:</u>

$P(\bar{A} \cap B) = P(B\backslash A) = P(B) - P(A \cap B)$

$P(\bar{A} \cap B) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Exercise 02:

Let us now assume the two events A and B such that:

$$P(A) = \frac{1}{2} \qquad P(B) = \frac{1}{3} \qquad P(A \cap B) = \frac{1}{4}$$

Calculate the following probabilities:

P(A/B) **P(B/A)** **P(A U B)**

Solution:

Calculate $P(A \cup B)$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$$

Calculate $P(B/A)$:

$$P(A \cap B) = P(A) \cdot P(B/A) \Rightarrow P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(B/A) = \frac{1/4}{1/2} = \frac{1}{2}$$

Calculate $P(A/B)$:

$$P(A \cap B) = P(AB) \cdot P(A/B) \Rightarrow P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A/B) = \frac{1/4}{1/3} = \frac{3}{4}$$

Exercise 03:

Let us now assume the two events A and B such that:

$$P(A) = \frac{1}{2} \qquad P(B) = \frac{1}{3} \qquad P(B/\bar{A}) = \frac{1}{4}$$

Calculate **P(A U B)**

Solution:

Based on the relationship between event $\bar{A}$ and event B, which states:

$$P(\bar{A} \cap B) = P(\bar{A}) \cdot P(B/\bar{A}) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

On the other hand, we have:

$$P(B) = P(\bar{A} \cap B) + P(A \cap B)$$

$$\Rightarrow P(A \cap B) = P(B) - P(\bar{A} \cap B)$$

$$\Rightarrow P(A \cap B) = \frac{1}{3} - \frac{1}{8} = \frac{5}{24}$$

Then P(A∪B) can be calculated as follows:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = \frac{1}{2} + \frac{1}{3} - \frac{5}{24} = \frac{5}{8}$$

Exercise 04:

The production process in an industrial workshop depends on three machines. The probability that the first machine will break down on a given day is 0.05, the second will break down with a probability of 0.10, and the third with a probability of 0.15. What is the probability that the following events will occur during this day:

1. Only one machine will break down?
2. Only two machines will break down?
3. No machine will break down?
4. At least one machine will break down?

Solution:

Let us call the event H_i when machine I breaks down, since: **i = 1, 2, 3**, meaning we know the following events:

- H_1 Event occurs when the first machine breaks down with probability $P(H_1) = 0.05$, and therefore the probability that this machine does not break down is equal to: $P(\overline{H}_1) = 1 - P(H_1) = 1 - 0.05 = 0.95$
- H_2 Event occurs when the second machine breaks down with probability $P(H_2) = 0.10$, and therefore the probability that this machine does not break down is equal to: $P(\overline{H}_2) = 1 - P(H_2) = 1 - 0.10 = 0.90$.
- H_3 Event occurs when the third machine breaks down with probability $P(H_3) = 0.15$, and therefore the probability that this machine does not break down is equal to: $P(\overline{H}_3) = 1 - P(H_3) = 1 - 0.15 = 0.85$

1. Calculate P(A):

The event **A** occurs when one machine and only one machine break down, that is, when the first machine breaks down (H_1 is achieved) and the second machine does not break down ($\overline{H}_2$ is achieved) and the third machine does not break down ($\overline{H}_3$ is achieved), or the second machine breaks down (H_2 is achieved) and the machine does not break down The first machine ($\overline{H}_1$ is achieved) and the third machine does not break down ($\overline{H}_3$ is achieved), or the third machine breaks down (H_3 is achieved) and the first machine does not break down ($\overline{H}_1$ is achieved) and the second machine does not break down ($\overline{H}_2$ is achieved), and incident **A** can be expressed by the following relationship :

$$A = (H_1 \cap \overline{H}_2 \cap \overline{H}_3) \cup (\overline{H}_1 \cap H_2 \cap \overline{H}_3) \cup (\overline{H}_1 \cap \overline{H}_2 \cap H_3)$$

Hence, the probability of event **A** occurring is equal to:

$$P(A) = P\big((H_1 \cap \bar{H}_2 \cap \bar{H}_3) \cup (\bar{H}_1 \cap H_2 \cap \bar{H}_3) \cup (\bar{H}_1 \cap \bar{H}_2 \cap H_3)\big)$$

However, we note that event **A** consists of the union of a group of mutually exclusive events two by two, meaning that:

$$\begin{cases} (H_1 \cap \bar{H}_2 \cap \bar{H}_3) \cap (\bar{H}_1 \cap H_2 \cap \bar{H}_3) = \emptyset \\ (H_1 \cap \bar{H}_2 \cap \bar{H}_3) \cap (\bar{H}_1 \cap \bar{H}_2 \cap H_3) = \emptyset \\ (\bar{H}_1 \cap H_2 \cap \bar{H}_3) \cap (\bar{H}_1 \cap \bar{H}_2 \cap H_3) = \emptyset \end{cases}$$

Hence, the probability of the union of these events is equal to the sum of the probabilities of each one of them. In other words, we write:

$$P(A) = P(H_1 \cap \bar{H}_2 \cap \bar{H}_3) + P(\bar{H}_1 \cap H_2 \cap \bar{H}_3) + P(\bar{H}_1 \cap \bar{H}_2 \cap H_3)$$

On the other hand, we note that the probability of event **A** is equal to the sum of the probabilities of a group of intersecting incidents, and since the machines **I** are independent among themselves, the probability of the intersection of these events is equal to the product of their probabilities, and we write the following expression:

$$P(A) = P(H_1)P(\bar{H}_2)P(\bar{H}_3) + P(\bar{H}_1)P(H_2)P(\bar{H}_3) + P(\bar{H}_1)P(\bar{H}_2)P(H_3)$$

By numerical application we find:

$$P(A) = (0.05)(0.90)(0.85) + (0.95)(0.10)(0.15) + (0.95)(0.90)(0.15)$$

$$P(A) = (0.03825) + (0.8075) + (0.12825)$$

$$P(A) = 0.24725$$

2. **Calculate P(B):**

The event **B** occurs when two machines break down, that is, when the first and second machine break down (**H_1** and **H_2** are achieved) and the third machine does not break down ($\bar{H}_3$ is achieved), or the second and the third machine break down (**H_2** and **H_3** are achieved) and the first machine does not ($\bar{H}_1$ is achieved), or the first and third machine break down (**H_1 and H_3** are achieved) and the second machine does not break down ($\bar{H}_2$ is achieved), and incident **B** can be expressed by the following relationship :

$$B = (H_1 \cap H_2 \cap \bar{H}_3) \cup (H_1 \cap \bar{H}_2 \cap H_3) \cup (\bar{H}_1 \cap H_2 \cap H_3)$$

Hence, the probability of event **B** occurring is equal to:

$$P(B) = P\big((H_1 \cap H_2 \cap \bar{H}_3) \cup (H_1 \cap \bar{H}_2 \cap H_3) \cup (\bar{H}_1 \cap H_2 \cap H_3)\big)$$

However, we note that event **B** consists of the union of a group of mutually exclusive events two by two, meaning that:

$$\begin{cases} (H_1 \cap H_2 \cap \bar{H}_3) \cap (H_1 \cap \bar{H}_2 \cap H_3) = \emptyset \\ (H_1 \cap H_2 \cap \bar{H}_3) \cap (\bar{H}_1 \cap H_2 \cap H_3) = \emptyset \\ (H_1 \cap \bar{H}_2 \cap H_3) \cap (\bar{H}_1 \cap H_2 \cap H_3) = \emptyset \end{cases}$$

Hence, the probability of the union of these events is equal to the sum of the probabilities of each one of them. In other words, we write:

$$P(B) = P(H_1 \cap H_2 \cap \overline{H}_3) + P(H_1 \cap \overline{H}_2 \cap H_3) + P(\overline{H}_1 \cap H_2 \cap H_3)$$

Since the machines **i** are independent among themselves, the probability of these events intersecting is equal to the product of their probabilities, and we write the following expression:

$$P(B) = P(H_1)P(H_2)P(\overline{H}_3) + P(H_1)P(\overline{H}_2)P(H_3) + P(\overline{H}_1)P(H_2)P(H_3)$$

By numerical application we find:

$$P(B) = (0.05)(0.10)(0.85) + (0.05)(0.90)(0.15) + (0.95)(0.10)(0.15)$$

$$P(B) = (0.00425) + (0.00675) + (0.01425)$$

$$P(B) = 0.02525$$

3. **Calculate P(C):**

The event C occurs when the three machines fail at the same time, i.e. the first machine crashes (**H_1** checks out), the second machine crashes (**H_2** checks out), and the third machine also crashes (**H_3** checks out). We write as follows:

$$C = H_1 \cap H_2 \cap H_3$$

$$P(C) = P(H_1 \cap H_2 \cap H_3)$$

Since the events **H_i** are independent of each other, the probability of the occurrence of event C is equal to the product of the probabilities **$P(H_i)$** and we write:

$$P(C) = P(H_1)\,P(H_2)P(H_3)$$

By numerical application we find:

$$P(C) = (0.05)(0.10)(0.15)$$

$$P(C) = 0.00075$$

4. **Calculate P(D):**

The event D occurs when none of the three machines malfunctions, that is, the first machine does not malfunction ($\overline{H}_1$ is achieved), the second machine does not malfunction ($\overline{H}_2$ is achieved), and the third machine does not malfunction ($\overline{H}_3$ is achieved), and we write in the following form:

$$C = \overline{H}_1 \cap \overline{H}_2 \cap \overline{H}_3$$

Hence, the probability of the occurrence of the event D is equal to the probability of the intersection of the events $\overline{H}_i$ is:

$$P(C) = P(\overline{H}_1 \cap \overline{H}_2 \cap \overline{H}_3)$$

Since the events $\overline{H}_i$ are independent of each other, the probability of the event D occurring is equal to the product of the probabilities P($\overline{H}_i$) and we write:

$$P(D) = P(\overline{H}_1)\,P(\overline{H}_2)P(\overline{H}_3)$$

By numerical application we find:

$$P(D) = (0.95)(0.90)(0.85)$$

$$\boldsymbol{P(D) = 0.72675}$$

5. Calculate P(E):

The event **E** occurs when at least one machine fails, i.e. only one machine breaks down (incident **A** occurs), two machines fail (incident **B** occurs), or all three machines fail simultaneously (incident **C** occurs). Here, incident E can be expressed as follows:

$$\boldsymbol{E = A \cup B \cup C}$$

Since events **A**, **B** and **C** are mutually exclusive, the probability of occurrence E is equal to the sum of the probabilities of events **A**, **B** and **C**, and we write:

$$\boldsymbol{P(E) = P(A) + P(B) + P(C)}$$

By numerical application we find:

$$P(E) = (0.24725) + (0.02525) + (0.00075)$$

$$\boldsymbol{P(E) = 0.27325}$$

Exercise 05:

One of the planes is fired with three shots: $\mathbf{A_3}$, $\mathbf{A_2}$, $\mathbf{A_1}$. If we know that the probability of it being hit by the first shot is **0.4**, by the second shot is **0.5**, and by the third shot is **0.7**. It is believed that three shots are sufficient to bring down the plane, just as the probability of it falling if it is hit by one shot is **0.2** and if it is hit by two shots is **0.6**. What is the probability that the plane will fall as a result of the three shots that may or may not hit it?

Solution :

Let us call the events:

$\mathbf{H_i}$: the plane is hit by **i** shots, **i =1, 2, 3**, which that means:

- $\mathbf{H_1}$ the plane is hit by one shot
- $\mathbf{H_2}$ the plane is hit by two shots
- $\mathbf{H_3}$ the plane is shot by three shots.

Also, the events:

$\mathbf{A_i}$: the plane is hit by the $\mathbf{i^{th}}$ shot, which that means:

- $\mathbf{A_1}$: the plane is hit by the $\mathbf{1^{st}}$ shot
- $\mathbf{A_2}$: the plane is hit by the $\mathbf{2^{nd}}$ shot
- $\mathbf{A_3}$: the plane is hit by the $\mathbf{3^{rd}}$ shot

Exercise 06:

To reach his workplace daily, a specific person can take three different routes. He takes route $\mathbf{H_1}$ on **40%** of the days, route $\mathbf{H_2}$ on **35%** of the days, and route $\mathbf{H_3}$ on **25%** of the days. The person has observed that the probability of arriving late to his workplace is **3%** when using route **H_1**, **2%** when using route **H_2**, and only **1%** when using route **H_3**. Let's denote the event of arriving late as **A**.

- What is the probability that this person arrives late to his workplace on a given day?
- If it is known that this person arrived late, what is the probability that he used route **H_1**?

Solution:

Let us know the following events:

- **A**: this person arrives late to his workplace.
- **H_1** this person takes route **H_1** to go to his workplace;
- **H_2** this person takes route **H_2** to go to his workplace;
- **H_3** this person takes route **H_3** to go to his workplace;

To answer the first question, we use the principle of total probability as following:

$$P(A) = P(H_1).P(A/H_1) + P(H_2).P(A/H_2) + P(H_3).P(A/H_3)$$

Substituting the value of the probabilities with their equal value, we find:

$$P(A) = \big((0.4) \times (0.03)\big) + \big((0.35) \times (0.02)\big) + \big((0.25) \times (0.01)\big)$$

$$P(A) = (0.012) + (0.007) + (0.0025) = 0.0215$$

To answer the second question, we are going to use the Bays' Conditional Probability rule as following:

$$P(H_1/A) = \frac{P(H_1)P(A/H_1)}{P(A)} = \frac{(0.4) \times (0.03)}{0.0215} = 0.558$$

Chapter three:

Discrete random variables and their probability distribution

1. Definition of random variable:

We have seen before that ample space (Ω) is simply set which its elements describe the outcomes of the experiment, and each element represent a possible result of this experiment.

A random variable is a function that assigns or attribute a real number to each element of the sample space of a random experiment.

$$\mathbf{X : \Omega \rightarrow \Re}$$

A random variable is denoted by an uppercase letter such as **X**. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as **x**

Example:

Let define an experiment by tossing two coins and observing the results. Let X equal the number of heads obtained.

Identify the sample points in S, assign a value of X to each sample point, and identify the sample points associated with each value of the random variable X.

Solution:

Let **H** and **T** represent head and tail, respectively, and let an ordered pair of symbols identify the outcome for the first and second coins. (Thus, HT implies a head on the first coin and a tail on the second.)

Then the four sample points in (Ω) are:

(Ω) = {HH, HT, TH, TT}

We are going to attribute to each outcome of our experiment $\mathbf{E_i}$ a real number which correspond to a number of head obtained.

$\mathbf{E_i}$	HH	HT	TH	TT
X	2	1	1	0

Summarizing, the random variable X can take three values, X = 0, 1, and 2, which are events defined by specific collections of sample points:

{Y = 0} = {E4}, {Y = 1} = {E2, E3}, {Y = 2} = {E1}.

Notice:

Since that we can’t know the outcome of our experiment before executing, we can’t know the value of random variable that it will take in advance, so a random variable is a variable that transforms between a set of values or within a range of values in an unorganized manner because of a random aspect of the experiment.

2. Discrete random variables:

A random variable is discrete if it can assume spaces values with a countably finite or infinite number of values, Examples of this are:

- The number of times an even number can be obtained when rolling a dice three times,

X= {0, 1, 2, 3}

- The number of male children in a sample of families consisting of four children.

X= {0, 1, 2, 3, 4}

- The number of people requesting service at a particular counter.

X= {0, 1, 2, 3, ….}

- Number of defective parts in total daily production,

X= {0, 1, 2, 3, …. n}

- The number of ships that dock at the port or the number of planes that land at the airport...

X= {0, 1, 2, 3, ….}

3. The probability distribution of a random variable

The probability distribution of a random variable **X** is a description of the probabilities associated with the possible values of **X**. For a discrete random variable, the distribution is often specified by just a list of the possible values along with the probability of each. In some cases, it is convenient to express the probability in terms of a table or formula. Let us have a random variable (X) that has n possible values, forming a subset X of the set of real numbers $\mathfrak{R}$.

$$X = \{x_1, x_2, \ldots x_n\}$$

Each of these values x_i corresponds to a possible outcome of the experiment, and therefore represents an event that can be verified during the experiment and with an imposed and specific probability P_i, so that:

$$P(X = x_1) = P_1, \quad P(X = x_2) = P_2, \quad \ldots. \quad P(X = x_n) = P_n$$

$$P(X = x_i) = \begin{cases} P_1 & if \quad X = x_1 \\ P_2 & if \quad X = x_2 \\ & \vdots \\ P_n & if \quad X = x_n \\ 0 & if \quad else \end{cases}$$

We can also represent the relationship between each value of the variable and its corresponding probabilities in the form of a table consisting of two lines (or two columns). The first line (or first column) represents the values of the variable, and the second line (or second column) represents the probabilities corresponding to each value, as represented by the next table.

X	x_1	x_2	……………………	x_n
P(X=xi)	P_1	P_2	……………………	P_n

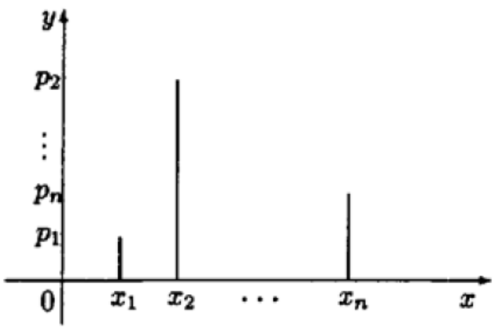
So that the following two conditions must be met:

$$\mathbf{0 \leq P(X = xi) \leq 1}$$

$$\sum_{i=1}^{n} P(X = x_i) = 1$$

4. Graphical representation of the probability distribution law:

We represent the law of probability distribution for a discrete random variable through bar graph, where the interval axis represents the possible values of the random variable and the ordinal axis represents the probability values, where we drop a column in front of each value of the variable whose length corresponds to the probability value as in the following figure:



5. The cumulative distribution function:

The cumulative distribution function of a discrete random variable **X**, denoted as **F(x)**, is It is the function that gives us the probability that the value of the random variable will be less than or equal to $\mathbf{x_f}$.

$$F(x_f) = P(X \leq x_f) = \sum_{i=1}^{f} P(X = x_i)$$

$$F(x) = \begin{cases} P_1 & \text{if} \quad X \leq x_1 \\ P_1 + P_2 & \text{if} \quad x_1 \leq X \leq x_2 \\ P_1 + P_2 + P_3 & \end{cases}$$

$$if \quad x_2 \leq X \leq x_2$$

$$if \quad x_{n-2} \leq X \leq x_{n-1}$$

$$if \quad x_{n-1} \leq X \leq x_n$$

6. Calculating probabilities:

The process of calculating a random variable X to take a specific value or a set of possible values for it is through collecting or summing the probabilities corresponding to those values, and here we must make sure to take into account or not the value within the range depending on the case, since if it is required to take into account values that are larger or equal (or smaller or equal) than a certain value, the probability corresponding to that value is taken into account, while it is excluded and not calculated if values that are completely larger (or completely smaller) than that value are requested, and here according to the required cases. The following rules can be relied upon:

- $P(X \leq x_f) = F(x_f) = \sum_{i=1}^{i=f} P(X = x_i)$
- $P(X < x_f) = \sum_{i=1}^{i=f-1} P(X = x_i) = F(x_f) - P(X = x_f)$
- $P(X \geq x_f) = \sum_{i=f}^{n} P(X = x_i) = 1 - \sum_{i=1}^{i=f-1} P(X = x_i)$

$$P(X \geq x_f) = 1 - \left(F(x_f) - P(X = x_f)\right) = 1 - P(X < x_f)$$

- $P(X > x_f) = \sum_{i=f+1}^{n} P(X = x_i) = 1 - \sum_{i=1}^{i=f} P(X = x_i)$

$$P(X > x_f) = 1 - F(x_f) = 1 - P(X \leq x_f)$$

- $P(x_{f'} \leq X \leq x_f) = \sum_{i=f'}^{i=f} P(X = x_i)\, x_{f'} \leq x_f$

$$P(x_{f'} \leq X \leq x_f) = P(X \leq x_f) - P(X < x_{f'})$$

$$P(x_{f'} \leq X \leq x_f) = F(x_f) - F(x_{f'}) + P(X = x_{f'})$$

- $P(x_{f'} < X \leq x_f) = \sum_{i=f'+1}^{f} P(X = x_i) = P(X \leq x_f) - (X \leq x_{f'})$

$$P(x_{f'} < X \leq x_f) = F(x_f) - (x_{f'})$$

- $P(x_{f'} \leq X < x_f) = \sum_{i=f'}^{i=f-1} P(X = x_i) = P(X < x_f) - (X < x_{f'})$

$$P(x_{f'} \leq X < x_f) = \left(F(x_f) - P(X = x_f)\right) - \left(F(x_{f'}) - P(X = x_{f'})\right)$$

$$P(x_{f'} \leq X < x_f) = \left(F(x_f) - F(x_{f'})\right) - \left(P(X = x_{f'}) - P(X = x_f)\right)$$

7. Numerical caracteristics:

7.1 Mathematical expectation:

It is symbolized by the symbol **E(X)**, which is a very important concept in probability and statistics is that of *mathematical expectation*, *expected value*, or briefly the *expectation*, which represent the expected or hoped-for value of the random variable of a random variable. For a discrete random variable **X** having the possible values $\mathbf{x_1}$, $\mathbf{x_2}$, …, $\mathbf{x_n}$, the expectation of **X** is calculated by summing each value of the variable xi multiplied by the corresponding probability value defined as:

$$E(X) = \sum_{i=1}^{n} x_i P(X = x_i)$$

Notice:

Given a constant number C, some mathematical properties of the expected value can be deduced as follows:

1. $E(C) = \sum_{i=1}^{n} C.P(X = x_i) = C.\sum_{i=1}^{n} P(X = x_i) = C \times 1 = C$

2. $E(CX) = \sum_{i=1}^{n} C.x_i P(X = x_i) = C.\sum_{i=1}^{n} x_i P(X = x_i)$

$E(CX) = C.E(X)$

3. $E(C + X) = \sum_{i=1}^{n}(C + x_i)P(X = x_i)$

$$E(C + X) = \sum_{i=1}^{n} C.P(X = x_i) + \sum_{i=1}^{n} x_i P(X = x_i)$$

$$E(C + X) = \sum_{i=1}^{n} C.P(X = x_i) + \sum_{i=1}^{n} x_i P(X = x_i)$$

$$E(C + X) = C.\sum_{i=1}^{n} P(X = x_i) + \sum_{i=1}^{n} x_i P(X = x_i)$$

$$E(C + X) = C.1 + E(X)$$

7.2 The simple moments:

The simple moment of degree **s** of a discrete random variable **X** denoted $\mathbf{M_s(X)}$ is defined as the sum of the product of each value of the variable $\mathbf{x_i}$ raised to the power **s** with its corresponding probability value $\mathbf{P(X=x_i)}$ and is calculated by the following relationship:

$$M_s(X) = \sum_{i=1}^{n} x_i^s P(X = x_i)$$

- The simple moment of degree 0 equals to:

$$M_0(X) = \sum_{i=1}^{n} x_i^0 P(X = x_i) = \sum_{i=1}^{n} P(X = x_i) = 1$$

- The simple moment of degree 1 equals to:

$$M_1(X) = \sum_{i=1}^{n} x_i P(X = x_i) = E(X)$$

- The simple moment of degree 2 equals to:

$$M_2(X) = \sum_{i=1}^{n} x_i^2 P(X = x_i) = E(X^2)$$

- The simple moment of degree 3 equals to:

$$M_3(X) = \sum_{i=1}^{n} x_i^3 P(X = x_i) = E(X^3)$$

7.3 The centered moments:

The centered moment of degree s (or the sth centered moment) of a discrete random variable **(X)** is defined as the sum of the product of the deviation of each value of the variable x_i relative to its mathematical expectation, raised to the power s with its corresponding probability value $P(X=x_i)$, it is denoted $\mu_s(X)$ and is calculated by the following relationship:

$$\mu_s(X) = \sum_{i=1}^{n} \left(x_i - E(X)\right)^s . P(X = x_i)$$

Depending on the relationship known for the central moment of degree **s**, the following central moments can be calculated:

- The first-order centered moment $\mu_1(X)$ is equal to zero:

$$\mu_1(X) = \sum_{i=1}^{n} \left(x_i - E(X)\right)^1 . P(X = x_i) = \sum_{i=1}^{n} x_i . P(X = x_i) - \sum_{i=1}^{n} E(X) . P(X = x_i)$$

$$= E(X) - E(X) \sum_{i=1}^{n} P(X = x_i) = E(X) - E(X) . 1$$

$$= E(X) - E(X) = 0$$

- The second-order centered moment $\mu_2(X)$ is equal to the difference between the second-order simple moment and the square of the first-order simple moment:

$$\mu_2(X) = \sum_{i=1}^{n} \left(x_i - E(X)\right)^2 . P(X = x_i)$$

$$\mu_2(X) = \sum_{i=1}^{n} \left(x_i^2 + (E(X))^2 - 2E(X)x_i\right) P(X = x_i)$$

$$\mu_2(X) = \sum_{i=1}^{n} x_i^2 P(X = x_i) + (E(X))^2 \sum_{i=1}^{n} P(X = x_i) - 2E(X) \sum_{i=1}^{n} x_i P(X = x_i)$$

$$\mu_2(X) = M_2(X) + \left(E(X)\right)^2 . 1 - 2E(X). E(X)$$

$$\mu_2(X) = M_2(X) + \left(E(X)\right)^2 - 2\left(E(X)\right)^2$$

$$\mu_2(X) = M_2(X) - \left(E(X)\right)^2$$

$$\mu_2(X) = M_2(X) + M_1^2(X)$$

7.4 Variance and Standard Deviation:

The variance denoted as **V(X)** of a random variable **X** is defined as the second-order centered moment, while the standard deviation $\boldsymbol{\delta}_X$ is the square root of the variance.

$$V(X) = \partial_X^2 = \mu_2(X) = \sum_{i=1}^{n} \left(x_i - E(X)\right)^2 . P(X = x_i)$$

$$V(X) = M_2(X) + M_1^2(X)$$

$$\partial_X = \sqrt{V(X)}$$

Notice:

In the presence of a fixed number **C**, some mathematical properties of variance can be deduced as follows:

1. $V(C) = \sum_{i=1}^{n} \left(C - E(C)\right)^2 \bullet P_i = \sum_{i=1}^{n} (C - C)^2 \bullet P_i = \sum_{i=1}^{n} (0)^2 \bullet P_i = 0$

2. $V(C \cdot X) = \sum_{i=1}^{n} \left(Cx_i - E(C \cdot X)\right)^2 \bullet P_i = \sum_{i=1}^{n} \left(C \cdot x_i - C \cdot E(X)\right)^2 \bullet P_i = C^2 V(X)$

3. $V(C + X) = \sum_{i=1}^{n} \left((C + x_i) - E(C + X)\right)^2 \bullet P = V(X)$

Solved exercises related to the chapter

<u>Exercise 01:</u>

Let **X** be a random variable defined by the following probability law:

X	**1**	**2**	**3**	**4**	**5**	**6**
P(X=xᵢ)	**1/10**	**3/10**	**4/10**	**1/10**	**0.5/10**	**0.5/10**

- Calculate and draw the graph of the distribution function **F(X).**
- Calculate the following probabilities $P(2 \leq X \leq 4)$، $P(X \geq 2)$،$P(X \leq 4)$.
- Calculate the value of mathematical expectation **E(X)**.
- Calculate $E(X^2)$, $E(X^3)$, $E[X-E(X)]^2$.

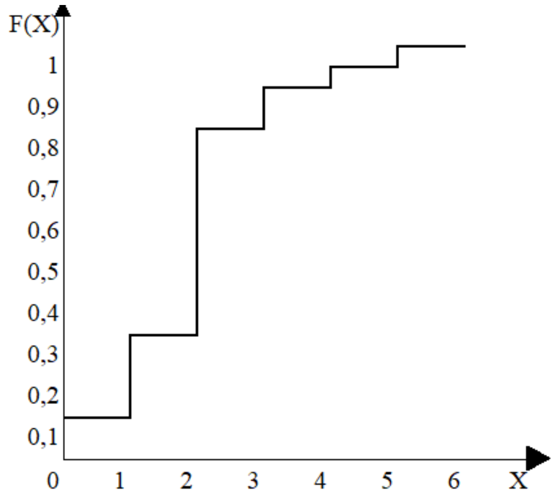
Solution:

X	1	2	3	4	5	6	∑
$P(X=x_i)$	**0,1**	**0,3**	**0,4**	**0,1**	**0,05**	**0,05**	-
$P(X \leq x_f)$	0,1	0,4	0,8	0,9	0,95	1	-
$x_i.P(X=x_i)$	0,1	0,6	1,2	0,4	0,25	0,3	**2,85**
x_i^2	1	4	9	16	25	36	-
$x_i^2.P(X=x_i)$	0,1	1,2	3,6	1,6	1,25	1,8	**9,55**
x_i^3	1	8	27	64	125	216	-
$x_i^3.P(X=x_i)$	0,1	2,4	10,8	6,4	6,25	10,8	**36,75**
$xi - E(x)$	-1,85	-0,85	0,15	1,15	2,15	3,15	**3,9**
$(xi - E(x))^2$	3,4225	0,7225	0,0225	1,3225	4,6225	9,9225	-
$(xi - E(x))^2.P(X=xi)$	0,34225	0,21675	0,009	0,13225	0,231125	0,496125	**1,4275**

The cumulative function:

$$F(X) = P(X \leq x_f) = \begin{cases} 0.1 & if \quad X \leq 1 \\ 0.4 & if \quad 1 < X \leq 2 \\ 0.8 & if \quad 2 < X \leq 3 \\ 0.9 & if \quad 3 < X \leq 4 \\ 0.95 & if \quad 4 < X \leq 5 \\ 1 & if \quad 5 < X \leq 6 \end{cases}$$

The graphical representation of the cumulative function:



Calculating the probabilities:

$\boldsymbol{P(2 \leq X \leq 4)} = P(X=2) + P(X=3) + P(X=4)$

$\boldsymbol{P(2 \leq X \leq 4)} = 0.3 + 0.4 + 0.1$

$$\boldsymbol{P(2 \leq X \leq 4)} = 0.8$$

$\boldsymbol{P(X \geq 2)} = P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$\boldsymbol{P(X \geq 2)} = 0.3 + 0.4 + 0.1 + 0.05 + 0.05$

$$\boldsymbol{P(X \geq 2)} = 0.9$$

$\boldsymbol{P(X \leq 4)} = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$\boldsymbol{P(X \leq 4)} = 0.1 + 0.3 + 0.4 + 0.1$

$$\boldsymbol{P(X \leq 4)} = 0.9$$

Calculate E(X):

$\boldsymbol{E(X)} = \boldsymbol{\sum_{i=1}^{6} x_i . P(X = x_i)}$

$\boldsymbol{E(X)} = \big(1 \times (0.1)\big) + \big(2 \times (0.3)\big) + \big(3 \times (0.4)\big) + \big(4 \times (0.1)\big) + \big(5 \times (0.05)\big) + \big(56 \times (0.05)\big)$

$$\boldsymbol{E(X) = 2.85}$$

Calculate E(X²), E(X³), E[X-E(X)]² :

$\boldsymbol{E(X^2) = M_2(X) = \sum_{i=1}^{6} x_i^2 P(X = x_i)}$

$\boldsymbol{E(X^2) = ((1)^2 \times 0.1) + ((2)^2 \times 0.3) + ((3)^2 \times 0.2) + ((4)^2 \times 0.1) + ((5)^2 \times 0.05) + ((6)^2 \times 0.05)}$

$$\boldsymbol{E(X^2) = 9.55}$$

$$E(X^3) = M_3(X) = \sum_{i=1}^{6} x_i^3 P(X = x_i)$$

$$E(X^3) = ((1)^3 \times 0.1) + ((2)^3 \times 0.3) + ((3)^3 \times 0.2) + ((4)^3 \times 0.1) + ((5)^3 \times 0.05) + ((6)^3 \times 0.05)$$

$$\boldsymbol{E(X^3) = 36.75}$$

$$\boldsymbol{E\left(X - E(X)\right)^2 = V(X) = \partial_X^2 = \mu_2(X) = \sum_{i=1}^{n}\left(x_i - E(X)\right)^2 . P(X = x_i)}$$

$$\boldsymbol{V}(\boldsymbol{X}) = \left((1 - 2.85)^2.(0.1)\right) + \left((2 - 2.85)^2.(0.3)\right) + \left((3 - 2.85)^2.(0.2)\right) + \left((4 - 2.85)^2.(0.1)\right) + \left((5 - 2.85)^2.(0.05)\right) + \left((6 - 2.85)^2.(0.05)\right)$$

$$\boldsymbol{V(X) = 1.4275}$$

In another way:

$$\boldsymbol{V(X) = M_2(X) + M_1^2(X)}$$

$$\boldsymbol{V}(\boldsymbol{X}) = 9.55 - (2.85)^2 = \boldsymbol{1.4275}$$

<u>Exercise 02:</u>

A player rolls a dice, he gives the following probabilities for various events:

X	**1**	**2**	**3**	**5 أو 1**	**6 أو 2**	**even number**
P(X=x_i)	**0.1**	**0.2**	**0.2**	**0.2**	**0.3**	**0.6**

The unequal probability values for various events are due to several reasons: the dice room is unbalanced, the player has a special method when throwing the stone... etc.

- Calculate P(X=4), P(X=5) and P(X=6), then prove that the sum of the probabilities equals one.
- Let event A be: obtaining a number greater than 3, and let event B be the event: obtaining an even number. Prove that the following two statements are true:

$$\boldsymbol{P(\bar{A}) = 1 - P(A)}$$

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Solution :

Calculate the patobabilities :

$\boldsymbol{P}[(\boldsymbol{X} = 2) \cup (\boldsymbol{X} = 6)] = 0.3$

$\Rightarrow \boldsymbol{P}(\boldsymbol{X} = 2) + \boldsymbol{P}(\boldsymbol{X} = 6) = 0.3$

$\Rightarrow \boldsymbol{P}(\boldsymbol{X} = 6) = \boldsymbol{P}[(\boldsymbol{X} = 2) \cup (\boldsymbol{X} = 6)] - \boldsymbol{P}(\boldsymbol{X} = 2)$

$\Rightarrow \boldsymbol{P}(\boldsymbol{X} = 6) = 0.3 - 0.2$

$$\boldsymbol{P(X = 6) = 0.1}$$

$P[(X = 2) \cup (X = 4) \cup (X = 6)] = 0.6$

$\Rightarrow P(X = 2) + P(X = 4) + P(X = 6) = 0.6$

$\Rightarrow P(X = 4) = P[(X = 2) \cup (X = 4) \cup (X = 6)] - [P(X = 2) + P(X = 6)]$

$\Rightarrow P(X = 4) = 0.6 - (0.3)$

$$P(X = 4) = 0.3$$

$P[(X = 1) \cup (X = 5)] = 0.2$

$\Rightarrow P(X = 1) + P(X = 5) = 0.2$

$\Rightarrow P(X = 5) = P[(X = 1) \cup (X = 5)] - P(X = 1)$

$\Rightarrow P(X = 5) = 0.2 - 0.1$

$$P(X = 5) = 0.1$$

We place the obtained results in the following table

X	**1**	**2**	**3**	**4**	**5**	**6**	$\sum$
P(X)	**0.1**	**0.2**	**0.2**	**0.3**	**0.1**	**0.1**	**1**

We note that:

$$0 \leq P(X = xi) \leq 1$$

$$\sum_{i=1}^{n} P(X = x_i) = 1$$

Hence, the variable X follows the probability distribution.

Prove that $P(A) = 1 - P(\bar{A})$:

$P(A) = P(X > 3) = P[(X = 4) \cup (X = 5) \cup (X = 6)]$

$P(A) = P(X = 4) + P(X = 5) + P(X = 6)$

$P(A) = 0.3 + 0.1 + 0.1$

$$\boldsymbol{P}(\mathbf{A}) = \mathbf{0.5}$$

On the other hand, we have:

$\mathbf{P}(\bar{\mathbf{A}}) = \boldsymbol{P}(\boldsymbol{X} \leq 3) = \boldsymbol{P}[(\boldsymbol{X} = 1) \cup (\boldsymbol{X} = 2) \cup (\boldsymbol{X} = 3)]$

$\mathbf{P}(\bar{\mathbf{A}}) = \boldsymbol{P}(\boldsymbol{X} = 1) + \boldsymbol{P}(\boldsymbol{X} = 2) + \boldsymbol{P}(\boldsymbol{X} = 3)$

$\mathbf{P}(\bar{\mathbf{A}}) = 0.1 + 0.2 + 0.2$

$$\mathbf{P}(\bar{\mathbf{A}}) = 0.5 = 1 - \boldsymbol{P}(\mathrm{A})$$

Prove that $\boldsymbol{P}(\boldsymbol{A} \cup \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}) + \boldsymbol{P}(\boldsymbol{B}) - \boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B})$:

$\boldsymbol{P}(\mathrm{B}) = \boldsymbol{P}[(\boldsymbol{X} = 2) \cup (\boldsymbol{X} = 4) \cup (\boldsymbol{X} = 6)]$

$\boldsymbol{P}(\mathrm{B}) = \boldsymbol{P}(\boldsymbol{X} = 2) + \boldsymbol{P}(\boldsymbol{X} = 4) + \boldsymbol{P}(\boldsymbol{X} = 6)$

$\boldsymbol{P}(\mathrm{B}) = 0.2 + 0.3 + 0.1$

$$\boldsymbol{P}(\mathbf{B}) = \mathbf{0.6}$$

Also, we have:

$\boldsymbol{P}(\mathrm{A} \cup \mathrm{B}) = \boldsymbol{P}[(\boldsymbol{X} = 2) \cup (\boldsymbol{X} = 4) \cup (\boldsymbol{X} = 5) \cup (\boldsymbol{X} = 6)]$

$\boldsymbol{P}(\mathrm{A} \cup \mathrm{B}) = \boldsymbol{P}(\boldsymbol{X} = 2) + \boldsymbol{P}(\boldsymbol{X} = 4) + \boldsymbol{P}(\boldsymbol{X} = 5) + \boldsymbol{P}(\boldsymbol{X} = 6)$

$\boldsymbol{P}(\mathrm{A} \cup \mathrm{B}) = 0.2 + 0.3 + 0.1 + 0.1$

$$\boldsymbol{P}(\mathbf{A} \cup \mathbf{B}) = \mathbf{0.7}$$

On the other hand, we have:

$\boldsymbol{P}(\mathrm{A} \cap \mathrm{B}) = \boldsymbol{P}[(\boldsymbol{X} = 4) \cup (\boldsymbol{X} = 6)]$

$\boldsymbol{P}(\mathrm{A} \cap \mathrm{B}) = \boldsymbol{P}(\boldsymbol{X} = 4) + \boldsymbol{P}(\boldsymbol{X} = 6)$

$\boldsymbol{P}(\mathrm{A} \cap \mathrm{B}) = 0.3 + 0.1$

$$\boldsymbol{P}(\mathbf{A} \cap \mathbf{B}) = \mathbf{0.4}$$

We note that:

$\boldsymbol{P}(\boldsymbol{A}) + \boldsymbol{P}(\boldsymbol{B}) - \boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \mathbf{0.5 + 0.6 - 0.4 = 0.7}$

Hence :

$$\boldsymbol{P}(\boldsymbol{A} \cup \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}) + \boldsymbol{P}(\boldsymbol{B}) - \boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B})$$

Exercise 03:
We roll two dice such that $\mathbf{A_1}$ represents the obtained number of the first dice, and $\mathbf{A_2}$ represents the obtained number of the second dice, and $\mathbf{X=A_1+>A_2}$.

- Find the law of probability (X) and represent it graphically.
- Find the distribution function F(X) and represent it graphically.
- Calculate the first and the second ordred simple moments and deduct the value of the mathematical expectation and the variance.

Solution:

To determine the range of definition of **X** , we rely on a two-dimensional table, the first line represents the number shown by the first die, $\mathbf{A_1}$, where **6** possible cases can be obtained. The first column represents the number shown by the second die, $\mathbf{A_2}$, where **6** possible cases can also be obtained here. Therefore, the number of possible cases is for the experiment, it is equal to **36** cases, meaning that the total blank of the initial events contains **36** elements, and we write:

$$\mathrm{Car}(\Omega) = 36$$

The intersections of the lines and the columns represent the sum of the two numbers shown, which expresses the value of the variable X as the following table shows:

A_2 \ A_1	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Since the throwing process takes place randomly on one side and the two dice are balanced, each side has the same possibility of appearing, and therefore the probability of obtaining each value is equal to the number of cases corresponding to obtaining that element over the total number of cases, and from this we obtain the following table:

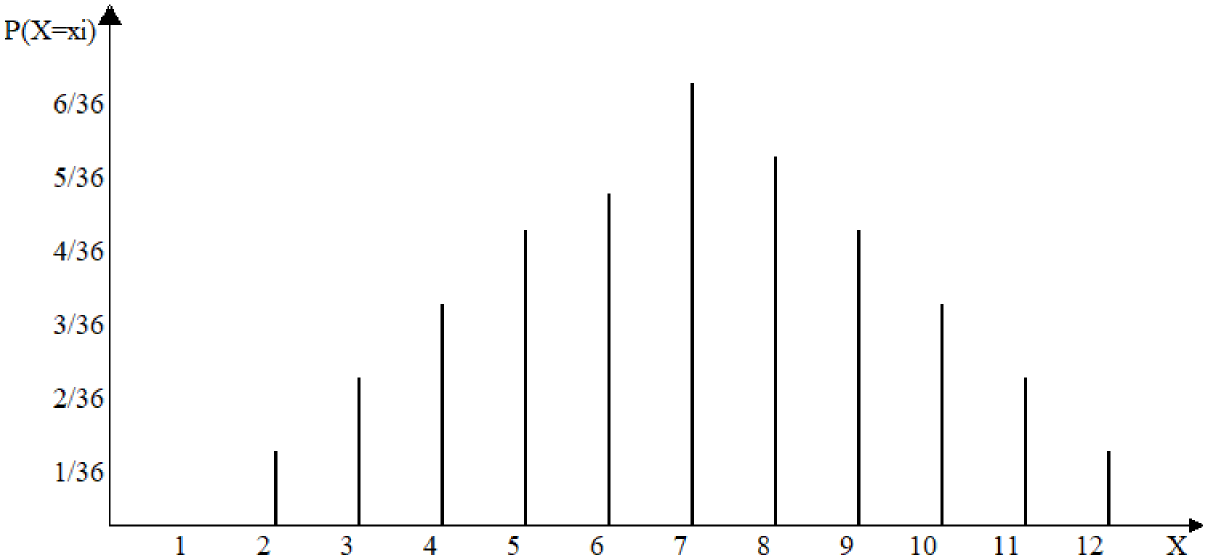
X	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	$\sum$
P(X=x_i)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36	1

We note that:

$$\mathbf{0 \leq P(X = xi) \leq 1}$$

$$\sum_{i=1}^{n} P(X = x_i) = 1$$

Hence, the variable **X** follows the probability distribution, and its graphical representation is shown in the following figure.

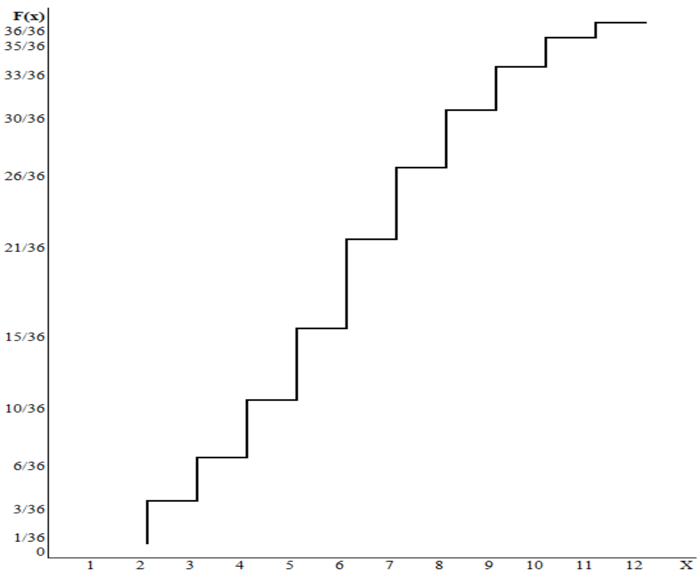


The distribution function F(X):

The distribution function is the cumulative function of the probability distribution, and it's given in the next table:

X	2	3	4	5	6	7	8	9	10	11	12
P(X=x$_i$)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36
F(x)	36/1	36/3	36/6	36/10	36/15	36/21	36/26	36/30	36/33	36/35	1

The graphical representation of the function of distribution is shown in the following figure.



Calculate the moments:

- The simple moment of degree 1 equals to:

$$M_1(X) = \sum_{i=1}^{n} x_i P(X = x_i) = 7$$

- The simple moment of degree 2 equals to:

$$M_2(X) = \sum_{i=1}^{n} x_i^2 P(X = x_i) = \frac{1974}{36}$$

- The mathematical expectation is equal to the simple moment of degree 1

$$E(X) = M_1(X) = 7$$

- The variance:

$$V(X) = M_2(X) + M_1^2(X) = \frac{1974}{36} - (7)^2 = \frac{35}{6}$$

Chapter four:

Continuous random variable and their probability distribution

1. Continuous random variables:

It is a variable that takes the form of an infinite number of values in a finite or infinite interval of numbers. Example for that:

- The distance traveled by a means of transportation before it breaks down;
- The shelf life of the equipment;
- The ages of people who develop a particular disease;
- Production quantity, turnover.

2. Probability density function:

A continuous random variable is defined in a field that has infinity of possible values. Therefore, its distribution law cannot be given in the form of a table as we saw previously in a discrete random variable, but rather it is given in the form of a function for (x) called the density function and it is as follows:

$$\boldsymbol{f(x)} = \begin{cases} \boldsymbol{f(x)}, & x \in [\boldsymbol{a}, \boldsymbol{b}] \\ \boldsymbol{0}, & x \notin [\boldsymbol{a}, \boldsymbol{b}] \end{cases}$$

In order for the function ***f(x)*** to be a probability density function, it must meet two basic conditions:

- It must be positive throughout its range of definition so that:

$$\boldsymbol{f(x) \geq 0} \qquad \forall\, \boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$$

- The value of the space that it limits in this field must be equal to one, that is, its limited integral in the field of its definition is equal to one.

$$\int_{-\infty}^{+\infty} \boldsymbol{f(x)dx} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f(x)dx} = \boldsymbol{1}$$

3. Distribution function (Cumulative distribution functions) :

The cumulative distribution function of a continuous random variable **X** is:

$$\boldsymbol{F(x) = P(X \leq x)} = \int_{-\infty}^{\boldsymbol{x}} \boldsymbol{f(x)dx}$$

Notice:

The distribution function has the following characteristics:

- The distribution function **F(X)** is a positive continuous function and takes values limited between **0** and **1**:

$$\boldsymbol{0 \leq F(x) \leq 1}$$

- The distribution function **F(X)** is an increasing function, so if **x < y** then **F(x) ≤ F(y)**.
- When **(X)** approaches **(+∞)**, the limit of the distribution function is equal to **1**, i.e.

$$\lim_{x \to +\infty} \boldsymbol{F(x) = 1}$$

- When **(X)** approaches **(-∞)**, the limit of the distribution function is equal to **0** i.e.

$$\lim_{x \to -\infty} F(x) = 0$$

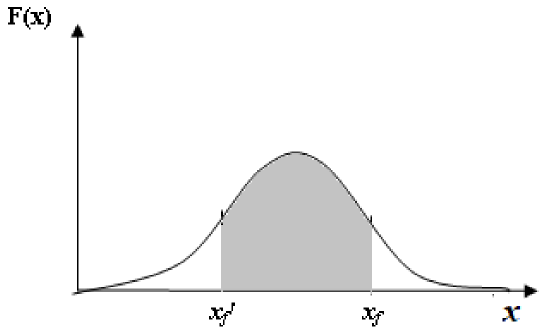
4. Calculating probabilities:

If we want to calculate the probability that a random variable takes values between two values (x_f and x'_f), where $x_f \geq x'_f$, we have to calculate the value of the area enclosed by the curve of the probability density function ***f(x)*** and the abscissa axes within the range **[x_f, x'_f],** that is:

$$P(x_{f'} \leq x \leq x_f) = \int_{x_{f'}}^{x_f} f(x)dx$$

$$P(x_{f'} \leq x \leq x_f) = \int_{-\infty}^{x_f} f(x)dx - \int_{-\infty}^{x_{f'}} f(x)dx$$

$$P(x_{f'} \leq x \leq x_f) = F(x_f) - F(x_{f'})$$



Notice:

Since calculating probability in a continuous random variable is calculating the enclosed area of the density function curve and the ordinal axis at the boundaries of two points in the definition field, which is only the end of the limited integration in that field, therefore we do not distinguish between values being greater or equal to **(≤)** and being Exactly greater **(>)**, and we do not distinguish between values being less than or equal to **(≥)** and being exactly greater **(<)**. It is also worth noting the following points:

- $P(X = x) = \int_x^x f(x) = 0$
- $P(X > x_f) = \int_{x_f}^{+\infty} f(x)\,dx = 1 - \int_{-\infty}^{x_f} f(x)dx = 1 - F(x_f)$

5. Numerical caracteristics:

5.1 Mathematical expectation:

It is equal to the value of the finite integral of the probability density function multiplied by the value of the variable in the field of its definition. Mathematical expectation denoted **E(X)**, is defined by the following relationship:

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx$$

Notice:

Given a constant number **C**, some mathematical properties of the expected value can be deduced as follows:

1. ***E(C.X) = C.E(X)***

$$E(C.X) = \int_{-\infty}^{+\infty} Cxf(x)dx = C\int_{-\infty}^{+\infty} xf(x)dx = C.E(X)$$

2. $E(C + X) = C + E(X)$

$$E(C + X) = \int_{-\infty}^{+\infty} (C + x)f(x)dx$$

$$= \int_{-\infty}^{+\infty} cf(x)dx + \int_{-\infty}^{+\infty} xf(x)dx$$

$$= C\int_{-\infty}^{+\infty} f(x)dx + E(X) = C \times 1 + E(X) = C + E(X)$$

5.2 The simple moments:

The simple moment of degree **s** (or the **s**th simple moment) denoted **Ms(X)** of the continuous random variable **X** is equal to the value of the finite integral of the probability density function multiplied by the value of the variable raised to the power **s** in the field of its definition, and the relationship of the initial moment of degree **s** is given by the relationship:

$$M_s(X) = \int_{-\infty}^{+\infty} x^s f(x)dx$$

- The simple moment of degree 0 equals to:

$$M_0(X) = \int_{-\infty}^{+\infty} x^0 f(x)dx = \int_{-\infty}^{+\infty} f(x)dx = 1$$

- The simple moment of degree 1 equals to:

$$M_1(X) = \int_{-\infty}^{+\infty} x^1 f(x)dx = E(X)$$

- The simple moment of degree 2 equals to:

$$M_2(X) = \int_{-\infty}^{+\infty} x^2 f(x)dx = E(X^2)$$

- The simple moment of degree 3 equals to:

$$M_3(X) = \int_{-\infty}^{+\infty} x^3 f(x)dx = E(X^3)$$

5.3 The centered moments:

The centered moment of degree **s** of a continuous random variable (or the **s^th^** centered moment) is equal to the value of the finite integral of the probability density function ***f(x)*** multiplied by the value of the deviation of the variable with respect to its mathematical expectation **E(X)**, raised to the power **s** in the field of its definition, it is denoted $\mu_s(X)$ and is calculated by the following relationship:

$$\mu_s(X) = \int_{-\infty}^{+\infty} \left(x - E(X)\right)^s f(x)dx$$

Depending on the relationship known for the central moment of degree **s**, the following central moments can be calculated:

- The first-order centered moment $\mu_1(X)$ is equal to zero:

$$\mu_1(X) = \int_{-\infty}^{+\infty} \left(x - E(X)\right) f(x)dx = \int_{-\infty}^{+\infty} x f(x)dx - \int_{-\infty}^{+\infty} E(X) f(x)dx$$

$$\mu_1(X) = \int_{-\infty}^{+\infty} x f(x)dx - E(X) \int_{-\infty}^{+\infty} f(x)dx$$

$$\mu_1(X) = E(X) - E(X) = 0$$

- The second-order centered moment $\mu_2(X)$ is equal to the difference between the second-order simple moment and the square of the first-order simple moment:

$$\mu_2(X) = \int_{-\infty}^{+\infty} \left(x - E(X)\right)^2 f(x)dx$$

$$\mu_2(X) = \int_{-\infty}^{+\infty} \left(x^2 + \left(E(X)\right)^2 - 2xE(X)\right) f(x)dx$$

$$\mu_2(X) = \int_{-\infty}^{+\infty} x^2 f(x)dx + \int_{-\infty}^{+\infty} \left(E(X)\right)^2 f(x)dx - \int_{-\infty}^{+\infty} 2xE(X) f(x)dx$$

$$\mu_2(X) = M_2(X) - 2E(X) \int_{-\infty}^{+\infty} x f(x)dx$$

$$\mu_2(X) = M_2(X) - 2E(X)E(X) = M_2(X) - \left(E(X)\right)^2$$

$$\mu_2(X) = M_2(X) - M_1^2(X)$$

5.4 Variance and Standard Deviation:

The variance denoted as **V(X)** of a random variable **X** is defined as the second-order centered moment, while the standard deviation δ_X is the square root of the variance.

$$V(X) = \partial_X^2 = \mu_2(X) = \int_{-\infty}^{+\infty} \left(x - E(X)\right)^2 f(x)dx$$

$$V(X) = M_2(X) - M_1^2(X)$$

$$\partial_X = \sqrt{V(X)}$$

Solved exercises related to the chapter

<u>Exercise 01:</u>

Let **X** be a random variable defined by the following probability law:

$$f(x) = \begin{cases} \frac{x}{2} & if\ 0 \leq x \leq 2 \\ 0 & ifelse \end{cases}$$

- Calculate and draw the graph of the distribution function **F(X).**
- Calculate the value of the mathematical expectation **E(X).**
- Calculate the simple moments end the centered moments.
- Find the value of the variance **V(X).**

Solution:

Calculate The mathematical expectation:

The mathematical expectation is equal to the value of the finite integral of the probability density function multiplied by the value of the variable in the field of its definition as following:

$$E(x) = \int_{-\infty}^{+\infty} xf(x)dx = \int_0^2 xf(x)dx$$

$$E(x) = \int_{-\infty}^{+\infty} x.\frac{x}{2}dx$$

$$E(x) = \int_{-\infty}^{+\infty} \frac{x^2}{2}dx = \frac{x^3}{6}\bigg|_0^2$$

$$E(x) = \lim_{x\to 2}\left(\frac{x^3}{6}\right) - \lim_{x\to 0}\left(\frac{x^3}{6}\right) = \frac{(2)^3}{6} - \frac{(0)^3}{6} = \frac{8}{6}$$

$$E(x) = \frac{4}{3}$$

Calculate the simple moments:

The simple moment of degree **s** is calculated by the following relationship:

$$M_s(X) = \int_{-\infty}^{+\infty} x^s f(x)dx = \int_0^2 x^s f(x)dx$$

$$M_s(X) = \int_0^2 x^s \frac{x}{2} dx = \frac{1}{2}\int_0^2 x^{s+1} dx$$

$$M_s(X) = \left.\frac{x^{s+2}}{2(s+2)}\right|_0^2 = \lim_{x\to 2} \frac{x^{s+2}}{2(s+2)} - \lim_{x\to 0} \frac{x^{s+2}}{2(s+2)}$$

$$M_s(X) = \frac{2^{s+2}}{2(s+2)} - \frac{0^{s+2}}{2(s+2)}$$

$$M_s(X) = \frac{2^{s+1}}{(s+2)}$$

The centered moments:

The centered moment of degree **s** denoted $\mu_s(X)$ and is calculated by the following relationship:

$$\mu_s(X) = \int_{-\infty}^{+\infty} (x - E(X))^s f(x)dx$$

$$\mu_s(X) = \int_0^2 \left(x - \frac{4}{3}\right)^s \frac{x}{2} dx$$

By using integration by parts, we have:

$$V = \frac{x}{2} \Longrightarrow dV = \frac{dx}{2}$$

$$dU = \left(x - \frac{4}{3}\right)^s dx \Longrightarrow U = \frac{\left(x - \frac{4}{3}\right)^{s+1}}{(s+1)}$$

Hence :

$$\mu_s(X) = \left.\frac{\left(x - \frac{4}{3}\right)^{s+1}}{(s+1)} \cdot \frac{dx}{2}\right|_0^2 - \int_0^2 \frac{\left(x - \frac{4}{3}\right)^{s+1}}{(s+1)} \cdot \frac{dx}{2}$$

$$\mu_s(X) = \frac{\left(2 - \frac{4}{3}\right)^{s+1}}{2(s+1)} - \frac{\left(-\frac{4}{3}\right)^{s+1}}{2(s+1)} - \left.\frac{\left(x - \frac{4}{3}\right)^{s+2}}{2(s+1)(s+2)}\right|_0^2$$

$$\mu_s(X) = \frac{\left(2 - \frac{4}{3}\right)^{s+1}}{2(s+1)} - \frac{\left(-\frac{4}{3}\right)^{s+1}}{2(s+1)} - \frac{\left(2 - \frac{4}{3}\right)^{s+2} - \left(0 - \frac{4}{3}\right)^{s+2}}{2(s+1)(s+2)}$$

$$\mu_s(X) = \frac{\left(\frac{2}{3}\right)^{s+1} - \left(-\frac{4}{3}\right)^{s+1}}{2(s+1)} - \frac{\left(\frac{2}{3}\right)^{s+2} - \left(-\frac{4}{3}\right)^{s+2}}{2(s+1)(s+2)}$$

Calculate the variance and Standard Deviation

$$V(X) = \partial_X^2 = M_2(X) - M_1^2(X)$$

$$M_2(X) = \frac{2^{2+1}}{(2+2)} = \frac{8}{4} = 2$$

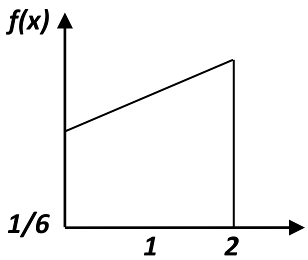
$$V(X) = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}$$

Henc the standard deviation is equal:

$$\partial_x = \sqrt{V(X)} = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$$

Exercise 02:

If we have the probability density function ***f(x)*** defined as the corresponding graph, answer the following questions:



- What type is this variable? find its probability density function ***f(x)*** and its definition domain.
- Find the distribution function **F(X)** and then draw its graph.
- Find the mathematical expectation **E(X)** and the variance **V(X).**

Solution:

1. The probability density function

The studied variable is a continuous random variable, and from the graph we notice that its probability density function is defined in the field **[0,2]**, and it is a linear function of the form ***f(x)=a+bx*** where ***a=f(0)=1/6***, and to find Form of the function We know that:

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \Rightarrow \int_0^2 (a+bx) = 1$$

$$\Rightarrow \int_0^2 \left(\frac{1}{6} + bx\right) = 1$$

$$\Rightarrow \left(\frac{1}{6}x + \frac{b}{2}x^2\right)\Bigg|_0^2 = 1$$

$$\Rightarrow \lim_{x\to 2}\left(\frac{1}{6}x+\frac{b}{2}x^2\right)-\lim_{x\to 0}\left(\frac{1}{6}x+\frac{b}{2}x^2\right)=1$$

$$\Rightarrow \left(\left(\frac{1}{6}\times 2\right)+\left(\frac{b}{2}\times 2^2\right)\right)+0=1$$

$$\Rightarrow b=\frac{1}{3}$$

Hence, the probability density function for the variable x is defined as follows:

$$f(x)=\begin{cases}\frac{1}{6}+\frac{1}{3}x, & x\in[0,2]\\ 0, & sinon\end{cases}$$

2. The distribution function:

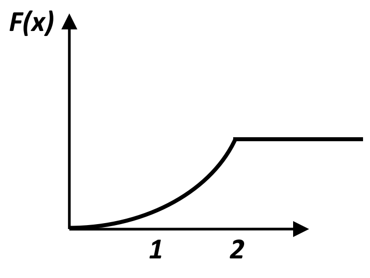
$$F(x)=\int_{-\infty}^{x} f(x)dx=\int_{0}^{x}\left(\frac{1}{6}+\frac{1}{3}x\right)dx$$

$$F(x)=\left(\frac{1}{6}x+\frac{1}{3}x^2\right)\Big|_0^x$$

$$F(x)=\lim_{x\to x}\left(\frac{1}{6}x+\frac{1}{6}x^2\right)-\lim_{x\to 0}\left(\frac{1}{6}x+\frac{1}{6}x^2\right)$$

$$F(x)=\frac{1}{6}x(x+1)$$

It is a polynomial function of the second degree passing through the center, and its shape is a parabola as follows:



3. The mathematical expectation and the variance:

$$E(X)=M_1(X)=\int_{-\infty}^{+\infty} xf(x)dx=\int_{0}^{2} x\left(\frac{1}{6}+\frac{1}{3}x\right)dx$$

$$E(X)=\int_{0}^{2}\left(\frac{x}{6}+\frac{x^2}{3}\right)dx=\left(\frac{x^2}{12}+\frac{x^3}{9}\right)\Big|_0^2$$

$$E(X)=\lim_{x\to 2}\left(\frac{x^2}{12}+\frac{x^3}{9}\right)-\lim_{x\to 0}\left(\frac{x^2}{12}+\frac{x^3}{9}\right)$$

$$E(X) = \frac{4}{12} + \frac{8}{9}$$

$$E(X) = \frac{11}{9}$$

The variance is given by the following relation:

$$V(x) = M_2(X) - M_1^2(X)$$

Let us calculate the second-order simple moment, which is given by the following relationship:

$$M_2(X) = \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_0^2 x^2\left(\frac{1}{6} + \frac{1}{3}x\right)dx$$

$$M_2(X) = \int_0^2 \left(\frac{x^2}{6} + \frac{x^3}{3}\right)dx = \left(\frac{x^3}{18} + \frac{x^4}{12}\right)\Big|_0^2$$

$$M_2(X) = \lim_{x\to 2}\left(\frac{x^3}{18} + \frac{x^4}{12}\right) - \lim_{x\to 0}\left(\frac{x^3}{18} + \frac{x^4}{12}\right)$$

$$M_2(X) = \frac{8}{18} + \frac{16}{12}$$

$$M_2(X) = \frac{16}{9}$$

Hence, the variance is equal to:

$$V(X) = \frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{144}{81} - \frac{121}{81}$$

$$V(X) = \frac{23}{81}$$

<u>Exercise 03:</u>

We say that a continuous random variable follows Pareto's distribution law if its density function is written as follows:

$$f(x) = \begin{cases} \dfrac{\alpha C^{\alpha}}{x^{\alpha+1}} & si \quad x \geq C \\ 0 & si \quad x < C \end{cases}$$

Where **α** and **C** are real constants

- Prove that this law is a probability distribution law.
- Find the simple moment of degree **k**.
- Deduce the mathematical expectation and variance.

Solution:

1. Prove that this law is a probability distribution

For this function to be a probability distribution law, the definite integral in the field of its definition must be equal to one. In other words, we write:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{c}^{+\infty} \frac{\alpha C^{\alpha}}{x^{\alpha+1}} dx$$

$$= \alpha C^{\alpha} \int_{c}^{+\infty} x^{-(\alpha+1)} dx$$

$$= -C^{\alpha} x^{-\alpha}|_{C}^{+\infty}$$

$$= \lim_{x \to +\infty} (-C^{\alpha} x^{-\alpha}) - \lim_{x \to C} (-C^{\alpha} x^{-\alpha})$$

$$= 0 - (-C^{\alpha} C^{\alpha}) = 1$$

Hence, Paretto's distribution is a probability distribution.

2. The simple moment of degree **k**

$$M_k(X) = \int_{-\infty}^{+\infty} x^k f(x)dx = \int_{c}^{+\infty} x^k \left(\frac{\alpha C^{\alpha}}{x^{\alpha+1}}\right) dx$$

$$M_k(X) = \alpha C^{\alpha} \int_{-\infty}^{+\infty} x^{k-\alpha-1} dx$$

$$M_k(X) = \alpha C^{\alpha} \left(\frac{x^{k-\alpha}}{k-\alpha}\right)\Bigg|_{C}^{+\infty}$$

$$M_k(X) = \lim_{x \to +\infty} \alpha C^{\alpha} \left(\frac{x^{k-\alpha}}{k-\alpha}\right) - \lim_{x \to C} \alpha C^{\alpha} \left(\frac{x^{k-\alpha}}{k-\alpha}\right)$$

$$M_k(X) = 0 - \alpha C^{\alpha} \left(\frac{c^{k-\alpha}}{k-\alpha}\right) \qquad k < \alpha$$

$$M_k(X) = \frac{\alpha}{\alpha - k} C^k$$

3. Deduce the Mathematical Expectancy and Variance:

$$E(X) = M_1(X) = \frac{\alpha}{\alpha - 1} C$$

$$V(x) = M_2(X) - M_1^2(X)$$

$$V(x) = M_2(X) - M_1^2(X) = \frac{\alpha}{\alpha - 2} C^2 - \left(\frac{\alpha}{\alpha - 1} C\right)^2$$

$$V(x) = C^2 \left(\frac{\alpha}{\alpha - 2} - \frac{\alpha^2}{(\alpha - 1)^2}\right)$$

Chapter five:

Usual probability distributions for discrete random variables

In many cases, we find that random experiments are similar in terms of their content and the conditions that determine the parameters of the studied random variable and also determine the probability of it being achieved or not, such as the number of times the experiment is repeated, the independence of events and the results of the experiment from time to time. Therefore, statisticians were able to deduce the statistical distributions that these variables follow according to each experiment and its nature. They formulated a mathematical relationship for each distribution, as well as extracting the numerical features.

Through this chapter, we will try to look at the most important distributions for the discrete random variable. We will formulate the mathematical function of the law based on the general conditions of the experiment corresponding to it. We will also draw conclusions. Numerical features represented by mathematical expectation and variance.

1. The Bernoulli distribution:

The Bernoulli distribution is used to model an experiment with only two possible outcomes, often referred to as “success” and “failure”, usually encoded as 1 and 0.

The random variable **X** is assigned the value 1 if the event occurs “success”, and 0 if it does not occur “failure”, so the possible values it takes are 1 if it succeeds and the event occurs with probability p, and 0 if it fails and the event does not occur with probability q, where:

$$P(X = 1) = p$$

$$\Omega = \{0, 1\}$$

$$P(X = 0) = q = 1 - p$$

$$P(X = 1) = p$$

We say that the random variable **X** follows probability distribution law if:

$$\sum_{i=0}^{n} P(X = x_i) = 1$$

In other words, we write:

$$\sum_{i=0}^{n} P(X = x_i) = P(X = 0) + P(X = 1)$$

We know that:

$P(X = 0) = q = 1 - p$ and $P(X = 1) = p$

Hence:

$$\sum_{i=0}^{n} P(X = x_i) = p + q = p + 1 - p = 1$$

Hence, Bernoulli's distribution law is a probability distribution.

1.1 Mathematical expectation:

Based on the definitional relationship of the mathematical expectation of a discrete random variable, it is:

$$E(X) = \sum_{i=0}^{1} x_i P(X = x_i)$$

$$E(X) = 0 \times P(X = 0) + 1 \times P(X = 1)$$

$$E(X) = 0 \times q + 1 \times p = p$$

Hence, the mathematical expectation for a discrete random variable that follows Bernoulli's distribution law is equal to:

$$E(X) = p$$

1.2 Variance:

The variance calculation relationship is given as follows:

$$V(X) = M_2(X) + M_1^2(X)$$

Let us first calculate the second-order simple moment, which is given by the following relationship:

$$M_2(X) = E(X^2) = \sum_{i=1}^{n} x_i^2 P(X = x_i)$$

$$M_2(X) = (0)^2 \times P(X = 0) + (1)^2 \times P(X = 1)$$

$$M_2(X) = 0 \times q + 1 \times p$$

$$M_2(X) = p$$

By substituting the value of the mathematical expectation and the value of the second-order simple moment into the variance calculation relationship, we find:

$$V(X) = p - p^2 = p(1 - p) = pq$$

Hence, the variance for a discrete random variable that follows Bernoulli's distribution law is equal to:

$$V(X) = pq$$

2. The binomial distribution:

The binomial distribution applies when we repeat Bernoulli's experiment **n** times under the same conditions, so that at each time the experiment has two possible outcomes: success if the event is achieved and loss if it does not occur, and given the independence of the events among them when the experiment is repeated and the conditions for carrying out the experiment that affect the experiment do not change. The possibility of the event being achieved or not. The value of the probability of success, **p**, remains constant no matter how much the experiment is repeated. Conversely, the probability of the event not being achieved, **q**, also remains constant.

To give the binomial distribution law relationship, we will study the following example:

We roll a dice three **(03)** consecutive times in a random manner, and we will extract the probability distribution law for the variable **X**, which represents the number of times the number **(5)** is obtained.

We notice that the probability of getting the number **(5)**, i.e. the probability of the event being achieved in each throw, remains constant and equals **p=1/6**, and the probability of the event not being achieved also remains constant and equals **q=5/6**.

When we throw a dice three times, we may not get the number **(5)** any time in this case **(X=0)**, or we may get the number **(5)** one time in this case (X=1), or we may get the number **(5)** twice, in this case **(X=2)**, or we get the number **(5)** three times, in this case **(X=3)**, meaning that **X = {0, 1, 2, 3}**

Let's call the events E_i getting the number **(5)** on the i^{th} throw:

- E_1 Get the number **(5)** on the first throw.
- E_2 Get the number **(5)** on the second throw.
- E_3 Get the number **(5)** on the third throw.

We also call accidents: $\overline{E_i}$not getting the number **(5)** on the i^{th} throw:

- $\overline{E_1}$ Not getting the number **(5)** on the first throw.
- $\overline{E_2}$ Not getting the number **(5)** on the second throw.
- $\overline{E_3}$ Not getting the number **(5)** on the third throw.

Note that the result of any throw is independent of the result of another throw, since all throws are random and the dice are assumed to be balanced.

Calculating P(X=0):

$$P(X=0)=P(\overline{E_1}\cap\overline{E_2}\cap\overline{E_3})$$

Since the events are independent, the probability of these events intersecting is equal to the product of the probabilities, i.e.:

$$P(X=0)=P(\overline{E_1}).P(\overline{E_2}).P(\overline{E_3})$$

$$P(X=0)=(1-p).(1-p).(1-p)=q.q.q$$

$$P(X=0)=q^3$$

Calculating P(X=1):

$$P(X=1)=P((E_1\cap\overline{E_2}\cap\overline{E_3})\cup(\overline{E_1}\cap E_2\cap\overline{E_3})\cup(\overline{E_1}\cap\overline{E_2}\cap E_3))$$

Since these incidents are mutually exclusive, that is:

$$\begin{cases}(E_1\cap\overline{E_2}\cap\overline{E_3})\cap(\overline{E_1}\cap E_2\cap\overline{E_3})=\emptyset\\(E_1\cap\overline{E_2}\cap\overline{E_3})\cap(\overline{E_1}\cap\overline{E_2}\cap E_3)=\emptyset\\(\overline{E_1}\cap\overline{E_2}\cap E_3)\cap(\overline{E_1}\cap E_2\cap\overline{E_3})=\emptyset\end{cases}$$

The probability of the union of mutually exclusive events is equal to the sum of the probabilities, so:

$$P(X=1)=P(E_1\cap\overline{E_2}\cap\overline{E_3})+P(\overline{E_1}\cap E_2\cap\overline{E_3})+P(\overline{E_1}\cap\overline{E_2}\cap E_3)$$

Since the events are independent, the probability of these events intersecting is equal to the product of the probabilities, i.e.:

$$P(X=1)=\left(P(E_1).P(\overline{E_2}).P(\overline{E_3})\right)+\left(P(\overline{E_1}).P(E_2).P(\overline{E_3})\right)+\left(P(\overline{E_1}).P(\overline{E_2}).P(E_3)\right)$$

$$P(X=1)=p.(1-p).(1-p)+(1-p).p.(1-p)+(1-p).(1-p).p$$

$$P(X=1)=p.q.q+q.p.q+q.q.p$$

$$P(X=1)=3pq^2$$

Calculating P(X=2):

$$P(X=2)=P((E_1\cap E_2\cap\overline{E_3})\cup(E_1\cap\overline{E_2}\cap E_3)\cup(\overline{E_1}\cap E_2\cap E_3))$$

Since these incidents are mutually exclusive, that is:

$$\begin{cases}(E_1\cap E_2\cap\overline{E_3})\cap(E_1\cap\overline{E_2}\cap E_3)=\emptyset\\(E_1\cap E_2\cap\overline{E_3})\cap(\overline{E_1}\cap E_2\cap E_3)=\emptyset\\(E_1\cap\overline{E_2}\cap E_3)\cap(\overline{E_1}\cap E_2\cap E_3)=\emptyset\end{cases}$$

The probability of the union of mutually exclusive events is equal to the sum of the probabilities, so:

$$P(X=2)=P(E_1\cap E_2\cap\overline{E_3})+P(E_1\cap\overline{E_2}\cap E_3)+P(\overline{E_1}\cap E_2\cap E_3)$$

Since the events are independent, the probability of these events intersecting is equal to the product of the probabilities, i.e.:

$$P(X=2)=\left(P(E_1).P(E_2).P(\overline{E_3})\right)+\left(P(E_1).P(\overline{E_2}).P(E_3)\right)+\left(P(\overline{E_1}).P(E_2).P(E_3)\right)$$

$$P(X=2)=p.p.(1-p)+(1-p).(1-p).p+(1-p).p.p$$

$$P(X=2)=p.p.q+q.qp+q.p.p$$

$$P(X=2)=3p^2q$$

Calculating P(X=3):

$$P(X=3)=P(E_1\cap E_2\cap E_3)$$

Since the events are independent, the probability of these events intersecting is equal to the product of the probabilities, i.e.:

$$P(X=3)=P(E_1).P(E_2).P(E_3)$$

$$P(X=3)=p.p.p$$

$$P(X=3)=p^3$$

Let us write the results obtained as follows:

- $P(X=0)=q^3=1.1.q^3=C_3^0p^0q^3=C_3^0p^0q^{3-0}$
- $P(X=1)=3pq^2=C_3^1p^1q^2=C_3^1p^1q^{3-1}$
- $P(X=2)=3p^2q=C_3^2p^2q^1=C_3^2p^2q^{3-2}$
- $P(X=3)=p^3=1.p^3.1=C_3^3p^3q^0=C_3^3p^3q^{3-3}$

If we roll the dice **n** times and look for the probability of the event occurring **x** times, the law will be as follows:

$$P(X=x_i)=C_n^xp^xq^{n-x}$$
$$0\leq x\leq n$$
$$p+q=1$$

But does the binomial distribution represent a probability distribution? Whereas, to be a probability distribution, it must satisfy the following statement:

$$\sum_{x=0}^{n}P(X=x_i)=\sum_{x=0}^{n}C_n^xp^xq^{n-x}=1$$

We know from Newton's Binomial Expansion Formula that:

$$(a+b)^n=C_n^0a^0b^n+C_n^1a^1b^{n-1}+C_n^2a^2b^{n-2}+\cdots+C_n^na^nb^0$$

In other term:

$$(a+b)^n=\sum_{x=0}^{n}C_n^xa^xb^{n-x}$$

By matching it can be concluded that:

$$(p+q)^n=\sum_{x=0}^{n}C_n^xp^xq^{n-x}$$

Since the:

$$(p+q)=1\Rightarrow(p+q)^n=1$$

That is:

$$\sum_{x=0}^{n}P(X=x_i)=\sum_{x=0}^{n}C_n^xp^xq^{n-x}=1$$

Hence, the binomial distribution is the law of probability distribution.

2.1 Mathematical expectation:

Based on the definitional relationship of the mathematical expectation of a discrete random variable, it is:

$$E(X) = \sum_{x=0}^{n} xP(X = x_i) = \sum_{x=0}^{n} xC_n^x p^x q^{n-x}$$

$$E(X) = \sum_{x=0}^{n} x\frac{n!}{x!\,(n-x)!}p^x q^{n-x}$$

$$E(X) = \sum_{x=1}^{n} x\frac{n(n-1)!}{x(x-1)!\,(n-x)!}p.p^{x-1} q^{n-x}$$

$$E(X) = np\sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!}p^{x-1} q^{n-x}$$

By placing: ***m* = *n*-1** ⇒ ***n* = *m*+1** *and* ***k* = *x*-1** ⇒ ***x* = *k*+1** we find:

$$E(X) = np\underbrace{\sum_{k=0}^{m} \frac{m!}{k!\,(m-k)!}p^{k} q^{m-k}}_{=1}$$

Hence:

$$E(X) = np$$

In another way we know that:

$$(p+q)^n = \sum_{x=0}^{n} C_n^x p^x q^{n-x}$$

We do derivative for **p**:

$$n(p+q)^{n-1} = \sum_{x=0}^{n} xC_n^x p^{x-1} q^{n-x}$$

$$n(p+q)^{n-1} = \sum_{x=0}^{n} xC_n^x p^{x} q^{n-x}p^{-1}$$

$$np\underbrace{(p+q)^{n-1}}_{=1} = \sum_{x=0}^{n} xC_n^x p^{x} q^{n-x} = E(X)$$

Hence:

$$E(X) = np$$

2.2 Variance:

The variance calculation relationship is given as follows:

$$V(X) = M_2(X) + M_1^2(X)$$

Let us first calculate the second-order simple moment, which is given by the following relationship:

$$M_2(X) = \sum_{x=0}^{n} x^2 P(X = x_i) = \sum_{x=0}^{n} x^2 C_n^x p^x q^{n-x}$$

$$M_2(X) = \sum_{x=0}^{n} (x(x-1) + x) C_n^x p^x q^{n-x}$$

$$M_2(X) = \underbrace{\sum_{x=0}^{n} x(x-1) C_x^n p^x q^{n-x}}_{=A} + \underbrace{\sum_{x=0}^{n} x C_n^x p^x q^{n-x}}_{=E(X)=np}$$

Let's calculate part **A** separately:

$$A = \sum_{x=0}^{n} x(x-1) C_n^x p^x q^{n-x} = \sum_{x=0}^{n} x(x-1) \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$A = \sum_{x=2}^{n} x(x-1) \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\,(n-x)!} p^{x-2} q^{n-x} p^2$$

$$A = n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} q^{n-x}$$

By placing $m = n-2 \Rightarrow n = m+2$ and $k = x-2 \Rightarrow x = k-2$

$$A = n(n-1)p^2 \underbrace{\sum_{k=0}^{m} \frac{m!}{k!\,(m-k)!} p^k q^{m-k}}_{=1}$$

$$A = n(n-1)p^2$$

Hence, the second-degree simple moment is equal to:

$$M_2(X) = n(n-1)p^2 + np$$

$$M_2(X) = n^2p^2 - np^2 + np$$

And the variance is equal to:

$$V(X) = n^2p^2 - np^2 + np - n^2p^2$$

$$V(X) = np - np^2$$

$$V(X) = np\,\underbrace{(1-p)}_{=q}$$

$$V(X) = npq$$

3. Poisson Distribution:

This law is concerned with phenomena that have two mutually exclusive states: the occurrence of the event (success) or its failure to occur (loss), as we saw previously in the law of binomial distribution, but in this law it is stipulated:

- The number of experiments **n** that we perform is very large, i.e. **n⟶+∞.**
- The probability of the phenomenon **p** occurring is very small, i.e. p⟶0.
- So that the limit of multiplying **p** by n is equal to a positive real number, let it be **λ**.

In other words:

$$\lim_{\substack{n\to+\infty \\ p\to 0}} np = \lambda$$

We can say that when **(n)** is very large and **(p)** is very small, the binomial distribution approaches the Poisson distribution, so we say that a discrete random variable with an intermediate **(λ)** is subject to the Poisson distribution if the formula of its law is as follows:

$$P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!}$$

$$X=0,1,2,\dots,+\infty$$

3.1 Mathematical expectation:

Based on the definitional relationship of the mathematical expectation of a discrete random variable, it is:

$$E(X)=\sum_{x=0}^{+\infty} xP(X=x_i)=\sum_{x=0}^{+\infty} x\frac{\lambda^x e^{-\lambda}}{x!}$$

$$E(X)=e^{-\lambda}\sum_{x=0}^{+\infty} x\frac{\lambda.\lambda^{x-1}}{x(x-1)!}$$

$$E(X)=\lambda e^{-\lambda}\sum_{x=0}^{+\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

By placing :

$$\begin{cases} s=x-1 \\ x=s+1 \end{cases} \qquad \begin{matrix} s=0,1,2,\dots,+\infty \\ x=1,2,\dots,+\infty \end{matrix}$$

$$E(X)=\lambda e^{-\lambda}\sum_{s=0}^{+\infty}\frac{\lambda^s}{s!}$$

$$\sum_{s=0}^{+\infty} \frac{\lambda^s}{s!} = e^{\lambda}$$

$$E(X) = \lambda e^{-\lambda}.e^{\lambda} = \lambda$$

$$\boxed{E(X) = \lambda}$$

3.2 Variance:

The variance calculation relationship is given as follows:

$$V(X) = M_2(X) + M_1^2(X)$$

Let us first calculate the second-order simple moment, which is given by the following relationship:

$$M_2(X) = \sum_{x=0}^{+\infty} x^2 P(X = x_i) = \sum_{x=0}^{+\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!}$$

$$M_2(X) = \sum_{x=0}^{+\infty} (x(x-1)+x) \frac{\lambda^x e^{-\lambda}}{x!}$$

$$M_2(X) = \underbrace{\sum_{x=0}^{+\infty} \big(x(x-1)\big) \frac{\lambda^x e^{-\lambda}}{x!}}_{=A} + \underbrace{\sum_{x=0}^{+\infty} x \frac{\lambda^x e^{-\lambda}}{x!}}_{=E(X)=\lambda}$$

Let us first consider side A:

$$A = \sum_{x=0}^{+\infty} \big(x(x-1)\big) \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=2}^{+\infty} \big(x(x-1)\big) \frac{\lambda^2 \lambda^{x-2} e^{-\lambda}}{x(x-1)(x-2)!}$$

We put :

$$\begin{cases} s = x - 2 \\ x = s + 2 \end{cases} \qquad \begin{matrix} s = 0, 1, 2, \dots, +\infty \\ x = 2, 3, \dots, +\infty \end{matrix}$$

$$A = \lambda^2 \underbrace{\sum_{s=0}^{+\infty} \frac{\lambda^s e^{-\lambda}}{s!}}_{=1} = \lambda^2$$

Hence, the second simple moment is equal to:

$$M_2(X) = \lambda^2 + \lambda$$

The variance is equal to:

$$V(X) = \lambda^2 + \lambda - \lambda^2$$

$$\boxed{V(X) = \lambda}$$

4. Geometric distribution:

Let there be an experiment related to event A, where the probability of the event occurring (success) is equal to p and the probability of it not occurring is equal to q, where q = 1 – p. We repeat the experiment until event A is achieved, with the probability of success remaining constant no matter how often the experiment is repeated (independence of incidents). Let

- **A** appears on the first throw $\quad P(X=1)=P(A)=p=p\times 1=pq^{1-1}$
- **A** appears on the second throw $\quad P(X=2)=P(A\cap\bar{A})=pq=pq^{2-1}$
- **A** appears on the third throw $\quad P(X=3)=P(A\cap\bar{A}\cap\bar{A})=pq^{2}=pq^{3-1}$
- **A** appears on the fourth throw $\quad P(X=4)=P(A\cap\bar{A}\cap\bar{A}\cap\bar{A})=pq^{3}=pq^{4-1}$

.
.
.

- **A** appears on the x^th throw $\quad P(X=x)=P\left(A\cap\underbrace{\bar{A}\cap\bar{A}\cap\ldots\cap\bar{A}}_{\text{مرة }(x-1)}\right)=pq^{x-1}$

Hence, the general formula of Pascal's distribution law is written in the form:

$$P(X=x)=pq^{x-1}$$

$$X=1,2,\ldots,+\infty$$

Pascal's distribution law is a probability distribution law:

$$\sum_{1}^{+\infty}P(X=x)=\sum_{1}^{+\infty}pq^{x-1}$$

$$\sum_{1}^{+\infty}P(X=x)=p\sum_{1}^{+\infty}q^{x-1}$$

$$\sum_{1}^{+\infty}P(X=x)=p(1+q^{1}+q^{2}+q^{3}+\cdots)$$

This sum represents the sum of terms of a geometric sequence whose first term is **1** and whose base is **q**, which is given in the form:

$$(1+q^{1}+q^{2}+q^{3}+\cdots)=\frac{1}{1-q}=\frac{1}{p}$$

Hence :

$$\sum_{1}^{+\infty}P(X=x)=\sum_{1}^{+\infty}pq^{x-1}=p.\frac{1}{1-q}=\frac{p}{p}=1$$

4.1 Mathematical expectation:

Based on the mathematical expectation calculation relationship:

$$E(X)=\sum_{x=1}^{+\infty}xP(X=x_i)=\sum_{x=1}^{+\infty}xpq^{x-1}$$

$$E(X) = (p + 2pq + 3pq^2 + 4pq^3 + 5pq^4 + \ldots\ldots)$$

$$E(X) = p\,(1 + 2q + 3q^2 + 4q^3 + 5q^4 + \ldots\ldots)$$

$$E(X) = p\,(1 + q + q^2 + q^3 + q^4 + \ldots\ldots)^2$$

$$E(X) = p.\frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$E(X) = \frac{1}{p}$$

4.2 Variance:

Based on the varaiance calculation relationship:

$$M_2(X) = \sum_{x=1}^{+\infty} x^2 P(X = x_i) = \sum_{x=1}^{+\infty} (x(x-1) + x) P(X = x_i)$$

$$M_2(X) = \underbrace{\sum_{x=1}^{+\infty} \big(x(x-1)\big) P(X = x_i)}_{=A} + \underbrace{\sum_{x=1}^{+\infty} x P(X = x_i)}_{=E(X)=\frac{1}{p}}$$

Let's take care of part A first:

$$A = \sum_{x=1}^{+\infty} \big(x(x-1)\big) P(X = x_i) = 0 + 2pq + 6pq^2 + 12pq^3 + \cdots$$

$$A = 2pq(1 + 3q + 6q^2 + \cdots)$$

$$A = 2pq(1 + q + q^2 + q^3 + \cdots)^3$$

$$A = \frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}$$

Hence, the second simple moment of the second degre is equal to:

$$M_2(X) = \frac{2q}{p^2} + \frac{1}{p} = \frac{2q}{p^2} + \frac{p}{p^2}$$

$$M_2(X) = \frac{2q + p}{p^2} = \frac{2(1-p) + p}{p^2}$$

$$M_2(X) = \frac{2 - 2p + p}{p^2} = \frac{2 - p}{p^2}$$

Hence, the variance is equal to :

$$V(X) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

$$V(X) = \frac{q}{p^2}$$

Solved exercises related to the chapter

Exercise 01:

1. What is the probability of getting 4 times head if we throw a balanced coin 6 times? What is the probability of getting the number 6 three times if we roll a die four times?

2. If you know that 20% of the students who join the university leave before completing their studies, from (20) students who were randomly selected, what is the probability that (03) of them will leave the university before completing their studies?

3. If you know that 40% of an organization's workers want to have a union representation, and you take a sample of 10 workers, what is the probability that this idea will get a majority in the sample selected?

Solution:

1. When a coin is tossed 4 times, i.e. n = 4, the events are independent of each other at each toss, with the probability of getting heads (the event occurring) remaining constant p = 0.5 and the probability of not getting heads (the event not occurring) also remaining constant q = 1 – p = 0.5 Therefore, the random variable that expresses the number of times the image is obtained follows the binomial distribution law, which is given by:

$$P(X = x) = C_n^x p^x q^{n-x}$$

Therefore, the probability of getting 4 heads if we toss a balanced coin 6 times is equal to:

$$P(X = 4) = C_6^4 (0,5)^4 (0,5)^{6-4}$$

$$P(X = 4) = 0,234$$

In the same way in this case:

q = 1 – p = 5/6 **p = 1/6 = <u>cte</u>** **n = 4**

Therefore, the probability of getting the number 6 three times if we roll a die four times is equal to:

$$P(X = 3) = C_4^3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{4-3}$$

$$P(X = 4) = 0,015$$

2. In the same way in this case:

q = 1 – p = 0,8 **p = 0,2 = <u>cte</u>** **n = 20**

Therefore, the probability that (03) of them will leave the university before completing their studies is equal to

$$P(X = 3) = C_{20}^3 (0,2)^3 (0,8)^{20-3}$$

$$P(X = 4) = 0,205$$

3. In the same way in this case:

q = 1 – p = 0,6 **p = 0,4 = <u>cte</u>** **n = 10**

Therefore, the probability that this idea will obtain a majority in the selected sample is equal to P(X > 5), i.e.:

$$P(X > 5) = P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$P(X=6)=C_{10}^{6}(0,4)^{6}(0,6)^{10-6}=0,111$$

$$P(X=7)=C_{10}^{7}(0,4)^{7}(0,6)^{10-7}=0,042$$

$$P(X=8)=C_{10}^{8}(0,4)^{8}(0,6)^{10-8}=0,010$$

$$P(X=9)=C_{10}^{9}(0,4)^{9}(0,6)^{10-9}=0,001$$

$$P(X=10)=C_{10}^{10}(0,4)^{10}(0,6)^{10-10}=0,0001$$

Hence :

$$P(X>5)=0,166$$

Exercise 02:

The probability of graduating a student enrolled in the college is 0.4. From 5 students chosen at random, what are the following probabilities:

1. Don't any of them graduate?
2. Should one of them graduate?
3. Should at least one of them graduate?

Solution:

Here we also use the binomial distribution law relationship, where:

q = 1 – p = 0,6 **p = 0,4 = <u>cte</u>** **n = 5**

Therefore, the random variable that expresses the number of graduating students follows the binary distribution law, which is given by:

$$P(X=x)=C_{n}^{x}p^{x}q^{n-x}$$

1. The probability that none of them will graduate is equal to:

$$P(X=0)=C_{5}^{0}(0,4)^{0}(0,6)^{5-0}=0,077$$

2. The probability that one of these students will graduate is equal to:

$$P(X=1)=C_{5}^{1}(0,4)^{1}(0,6)^{5-1}=0,259$$

3. The probability that at least one student will graduate is equal to:

$$P(X\geq 1)=P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$$

$$P(X=1)=0,259$$

$$P(X=2)=C_{5}^{2}(0,4)^{2}(0,6)^{5-2}=0,345$$

$$P(X=3)=C_{5}^{3}(0,4)^{3}(0,6)^{5-3}=0,230$$

$$P(X=4)=C_{5}^{4}(0,4)^{4}(0,6)^{5-4}=0,076$$

$$P(X=5)=C_{5}^{5}(0,4)^{5}(0,6)^{5-5}=0,010$$

Hence :

$$P(X\geq 1)=0,923$$

This probability can be calculated based on the method of calculating the probability of the complementary event, i.e.:

$P(X \geq 1) = 1 - P(X < 1)$

$P(X \geq 1) = 1 - P(X = 0)$

$P(X \geq 1) = 1 - 0,077$

$$P(X \geq 1) = 0,923$$

Exercise 03:

The probability that a particular electronic piece will operate at a high temperature is 0.9. We have a device that uses four pieces of this type. Find the following probabilities:

1. Does the device work because all parts are valid?
2. That the device malfunctions because one piece is out of use?
3. That the device malfunctions because at least one part is out of use?

Solution:

We use the binomial distribution law relationship, where the studied variable is the number of valid parts that make the device work, so that:

q = 1 – p = 0,1 **p = 0,9 = <u>cte</u>** **n = 4**

1. The probability that the device will work because all parts are valid, i.e. **P(X=4)**, is equal to:

$P(X = 4) = C_4^4 (0,9)^4 (0,1)^{4-4} = 0,6561$

2. The probability that the device will malfunction because one piece has malfunctioned, in other words 03 valid parts, and we write **P(X=3)** equal to:

$P(X = 3) = C_4^3 (0,9)^3 (0,1)^{4-3} = 0,2916$

3. The probability of the device malfunctioning because at least one part has malfunctioned. In other words, at most 03 parts are valid, and we write $P(X \leq 3)$ equal to:

$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$P(X = 0) = C_4^0 (0,9)^0 (0,1)^{4-0} = 0,0001$

$P(X = 1) = C_4^1 (0,9)^1 (0,1)^{4-1} = 0,0036$

$P(X = 2) = C_4^2 (0,9)^2 (0,1)^{4-2} = 0,0486$

$P(X = 3) = C_4^3 (0,9)^3 (0,1)^{4-3} = 0,2916$

$$P(X \leq 3) = 0,3439$$

Exercise 04:

An urn containing twenty (20) balls, ten (10) white balls and ten (10) black balls. We randomly drew four (04) balls and the drawn ball is put back into the urn, so calculate:

1 - The probability of not drawing any white balls.

2 - The probability of drawing two (02) white balls.

3 - The probability of drawing at least two (02) white balls.

4 - Calculate the odds if the drawn ball is not returned to the urn.

Solution:

Since the drawn ball is returned to the bowl, the result of each draw is independent of the result of the drawing that precedes it, as well as the result of the drawing that follows it, and the probability of the event occurring (obtaining a white ball) remains constant, and we use the binomial distribution law relationship to calculate the probabilities, as:

$$P(X = x) = C_n^x p^x q^{n-x}$$

q = 1 – p = 0,5 **p = 0,5 = cte** **n = 4**

1. The probability of drawing a white ball, i.e. **P(X=0)**, is equal to:

$$P(X = 0) = C_4^0(0,5)^0(0,5)^{4-0} = 0,0625$$

2. The probability of drawing two **(02)** white balls, i.e. **P(X=2)**, is equal to:

$$P(X = 2) = C_4^2(0,5)^2(0,5)^{4-2} = 0,375$$

3. The probability of drawing at least two **(02)** white balls, i.e. **P(X≥2)**, is equal to:

$P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$

$P(X = 2) = 0,375$

$P(X = 3) = C_4^3(0,5)^2(0,5)^{4-3} = 0,25$

$P(X = 4) = C_4^4(0,5)^4(0,5)^{4-4} = 0,0625$

$$P(X \geq 2) = 0,6875$$

In the case of a drawing of balls is done without replacement, the result of each drawing is linked to the result of the drawing that precedes it and affects the result of the drawing that follows it.

Therefore, we use here the law of hypergeometric distribution, which is given by the relationship:

$$P(X = x) = \frac{C_P^x . C_{N-P}^{n-x}}{C_N^n}$$

Whereas, according to this exercise:

P: The number of white blood cells (which have the studied characteristic)

n: The number of pellets drawn, i.e. 4.

N: is the sample size, which here equals 20.

Hence, the probabilities are equal to:

$$P(X = 0) = \frac{C_{10}^0 . C_{(20-10)}^{(4-0)}}{C_{20}^4} = \frac{(1)(210)}{(4845)} = 0,043$$

$$P(X = 2) = \frac{C_{10}^2 . C_{(20-10)}^{(4-2)}}{C_{20}^4} = \frac{(45)(45)}{(4845)} = 0,417$$

$P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$

$P(X = 2) = 0,417$

$$P(X=3)=\frac{C_{10}^{3}.C_{(20-10)}^{(4-3)}}{C_{20}^{4}}=\frac{(120)(10)}{(4845)}=0,247$$

$$P(X=4)=\frac{C_{10}^{4}.C_{(20-10)}^{(4-4)}}{C_{20}^{4}}=\frac{(210)(1)}{(4845)}=0,043$$

$$P(X\geq 2)=0,7089$$

Exercise 05:

An experience consists to throw a die many times and we stop once the number (05) is obtained.

The variable **X** was the number of times needed to obtain the number (05).

1. Write the x distribution law while giving its numerical features.
2. What is the probability that we get the number (05) in the fourth throw?
3. What is the probability that we do not get the number (05) except after throwing the dice at least four times?

Solution:

1. The variable **X** that expresses the number of times the dice is thrown which is necessary to get the number **(05)**, where the probability of the event occurring at each throw **p = 1/6** and the probability of the event not occurring is equal to *q = 1 – p = 5/6* and it follows the law The geometric distribution whose relationship is written as follows:

$P(X=x)=pq^{x-1}$	whereas	**X = {1 , 2, …+∞**
p = 1/6	And	**q = 1 – p = 5/6**

The mathematical expectation is equal to: $E(X)=\frac{1}{p}=\frac{1}{\left(\frac{1}{6}\right)}=6$

2. The probability that we get the number (05) on the fourth throw, i.e**. P(X=4)**, is equal to:

$$P(X=4)=\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{4-1}=0,096$$

3. The probability that we will not get the number **(05)** until after rolling the dice at least four times, i.e. **P(X≥4)**, is equal to:

$$P(X\geq 4)=P(X=4)+P(X=5)+P(X=6)+\cdots$$

Since this series is infinite, we will use the special relationship to calculate the completing event, i.e.:

$$P(X\geq 4)=1-P(X<4)$$

$$P(X\geq 4)=1-\big(P(X=1)+P(X=2)+P(X=3)\big)$$

Let's calculate each term:

$$P(X=1)=\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{1-1}=\frac{1}{6}$$

$$P(X=2)=\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{2-1}=\frac{5}{36}$$

$$P(X=3)=\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{3-1}=\frac{25}{216}$$

Hence :

$$P(X\geq 4)=1-\left(\frac{1}{6}+\frac{5}{36}+\frac{25}{216}\right)$$

$$P(X\geq 4)=1-\frac{91}{216}$$

$$P(X\geq 4)=\frac{125}{216}$$

Exercise 06:

The percentage of the defect when producing a piece of spare parts was (0.01), a sample was chosen with a hundred (100) pieces of total production, specify the following probabilities:

1. To be a corrupt piece in the sample.
2. That two pieces be corrupt.
3. That is at least two corrupt pieces.

Solution :

We note from the exercise data that the probability of the event occurring (the presence of a defect in the selected piece) **p = 0.01**, which is a very weak probability, while the sample size **n = 100** is very large, and therefore the studied variable **X** that expressed the number of defective pieces withdrawn follows the Poisson distribution law and we write:

$$\left.\begin{matrix} p=0.01\to 0 \\ n=100\to +\infty \end{matrix}\right\}\Rightarrow X\sim \mathcal{P}(\lambda)$$

whereas:

$$P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!} \qquad \lambda=np=0.01\times 100=1$$

1. The probability that one piece is corrupt, i.e. **P(X = 1)**, is equal to:

$$P(X=1)=\frac{(1)^1 e^{-1}}{1!}=0.367$$

2. The probability that there are two bad pieces, i.e. **P(X = 2)**, is equal to:

$$P(X=2)=\frac{(1)^2 e^{-1}}{2!}=0.183$$

3. The probability that at least two pieces are corrupt, i.e. **P(X≥2)**, is equal to:

$$\mathbf{P}(\mathbf{X}\geq 2)=\mathbf{P}(\mathbf{X}=2)+\mathbf{P}(\mathbf{X}=3)+\mathbf{P}(\mathbf{X}=4)+\cdots$$

$$\mathbf{P}(\mathbf{X}\geq 2)=1-\mathbf{P}(\mathbf{X}<2)$$

$$\mathbf{P(X \geq 2) = 1 - (P(X = 0) + P(X = 1))}$$

$$P(X = 0) = \frac{(1)^0 e^{-1}}{0!} = 0.367$$

Hence :

$$\mathbf{P(X \geq 2) = 1 - (0.367 + 0.367)}$$

$$\mathbf{P(X \geq 2) = 0.264}$$

Exercise 07:

The health experience showed that 0.03 of the total workforces suffers from a serious illness during the year, if a sample consisting of 100 workers is chosen, then find the expected value of the number of workers who will develop this disease, and What is the probability that 5 workers will contract this disease?

Solution:

We note from the exercise data that the probability of the event occurring (the worker contracting the disease) **p = 0.03**, which is a very weak probability, while the sample size **n = 100** is very large, and therefore the studied variable:

$$\left.\begin{matrix} p = 0.03 \rightarrow 0 \\ n = 100 \rightarrow +\infty \end{matrix}\right\} \Rightarrow X \sim \mathcal{P}(\lambda)$$

Wherase :

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad \lambda = np = 0.03 \times 100 = 3$$

Then the expected value is the mathematical expectation **E(X)** equal to:

$$E(X) = \lambda = 3$$

In other words, we expect that in the selected sample there will be **03** workers infected with this disease.

The probability that 5 workers will become infected with this disease, i.e. **P(X = 5),** is equal to:

$$P(X = 5) = \frac{(3)^5 e^{-3}}{5!} = 0.100$$

Exercise 08:

1. If **X** follows the binomial distribution with an expected value equals to **1** and variance equals to **3/2**, then find **P (X≥1).**
2. If **X** follows Poisson distribution and it was **P (X = 1) = 2p (X = 0)**, calculate **P(X = 4).**

Solution:

1. Based on our exercise data:

$$E(X) = np = 1$$

$$V(X) = npq = \frac{3}{2}$$

Thus, it can be written:

$$V(X) = (np)q = 1 \times q = \frac{3}{2} \Rightarrow q = \frac{2}{3}$$

Then we can find p which equals:

$$p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$$

Likewise, we can find n which equals:

$$E(X) = np = 1 \Rightarrow n = \frac{1}{p} = \frac{1}{\left(\frac{1}{3}\right)} = 3$$

Now we can calculate the probability P(X≥1), which is equal to:

$$P(X = 1) = C_3^1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{3-1} = 0,444$$

$$P(X = 2) = C_3^2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{3-2} = 0,222$$

$$P(X = 3) = C_3^3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{3-3} = 0,0.37$$

Hence :

$$P(X \geq 1) = 0.703$$

2. Based on our exercise data:

$$P(X = 1) = 2P(X = 0)$$

On the other hand, we also know that:

$$P(X = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda}$$

$$P(X = 1) = \frac{\lambda^1 e^{-\lambda}}{1!} = \lambda e^{-\lambda}$$

Therefore:

$$\lambda e^{-\lambda} = 2e^{-\lambda} \Rightarrow \lambda = 2$$

Hence:

$$P(X = 4) = \frac{(2)^4 e^{-2}}{4!} = 0.195$$

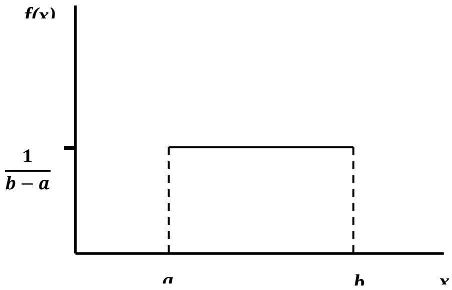
Chapter six: Usual probability distributions for continuous random variables

1. Uniform distribution:

We say that the variable **x** defined in the infinite field **[a, b]** follows the uniform distribution if its density function is written in the following form:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & si\ x \in [a, b] \\ 0 & sinon \end{cases}$$

The density function curve of this distribution is as follows:



The uniform distribution is a probability distribution law because the finite integral of the probability density function in the domain of its definition is equal to one, where:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_a^b \frac{1}{b-a}dx$$

$$\int_a^b \frac{1}{b-a}dx = \frac{x}{b-a}\Big|_a^b$$

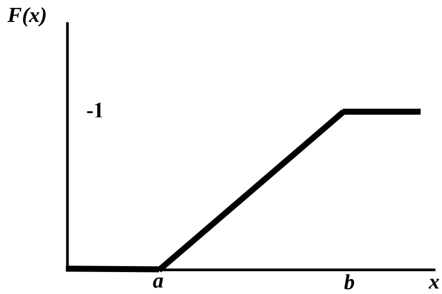
$$\int_a^b \frac{1}{b-a}dx = \lim_{x\to b}\frac{x}{b-a} - \lim_{x\to a}\frac{x}{b-a}$$

$$\int_a^b \frac{1}{b-a}dx = \frac{b}{b-a} - \frac{a}{b-a} = \frac{b-a}{b-a} = 1$$

Hence, the uniform distribution is a probability distribution law, and the probability distribution function **F(X)** is as follows:

$$F(X) = \int_{-\infty}^{x} f(x)dx = \begin{cases} 0 & si & x < a \\ \dfrac{x-a}{b-a} & si & x \in [a, b] \\ 1 & si & x > b \end{cases}$$

The distribution function curve is as follows:



Mathematical expectation:

Based on the definitional relationship of the mathematical expectation of a continuous random variable, it is:

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b x\frac{1}{b-a}dx$$

$$E(X) = \frac{1}{b-a}\int_a^b xdx$$

$$E(X) = \frac{x^2}{2(b-a)}\bigg|_a^b = \lim_{x\to b}\frac{x^2}{2(b-a)} - \lim_{x\to a}\frac{x^2}{2(b-a)}$$

$$E(X) = \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{b^2-a^2}{2(b-a)}$$

$$E(X) = \frac{(b-a)(b+a)}{2(b-a)}$$

$$E(X) = \frac{(b+a)}{2}$$

Variance:

The variance calculation relationship is given as follows:

$$V(X) = M_2(X) + M_1^2(X)$$

Let us first calculate the second-order simple moment, which is given by the following relationship:

$$M_2(X) = \int_{-\infty}^{+\infty} x^2f(x)dx = \int_a^b x^2\frac{1}{b-a}dx$$

$$M_2(X) = \frac{1}{b-a}\int_a^b x^2dx$$

$$M_2(X) = \frac{x^3}{3(b-a)}\bigg|_a^b = \lim_{x\to b}\frac{x^3}{3(b-a)} - \lim_{x\to a}\frac{x^3}{3(b-a)}$$

$$M_2(X) = \frac{b^3}{3(b-a)} - \frac{a^3}{3(b-a)} = \frac{b^3-a^3}{3(b-a)}$$

$$M_2(X) = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}$$

$$M_2(X) = \frac{(b^2+ab+a^2)}{3}$$

By substituting the value of the mathematical expectation and the value of the second-order simple moment into the variance calculation relationship, we find:

$$V(X) = \frac{(b^2+ab+a^2)}{3} - \left(\frac{b+a}{2}\right)^2$$

Hence, the variance for a continuous random variable that follows uniform distribution is equal to:

$$V(X) = \frac{(b-a)^2}{12}$$

Exponential distribution:

We say that the variable **x** defined in the infinite domain **[0, +∞ [**follows an exponential distribution with a positive real parameter **λ** if its probability density function is written in the following form:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & si\ x \geq 0 \\ 0 & sinon \end{cases}$$

The exponential distribution is a probability distribution because the finite integral of the probability density function in its definition domain is equal to one, where:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{0}^{+\infty} \lambda e^{-\lambda x} dx$$

$$\int_{-\infty}^{+\infty} f(x)dx = -e^{-\lambda x}\Big|_{0}^{+\infty}$$

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{x\to+\infty}\left(-e^{-\lambda x}\right) - \lim_{x\to 0}\left(-e^{-\lambda x}\right)$$

$$\int_{-\infty}^{+\infty} f(x)dx = 0 - (-1) = 1$$

Mathematical expectation:

Based on the definitional relationship of the mathematical expectation of a continuous random variable, it is:

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \int_{0}^{+\infty} x\lambda e^{-\lambda x} dx$$

$$E(X) = \lambda \int_{0}^{+\infty} xe^{-\lambda x} dx$$

We use integration by parts where we put:

$$U = x \Rightarrow dU = dx$$

$$dV = e^{-\lambda x}dx \Rightarrow V = -\frac{e^{-\lambda x}}{\lambda}$$

Hence the mathematical expectation is equal to :

$$E(X) = \lambda\left[\underbrace{-\frac{xe^{-\lambda x}}{\lambda}\Bigg|_{0}^{+\infty}}_{=0} - \int_{0}^{+\infty} -\frac{e^{-\lambda x}}{\lambda}dx\right]$$

$$E(X) = \lambda\left[0 + \frac{e^{-\lambda x}}{\lambda^2}\Big|_0^{+\infty}\right]$$

$$E(X) = \frac{\lambda}{\lambda^2}$$

$$E(X) = \frac{1}{\lambda}$$

Variance:

The variance calculation relationship is given as follows:

$$V(X) = M_2(X) + M_1^2(X)$$

Let us first calculate the second-order simple moment, which is given by the following relationship:

$$M_2(X) = \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_0^{+\infty} x^2\lambda e^{-\lambda x}dx$$

We use integration by parts where we put:

$$U = x^2 \Rightarrow dU = 2xdx$$

$$dV = e^{-\lambda x}dx \Rightarrow V = -\frac{e^{-\lambda x}}{\lambda}$$

$$M_2(X) = \underbrace{-\frac{x^2e^{-\lambda x}}{\lambda}\Big|_0^{+\infty}}_{=0} - 2\int_0^{+\infty} -\frac{xe^{-\lambda x}}{\lambda}dx$$

$$M_2(X) = \frac{2}{\lambda}\underbrace{\int_0^{+\infty} xe^{-\lambda x}dx}_{=E(X)=\frac{1}{\lambda}}$$

$$M_2(X) = \frac{2}{\lambda}\cdot\frac{1}{\lambda}$$

$$M_2(X) = \frac{2}{\lambda^2}$$

By substituting the value of the mathematical expectation and the value of the second-order simple moment into the variance calculation relationship, we find:

$$V(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

Hence, the variance for a continuous random variable that follows exponential distribution is equal to:

$$V(X) = \frac{1}{\lambda^2}$$

The normal distribution:

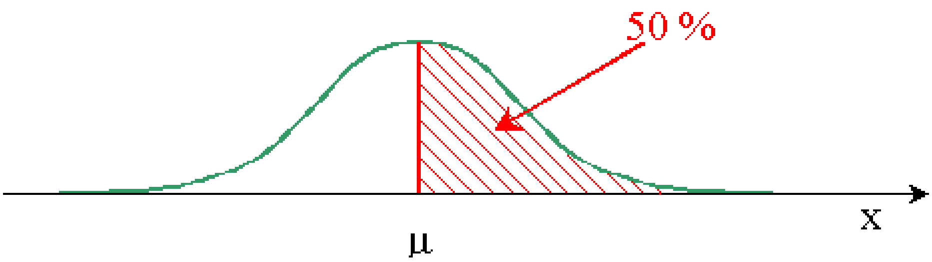
We say that a continuous random variable **x** defined in the infinite domain **[-∞, +∞ [**follows normal distribution or Laplace-Gausse distribution if its density function is written in the following form:

$$\boldsymbol{f(x)} = \frac{\mathbf{1}}{\boldsymbol{\delta\sqrt{2\pi}}} \quad \boldsymbol{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2} \quad \forall \boldsymbol{x} \in \left]-\infty, +\infty\right[$$

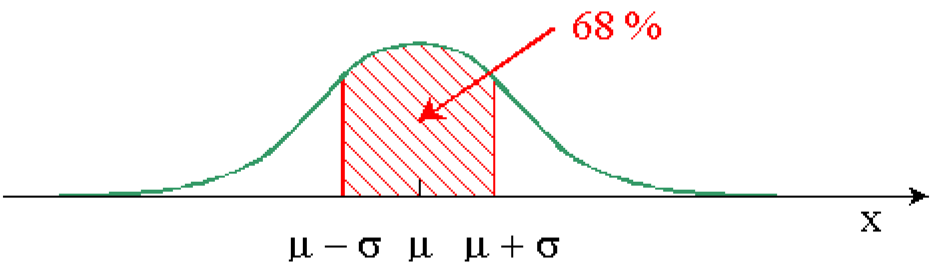
Whereas $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ are optional constants.

This distribution is considered one of the most important probability distributions due to the presence of a large number of random variables that are subject to it. Its density function curve is given the shape of a symmetrical bell to the line **X = μ**, and it achieves a set of characteristics that we mention below:

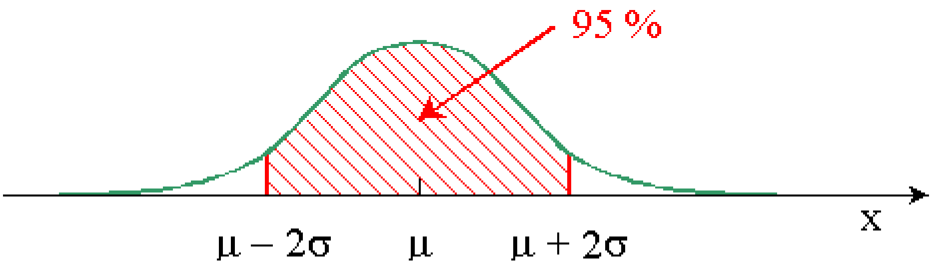
1. In statistical populations that follow a normal distribution, **50%** of the statistical units have a value above the mathematical expectation value **μ**, and the remaining **50%** have values less than the value **μ**.



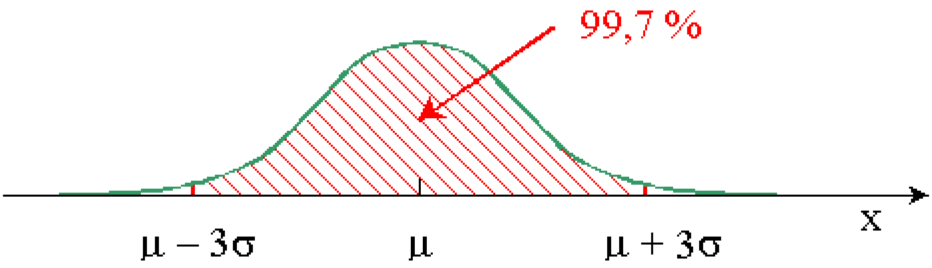
2. In statistical populations that follow a normal distribution, **68%** of statistical units are in the range **[μ-δ,μ+δ].**



3. In statistical populations that follow a normal distribution, **95%** of statistical units are in the range$[\boldsymbol{\mu} - \mathbf{2}\boldsymbol{\delta}, \boldsymbol{\mu} + \mathbf{2}\boldsymbol{\delta}]$**.**



4. In statistical populations that follow a normal distribution, **99.7%** of statistical units are in the range$[\boldsymbol{\mu - 3\delta, \mu + 3\delta}]$.



From the various curves, we notice the role and importance of the two chosen constants. Whenever $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ change, there is a change in the shape of the density function curve ***f(x)***.

The normal distribution is a probability distribution law because the finite integral of the probability density function in the domain of its definition is equal to one, where:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{+\infty} \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2} dx$$

$$\int_{-\infty}^{+\infty} f(x)dx = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2} dx$$

We put :

$$t^2 = \frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2 \Rightarrow t = \frac{1}{\sqrt{2}}.\frac{x-\mu}{\delta}$$

$$\Rightarrow x = t\delta\sqrt{2} + \mu \Rightarrow dx = \delta\sqrt{2}dt$$

Hence:

$$\int_{-\infty}^{+\infty} f(x)dx = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}t^2} \delta\sqrt{2}dt$$

$$\int_{-\infty}^{+\infty} f(x)dx = \frac{\delta\sqrt{2}}{\delta\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}t^2} dt$$

$$\int_{-\infty}^{+\infty} f(x)dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}t^2} dt$$

Using Fourrier integrals, we find that:

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}t^2} dt = \sqrt{\pi} \Rightarrow \int_{-\infty}^{+\infty} f(x)dx = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

Mathematical expectation:

Based on the definitional relationship of the mathematical expectation of a continuous random variable, it is:

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{+\infty} [(x-\mu)+\mu]f(x)dx$$

$$E(X) = \underbrace{\int_{-\infty}^{+\infty} (x-\mu)f(x)dx}_{A} + \underbrace{\int_{-\infty}^{+\infty} \mu f(x)dx}_{B}$$

Let us take care of part **A** first:

$$A = \int_{-\infty}^{+\infty} \frac{(x-\mu)}{\delta\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2} dx$$

We put : $y = x - \mu$

$$A = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{+\infty} y\, e^{-\frac{1}{2}\left(\frac{y}{\delta}\right)^2} dy$$

We also put :

$$y^2 = z \Rightarrow dz = 2ydy \Rightarrow dy = \frac{dz}{2y}$$

$$A = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{+\infty} y\, e^{-\frac{z}{2\delta^2}} \frac{dz}{2y}$$

$$A = \frac{1}{2\delta\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{z}{2\delta^2}} dz$$

$$A = \frac{1}{2\delta\sqrt{2\pi}} \cdot \left. \frac{e^{-\frac{z}{2\delta^2}}}{\frac{-1}{2\delta^2}} \right|_{-\infty}^{+\infty} = \frac{-\delta}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y}{\delta}\right)^2} \Big|_{-\infty}^{+\infty} = 0$$

$$A = 0$$

Part **B** is equal to:

$$B = \int_{-\infty}^{+\infty} \mu f(x)dx = \mu \int_{-\infty}^{+\infty} f(x)dx = \mu \times 1$$

$$B = \mu$$

Hence, the mathematical expectation is equal to:

$$E(X) = A + B = 0 + \mu = \mu$$

$$E(X) = \mu$$

Variance:

The variance calculation relationship is given as follows:

$$V(X)=\int_{-\infty}^{+\infty}\left(x-E(X)\right)^2 f(x)\,dx$$

$$V(X)=\int_{-\infty}^{+\infty}\left(x-E(X)\right)^2\frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2}}{\delta\sqrt{2\pi}}\,dx$$

$$V(X)=\frac{1}{\delta\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(x-E(X)\right)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2}\,dx$$

We put:

$$t^2=\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2 \Rightarrow t=\frac{x-\mu}{\delta\sqrt{2}}$$

$$\Rightarrow x=t\delta\sqrt{2}+\mu \Rightarrow dx=\delta dt\sqrt{2}$$

$$\Rightarrow (x-\mu)^2=2\delta^2t^2$$

$$V(X)=\frac{1}{\delta\sqrt{2\pi}}\int_{-\infty}^{+\infty}t^2\,e^{-t^2}\delta dt\sqrt{2}$$

$$V(X)=\frac{2\delta^2}{\sqrt{\pi}}\int_{-\infty}^{+\infty}t^2\,e^{-t^2}dt$$

We use integration by parts where we put:

$$U=t\Rightarrow dU=dt$$

$$dV=te^{-t^2}dt\Rightarrow V=-\frac{1}{2}e^{-t^2}$$

$$V(X)=\frac{2\delta^2}{\sqrt{\pi}}\left[\underbrace{t-\frac{1}{2}e^{-t^2}\Big|_{-\infty}^{+\infty}}_{=0}+\int_{-\infty}^{+\infty}\frac{1}{2}e^{-t^2}dt\right]$$

$$V(X)=\frac{2\delta^2}{\sqrt{\pi}}.\frac{1}{2}\int_{-\infty}^{+\infty}e^{-t^2}dt=\frac{2\delta^2}{\sqrt{\pi}}.\frac{\sqrt{\pi}}{2}=\delta^2$$

Hence, the variance for a continuous random variable that follows normal distribution is equal to:

$$V(X)=\delta^2$$

Standard normal distribution and probability calculation:

Calculating the probability that the variable **x** defined in the range **]-∞,+∞[**, which follows the general normal distribution with mathematical expectation **μ** and variance **δ²**, takes a value confined between **[x_1, x_2]** where $x_1 \le x \le x_2$, we will calculate the area enclosed by the density function in the range **[x_1, x_2]** ,in other words, we will calculate the finite integral of the density function in this field, which is written in the following form:

$$P(x_1 \le x \le x_2) = \int_{x_1}^{x_2} \frac{1}{\delta\sqrt{2\pi}} \quad e^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2}$$

Calculating this integration is a difficult and complex process, so mathematicians and statisticians resort to doing some mathematical transformations, where we calculate the ***z*** value defined as follows:

$$z = \frac{x-\mu}{\delta}$$

Since this variable follows a normal distribution with mathematical expectation **μ= 0** and variance **δ ² = 1**, it is written as follows:

$$f(z) = \frac{1}{\sqrt{2\pi}} \quad e^{-\frac{z^2}{2}} \quad \forall z \in]-\infty, +\infty[$$

These two properties can be proven based on the properties of mathematical expectation, as we have:

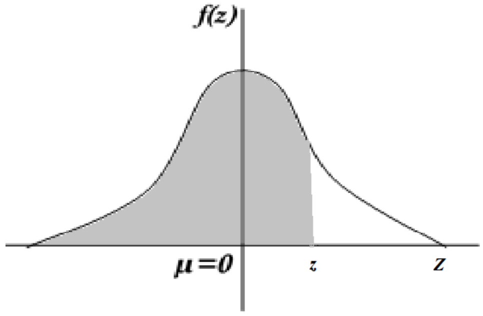
$$E(X) = \mu \Rightarrow E(X-\mu) = E(X) - \mu = 0$$

On the other hand, we also have for the variance:

$$V(X) = \delta^2 \Rightarrow V(X-\mu) = \delta^2$$

$$V\left(\frac{X-\mu}{\delta}\right) = \frac{V(X-\mu)}{\delta^2} = \frac{\delta^2}{\delta^2} = 1$$

The density function curve takes the shape of a symmetrical bell to the line **z = 0**. In this case, it is called **the standard normal distribution**, and it comes in the following form:



Then, statisticians and mathematicians calculated the area enclosed by the variable z, that is, the aggregate function, and recorded the values of these probabilities in a table.

$$F(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \quad e^{-\frac{z^2}{2}} dz$$

z	0	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,50000000	0,50398936	0,50797831	0,51196647	0,51595344	0,51993881	0,52392218	0,52790317	0,53188137	0,53585639
0,1	0,53982784	0,54379531	0,54775843	0,55171679	0,55567000	0,55961769	0,56355946	0,56749493	0,57142372	0,57534543
0,2	0,57925971	0,58316616	0,58706442	0,59095412	0,59483487	0,59870633	0,60256811	0,60641987	0,61026125	0,61409188
0,3	0,61791142	0,62171952	0,62551583	0,62930002	0,63307174	0,63683065	0,64057643	0,64430875	0,64802729	0,65173173
0,4	0,65542174	0,65909703	0,66275727	0,66640218	0,67003145	0,67364478	0,67724189	0,68082249	0,68438630	0,68793305
0,5	0,69146246	0,69497427	0,69846821	0,70194403	0,70540148	0,70884031	0,71226028	0,71566115	0,71904269	0,72240468
0,6	0,72574688	0,72906910	0,73237111	0,73565271	0,73891370	0,74215389	0,74537309	0,74857110	0,75174777	0,75490291
0,7	0,75803635	0,76114793	0,76423750	0,76730491	0,77035000	0,77337265	0,77637271	0,77935005	0,78230456	0,78523612
0,8	0,78814460	0,79102991	0,79389195	0,79673061	0,79954581	0,80233746	0,80510548	0,80784980	0,81057035	0,81326706

0,9	0,81593987	0,81858875	0,82121362	0,82381446	0,82639122	0,82894387	0,83147239	0,83397675	0,83645694	0,83891294
1,0	0,84134475	0,84375235	0,84613577	0,84849500	0,85083005	0,85314094	0,85542770	0,85769035	0,85992891	0,86214343
1,1	0,86433394	0,86650049	0,86864312	0,87076189	0,87285685	0,87492806	0,87697560	0,87899952	0,88099989	0,88297680
1,2	0,88493033	0,88686055	0,88876756	0,89065145	0,89251230	0,89435023	0,89616532	0,89795768	0,89972743	0,90147467
1,3	0,90319952	0,90490208	0,90658249	0,90824086	0,90987733	0,91149201	0,91308504	0,91465655	0,91620668	0,91773556
1,4	0,91924334	0,92073016	0,92219616	0,92364149	0,92506630	0,92647074	0,92785496	0,92921912	0,93056338	0,93188788
1,5	0,93319280	0,93447829	0,93574451	0,93699164	0,93821982	0,93942924	0,94062006	0,94179244	0,94294657	0,94408260
1,6	0,94520071	0,94630107	0,94738386	0,94844925	0,94949742	0,95052853	0,95154277	0,95254032	0,95352134	0,95448602
1,7	0,95543454	0,95636706	0,95728378	0,95818486	0,95907049	0,95994084	0,96079610	0,96163643	0,96246202	0,96327304
1,8	0,96406968	0,96485211	0,96562050	0,96637503	0,96711588	0,96784323	0,96855724	0,96925809	0,96994596	0,97062102
1,9	0,97128344	0,97193339	0,97257105	0,97319658	0,97381016	0,97441194	0,97500210	0,97558081	0,97614824	0,97670453
2,0	0,97724987	0,97778441	0,97830831	0,97882173	0,97932484	0,97981778	0,98030073	0,98077383	0,98123723	0,98169110
2,1	0,98213558	0,98257082	0,98299698	0,98341419	0,98382262	0,98422239	0,98461367	0,98499658	0,98537127	0,98573788
2,2	0,98609655	0,98644742	0,98679062	0,98712628	0,98745454	0,98777553	0,98808937	0,98839621	0,98869616	0,98898934
2,3	0,98927589	0,98955592	0,98982956	0,99009692	0,99035813	0,99061329	0,99086253	0,99110596	0,99134368	0,99157581
2,4	0,99180246	0,99202374	0,99223975	0,99245059	0,99265637	0,99285719	0,99305315	0,99324435	0,99343088	0,99361285
2,5	0,99379033	0,99396344	0,99413226	0,99429687	0,99445738	0,99461385	0,99476639	0,99491507	0,99505998	0,99520120
2,6	0,99533881	0,99547289	0,99560351	0,99573076	0,99585470	0,99597541	0,99609297	0,99620744	0,99631889	0,99642740
2,7	0,99653303	0,99663584	0,99673590	0,99683328	0,99692804	0,99702024	0,99710993	0,99719719	0,99728206	0,99736460
2,8	0,99744487	0,99752293	0,99759882	0,99767260	0,99774432	0,99781404	0,99788179	0,99794764	0,99801162	0,99807379
2,9	0,99813419	0,99819286	0,99824984	0,99830519	0,99835894	0,99841113	0,99846180	0,99851100	0,99855876	0,99860511

How to use the table and calculate probabilities:

To calculate the probability that the variable **x** takes the form defined in the field **[-∞,+∞]** and which follows the general normal distribution with mathematical expectation **μ** and variance **δ²** for a value less than or equal to x_1, i.e. calculating **P(X ≤ x_1)**, we follow the following steps:

1. Calculate z_0 using the relationship: $z_0 = \frac{x_0-\mu}{\delta}$

2. Searching for the value of z_0 in the table, where the first column gives us the integer number and the first decimal number (the first number after the comma) in the value of z_0, while the first line gives us the second decimal number, i.e. the second number after the comma in the value of z_0.

3. Read the probability value in the table that is found at the intersection of the line and column specified for the value of z_0, and we denote it with the symbol $\phi(z_0)$.

4. To calculate the probability **P(Z> z_0)** we do the following:

$$P(Z > z_0) = 1 - P(Z < z_0) = 1 - \Phi(z_0)$$

5. If the value of z_0 is negative, i.e. **($-z_0$)**, we perform the following operation:

$$P(Z < -z_0) = P(Z > z_0) = 1 - P(Z < z_0) = 1 - \Phi(z_0)$$

6. Calculating the probability that z is confined between two values z_1 and z_2, where $z_1<z<z_2$, that is, calculating the probability **P($z_1<z<z_2$)**, we perform the following process:

$$P(z_1 < z < z_2) = P(z < z_2) - P(z < z_1) = \Phi(z_2) - \Phi(z_1)$$

Examples:

- **Let us calculate P (Z < 1.48):**

To read the probability value that **Z** will take a value less than **1,48**, we search in the first column **1,4** corresponding to the first integer and the first decimal number forming the value **1,48**, and we search in the first line for the value **0,08** corresponding to the second decimal number forming the value **1,48**, then we read the probability value in the box representing the intersection of that line and that column as shown in the table, which is: **ϕ (1,48) = 0.93056338**.

z	0	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,50000000	0,50398936	0,50797831	0,51196647	0,51595344	0,51993881	0,52392218	0,52790317	0,53188137	0,53585639
0,1	0,53982784	0,54379531	0,54775843	0,55171679	0,55567000	0,55961769	0,56355946	0,56749493	0,57142372	0,57534543
0,2	0,57925971	0,58316616	0,58706442	0,59095412	0,59483487	0,59870633	0,60256811	0,60641987	0,61026125	0,61409188
0,3	0,61791142	0,62171952	0,62551583	0,62930002	0,63307174	0,63683065	0,64057643	0,64430875	0,64802729	0,65173173
0,4	0,65542174	0,65909703	0,66275727	0,66640218	0,67003145	0,67364478	0,67724189	0,68082249	0,68438630	0,68793305
0,5	0,69146246	0,69497427	0,69846821	0,70194403	0,70540148	0,70884031	0,71226028	0,71566115	0,71904269	0,72240468
0,6	0,72574688	0,72906910	0,73237111	0,73565271	0,73891370	0,74215389	0,74537309	0,74857110	0,75174777	0,75490291
0,7	0,75803635	0,76114793	0,76423750	0,76730491	0,77035000	0,77337265	0,77637271	0,77935005	0,78230456	0,78523612
0,8	0,78814460	0,79102991	0,79389195	0,79673061	0,79954581	0,80233746	0,80510548	0,80784980	0,81057035	0,81326706
0,9	0,81593987	0,81858875	0,82121362	0,82381446	0,82639122	0,82894387	0,83147239	0,83397675	0,83645694	0,83891294
1,0	0,84134475	0,84375235	0,84613577	0,84849500	0,85083005	0,85314094	0,85542770	0,85769035	0,85992891	0,86214343
1,1	0,86433394	0,86650049	0,86864312	0,87076189	0,87285685	0,87492806	0,87697560	0,87899952	0,88099989	0,88297680
1,2	0,88493033	0,88686055	0,88876756	0,89065145	0,89251230	0,89435023	0,89616532	0,89795768	0,89972743	0,90147467
1,3	0,90319952	0,90490208	0,90658249	0,90824086	0,90987733	0,91149201	0,91308504	0,91465655	0,91620668	0,91773556
1,4	0,91924334	0,92073016	0,92219616	0,92364149	0,92506630	0,92647074	0,92785496	0,92921912	0,93056338	0,93188788
1,5	0,93319280	0,93447829	0,93574451	0,93699164	0,93821982	0,93942924	0,94062006	0,94179244	0,94294657	0,94408260
1,6	0,94520071	0,94630107	0,94738386	0,94844925	0,94949742	0,95052853	0,95154277	0,95254032	0,95352134	0,95448602
1,7	0,95543454	0,95636706	0,95728378	0,95818486	0,95907049	0,95994084	0,96079610	0,96163643	0,96246202	0,96327304
1,8	0,96406968	0,96485211	0,96562050	0,96637503	0,96711588	0,96784323	0,96855724	0,96925809	0,96994596	0,97062102
1,9	0,97128344	0,97193339	0,97257105	0,97319658	0,97381016	0,97441194	0,97500210	0,97558081	0,97614824	0,97670453
2,0	0,97724987	0,97778441	0,97830831	0,97882173	0,97932484	0,97981778	0,98030073	0,98077383	0,98123723	0,98169110
2,1	0,98213558	0,98257082	0,98299698	0,98341419	0,98382262	0,98422239	0,98461367	0,98499658	0,98537127	0,98573788
2,2	0,98609655	0,98644742	0,98679062	0,98712628	0,98745454	0,98777553	0,98808937	0,98839621	0,98869616	0,98898934
2,3	0,98927589	0,98955592	0,98982956	0,99009692	0,99035813	0,99061329	0,99086253	0,99110596	0,99134368	0,99157581
2,4	0,99180246	0,99202374	0,99223975	0,99245059	0,99265637	0,99285719	0,99305315	0,99324435	0,99343088	0,99361285
2,5	0,99379033	0,99396344	0,99413226	0,99429687	0,99445738	0,99461385	0,99476639	0,99491507	0,99505998	0,99520120
2,6	0,99533881	0,99547289	0,99560351	0,99573076	0,99585470	0,99597541	0,99609297	0,99620744	0,99631889	0,99642740
2,7	0,99653303	0,99663584	0,99673590	0,99683328	0,99692804	0,99702024	0,99710993	0,99719719	0,99728206	0,99736460
2,8	0,99744487	0,99752293	0,99759882	0,99767260	0,99774432	0,99781404	0,99788179	0,99794764	0,99801162	0,99807379
2,9	0,99813419	0,99819286	0,99824984	0,99830519	0,99835894	0,99841113	0,99846180	0,99851100	0,99855876	0,99860511

- **Let's calculate the probability P(Z < 0.56):**

To read the probability value that **Z** will take a value less than **0.56**, we search in the first column **0,5** corresponding to the first whole number and the first decimal number forming the value **0,56**, and we search in the first line for the value **0,06** corresponding to the second decimal number forming the value **0,56**, then we read the probability value in the box representing the intersection of that line and that column as shown in the table, which is: **ϕ (0.56) = 0.71226028**.

z	0	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,50000000	0,50398936	0,50797831	0,51196647	0,51595344	0,51993881	0,52392218	0,52790317	0,53188137	0,53585639
0,1	0,53982784	0,54379531	0,54775843	0,55171679	0,55567000	0,55961769	0,56355946	0,56749493	0,57142372	0,57534543
0,2	0,57925971	0,58316616	0,58706442	0,59095412	0,59483487	0,59870633	0,60256811	0,60641987	0,61026125	0,61409188
0,3	0,61791142	0,62171952	0,62551583	0,62930002	0,63307174	0,63683065	0,64057643	0,64430875	0,64802729	0,65173173
0,4	0,65542174	0,65909703	0,66275727	0,66640218	0,67003145	0,67364478	0,67724189	0,68082249	0,68438630	0,68793305
0,5	0,69146246	0,69497427	0,69846821	0,70194403	0,70540148	0,70884031	0,71226028	0,71566115	0,71904269	0,72240468
0,6	0,72574688	0,72906910	0,73237111	0,73565271	0,73891370	0,74215389	0,74537309	0,74857110	0,75174777	0,75490291
0,7	0,75803635	0,76114793	0,76423750	0,76730491	0,77035000	0,77337265	0,77637271	0,77935005	0,78230456	0,78523612
0,8	0,78814460	0,79102991	0,79389195	0,79673061	0,79954581	0,80233746	0,80510548	0,80784980	0,81057035	0,81326706
0,9	0,81593987	0,81858875	0,82121362	0,82381446	0,82639122	0,82894387	0,83147239	0,83397675	0,83645694	0,83891294
1,0	0,84134475	0,84375235	0,84613577	0,84849500	0,85083005	0,85314094	0,85542770	0,85769035	0,85992891	0,86214343
1,1	0,86433394	0,86650049	0,86864312	0,87076189	0,87285685	0,87492806	0,87697560	0,87899952	0,88099989	0,88297680
1,2	0,88493033	0,88686055	0,88876756	0,89065145	0,89251230	0,89435023	0,89616532	0,89795768	0,89972743	0,90147467
1,3	0,90319952	0,90490208	0,90658249	0,90824086	0,90987733	0,91149201	0,91308504	0,91465655	0,91620668	0,91773556
1,4	0,91924334	0,92073016	0,92219616	0,92364149	0,92506630	0,92647074	0,92785496	0,92921912	0,93056338	0,93188788

1,5	0,93319280	0,93447829	0,93574451	0,93699164	0,93821982	0,93942924	0,94062006	0,94179244	0,94294657	0,94408260
1,6	0,94520071	0,94630107	0,94738386	0,94844925	0,94949742	0,95052853	0,95154277	0,95254032	0,95352134	0,95448602
1,7	0,95543454	0,95636706	0,95728378	0,95818486	0,95907049	0,95994084	0,96079610	0,96163643	0,96246202	0,96327304
1,8	0,96406968	0,96485211	0,96562050	0,96637503	0,96711588	0,96784323	0,96855724	0,96925809	0,96994596	0,97062102
1,9	0,97128344	0,97193339	0,97257105	0,97319658	0,97381016	0,97441194	0,97500210	0,97558081	0,97614824	0,97670453
2,0	0,97724987	0,97778441	0,97830831	0,97882173	0,97932484	0,97981778	0,98030073	0,98077383	0,98123723	0,98169110
2,1	0,98213558	0,98257082	0,98299698	0,98341419	0,98382262	0,98422239	0,98461367	0,98499658	0,98537127	0,98573788
2,2	0,98609655	0,98644742	0,98679062	0,98712628	0,98745454	0,98777553	0,98808937	0,98839621	0,98869616	0,98898934
2,3	0,98927589	0,98955592	0,98982956	0,99009692	0,99035813	0,99061329	0,99086253	0,99110596	0,99134368	0,99157581
2,4	0,99180246	0,99202374	0,99223975	0,99245059	0,99265637	0,99285719	0,99305315	0,99324435	0,99343088	0,99361285
2,5	0,99379033	0,99396344	0,99413226	0,99429687	0,99445738	0,99461385	0,99476639	0,99491507	0,99505998	0,99520120
2,6	0,99533881	0,99547289	0,99560351	0,99573076	0,99585470	0,99597541	0,99609297	0,99620744	0,99631889	0,99642740
2,7	0,99653303	0,99663584	0,99673590	0,99683328	0,99692804	0,99702024	0,99710993	0,99719719	0,99728206	0,99736460
2,8	0,99744487	0,99752293	0,99759882	0,99767260	0,99774432	0,99781404	0,99788179	0,99794764	0,99801162	0,99807379
2,9	0,99813419	0,99819286	0,99824984	0,99830519	0,99835894	0,99841113	0,99846180	0,99851100	0,99855876	0,99860511

- **Let's calculate the probability P(Z > 2.23):**

Since the table gives us the probability that **Z** takes a value less than **2,23** and does not give us the probability that **Z** takes a value greater than **2,23**, we will use the probability property of the complementary event as follows:

$$P(Z > 2.23) = 1 - P(Z < 2.23) = 1 - \Phi(2.23)$$

To read the probability value that **Z** will take a value less than **2,23**, we search in the first column **2,2** corresponding to the first integer number and the first decimal number forming the value **2,23**, and we search in the first line for the value **0,03** corresponding to the second decimal number forming the value **2,23**, then we read the probability value in the box representing the intersection of that line and that column as shown in the table, which is: **ϕ (2,23) = 0.98712628**.

Hence:

$$P(Z > 2.23) = 1 - \Phi(2.23) = 1 - 0.98712628 = 0.01287372$$

- **Let's calculate the probability P(Z > -1.33):**

Since the table does not give us the probability that **Z** will take a value greater than negative one, and based on the property of symmetry of the curve, we have:

$$P(Z < -1.33) = P(Z > 1.33) = 1 - P(Z < 1.33) = 1 - \Phi(1.33)$$

To read the probability value that **Z** will take a value less than **1,33**, we search in the first column **1,3** corresponding to the first integer number and the first decimal number forming the value **1,33**, and we search in the first line for the value **0,03** corresponding to the second decimal number forming the value **1,33**, then we read the probability value in the box representing the intersection of that line and that column as shown in the table, which is: **ϕ (1,33) = 0.90824086**.

z	0	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,50000000	0,50398936	0,50797831	0,51196647	0,51595344	0,51993881	0,52392218	0,52790317	0,53188137	0,53585639
0,1	0,53982784	0,54379531	0,54775843	0,55171679	0,55567000	0,55961769	0,56355946	0,56749493	0,57142372	0,57534543
0,2	0,57925971	0,58316616	0,58706442	0,59095412	0,59483487	0,59870633	0,60256811	0,60641987	0,61026125	0,61409188
0,3	0,61791142	0,62171952	0,62551583	0,62930002	0,63307174	0,63683065	0,64057643	0,64430875	0,64802729	0,65173173
0,4	0,65542174	0,65909703	0,66275727	0,66640218	0,67003145	0,67364478	0,67724189	0,68082249	0,68438630	0,68793305
0,5	0,69146246	0,69497427	0,69846821	0,70194403	0,70540148	0,70884031	0,71226028	0,71566115	0,71904269	0,72240468
0,6	0,72574688	0,72906910	0,73237111	0,73565271	0,73891370	0,74215389	0,74537309	0,74857110	0,75174777	0,75490291

0,7	0,75803635	0,76114793	0,76423750	0,76730491	0,77035000	0,77337265	0,77637271	0,77935005	0,78230456	0,78523612
0,8	0,78814460	0,79102991	0,79389195	0,79673061	0,79954581	0,80233746	0,80510548	0,80784980	0,81057035	0,81326706
0,9	0,81593987	0,81858875	0,82121362	0,82381446	0,82639122	0,82894387	0,83147239	0,83397675	0,83645694	0,83891294
1,0	0,84134475	0,84375235	0,84613577	0,84849500	0,85083005	0,85314094	0,85542770	0,85769035	0,85992891	0,86214343
1,1	0,86433394	0,86650049	0,86864312	0,87076189	0,87285685	0,87492806	0,87697560	0,87899952	0,88099989	0,88297680
1,2	0,88493033	0,88686055	0,88876756	0,89065145	0,89251230	0,89435023	0,89616532	0,89795768	0,89972743	0,90147467
1,3	0,90319952	0,90490208	0,90658249	0,90824086	0,90987733	0,91149201	0,91308504	0,91465655	0,91620668	0,91773556
1,4	0,91924334	0,92073016	0,92219616	0,92364149	0,92506630	0,92647074	0,92785496	0,92921912	0,93056338	0,93188788
1,5	0,93319280	0,93447829	0,93574451	0,93699164	0,93821982	0,93942924	0,94062006	0,94179244	0,94294657	0,94408260
1,6	0,94520071	0,94630107	0,94738386	0,94844925	0,94949742	0,95052853	0,95154277	0,95254032	0,95352134	0,95448602
1,7	0,95543454	0,95636706	0,95728378	0,95818486	0,95907049	0,95994084	0,96079610	0,96163643	0,96246202	0,96327304
1,8	0,96406968	0,96485211	0,96562050	0,96637503	0,96711588	0,96784323	0,96855724	0,96925809	0,96994596	0,97062102
1,9	0,97128344	0,97193339	0,97257105	0,97319658	0,97381016	0,97441194	0,97500210	0,97558081	0,97614824	0,97670453
2,0	0,97724987	0,97778441	0,97830831	0,97882173	0,97932484	0,97981778	0,98030073	0,98077383	0,98123723	0,98169110
2,1	0,98213558	0,98257082	0,98299698	0,98341419	0,98382262	0,98422239	0,98461367	0,98499658	0,98537127	0,98573788
2,2	0,98609655	0,98644742	0,98679062	0,98712628	0,98745454	0,98777553	0,98808937	0,98839621	0,98869616	0,98898934
2,3	0,98927589	0,98955592	0,98982956	0,99009692	0,99035813	0,99061329	0,99086253	0,99110596	0,99134368	0,99157581
2,4	0,99180246	0,99202374	0,99223975	0,99245059	0,99265637	0,99285719	0,99305315	0,99324435	0,99343088	0,99361285
2,5	0,99379033	0,99396344	0,99413226	0,99429687	0,99445738	0,99461385	0,99476639	0,99491507	0,99505998	0,99520120
2,6	0,99533881	0,99547289	0,99560351	0,99573076	0,99585470	0,99597541	0,99609297	0,99620744	0,99631889	0,99642740
2,7	0,99653303	0,99663584	0,99673590	0,99683328	0,99692804	0,99702024	0,99710993	0,99719719	0,99728206	0,99736460
2,8	0,99744487	0,99752293	0,99759882	0,99767260	0,99774432	0,99781404	0,99788179	0,99794764	0,99801162	0,99807379
2,9	0,99813419	0,99819286	0,99824984	0,99830519	0,99835894	0,99841113	0,99846180	0,99851100	0,99855876	0,99860511

Solved exercises related to the chapter

<u>Exercise 01:</u>

Let the random variable defined by the density function be as follows:

$$f(x) = \begin{cases} \dfrac{3}{4} & si \quad 0 \leq x \leq 1 \\ \dfrac{1}{4} & si \quad 2 \leq x \leq 3 \\ 0 & sinon \end{cases}$$

1- Prove that f is a probability density function and represent it graphically.

2- Find the distribution function and draw its curve.

Solution:

In order for f to be a probability density function, its finite integral in the domain of its definition must be equal to one, that is, it fulfills the following relationship:

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

Let us calculate this definite integral, i.e.:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{0}^{1} \frac{3}{4}dx + \int_{2}^{3} \frac{1}{4}dx$$

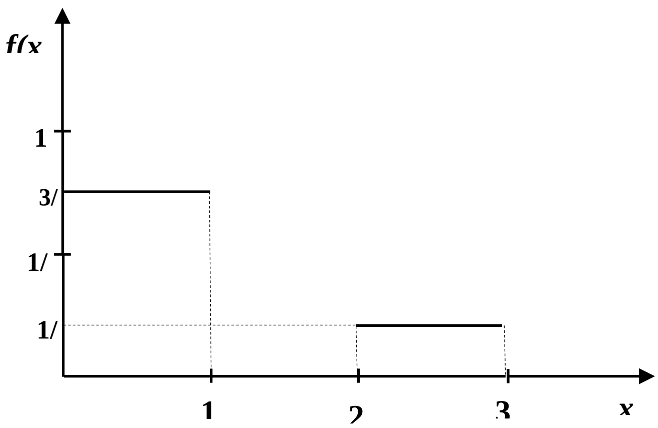
$$\int_{-\infty}^{+\infty} f(x)dx = \frac{3}{4}x\Big|_{0}^{1} + \frac{1}{4}x\Big|_{2}^{3}$$

$$\int_{-\infty}^{+\infty} f(x)dx = \left[\lim_{x\to 1}\frac{3}{4}x - \lim_{x\to 0}\frac{3}{4}x\right] + \left[\lim_{x\to 3}\frac{1}{4}x - \lim_{x\to 2}\frac{1}{4}x\right]$$

$$\int_{-\infty}^{+\infty} \boldsymbol{f(x)dx} = \left[\frac{3}{4} - 0\right] + \left[\frac{3}{4} - \frac{2}{4}\right]$$

$$\int_{-\infty}^{+\infty} \boldsymbol{f(x)dx} = \left[\frac{3}{4}\right] + \left[\frac{1}{4}\right] = 1$$

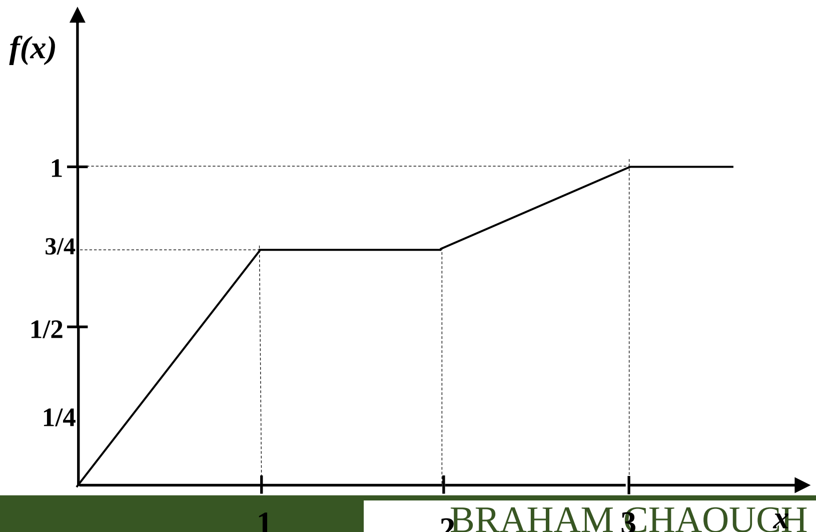
Hence, ***f*** represents a probability distribution and its curve is as follows:



Finding the distribution function:

$$F(x) = \int_0^x f(x)dx = \begin{cases} \frac{3}{4}x & si \quad 0 \le x \le 1 \\ \frac{3}{4} & si \quad 1 \le x \le 2 \\ \frac{3}{4}x + \frac{1}{4}x & si \quad 2 \le x \le 3 \\ 1 & si \quad x \ge 3 \end{cases}$$

The curve of the distribution function is as follows:



Exercise 02:

The shelf life of an electronic piece, estimated in years, is a random variable T whose distribution function is written as follows:

$$\begin{cases} F(t) = 0 & si \quad t < 0 \\ F(t) = 1 - e^{-\frac{t}{2}} & si \quad t \geq 0 \end{cases}$$

1- Find the probability density function for the variable **T** and give the name of this distribution and its numerical features.

2- If you know that this piece has been valid for a full year, what is the probability that it will remain valid for at least another two years?

Solution:

1. The probability density function is the derivative function of the distribution function, and therefore the probability density function followed by the variable **T** is equal to:

$$f(t) = \begin{cases} 0 & si \quad t < 0 \\ \frac{1}{2} e^{-\frac{t}{2}} & si \quad t \geq 0 \end{cases}$$

We note that the probability density function is of the form:

$$\lambda = \frac{1}{2} \quad \text{أن حيث} \quad f(x) = \begin{cases} \lambda e^{-\lambda x} & si\ x \geq 0 \\ 0 & sinon \end{cases}$$

Hence, **T** follows the exponential distribution law with a parameter **λ=1/2**, where the numerical features are equal to:

$$E(X) == \frac{1}{\lambda} = 2$$

$$V(X) = \frac{1}{\lambda^2} = 4$$

2. Calculate the probability of **P(1≤T≤3):**

$$P(1 \leq T \leq 3) = F(3) - F(1)$$

$$P(1 \leq T \leq 3) = \left(1 - e^{-\frac{3}{2}}\right) - \left(1 - e^{-\frac{1}{2}}\right)$$

$$P(1 \leq T \leq 3) = -e^{-\frac{3}{2}} + e^{-\frac{1}{2}}$$

$$P(1 \leq T \leq 3) = 0,3834$$

Exercise 03:

Find the area under the standard normal distribution between:

$t=\pm1.64$ $t=\pm1.96$, $t=\pm2.58$, (t=0.9,t=2.10) , $t\leq0.9$, $t\geq2.1$,

Solution:

$$P(-1,64 \leq t \leq +1,64) = P(t \leq +1,64) - P(t \leq -1,64)$$

$$P(-1,64 \leq t \leq +1,64) = \Phi(1,64) - \Phi(-1,64)$$

$$P(-1,64 \leq t \leq +1,64) = \Phi(1,64) - [1 - \Phi(1,64)]$$

$P(-1,64 \leq t \leq +1,64) = 2\Phi(1,64) - 1$

$P(-1,64 \leq t \leq +1,64) = 2 \times (0,94949742) - 1$

$P(-1,64 \leq t \leq +1,64) = 0,89899484$

$P(t \geq 2,1) = 1 - P(t \leq 2,1)$

$P(t \geq 2,1) = 1 - \Phi(2,1)$

$P(t \geq 2,1) = 1 - 0,98213558$

$P(t \geq 2,1) = 0,01786442$

$P(t \leq 0,9) = \Phi(0,9)$

$P(t \leq 0,9) = 0,81593987$

$P(0.9 \leq t \leq 2.1) = P(t \leq 2.1) - P(t \leq 0.9)$

$P(0.9 \leq t \leq 2.1) = \Phi(2.1) - \Phi(0.9)$

$P(0.9 \leq t \leq 2.1) = 0,98213558 - 0,81593987$

$P(0.9 \leq t \leq 2.1) = 0,16619571$

$P(-2,58 \leq t \leq +2,58) = P(t \leq +2,58) - P(t \leq -2,58)$

$P(-2,58 \leq t \leq +2,58) = \Phi(2,58) - \Phi(-2,58)$

$P(-2,58 \leq t \leq +2,58) = \Phi(2,58) - [1 - \Phi(2,58)]$

$P(-2,58 \leq t \leq +2,58) = 2\Phi(2,58) - 1$

$P(-2,58 \leq t \leq +2,58) = 2 \times (0,99505998) - 1$

$P(-2,58 \leq t \leq +2,58) = 0,99011996$

$P(-1,96 \leq t \leq +1,96) = P(t \leq +1,96) - P(t \leq -1,96)$

$P(-1,96 \leq t \leq +1,96) = \Phi(1,96) - \Phi(-1,96)$

$P(-1,96 \leq t \leq +1,96) = \Phi(1,96) - [1 - \Phi(1,96)]$

$P(-1,96 \leq t \leq +1,96) = 2\Phi(1,96) - 1$

$P(-1,96 \leq t \leq +1,96) = 2 \times (0,97500210) - 1$

$P(-1,96 \leq t \leq +1,96) = 0,9500042$

Exercise 04:

The average usage of a certain type of rubber wheel is **38,000** km with a standard deviation of **3,000**. What is the probability that a randomly selected wheel will have a lifespan of at least **35,000** km? What is the probability that the period of use will exceed **45,000** km, given that the age of the wheels is normally distributed?

Solution:

The variable **X** representing the lifespan of rubber wheels follows a normal distribution with mathematical expectation **μ = 38000** and standard deviation **δ = 3000**, and we write:

$$X \sim \mathcal{N}(38000, (3000)^2)$$

To calculate the probabilities, we move from the normal distribution towards the standard normal variable by subtracting the mathematical expectation value and dividing by the standard deviation value, i.e.:

$$P(X \geq 35000) = P\left(Z \geq \frac{35000 - 38000}{3000}\right)$$

Whereas the variable Z follows the standard normal distribution, i.e.:

$$Z \sim \mathcal{N}(0, 1)$$

$$P(X \geq 35000) = P(Z \geq -1)$$

$$P(X \geq 35000) = P(Z \leq 1)$$

$$P(X \geq 35000) = \Phi(1)$$

$$P(X \geq 35000) = 0,84134475$$

$$P(X \geq 45000) = P\left(Z \geq \frac{45000 - 38000}{3000}\right)$$

$$P(X \geq 45000) = P(Z \geq 2.33)$$

$$P(X \geq 45000) = 1 - P(Z \leq 2.33)$$

$$P(X \geq 45000) = 1 - \Phi(2.33)$$

$$P(X \geq 45000) = 1 - 0,99009692$$

$$P(X \geq 45000) = 0,00990308$$

Exercise 05:

To reach his workplace, a worker can take two different paths. It takes an average of **27** minutes to reach via the first path with a standard deviation of **2.5** minutes, and it takes an average of **29** minutes to reach via the second path with a standard deviation of **1** minute. In both cases, the time to reach the workplace follows normal distribution, and this worker follows the path that allows him to reach his workplace at the right time and with the greatest probability, so what is the best path:

- He has 32 minutes to reach his workplace.
- He has 28 minutes to reach his workplace.

Solution:

Let us call X_1 the variables the time it takes for this worker to reach his workplace via the first route, which follows a general normal distribution with a mathematical expectation **μ = 27** and a standard deviation **δ = 2.5**, and we write:

$$X_1 \sim \mathcal{N}(27, (2,5)^2)$$

Let us call **X_2** the variable the time it takes for this worker to reach his workplace via the second route, which follows a general normal distribution with a mathematical expectation **μ = 29** and a standard deviation **δ = 1**, and we write:

$$X_2 \sim \mathcal{N}(29, (1)^2)$$

Here as well, to calculate the probabilities, we move from the general normal distribution towards the standard normal variable by subtracting the value of the mathematical expectation and dividing by the value of the standard deviation. In the case of a worker who has **32** minutes, the probability is that he will not arrive late via the first route, that is, that the arrival time will not exceed the available time, and we write:

$$P(X \leq 32) = P\left(Z \geq \frac{32-27}{2,5}\right)$$

$$P(X \leq 32) = P(Z \leq 2)$$

$$P(X \leq 32) = \Phi(2)$$

$$P(X \leq 32) = 0,97724987$$

When using the second route, the probability of not arriving late is as follows:

$$P(X \leq 32) = P\left(Z \geq \frac{32-29}{1}\right)$$

$$P(X \leq 32) = P(Z \leq 3)$$

$$P(X \leq 32) = \Phi(3)$$

$$P(X \leq 32) = 0,9986501$$

We note that the probability of using the second route is greater than the probability of using the first route, and therefore, when **32** minutes are available, it is advisable to use the second route.

In the case that the worker has **28** minutes, the probability is that he will not arrive late via the first route, meaning that the arrival time will not exceed the available time, and we write:

$$P(X \leq 28) = P\left(Z \geq \frac{28-27}{2.5}\right)$$

$$P(X \leq 28) = P(Z \leq 0,4)$$

$$P(X \leq 28) = \Phi(0,4)$$

$$P(X \leq 28) = 0,65542174$$

When using the second route, the probability of not arriving late is as follows:

$$P(X \leq 28) = P\left(Z \geq \frac{28-29}{1}\right)$$

$$P(X \leq 28) = P(Z \leq -1)$$

$$P(X \leq 28) = \Phi(-1)$$

$$P(X \leq 28) = 1 - \Phi(1)$$

$$P(X \leq 28) = 1 - 0,84134475$$

$$P(X \leq 28) = 0,15865525$$

We note that the probability of using the first route is greater than the probability of using the second route, and therefore, when **28** minutes are available, it is advisable to use the first route.

Exercise 06:

A medical study has proven that the percentage of CHOLESTEROL in the blood of a group of randomly selected people was distributed normally according to the following table:

Cholesterol levels in the blood	Percentage of people
CHOLESTEROL ratio less than **165 cg**	**%58**
The percentage of CHOLESTEROL is limited to **165cg - 180cg**	**38 %**
CHOLESTEROL ratio more than **180cg**	**4%**

1 - Calculate the mean value μ and the standard deviation δ.

2 - If we assume that people who have a cholesterol level in the blood of more than **183cg** must be subjected to medical treatment, what is the number of people who will be treated in a normally distributed population containing **10,000** people?

Solution:

3. To find the value of the mathematical expectation μ and the value of the standard deviation **δ**, we will search for the values of **Z** corresponding to the different probabilities given in the table above, as:

$$P(X \leq 165) = P\left(Z \leq \frac{165 - \mu}{\delta}\right) = 0,58$$

When searching in the probability table for the standard normal distribution, we will find the probability value closest to **0.58**, corresponding to **Z = 0.20**, meaning that:

$$\frac{165 - \mu}{\delta} = 0,20 \Longrightarrow 0,2\delta + \mu = 165$$

On the other hand, we have:

$$P(X \geq 180) = P\left(Z \geq \frac{180 - \mu}{\delta}\right) = 0,04 \Longrightarrow$$

$$P(X \leq 180) = P\left(Z \leq \frac{180 - \mu}{\delta}\right) = 0,96$$

Likewise, when searching the probability table for the standard normal distribution, we will find the probability value closest to **0.96**, corresponding to **Z = 1.75**, meaning that:

$$\frac{180-\mu}{\delta} = 1,75 \Rightarrow 1,75\delta + \mu = 180$$

Then we solve the following set of equations:

$$\begin{cases} 1,75\delta + \mu = 180 \\ 0,2\delta + \mu = 165 \end{cases} \Rightarrow \begin{cases} \mu = 163.0 \\ \delta = 9,67 \end{cases}$$

1. To calculate the percentage of people whose blood cholesterol exceeds **183 cg**, we calculate the following probability:

$$P(X \geq 183) = P\left(Z \geq \frac{183-163}{9,67}\right)$$

$$P(X \geq 183) = P(Z \geq 2,06)$$

$$P(X \geq 183) = 1 - P(Z \leq 2,06)$$

$$P(X \geq 183) = 1 - \Phi(2,06)$$

$$P(X \geq 183) = 1 - 0,98030073$$

$$P(X \geq 183) = 0,01969927 = 1,969\%$$

Thus, in a population of **10,000** people, we will find **197** people who will undergo medical treatment.

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