République Algérienne Démocratique et Populaire

Ministère de l'enseignement Supérieur et de la Recherche Scientifique

Université de Mohamed El Bachir El Ibrahimi–Bordj Bou Arreridj

Faculté des Mathématiques et de l'informatique

Département de Mathématiques

THÈSE

Présentée pour l'obtention du diplôme

de Doctorat troisième cycle

Spécialité:

Mathématiques

Option:

Mathématiques Appliquées

Par:

Ahmed Guechi

Thème

Analysis and Approximate Solution of a Class of Singular Integral Equations

Soutenue publiquement le: 28 / 06 /2018, devant le jury:

H. BENSERIDI	Prof.,	Université de Sétif 1	Président
M. NADIR	Prof.,	Université de M'sila	Rapporteur
A. RAHMOUNE	M.C.A,	Université de B.B.A	Co-rapporteur
A. MEROUANI	Prof.,	Université de B.B.A	Examinateur
L. ZEDAM	Prof.,	Université de M'sila	Examinateur

Acknowledgments

A special thanks to my co-supervisor Dr. Rahmoune Azedine, who worked hard with me from the beginning till the completion of the present research, and has been always generous during all phases of the research. I would like to thank my supervisor, Prof. Nadir Mostefa for the interesting subject that he proposed to me.

I also express my deep gratitude to, Prof. Benseridi Hamid to chair of the jury of the thesis. I warmly thank Professors; Merouani Abdelbaki and Zedam Lamnouar who have accepted to be the examiners of the thesis.

I do not forget my thanks to my family who helped me and supported me along ma work. I also thank all members of our faculty and the Mathematics Department.

Contents

List of Symbols and Abbreviations

The table below is a short list of symbols and notation used in this thesis

$C[a,b]$	:	Set of continuously functions on $[a,b]$.
$BC(\mathbb{R})$	:	Space of bounded continuous function on $\mathbb{R}$.
P_N	:	Set of all real polynomials of degree $\leq N$.
$\lVert . \rVert_\infty$	:	Norm defined by $\lVert \varphi(x) \rVert_\infty = \max_{x\in G} \lvert \varphi(x) \rvert$.
X,Y	:	Banach or Hilbert spaces.
δ_{ij}	:	Symbol of Kronecker define by $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
BVP	:	Boundary value problem.
IVP	:	Initial value problem.
GLPs	:	Generalized Laguerre polynomials.
GLFs	:	Generalized Laguerre functions.
$L^2_\omega(\Omega)$	:	Space of all measurable functions, with $\lVert u \rVert_\omega = \left(\int_\Omega \lvert u(x) \rvert^2 \omega dx\right)^{1/2}$.
$B^m_\alpha(\Lambda)$	:	Sobolev spaces defined by $B^m_\alpha(\Lambda) := \{u : \partial^k_x u \in L^2_{\omega_{\alpha+k}}(\Lambda), \ \ 0 \leq k \leq m\}$, equipped with the norm $\lVert u \rVert_{B^m_\alpha} = \left(\sum_{k=0}^m \lVert \partial^k_x u \rVert^2_{\omega_{\alpha+k}}\right)^{1/2}$.
Γ	:	Gamma function.
JG, JGR	:	Jacobi-Gauss and Jacobi-Gauss-Radau, respectively.
CG	:	Chebyshev-Gauss.
CGR	:	Chebyshev-Gauss-Radau.
CGL	:	Chebyshev-Gauss–Lobatto.
Log.map	:	Logarithmic mapping.
Alg.map	:	Algebraic mapping.
Card.func.	:	Cardinal function.

List of Tables

List of Figures

Introduction

Integral equation is encountered in a variety of applications in many fields of science and engineering, such as the radiative transfer [10, 28], acoustic resonance scattering [7, 8], population dynamics [6, 23], electromagnetic and elastic waves [12], fluid dynamics [33] and on many other problems in science and engineering which are set in unbounded domains.

There are several methods to solve integral equations on a bounded domain, for example (successive approximations method, degenerate kernel method, projection method, Nyström method, modified Simpson method,...), however what is not addressed is the solution of these equations on an unbounded domain i.e. on the real line, such as equation of the form:

$$\varphi(x) - \int_{-\infty}^{\infty} K(x,t)\varphi(t)dt = f(x) \tag{1}$$

where the function $K : \mathbb{R}^2 \to \mathbb{C}$ and $f \in Y := BC(\mathbb{R})$, the space of bounded continuous function on $\mathbb{R}$, are assumed known, and $\varphi \in Y$ is the solution to be determined (see,e.g., [7, 8, 9]).

In this work, we will search for solutions to this type of equation, here are two methods to solve the integral equation. These methods will be explained later and have been used in differential equations (see, e.g., [2, 22, 27, 36, 35, 34]).

So it is necessary to solve them over bounded or unbounded domains. In recent years spectral methods for bounded domains have been found to be a powerful tool for the solution of differential and integral equations due to their bigger accuracy when compared to standard methods. The rate of convergence of spectral approximations depends only on the smoothness of the solution, yielding the ability to achieve high precision with a small number of data set (see, e.g., [2, 4, 5, 11, 36, 39]).

Spectral methods in unbounded domains have also been used and discussed by many authors with different approaches, (see, e.g., [3, 4, 27, 31, 34, 36]). In general, they are essentially classified into three procedures. First, impose artificial boundaries at a large but finite distance (a procedure that will henceforth be called *Domain truncation* and solve the problem on the interval $[-L, L]$ instead of $(-\infty, \infty)$). Second on approximation by classical orthogonal systems on unbounded domains (e.g., Laguerre or Hermite polynomials and functions,...), or approximation by non-classical orthogonal systems. Third, one can employ a mapping for transform $(-\infty, \infty)$ into the finite interval $[-1, 1]$. We should note here that, the domain truncation approach is only a viable option for problems with rapidly (exponentially) decaying solutions. However, with proper choices of mapping or scaling parameters, the other three approaches can all be effectively applied to a variety of problems with rapid or slow decaying solutions (see e.g., [2, 4, 17, 27, 36, 38]). The thesis is organized as follows:

In chapter 1, we present integral equations and illustrate different criteria of classification of these equations. We also we mention some theorems of existence and uniqueness of the solution of equation (1). We describe conversion of initial value problems to Volterra integral equations, and finally the conversion of boundary value problem to Fredholm integral equations.

In chapter 2, we present technique of applications of the numerical methods for the integral equations on the unbounded intervals where we starts with the method the Laguerre polynomials and its generalization with the introduction of quadrature formulas and interpolation of the Laguerre-Gauss. The same study was made for the Hermite and Chebyshev polynomials and their generalization. Finally, we study by so-called mapping for the unbounded intervals.

The last chapter presents the aim of this thesis, fredholm integral equation of the second kind on the real-line is solved by the mapped Chebyshev spectral methods. Convergence of the presented methods is analyzed and tested. Also we give some examples utilizing these methods and compare them to classical results.

CHAPTER 1

Introductory Concepts

This chapter is dedicated to recalling some preliminary concepts that we use during the completion of this work to exploit and use them as a means of clarifying the new notions.

1.1 Definitions

An integral equation is an equation in which the unknown to be determined function $u(x)$ appears under the integral sign. A typical form of an integral equation is of the form

$$u(x) = f(x) + \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt, \tag{1.1}$$

where $K(x,t)$ is called the kernel of the integral equation, and $\alpha(x)$ and $\beta(x)$ are the limits of integration. In (1.1), it is easily observed that the unknown function $u(x)$ appears under the integral sign as stated above, and out of the integral sign in most other cases as will be discussed later. It is important to point out that the kernel $K(x,t)$ and the function $f(x)$ in (1.1) are given in advance. If we set $f(x) = 0$ in (1.1), the resulting equation is called a *homogeneous* integral equation, otherwise it is called *nonhomogeneous* or *inhomogeneous* integral equation. Our goal is to determine $u(x)$ that will satisfy (1.1), and this may be achieved by using different techniques that will be discussed in the forthcoming chapters. The primary concern of this text will be focused on introducing these methods and techniques supported by illustrative and practical examples. Integral equations arise naturally in physics, chemistry, biology and engineering applications modelled by initial value problems on a finite interval $[a, b]$. More details about the history origins of integral equations can be found in [14] and [20]. In the following example we will discuss how an initial value problem will be converted to the form of an integral equation.

We will further discuss the algorithms of converting initial value problems and boundary value problems in detail to equivalent integral equations in the forthcoming sections. As stated above, our task is to determine the unknown function $u(x)$ that appears under the integral sign as in (1.1) and that will satisfy the given integral equation.
We further point out that integral equations as (1.1) are called linear integral equations. This classification is used if the unknown function $u(x)$ under the integral sign occurs linearly i.e. to the first power. However, if $u(x)$ under the integral sign is replaced by a nonlinear function in $u(x)$, such as $u^2(x)$, $\cos u(x)$, $\cosh u(x)$ and $e^{u(x)}$, etc., the integral equation is called in this case a nonlinear integral equation.

1.2 Classification of Integral Equations

The most frequently used integral equations fall under two main classes namely Fredholm and Volterra integral equations. However, we distinguish four more related types of integral equations to the two main classes. In what follows, we give a list of the Fredholm and Volterra integral equations, and the four related types:

1. Fredholm integral equations.
2. Volterra integral equations.
3. Integro-differential equations.
4. Singular integral equations.

In the following we outline the basic definitions and properties of each type.

1.2.1 Fredholm Integral Equations

Linear Fredholm Integral Equations

The standard form of linear Fredholm integral equations, where the limits of integration a and b are constants, are given by the form

$$\phi(x)u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \tag{1.2}$$

where the kernel of the integral equation $K(x,t)$ and the function $f(x)$ are given in advance, and λ is a parameter. The equation (1.2) is called linear because the unknown function $u(x)$ under the integral sign occurs linearly, i.e. the power of $u(x)$ is one. The value of $\phi(x)$ will give the following kinds of Fredholm linear integral equations:

- When $\phi(x) = 0$, equation (1.2) becomes

$$f(x) + \lambda \int_a^b K(x,t)u(t)dt = 0, \tag{1.3}$$

  is known as Fredholm linear integral equation of first kind.

- When $\phi(x) = 1$, equation (1.2) becomes

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \tag{1.4}$$

  and the integral equation is called Fredholm linear integral equation of the second kind. In fact, the equation (1.4) can be obtained from (1.2) by dividing both sides of (1.2) by $\phi(x)$ provided that $\phi(x) \neq 0$.

Nonlinear Fredholm Integral Equations

- Of the first kind

$$f(x) = \lambda \int_a^b K(x,t,u(t))dt. \tag{1.5}$$

- Of the second kind

$$u(x) = f(x) + \lambda \int_a^b K(x,t,u(x),u(t))dt. \tag{1.6}$$

In summary, the Fredholm integral equation is of the first kind if the unknown function $u(x)$ appears only under the integral sign. However, the Fredholm integral equation is of the second kind if the unknown function $u(x)$ appears inside and outside the integral sign.

1.2.2 Volterra Integral Equations

Linear Volterra integral equations

The standard form of Volterra linear integral equations, where the limits of integration are functions of x rather than constants, are of the form

$$\phi(x)u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt, \tag{1.7}$$

where the unknown function $u(x)$ under the integral sign occurs linearly as stated before. It is worth noting that (1.7) can be viewed as a special case of Fredholm integral equation when the kernel $K(x,t)$ vanishes for $t > x$, x is in the range of integration $[a, b]$.

As in Fredholm equations, Volterra integral equations fall under two kinds, depending on the value of $\phi(x)$, namely:

- When $\phi(x) = 0$, equation (1.7) becomes

$$f(x) + \lambda \int_0^x K(x,t)u(t)dt = 0, \tag{1.8}$$

  and in this case the integral equation is called Volterra integral equation of the first kind.

- When $\phi(x) = 1$, equation (1.7) becomes

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{1.9}$$

  and in this case the integral equation is called Volterra integral equation of the second kind.

Nonlinear Volterra Integral Equations

- Of the first kind

$$f(x) = \lambda \int_a^x K(x,t,u(t))dt. \tag{1.10}$$

- Of the first kind

$$u(x) = f(x) + \lambda \int_a^x K(x, t, u(x), u(t))dt. \tag{1.11}$$

For example, consider the first order equation

$$\frac{du(t)}{dt} = f(x, u), \qquad u(0) = u_0.$$

Under suitable continuity on $f(x, u)$ we can 'solve' this initial value problem to yield

$$u(x) = u_0 + \int_0^x f(t, u(t))dt.$$

Examining the equations (1.2)–(1.9) carefully, the following remarks can be concluded: In summary, the Volterra integral equation is of the first kind if the unknown function $u(x)$ appears only under the integral sign. However, the Volterra integral equation is of the second kind if the unknown function $u(x)$ appears inside and outside the integral sign.

1.2.3 Integro-Differential Equations

Volterra, in the early 1900, studied the population growth, where new type of equations have been developed and was termed as integro-differential equations. In this type of equations, the unknown function $u(x)$ occurs in one side as an ordinary derivative, and appears on the other side under the integral sign. Several phenomena in physics and biology [20], [22] and [30] give rise to this type of integro-differential equations.

Fredholm Integro-Differential Equations

Fredholm integro-differential equations appear when we convert differential equations to integral equations. The Fredholm integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x)$, $n \geq 1$ inside and outside the integral sign respectively. The limits of integration in this case are fixed as in the Fredholm integral equations. The equation is labeled as integro-differential because it contains differential and integral operators in the same equation. It is important to note that initial conditions should be given for Fredholm integro-differential equations to obtain the particular solutions. The Fredholm integro-differential equation appears in the form:

$$u^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt,$$

where $u^{(n)}$ indicates the n^{th} derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ at the left side.

Volterra Integro-Differential Equations

Volterra integro-differential equations appear when we convert initial value problems to integral equations. The Volterra integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x)$, $n \geq 1$ inside and outside the integral sign. At least one of the limits of integration in this case is a variable as in the Volterra integral equations. The equation is called integro-differential because differential and integral operators are involved in the same equation. It is important to note that initial conditions should be given for Volterra integro-differential equations to determine the particular solutions. The Volterra integro-differential equation appears in the form:

$$u^{(n)}(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt,$$

where $u^{(n)}$ indicates the n^{th} derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ at the left side.

1.2.4 Singular Integral Equations

An integral equation is called singular if the integration is improper. This usually occurs if the interval of integration is *infinite*, or if the kernel becomes *unbounded* at one or more points of the interval of consideration $a \leq t \leq b$. Examples of singular integral equations are given by the following examples:

$$f(x) = \int_0^\infty K(x,t)u(t)dt,$$

$$u(x) = f(x) + \int_{-\infty}^0 K(x,t)u(t)dt,$$

$$u(x) = f(x) + \int_{-\infty}^\infty K(x,t)u(t)dt,$$

where the singular behavior in these examples has resulted from the range of integration becoming *infinite*. Based on the nature of unboundedness of the kernel, one can have *weakly singular* integral equation and *strongly singular* integral equation.

Weakly singular integral equation

If $K(x,t)$ is of the form

$$K(x,t) = \frac{L(x,t)}{|x-t|^{\alpha}},$$

where $L(x,t)$ is bounded in $[a,b] \times [a,b]$ with $L(x,x) \neq 0$, and α is a constant such that $0 < \alpha < 1$, then the integral $\int_a^b K(x,t)dt \quad (a < x < b)$ exists in the sense of Riemann, and the kernel is weakly singular, and the corresponding integral equation (first or second kind) is called a *weakly singular integral equation.* Also the *logarithmically singular* kernel

$$K(x,t) = L(x,t)\ln|x-t|,$$

where $L(x,t)$ is bounded with $L(x,x) \neq 0$, is also regarded as a weakly singular kernel.

If the kernel $K(x,t)$ is of the form

$$K(x,t) = \frac{L(x,t)}{\sqrt{x-t}}, \quad a < x < b,$$

the corresponding integral equation (first or second kind) is called a *Abel's integral equation.* The most general form Abel's integral equation is given by

$$f(x) = \int_0^x \frac{1}{(x-t)^{\alpha}} u(t)dt, \qquad 0 < \alpha < 1.$$

Strong singular integral equation

If the kernel $K(x,t)$ is of the form

$$K(x,t) = \frac{L(x,t)}{x-t}, \quad a < x < b,$$

where $L(x,t)$ is a differentiable function with $L(x,x) \neq 0$ (the function L can still be weaker!), then the kernel $K(x,t)$ has a *strong singularity* at $t = x$, or rather it has a *singularity of Cauchy* type at $t = x$.

If the kernel $K(x,t)$ is of the form

$$K(x,t) = \frac{L(x,t)}{(x-t)^2}, \quad a < x < b,$$

where $L(x,t)$ is continuous and $L(x,x) \neq 0$, then $K(x,t)$ has a *very strong singularity* at $t = x$. It is clear that the kernel in each equation becomes infinite at the upper limit $t = x$.

1.3 Existence Solutions of Integral Equations

It is also known to be an important step before solving each equation. in this section to prove the existence of solutions of linear integral equations (1), we will apply Arzelà-Ascoli, Riesz theory and Fredholm alternative,

Definition 1.1. *A set G in $\mathbb{R}^m$ is called Jordan measurable if the characteristic function χ_G, given by $\chi_G(x) = 1$ for $x \in G$ and $\chi_G(x) = 0$ for $x \notin G$, is Riemann integrable.*

Theorem 1.2 (Arzelà-Ascoli)**.** *Each sequence from a subset $U \subset C[a,b]$ contains a uniformly convergent subsequence; i.e., U is relatively sequentially compact, if and only if it is bounded and equicontinuous, i.e., if there exists a constant C such that*

$$|\varphi(x)| \leq C$$

for all $x \in [a,b]$ and all $\varphi \in U$. and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(y)| < \varepsilon,$$

for all $x, y \in [a,b]$ with $|x - y| < \delta$ and all $\varphi \in U$.

Theorem 1.3 (Riesz)**.** *Given a compact linear operator $A : X \to X$ on a normed space X, we define $L := I - A$, where I denotes the identity operator*

1. *The nullspace of the operator L, i.e.,*

$$N(L) := \{\varphi \in X : L\varphi = 0\}$$

is a finite-dimensional subspace.

2. *The range of the operator L, i.e.,*

$$L(X) := \{L\varphi : \varphi \in X\}$$

is a closed linear subspace.

3. *There exists a uniquely determined nonnegative integer r, called the Riesz number of the operator A, such that*

$$0 = N(L^0) \subsetneqq N(L^1) \subsetneqq \cdots \subsetneqq N(L^r) = N(L^{r+1}) = \ldots \quad , \tag{1.12}$$

and

$$X = L^0(X) \supsetneqq L^1(X) \supsetneqq \cdots \supsetneqq L^r(X) = L^{r+1}(X) = \ldots \quad . \tag{1.13}$$

Furthermore, we have the direct sum

$$X = N(L^r) \oplus L^r(X),$$

i.e., for each $\varphi \in X$ there exist uniquely determined elements $\psi \in N(L^r)$ and $\chi \in L^r(X)$ such that $\varphi = \psi + \chi$.

We now consider the integral equations of the form

$$\varphi(x) - \int_{-\infty}^{\infty} K(x,t)\varphi(t)dt = f(x), \tag{1.14}$$

where the functions $K : \mathbb{R}^2 \to \mathbb{C}$ and $f \in Y := BC(\mathbb{R})$, the space of bounded continuous on $\mathbb{R}$, and are assumed known, and $\varphi \in Y$ is the solution to be determined. Define the integral operator $\mathcal{A}$ by

$$\mathcal{A}\varphi(x) = \int_{\mathbb{R}} K(s,t)\varphi(t)dt, \quad x \in \mathbb{R}. \tag{1.15}$$

Let $X := L_\infty(\mathbb{R})$. Then $\mathcal{A} : X \to X$ and is bounded if and only if K is measurable, $K(x,.) \in L_1(\mathbb{R})$ for almost all $x \in \mathbb{R}$, and

$$|||K||| := ess.\sup_{x\in\mathbb{R}} \|K(x,.)\|_1 < \infty. \tag{1.16}$$

Clearly, $\mathcal{A} : X \to Y$ and is bounded if and only if, in addition, $K(x,.) \in L_1(\mathbb{R})$ for all $x \in \mathbb{R}$, and $\mathcal{A}\varphi \in C(\mathbb{R})$ for every $\varphi \in X$, in which case it holds also that

$$|||K||| = \sup_{x\in\mathbb{R}} \|K(x,.)\|_1.$$

If $\mathcal{A} : X \to Y$ and is bounded, then $\|\mathcal{A}\| = |||K|||$,(see [9]).
In the case that K is measurable and (1.16). holds it is convenient to identify $K : \mathbb{R}^2 \to \mathbb{C}$ with the mapping $x \to K(x, .)$ in $Z := L_\infty(\mathbb{R}, L_1(\mathbb{R}))$, which is measurable and essentially bounded with norm $|||K|||$. Let $\mathbb{K}$ denote the set of those functions $K \in Z$ having the property that $\mathcal{A}\varphi \in C(\mathbb{R})$ for every $\varphi \in X$, where $\mathcal{A}$ is the integral operator (1.15). It is easy to see that Z is a Banach space with the norm $|||.|||$ and that $\mathbb{K}$ is a closed subspace of Z. Further, $\mathcal{A} : X \to Y$ and is bounded if and only if $K \in \mathbb{K}$.

Then the integral equation (1.14) can be written in terms of the operator $\mathcal{A}$ and I, the identity operator, as

$$(I - \mathcal{A})\varphi = f. \tag{1.17}$$

Definition 1.4. *Let E and F be Banach spaces and let $T : E \to F$ be a bounded linear operator. T is said to be Fredholm if the following hold.*

1. *$ker(T)$ is finite dimensional.*
2. *$Ran(T)$ is closed.*
3. *$Coker(T)$ is finite dimensional.*

If T is Fredholm define the index of T denoted $Ind(T)$ by

$$Ind(T) = \dim(ker(T)) - \dim(Coker(T)).$$

Theorem 1.5 (Fredholm alternative)**.** *Let $W \subset BC(\mathbb{R}, L_1(\mathbb{R}))$. If W satisfies these conditions and $I - \mathcal{A}$ is injective for all $K \in W$ then $I - \mathcal{A}$ is also surjective for all $K \in W$ and, moreover, the inverse operators $(I - \mathcal{A})^{-1}$ on Y are uniformly bounded for $K \in W$.*

Remark 1.6 ([9])**.** *We note certainly $BC(\mathbb{R}, L_1(\mathbb{R})) \subset \mathbb{K}$; i.e., $K \in \mathbb{K}$ if $K \in Z$ and if, additionally, for all $x \in \mathbb{R}$,*

$$\|K(x, .) - K(x', .)\|_1 \to 0 \quad as \quad x' \to x. \tag{1.18}$$

Not much is known about the solvability in Y (or in other function spaces) of general integral equations of the form (1.14), even when the kernel $K \in Z$ satisfies (1.18).

Corollary 1.7. *Let $K \in Z$. If K satisfies (1.18) sufficient additional conditions so that $\mathcal{A}$ is compact, for*

$$\int_{\mathbb{R}} |K(x,t)|dt \to 0 \quad as \quad x \to \infty, \tag{1.19}$$

Theorem 1.8. *Let $K \in Z$. If K satisfies (1.18)–(1.19) sufficient additional conditions and $x \in \mathbb{R}$ then $(I - \mathcal{A})$ is Fredholm of index zero on Y so that*

$$(I - \mathcal{A}) \ \ injective \Rightarrow (I - \mathcal{A}) \ \ surjective, \quad (I - \mathcal{A})^{-1} \in B(Y),$$

where $B(Y)$ denotes the Banach space of bounded linear operators on Y.

Proof. (see, [8]) □

For more detail see [7, 8, 9].

1.4 Converting IVP to Volterra Integral Equation

In this section, we will study the method that converts an initial value problem (IVP) to an equivalent Volterra integral equation. Before outlining the method needed, we wish to recall the useful transformation formula (1.20). This is an essential and useful formula that will be employed in the method that will be used in the conversion technique. We wish to recall the useful transformation formula:

$$\int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_n)dx_n \cdots dx_1 = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt, \tag{1.20}$$

we apply this technique to a second order initial value problem given by:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x), \tag{1.21}$$

with the initial conditions

$$y(a) = \alpha, \quad y'(a) = \beta,$$

we first set

$$y''(x) = u(x), \tag{1.22}$$

where $u(x)$ is a continuous function on the interval of discussion. This can be simply performed by integrating both sides of (1.22) from a to x where we find:

$$y'(x) = \beta + \int_a^x u(t)dt. \tag{1.23}$$

Integrating both sides of (1.23) from a to x gives

$$y(x) = \alpha + \beta(x-a) + \int_a^x \int_a^x u(t)dtdt. \tag{1.24}$$

Using the conversion formulas (1.20),we obtain

$$y(x) = \alpha + \beta(x-a) + \int_a^x (x-t)u(t)dt. \tag{1.25}$$

Substituting (1.22), (1.23) and (1.25) into (1.21) leads to the following Volterra integral equation of the second kind

$$u(x) = F(x) + \int_a^x K(x,t)u(t)dt, \tag{1.26}$$

where

$$K(x,t) = -P(x) - Q(x)(x-t),$$

and

$$F(x) = f(x) - \beta P(x) - (\alpha + \beta(x-a))Q(x).$$

Example 1.9. *We want to derive an equivalent Volterra integral equation to the following initial value problem*

$$y'' + 5y' + 6y = 0, \quad \textit{with} \;\; y(0) = 1, \quad y'(0) = 1, \tag{1.27}$$

where

$$P(x) = 5, \quad Q(x) = 6, \quad f(x) = 0, \quad a = 0, \quad \alpha = \beta = 1.$$

Consequently, the desired Volterra integral equation of the second kind is given by

$$u(x) = -6x - 11 + \int_0^x (6t - 6x - 5)u(t)dt, \tag{1.28}$$

1.5 Converting BVP to Fredholm Integral Equations

We present in this section how a boundary value problem (BVP) can be converted to an equivalent Fredholm integral equation. The method is similar to that discussed in the previous section with some exceptions that are related to the boundary conditions. It is to be noted here that the method of reducing a BVP to a Fredholm integral equation is complicated and rarely used. Let us consider the following second-order ordinary differential with the given boundary conditions:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x), \quad a < x < b, \tag{1.29}$$

with the boundary conditions:

$$y(a) = \alpha, \quad y(b) = \beta. \tag{1.30}$$

We first set

$$y''(x) = u(x). \tag{1.31}$$

Integrating both sides of (1.31) from a to x gives

$$y'(x) = y'(a) + \int_a^x u(t)dt. \tag{1.32}$$

Note that $y'(a)$ is not prescribed yet. Integrating both sides of equation (1.32) with respect to x from a to x and applying the given boundary condition at $x = a$, we find

$$y(x) = y(a) + (x-a)y'(a) + \int_a^x \int_a^x u(t)dtdt$$

$$= \alpha + (x-a)y'(a) + \int_a^x (x-t)u(t)dt, \tag{1.33}$$

and using the boundary condition at $x = b$ yields

$$y(b) = \beta = \alpha + (b-a)y'(a) + \int_a^b (b-t)u(t)dt, \tag{1.34}$$

and the unknown constant $y'(a)$ is determined as

$$y'(a) = \frac{\beta-\alpha}{b-a} - \frac{1}{b-a}\int_a^x (b-t)u(t)dt - \frac{1}{b-a}\int_x^b (b-t)u(t)dt. \tag{1.35}$$

Hence the solution (1.33) can be rewritten as

$$y(x)=\alpha+(x-a)\left(\frac{\beta-\alpha}{b-a}-\frac{1}{b-a}\int_a^x(b-t)u(t)dt-\frac{1}{b-a}\int_x^b(b-t)u(t)dt\right)+\int_a^x(x-t)u(t)dt.$$

Therefore, equation (1.29) can be written in terms of $u(x)$ as

$$u(x)=F(x)+\int_a^b K(x,t)u(t)dt, \tag{1.36}$$

where

$$K(x,t)=\begin{cases}\frac{P(x)(a-t)-Q(x)(b-x)(a-t)}{b-a} & \text{if} \quad a\le t\le x\\ \frac{P(x)(b-t)-Q(x)(a-x)(b-t)}{b-a} & \text{if} \quad x\le t\le b\end{cases}$$

and

$$F(x)=f(x)-P(x)\frac{\beta-\alpha}{b-a}-Q(x)\left(\alpha+(x-a)\frac{\beta-\alpha}{b-a}\right).$$

This is a complicated procedure to determine the solution of a BVP by equivalent Fredholm integral equation.

Example 1.10. *We want to derive an equivalent Fredholm integral equation to the following boundary value problem*

$$y''+4y=\sin x, \quad 0<x<1 \quad \textit{with} \quad y(0)=y(1)=0, \tag{1.37}$$

where

$$P(x)=0, \quad Q(x)=4, \quad f(x)=\sin x, \quad a=0, \quad b=1, \quad \alpha=\beta=0.$$

Consequently, the desired Fredholm integral equation of the second kind is given by

$$u(x)=\sin x+\int_a^b K(x,t)u(t)dt, \tag{1.38}$$

where the kernel $K(x,t)$ is defined by

$$K(x,t)=\begin{cases}4t(1-x) & \textit{if} \quad a\le t\le x\\ 4x(1-t) & \textit{if} \quad x\le t\le b\end{cases}$$

which can be easily shown that the kernel is symmetric.

CHAPTER 2

Methods for Unbounded Intervals

Introduction

In this chapter, we discuss a spectral methods in unbounded domains have also been used and discussed by many authors with different approaches, (see, e.g., [3, 4, 27, 31, 36]). It's been more than forty years since Grosch and Orszag [17] showed that differential equations on a semi-infinite interval can be solved very effectively by mapping the interval into $[-1, 1]$ using an algebraic function for the map. Boyd [4] generalized their technique and provided in Boyd [2, Ch. 17, p. 338] an excellent extensive review on general properties and practical implementations for many of these approaches. The many options for unbounded domains fall into three broad categories:

1. Domain truncation: approximation of $y \in [-\infty, \infty]$ or $y \in [0, \infty]$ by $[-L, L]$ or $[0, L]$, respectively, with $L >> 1$ (see, e.g., [13, 15, 16, 37]).
2. Basis functions intrinsic to an infinite interval (*sinc*, Hermite) or semi-infinite (Laguerre) (see, e.g., [3, 5, 29]).
3. Change-of-coordinate (mapping) of the unbounded interval to a finite one, followed by application of Chebyshev polynomials or a Fourier series (see, e.g., [2, 27, 35, 38]).

2.1 Laguerre Polynomials and Functions

We present in this section some basic properties of Laguerre polynomials and functions, and introduce the Laguerre-Gauss-type quadrature formulas and the associated interpolation.

The Laguerre polynomials $\mathcal{L}_k = \mathcal{L}_k(x)$ satisfy the second-order linear ordinary differential equation

$$xy''(x) + (1-x)y'(x) + ky(x) = 0, \tag{2.1}$$

and are defined by

$$\begin{aligned} \mathcal{L}_k(x) &= \frac{1}{k!}e^x\frac{d^k}{dx^k}(x^k e^{-x}) \\ &= \frac{(-1)^k}{k!}\left[x^k - k^2x^{k-1} + \frac{k^2(k-1)^2}{2!}x^{k-2} + ...\right], \end{aligned}$$

gives the recursion relation

$$\mathcal{L}_{k+1}(x) = \frac{1}{k+1}\left[(2k+1-x)\mathcal{L}_k(x) - k\mathcal{L}_{k-1}(x)\right], \quad k = 1, 2, 3, ..., \tag{2.2}$$

with $\mathcal{L}_0(x) = 1$ and $\mathcal{L}_1(x) = -x + 1$. Write down few first Laguerre polynomials; we determine them directly from formula (2.2):

$$\begin{aligned} \mathcal{L}_0(x) &= 1, \\ \mathcal{L}_1(x) &= -x+1, \\ \mathcal{L}_2(x) &= \frac{1}{2}(x^2 - 4x + 2), \\ \mathcal{L}_3(x) &= \frac{1}{6}(-x^3 + 9x^2 - 18x + 6), \\ \mathcal{L}_4(x) &= \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24), \\ \mathcal{L}_5(x) &= \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120), \\ \mathcal{L}_6(x) &= \frac{1}{720}(x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720). \end{aligned}$$

They are orthogonal in $(0, +\infty)$ with respect to the weight $\omega(x) = e^{-x}$

$$\int_0^\infty \mathcal{L}_n(x)\mathcal{L}_m(x)e^{-x}dx = \delta_{nm}. \tag{2.3}$$

2.1.1 Generalized Laguerre Polynomials

The generalized Laguerre polynomials (GLPs) $\mathcal{L}_n^{(\alpha)} = \mathcal{L}_n^{(\alpha)}(x)$ $(\alpha > -1)$ satisfy the equation

$$xy''(x) + (\alpha + 1 - x)y'(x) + ky(x) = 0. \tag{2.4}$$

and are defined by the formulas

$$\begin{aligned} \mathcal{L}_k^{(\alpha)}(x) &= \frac{1}{k!} x^{-\alpha} e^x \frac{d^k}{dx^k}(x^{k+\alpha} e^{-x}) \\ &= \sum_{i=0}^{k} C_{k+\alpha}^{k-i} \frac{(-x)^i}{i!}, \end{aligned}$$

gives the recursion relation

$$\mathcal{L}_{k+1}^{(\alpha)}(x) = \frac{1}{k+1}\left[(2k+\alpha+1-x)\mathcal{L}_k^{(\alpha)}(x) - (k+\alpha)\mathcal{L}_{k-1}^{(\alpha)}(x)\right], \quad k \geq 1. \tag{2.5}$$

with $\mathcal{L}_0^{(\alpha)}(x) = 1$ and $\mathcal{L}_1^{(\alpha)}(x) = -x + 1 + \alpha$.

The first generalized polynomials of Laguerre are

$$\begin{aligned} \mathcal{L}_0^{(\alpha)}(x) &= 1, \\ \mathcal{L}_1^{(\alpha)}(x) &= -x + \alpha + 1, \\ \mathcal{L}_2^{(\alpha)}(x) &= \frac{x^2}{2} - (\alpha+2)x + \frac{(\alpha+2)(\alpha+1)}{2}, \\ \mathcal{L}_3^{(\alpha)}(x) &= \frac{-x^3}{6} + \frac{(\alpha+3)x^2}{2} - \frac{(\alpha+3)(\alpha+2)x}{2} + \frac{(\alpha+3)(\alpha+2)(\alpha+1)}{6}. \end{aligned}$$

They are orthogonal in $(0, +\infty)$ with respect to the weight $\omega_\alpha(x) = x^\alpha e^{-x}$:

$$\int_0^\infty \mathcal{L}_n^{(\alpha)}(x)\mathcal{L}_m^{(\alpha)}(x) x^\alpha e^{-x} dx = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{nm}. \tag{2.6}$$

2.1.2 Generalized Laguerre Functions

In many applications, the underlying solutions decay algebraically or exponentially at infinity, it is certainly not a good idea to approximate these functions by GLPs which grow rapidly at infinity. It is advisable to approximate them by spectral expansions of generalized Laguerre functions (GLFs).

The generalized Laguerre functions (GLFs) are defined by

$$\hat{\mathcal{L}}_k^{(\alpha)}(x) = e^{-x/2}\mathcal{L}_k^{(\alpha)}(x),$$

three-term recurrence relation

$$(k+1)\hat{\mathcal{L}}_{k+1}^{(\alpha)}(x) = (2k+\alpha+1-x)\hat{\mathcal{L}}_k^{(\alpha)}(x) - (k+\alpha)\hat{\mathcal{L}}_{k-1}^{(\alpha)}(x), \tag{2.7}$$

$$\hat{\mathcal{L}}_0^{(\alpha)}(x) = e^{-x/2}, \quad \hat{\mathcal{L}}_1^{(\alpha)}(x) = (\alpha+1-x)e^{-x/2}.$$

the GLFs are orthogonal with respect to the weight function $\tilde{\omega}_\alpha(x) = x^\alpha$

$$\int_0^\infty \hat{\mathcal{L}}_n^{(\alpha)}(x)\hat{\mathcal{L}}_m^{(\alpha)}(x)x^\alpha dx = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{nm}. \tag{2.8}$$

In order to avoid such a problem for approximating functions that vanish at $+\infty$, it may be more appropriate to expand in the Laguerre functions defined as $\hat{\mathcal{L}}_k(x) = e^{-x/2}\mathcal{L}_k(x)$. Thanks to (2.3), they satisfy

$$\int_0^\infty \hat{\mathcal{L}}_k(x)\hat{\mathcal{L}}_m(x)dx = \delta_{km}, \quad k,m \geq 0. \tag{2.9}$$

Computation of Nodes

It suffices to compute the Laguerre-Gauss quadrature nodes and weights. We find the zeros $\{x_j^{(\alpha)}\}_{j=0}^N$ of $\mathcal{L}_{N+1}^{(\alpha)}(x)$ are the eigenvalues of the symmetric tridiagonal matrix

$$A_{N+1} = \begin{bmatrix} a_0 & -\sqrt{b_1} & & & \\ -\sqrt{b_1} & a_1 & -\sqrt{b_2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\sqrt{b_{n-1}} & a_{n-1} & -\sqrt{b_{n-1}} \\ & & & -\sqrt{b_n} & a_n \end{bmatrix}$$

whose entries are derived from (2.5):

$$a_j = 2j+\alpha+1, \quad 0 \leq j \leq N; \quad b_j = j(j+\alpha), \quad 1 \leq j \leq N. \tag{2.10}$$

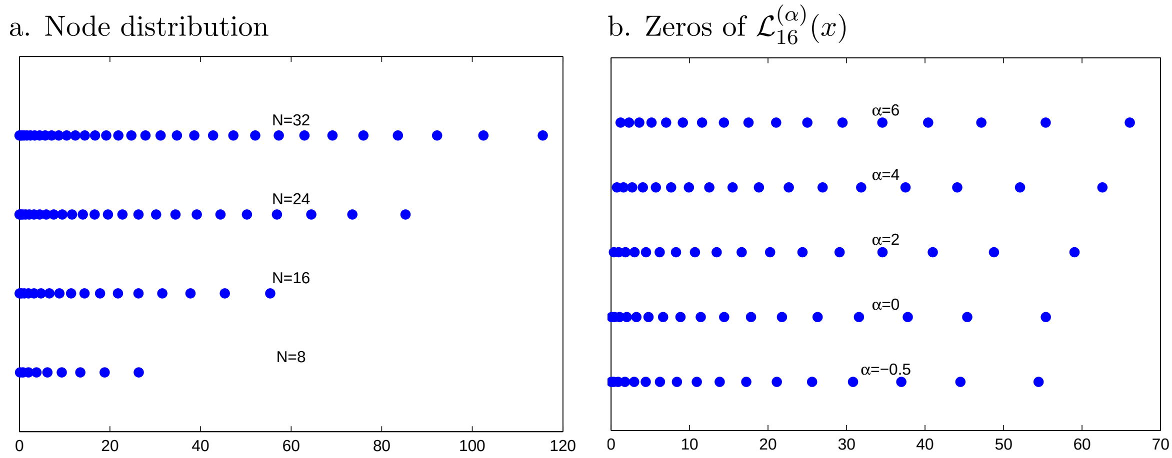

Figure 2.1: a. Distribution of Laguerre-Gauss nodes $\{x_j\}_{j=0}^N$ with N=8,16,24,32 with $\alpha = 0$; b. Distribution of $\mathcal{L}_{16}^{(\alpha)}(x)$ zeros with various α.

Definition 2.1. *Gauss-type quadrature is to seek the best numerical approximation of an integral by selecting optimal nodes at which the integrand is evaluated. It belongs to the family of the numerical quadratures:*

$$\int_a^b f(x)\omega(x)dx = \sum_{j=0}^{N} f(x_j)\omega_j + E_N[f], \tag{2.11}$$

where $\{x_j, \omega_j\}_{j=0}^N$ are the quadrature nodes and weights, and $E_N[f]$ is the quadrature error.

Theorem 2.2. *Let $\{x_j^\alpha, \omega_j^\alpha\}_{j=0}^N$ be a set of Laguerre-Gauss or Laguerre-Gauss-Radau quadrature nodes and weights.*

- *For Laguerre-Gauss quadrature,*

$$\{x_j^{(\alpha)}\}_{j=0}^N \text{ are the zeros of } \mathcal{L}_{N+1}^{(\alpha)}(x)$$

$$\begin{aligned} \omega_j^{(\alpha)} &= -\frac{\Gamma(N+\alpha+1)}{(N+1)!}\frac{1}{\mathcal{L}_N^{(\alpha)}(x_j^{(\alpha)})\partial_x \mathcal{L}_{N+1}^{(\alpha)}(x_j^{(\alpha)})} \\ &= -\frac{\Gamma(N+\alpha+1)}{(n+\alpha+1)(N+1)!}\frac{x_j^{(\alpha)}}{[\mathcal{L}_N^{(\alpha)}(x_j^{(\alpha)})]^2}, \end{aligned}$$

- *For Laguerre-Gauss-Radau quadrature,*

$$x_0^{(\alpha)} = 0 \text{ and } \{x_j^{(\alpha)}\}_{j=1}^N \text{ are the zeros of } \partial\mathcal{L}_{N+1}^{(\alpha)}(x),$$

$$\begin{aligned}
\omega_0^{(\alpha)} &= \frac{(\alpha+1)\Gamma^2(\alpha+1)N!}{\Gamma(N+\alpha+2)}, \\
\omega_j^{(\alpha)} &= \frac{\Gamma(N+\alpha+1)}{N!(N+\alpha+1)} \frac{1}{[\partial_x \mathcal{L}_N^{(\alpha)}(x_j^{(\alpha)})]^2} \\
&= \frac{\Gamma(N+\alpha+1)}{N!(N+\alpha+1)} \frac{1}{[\mathcal{L}_N^{(\alpha)}(x_j^{(\alpha)})]^2}, \quad 1 \le j \le N.
\end{aligned}$$

With the above nodes and weights, we have

$$\int_0^\infty p(x) x^\alpha e^{-x} dx = \sum_{j=0}^{N} p(x_j^{(\alpha)}) \omega_j^{(\alpha)}, \quad \forall p \in P_{2N+\delta}, \tag{2.12}$$

where $\delta = 1, 0$ for the Laguerre-Gauss quadrature and Laguerre-Gauss-Radau quadrature, respectively. And

$$\int_a^b f(x) e^{-x} dx = \sum_{j=0}^{N-1} f(x_j)\omega_j + E_N[f], \quad \textit{with} \quad E_N[f] = \frac{(N!)^2}{(2N)!} f^{(2N)}(\xi).$$

Proof. ([see [36],p ,244]) □

Remark 2.3. *If $\alpha = 0$, we simply denote the usual Laguerre-Gauss-type nodes and weights by $\{x_j, \omega_j\}_{j=0}^N$.*

- *In the Gauss case, $\{x_j\}_{j=0}^N$ are the zeros of $\mathscr{L}_{N+1}(x)$, and the weights have the representations:*

$$\omega_j = \frac{x_j}{(N+1)^2[\mathcal{L}_N(x_j)]^2}, \quad 0 \le j \le N. \tag{2.13}$$

- *In the Gauss-Radau case, $x_0 = 0$ and $\{x_j\}_{j=1}^N$ are the zeros of $\partial_x \mathcal{L}_{N+1}(x)$, and the weights are expressed by*

$$\omega_j = \frac{1}{(N+1)[\mathcal{L}_N(x_j)]^2}, \quad 0 \le j \le N. \tag{2.14}$$

With a slight modification of the quadrature weights in Theorem 2.2, we can derive the quadrature formulas associated with the generalized Laguerre functions.

Theorem 2.4. *Let $\{x_j^\alpha, \omega_j^\alpha\}_{j=0}^N$ be the set of Laguerre-Gauss or Laguerre-Gauss-Radau quadrature nodes and weights in Theorem 2.2. Define*

$$\tilde{\omega}_j^{(\alpha)} = e^{x_j^{(\alpha)}} \omega_j^{(\alpha)}, \quad 0 \leq j \leq N,$$

and

$$\hat{P}_N := \{\varphi : \varphi = e^{-x/2}\phi, \quad \forall \phi \in P_N\}.$$

Then we have the modified quadrature formula

$$\int_0^\infty p(x)q(x)x^\alpha dx = \sum_{j=0}^N p(x_j^{(\alpha)})\omega_j^{(\alpha)}, \quad \forall p \in P_{2N+\delta},$$

where $\delta = 1, 0$ for the Laguerre-Gauss quadrature and Laguerre-Gauss-Radau quadrature, respectively.

2.1.3 Interpolations and Approximations

Let $\{x_j, \omega_j\}_{j=0}^N$ be a set of Gauss, Gauss-Radau or Gauss-Lobatto quadrature nodes and weights. We define the corresponding discrete inner product and norm as

$$\langle u, v\rangle_{N,\omega} = \sum_{j=0}^N u(x_j)v(x_j)\omega, \quad \| u \|_{N,\omega} = \sqrt{\langle u, u\rangle_{N,\omega}}. \tag{2.15}$$

Note that $\langle ., .\rangle_{N,\omega}$ is an approximation to the continuous inner product $(u, v)_\omega$, and the exactness of Gauss-type quadrature formulas implies

$$\langle u, v\rangle_{N,\omega} = (u, v)_\omega, \quad \forall u, \quad v \in P_{2N+\delta}, \tag{2.16}$$

where $\delta = 1, 0 \quad and \quad -1$ for the Gauss, Gauss-Radau and Gauss-Lobatto quadrature, respectively.

Definition 2.5. *For any $u \in C(\Lambda)$, The definition of the interpolation operator $\mathcal{I}_N : C(\Lambda) \to P_N$ such that*

$$(\mathcal{I}_N u)(x_j) = u(x_j) = u(x_j), \quad 0 \leq j \leq N, \tag{2.17}$$

where $\Lambda = (a, b)$, $[a, b)$, $[a, b]$ for the Gauss, Gauss-Radau and Gauss-Lobatto quadrature, respectively.

Consider the interpolation using generalized Laguerre functions, let $\{x_j^{(\alpha)}, \tilde{\omega}_j^{(\alpha)}\}_{j=0}^N$ be a set of Laguerre-Gauss or Laguerre-Gauss-Radau quadrature nodes and weights. We define the corresponding discrete inner product and norm as

$$\langle u, v\rangle_{N,\tilde{\omega}_\alpha} = \sum_{j=0}^{N} u(x_j^{(\alpha)}) v(x_j^{(\alpha)}) \tilde{\omega}_j^{(\alpha)}, \quad \| u \|_{N,\tilde{\omega}_\alpha} = \sqrt{\langle u, u\rangle_{N,\tilde{\omega}_\alpha}}. \tag{2.18}$$

where $\tilde{\omega}_j^{(\alpha)} = e^{x_j^{(\alpha)}} \omega_j^{(\alpha)}$, $0 \le j \le N$.

Definition 2.6. *Let $\mathbb{P}_N$ be the finite dimensional space defined by*

$$\mathbb{P}_N := \{\phi : \quad \phi = e^{-x/2}\varphi, \quad \forall \varphi \in P_N\}.$$

Define the corresponding interpolation operator $\mathcal{I}_N^{(\alpha)} : C[0, +\infty) \to \mathbb{P}_N$ such that

$$(\mathcal{I}_N^{(\alpha)} u)(x_j^{(\alpha)}) = u(x_j^{(\alpha)}), \quad 0 \le j \le N.$$

which can be expressed by

$$(\mathcal{I}_N^{(\alpha)} u)(x) = \sum_{k=0}^{N} \tilde{u}_k^{(\alpha)} \hat{\mathcal{L}}_k^{(\alpha)}(x) \in \mathbb{P}_N.$$

where

$$\tilde{u}_k^{(\alpha)} = \frac{k!}{\Gamma(k+\alpha+1)} \sum_{j=0}^{N} u(x_j^{(\alpha)}) \hat{\mathcal{L}}_k^{(\alpha)}(x_j^{(\alpha)}) \tilde{\omega}_j^{(\alpha)}, \quad 0 \le j \le N.$$

Theorem 2.7. *Let $\alpha > -1$. If $u \in C(\mathbb{R}_+) \cap B_\alpha^m(\mathbb{R}_+)$ and $\partial_x u \in B_\alpha^m(\mathbb{R}_+)$ with $0 \le m \le N+1$, then*

$$\|\mathcal{I}_N^{(\alpha)} u - u\|_{\omega_\alpha} \le c\sqrt{\frac{(N-m+1)!}{N!}} (\|\partial_x^m u\|_{\omega_{\alpha+m-1}} + (\ln N)^{1/2} \|\partial_x^m u\|_{\omega_{\alpha+m}}),$$

where c is a positive constant independent of m, N and u.

Now consider the approximations by (generalized) Laguerre polynomials or functions. Consider the $L^2_{\omega_\alpha}$-orthogonal projection $\mathcal{P}_{N,\alpha} : L^2_{\omega_\alpha}(\mathbb{R}_+) \to P_N$, defined by

$$(\mathcal{P}_{N,\alpha} u - u, v_N)_{\omega_\alpha} = 0, \quad \forall v_N \in P_N, \tag{2.19}$$

so we have

$$\mathcal{P}_{N,\alpha}u(x) = \sum_{k=0}^{N} \tilde{u}_k^{(\alpha)} \mathcal{L}_k^{(\alpha)} \text{ with } \tilde{u}_k^{(\alpha)}(x) = \frac{k!}{\Gamma(k+\alpha+1)} \int_{\mathbb{R}_+} u(x)\mathcal{L}_k^{(\alpha)}(x)\omega_\alpha(x)dx.$$

we introduce the space

$$B_\alpha^m(\mathbb{R}_+) := \{u : \partial_x^k u \in L^2_{\omega_{\alpha+k}}(\mathbb{R}_+), \quad 0 \le k \le m\},$$

equipped with the norm and semi-norm

$$\|u\|_{B_\alpha^m} = \left(\sum_{k=0}^{m} \|\partial_x^k u\|^2_{\omega_{\alpha+k}}\right)^{1/2}, \quad |u|_{B_\alpha^m} = \|\partial_x^m u\|_{\omega_{\alpha+m}}.$$

As usual, we will drop the subscript α if $\alpha = 0$. Notice that the weight function corresponding to the derivative of different order is different in $B_\alpha^m(\mathbb{R}_+)$, as opposed to the Sobolev space $H_\alpha^m(\mathbb{R}_+)$.

Theorem 2.8. *Let $\alpha > -1$. If $u \in B_\alpha^m(\mathbb{R}_+)$ and $0 \le m \le N+1$, we have*

$$\|\partial_x^l(\mathcal{P}_{N,\alpha}u - u)\|_{\omega_{\alpha+l}} \le \sqrt{\frac{(N-m+1)!}{(N-l+1)!}} \|\partial_x^m u\|_{\omega_{\alpha+m}}, \quad 0 \le l \le m.$$

we consider the projection of the Laguerre function, recall that $\hat{\omega}_\alpha = x^\alpha$. For any $u \in L^2_{\hat{\omega}_\alpha}(\mathbb{R}_+)$, we have $ue^{x/2} \in L^2_{\omega_\alpha}(\mathbb{R}_+)$. Define the operator

$$\hat{\mathcal{P}}_{N,\alpha}u = e^{-x/2}\mathcal{P}_{N,\alpha}(ue^{x/2}) \in \hat{P}_N.$$

Clearly

$$(\hat{\mathcal{P}}_{N,\alpha}u - u, v_N)_{\hat{\omega}_\alpha} = (\mathcal{P}_{N,\alpha}(ue^{x/2}) - (ue^{x/2}), (ve^{x/2}))_{\omega_\alpha} = 0, \quad \forall v_N \in \hat{P}_N.$$

Let us define

$$\hat{B}_\alpha^m(\mathbb{R}_+) := \{u : \hat{\partial}_x^k u \in L^2_{\hat{\omega}_{\alpha+k}}(\mathbb{R}_+), \quad 0 \le k \le m\},$$

equipped with the norm and semi-norm

$$\|u\|_{\hat{B}_\alpha^m} = \left(\sum_{k=0}^{m} \|\hat{\partial}_x^k u\|^2_{\hat{\omega}_{\alpha+k}}\right)^{1/2}, \quad |u|_{\hat{B}_\alpha^m} = \|\hat{\partial}_x^m u\|_{\hat{\omega}_{\alpha+m}}.$$

Theorem 2.9. *Let $\hat{\partial}_x = \partial_x + \frac{1}{2}$ and $\alpha > -1$. Then for any $u \in \hat{B}_\alpha^m(\mathbb{R}_+)$ and $0 \le m \le N+1$,*

$$\|\hat{\partial}_x^l(\hat{\mathcal{P}}_{N,\alpha}u - u)\|_{\hat{\omega}_{\alpha+l}} \le \sqrt{\frac{(N-m+1)!}{(N-l+1)!}} \|\hat{\partial}_x^m u\|_{\hat{\omega}_{\alpha+m}}, \quad 0 \le l \le m.$$

For more detail see [36].

2.2 Hermite Polynomials and Functions

We present in this section basic properties of the Hermite polynomials and functions, and derive Hermite-Gauss quadrature and the associated interpolation.

2.2.1 Hermite Polynomials

The Hermite polynomial $H_k = H_k(x)$ satisfies the second-order linear ordinary differential equation

$$y''(x) - 2xy'(x) + 2ky(x) = 0, \tag{2.20}$$

and is defined by the formulas

$$\begin{aligned} H_k(x) &= (-1)^k e^{x^2} \frac{d^k}{dx^k}(e^{-x^2}) \\ &= \sum_{m=0}^{[k/2]} \frac{(-1)^m k!}{m!(k-2m)!}(2x)^{k-2m}. \end{aligned}$$

where $[k/2]$ denotes again the integral part of $k/2$. Then the Hermite polynomials satisfy the three-term recurrence relation:

$$H_{k+1}(x) = 2xH_k(x) - 2kH_{k-1}(x), \quad k \geq 1, \tag{2.21}$$

with $H_0(x) = 1$ and $H_1(x) = 2x$.

and the first few members are

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= 2x, \\ H_2(x) &= 4x^2 - 2, \\ H_3(x) &= 8x^3 - 12x. \end{aligned}$$

some properties of the Hermite polynomials are

$$\begin{aligned} H_k(-x) &= (-1)^k H_k(x), \\ H_k'(x) &= 2xH_k(x) - H_{k+1}(x), \quad k \geq 0. \\ \partial_x^n H_k(x) &= \frac{2^n k!}{(k-n)!} H_{k-n}(x), \quad k \geq n, \end{aligned}$$

The Hermite polynomials, defined on the whole line $\mathbb{R}$, are orthogonal with respect to the weight function $\omega(x) = e^{-x^2}$, namely,

$$\int_{-\infty}^{+\infty} H_k(x)H_m(x)\omega(x)dx = \begin{cases} 0 & k \neq m \\ \sqrt{\pi}2^k k! & k = m \end{cases} \tag{2.22}$$

2.2.2 Hermite Functions

The Hermite functions $h_k(x)$ are introduced by the formula

$$h_k(x) = \frac{1}{\pi^{1/4}\sqrt{2^k k!}} e^{-x^2/2} H_k(x), \quad k \geq 0, \quad x \in \mathbb{R}, \tag{2.23}$$

which are normalized so that

$$\int_{-\infty}^{+\infty} h_k(x)h_m(x)dx = \delta_{km}.$$

Computation of Nodes

The zeros $\{x_j\}_{j=0}^{N}$ of $H_{N+1}(x)$ are the eigenvalues of the symmetric tridiagonal matrix

$$A_{N+1} = \begin{bmatrix} a_0 & -\sqrt{b_1} & & & \\ -\sqrt{b_1} & a_1 & -\sqrt{b_2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\sqrt{b_{n-1}} & a_{n-1} & -\sqrt{b_{n-1}} \\ & & & -\sqrt{b_n} & a_n \end{bmatrix}$$

whose entries are derived from (2.21):

$$a_j = 0, \quad 0 \leq j \leq N; \quad b_j = j/2, \quad 1 \leq j \leq N.$$

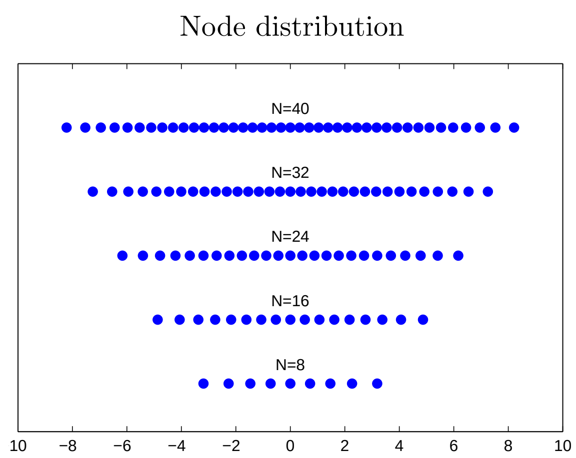

Figure 2.2: Distribution of the Hermite-Gauss nodes $\{x_j\}_{j=0}^N$ with N=8,16,24,32.

Theorem 2.10. *Let $\{x_j\}_{j=0}^N$ be the zeros of $H_{N+1}(x)$, and let $\{\omega_j\}_{j=0}^N$ be given by*

$$\omega_j = \frac{\sqrt{\pi}2^N N!}{(N+1)[H_N(x_j)]^2}, \quad 0 \leq j \leq N. \tag{2.24}$$

Then, we have

$$\int_{-\infty}^{+\infty} p(x)e^{-x^2}dx = \sum_{j=0}^{N} p(x_j)\omega_j, \quad \forall p \in P_{2N+1}, \tag{2.25}$$

and

$$\int_a^b f(x)e^{-x^2}dx = \sum_{j=0}^{N-1} f(x_j)\omega_j + E_N[f], \quad \textit{with} \quad E_N[f] = \frac{N!\sqrt{\pi}}{2^N(2N)!}f^{(2N)}(\xi).$$

Proof. ([see [36],p ,258]) □

2.2.3 Interpolations and Approximations

Let $\mathcal{I}_N^h$ be the interpolation operator associated with the Hermite-Gauss points $\{x_j\}_{j=0}^N$ such that for any $u \in C(\mathbb{R})$,

$$\mathcal{I}_N^h u \in P_N; \quad (\mathcal{I}_N^h u)(x_j) = u(x_j), \quad 0 \leq j \leq N, \tag{2.26}$$

which can be expanded as

$$(\mathcal{I}_N^h u)(x) = \sum_{k=0}^{N} \tilde{u}_k H_k(x), \tag{2.27}$$

where

$$\tilde{u}_k = \frac{1}{2k}\sum_{i=0}^{N} u(x_i)H_k(x_i)\omega_i, \quad 0 \le k \le N. \tag{2.28}$$

Let

$$L^2_\omega(\mathbb{R}) = \{u| \;\; u \text{ is measurable and } \|u\|_{L^2_\omega} < \infty\},$$

is a Hilbert space with the inner product

$$(u, v)_{L^2_\omega(\mathbb{R})} = \int_{\mathbb{R}} u(x)v(x)\omega(x)dx.$$

Theorem 2.11. *For any $u \in B^m_\omega(\mathbb{R})$ and $m \ge 1$ with $0 \le k \le m$, then*

$$\|\mathcal{I}^h_N u - u\|_{B^k_\omega} \le cN^{\frac{1}{3}+\frac{m-k}{2}}\|u\|_{B^m_\omega}.$$

Proof. see [19] □

Now consider approximations by Hermite polynomials or functions. Let $\omega = e^{-x^2}$ be the Hermite weight function as before. Consider the L^2_ω-orthogonal projection $\mathcal{P}_N : L^2_\omega(\mathbb{R}) \to P_N$, defined by

$$(u - \mathcal{P}_N u, v_N)_\omega = 0, \quad \forall v_N \in P_N, \tag{2.29}$$

It is clear that

$$\mathcal{P}_N u(x) = \sum_{k=0}^{N} \tilde{u}_k H_k \text{ with } \tilde{u}_k(x) = \frac{1}{\sqrt{\pi}2^k k!}\int_{\mathbb{R}} u(x)H_k(x)\omega(x)dx.$$

Theorem 2.12. *For any $u \in B^m_\omega(\mathbb{R})$ with $0 \le m \le N+1$,*

$$\|\partial^l_x(\mathcal{P}_N u - u)\|_\omega \le 2^{(l-m)/2}\sqrt{\frac{(N-m+1)!}{(N-l+1)!}}\|\partial^m_x u\|_\omega, \quad 0 \le l \le m.$$

Lemma 2.13 (see [19])**.** *For any $u \in B^m_\omega(\mathbb{R})$ and $m \ge 1$ with $0 \le k \le m$, then*

$$\|\mathcal{P}_N u - u\|_{B^k_\omega} \le cN^{\frac{m-k}{2}}\|u\|_{B^m_\omega}.$$

For the Hermite function approximations, notice that for any $u \in L^2(\mathbb{R})$, we have $ue^{x^2/2} \in L^2_\omega(\mathbb{R})$. Define

$$\hat{\mathcal{P}}_N u = e^{-x^2/2}\mathcal{P}_N(ue^{x^2/2}) \in \hat{P}_N,$$

which satisfies

$$(u - \hat{\mathcal{P}}_N u, v_N) = (ue^{x/2} - \mathcal{P}_N(ue^{x/2}), ve^{x/2})_\omega = 0, \quad \forall v_N \in \hat{P}_N.$$

The following result is a direct consequence of Theorem 2.12.

Corollary 2.14. *Let $\hat{\partial}_x = \partial_x + x$. For any $\hat{\partial}_x^m \in L^2(\mathbb{R})$ with $0 \le m \le N+1$,*

$$\|\hat{\partial}_x^l(\hat{\mathcal{P}}_N u - u)\|_\omega \le c2^{(l-m)/2}\sqrt{\frac{(N-m+1)!}{(N-l+1)!}}\|\hat{\partial}_x^m u\|_\omega, \quad 0 \le l \le m.$$

where c is a positive constant independent of m, N and u.

2.3 Jacobi Polynomials

In this section we collect some useful formulas for these polynomials (for more details, see, e.g., [5], [36] and [29]).

2.3.1 Basic Properties

The Jacobi polynomials, $J_k^{\alpha,\beta}(x)$, are solutions of the second-order linear ordinary differential equation

$$(1-x^2)y''(x) + [\beta - \alpha - (\alpha+\beta+2)x]y'(x) + k(k+\alpha+\beta+1)y(x) = 0, \tag{2.30}$$

and are defined by the formulas

$$\begin{aligned} J_k^{\alpha,\beta}(x) &= \frac{(-1)^k}{2^k k!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^k}{dx^k}[(1-x)^{\alpha+k}(1+x)^{\beta+k}] & (2.31)\\ &= 2^{-k}\sum_{m=0}^{k} C_{k+\alpha}^m C_{k+\beta}^{k-m}(x-1)^{k-m}(x+1)^m, & (2.32)\end{aligned}$$

where the C_b^a are binomial coefficients. The Jacobi polynomials are generated by the three-term recurrence relation

$$\begin{aligned} J_{k+1}^{\alpha,\beta}(x) &= (a_k^{\alpha,\beta}x - b_k^{\alpha,\beta})J_k^{\alpha,\beta}(x) - c_k^{\alpha,\beta}J_{k-1}^{\alpha,\beta}(x), \quad n \ge 1,\\ J_0^{\alpha,\beta}(x) &= 1, \quad J_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta),\end{aligned}$$

where

$$
\begin{aligned}
a_k^{\alpha,\beta} &= \frac{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)}{2(k+1)(k+\alpha+\beta+1)},\\
b_k^{\alpha,\beta} &= \frac{(\beta^2-\alpha^2)(2k+\alpha+\beta+1)}{2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)},\\
c_k^{\alpha,\beta} &= \frac{(k+\alpha)(k+\beta)(2k+\alpha+\beta+2)}{(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)},
\end{aligned}
$$

Jacobi polynomials for which $\alpha = \beta$ are called *ultraspherical polynomials* and are denoted simply by $J_k^{\alpha}(x)$, than the formula (2.31) is defined by:

$$J_k^{\alpha}(x) = \frac{(-1)^k}{2^k k!}(1-x^2)^{-\alpha}\frac{d^k}{dx^k}[(1-x^2)^{\alpha+k}].$$

and the norm is

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}[J_k^{\alpha,\beta}(x)]^2dx = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta)k!\Gamma(k+\alpha+\beta+1)}.$$

Theorem 2.15. *The Jacobi polynomials $J_k^{\alpha,\beta}(x)$ are the orthogonal polynomials with the weight function $\omega(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha > -1$, $\beta > -1$. The Gauss-Jacobi rule is defined by*

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}f(x)dx \approx \sum_{i=0}^{k}\omega_i f(x_i),$$

where the nodes and weight given by

- *For Jacobi-Gauss*

$$
\begin{aligned}
&\{x_i\}_{i=0}^{k} \text{ are the zeros of } J_k^{\alpha,\beta}(x),\\
&\omega_i = \frac{G_k^{\alpha,\beta}}{J_k^{\alpha,\beta}(x_i)\partial J_{k+1}^{\alpha,\beta}(x_i)}.
\end{aligned}
$$

- *For Jacobi-Gauss-Radau*

$$
\begin{aligned}
&x_0 = -1 \text{ and } \{x_i\}_{i=1}^{k} \text{ be the zeros of } J_k^{\alpha,\beta+1}(x),\\
&\omega_0 = \frac{2^{\alpha+\beta+1}(\beta+2)\Gamma^2(\beta+1)k!\Gamma(k+\alpha+1)}{\Gamma(k+\beta+2)\Gamma(k+\alpha+\beta+2)},\\
&\omega_i = \frac{1}{1+x_i}\frac{G_{k-1}^{\alpha,\beta+1}}{J_{k-1}^{\alpha,\beta+1}(x_i)\partial J_k^{\alpha,\beta+1}(x_i)}.
\end{aligned}
$$

- *For Jacobi- Gauss-Lobatto*

$$
\begin{aligned}
&x_0 = -1, \quad x_k = 1 \text{ and } \{x_i\}_{i=1}^{k-1} \text{ be the zeros of } \partial J_k^{\alpha,\beta}(x),\\
&\omega_0 = \frac{2^{\beta+1}(\beta+2)\Gamma^2(\beta+1)\Gamma(k)\Gamma(k+\alpha+1)}{\Gamma(k+\beta+1)\Gamma(k+\alpha+\beta+2)},\\
&\omega_0 = \frac{2^{\alpha+1}(\beta+2)\Gamma^2(\alpha+1)\Gamma(k)\Gamma(k+\beta+1)}{\Gamma(k+\alpha+1)\Gamma(k+\alpha+\beta+2)},\\
&\omega_i = \frac{1}{1-x_i^2}\frac{G_{k-2}^{\alpha+1,\beta+1}}{J_{k-2}^{\alpha+1,\beta+1}(x_i)\partial J_{k-1}^{\alpha+1,\beta+1}(x_i)},
\end{aligned}
$$

where

$$
G_k^{\alpha,\beta} = \frac{2^{\alpha+\beta}(2k+\alpha+\beta+2)\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(k+1)!\Gamma(k+\alpha+\beta+2)}.
$$

Proof. see [36]. □

2.3.2 Computation of Nodes

The explicit expressions of the nodes and weights of the general Jacobi-Gauss quadrature are not available, so they have to be computed by numerical means.the zeros of the Jacobi polynomial $J_{k+1}^{\alpha,\beta}(x)$ are the eigenvalues of the following symmetric tridiagonal matrix

$$
A_{k+1} = \begin{bmatrix}
a_0 & -\sqrt{b_1} & & & \\
-\sqrt{b_1} & a_1 & -\sqrt{b_2} & & \\
 & \ddots & \ddots & \ddots & \\
 & & -\sqrt{b_{k-1}} & a_{k-1} & -\sqrt{b_{k-1}} \\
 & & & -\sqrt{b_k} & a_k
\end{bmatrix}
$$

where

$$
a_j = \frac{\beta^2-\alpha^2}{(2j+\alpha+\beta)(2j+\alpha+\beta+2)},
$$

$$
b_j = \frac{4j(j+\alpha)(j+\beta)(j+\alpha+\beta)}{(2j+\alpha+\beta-1)(2j+\alpha+\beta)^2(2j+\alpha+\beta+1)}.
$$

2.3.3 Interpolations and Approximations

Let $\{x_j,\omega_j\}_{j=0}^N$ be a set of Jacobi-Gauss-type nodes and weights. As in Sect.2.1.3 we can define the corresponding interpolation operator, discrete inner product and discrete norm, denoted by $\mathcal{I}_N^{\alpha,\beta}$, $\langle .,. \rangle_{N,\omega^{\alpha,\beta}}$ and $\|.\|_{N,\omega^{\alpha,\beta}}$, respectively.

The exactness of the quadratures implies

$$\langle u, v\rangle_{N,\omega^{\alpha,\beta}} = (u, v)_{\omega^{\alpha,\beta}}, \quad \forall u, \quad v \in P_{2N+\delta}, \tag{2.33}$$

where $\delta = 1, \quad 0 \quad and \quad -1$ for the Jacobi-Gauss, Jacobi-Gauss-Radau and Jacobi-Gauss-Lobatto, respectively. Accordingly, we have

$$\|u\|_{N,\omega^{\alpha,\beta}} = \|u\|_{\omega^{\alpha,\beta}}, \quad \forall u \in P_N. \text{ for JG and JGR.}$$

Definition 2.16. *The interpolation polynomials $\mathcal{I}_N^{\alpha,\beta} u \in P_N$ are defined by*

$$(\mathcal{I}_N^{\alpha,\beta} u)(x) = \sum_{k=0}^{N} \tilde{u}_k^{\alpha,\beta} J_k^{\alpha,\beta}(x), \tag{2.34}$$

where the coefficients $\{\tilde{u}_k^{\alpha,\beta}\}_{k=0}^{N}$ are determined by the forward discrete Jacobi transform.

Theorem 2.17.

$$\tilde{u}_k^{\alpha,\beta} = \frac{1}{\delta_k^{\alpha,\beta}} \sum_{j=0}^{N} u(x_j) J_k^{\alpha,\beta}(x_j)\omega_j, \tag{2.35}$$

where $\delta_k^{\alpha,\beta} = \gamma_k^{\alpha,\beta}$ for $0 \le N \le N-1$, and

$$\delta_k^{\alpha,\beta} = \begin{cases} \gamma_k^{\alpha,\beta}, & \textit{for JG and JGR,} \\ (2 + \frac{\alpha+\beta+1}{N})\gamma_k^{\alpha,\beta}, & \textit{for JGL.} \end{cases}$$

Proof. see [36]. □

2.4 Chebyshev Polynomials

In this section, we consider Chebyshev polynomials of the first kind, which are proportional to Jacobi polynomials $\{J_n^{1/2,1/2}\}$ and are orthogonal with respect to the weight function $\omega(x) = (1 - x^2)^{-1/2}$.

2.4.1 Chebyshev Polynomials of the First Kind

Classical references on the Chebyshev polynomials are Fox and Parker (1968) and Rivlin (1974) provided in [5]. The Chebyshev polynomials of the first kind $T_k = T_k(x)$ satisfy the equation

$$(1-x^2)y''(x) - xy'(x) + k^2 y(x) = 0, \tag{2.36}$$

and are defined by

$$\begin{aligned} T_k(x) &= \cos(k \arccos x) \\ &= \frac{(-2)^k k!}{(2k)!}\sqrt{1-x^2}\frac{d^k}{dx^k}[(1-x^2)^{k-\frac{1}{2}}], \end{aligned}$$

then, for any k, $T_k(x)$ is even if k is even, and odd if k is odd. If T_k is normalized so that $T_k(1) = 1$.

Thus, the Chebyshev polynomials are nothing but cosine functions after a change of independent variable. This property is the origin of their widespread popularity in the numerical approximation of non periodic boundary value problems.

The Chebyshev polynomials can be expanded in power series as

$$T_k(x) = \frac{k}{2}\sum_{i=0}^{[k/2]}(-1)^k \frac{k-i-1}{i!(k-2i)!}(2x)^{k-2i}. \tag{2.37}$$

Moreover, the trigonometric relation

$$\cos(k+1)\theta + \cos(k-1)\theta = 2\cos\theta\cos k\theta,$$

gives the recursion relation

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x), \quad k = 2, 3, \ldots \ , \tag{2.38}$$

with $T_0(x) \equiv 1$ and $T_1(x) \equiv x$.

Write down few first Chebyshev polynomials of the first kind; we determine them directly from formula (2.38):

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, \\ T_5(x) &= 16x^5 - 20x^3 + 5x, \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1. \end{aligned} \tag{2.39}$$

Some properties of the Chebyshev polynomials are

$$
\begin{aligned}
|T_k(x)| &\leq 1, \quad -1 \leq x \leq 1, && (2.40)\\
T_k(\pm 1) &= (\pm 1)^k, && (2.41)\\
|T_k'(x)| &\leq k^2, \quad -1 \leq x \leq 1, && (2.42)\\
T_k'(\pm 1) &= (\pm 1)^{k+1} k^2. && (2.43)
\end{aligned}
$$

It is also easy to show that

$$\int_{-1}^{1} T_k(x) T_l(x) \frac{dx}{\sqrt{1-x^2}} = c_k \frac{\pi}{2} \delta_{kl}, \tag{2.44}$$

where

$$c_k = \begin{cases} 2, & k = 0 \\ 1, & k \geq 1 \end{cases}$$

The Chebyshev expansion of a function $u \in L^2_\omega(-1,1)$ is

$$u(x) = \sum_{k=0}^{\infty} \hat{u}_k T_k(x), \quad \hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^{1} u(x) T_k(x) \omega(x) dx. \tag{2.45}$$

Theorem 2.18 (see [36], p. 108)**.** *Let $\{x_j, \omega_j\}_{j=0}^N$ be a set of Chebyshev-Gauss type quadrature nodes and weights.*

- *For Chebyshev-Gauss (CG) quadrature*

$$x_j = -\cos \frac{(2j+1)\pi}{2N+2}, \quad \omega_j = \frac{\pi}{N+1}, \quad 0 \leq j \leq N.$$

- *For Chebyshev-Gauss-Radau (CGR) quadrature,*

$$
\begin{aligned}
x_j &= -\cos \frac{(2j+1)\pi}{2N+2}, \quad 0 \leq j \leq N,\\
\omega_0 &= \frac{\pi}{2N+1}, \quad \omega_j = \frac{2\pi}{2N+1}, \quad 1 \leq j \leq N.
\end{aligned}
$$

- *For Chebyshev-Gauss-Lobatto (CGL) quadrature,*

$$x_j = -\cos \frac{j\pi}{N}, \quad \omega_j = \frac{\pi}{\tilde{c}_j N}, \quad 0 \leq j \leq N.$$

where $\tilde{c}_0 = \tilde{c}_N = 2$ and $\tilde{c}_j = 1$ for $j = 1, 2, ..., N-1$.

With the above choices, there holds

$$\int_{-1}^{1} p(x)\frac{1}{\sqrt{1-x^2}}dx = \sum_{j=1}^{N} p(x_j)\omega_j, \quad \forall p \in P_{2N+\delta}, \tag{2.46}$$

where $\delta = 1, 0, -1$ *for the CG, CGR and CGL, respectively.*

Theorem 2.19 (see [21], p. 94)**.** *Given* $f \in C^{2N}[-1,1]$ *and let* $x_0, x_1, ..., x_N$ *be the* $N+1$ *zeros of* $T_{N+1}(x)$*, then*

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx \approx \frac{\pi}{N+1}\sum_{i=0}^{N} f(x_i). \tag{2.47}$$

2.4.2 Interpolations and Approximations

Let $\{x_j, \omega_j\}_{j=0}^{N}$ be a set of Gauss, Gauss-Radau or Gauss-Lobatto quadrature nodes and weights. We define the corresponding discrete inner product and norm as

$$\langle u, v\rangle_{N,\omega} := \sum_{j=0}^{N} u(x_j)v(x_j)\omega_j, \quad \|u\|_{N,\omega} = \sqrt{\langle u, u\rangle_{N,\omega}}. \tag{2.48}$$

Definition 2.20. *For any* $u \in C[-1,1]$*, The definition of the interpolation operator* $\mathcal{I}_N : C[-1,1] \to P_N$ *such that*

$$(\mathcal{I}_N u)(x_j) = u(x_j), \quad 0 \le j \le N. \tag{2.49}$$

Theorem 2.21. *For the approximation using Chebyshev polynomials, we have*

$$(\mathcal{P}_N u)(x) = \sum_{i=0}^{N} c_i T_i(x),$$

where

$$\begin{aligned} c_0 &\approx \frac{1}{N+1}\sum_{i=0}^{N} u(x_i) \\ c_j &\approx \frac{2}{N+1}\sum_{i=0}^{N} u(x_i)T_j(x_i), \quad 1 \le j \le N. \end{aligned}$$

Lemma 2.22 (see [5], p. 296)**.** *Let* $u \in H_\omega^m(-1,1)$ *with* $m \ge 1$*. The truncation error* $u - \mathcal{P}_N u$*, satisfies the inequality*

$$\|u - \mathcal{P}_N u\|_{L^2_\omega(-1,1)} \le CN^{-m}|u|_{H_\omega^{m,N}(-1,1)} \tag{2.50}$$

where,

$$|u|_{H^{m,N}_{\omega}(-1,1)} = \left(\sum_{k=\min(m,N+1)}^{m} \|u^{(k)}\|^2_{L^2_{\omega}(-1,1)} \right)^{1/2}. \tag{2.51}$$

2.5 Cardinal Functions

Sir Edmund Whittaker (1915) showed that for an infinite interval, the analogues to the *fundamental polynomials of Lagrangian interpolation* are what he called the *cardinal functions.* The collocation points are evenly spaced:

$$x_i = ih, \quad i = 0, \pm 1, \pm 2, \pm 3, ... \ , \tag{2.52}$$

Whittaker's cardinal functions are

$$S(k,h)(x) = sinc(\frac{x - kh}{h}), \tag{2.53}$$

where

$$sinc(x) = \frac{\sin(\pi x)}{\pi x},$$

and have the property

$$S(k,h)(x_i) = \delta_{ik}.$$

Definition 2.23. *An approximation to a function $f(x)$ of the form*

$$f(x) \approx \sum_{k=-n}^{n} f(x_k) S(k,h)(x), x \in \mathbb{R},$$

and we have

$$\int_{-\infty}^{\infty} sinc(x)dx = 1,$$

then

$$\int_{-\infty}^{\infty} f(x)dx \approx \sum_{k=-n}^{n} f(x_k) \int_{-\infty}^{\infty} S(k,h)(x)dx = h \sum_{k=-n}^{n} f(x_k),$$

where h is the grid spacing.

For example

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2} dx &\approx \int_{-\infty}^{\infty} \sum_{k=-n}^{n} e^{-x_k^2} S(k,h)(x) dx \\ &= \sum_{k=-n}^{n} e^{-x_k^2} \int_{-\infty}^{\infty} sinc(\frac{x-kh}{h}) dx \\ &= h \sum_{k=-n}^{n} e^{-(kh)^2}.\end{aligned}$$

2.6 Mappings for Unbounded Domains

The purpose of this section is to present a general framework for the analysis and implementation of the mapped spectral methods. A common and effective strategy in dealing with unbounded domains is to use a suitable mapping that transforms an infinite domain to a finite domain. Then, images of classical orthogonal polynomials under the inverse mapping will form a set of orthogonal basis functions which can be used to approximate solutions of PDEs in the infinite domains.

Mappings

Consider a family of mappings of the form

$$x = x(y; s), \quad s > 0, \quad y \in I := (-1, 1), \quad x \in \Lambda := (0, \infty) \text{ or } (-\infty, \infty), \tag{2.54}$$

such that

$$\begin{aligned}&\frac{dx}{dy} = x'(y; s) > 0, \quad s > 0, \quad y \in I, \\ &x(-1, s) = 0, \quad x(1; s) = +\infty, \quad \text{if } \Lambda = (0, +\infty), \\ &x(\pm 1; s) = \pm\infty, \quad \text{if } \Lambda = (-\infty, +\infty).\end{aligned} \tag{2.55}$$

Without loss of generality, we assume that the mapping is explicitly invertible, and denote its inverse mapping by

$$y = y(x; s), \quad \forall x \in \Lambda. \ \forall y \in I.$$

In this one-to-one transform, the parameter s is a positive scaling factor.

2.6.1 Semi-Infinite Intervals

The most frequently used mappings are algebraic, exponential and logarithmic, given by the following formulas, between $x \in \Lambda = (0, +\infty)$ and $y \in I = (-1, 1)$ with $s > 0$:

- Algebraic Mapping

$$x = s\frac{1+y}{1-y}, \quad y = \frac{x-s}{x+s}. \tag{2.56}$$

- Logarithmic Mapping

$$x = s\tanh^{-1}(\frac{1+y}{2}), \quad y = 1 - 2\tanh(\frac{x}{s}). \tag{2.57}$$

- Exponential Mapping

$$x = \sinh(\frac{s}{2}(1+y)), \quad y = \frac{2}{s}\ln(x + \sqrt{x^2+1}) - 1, \tag{2.58}$$

where $y \in (-1, 1)$ and $x \in (0, L_s)$ with $L_s = \sinh(s)$.

Hence, The intensity of stretching essentially depends on the derivative values of the mapping. For the mappings (2.56), (2.57) and (2.58), we have

$$\frac{dx}{dy} = \frac{2s}{(1-y)^2}, \quad \frac{2s}{(3+y)(1-y)}, \quad \frac{s}{2}\cosh(\frac{s}{2}(1+y)),$$

respectively. Therefore, the grid is stretched more and more as s increases.

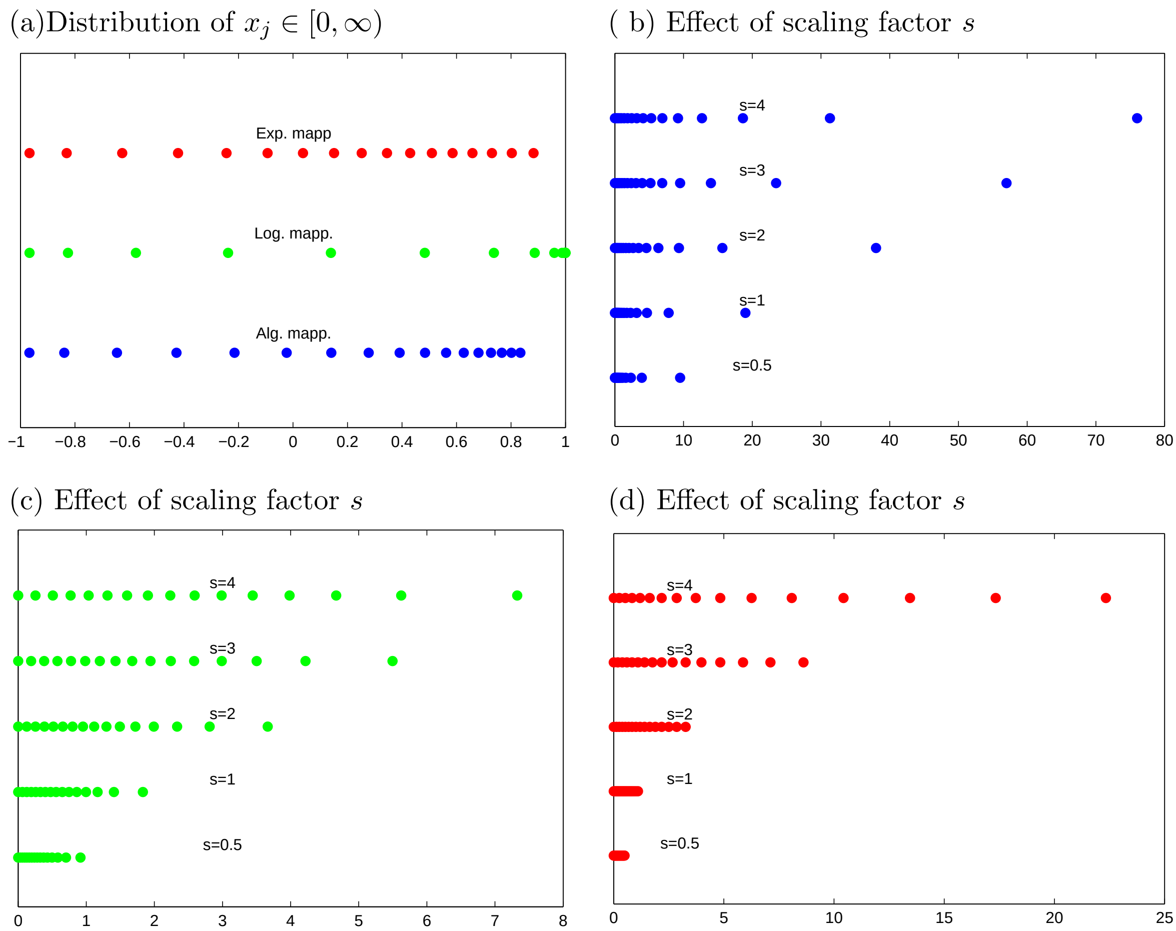

Figure 2.3: (a).Distribution of $\mathcal{L}_{16}$ zeros by used mappings for s=5. (b), (c) and (d) distribution a 16 point of the interval $[-1, 0.9]$ before used Alg.map, Log.map and Exp.map inverse, respectively.

2.6.2 The Real Line

Similar considerations apply to expansions on $(-\infty, +\infty)$ as on semi-infinite intervals, given by the following formulas, between $x \in \Lambda = (-\infty, +\infty)$ and $y \in I = (-1, 1)$ with $s > 0$:

- Algebraic Mapping

$$x = \frac{sy}{\sqrt{1-y^2}}, \quad y = \frac{x}{\sqrt{x^2+s^2}}. \tag{2.59}$$

- Logarithmic Mapping

$$x = s\tanh^{-1}(y), \quad y = \tanh(\frac{x}{s}). \tag{2.60}$$

- Exponential Mapping

$$x = \sinh(sy), \quad y = \frac{1}{s}\ln(x + \sqrt{x^2+1}), \tag{2.61}$$

where $y \in (-1, 1)$ and $x \in (-L_s, L_s)$ with $L_s = \sinh(s)$.

Hence, The intensity of stretching essentially depends on the derivative values of the mapping. For the mappings (2.59), (2.60) and (2.61), we have

$$\frac{dx}{dy} = \frac{s}{(1-y^2)^{3/2}}, \quad \frac{s}{1-y^2}, \quad s\cosh(sy),$$

respectively. Therefore, the grid is stretched more and more as s increases.

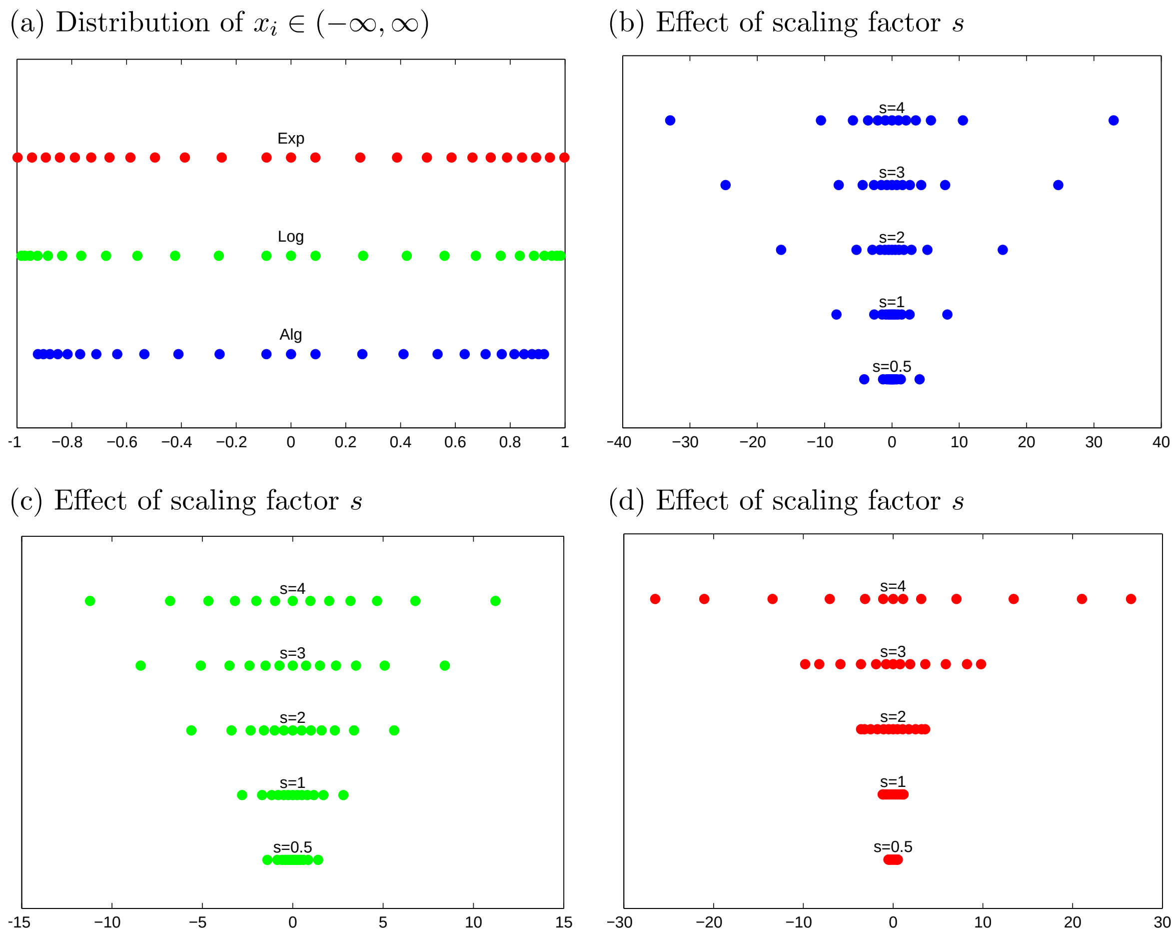

Figure 2.4: (a) Distribution of the zeros of H_{16} by used mappings for s=2.5; (b), (c) and (d) Mapped Chebyshev-Gauss with $n = 24$ using the algebraic map (2.59), logarithmic map (2.60) and exponential map (2.61) respectively, various scaling factor s.

CHAPTER 3

Mapped Chebyshev Spectral Methods for Solving Second Kind Integral Equations on the Real Line

Introduction

In this chapter we investigate the utility of mappings to solve numerically an important class of integral equations on the real line. The main idea is to map the infinite interval to a finite one and use Chebyshev spectral-collocation method to solve the mapped integral equation in the finite interval. Numerical examples are presented to illustrate the accuracy of the method.

3.1 Solution Methods

3.1.1 Logarithmic Mapping

Consider integral equations of the form

$$\varphi(x) - \int_{-\infty}^{+\infty} K(x,t)\varphi(t)dt = f(x), \tag{3.1}$$

where the functions $K(x,t)$ and $f(x)$ are given, and $\varphi(x)$ is an unknown function to be determined. The regular part of (3.1) is assumed to be exists in the Riemann sense. Which is considered becomes

$$\lim_{L\to\infty} \int_{-L}^{L} K(x,t)\varphi(t)dt. \tag{3.2}$$

We use Logarithmic mapping then (3.1) becomes

$$\varphi(x) - \int_{-1}^{1} K(x, s\tanh^{-1} y)\varphi(s\tanh^{-1} y)\frac{s}{1-y^2}dy = f(x). \quad (3.3)$$

If we pose $x = s\tanh^{-1} z$, then

$$\varphi(s\tanh^{-1} z) - \int_{-1}^{1} K(s\tanh^{-1} z, s\tanh^{-1} y)\varphi(s\tanh^{-1} y)\frac{s}{1-y^2}dy = f(s\tanh^{-1} z),$$

and let $u(z) = \varphi(s\tanh^{-1} z)$, so we obtain

$$u(z) - \int_{-1}^{1} K(s\tanh^{-1} z, s\tanh^{-1} y)u(y)\frac{s}{(1-y^2)}dy = f(s\tanh^{-1} z), \quad (3.4)$$

by setting

$$M(z,y) = \frac{K(s\tanh^{-1} z, s\tanh^{-1} y)}{\sqrt{1-y^2}},$$

$$\omega(y) = \frac{1}{\sqrt{1-y^2}} \quad \text{and} \quad F(z) = f(s\tanh^{-1} z),$$

equation (3.4) can be written as follows

$$u(z) - s\int_{-1}^{1} M(z,y)u(y)\omega(y)dy = F(z). \quad (3.5)$$

3.1.2 Algebraic Mapping

For using Algebraic mapping then (3.1) becomes

$$\varphi(x) - \int_{-1}^{1} K(x, \frac{sy}{\sqrt{1-y^2}})\varphi(\frac{sy}{\sqrt{1-y^2}})\frac{s}{(1-y^2)^{3/2}}dy = f(x). \quad (3.6)$$

If we pose $x = \frac{sz}{\sqrt{1-z^2}}$, then

$$\varphi(\frac{sz}{\sqrt{1-z^2}}) - \int_{-1}^{1} K(\frac{sz}{\sqrt{1-z^2}}, \frac{sy}{\sqrt{1-y^2}})\varphi(\frac{sy}{\sqrt{1-y^2}})\frac{s}{(1-y^2)^{3/2}}dy = f(\frac{sz}{\sqrt{1-z^2}}),$$

and let $u(z) = \varphi(\frac{sz}{\sqrt{1-z^2}})$, so we obtain

$$u(z) - \int_{-1}^{1} K(\frac{sz}{\sqrt{1-z^2}}, \frac{sy}{\sqrt{1-y^2}})u(y)\frac{s}{(1-y^2)^{3/2}}dy = f(\frac{sz}{\sqrt{1-z^2}}), \quad (3.7)$$

by setting

$$M(z,y) = \frac{K(\frac{sz}{\sqrt{1-z^2}}, \frac{sy}{\sqrt{1-y^2}})}{1-y^2},$$

and

$$F(z) = f(\frac{sz}{\sqrt{1-z^2}}),$$

equation (3.7) can be written as follows

$$u(z) - s\int_{-1}^{1} M(z,y)u(y)\omega(y)dy = F(z). \tag{3.8}$$

Therefore, in both cases, the problem is to solve approximately the mapped integral equation (3.8) which can be written in the operator form

$$(I - s\mathcal{K})u = F. \tag{3.9}$$

To do this, we assume that the operator $\mathcal{K}$ is compact on the space $L^2_\omega(-1,1)$, and let $\mathbb{P}_n$ the $n+1$-dimensional subspace spanned by the Chebyshev polynomials $T_0, \cdots, T_n$. Let $\mathcal{P}_n : L^2_\omega(-1,1) \longrightarrow \mathbb{P}_n$ be a bounded projection operator. Our motivation is to approximate (3.8) by attempting to solve the problem

$$(I - s\mathcal{P}_n\mathcal{K})u_n = \mathcal{P}_n F, \quad u_n \in \mathbb{P}_n. \tag{3.10}$$

Using Gauss-Chebyshev quadrature formula given by Theorem 2.19 to approximate the integral part of the equation (3.8) gives

$$\tilde{u}(z) - \frac{s\pi}{n+1}\sum_{i=0}^{n} M(z,y_i)\tilde{u}(y_i) = F(z). \tag{3.11}$$

Now, forcing this semi-discrete equation to be almost exact in the sense that the residual

$$r_n(z) = \tilde{u}(z) - \frac{s\pi}{n+1}\sum_{i=0}^{n} M(z,y_i)\tilde{u}(y_i) - F(z), \tag{3.12}$$

is zero at collocation points $z_j, \quad j = 0, ..., n$. Thus, the condition $r(z_j) = 0$, for j from 0 to n, lead to the following system of linear equations

$$\tilde{u}(z_j) - \frac{s\pi}{n+1}\sum_{i=0}^{n} M(z_j,y_i)\tilde{u}(y_i) = F(z_j). \tag{3.13}$$

Therefore, using the Chebyshev polynomial approximation formula given by Theorem 2.21 then

$$\mathcal{P}_n\tilde{u}(z) = \sum_{i=0}^{n} c_i T_i(z), \tag{3.14}$$

where

$$c_0 \approx \frac{1}{n+1}\sum_{i=0}^{n} \tilde{u}(z_i), \tag{3.15}$$

$$c_j \approx \frac{2}{n+1}\sum_{i=0}^{n} \tilde{u}(z_i)T_j(z_i), \quad j = 1, \ldots, n. \tag{3.16}$$

Consequently, the approximate solution of equation (3.1) in the real line is given by

$$\varphi_n(x) = \sum_{i=0}^{n} c_i T_i \left(\tanh(s^{-1}x)\right), \tag{3.17}$$

using Logarithmic map, and

$$\varphi_n(x) = \sum_{i=0}^{n} c_i T_i \left(x/\sqrt{x^2+s^2}\right), \tag{3.18}$$

in the case of algebraic map.

3.1.3 Spectral Convergence Analysis

In this subsection, a convergence analysis for the numerical schemes for the mapped integral equation (3.8) will be provided. The goal is to show that the rate of convergence depends on the regularity properties of the corresponding exact solution. Also, we note that the following results are based on framework in [1, 5].

Theorem 3.1 (see [1], p. 55)**.** *Assume $\mathcal{K} : L^2_\omega(-1,1) \longrightarrow L^2_\omega(-1,1)$ is bounded, and assume $I - s\mathcal{K}$ is one-to-one and onto operator. Further assume*

$$\|\mathcal{K} - \mathcal{P}_n\mathcal{K}\| \to 0 \text{ as } n \to \infty. \tag{3.19}$$

Then for all sufficiently large n, the operator $(I - s\mathcal{P}_n\mathcal{K})^{-1}$ exists as a bounded operator. Moreover, it is uniformly bounded:

$$\sup_n \|(I - s\mathcal{P}_n\mathcal{K})^{-1}\| \le M. \tag{3.20}$$

For the solution of (3.9) and (3.10),

$$\|u - u_n\| \le s^{-1}M\|u - \mathcal{P}_n u\|. \tag{3.21}$$

Lemma 3.2. *Assume* $\mathcal{K} : L^2_\omega(-1,1) \longrightarrow L^2_\omega(-1,1)$ *is a compact operator and assume* $\mathcal{P}_n u \to u$ *for all sufficiently large* n*. Then,*

$$\|\mathcal{K} - \mathcal{P}_n\mathcal{K}\| \to 0. \tag{3.22}$$

Proof. From the definition of operator norm,

$$\|\mathcal{K} - \mathcal{P}_n\mathcal{K}\| = \sup_{\|u\|\leq 1} \|\mathcal{K}u - \mathcal{P}_n\mathcal{K}u\| = \sup_{v\in\mathcal{K}(U)} \|v - \mathcal{P}_n v\|,$$

with $\mathcal{K}(U) = \{\mathcal{K}u \quad | \quad \|u\| \leq 1\}$. Since the set $\mathcal{K}(U)$ is compact. Therefore, by the Lemma 2.22,

$$\sup_{v\in\mathcal{K}(U)} \|v - \mathcal{P}_n v\| \to 0,$$

for all sufficiently large n. □

3.2 Illustrative Examples

Example 3.3. *Consider Fredholm integral equation (3.1) with*

$$K(x,t) = x^2 t, \quad f(x) = xe^{-x^2} - \frac{\sqrt{\pi}}{2}x^2.$$

Whose exact solution is $\varphi(x) = xe^{-x^2}$, which is a smooth function and decay exponentially at infinity. In Figure 3.1, a. we comparison of the solution determined by the approximate solution using Logarithmic map, b. we give maximum absolute errors at 1000 selected equally spaced points on the interval [-200, 200] using logarithmic map against various N and different scaling factor s.

a. Numerical vs. exact b. Maximum errors

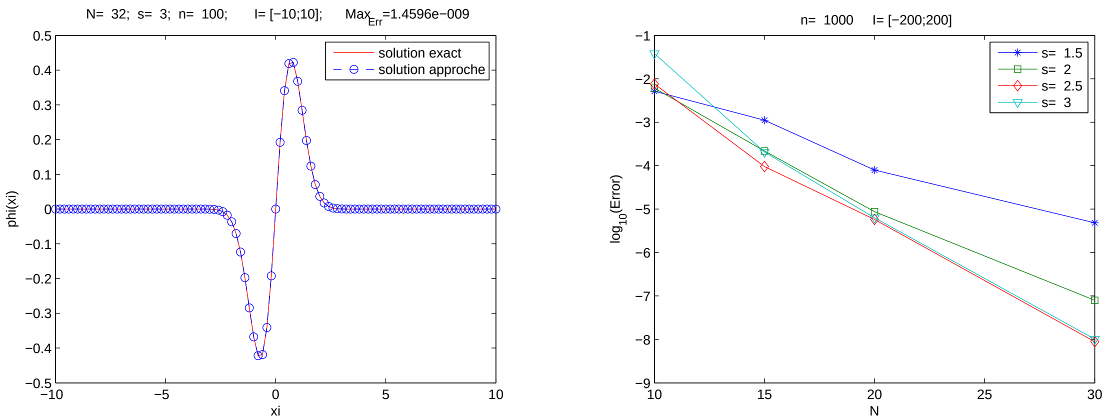

Figure 3.1: a. Comparison of the solution exact by the approximate solution using Logarithmic map. b. Relative factor s versus the degree of approximate solution N.

Example 3.4. *Consider Fredholm integral equation (3.1) with*

$$K(x,t) = \sin(xt), \quad f(x) = \sin(\frac{1}{1+x^2}).$$

Whose exact solution is $\varphi(x) = \sin(\frac{1}{1+x^2})$, which is a smooth function and decay exponentially at infinity. In Figure 3.2, a. we comparison of the solution determined by the approximate solution using algebraic map, b. we give maximum absolute errors at 1000 selected equally spaced points on the interval [-250, 250] using algebraic map against various N and different scaling factor s.

a. Numerical vs. exact b. Maximum errors

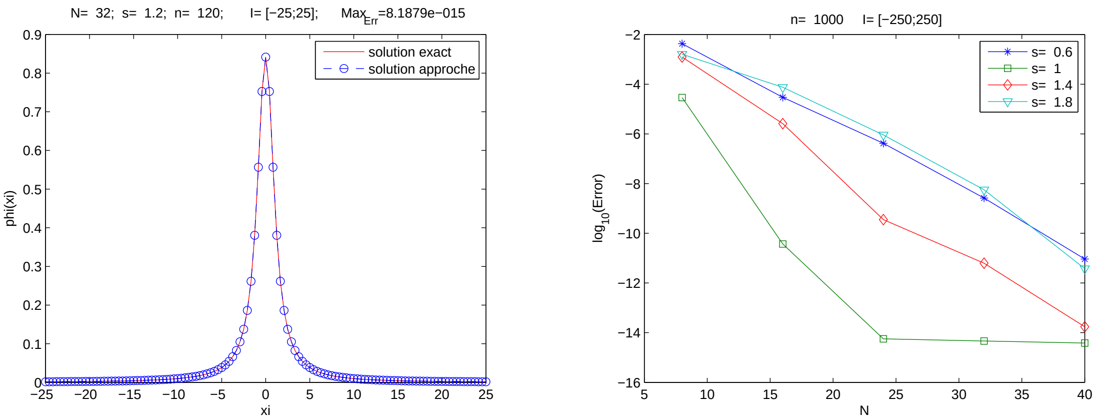

Figure 3.2: a. Comparison of the solution exact by the approximate solution using Algebraic map. b. Relative factor s versus the degree of approximate solution N.

3.3 Comparison of Results

Example 3.5. *Consider Fredholm integral equation (3.1) with*

$$K(x,t) = \cos(x-t), \quad f(x) = e^{-x^2} - \sqrt{\pi}e^{-0.25}\cos(x).$$

Where the exact solution is $\varphi(x) = e^{-x^2}$. In Table (3.2) we give maximum absolute errors at 1000 selected equally spaced points on the interval $[-500, 500]$, using algebraic map and logarithmic map against various N and different scaling factor s. Table (3.1) shows the results of the previous methods and another method Card. Func. which will be explained in the near future.

Table 3.1: Maximum absolute error at 50, 100 or 150 equally spaced points on the interval $[-100, 100]$ for example 3.5.

interval	N	Log.map	Alg.map	Card.func.
$[-100, 100]$	50	1.54E-09	6.95E-06	4.32E-006
	100	1.71E-09	1.31E-09	8.50E-011
	150	2.10E-09	1.31E-09	8.50E-011

Table 3.2: Maximum absolute error at 500 equally spaced points on the interval $[-100, 100]$ for example 3.5.

		s			
Mapping	N	1	2.5	5	7.5
Log.map	20	4.30E-04	2.86E-06	8.48E-03	1.19E-01
	30	9.65E-05	2.92E-09	1.21E-05	8.98E-03
	40	1.68E-05	3.81E-11	3.96E-10	2.58E-04
	50	2.41E-06	5.61E-13	1.62E-13	2.34E-06
	60	1.90E-07	3.94E-14	2.89E-15	4.60E-09
Alg.map	20	9.37E-04	5.82E-05	6.95E-03	1.19E-01
	30	5.44E-05	5.59E-07	1.04E-05	8.34E-03
	40	9.79E-06	2.20E-08	1.78E-08	1.79E-04
	50	1.44E-06	6.66E-10	1.02E-10	4.33E-07
	60	1.90E-07	5.22E-11	1.07E-12	5.23E-10

Example 3.6. *Consider Fredholm integral equation (3.1) with*

$$K(x,t) = x^2t^2, \quad f(x) = \cos(x)e^{-x^2} - \frac{\sqrt{\pi}}{4}e^{-0.25}x^2.$$

The corresponding exact solution is $\varphi(x) = \cos(x)e^{-x^2}$. In Table (3.3) we give maximum absolute errors at 200 selected equally spaced points on the interval $[-50, 50]$, using algebraic map and logarithmic map against various N and different scaling factor s. Table (3.4) shows the results of the previous methods and another method Card. Func. we give maximum absolute errors at 50, 100 and 200 selected equally spaced points on the interval $[-100, 100]$.

Table 3.3: Maximum absolute error at 200 equally spaced points on the interval $[-50, 50]$ for example 3.6.

		s					
Methods	N	1	2	3	4	5	6
Log.map	20	1.88E-04	6.95E-06	4.32E-06	9.70E-04	1.17E-02	3.71E-02
	40	2.83E-05	1.31E-09	8.50E-11	1.22E-10	9.95E-09	8.36E-06
	60	8.02E-07	1.19E-11	8.38E-15	7.81E-15	1.81E-14	3.29E-13
	80	4.41E-07	8.60E-14	4.66E-15	7.74E-15	1.95E-14	2.87E-14
Alg.map	20	1.43E-03	1.03E-04	1.55E-04	4.88E-04	1.10E-02	4.08E-02
	40	1.03E-05	2.75E-07	4.30E-08	3.27E-08	1.60E-07	1.53E-06
	60	5.27E-07	1.05E-09	3.46E-11	7.32E-12	8.31E-12	3.64E-11
	80	4.51E-08	1.41E-11	2.38E-13	8.21E-14	1.07E-13	1.35E-13

Table 3.4: Maximum absolute error at 50, 100 and 150 equally spaced points on the interval $[-100, 100]$ for example 3.6.

Interval	N	Log.map	Alg.map	Card.func.
	50	1.54E-09	6.95E-06	4.32E-06
$[-100, 100]$	100	1.71E-09	1.31E-09	8.50E-11
	150	2.10E-09	1.31E-09	8.50E-11

Example 3.7. *Consider Fredholm integral equation (3.1) with*

$$K(x,t) = x^2t^2, \quad f(x) = \frac{1}{1+x^2} - \pi x^2.$$

The corresponding exact solution is $\varphi(x) = \frac{1}{1+x^2}$. Table (3.1) shows the results of the previous methods and another method Card.func. we give maximum absolute errors at 40, 100 and 160 selected equally spaced points on the interval $[-100, 100]$.

Table 3.5: Maximum absolute error at 40, 100 and 160 equally spaced points on the interval $[-100, 100]$ for example 3.7.

Interval	N	Log.map	Alg.map	Card.func.
	40	1.54E-09	6.95E-06	4.34E-04
$[-100, 100]$	100	1.71E-09	1.31E-09	4.65E-04
	160	2.10E-09	1.31E-09	4.72E-04

Conclusions and Future Lines of Research

The integral equations of the type (1) are difficult to study for two reasons, the first reason is , from one side, the singularity, which is the type of the interval of integration is infinite, while the second reason is the lack or the absence of the theories of this type. Two methods were found to solve these equations, which we have explained, given and have compared in Chapter 3.
The proposed methods provides a good efficiency for smooth solutions decaying exponentially or algebraically at infinity as shown in Tables (3.2)–(3.3). The scaling factor s offers great flexibility to improve the numerical resolution. However, the best choice of s is one that gives a good adjustment of the collocation points.

The Card.func method is quadrature which is in preparation. This leads us to ask the following questions:

(i) What is the optimal value of the scaling factor s?

(ii) Are there other methods to solve this type of equation?

A part of these questions will discuss in a future work.

Bibliography

[1] K.E. Atkinson,The Numerical Solution of Integral Equations of the Second Kind. Cambridge University Press, 1997.

[2] J.P. Boyd, Chebyshev and Fourier Spectral Methods. Dover Publication Inc., Mineola, NY, 2d edition, 2001.

[3] J.P. Boyd, Spectral Methods Using Rational Basis Functions on an Infinite Interval. J. Comp. Phys. 69(1987), 112–142.

[4] J.P. Boyd, The Optimisation of Convergence for Chebyshev Polynomial Methods in an Unbounded Domain. J. Comp. Phys. 45(1982), 43–79.

[5] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, Spectral Methods Fundamentals in Single Domains, Springer-Verlag Berlin Heidelberg, 2006.

[6] P. Cerone, On the effects of the generalised Renewal integral equation model of population dynamics. Genus 52(1996), 53–70.

[7] S.N. Chandler-Wilde, M. Rahman and C.R. Ross, A fast two-grid and finite section method for a class of integral equations on the real line with application to an acoustic scattering problem in the half-plane. Numer. Math. 93(2002), 1–51.

[8] S.N. Chandler-Wilde and B. Zhang, On the Solvability of a Class of Second Kind Integral Equations on Unbounded Domains. J. Math. Anal. Appl. 214(1997), 482–502.

[9] S.N. Chandler-Wilde, B. Zhang and C.R. Ross, On the Solvability of Second Kind Integral Equations on the Real line. J. Math. Anal. Appl. 245(2000), 28–51.

[10] S. Chandrasekhar, Radiative transfer, Dover publications, Inc, New York, 1960.

[11] Y. Chen and T. Tang, Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel. Math. Comput. 79(2010), 147–167.

[12] W.C. Chew, M.S. Tong and B. Hu, Recent Advances in Integral Equation Solvers in Electromagnetics Synthesis Lectures on Computati, Morgan and Claypool Publishers series 2009.

[13] B. Engquist and A. Majda. Absorbing boundary conditions for numerical simulation of waves. Proc. Nat. Acad. Sci. U.S.A., 74(1977), 1765–1766.

[14] R. Estrada and R. Kanwal, Singular Integral Equations, Birkhauser, Berlin, 2000.

[15] D. Givoli. Numerical Methods for Problems in Infinite Domains. Elsevier, Amsterdam, 1992.

[16] D. Givoli and J.B. Keller. A finite element method for large domains. Comput.Methods Appl. Mech. Engrg., 76(1989), 41–66.

[17] C.E. Grosch and S.A. Orszag, Numerical Solution of Problems in Unbounded Regions. J. Comp. Phys. 25(1977), 273–295.

[18] A. Guechi and A. Rahmoune, Mapped Chebyshev Spectral Methods for Solving Second Kind Integral Equations on the Real Line, AJMAA, 14(2017), 1–8.

[19] Y. Guo and C.L. Xu, Hermite pseudospectral method for nonlinear partial differential equations. Math. Model. Numer. Anal. 34(2000), 859–872.

[20] J.H. He, Some Asymptotic Methods for Strongly Nonlinear Equations, International Journal of Modern Physics B, 20(2006), 1141–1199.

[21] F. Jedrzejewski, Introduction aux méthodes numériques. Springer-Verlag, France, Deuxième édition, 2005.

[22] R.P. Kanwal, Linear Integral Equations, Birkhauser, Boston, 1997.

[23] D.A. Kay, M. Sagheer and Q. Tang, Mathematical analysis of an integral equation arising from population dynamics. Mathematical Bioscieces, 210(2007), 415–435.

[24] R. Kress, Linear Integral Equations. Springer-Verlag, New York, 3rd ed, 2014.

[25] P.K. Kythe, Michael R. Schäferkotter, Handbook of computational methods for integration, CRC Press, 2005.

[26] B.N. Mandal and A. Chakrabarti, Applied Singular Integral Equations, CRC Press, 2011

[27] T. Matsushima and P.S. Marcus, A Spectral Method for Unbounded Domains. J. Comput. Phys. 137(1997), 321–345.

[28] G. Panasenko, B. Rutly and O. Titaud, Asymptotic analysis of integral equations for a great interval and its application to stellar radiative transfer, C.R. Mecanique 330(2002), 735–740.

[29] A.D. Polyanin and A.V. Manzhirov, Handbook of Mathematics for Engineers and Scientists, Chapman and Hall/CRC Press, Boca Raton, 2007.

[30] M. Rahman, Integral Equations and their Applications, Southampton, Britain, 2007.

[31] A. Rahmoune, Spectral Collocation Method for Solving Fredholm Integral Equations on the Half-Line. J. App. Math. Comp., 219(2013), 9254–9260.

[32] A. Rahmoune, Sur la Résolution Numérique des Équations Intégrales en Utilisant des Fonctions Spéciales, Thesis of Doctorate, University of Batna, 2011.

[33] H. Schippers, Application of multigrid methods for integral equations to two problems from fluid dynamics, J. Comput. Phys. 48(1982), 441–461.

[34] J. Shen, Stable and efficient spectral methods in unbounded domains using Laguerre functions. SIAM J. Numer. Anal. 38(2000), 1113–1133.

[35] J. Shen, T. Tang and L-L. Wang, Approximations by orthonormal mapped Chebyshev functions for higher-dimensional problems in unbounded domains, J. App. Math. Comp., 265(2014), 264–275.

[36] J. Shen, T. Tang and L.-L. Wang, Spectral Methods. Algorithms, Analysis and Applications. Springer-Verlag Berlin Heidelberg, 2011.

[37] J. Shen and L.-L. Wang. Analysis of a spectral-Galerkin approximation to the Helmholtz equation in exterior domains. SIAM J. Numer. Anal., 45(2007), 1954–1978.

[38] J. Shen and L.-L. Wang. Some Recent Advances on Spectral Methods for Unbounded Domains. J. Comput. Phys, 5(2009), 195–241.

[39] L.N. Trefethen, Spectral Methods in Matlab. SIAM., Philadelphia, 2000.

[40] A.M. Wazwaz, A First Course in Integral Equations, World Scientific, 2005.

ملخص

في هذه الأطروحة قمنا بدراسة المعادلة التكاملية الشاذة، التي لها الشكل العام

$$\varphi(x) - \int_{-\infty}^{\infty} K(x,y)\,\varphi(y)dy = f(x), \qquad x \in \mathbb{R}$$

الهدف من هذه الأطروحة هو إيجاد حل هذه المعادلة، فالفكرة هي تغيير المجال من غير محدود إلى مجال محدود، ثمّ استخدام الطرق الطيفية لتشيبيشيف من أجل حل المعادلة التكاملية في المجال المحدود.

الكلمات المفتاحية: المعادلات التكاملية، كثيرات حدود تشيبيشيف، تكامل غير محدود، والطرق الطيفية.

Abstract

In this thesis, we have studied the singular integral equation, which has the general form

$$\varphi(x) - \int_{-\infty}^{\infty} K(x,y)\,\varphi(y)dy = f(x), \qquad x \in \mathbb{R}.$$

The aim of this thesis is to provide the solution of this equation. The main idea is to change the unbounded interval to a bounded one and use the Chebyshev spectral methods to solve the integral equation in the finite interval.

Keywords: *Integral equations, Chebyshev polynomials, Infinite integral, Spectral methods.*

Résumé

Dans cette thèse nous avons étudié l'équation intégrale singulière, qui a la forme générale

$$\varphi(x) - \int_{-\infty}^{\infty} K(x,y)\,\varphi(y)dy = f(x), \qquad x \in \mathbb{R}.$$

Le but de cette thèse est trouvé la solution de cette équation. L'idée principale est de changé l'intervalle non borné au intervalle borné et d'utiliser les méthodes spectrale de Tchebychev pour résoudre l'équation intégrale dans l'intervalle fini.

Mots-clés: *Equations intégrale, polynômes de Tchebychev, intégrale infinie, méthodes spectrale.*