Acknowledgement

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Introduction

The nonlinear differential systems establish a very important branch of applied mathematics, they represent a very important tool for studying a various natural phenomena that are employed extensively in natural sciences, engineering, technology, physics, biology, economics, ... In general, most of the nonlinear differential equations cannot be solved analytically. The numerical methods only allw to compute an approximate solution with initial conditions on a finite time interval. However, the qualitative or the geometrical theory of differential equations is being used to analyze differential equations when the explicit solutions of these systems are very difficult to find. These tools are originated by Henri Poincaré's work on differential equations at the end of the nineteenth century [87], which consits of analysing the characteristics of the solutions of this system. This study will help us to obtain information on the behavior of the solutions of many classes of nonlinear differential systems without the need to get their explicit solutions.

Quadratic polynomial differential systems (or simply **QS**) which are the differential systems of the form

$$\begin{aligned} \dot{x} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \dot{y} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2. \end{aligned} \tag{1}$$

Here the dot denotes the derivative with respect to the independent variable t, usually called time. Differential quadratic systems appear frequently in many areas of applied mathematics, electrical circuits, astrophysics, in population dynamics, chemistry, biology, hydrodynamics, etc. Although these differential systems are the simplest nonlinear polynomial systems, they are very important because they serve as a basic for testing the general theory of nonlinear differential systems. In the literature there are hundreds papers published on **QS**.

Its well known that quadratic differential systems contain twelve parameters in their expressions, which makes their study very difficult. For this reason, the authors have studied the global phase portraits of particular classes of these systems. For example, the authors of the papers [18, 20, 41, 58, 59, 39, 97] classified the quadratic centers. For the class of Hamiltonian quadratic systems, see [2, 20, 57]. In [11, 49], authors provided a complete classification in the coefficient space $\mathbb{R}^{12}$ of quadratic systems with a rational first integral and polynomial first integral. More recently, the classification of some families of quadratic systems has been made using more modern methods such as the algebraic and geometric invariants; see, for instance, the classification of the quadratic systems with a weak focus of second order [7], the classification of the quadratic systems with a weak focus and an invariant straight line [8], and the classification of the geometric configurations of singularities for quadratic systems [5, 16, 14, 15].

Recently, in 2019 Benterki and Llibre [26] classified the global phase portraits of **QS** systems having some relevant classic quadratic algebraic curves as invariant algebraic curves, i.e. these curves are formed by orbits of the quadratic polynomial differential system. More precisely, they realized 14 different well-known algebraic curves of degree 4 as invariant curves inside the quadratic polynomial differential systems. These realizations produced 28 topologically different phase portraits in the Poincaré disc for such quadratic polynomial differential systems.

The study of the existence of limit cycles for planar nonlinear differential systems is one of the most important and difficult problems. We recall that this notion was defined by Poincaré [87]. The importance of the study of the existence of limit cycles for differential systems comes from the fact that they model several phenomena, see for instance as [39, 80]. Later, due to physicists, engineers and recently biologists have been interested in studying the maximum number and other properties of limit cycles, see for instance [32, 98].

To determine if there exist a limit cycle for a particular differential equation and to examine the properties of limit cycles, Poincaré introduced an important theoretical concepts like the topographic system method, the successor function, the small parameter method which appeared in his book "New methods in celestial mechanics" and the ring region theorem, and he built many examples to check the impact of these methods. At the same time, he had also noticed the close correlation between studying limit cycles and the solutions of global structural problems of a family of integral curves of differential equations.

Among the most effective tools for obtaining limit cycles is to perturb the periodic orbits of a center. This was studied widely by perturbing the periodic orbits of centers of quadratic polynomial differential systems; see Christopher and Li's book [37], and the references cited therein.

The uses of averaging theory to study the existence of periodic solutions of differential systems has a very long history (see for example Marsden and McCracken [79], Chow and Hale [36] and their references). We know also that the examination of the number and and the assignment of limit cycles for arbitrary systems remains a big problem (see e.g. Ye et al. [98], Christopher and Li [37] and references therein). We will show that among useful techniques for studying the maximal number of limit cycles for a given differential system is the averaging theory. Furthermore, this technique can be used to get the form, stability, and close expressions of limit cycles.

In this thesis we are also interesting in studying piecewise linear differential systems, where the first works studying this kind of differential systems in the plane is due to Andronov, Vitt and Khaikin in [1]. Later on these systems became a topic of great interest in the mathematical community due to their applications for modeling real phenomena, see for instance the books [32, 93] and references there quoted.

As for the general case of planar differential systems one of the main problems for the case of the piecewise differential systems is to determine the existence and the maximum number of crossing limits cycles that these systems can exhibit, that is the version of Hilbert's 16th problem for piecewise differential systems (or simply PWLS) in the plane [53]. Here we consider this problem for the planar discontinuous piecewise linear Hamiltonian systems without equilibrium points and separated by irreducible cubics.

Recently an increasing interest appeared for the piecewise differential systems, mainly due to its applications in engineering, mechanics, electric circuits, see for instance the books of [1, 32, 93] and the hundreds of references therein. A good deal of that interest is placed in studying the limit cycles of these piecewise differential systems. See for instance the papers dedicated to study the limit cycles of the piecewise linear differential systems separated by a straight line or by other kind of curves. See, without trying to be exhaustive, for instance [4, 34, 43, 45, 46, 47, 52, 64, 67, 65, 70, 83].

This work consists of four chapters, where in the first Chapter we briefly present some of the basic concepts, definitions and results used through this work.

In the second chapter, we classify the global phase portraits of quadratic polynomial differential systems exhibiting some relevant classic cubic curves as invariant algebraic ones, i.e. these curves are formed by orbits of the quadratic polynomial differential system. More precisely, we realize 5 different well-known algebraic curves of degree 3 as invariant curves inside the quadratic polynomial differential systems. These realizations produce 29 topologically different phase portraits in the Poincaré disc for such quadratic polynomial differential systems.

In Chapter 3 we classify the global phase portraits of the polynomial differential systems

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + ax^8 + bx^4y^4 + cy^8. \end{aligned} \tag{2}$$

The first main objective of this work is to study the phase portraits on the Poincaré disc of the differential systems (2). The second objectif, to solve the second part of 16th Hilbert problem for these classes where we give the optimal upper bounds of the number of limit cycles of these systems.

And finally, in the fourth chapter we solve the second part of 16–th Hilbert problem for a discontinuous piecewise linear Hamiltonian systems (PLHS for short) in $\mathbb{R}^2$ separated by an irreducible cubics.

Chapter 1

Some Basic Concepts on Qualitative Theory of Ordinary Differential Equations

In this chapter we present some basic results on the qualitative theory of ordinary differential equations. It is also a question of citing some bibliographical references for the reader who wishes to have a more complete presentation on the subject.

First, we discuss some defnitions about dynamical systems such as: singular points, stability of singular points, phase portraits, invariant algebraic curves, limit cycles, planar discontinuous vector fields, ...

We also describe the averaging method, which consists of searching for number of limit cycles that appear after the pertarbation of a differential system.

Section 1.1 Vector Fields and Flows

Definition 1.1 (Vector fields)
Let $\mathcal{D}$ be an open subset in $\mathbb{R}^n$. We define a vector field of class C^r on $\mathcal{D}$ as a C^r map $\mathcal{X} : \mathcal{D} \longrightarrow \mathbb{R}^n$, where $\mathcal{X}(x)$ is meant to represent the free part of a vector attached at the point $x \in \mathcal{D}$. Here the r of C^r denotes a positive integer or $+\infty$.

The graphical representation of a vector field on the plane consists in drawing a number of well chosen vectors $(x, \mathcal{X}(x))$ as in Figure 1.1. [42]

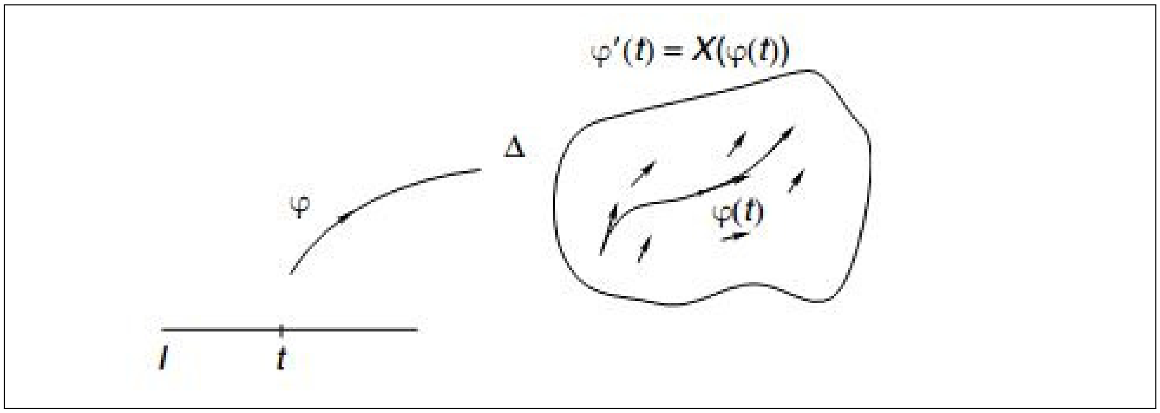


Figure 1.1: Vector fields.

Remark 1 *Integrating a vector field means that we look for curves $x(t)$, with t belonging to some interval in $\mathbb{R}$, that are solutions of the differential equation*

$$\dot{x} = \mathcal{X}(x), \tag{1.1}$$

where $x \in \mathcal{D}$, and $\dot{x}$ denotes dx/dt. The variables x and t are called the dependent variable and the independent variable of the differential equation (1.1), respectively. Usually t is also called the time.

Since $\mathcal{X} = \mathcal{X}(x)$ does not depend on t, we say that the differential equation (1.1) is autonomous.

Definition 1.2 (Flow)
Let $\mathcal{X}$ be the vector feld defned in (1.1). The flow is a $\mathcal{C}^1$ function

$$\varphi : \mathbb{R} \times \mathcal{D} \to \mathcal{D},$$

where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ and if $\varphi_t(x) = \phi(t,x)$, then φ_t satisfies

(i) $\varphi_0(x) = x \quad \forall x \in \mathcal{D}$,

(ii) $\dfrac{d\varphi}{dt}(x) = \mathcal{X}(\varphi_t(x))$,

(iii) $\varphi_t \circ \varphi_s(x) = \varphi_{t+s}(x) \quad \forall s,t \in \mathbb{R}$ *and* $x \in \mathcal{D}$. *[86]*

Remark 2 *The same properties preserve for a linear system have the flow $\phi_t = e^{At}$ defined from $\mathbb{R}^n$ to $\mathbb{R}^n$.*

Section 1.2 Singular Points

Let $f : \mathcal{D} \to \mathbb{R}^n$ be a continuous function in an open set $\mathcal{D} \subset \mathbb{R}^n$ and consider the autonomous equation

$$\dot{x} = f(x). \tag{1.2}$$

The set $\mathcal{D}$ is called the phase space of the differntial equation (1.2).

Definition 1.3 (Singular points)
A point $x_0 \in \mathbb{R}^n$ is called a singular point (or an equilibrium point) of (1.2) if $f(x_0) = 0$. If a singular point has a neighborhood that does not contain any other singular points, then that singular point is called an isolated equilibrium point. [44]

Remark 3 *A singular point can also be called either a singularity, or critical point, or an equilibrium point. In the chapter two, three and four of this thesis we use the denomination "singular point".*

1.2.1 Types of singular points

Definition 1.4 *Let p be a singular point of a planar C^r vector field $\mathcal{X} = (P, Q)$. In general the study of the local behavior of the flow near p is quite complicated. Already the linear systems show different classes, even for local topological equivalence.*

We say that

$$D\mathcal{X}(p) = \begin{pmatrix} \frac{\partial P}{\partial x}(p) & \frac{\partial P}{\partial y}(p) \\ \frac{\partial Q}{\partial x}(p) & \frac{\partial Q}{\partial y}(p) \end{pmatrix},$$

is the linear part of the vector field $\mathcal{X}$ at the singular point p. [42]

We classify the singular points of a planar differential system in *hyperbolic*, *semi-hyperbolic*, *nilpotent* and *linearly zero*.

The *hyperbolic* ones are the singular points such that its linear part of the differential system has eigenvalues with nonzero real part see for instance Theorem 2.15 of [42] for the classification of their local phase portrait.

The *semi-hyperbolic* are the ones having a unique eigenvalue equal to zero, their phase portraits are characterized in Theorem 2.19 of [42]. The *nilpotent* singular points have both eigenvalues zero but their linear part is not identically zero.

Finally the *linearly zero* singular points are the ones such that their linear part is identically zero, and their local phase portraits must be studied using the change of variables called Blow-ups, see for instance chapter 2 and 3 of [42].

1.2.2 Stability of singular points

The behavior of solutions in the neighbourhood towards or away from the fxed points as t goes to infnity is what stability of singular points means.

Definition 1.5 (Stable singular point)
Let φ_t denote the flow of the differential equation (1.1) defined for all $t \in \mathbb{R}$. An equilibrium point x_0 of (1.1) is stable if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \|\varphi_t - x_0\| < \delta \Longrightarrow \forall t > 0 :\ \|\varphi_t(x) - x_0\| < \varepsilon. \quad [86]$$

Definition 1.6 (Asymptotically stable singular point)
An equilibrium point x_0 of (1.1) is asymptotically stable if it is stable and

$$\lim_{t\to\infty} \|\varphi_t(x) - x_0\| = 0. \quad [86]$$

Remark 4 *The singular point x_0 is unstable if it is not stable.*

Section 1.3 Phase Portrait of a Vector Fields

Although it is often impossible (or very difficult) to determine explicitly the solutions of a differential equation, it is still important to obtain information about these solutions, at least of qualitative nature. To a considerable extent, this can be done describing the phase portrait of the differential equation.

Definition 1.7 (Orbits)
Let $\varphi : I_{x_0} \to \mathcal{D}$ be a maximal solution; it can be regular or constant. Its image $\gamma_\varphi = \{\varphi(t) : t \in I_{x_0}\} \subset \mathcal{D}$, endowed with the orientation induced by φ, in case φ is regular, is called the trajectory, orbit or (maximal) integral curve associated to the maximal solution φ. [42]

We recall that for a solution defining an integral curve the tangent vector $\dot{\varphi}(t)$ at $\varphi(t)$ coincides with the value of the vector field $\mathcal{X}$ at the point $\varphi(t)$; see Fig 1.1.

Definition 1.8 *A phase portraits of the vector field given by (1.2) is the set of all trajectories (solution or orbits) of the system. [94]*

Exemple 1.1 *Let us construct the phase portrait for the undamped pendulum*

$$\ddot{x} + \sin x = 0, \tag{1.3}$$

Equation (1.3) can be written as a Newtonian system $\dot{x} = y, \quad \dot{y} = -\sin x$.

The differential equation for the phase portraits is

$$\frac{dy}{dx} = -\frac{\sin x}{x}. \tag{1.4}$$

This equation is separable, leading to

$$\int y dy = -\int \sin x dx,$$

or $\frac{1}{2}y^2 = \cos x + C$, *where* C *is the parameter of the phase portraits. Therefore the equation of the phase portraits is*

$$y(x) = \pm\sqrt{2(\cos x + C)}.$$

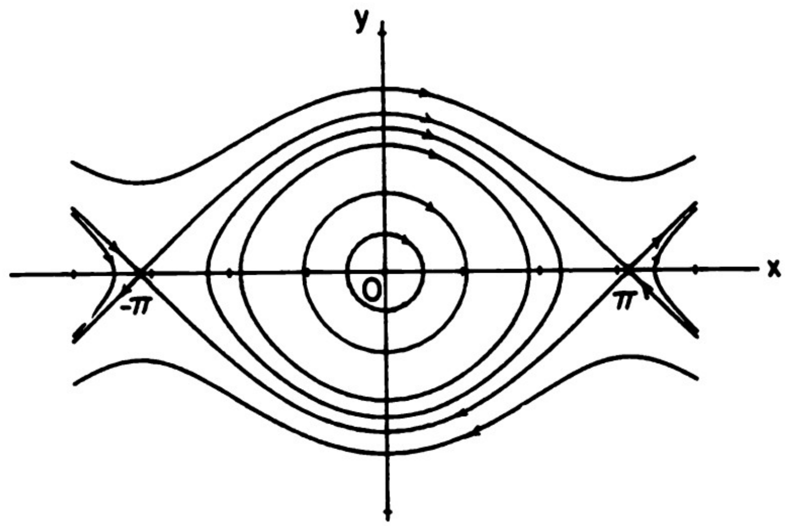


Figure 1.2: The phase portrait for the undamped pendulum.

Definition 1.9 (Periodic solutions)
A solution $\varphi(t,x)$ *of* (1.2) *is periodic if there exists a finite time* $T > 0$ *such that* $\varphi(t+T,x) = \varphi(t,x)$ *for all* $t \in \mathbb{R}$. *The minimal* T *for which the solution* $\varphi(t,x)$ *of* (1.2) *is periodic is called the period. [94]*

Definition 1.10 (Limit cycle)
A limit cycle of a planar vector feld given by (1.1) *is an isolated periodic trajectory. In other word, a periodic trajectory of a vector field is a limit cycles, if it has annular neighborhood free from other periodic trajectories. [94]*

Section 1.4 Invariant Curve and First Integral

The aim of this section is to introduce the terminology of the notions of invariant curve and the first integral for real planar polynomial differential systems. For a detailed discussion of this theory see [42].

We consider the system

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y). \end{cases} \tag{1.5}$$

Definition 1.11 (Invariant curve)
Let $f \in \mathbb{C}[x,y]$. We say taht the algebraic curve $f(x,y) = 0$ is an invariant algebraic curve of systems (1.5), if there exists $K \in \mathbb{C}[x,y]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial x} = Kf.$$

The polynomial K is called the cofactor of the invariant algebraic curve. [90]

Exemple 1.2 *The algebraic cubic **Cubical Hyperbola** curve given by $f(x,y) = xy^2 - (x+1)^2 = 0$ is an invariant algebraic curve of the quadratic system*

$$\dot{x} = ax + ax^2 - 2xy,$$
$$\dot{y} = -2 - 2x - \frac{ay}{2} + \frac{axy}{2} + y^2.$$

So, its associated cofactor $K(x,y) = 2ax$.

Definition 1.12 (First integral)
Let U be an open subset of $\mathbb{R}^n$. A non–locally constant function $H : U \to \mathbb{R}$ is a first integral of the differential system (1.5) if H is constant on the orbits of (1.5) contained in

U, i.e.

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}(x,y)P(x,y) + \frac{\partial H}{\partial y}(x,y)Q(x,y) = 0, \quad \text{in the points } (x,y) \in U.$$

We say that a quadratic system is integrable if it has a first integral $H : U \to \mathbb{R}$. [71]

Exemple 1.3 *The algebraic cubic **Egg** curve given by $H(x,y) = x^2 + (1+x)y^2 - 1 = 0$ is a first integral of the quadratic system*

$$\dot{x} = -1 + x^2 + ay + axy,$$
$$\dot{y} = \frac{a}{2} + (b - \frac{a}{2})x + \frac{y}{2} + \frac{xy}{2} + by^2.$$

So, its associated cofactor $K(x,y) = 0$.

Section 1.5 Hamiltonian Systems

In this part we study one interesting type of system which arise in physical problems.

Definition 1.13 *Let $\mathcal{D}$ be an open subset of $\mathbb{R}^{2n}$ and let $H \in \mathcal{C}^2(\mathcal{D})$ where $H = H(x,\ y)$ with $x,\ y \in \mathbb{R}$. A system of the form*

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}, \\ \dot{y} &= -\frac{\partial H}{\partial x}, \end{aligned} \tag{1.6}$$

where

$$\frac{\partial H}{\partial x} = \left(\frac{\partial H}{\partial x_1}, \cdots, \frac{\partial H}{\partial x_n}\right)^T \quad \text{and} \quad \frac{\partial H}{\partial y} = \left(\frac{\partial H}{\partial y_1}, \cdots, \frac{\partial H}{\partial y_n}\right)^T$$

is called a Hamiltonian system with n degrees of freedom on $\mathcal{D}$. [86]

Exemple 1.4 *(Undamped Harmonic Oscillator) Recall that this system is given by*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -kx, \end{aligned} \tag{1.7}$$

where $k > 0$. *A Hamiltonian function for this system is*

$$H(x,y) = \frac{1}{2}y^2 + \frac{k}{2}x^2.$$

Section 1.6 Poincaré Compactification.

In this subsection we give some basic results which are necessary for studying the behavior of the trajectories of a planar polynomial differential systems near infinity. Let $\mathcal{X}(x,y) = (P(x,y), Q(x,y))$ represent a vector field to each system which we are going to study its phase portraits, then for doing this we use the so called a Poincaré compactification. We consider the Poincaré sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$, and we define the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ (with $T_{(0,0,1)}\mathbb{S}^2$ the tangent space of $\mathbb{S}^2$ at the point $(0,0,1)$), such that for each point $q \in T_{(0,0,1)}\mathbb{S}^2$, $T_{(0,0,1)}\mathbb{S}^2(q)$ associates the two intersection points of the straight line which connects the point q and $(0,0)$). The equator $\mathbb{S}^1 = \{(x,y,z) \in \mathbb{S}^2 : z = 0\}$ represent the infinity points of $\mathbb{R}^2$. In summary we get a vector field $\mathcal{X}'$ defined in $\mathbb{S}^2 \setminus \mathbb{S}^1$, which is formed by to symmetric copies of $\mathcal{X}$, and we prolong it to a vector field $p(\mathcal{X})$ on $\mathbb{S}^2$. By studiying the dynamics of $p(\mathcal{X})$ near $\mathbb{S}^1$ we get the dynamics of $\mathcal{X}$ at infinity. We need to do the calculations on the Poincaré sphere near the local charts $U_i = \{Y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{Y \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1,2,3$; with the associated diffeomorphisms $F_i : U_i \longrightarrow \mathbb{R}^2$ and $G_i : V_i \longrightarrow \mathbb{R}^2$ for $i = 1,2,3$. After a rescaling in the independent variable in the local chart (U_1, F_1) the expression of $p(\mathcal{X})$ is

$$\dot{u} = v^n\left[-uP\left(\frac{1}{v},\frac{u}{v}\right) + Q\left(\frac{1}{v},\frac{u}{v}\right)\right], \quad \dot{v} = -v^{n+1}P\left(\frac{1}{v},\frac{u}{v}\right);$$

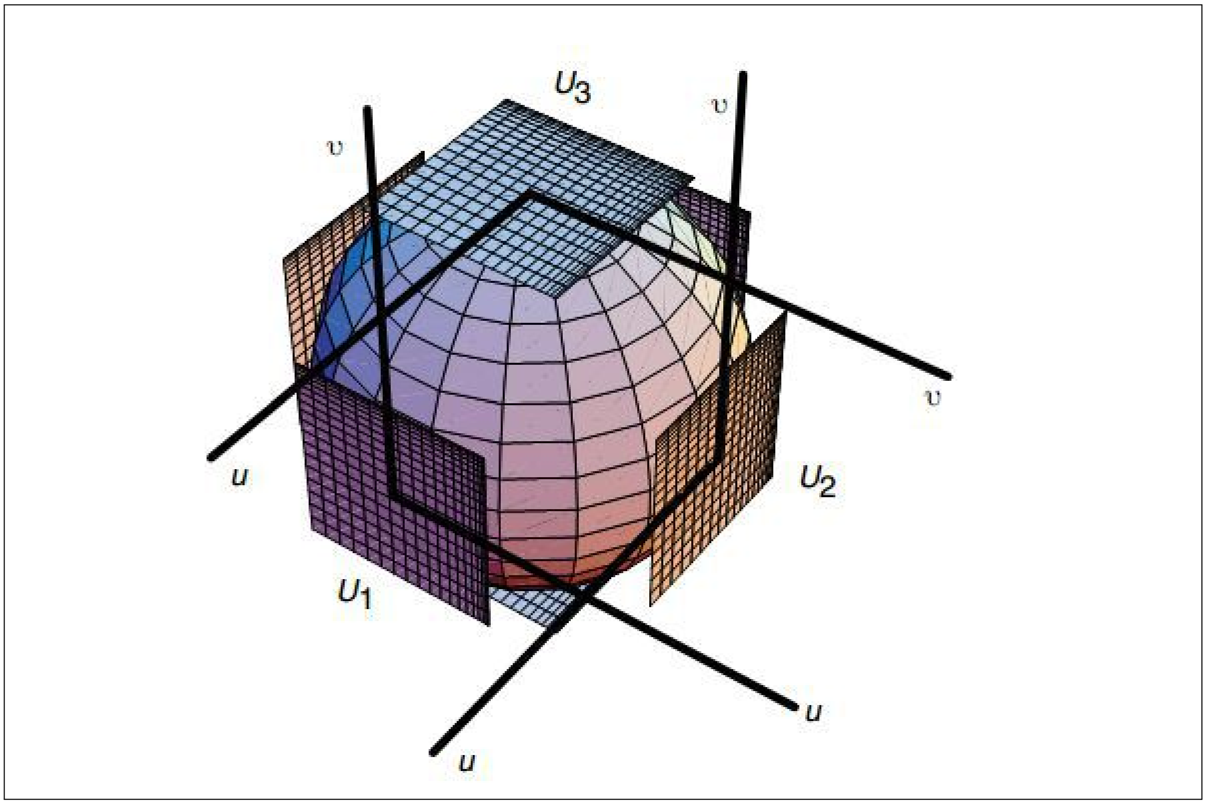


Figure 1.3: The local charts in the Poincaré sphere

in the local chart (U_2, F_2) the expression of $p(\mathcal{X})$ is

$$\dot{u} = v^n \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \quad \dot{v} = -v^{n+1} Q\left(\frac{u}{v}, \frac{1}{v}\right);$$

and for the local chart (U_3, F_3) the expression of $p(\mathcal{X})$ is

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v).$$

Note that for studying the singular points at infinity we only need to study the infinite singular points of the chart U_1 and the origin of the chart U_2, because the singular points at infinity appear in pairs diametrally opposite.

For more details on the Poincaré compactification see Chapter 5 of [42].

1.6.1 Phase portraits on the Poincaré disc

In this subsection we shall see how to characterize the global phase portraits in the Poincaré disc of quadratic polynomial differential systems. We shall determine the local phase portrait at all its finite and infinite singular points, then we have to study the proprities of its separatrices, where a *separatrix* of $p(\mathcal{X})$ is an orbit which is either a singular point, or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a singular point. Neumann [81] proved that the set formed by all separatrices of $p(\mathcal{X})$; denoted by $S(p(\mathcal{X}))$ is closed. We denote by S for the number of separatrices.

The open connected components of $\mathbb{D}^2 \setminus S(p(\mathcal{X}))$ are called *canonical regions* of $p(\mathcal{X})$: We define a *separatrix configuration* as a union of $S(p(\mathcal{X}))$ plus one solution chosen from each canonical region. Two separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are said to be *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S(p(\mathcal{X}))$ into the trajectories of $S(p(\mathcal{Y}))$. The following result is due to Markus [78], Neumann [81] and Peixoto [85]. We denote by R for the number of canonical regions.

Theorem 1.1 *The phase portraits in the Poincaré disc of the two compactified polynomial differential systems $p(\mathcal{X})$ and $p(\mathcal{Y})$ are topologically equivalent if and only if their separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are topologically equivalent. [26]*

This theorem implies that once separatrix configurations of a vector field in the Poincaré disc is determined, the global phase portrait of that vector field is obtained up to topological equivalence.

Similarly a parabolic sector, a hyperbolic sector and an elliptic sector are defined in the standard way.

Definition 1.14 *A sector which is topologically equivalent to the sector shown in Figure 1.4(a) is called a hyperbolic sector. A sector which is topologically equivalent to the sector shown in Figure 1.4(b) is called a parabolic sector. And a sector which is topologically equivalent to the sector shown in Figure 1.4(c) is called an elliptic sector. [86]*

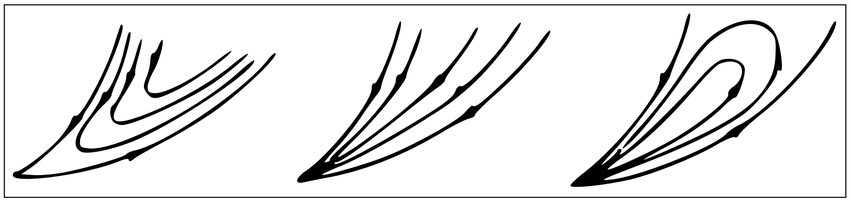


Figure 1.4: (*a*) A hyperbolic sector. (*b*) A parabolic sector. (*c*) An elliptic sector.

Section 1.7 Invariant Theory of Planar Polynomial Vector Fields

The tools for studying polynomial quadratic differential systems, have been greatly improved in the last years. Time ago, it was needed a detailed study of the finite and infinite singularities to complete a bifurcation diagram in one or several dimensions plus some more analytical and numerical study. Recently, Artés, Llibre, Schlomiuk, and Vulpe have published a series of articles [12, 14, 13, 6, 16, 19] in which they classify the different configurations of singularities (finite and infinite) that a quadratic differential system may have by means of algebraic invariants. That is, from the 12 coefficients of a quadratic system one can obtain a set of some dozens of invariants which can classify up to a very deep geometrical detail the properties of all the singular points of a quadratic system, without need of computing the coordinates of the singularities.

In the book [18] there is done the global geometrical classification of singularities and it was detected 1765 distinct configurations of singularities. This book comprise the whole classification of singularities when the number of finite singularities is 4 or less.

The use of invariants can also be applied to determine the number and configuration of invariant straight lines [91], the existence of first integrals [10], and in general, to anything which can be checked algebraically. For the moment these invariants are only available for quadratic systems but they can be extended to any polynomial differential system with a fixed degree.

Since the introduction of all the invariant theory needs huge space, we will give the geometric properties of some of the most important invariants that will appear:

The invariant μ_0 is related with finite singularities going to infinity. Indeed, if $\mu_0 = 0$ then at least one finite singularity has collided with an infinite one. Other invariants as μ_i for $i = 1,\dots,4$ control if more than one finite singularity has gone to infinity. The invariant κ controls if several finite singularities go to a single infinite singularity or to several.

The invariant $\mathbf{D}$ is related with the collision of finite singularities. Generically, if $\mathbf{D} = 0$ then two finite singularities have collided forming a saddle–node. But if more invariants vanish, then the collision may be even more degenerated, or even the collision may take place at infinity.

The invariant η is related with the collision of infinite singularities. Indeed, if $\eta = 0$ then at least two infinite singularities have collided. It may happen also that all three infinite singularities coalesce is a triple one, or even the infinite can be fulfilled by singularities. Invariants $\widetilde{M}$ and C_2 control these situations.

The invariant B_1 is related with the existence of invariant straight lines. Indeed, if $B_1 \neq 0$ then there are no invariant straight lines. The condition $B_1 = 0$ is necessary for the existence of invariant straight lines (but not sufficient). If some other invariants also vanish, then the number of invariant straight lines may be even greater. The conditions for the existence of 4 or more invariant straight lines (including infinity and counting multiplicity) are already known and can be applied in some cases of this work. For more details for the classification of the configurations of singularities, see book [18].

Section 1.8 Discontinuous Vector Fields

Definition 1.15 (Piecewise linear differential systems)
A differential system defined on an open region $S \subseteq \mathbb{R}^2$ is said to be a piecewise linear differential system (PWLS) on S if there exists a set of 3–tuples $\{(A_i; b_i; S_i)\}_{i\in I}$ such that: A_i is a 2×2 real matrix; $b_i \in \mathbb{R}^2$; $S_i \subseteq S$ is an open set in $\mathbb{R}^2$ satisfying that $S_i \cap S_j = \emptyset$ if $i \neq j$ and $\cup_{i\in I} Cl(S_i) = S$; and $A_i x + b_i$ is the vector field defined by the system when $x \in S_i$. As usual $Cl(S_i)$ denotes the closure of S_i. [66]

Remark 5 *Thus the vector field defined by a PWLS is a linear map on each of the disjoint regions S_i, but is not globally linear on the whole S.*

Section 1.9 The Averaging Theory Up to Sixth Order

To study the limit cycles that can bifurcate from the periodic orbits of centers, we can use one of the three following techniques; Abelian integral; averaging theory; or Melnikov function, which produce the same results in the plane.

The averaging theory is fundamental to our study, so we introduce the main result in order to apply it, see [63].

Theorem 1.2 *Consider the differential system*

$$\dot{x} = \sum_{i=1}^{6} \varepsilon^i F_i(t,x) + \varepsilon^7 R(t,x,\varepsilon), \tag{1.8}$$

where $F_i : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ for $i = 1, \cdots, 6$, and $R : \mathbb{R} \times \mathcal{D} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ are $T-$periodic in the first variable and continuous functions, $\mathcal{D}$ is an open interval of $\mathbb{R}^n$, and ε a small parameter. Assume that the following hypotheses (i) and (ii) hold:

(i) $F_1(t,.) \in \mathcal{C}^5(\mathcal{D})$, $F_2(t,.) \in \mathcal{C}^4(\mathcal{D})$, $F_3(t,.) \in \mathcal{C}^3(\mathcal{D})$, $F_4(t,.) \in \mathcal{C}^2(\mathcal{D})$, $F_5(t,.) \in \mathcal{C}^1(\mathcal{D})$ and $F_6(t,.) \in \mathcal{C}^0(\mathcal{D})$ for all $t \in \mathbb{R}$, F_1, F_2, F_3, F_4, F_5 , F_6, R, $\partial_x^5 F_1$, $\partial_x^4 F_2$, $\partial_x^3 F_3$, $\partial_x^2 F_5$ and $\partial_x^1 F_6$ are locally Lipschitz with respect to x, and R is five times differentiable with respect to ε.

For $i = 1, 2, \ldots, 6$ we define the averaging function $f_i : \mathcal{D} \to \mathbb{R}$ of order i as

$$f_i(z) = \frac{y_i(T,z)}{i!}, \tag{1.9}$$

where $y_i : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ for $i = 1, 2, \ldots, 6$ are defined recurrently by the following integral equations

$$y_i(t,z) = i! \int_0^t \Big(F_i(s,\varphi(s,z)) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} . \partial^L F_{i-l}(s,\varphi(s,z)) \prod_{j=1}^{l} y_j(s,z)^{b_j} \Big) ds.$$

Here $\partial^L G(\phi, z)$ denotes the derivative order L of a function G with respect to the variable z, and S_l is the set of all $l-$uples of non-negative integers $(b_1, b_2, \ldots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + l b_l = l$, and $L = b_1 + b_2 + \cdots + b_l$.

(ii) For $V \in \mathcal{D}$ an open and bounded set, and for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ there exists a_ε such that $f_1(a_\varepsilon) + \varepsilon f_2(a_\varepsilon) + \varepsilon^2 f_3(a_\varepsilon) + \varepsilon^3 f_4(a_\varepsilon) + \varepsilon^4 f_5(a_\varepsilon) + \varepsilon^5 f_6(a_\varepsilon) = 0$ and

$$d_B\Big(f_1(a_\varepsilon) + \varepsilon f_2(a_\varepsilon) + \varepsilon^2 f_3(a_\varepsilon) + \varepsilon^3 f_4(a_\varepsilon) + \varepsilon^4 f_5(a_\varepsilon) + \varepsilon^5 f_6(a_\varepsilon), V, a_\varepsilon\Big) \neq 0. \tag{1.10}$$

Then for $\mid \varepsilon \mid > 0$ sufficiently small there exists a $T-$periodic solution $\varphi(\cdot, \varepsilon)$ of the system such that $\varphi(0, \varepsilon) = a_\varepsilon$.

The expression (1.10) means that the Brouwer degree of the function $f_1 + \varepsilon f_2 + \varepsilon^2 f_3 + \varepsilon^3 f_4 + \varepsilon^4 f_5 + \varepsilon^5 f_6 : V \longrightarrow \mathbb{R}$ at the fixed point a_ε is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $f_1 + \varepsilon f_2 + \varepsilon^2 f_3 + \varepsilon^3 f_4 + \varepsilon^4 f_5 + \varepsilon^5 f_6$ at a_ε is not zero.

If f_1 is not identically zero, then the zeros of $f_1 + \varepsilon f_2 + \varepsilon^2 f_3 + \varepsilon^3 f_4 + \varepsilon^4 f_5 + \varepsilon^5 f_6$ are mainly the zeros of f_1 for ε sufficiently small. In this case, the previous result provides the averaging theory of first order.

If f_1 is identically zero and f_2 is not identically zero, then the zeros of $f_2 + \varepsilon f_3 + \varepsilon^2 f_4 + \varepsilon^3 f_5 + \varepsilon^4 f_6$ are mainly the zeros of f_2 for ε sufficiently small. In this case, the previous result provides the averaging theory of second order.

If f_1 and f_2 are both identically zero and f_3 is not identically zero, then the zeros of $f_3 + \varepsilon f_4 + \varepsilon^2 f_5 + \varepsilon^3 f_6$ are mainly the zeros of f_3 for ε sufficiently small. In this case, the previous result provides the averaging theory of third order.

If f_1, f_2 and f_3 are identically zero and f_4 is not identically zero, then the zeros of $f_4 + \varepsilon f_5 + \varepsilon^2 f_6$ are mainly the zeros of f_4 for ε sufficiently small. In this case, the previous result provides the averaging theory of fourth order.

If f_1, f_2 , f_3 and f_4 are identically zero and f_5 is not identically zero, then the zeros of $f_5 + \varepsilon f_6$ are mainly the zeros of f_5 for ε sufficiently small. In this case, the previous result provides the averaging theory of fifth order.

If f_1, f_2 , f_3 , f_4 and f_5 are identically zero and f_6 is not identically zero, then the zeros of f_6 is mainly the zeros of f_6 for ε sufficiently small. In this case, the previous result provides the averaging theory of sixth order. [63]

To know the number of zeros of a real polynomial, we are going to use the following Theorem.

Theorem 1.3 *[Descartes Theorem]*
Consider the real polynomial $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r}$ *with* $0 \leq i_1 < i_2 < \cdots < i_r$ *and* $a_{i_j} \neq 0$ *real constants for* $j \in \{1, 2, \cdots, r\}$. *When* $a_{i_j}a_{i_{j+1}} < 0$, *we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* m, *then* $p(x)$ *has at most* m *positive real roots. Moreover, it is always possible to choose the coefficients of* $p(x)$ *in such a way that* $p(x)$ *has exactly* $r-1$ *positive real roots. [63]*

Chapter 2

Global Phase Portraits of Some Families of Quadratic Polynomial Differential Systems

In this chapter, we will classify quadratic polynomial differential systems written as (1) exhibiting cubic invariant algebraic curve. The chapter is organized as follows. The first section is devoted to classify the dynamics of five classes of quadratic differential systems exhibiting five known different cubic invariant algebraic curves, i.e. these curves are formed by orbits of the quadratic polynomial differential system, by analysing their phase portraits in the Poincaré disc. More precisely, we realize 5 different well-known algebraic curves of degree 3 as invariant curves inside the quadratic polynomial differential systems. These realizations produce 29 topologically different phase portraits in the Poincaré disc for such quadratic polynomial differential systems. In the second section we aim to characterize the global phase portraits in the Poincaré disc of quadratic systems having the subsequent reducible cubic invariant curve. These realizations produce 13 topologically different phase portraits in the Poincaré disc for such quadratic polynomial differential systems.

In this section we study the qualitative behaviour of five classes of quadratic systems, exhibiting five known algebraic curves of degree 3, we do this by describing its global phase portraits in the Poincaré disc in function of its parameters. More precisely we analyse the quadratic differential systems having the cubic curves which are given in the following table.

Table 2.1: Classical algebraic curves of degree 3 realizable by quadratic systems.

Name	Curve
Right Serpentine 1	$f_1(x,y) = y(x+y)^2 - x$
Witch of Agnesi	$f_2(x,y) = x^2y - a^2(a-y)$, $a \neq 0$
Cissoid of Diocles	$f_3(x,y) = x(x^2+y^2) - ay^2$, $a \neq 0$
Crunodal Cubic	$f_4(x,y) = xy^2 - x - 1$
Longchamps or Mixed cubic	$f_5(x,y) = y^2x - ay^2 - bx^2$, $b \neq 0$

We consider the case where the polynomials P and Q are coprime.

2.1.1 Invariant algabraic curves

The five curves given in Table 2.1 are invariant for some classes of quadratic systems represented in the following Theorem.

Theorem 2.1

(*i*) *The cubic algebraic curve* ***Right Serpentine 1*** *with cofactor* $K_1(x,y) = (1-c/3)(x+3y)$ *is an invariant algebraic curve* ***QS***

$$\dot{x} = x^2 + (3+\frac{c}{3})xy + cy^2,$$

$$\dot{y} = \frac{c}{3} - (1+\frac{c}{3})xy + (1-c)y^2.$$

(*ii*) *The cubic algebraic curve* ***Witch of Agnesi*** *with cofactor* $K_2(x,y) = cy$ *is an invariant algebraic curve* ***QS***

$$\dot{x} = -\frac{a^2}{2} + \frac{acx}{2} - \frac{x^2}{2},$$

$$\dot{y} = -acy + xy + cy^2.$$

(*iii*) *The cubic algebraic curve* ***Cissoid of Diocles*** *with cofactor* $K_3(x,y) = x - a$ *is an invariant algebraic curve* ***QS***

$$\dot{x} = -\frac{ax}{3} + \frac{2acy}{3} + \frac{x^2}{3} - \frac{2cxy}{3},$$

$$\dot{y} = -\frac{ay}{2} + cx^2 + \frac{xy}{3} + \frac{cy^2}{3}.$$

(*iv*) *The cubic algebraic curve* ***Crunodal Cubic*** *with cofactor* $K_4(x,y) = x$ *is an invariant algebraic curve* ***QS***

$$\dot{x} = x + x^2 - 2cxy,$$

$$\dot{y} = -c - \frac{y}{2} + cy^2.$$

(*v*) *The cubic algebraic curve* ***Longchamps*** *with cofactor* $K_5(x,y) = x-a$ *is an invariant algebraic curve* ***QS***

$$\dot{x} = -\frac{ax}{2} + 2acy + \frac{x^2}{2} - 2cxy,$$

$$\dot{y} = -2bcx - \frac{ay}{2} + \frac{xy}{4} + cy^2.$$

Proof. *To prove this theorem we can easely verify that we have the partially differential equation:*

$$P(x,y)\frac{\partial f}{\partial x} + Q(x,y)\frac{\partial f}{\partial y} = Kf.$$

Proposition 2.1 (Reduction of the parameters)
Instead to study the systems given in Theorem 2.1 for any of their parameters, we can reduce the study by doing the following symmetries.

(a) *System* (ii) *is invariant under the changes* $(x,y,t,a,c) \to (-x,y,-t,a,-c)$ *and* $(x,y,t,a,c) \to (-x,-y,-t,-a,c)$, *then we study the system for* $c \geqslant 0$ *and* $a > 0$.

(b) *System* (iii) *is invariant under the changes* $(x,y,t,a,c) \to (-x,-y,-t,-a,c)$, *and* $(x,y,t,a,c) \to (x,-y,t,a,-c)$, *then we study the system for* $a > 0$ *and* $c \geqslant 0$.

(c) *System* (iv) *is invariant under the changes* $(x,y,t,c) \to (x,-y,t,-c)$, *then we study it for* $c \geqslant 0$.

(d) *System* (v) *is invariant under the change* $(x,y,t,a,b,c) \to (-x,-y,-t,-a,-b,c)$ *and* $(x,y,t,a,b,c) \to (x,-y,t,a,b,-c)$, *then we study the system for* $a \geqslant 0$, $c \geqslant 0$ *and* $b \neq 0$.

The study of the four different normal forms (i), (ii), (iii) and (iv) mentioned in Theorem 2.1 has been done by using the classical technique, where we looked for the finite and infinite singular points, and by determining their local behavior, and studying the direction of the flows on certain lines. For the fifth normal form in Theorem 2.1 which contains in its expression three parameters, the uses of the classical technique, may leave the study of some cases. The best current technique of studying such systems containing in their expression more than two parameters is the invariant theory, see book [18]. The presentation of our results by two different way allows the readers to make a comparison between the two techniques.

2.1.2 Finite and infinite singularities

This section is devoted to study the singular points (or simply **SP**) of all quadratic systems mentiened in Theorem 2.1.

Proposition 2.2 *The following statements hold for the quadratic systems of Theorem 2.1.*

(i) *If $c \in (-\infty, 0)$ system (i) has four hyperbolic singularities, an unstable node at $q_1 = ((c\sqrt{-c})/(\sqrt{3}(3-c)), (-\sqrt{3}\sqrt{-c})/(3-c))$, a stable node at $q_2 = ((-c\sqrt{-c})/(\sqrt{3}(3-c)), (\sqrt{3}\sqrt{-c})/(3-c))$, and two saddles at $q_3 = ((-\sqrt{3}\sqrt{-c})/2, \sqrt{-c}/(2\sqrt{3}))$ and $q_4 = -q_3$. If $c = 0$ it has one finite linearly zero* ***SP*** *at the origin, where its local phase portrait consists of two parabolic and two hyperbolic sectors. If $c \in (0, \infty)$ the system has no finite* ***SP****. For the infinite singularities system (i) has in the local chart U_1; $p_1 = (0,0)$ which is a saddle if $c \in (-\infty, -6)$, a stable node if $c \in (-6, \infty)$ and a semi-hyperbolic saddle-node of type $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ if $c = -6$; $p_2 = ((-6-c)/(3c), 0)$ which is a stable node if $c \in (-\infty, -6)$, a saddle if $c \in (-6, 0) \cup (0, 3) \cup (3, \infty)$ and p_2 becomes p_1 when $c = -6$, and a linearly zero singularity where its local phase portrait consists of two hyperbolic sectors if $c = 3$; the third singularity at $p_3 = (-1, 0)$ which is an unstable node if $c \in (-\infty, 3)$, a stable node if $c \in (3, \infty)$, and p_3 coincides with the singularity p_2 if $c = 3$.*

The origin of the local chart U_2 is not a singularity for all $c \neq 0$, and it is a saddle if $c = 0$.

(ii) *If $c \in [0, 2)$ and $a \neq 0$ system (ii) has no finite* ***SP****. If $c \in (2, \infty)$ and $a \neq 0$ the system has four hyperbolic singularities, a stable node at $q_1 = (a(c + \sqrt{c^2-4})/2, 0)$, an unstable node at $q_2 = (a(c - \sqrt{c^2-4})/2, a(c + \sqrt{c^2-4})/(2c))$, and two saddles at $q_3 = (a(c + \sqrt{c^2-4})/2, a(c - \sqrt{c^2-4})/2c)$ and $q_4 = (a(c - \sqrt{c^2-4})/2, 0)$. If $c = 2$ and $a \neq 0$ system (ii) has two semi-hyperbolic saddle-nodes at $q_1 = q_4 = (a, 0)$ and $q_2 = q_3 = (a, a/2)$.*

If $c \neq 0$ and $a = 0$ the system has one finite linearly zero ***SP*** *at the origin of coordinates and doing a blow-up change of variables, we know that its local phase portrait is formed by two parabolic and two hyperbolic sectors. If $c = 0$ and $a = 0$ the system has $x = 0$ as a line of singularities, and by doing the change of variable $xdt = ds$, we know that the system has a saddle at the origin of coordinates.*

For the infinite ***SP*** *and if $c \neq 0$ and $a \geqslant 0$, the system has an hyperbolic unstable node at $p_1 = (0,0)$ and an hyperbolic saddle at $p_2 = ((-3)/(2c), 0)$ and the origin of the local chart U_2 is a stable node. For $c = 0$ and $a \geqslant 0$, the system has an hyperbolic unstable node at p_1, and the origin of the local chart U_2 is a linearly zero singularity, where its local phase portrait consists of two hyperbolic sectors.*

(iii) *If $c \in (0, 1/\sqrt{48})$ and $a \neq 0$ system (iii) has four hyperbolic* ***SP****, a stable node at $q_1 = (0,0)$, two saddles at $q_2 = (a/(4c^2+1), a/(2c+8c^2))$ and $q_3 = (a, (a - a\sqrt{1-48c^2})/(4c))$, and an unstable node at $q_4 = (a, (a + a\sqrt{1-48c^2})/(4c))$. If $c \in (1/\sqrt{48}, \infty)$ and $a \neq 0$ it has two hyperbolic singularities, a stable node at q_1 and a saddle at q_2. If $c = 1/\sqrt{48}$ and $a \neq 0$ system (iii) has three finite* ***SP****, an hyperbolic stable node at q_1, a semi hyperbolic saddle-node at $q_3 = q_4 = (a, \sqrt{3}a)$ and an hyperbolic saddle at $q_2 = ((12a)/13, (24\sqrt{3}a)/13)$. If $c = 0$ and $a \neq 0$ the system has two hyperbolic singularities, a stable node at q_1 and a saddle at $q_3 = (a, 0)$.*

If $c \neq 0$ and $a = 0$ the system has one finite linearly zero ***SP*** *at the origin of coordinates and by doing a blow-up change of variables, we know that its local phase portrait is formed two hyperbolic sectors. If $c = 0$ and $a = 0$ the system has $x = 0$ as a line of singularities, and by doing the change of variable $xdt = ds$, we get that the system has an unstable node at the origin of coordinates.*

In the local chart U_1 and if $c \neq 0$ and $a \geqslant 0$ system (iii) has no singularity. If $c = 0$ and $a \neq 0$ the system has the infinity as a line of singularities. If $c = 0$ and $a = 0$ system (iii) has no singularity.

The origin of the local chart U_2 is an hyperbolic stable node if $c \neq 0$ and $a \geqslant 0$, an hyperbolic unstable node if $c = 0$ and $a \neq 0$, and it is not a singularity if $c = 0$ and $a = 0$.

(iv) *If $c \in (0, \infty)$ system (iv) has four hyperbolic* ***SP****, two saddles at $q_{1,2} = (0, (\pm\sqrt{16c^2+1} + 1)/(4c))$, a stable node at $q_3 = ((-\sqrt{16c^2+1} - 1)/2, (-\sqrt{16c^2+1} + 1)/(4c))$ and an unstable node at $q_4 = ((\sqrt{16c^2+1} - 1)/2, (\sqrt{16c^2+1} + 1)/(4c))$. If $c = 0$ the system has two hyperbolic singularities, a saddle at $q_2 = (0,0)$ and a stable node at $q_3 = (-1, 0)$.*

In the local chart U_1 and if $c \neq 0$ system (iv) has two hyperbolic singularities, a *stable node at $p_1 = (0,0)$ and a saddle at $p_2 = (1/(3c), 0)$. If $c = 0$ the system has one infinite singularity at p_1 which is a stable node.*

The origin of the local chart U_2 is a stable node if $c \neq 0$, and a linearly zero singularity if $c = 0$, where its local phase portrait consists of two parabolic and two hyperbolic sectors.

(v) *If $b \in ((-a)/(128c^2), \infty)$ and $ac \neq 0$ system (v) has four hyperbolic singularities; $q_1 = (0,0)$ which is a stable focus if $b \in (0, \infty)$ and a stable node if $b \in ((-a)/(128c^2), 0)$; $q_2 = (a + 16bc^2, (a + 16bc^2)/(4c))$ which is an unstable node if $b \in (0, \infty)$ and a saddle if $b \in ((-a)/(128c^2), 0)$, a saddle at $q_3 = (a, (a - \sqrt{a^2 + 128abc^2})/(8c))$; the fourth singularity at $q_4 = (a, (a + \sqrt{a^2 + 128abc^2})/(8c))$ which is a saddle if $b \in (0, \infty)$ and an unstable node if $b \in ((-a)/(128c^2), 0)$.*

If $b \in (-\infty, (-a)/(128c^2))$ and $ac \neq 0$ system (iv) has two singularities; q_1 which is a saddle if $b \in (-\infty, (-a)/(16c^2))$, a stable node if $b \in ((-a)/(16c^2), (-a)/(128c^2))$; q_2 which is a stable node if $b \in (-\infty, (-a)/(16c^2))$, a saddle if $b \in (-a/(16c^2), -a/(128c^2))$ and a semi-hyperbolic saddle-node if $b = (-a)/(16c^2)$.

If $b = (-a)/(128c^2)$ and $ac \neq 0$ the system has three singularities, a stable node at q_1, a saddle at $q_2 = ((7a)/8, (7a)/(32c))$ and a semi-hyperbolic saddle-node at $q_3 = (a, a/(8c))$.

*If $ab \neq 0$ and $c = 0$, or $b = 0$, $a \neq 0$ and $c = 0$ the system has two hyperbolic **SP**, a stable node at q_1 and a saddle at $q_3 = (a, 0)$.*

*If $bc \neq 0$ and $a = 0$ the system has two **SP**, a nilpotent saddle at q_1 and the second singularity at $q_2 = (16bc^2, 4bc)$ which is an hyperbolic unstable node if $b > 0$ and an hyperbolic stable node if $b < 0$.*

If $b \neq 0$, $a = 0$ and $c = 0$ the system has a line of singularities at $x = 0$, and by doing the change of variable $x\,dt = ds$, we obtain that the system has an unstable node at the origin of coordinates.

*If $b = 0$ and $ac \neq 0$ the system has three **SP**, a stable node at q_1, a saddle at $q_3 = (a, 0)$ and a semi-hyperbolic saddle-node at $q_2 = (a, 4c)$.*

If $b = 0$, $a = 0$ and $c \neq 0$ the system has one finite linearly zero ***SP*** *at the origin of coordinates and by doing a blow-up change of variables, we know that its local phase portrait is formed two parabolic and two hyperbolic sectors.*

If $b = 0$, $a = 0$ and $c = 0$ the system has $x = 0$ as a line of singularities, and by doing the change of variable $xdt = ds$. we know that the system has an unstable node at the origin of coordinates.

For the infinite ***SP*** *and if $b \neq 0$, $a \geqslant 0$ and $c \neq 0$, or $b = 0$, $a \geqslant 0$ and $c \neq 0$, or $b = 0$, $a = 0$ and $c \neq 0$, system (v) has three hyperbolic* ***SP****, two stable nodes and a saddle. If $b \neq 0$, $a = 0$ and $c = 0$, or $b = 0$, $a = 0$ and $c = 0$ the system has a stable node and a saddle. If $ab \neq 0$ and $c = 0$, or $b = 0$, $a \neq 0$ and $c = 0$ the system has a stable node and a linearly zero singularity where its local phase portrait consists of two parabolic and two hyperbolic sectors.*

Proof.

Proof of statement (i) of Proposition 2.2. *If $c \in (-\infty, 0)$ system (i) has four hyperbolic* ***SP****; $q_1 = ((c\sqrt{-c})/(\sqrt{3}(3-c)), (-\sqrt{3}\sqrt{-c})/(3-c))$ with eigenvalues $((9-c)\sqrt{-c})/(3\sqrt{3})$ and $(2\sqrt{-c})/\sqrt{3}$. Hence by using Theorem 2.15 of [42] we get that q_1 is an unstable node;*
$q_2 = ((-c\sqrt{-c})/\sqrt{3}(3-c), (\sqrt{3}\sqrt{-c})/(3-c))$ which is a stable node with eigenvalues $(-2\sqrt{-c})/\sqrt{3}$ and $((c-9)\sqrt{-c})/(3\sqrt{3})$ and two saddles at $q_3 = ((-\sqrt{3}\sqrt{-c})/2, \sqrt{-c}/(2\sqrt{3}))$ with eigenvalues $((9-c)\sqrt{-c})/(3\sqrt{3})$ and $(-2\sqrt{-c})/\sqrt{3}$ and $q_4 = -q_3$ with eigenvalues $(2\sqrt{-c})/\sqrt{3}$ and $((c-9)\sqrt{-c})/(3\sqrt{3})$.

If $c = 0$ system (i) becomes

$$\dot{x} = x^2 + 3xy, \qquad \dot{y} = y^2 - xy. \tag{2.1}$$

This system has one ***SP*** *at the origin which is a linearly zero singularity. We need to do a blow-up $y = zx$ for describing its local phase portrait. After eliminating the common factor x of $\dot{x}$ and $\dot{z}$, by doing the rescaling of the independent variable $ds = xdt$, we obtain the system $\dot{x} = x + 3xz$, $\dot{z} = -2z - 2z^2$. For $x = 0$ this system has two hyperbolic saddles at $(0,0)$ with eigenvalues (-2) and 1, and $(0,-1)$ with eigenvalues 2 and (-2). Going back through the two changes of variables $y = zx$ and $ds = xdt$ and by taking into account the direction of the flow of the system on the axes of coordinates, we obtain that the local phase portrait at the origin of system (2.1) is formed by two parabolic and two hyperbolic sectors.*

If $c \in (0,\infty)$ system (i) has no finite singularity.

System (i) in the local chart U_1 becomes

$$\begin{aligned} \dot{u} &= \frac{1}{3}(cv^2 - (c+6)u - 2(2c+3)u^2 - 3cu^3), \\ \dot{v} &= -\frac{v}{3}(3u+1)(cu+3). \end{aligned} \tag{2.2}$$

*System (2.2) has three singularities if $c \neq 0$; $p_1 = (0,0)$ with eigenvalues $(-2-(c/3))$ and (-1). Then, this point is a saddle if $c \in (-\infty,-6)$, a stable node if $c \in (-6,0)\cup(0,\infty)$ and a semi-hyperbolic singularity if $c = -6$. In order to obtain the local phase portrait at this point we use Theorem 2.19 of [42], and we obtain that p_1 is a saddle-node of type $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, and the second point $p_2 = ((-6-c)/(3c),0)$, such that the eigenvalues of its linear part are $(6-2c)/(3c)$ and $(2c-6)(c+6)/(9c)$. So, p_2 is a stable node if $c \in (-\infty,-6)$, a saddle if $c \in (-6,0)\cup(0,3)\cup(3,\infty)$ and p_2 becomes p_1 if $c = -6$, and a linearly zero **SP** if $c = 3$. Then, if $c = 3$ system (2.2) becomes*

$$\dot{u} = v^2 - 3u - 6u^2 - 3u^3, \quad \dot{v} = -v(u+1)(3u+1). \tag{2.3}$$

*System (2.3) has an hyperbolic stable node at $p_1 = (0,0)$ with eigenvalues (-3) and (-1) and a linearly zero **SP** at $p_3 = (-1,0)$. In order to know the nature of this singularity, first, we put these **SP** at the origin of coordinates by performing the translation $u = u_1 - 1$, $v = v_1$, and we get $\dot{u}_1 = 3u_1^2 - 3u_1^3 + v_1^2$, $\dot{v}_1 = 2u_1v_1 - 3u_1^2v_1$. Second, we need to do a blow-up $v_1 = wu_1$ for describing its local phase portrait. After eliminating the common factor u_1 of $\dot{u}_1$ and $\dot{w}$, by doing the rescaling of the independent variable $ds = u_1dt$, we obtain the system $\dot{u}_1 = 3u_1 - 3u_1^2 + u_1w^2$, $\dot{w} = -w^3 - w$. For $u_1 = 0$ this system has one hyperbolic singularity, a saddle at the origin with eigenvalues (-1) and 3. Going back through the two changes of variables $v_1 = wu_1$ and $ds = u_1dt$ and by taking into account the direction of the flow of the system on the axes of coordinates, we obtain that the local phase portrait at p_2 of system (2.2) is formed by two hyperbolic sectors; the third **SP** at $p_3 = (-1,0)$ with eigenvalues $\lambda_1 = \lambda_2 = 2-(2c)/3$ and a associated Jacobian matrix $\begin{pmatrix} \frac{2(3-c)}{3} & 0 \\ 0 & \frac{2(3-c)}{3} \end{pmatrix}$. We know that p_3 is an unstable star node if $c \in (-\infty,0)\cup(0,3)$, a stable star node if $c \in (3,\infty)$ and it becomes p_2 if $c = 3$.*

If $c = 0$ system (2.2) writes as

$$\dot{u} = -2u(u+1), \qquad \dot{v} = -v(3u+1). \tag{2.4}$$

This system has two hyperbolic singularities, a stable node at p_1 with eigenvalues (-2) and (-1) and an unstable star node at $p_3 = (-1, 0)$ with eigenvalues 2 and 2.

In the local chart U_2 system (i) writes as

$$\begin{aligned} \dot{u} &= \frac{1}{3}\Big(6u(u+1) + cu^2 - cu(v^2-4) + 3c\Big), \\ \dot{v} &= -\frac{v}{3}\Big(c(v^2-u-3) - 3u + 3\Big). \end{aligned} \tag{2.5}$$

If $c \neq 0$ it is clear that the origin is not a singularity.

If $c = 0$ system (2.5) is given by

$$\dot{u} = 2u(u+1), \qquad \dot{v} = v(u-1). \tag{2.6}$$

The origin of this system is an hyerbolic saddle with eigenvalues 2 and (-1). This completes the proof of Statement (i) of Proposition 2.2.

Proof of statement (ii) of Proposition 2.2. *If $c \in [0, 2)$ and $a \neq 0$ system (ii) has no finite **SP**. If $c \in (2, \infty)$ and $a \neq 0$ the system has four hyperbolic singularities; $q_1 = (a(c + \sqrt{c^2-4})/2, 0)$ with eigenvalues $(-a\sqrt{c^2-4})/2$ and $a(\sqrt{c^2-4} - c)/2$. Since $a > 0$ and by using Theorem 2.15 of [42] we get that q_1 is a stable node; $q_2 = (a(c - \sqrt{c^2-4})/2, a(c + \sqrt{c^2-4})/(2c))$ which is an unstable node with eigenvalues $(a\sqrt{c^2-4})/2$ and $a(\sqrt{c^2-4} + c)/2$, and two saddles at $q_3 = (a(c + \sqrt{c^2-4})/2, a(c - \sqrt{c^2-4})/(2c))$ with eigenvalues $(-a\sqrt{c^2-4})/2$ and $a(c - \sqrt{c^2-4})/2$ and $q_4 = (a(c - \sqrt{c^2-4})/2, 0)$ with eigenvalues $(a\sqrt{c^2-4})/2$ and $-a(c + \sqrt{c^2-4})/2$. If $c = 2$ and $a \neq 0$ it has two semi-hyperbolic **SP** at $q_1 = q_4 = (a, 0)$ with eigenvalues 0 and $(-a)$, and $q_2 = q_3 = (a, a/2)$ with eigenvalues 0 and a. In order to obtain the local phase portrait at these points we use Theorem 2.19 of [42], and we obtain that $q_1 = q_4$ and $q_2 = q_3$ are saddle-nodes.*

If $c \neq 0$ and $a = 0$ system (ii) becomes

$$\dot{x} = -\frac{x^2}{2}, \qquad \dot{y} = cy^2 + xy. \tag{2.7}$$

System (2.7) *has one* ***SP*** *at the origin which is a linearly zero singularity. We need to do a blow-up* $y = zx$ *for describing its local phase portrait. After eliminating the common factor* x *of* $\dot{x}$ *and* $\dot{z}$*, by doing the rescaling of the independent variable* $ds = xdt$*, we obtain the system* $\dot{x} = (-x)/2$, $\dot{z} = cz^2 + (3z)/2$*. For* $x = 0$ *this system has two hyperbolic singularities, a saddle at* $(0,0)$ *with eigenvalues* $(-1)/2$ *and* $3/2$*, and a stable node at* $(0, -3/(2c))$ *with eigenvalues* $(-1)/2$ *and* $(-3)/2$*. Going back through the two changes of variables* $y = zx$ *and* $ds = xdt$ *and by taking into account the direction of the flow of the system on the axes of coordinates, we obtain that the local phase portrait at the origin of system* (2.7) *is formed by two parabolic and two hyperbolic sectors.*

If $c = 0$ *and* $a = 0$ *system* (ii) *is given by* $\dot{x} = -\dfrac{x^2}{2}$, $\dot{y} = xy$*. This system has a line of singularities at* $x = 0$*, then by doing a change of variable* $xdt = ds$ *we obtain the system* $\dot{x} = -\dfrac{x}{2}$, $\dot{y} = y$*, which has a saddle at the origin with eigenvalues* 1 *and* $(-1)/2$*.*

System (ii) *in the local chart* U_1 *writes as*

$$\begin{aligned} \dot{u} &= \frac{u}{2}\Big(a^2v^2 - 3acv + 2cu + 3\Big), \\ \dot{v} &= \frac{v}{2}\Big(a^2v^2 - acv + 1\Big). \end{aligned} \tag{2.8}$$

This system has two hyperbolic singularities if $c \neq 0$ *and* $a \geqslant 0$*, an unstable node at* $p_1 = (0,0)$ *with eigenvalues* $3/2$ *and* $1/2$*, and a saddle at* $p_2 = (-3/(2c), 0)$ *with eigenvalues* $1/2$ *and* $(-3)/2$*.*

If $c = 0$ *and* $a \geqslant 0$ *system* (2.8) *is given by*

$$\dot{u} = u\Big(a^2v^2 + 3\Big)/2, \qquad \dot{v} = v\Big(a^2v^2 + 1\Big)/2. \tag{2.9}$$

This system has one hyperbolic unstable node at p_1 *with eigenvalues* $3/2$ *and* $1/2$*.*

In the local chart U_2 *system* (ii) *is give by*

$$\begin{aligned} \dot{u} &= \frac{1}{2}\Big(cu(3av - 2) - 3u^2 - a^2v^2\Big), \\ \dot{v} &= v\,(c(av - 1) - u). \end{aligned} \tag{2.10}$$

If $c \neq 0$ *and* $a \geqslant 0$ *the origin of this system is a stable star node with eigenvalues* $(-c)$ *and* $(-c)$*.*

If $c = 0$ and $a \geqslant 0$ system (2.10) writes as

$$\dot{u} = -(a^2v^2 + 3u^2)/2, \qquad \dot{v} = -vu. \tag{2.11}$$

*System (2.11) has one **SP** at the origin which is a linearly zero singularity. We need to do a blow-up $v = wu$ for describing its local phase portrait. After eliminating the common factor u of $\dot{u}$ and $\dot{w}$, by doing the rescaling of the independent variable $ds = udt$, we obtain the system $\dot{u} = (-a^2w^2u)/2-(3u)/2, \dot{w} = (a^2w^3)/2+w/2$. For $u = 0$ this system has one hyperbolic singularity, a saddle at the origin with eigenvalues $(-3)/2$ and $1/2$. Going back through the two changes of variables $v = wu$ and $ds = udt$ and by taking into account the direction of the flow of the system on the axes of coordinates, we obtain that the local phase portrait at the origin of system (2.11) is formed by two hyperbolic sectors. This completes the proof of Statement (ii) of Proposition 2.2.*

Proof of statement (iii) of Proposition 2.2. *If $c \in (0, 1/\sqrt{48})$ and $a \neq 0$ system (iii) has four hyperbolic **SP**, a stable node at $q_1 = (0,0)$ with eigenvalues $(-a)/2$ and $(-a)/3$, two saddles at $q_2 = (a/(4c^2+1), a/(2c+8c^2))$ with eigenvalues $(-4ac^2)/(4c^2+1)$ and $a/6$ and $q_3 = (a,(a-a\sqrt{1-48c^2})/(4c))$ with eigenvalues $(a\sqrt{1-48c^2}+1)/6$ and $(-a\sqrt{1-48c^2})/6$, and an unstable node at $q_4 = (a,(a+a\sqrt{1-48c^2})/(4c))$ with eigenvalues $(a-a\sqrt{1-48c^2})/6$ and $(a\sqrt{1-48c^2})/6$. If $c \in (1/\sqrt{48}, \infty)$ and $a \neq 0$ it has two hyperbolic singularities, a stable node at q_1 with eigenvalues $(-a)/2$ and $(-a)/3$ and a saddle at q_2 with eigenvalues $(-4ac^2)/(4c^2+1)$ and $(a/6)$. If $c = 1/\sqrt{48}$ and $a \neq 0$ system (iv) has an hyperbolic stable node at q_1 with eigenvalues $(-a)/2$ and $(-a)/3$, an hyperbolic saddle at $q_2 = ((12a)/13, (24\sqrt{3}a)/13)$ with eigenvalues $(-a)/13$ and $a/6$, and a semi-hyperbolic saddle-node at $q_3 = (a, \sqrt{3a})$ with eigenvalues $a/6$ and 0.*

*If $c = 0$ and $a \neq 0$ it has two hyperbolic **SP**, a stable node at q_1 with eigenvalues $(-a)/2$ and $(-a)/3$, and a saddle at $q_3 = (a, 0)$ with eigenvalues $(-a)/6$ and $a/3$.*

If $c \neq 0$ and $a = 0$ system (iii) becomes

$$\dot{x} = \frac{x^2}{3} - \frac{2cxy}{3}, \qquad \dot{y} = cx^2 + \frac{cy^2}{3} + \frac{xy}{3}. \tag{2.12}$$

*System (2.12) has one **SP** at the origin which is a linearly zero singularity. We need to do a blow-up $y = zx$ for describing its local phase portrait. After eliminating the common factor x*

of $\dot{x}$ and $\dot{z}$, by doing the rescaling of the independent variable $ds = xdt$, we obtain the system $\dot{x} = x - 2cxz$, $\dot{z} = z(2 + 5cz)/3$. For $x = 0$ this system has two hyperbolic singularities, an unstable node at $(0,0)$ with eigenvalues 1 and $2/3$, and a sadlle at $(0,-2/(5c))$ with eigenvalues $9/5$ and $(-3)/2$. Going back through the two changes of variables $y = zx$ and $ds = xdt$ and by taking into account the direction of the flow of the system on the axes of coordinates, we obtain that the local phase portrait at the origin of this system is formed by two hyperbolic sectors.

If $c = 0$ and $a = 0$ system (iii) is given by

$$\dot{x} = \frac{x^2}{3}, \qquad \dot{y} = \frac{xy}{3}. \tag{2.13}$$

System (2.13) has a line of singularities at $x = 0$, then by doing a change of variable $xdt = ds$ we obtain the system $\dot{x} = \frac{x}{3}$, $\dot{y} = \frac{y}{3}$, which has an unstable star node at the origin with eigenvalues $1/3$ and $1/3$.

System (iii) in the local chart U_1 writes as

$$\begin{aligned} \dot{u} &= c + cu^2(1 - (2av)/3) - (auv)/6, \\ \dot{v} &= -\frac{v}{3}(av - 1)(2cu - 1). \end{aligned} \tag{2.14}$$

If $c \neq 0$ and $a \geqslant 0$ this system has no singularity.

If $c = 0$ and $a \neq 0$ system (2.14) is given by

$$\dot{u} = (-avu)/6, \qquad \dot{v} = v(av - 1)/3. \tag{2.15}$$

Then $v = 0$ is a solution of system (2.15), which means that the infinity is a line of singularities. We do a change of variable $vdt = ds$, system (2.15) becomes

$$\dot{u} = (-au)/6, \qquad \dot{v} = (av - 1)/3. \tag{2.16}$$

This system has no singularity on $v = 0$.

If $c = 0$ and $a = 0$ system (2.14) is given by $\dot{u} = 0$, $\dot{v} = -v/3$. This system has no singularity.

In the local chart U_2 system (iii) writes as

$$\dot{u} = \frac{1}{6}\Big(c(4av - 6u^3 - 6u) + auv\Big), \quad \dot{v} = -\frac{v}{6}\Big(2u + c(6u^2 + 2) - 3av\Big). \tag{2.17}$$

If $c \neq 0$ and $a \geqslant 0$ the origin of this system is an hyperbolic stable node with eigenvalues $(-c)$ and $(-c)/3$.

If $c = 0$ and $a \neq 0$ system (2.17) writes as

$$\dot{u} = (avu)/6, \qquad \dot{v} = v(3av - 2u)/6. \tag{2.18}$$

We do the change of variable $vdt = ds$, system (2.18) becomes

$$\dot{u} = (au)/6, \qquad \dot{v} = (3av - 2u)/6. \tag{2.19}$$

The origin of the this system is an hyperbolic unstable node with eigenvalues $a/6$ and $a/2$.

If $c = 0$ and $a = 0$ system (2.17) is given by $\dot{u} = 0$, $\dot{v} = -uv/3$. This system has no singularity. This completes the proof of Statement (iii) of Proposition 2.2.

***Proof of statement (iv) of Proposition 2.2**. If $c \in (0, \infty)$ system (iv) has four hyperbolic singularities, two saddles at $q_1 = (0, (\sqrt{16c^2+1}+1)/(4c))$ with eigenvalues $(1-\sqrt{16c^2+1})/2$ and $\sqrt{16c^2+1}/2$, and $q_2 = (0, (-\sqrt{16c^2+1}+1)/(4c))$ with eigenvalues $(\sqrt{16c^2+1}+1)/2$ and $(-\sqrt{16c^2+1})/2$, a stable node at $q_3 = ((-\sqrt{16c^2+1}-1)/2, (-\sqrt{16c^2+1}+1)/(4c))$ with eigenvalues $(-\sqrt{16c^2+1}-1)/2$ and $(-\sqrt{16c^2+1})/2$, and an unstable node at $q_4 = (\sqrt{16c^2+1}-1)/2, (\sqrt{16c^2+1}+1)/(4c))$ with eigenvalues $(\sqrt{16c^2+1}-1)/2$ and $(\sqrt{16c^2+1})/2$. If $c = 0$ the system has two hyperbolic singularities, a saddle at $q_2 = (0, 0)$ with eigenvalues 1 and $(-1)/2$, and a stable node at $q_3 = (-1, 0)$ with eigenvalues (-1) and $(-1)/2$.*

System (iv) in the local chart U_1 writes as

$$\dot{u} = 3cu^2 - cv^2 - (3uv + 2u)/2, \quad \dot{v} = -v(v - 2cu + 1). \tag{2.20}$$

This system if $c \neq 0$, has a stable star node at $p_1 = (0, 0)$, with eigenvalues (-1) and (-1), and a saddle at $p_2 = (1/(3c), 0)$ with eigenvalues $(-1)/3$ and 1.

If $c = 0$ system (2.20) becomes

$$\dot{u} = -u(3v + 2)/2, \qquad \dot{v} = -v(v + 1). \tag{2.21}$$

This system has one hyperbolic stable star node at p_1 with eigenvalues (-1) and (-1).

In the local chart U_2 system (iv) writes as

$$\dot{u} = u^2 + cu(v^2 - 3) + (3uv)/2, \quad \dot{v} = \frac{v}{2}(2c(v^2 - 1) + v). \tag{2.22}$$

If $c \neq 0$ the origin of this system is an hyperbolic stable node with eigenvalues $(-3c)$ and $(-c)$.

If $c = 0$ system (2.22) is given by

$$\dot{u} = u^2 + (3uv)/2, \qquad \dot{v} = v^2/2. \tag{2.23}$$

*This system has one **SP** at the origin which is a linearly zero singularity. We need to do a blow-up $v = uw$ for describing its local phase portrait. After eliminating the common factor u of $\dot{u}$ and $\dot{w}$, by doing the rescaling of the independent variable $ds = udt$, we obtain the system $\dot{u} = u + (3uw)/2$, $\dot{w} = -w^2 - w$. For $u = 0$ this system has two hyperbolic saddles at the origin with eigenvalues (-1) and 1, and $(0, -1)$ with eigenvalues 1 and $(-1)/2$. Going back through the two changes of variables $v = uw$ and $ds = udt$ and by taking into account the direction of the flow of the system on the axes of coordinates, we obtain that the local phase portrait at the origin of system (2.23) is formed by two parabolic and two hyperbolic sectors. This completes the proof of Statement (iv) of Proposition 2.2.*

Proof of statement (v) of Proposition 2.2. *In what follows we shall illustrate the application of the invariant theory to the family of system (v).*

According to [18, Chapter 6, Diagram 6.1] (see also [84]) we have next table containing necessary and sufficient conditions for the number and multiplicity of finite singularities for an arbitrary quadratic system presented in any canonical form:

For system (v) we have:

$$\mu_0 = \frac{c^2}{2}, \ \ \mathbf{D} = -12a^3b^2c^6(a + 16bc^2)^2(a + 128bc^2), \ \ \text{and} \ \ \eta = \frac{9c^2}{16},$$

and we shall consider two cases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

Table 2.2: Number and multiplicity of the finite singular points of QS

No.	Zero–cycle $\mathcal{D}_{\mathbb{C}^2}(S)$	Invariant criteria	No.	Zero–cycle $\mathcal{D}_{\mathbb{C}^2}(S)$	Invariant criteria
1	$p+q+r+s$	$\mu_0\neq 0, \mathbf{D}<0, \mathbf{R}>0, \mathbf{S}>0$	10	$p+q+r$	$\mu_0=0, \mathbf{D}<0, \mathbf{R}\neq 0$
2	$p+q+r^c+s^c$	$\mu_0\neq 0, \mathbf{D}>0$	11	$p+q^c+r^c$	$\mu_0=0, \mathbf{D}>0, \mathbf{R}\neq 0$
3	$p^c+q^c+r^c+s^c$	$\mu_0\neq 0, \mathbf{D}<0, \mathbf{R}\leq 0$ $\mu_0\neq 0, \mathbf{D}<0, \mathbf{S}\leq 0$	12	$2p+q$	$\mu_0=\mathbf{D}=0, \mathbf{PR}\neq 0$
4	$2p+q+r$	$\mu_0\neq 0, \mathbf{D}=0, \mathbf{T}<0$	13	$3p$	$\mu_0=\mathbf{D}=\mathbf{P}=0, \mathbf{R}\neq 0$
5	$2p+q^c+r^c$	$\mu_0\neq 0, \mathbf{D}=0, \mathbf{T}>0$	14	$p+q$	$\mu_0=\mathbf{R}=0, \mathbf{P}\neq 0, \mathbf{U}>0$
6	$2p+2q$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0, \mathbf{PR}>0$	15	p^c+q^c	$\mu_0=\mathbf{R}=0, \mathbf{P}\neq 0, \mathbf{U}<0$
7	$2p^c+2q^c$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0, \mathbf{PR}<0$	16	$2p$	$\mu_0=\mathbf{R}=0, \mathbf{P}\neq 0, \mathbf{U}=0$
8	$3p+q$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0, \mathbf{P}=0, \mathbf{R}\neq 0$	17	p	$\mu_0=\mathbf{R}=\mathbf{P}=0, \mathbf{U}\neq 0$
9	$4p$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0, \mathbf{P}=\mathbf{R}=0$	18	0	$\mu_0=\mathbf{R}=\mathbf{P}=0, \mathbf{U}=0, \mathbf{V}\neq 0$

***1: The case* $\mathbf{D}\neq 0$.** *Then $c\neq 0$ and this implies $\mu_0\neq 0$ and $\eta\neq 0$ (more exactly we have $\mu_0>0$ and $\eta>0$). Considering the above table we examine two subcases: $\mathbf{D}<0$ and $\mathbf{D}>0$.*

***1.1: The subcase* $\mathbf{D}<0$.** *According to Table 2.2, system (v) in this case could possess either four real or four complex singularities. However we observe that this system possesses the real singularity $q_1=(0,0)$ and therefore we have obligatory four real distinct singularities. We observe that the condition $\mathbf{D}<0$ is equivalent to $a(a+128bc^2)>0$. Due to Proposition 2.1, we have $a+128bc^2>0$.*

On the other hand following the paper [17] (see Diagram 1 on page 5) due to the conditions $\mathbf{D}<0$ and $\mu_0>0$ we arrive at the configuration (10): $s,s,a,a;\ S,N,N$. In other words system (v) possesses at the finite part of the phase plane two saddles and two anti-saddles.

Examining the finite singularities of system (v) we determine the following coordinates:

$$q_1=(0,0);\quad q_2=\Big(a+16bc^2, (a+16bc^2)/(4c)\Big);\quad q_{3,4}=\Big(a, (a\mp\sqrt{a(a+128bc^2)})/(8c)\Big).$$

It is easy to detect that system (v) has the invariant straight line $x=a$ and on this line the finite singularities $q_{3,4}$ are located. Moreover, the finite singularities q_1 and q_2 are located on the invariant curve $f_5(x,y)=0$.

Since the condition $\mu_0 > 0$ is satisfied, then by [17] the quadrilateral formed by four finite singularities is convex and hence we have two saddles (respective two anti-saddles) on the opposite vertices. It remains now to examine the behavior of the separatrices of the saddles. In particular we have one saddle on the invariant line, which intersects the line at infinity $Z = 0$ at a singular point $N_1[0 : 1 : 0]$ and clearly in the case this infinite singularity is a saddle, we arrive at the existence of a separatrix connection and so on.

We have the next lemma.

Lemma *For the existence of an invariant straight line in one (respectively 2; 3 distinc directions in the affine plane it is necessary that $B_1 = 0$ (respectively $B_2 = 0$; $B_3 = 0$).* ■

For system (v) we obtain: $B_1 = 0$ (since we have already the invariant line $x = a$ in one direction) and

$$B_2 = \frac{1}{32}(-81)bc^2x^4\left(a-16bc^2\right)\left(3a-16bc^2\right)\left(a+144bc^2\right),$$
$$B_3 = \frac{1}{16}(-3)cx^2\left[-b(a-16bc^2)x^2 + 24bc(a-16bc^2)xy + a(3a+272bc^2)y^2\right].$$

So the condition $B_2 = 0$ due to $bc \neq 0$ gives

$$\left(a-16bc^2\right)\left(3a-16bc^2\right)\left(a+144bc^2\right) = 0, \tag{2.24}$$

and we have to consider each one of the possibilities given by this condition.

***1.1.1: If** $a - 16bc^2 = 0$**.** Then $b = a/(16c^2)$, and system (v) has two invariant lines: $x = a$ and $y = a/(2c)$ which is not a connection of separatrices. It is not too difficult to detect, that at the end of these invariant lines we have 2 infinite nodes and that the finite singularity $q_1 = (0,0)$ is a focus. As a result we get the phase portrait given by 22 of Figure 2.5.*

Since in this case we have $B_3 = -\dfrac{15}{4}a^2cx^2y^2 \neq 0$ we conclude that we could not have an invariant line in the third direction.

***1.1.2: If** $3a - 16bc^2 = 0$**.** Then $b = (3a)/(16c^2)$ and we obtain that system (v) has the following two invariant lines: $x - a = 0$, and $x - 12cy + 8a = 0$, and the global phase portrait is this case is given by 21 of Figure 2.5.*

A segment of the second invariant line is a sepparatrix connection of one finite saddle and one infinite saddle.

Now if we broke the separatrix connection perturbing the equality $b = (3a)/(16c^2)$ we obtain for $ac \neq 0$ and $b \in ((3a)/(16c^2), \infty)$, the phase portrait 20 of Figure 2.5, and for $ac \neq 0$ and $b \in (0, (3a)/(16c^2))$ the phase portrait 22 of Figure 2.5.

***1.1.3: The possibility** $a + 144bc^2 = 0$. We have $b = (-a)/(144c^2)$ and in this case system (v) possesses the invariant lines: $x - a = 0$, and $-x + 12cy = 0$, and the global phase portrait in this case is given by 23 of Figure 2.5.*

We again detect that the second invariant line connects one finite and one infinite saddle, and if we broke the separatrix connection perturbing the equality $b = (-a)/(144c^2)$, we arrive at the phase portrait 24 of Figure 2.5 for $ac \neq 0$ and $b \in ((-a)/(128c^2), (-a)/(144c^2))$, and at the phase portrait 25 of Figure 2.6 for $ac \neq 0$ and $b \in (0, (-a)/(144c^2))$.

***1.2: The subcase** $\mathbf{D} > 0$. According to Table 2.2, system (v) in this case has two real and two complex singularities. We observe that the condition $\mathbf{D} > 0$ is equivalent to $(a + 128bc^2) < 0$.*

Following the paper [17, Diagram 1] due to the conditions $\mathbf{D} < 0$, $\mu_0 > 0$ and $\eta > 0$ we arrive either at the configuration (25): $s, a; S, N, N$ or (26): $s, c; S, N, N$. According to this diagram the second case is realizable if and only if the conditions ($\mathfrak{C}_2$) for the existence of a center are satisfied, and we must have

$$(\mathfrak{C}_2): \quad \mathcal{T}_4 = \mathcal{T}_3 = 0,\ \mathcal{T}_2 > 0,\ \mathcal{B} < 0,\ \mathcal{F} = \mathcal{F}_1 = 0.$$

However for system (v) calculations yield:

$$\mathcal{T}_4 = -\frac{1}{128}a^3c^2(a + 80bc^2), \quad \mathcal{T}_3 = -\frac{1}{128}a^2c^2(11a + 560bc^2), \quad \mathcal{T}_3 = -\frac{1}{32}ac^2(9a + 160bc^2).$$

Due to $bc \neq 0$ evidently the conditions $\mathcal{T}_4 = \mathcal{T}_3 = 0$ imply $a = 0$ and this gives $\mathcal{T}_2 = 0$. This means that the conditions ($\mathfrak{C}_2$) could not be satisfied, i.e. we could not have a center. It can be easily detected that we have a saddle and a node and the phase portrait in this case is 27 of Figure 2.6.

We only point out that due to the conditions $a > 0$ and $a + 128bc^2 < 0$ (this implies $b < 0$) the condition $B_2 = 0$ from (2.24) could not be satisfied, because any one of the three conditions above gives $\mathbf{D} < 0$, which contradicts the condition $\mathbf{D} > 0$.

***2: The case* $\mathbf{D}=0$.** *This condition implies $abc(a+16bc^2)(a+128bc^2)=0$, and as $b\neq 0$ we get $ac(a+16bc^2)(a+128bc^2)=0$. Since for system (v) we have $\mu_0=\dfrac{c^2}{2}$ we consider two possibilities: $\mu_0\neq 0$ and $\mu_0=0$.*

***2.1: If* $\mu_0\neq 0$.** *Then $c\neq 0$ and the condition $\mathbf{D}=0$ in this case leads to the relation $a(a+16bc^2)(a+128bc^2)=0$. On the other hand considering Table 2.2 we observe that if $\mathbf{D}=0$ then the number and multiplicities of finite singularities in the case $\mu_0\neq 0$ depends on the value of the invariant polynomial $\mathbf{T}$. We have the next lemma.*

Lemma *Assume that for system (v) the conditions $\mu_0\neq 0$ and $\mathbf{D}=0$ are satisfied. Then we have $a+16bc^2=0$ (respectively $a+128bc^2=0$; $a=0$) if $\mathbf{T}>0$ (respectively $\mathbf{T}<0$; $\mathbf{T}=0$).* ∎

Proof. *To prove this lemma it is sufficient to observe that for $\mu_0\neq 0$ in the case $b=(-a)/(16c^2)$ $\Big($respectively $b=(-a)/(128c^2)$; $a=0$ $\Big)$ we obtain $\mathbf{T}=(21/16384)(a^6c^2x^2(8c^2y^2-2cxy+x^2)^2)>0$ $\Big($respectively $\mathbf{T}=-(147/1073741824)(a^6c^2(4cy-x)^2(4cy+3x)^2(8cy-x)^2)<0$; $\mathbf{T}=0$ $\Big)$. Thus we consider the three cases: $\mathbf{T}<0$, $\mathbf{T}>0$ and $\mathbf{T}=0$.*

2.1.1: **$\mathbf{T}<0$.** *According to Lemma 2 the condition $\mathbf{D}=0$ gives us $b=(-a)/(128c^2)$. So by Table 2.2, system (v) has one double and two simple finite singularities all real. On the other hand for this system in the considered case we calculate $B_2=-\dfrac{18225a^4x^4}{2097152}\neq 0$, due to $a\neq 0$. This means that we have not invariant lines in the second direction (different from $x=a$) and we have to investigate the phase portraits of the system (v) with $b=(-a)/(128c^2)$ and $B_2\neq 0$.*

Following [17, Diagram] since $\mathbf{D}=0$, $\mu_0=\dfrac{c^2}{2}>0$ and $\eta=\dfrac{9c^2}{16}>0$ we have to evaluate the invariant polynomial E_1 for the case under consideration. So we calculate $E_1=-\dfrac{49a^5c^2}{131072}\neq 0$ and by [17, Diagram 1] the double singularity is a saddle-node, whereas the simple finite singularities are a saddle and an anti-saddle which could not be a center. Moreover at infinity we have the same singularities: a saddle and two nodes. The global phase portrait in this case is 26 of Figure 2.6.

2.1.2: **$\mathbf{T}>0$.** *By Lemma 2 the condition $\mathbf{D}=0$ gives us $b=(-a)/(16c^2)$. So by Table 2.2 system (v) possesses one double real and two complex finite singularities. Since in this case $E_1=\dfrac{-7}{128}a^5c^2\neq 0$, according to [17, Diagram 1] the double singularity is a saddle-node and at infinity we have the same singularities: a saddle and two nodes.*

For system (v) *with* $b = (-a)/(16c^2)$ *we calculate* $B_2 = \frac{1}{8}(-81)a^4x^4 \neq 0$ *due to* $a \neq 0$. *Therefore by Lemma 1 we have not invariant lines in the second direction (different from* $x = a$ *) and hence the corresponding global phase portrait is 28 of Figure 2.6.*

2.1.3: $\mathbf{T} = 0$. *Considering Lemma 2 we have* $a = 0$ *and in accordance with Table 2.2 we evaluate for system* (v) *with* $a = 0$ *the invariant polynomials* $\mathbf{P}$ *and* $\mathbf{R}$*:*

$$\mathbf{P} = 0,\ \mathbf{R} = 12b^2c^6(4cy - x)^2.$$

Since $\mathbf{R} \neq 0$ *(due to* $bc \neq 0$*) according to Table 2 we deduce that for* $a = 0$, *system* (v) *has one triple and one simple real finite singularities. On the other hand following [17, see Diagram 1, page 6, block* $\mathcal{A}_4$*] since* $\mathbf{D} = \mathbf{T} = \mathbf{P} = 0$, $\mathbf{R} \neq 0$, $\mu_0 \neq 0$ *and* $\eta = \frac{9c^2}{16}$ *we have to evaluate the invariant polynomial* E_3 *for the case under consideration. So calculation yields* $E_3 = 8b^2c^6 > 0$ *we conclude that system* (v) *with* $a = 0$ *has a saddle and an anti-saddle, where the infinite singularities remaining the same. It could be easily detected that the saddle is nilpotent (triple) whereas the simple point is a node. Since for this system we have* $B_2 = -93312b^4c^8x^4 \neq 0$, *then the corresponding global phase portrait is 30 of Figure 2.6.*

2.2: $\mu_0 = 0$. *Then* $c = 0$ *and this implies* $\mathbf{D} = 0$. *We detect that in this case system* (v) *becomes as system*

$$\dot{x} = \frac{1}{2}x(x - a), \qquad \dot{y} = \frac{1}{4}y(x - 2a), \tag{2.25}$$

possessing the invariant curve $f_5(x, y) = 0$ *and the following invariant lines:* $x = 0$ *(double),* $x = a$, *and* $y = 0$. *So considering the line at infinity the above systems possess invariant line of total multiplicity 5. However such family of systems is completely investigated in [92] and [91]. Since we have two parallel invariant lines (one double and one simple) and another line* $y = 0$ *intersect them we arrive at the configuration of invariant lines given by Config. 5.14.*

On the other hand by [91] this configuration leads to two phase portraits: Picture 5.14(a) and Picture 5.14(b) and in order to detect which one of them is realizable in our case, we have to evaluate the following invariant polynomials:

$$\mu_0 = \mu_1 = \kappa = \kappa_1 = 0,\ \widetilde{M} = \frac{-x^2}{2} \neq 0,\ \mu_2 = \frac{a^2x^2}{32} > 0,\ L = x^2 > 0,\ \text{and}\ K_2 = 3a^2x^2 > 0.$$

So we obtain that there are satisfied necessary and sufficient conditions for the existence of the phase portrait Picture 5.14(a), which corresponds to the phase portrait 29 of Figure 2.6.

If $b \neq 0$, $a = 0$ and $c = 0$, or $b = a = c = 0$ the system has $x = 0$ as a line of singularities, and by doing the change of variable $xdt = ds$, we know that the system has an unstable node at the origin of coordinates with eigenvalues $1/2$ and $1/4$. For the infinite ***SP****, system (v) has a stable node and a saddle. The corresponding phase portrait of the system in this case is 31 of Figure 2.6.*

Finally, Assume $b = 0$ we distinguish three cases:

- *If $ac \neq 0$ the system has three* ***SP****, a stable node at q_1 with associated Jacobian matrix $\begin{pmatrix} \frac{-a}{2} & 2ac \\ 0 & \frac{-a}{2} \end{pmatrix}$ and eigenvalues $(-a)/2$ and $(-a)/2$, a saddle at $q_3 = (a, 0)$ with eigenvalues $(a/2)$ and $(-a)/4$, and a semi-hyperbolic* ***SP*** *at $q_2 = (a, /(4c))$ with eigenvalues 0 and $a/4$. In order to obtain the local phase portrait at this point we use Theorem 2.19 of [42], and we obtain that q_2 is a saddle-node. For the infinite* ***SP****, system (v) has two stable nodes and a saddle. The corresponding phase portrait is 32 of Figure 2.6.*

- *If $a = 0$ and $c \neq 0$ system (v) becomes*

$$\dot{x} = \frac{x^2}{2} - 2cxy, \quad \dot{y} = cy^2 + \frac{xy}{4}. \tag{2.26}$$

  System (2.26) has one ***SP*** *at the origin which is a linearly zero singularity. By doing a blow-up $y = zx$ we obtain that the local phase portrait at the origin of this system is formed by two parabolic and two hyperbolic sectors. For the infinite singular points, system (v) has two stable nodes and a saddle. The corresponding phase portrait is 33 of Figure 2.6.*

- *If $a \neq 0$ and $c = 0$ we get the phase portrait 29 of Figure 2.6. This completes the proof of Statement (v) of Proposition 2.2.* ∎

2.1.3 Local and global phase portraits

System (i) If $c \in (-\infty, -6) \cup (-6, 0)$ from statement (i) of Proposition 2.2 we obtain the local phase portrait of the finite and infinite **SP**. Due to the fact that the two singular points q_1 and q_2 belongs to the *Right Serpentine* 1 invariant curve of the system, we obtain some orbits on this invariant curves connecting those singular points, these connections vary if $c \in (-\infty, -6) \cup (-6, 0)$, (see local phase portrait 1 of Figure 2.1). Since $\dot{x}_{|x=0} = cy^2 < 0$, and $\dot{y}_{|y=0} = c/3 < 0$ the separatrices for which we do not know their α- or ω-limit can be easily determined from the mentioned figures, obtaining the global phase portrait 1 of Figure 2.4.

If $c = -6$ we get the local phase portrait 2 of Figure 2.1 for the finite and infinite **SP** of the system. Since $\dot{x}_{|x=0} = -6y^2 < 0$, and $\dot{y}_{|y=0} = -2$ we get that the global phase portrait in this case is 2 of Figure 2.4.

If $c = 0$ the system has one finite linearly zero **SP** at the origin, whose local phase portrait consists of two parabolic and two hyperbolic sectors; this one finite **SP** belong to the *Right Serpentine* 1 invariant curve of the system, and to the intersection of the three straight lines $x = 0$, $y = 0$, and $x + y = 0$ of the system. Then according to the local phase portraits 3 of Figure 2.1 we get the global phase portrait 3 of Figure 2.4.

If $c \in (0, 3) \cup (3, \infty)$ we get the local phase portrait 4 of Figure 2.1 for the finite and infinite **SP** of the system. Since $\dot{x}_{|x=0} = cy^2 > 0$, and $\dot{y}_{|y=0} = c/3 > 0$ we get that the global phase portrait in this case is 4 of Figure 2.4, respectively.

If $c = 3$ we get the local phase portrait 5 of Figure 2.1 for the finite and infinite **SP** of the system. Since $\dot{x}_{|x=0} = 3y^2 > 0$, and $\dot{y}_{|y=0} = 1$ we get the global phase portrait 5 of Figure 2.4.

System (ii) If $c = 0$ and $a \neq 0$ we get the local phase portrait 6 of Figure 2.1 and since $\dot{x}_{|x=0} = -a^2/2 < 0$, $\dot{y}_{|y=0} = 0$, and $y = 0$ is a straight invariant line for the system, then we get the global phase portrait 6 of Figure 2.4.

If $c \in (0, 2)$ and $a \neq 0$ we get the local phase portrait 7 of Figure 2.1 for the finite and infinite **SP** of the system. Since $\dot{x}_{|x=0} = -a^2/2 < 0$, $\dot{y}_{|y=0} = 0$, and $y = 0$ is an invariant straight line for the system, we get the global phase portrait 7 of Figure 2.4.

If $c = 2$ and $a \neq 0$ the system has two finite semi-hyperbolic saddle-nodes; one of them belongs to the *Witch of Agnesi* invariant curve, and the second belongs to the intersection of the two invariant straight lines $y = 0$ and $x - a = 0$ of the system . We have also $\dot{x}_{|x=0} = -a^2/2 < 0$, and $\dot{y}_{|y=0} = 0$, and from the local phase portrait 8 of Figure 2.1 we know that the global one is 8 in Figure 2.4.

If $c \in (2, \infty)$ and $a \neq 0$ from statement (ii) of Proposition 2.2, we obtain that the system has four **SP**, an unstable node at q_2 and a saddle at q_3 which belong to the *Witch of Agnesi* invariant curve of the system, a saddle at q_4 and a stable node at q_1 which belong to the intersection of the three invariant straight lines $y = 0$, $x - \frac{a}{2}(a\sqrt{c^2-4}+c) = 0$, and $x + \frac{a}{2}(c - a\sqrt{c^2-4}) = 0$. The system has two infinite **SP**, a saddle and an unstable node in the local chart U_1 and the origin of the local chart U_2 is a stable node. Since $\dot{x}_{|x=0} = -a^2/2 < 0$ and $\dot{y}_{|y=0} = 0$, (see the local phase portrait 9 of Figure 2.1). Hence the phase portrait 9 of Figure 2.4 is the global one of this system.

If $c = 0$ and $a = 0$ we get the local phase portrait 10 of Figure 2.1 and since $\dot{x}_{|x=0} = 0$, $\dot{y}_{|y=0} = 0$, and the fact that the system has two invariant straight lines $x = 0$ and $y = 0$ intersect at the **SP** of the system, then we get the global phase portrait 10 of Figure 2.4.

If $c \neq 0$ and $a = 0$ the system has one finite linearly zero **SP** at the origin, whose local phase portrait consists of two parabolic and two hyperbolic sectors; this finite **SP** belongs to the *Witch of Agnesi* invariant curve of the system. Since $x = 0$ and $y = 0$ are two invariants straight lines for the system containing in their intersection the origin, (see also the local phase portrait 11 of Figure 2.1) we get the global phase portrait 11 of Figure 2.4.

System (iii) If $c = 0$ and $a \neq 0$ the system has two finite **SP**, one of them belongs to the *Cissoid of Diocles* invariant curve of the system. Since each one of the straight lines: $x = 0$, $x - a = 0$ and $y = 0$ are invariant of the system containing in their intersection the two singular points, and according to the local phase portrait 12 of Figure 2.1, it results that phase portrait 12 of Figure 2.4 is the global one of the system.

If $c \in (0, 1/\sqrt{48})$ and $a \neq 0$ the system has four finite **SP**; where the stable node q_1 and the saddle q_2 belong to the *Cissoid of Diocles* invariant curve of the system. For the infinit **SP** it has a stable node at the origin of the local chart U_2. Taking into account the directions of the vector field of the system $\dot{x}_{|x=0} = (2acy)/3$, $\dot{y}_{|y=0} = cx^2 > 0$, and the fact that $x - a = 0$ is an invariant straight line for the system, then according to the local phase portrait 13 of Figure 2.2 we obtain the global phase portrait 13 of Figure 2.5.

If $c = 1/\sqrt{48}$ and $a \neq 0$ the system has three finite **SP**, two of them belong to the *Cissoid of Diocles* invariant curve of the system, and by using the same arguments as in the previous case (see also the local phase portrait 14 of Figure 2.2) we get the global phase portrait 14 of Figure 2.5.

If $c \in (1/\sqrt{48}, \infty)$ and $a \neq 0$ from statement (iii) of Proposition 2.2, we obtain that the system has two finite **SP** belong to the *Cissoid of Diocles* invariant curve of the system, so they connect each one to the other. For the infinite **SP** it has only an hyperbolic stable node at the origin of the local chart U_2. Due to the fact that $\dot{x}_{|x=0} = (2acy)/3$ and $\dot{y}_{|y=0} = cx^2 > 0$, (see the local phase portrait 15 of Figure 2.2). Hence, the phase portrait 15 of Figure 2.5 is the global phase portrait.

If $c \neq 0$ and $a = 0$ the system has one finite linearly zero **SP** at the origin, whose local phase portrait consists of two hyperbolic sectors; this finite **SP** belongs to the *Cissoid of Diocles* invariant curve of the system. Since $\dot{x}_{|x=0} = 0$, $\dot{y}_{|y=0} = x^2 > 0$ and $x = 0$ is a straight invariant line for the system, (see also the local phase portraits 16 of Figure 2.2) we get the global phase portrait 16 of Figure 2.5.

If $c = 0$ and $a = 0$ we get the local phase portrait 17 of Figure 2.2 and since $\dot{x}_{|x=0} = 0$, $\dot{y}_{|y=0} = 0$, and the fact that the system has two invariant straight lines $x = 0$ and $y = 0$ intersect at the **SP** of the system, then we get the global phase portrait 17 of Figure 2.5.

System (iv) If $c = 0$ the system has two finite **SP**, a stable node at $(-1, 0)$ which belongs to the *Crunodal cubic* and to the invariant straight line $x + 1 = 0$ of the system, and a saddle at $(0, 0)$ which belongs to the intersection of the two straight lines of the system $x = 0$ and $y = 0$. For the infinite **SP** it has a stable node at U_1 and the origin of U_2 is a linearly zero **SP** whose local phase portrait consists of two parabolic and two hyperbolic sectors. According to the local phase portrait 18 of Figure 2.2, it results the global phase portrait 18 of Figure 2.5.

If $c \in (0, \infty)$ from statement (iv) of Proposition 2.2; we obtain that the system has four **SP**, two nodes at q_3 and q_4 which belong to the *Crunodal cubic* invariant curve of the system, the four SP belong to the invariant straight lines $y - \frac{1-\sqrt{16c^2+1}}{4c} = 0$ and $y - \frac{1+\sqrt{16c^2+1}}{4c} = 0$, so they connect each one to the other. The system has two infinite **SP**, a saddle and a stable node in the local chart U_1, and a stable node at the origin of U_2. Taking into account the direction of the vector field of the system in the y–axes $\dot{y}_{|y=0} = -c < 0$ (see also the local phase portrait 19 of Figure 2.2), we know that the global phase portrait is 19 of Figure 2.5.

For systems (v), we have clearly explained and analyzed their global phase portraits in the proof of statement (v) of Proposition 4.2.

2.1.4 Statement of the main results

Now we provide the global phase portraits in the Poincaré disc of all these systems. More precisely our main result is the following

Theorem 2.2 *The global phase portraits in the Poincaré disc of the five systems of Theorem 2.1 are:*

(i.1) 1 *for system* (i) *when* $c \in (-\infty, -6) \cup (-6, 0)$*;*

(i.2) 2 *for system* (i) *when* $c = -6$*;*

(i.3) 3 *for system* (i) *when* $c = 0$*;*

(i.4) 4 *for system* (i) *when* $c \in (0, 3) \cup (3, \infty)$*. This phase portrait is topologically equivalent to the phase portrait* 7 *of system* (ii) *when* $c \in (0, 2)$ *with two different invariant algebraic curves;*

(i.5) 5 *for system* (i) *when* $c = 3$*. This phase portrait is topologically equivalent to the phase portrait* 6 *of system* (ii) *when* $c = 0$*, with two different invariant algebraic curves.*

(ii) System (ii) *exhibiting the cubic invariant algebraic curve Witch of Agnesi takes the form* (ii) *with* $a \neq 0$*, has four different phase portraits* 6*,* 7*,* 8 *and* 9*, more precisely it has the phase portrait;*

(ii.1) 6 *when* $c = 0$ *and* $a \neq 0$*;*

(ii.2) 7 *when* $c \in (0, 2)$ *and* $a \neq 0$*;*

(ii.3) 8 *when* $c = 2$ *and* $a \neq 0$*;*

(ii.4) 9 *when* $c \in (2, \infty)$ *and* $a \neq 0$*. While its closed normal form has in addition to the four previous phase portraits, the two following ones:*

(ii.5) 10 *when* $c = 0$ *and* $a = 0$;

(ii.6) 11 *when* $c \neq 0$ *and* $a = 0$. *This phase portrait is topologically equivalent to the phase portrait* 33 *of the closed normal form of system* (v) *when* $b = 0$, $a = 0$ *and* $c \neq 0$, *with two different invariant algebraic curves.*

(iii) *System* (iii) *exhibiting the cubic invariant algebraic curve Cissoid of Diocles takes the form* (iii) *with* $a \neq 0$, *has four different phase portraits* 12, 13, 14 *and* 15, *more precisely it has the phase portrait;*

(iii.1) 12 *when* $c = 0$ *and* $a \neq 0$;

(iii.2) 13 *when* $c \in (0, 1/\sqrt{48})$ *and* $a \neq 0$;

(iii.3) 14 *when* $c = 1/\sqrt{48}$ *and* $a \neq 0$;

(iii.4) 15 *when* $c \in (1/\sqrt{48}, \infty)$ *and* $a \neq 0$.

While its closed normal form has in addition to the four previous phase portraits, the two following ones:

(iii.5) 16 *when* $c \neq 0$ *and* $a = 0$;

(iii.6) 17 *when* $c = 0$ *and* $a = 0$.

(iv.1) 18 *for system* (iv) *when* $c = 0$. *This phase portrait is topologically equivalent to the phase portrait* 29 *of system* (v) *when* $a \neq 0$, $c = 0$ *and* $b \in \mathbb{R}$;

(iv.2) 19 *for system* (iv) *when* $c \in (0, \infty)$.

(v) *System* (v) *exhibiting the cubic invariant algebraic curve Longchamps, takes the form* (v) *with* $b \neq 0$, *has tweleve different phase portraits, more precisely it has the phase portrait;*

(v.1) 20 *when* $ac \neq 0$ *and* $b \in \left((3a)/(16c^2), \infty\right)$;

(v.2) 21 *when* $ac \neq 0$ *and* $b = (3a)/(16c^2)$;

(v.3) 22 *when* $ac \neq 0$ *and* $b \in \left(0, (3a)/(16c^2)\right)$;

(v.4) 23 *when* $ac \neq 0$ *and* $b = (-a)/(144c^2)$;

(v.5) 24 *when* $ac \neq 0$ *and* $b \in \left((-a)/(128c^2), (-a)/(144c^2)\right)$;

(v.6) 25 *when* $ac \neq 0$ *and* $b \in \left((-a)/(144c^2), 0\right)$;

(v.7) 26 *when* $ac \neq 0$ *and* $b = (-a)/(128c^2)$;

(v.8) 27 *when* $ac \neq 0$ *and* $b \in \left(-\infty, (-a)/(16c^2)\right) \cup \left((-a)/(16c^2), (-a)/(128c^2)\right)$;

(v.9) 28 *when* $ac \neq 0$ *and* $b = (-a)/(16c^2)$;

(v.10) 29 *when* $a \neq 0$, $c = 0$ *and* $b \neq 0$. *This phase portrait is topologically equivalent to the phase portrait of its closed normal form when* $b = c = 0$ *and* $a \neq 0$;

(v.11) 30 *when* $bc \neq 0$ *and* $a = 0$;

(v.12) 31 *when* $a = 0$, $c = 0$ *and* $b \neq 0$. *This phase portrait is topologically equivalent to the phase portrait of its closed normal form when* $b = a = c = 0$.

For the remaining closed normal form of this system, and in addition to the previous tweleve phase portraits, it has the two following ones:

(v.13) 32 *when* $b = 0$ *and* $ac \neq 0$;

(v.14) 33 *when* $b = 0$, $a = 0$ *and* $c \neq 0$.

2.1.5 Figure of local and global phase portraits

In what follows we give the local and the global phase portraits of systems mentioned in Theorem 2.2, respectively.

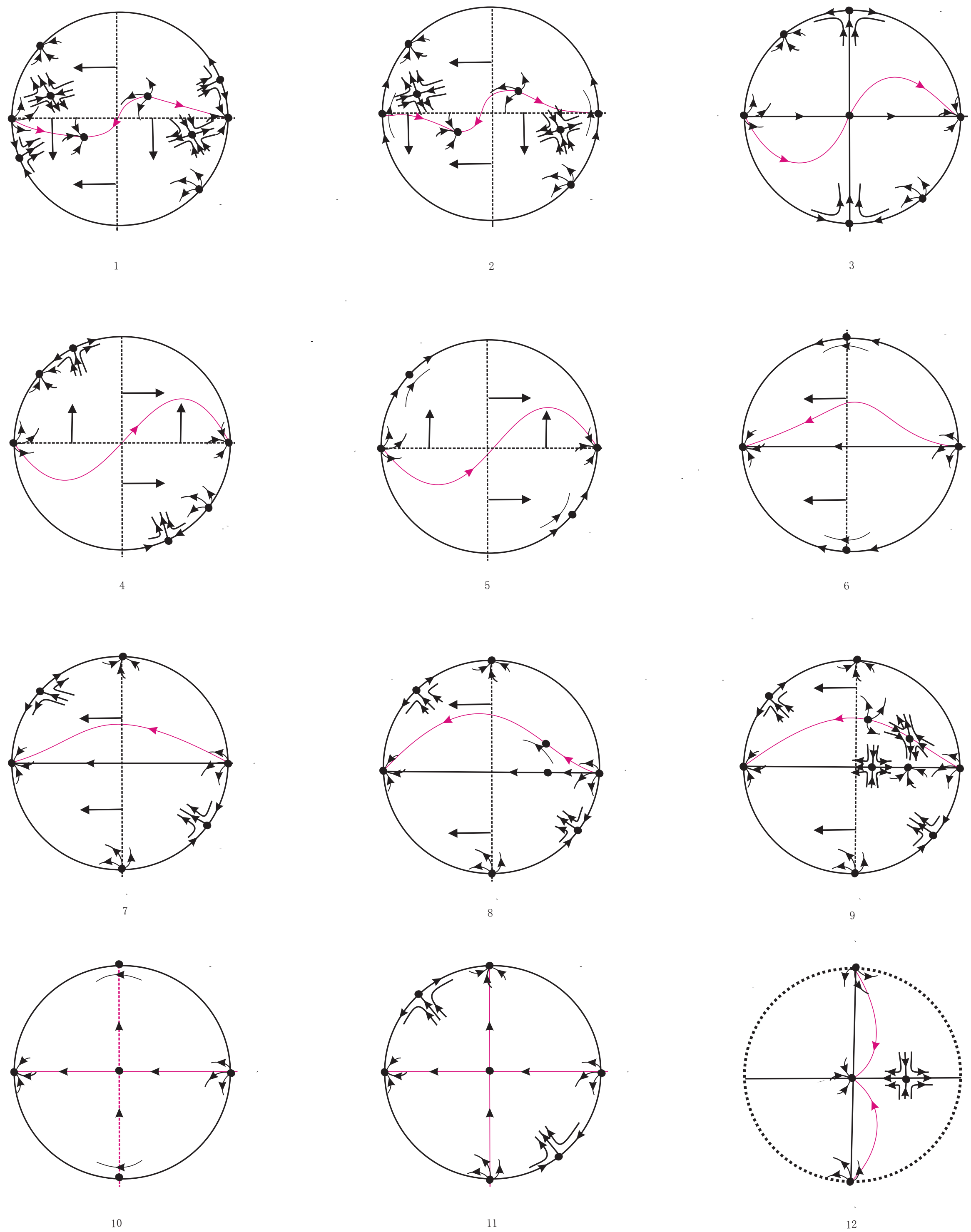


Figure 2.1: Local phase portraits in the Poincaré disc. The invariant algebraic curves of degree 3 are drawn in red color.

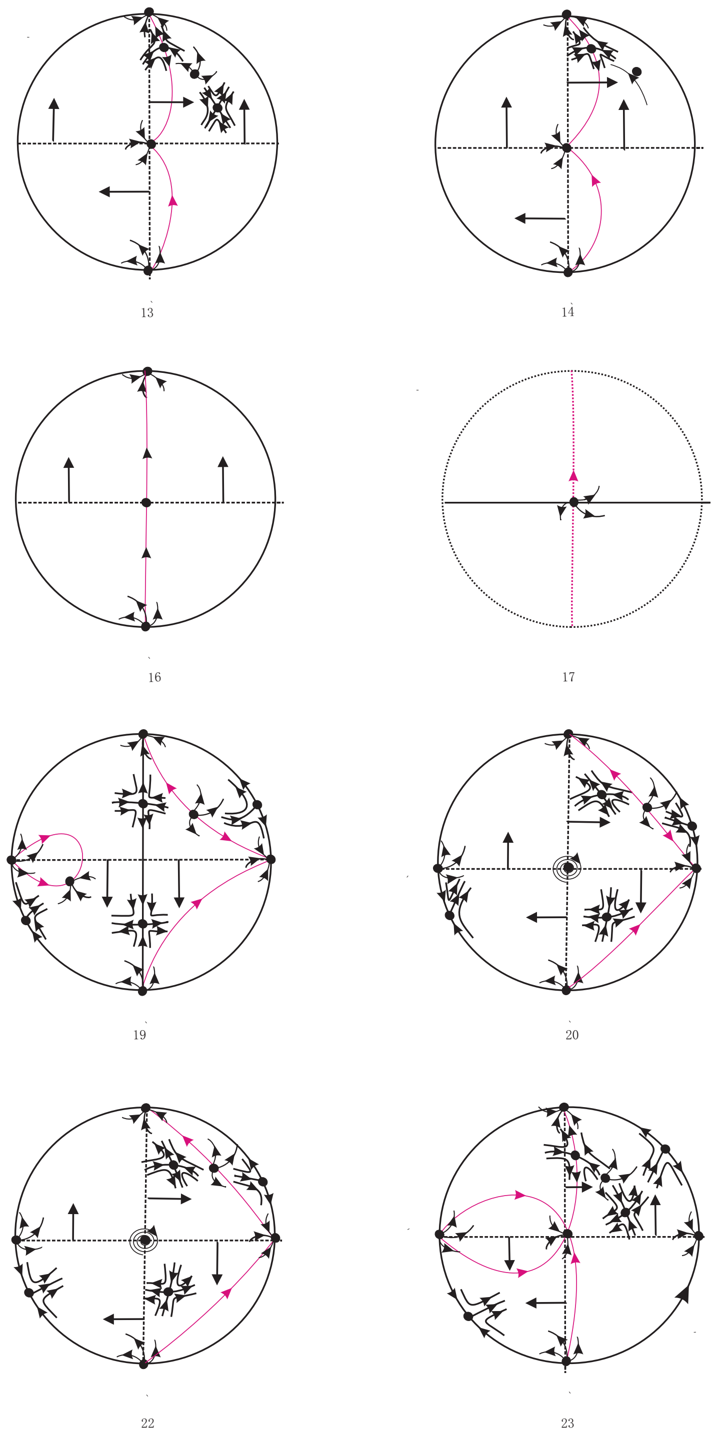


Figure 2.2: Continuation of Figure 2.1.

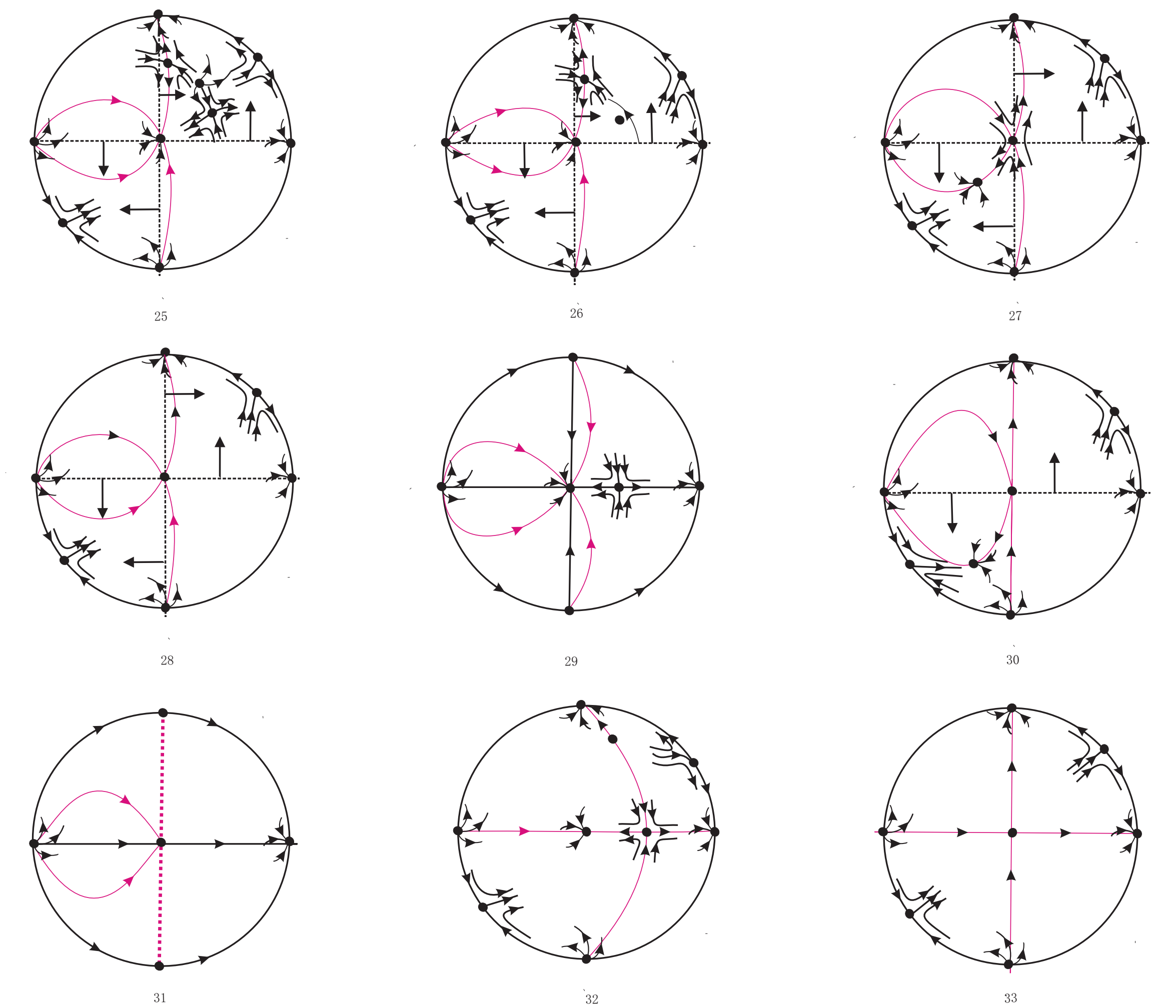


Figure 2.3: Continuation of Figure 2.1.

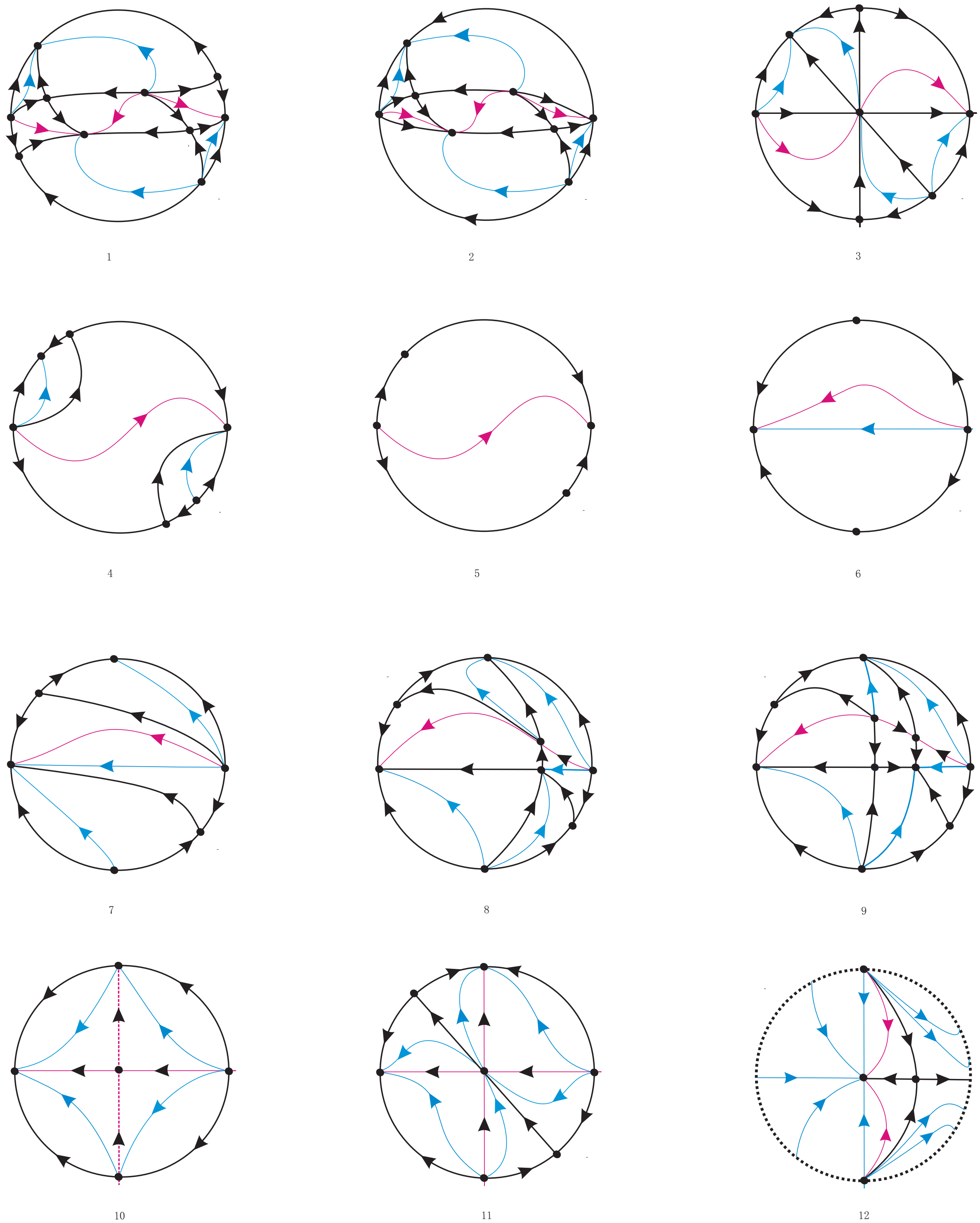


Figure 2.4: Global phase portraits in the Poincaré disc. The invariant algebraic curves of degree 3 are drawn in red color. An orbit inside a canonical region is drawn in blue except if it is contained in the invariant algebraic curve. The separatrices are drawn in black except if the separatrix is contained in the invariant algebraic curve then it is of red color but its arrow is black in order to indicate that is a separatrix.

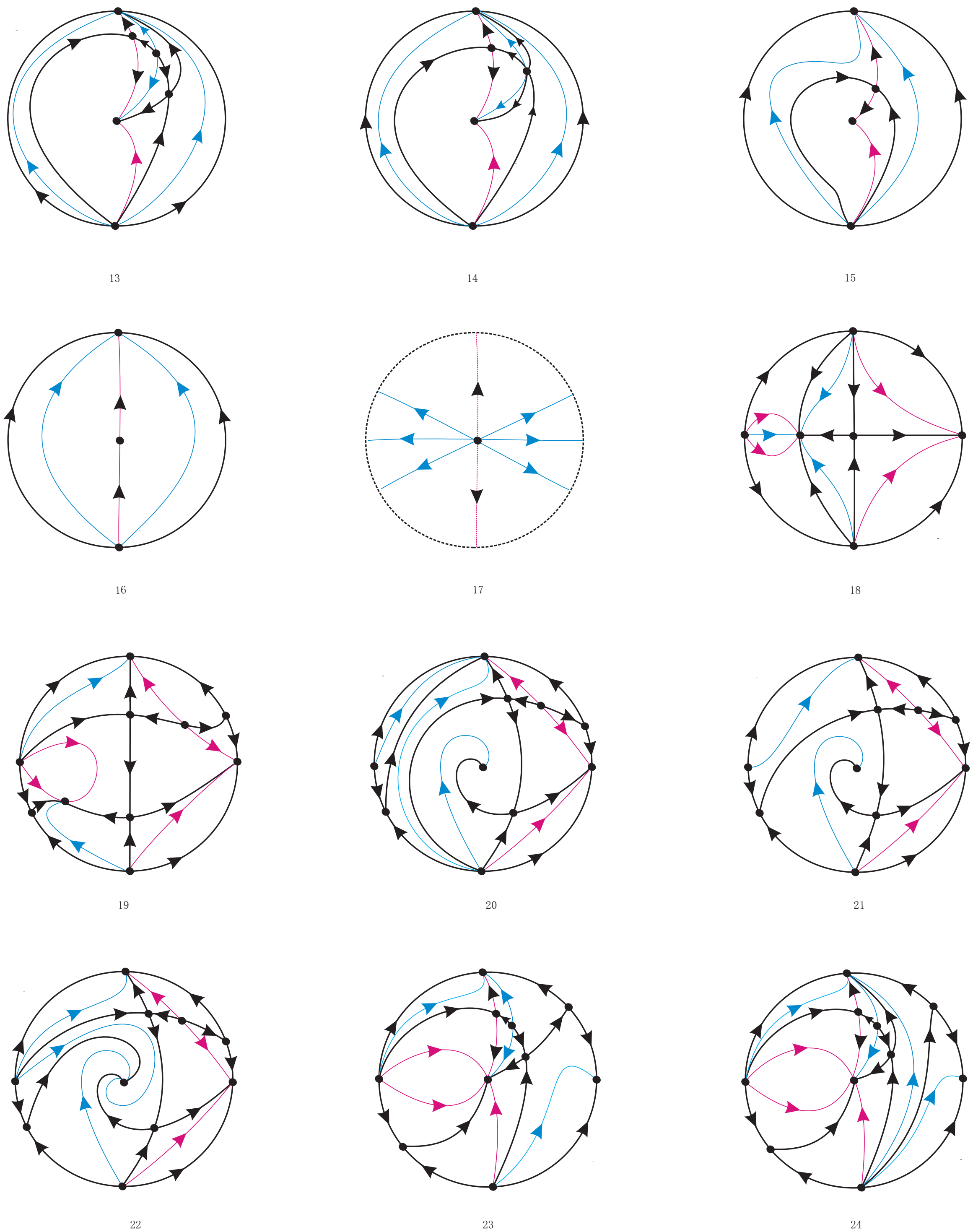


Figure 2.5: Continuation of Figure 2.4.

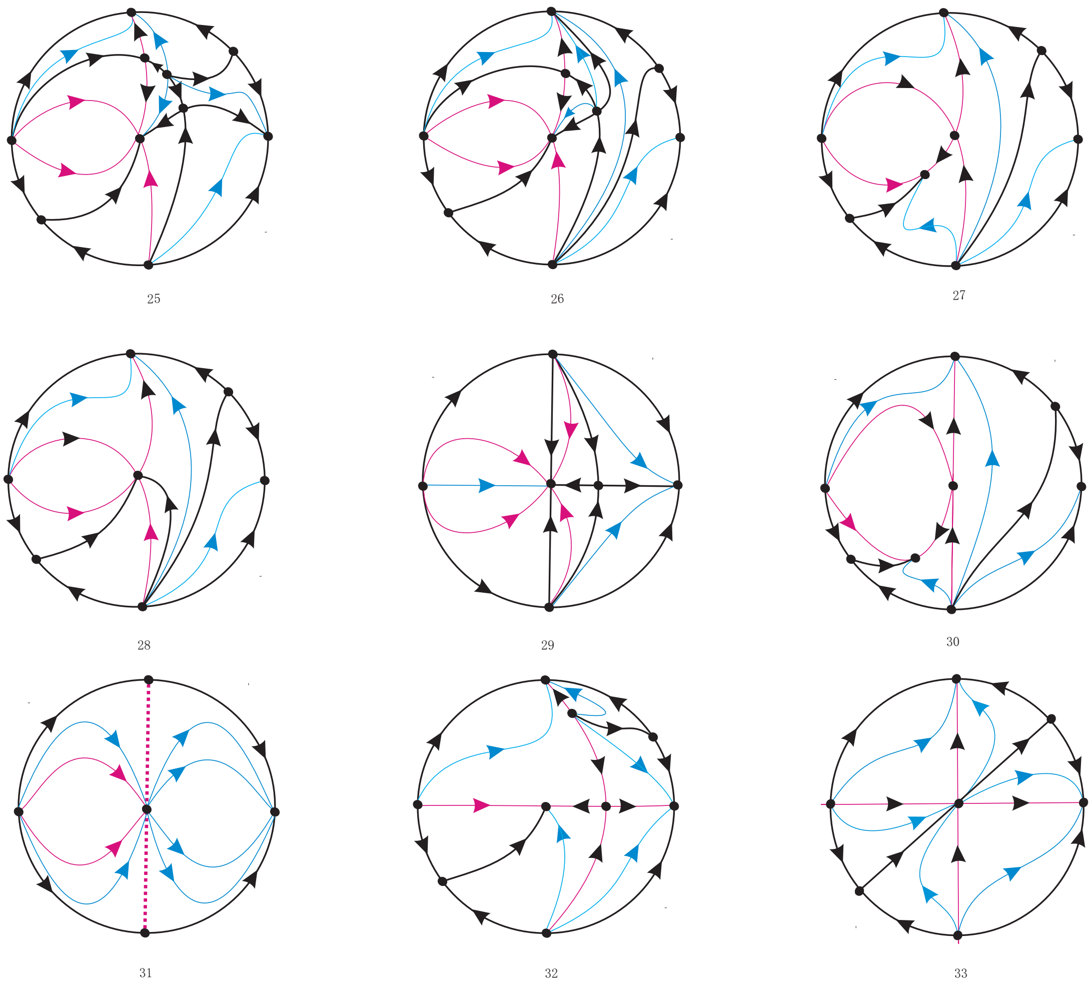


Figure 2.6: Continuation of Figure 2.4.

We intend in this section to classify the global phase portraits of a quadratic polynomial differential system exhibiting reducible invariant curve of degree three, by investigating their global phase portraits in the Poincaré disc. We realize that these systems produce 13 topologically different phase portraits.

2.2.1 Reducible invariant algabraic curve

The reducible cubic curve given by $f(x,y) = (y-k)(x^2+y^2-1) = 0$ is invariant of the quadratic systems represented in the following Theorem.

Theorem 2.3 *The reducible algebraic curve of degree three given by: $f(x,y) = 0$ with $f(x,y) = (y-k)(x^2+y^2-1)$ where $k \neq 0$, is an invariant algebraic curve with associated cofactor $K(x,y) = ax$ of* **QS**

$$\begin{aligned} \dot{x} &= \frac{1}{2}(a-1)x^2 + \frac{1}{2}(a-3)y^2 + ky + \frac{1-a}{2}, \\ \dot{y} &= x(y-k). \end{aligned} \tag{2.27}$$

Proof. *It is immediate that the function f on the orbits of systems (2.27) satisfy*

$$\frac{df}{dt} = \dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y} = Kf.$$

Remark 6 *Systems (2.27) are invariant under the change $(x,y,t,a,k) \longrightarrow (-x,-y,t,a,-k)$, then we only need to study them for $k > 0$.*

2.2.2 Finite singular points

For a planar polynomial differential systems (2.27) their finite singular points are characterized in the following proposition.

Proposition 2.3 *The following statements hold for the quadratic systems (2.27).*

(I) Assume $k \in (0,1)$

(i) *If $a \in (1, c_1) \cup (c_2, 3) \cup (3, \infty)$ where $c_1 = 2 - \sqrt{1-k^2}$ and $c_2 = 2 + \sqrt{1-k^2}$, then systems (2.27) have four singularities, an hyperbolic stable node at $p_1 = (-\sqrt{1-k^2}, k)$, an hyperbolic unstable node at $p_2 = (\sqrt{1-k^2}, k)$; the third singularity at $p_3 = \left(0, \dfrac{k + \sqrt{a^2 - 4a + k^2 + 3}}{3-a}\right)$ which is an hyperbolic saddle if $a \in (1, c_1) \cup (3, \infty)$ and a center if $a \in (c_2, 3)$; the fourth singular point at $p_4 = \left(0, \dfrac{k - \sqrt{a^2 - 4a + k^2 + 3}}{3-a}\right)$ which is an hyperbolic saddle if $a \in (c_2, 3) \cup (3, \infty)$ and a center if $a \in (1, c_1)$.*

(ii) *If $a \in (-\infty, 1)$ systems (2.27) have four singularities, two hyperbolic saddles at p_1 and p_2, and two centers at p_3 and p_4.*

(iii) *If $a \in (c_1, c_2)$ systems (2.27) have two hyperbolic singularities, a stable node at p_1 and an unstable node at p_2.*

(iv) *If $a = c_1$ systems (2.27) have three singularities, an hyperbolic stable node at p_1, an hyperbolic unstable node at p_2 and a nilpotent singularity at $p_3 = \left(0, \dfrac{1 - \sqrt{1-k^2}}{k}\right)$, where its local phase portrait formed by two hyperbolic sectors.*

(v) *If $a = c_2$ systems (2.27) have three singularities, an hyperbolic stable node at p_1, an hyperbolic unstable node at p_2 and a nilpotent singularity at $p_4 = \left(0, \dfrac{1 + \sqrt{1-k^2}}{k}\right)$, where its local phase portrait formed by two hyperbolic sectors.*

(vi) *If $a = 1$ systems (2.27) have $y = k$ as a line of singularities, and by doing the change of variables $(y-k)dt = ds$, we know that the systems have a center at the origin.*

(vii) *If $a = 3$ systems (2.27) have three hyperbolic singularities, a stable node at p_1, an unstable node at p_2 and a saddle at $p_4 = \left(0, \dfrac{1}{k}\right)$.*

(II) Assume $k \in (1,\infty)$

- (i) *If $a \in (-\infty,1)\cup(3,\infty)$ systems (2.27) have two singularities, an hyperbolic saddle at p_3 and a center at p_4.*
- (ii) *If $a \in (1,3)$ they have two centers at p_3 and p_4.*
- (iii) *If $a = 1$ they have $y = k$ as a line of singularities, and by doing the change of variables $(y-k)dt = ds$, we know that the systems have a center at the origin.*
- (iv) *If $a = 3$ they have one singularity at p_4 which is a center.*

(III) Assume $k = 1$

- (i) *If $a = 1$ system (2.27) has $y = 1$ as a line of singularities, and by doing the change of variables $(y-1)dt = ds$, we know that it has a center at the origin.*
- (ii) *If $a = 2$ the system has one linearly zero finite singularity at $p_3 = (0,1)$, and its local phase portrait formed by two elliptic sectors.*
- (iii) *If $a = 3$ it has one nilpotent finite singularity at p_3, and its local phase portrait formed by two parabolic and two hyperbolic sectors.*
- (iv) *If $a \in \mathbb{R}\setminus\{1, 2, 3\}$ systems (2.27) have two singularities; p_3 which is nilpotent, and its local phase portrait formed by two parabolic, two elliptic and two hyperbolic sectors if $a \in (3,\infty)$, and two hyperbolic sectors if $a \in (-\infty,1)\cup(1,2)\cup(2,3)$; the fourth singularity at $p_4 = \left(0, \dfrac{1-a}{a-3}\right)$ which is an hyperbolic saddle if $a \in (3,\infty)$ and a center if $a \in (-\infty,1)\cup(1,2)\cup(2,3)$.*

Proof.

Proof of statement (I) of Proposition 2.3.

If $a \in (1,c_1)\cup(c_2,3)\cup(3,\infty)$ the differential systems (2.27) have four singularities, an hyperbolic stable node at p_1 which has the eigenvalues $\lambda_1 = -(a-1)\sqrt{1-k^2}$ and $\lambda_2 = -\sqrt{1-k^2}$, and an hyperbolic unstable node at p_2 with its corresponding eigenvalues $\lambda_1 = \sqrt{1-k^2}$ and $\lambda_2 = (a-1)\sqrt{1-k^2}$, and the third singular point p_3 has $\lambda_{1,2} = \mp i B_1$ such that $B_1 = \sqrt{-S((a-2)k+S)/(a-3)}$ and $S = \sqrt{(a-4)a+k^2+3}$, then $\lambda_1.\lambda_2 = -\dfrac{S\,((a-2)k+S)}{a-3}$. According to the sign of the parameter a we know that p_3 is an hyperbolic saddle if $a \in (1,c_1)\cup(3,\infty)$ and a center if $a \in (c_2,3)$.

The fourth singularity p_4 *has* $\lambda_{1,2} = \mp iB_2$ *such that* $B_2 = \sqrt{S((a-2)k-S)/(a-3)}$ *and* $S = \sqrt{(a-4)a+k^2+3}$, *then* $\lambda_1.\lambda_2 = \dfrac{S\,((a-2)k-S)}{a-3}$. *According to the sign of the parameter* a *we know that* p_4 *is an hyperbolic sadlle if* $a \in (c_2, 3) \cup (3, \infty)$ *and a center if* $a \in (1, c_1)$. *Then the statement (i) holds.*

If $a \in (-\infty, 1)$ *systems* (2.27) *have four singularities, an hyperbolic stable node at* p_1 *which has the eigenvalues* $\lambda_1 = -(a-1)\sqrt{1-k^2}$ *and* $\lambda_2 = -\sqrt{1-k^2}$, *and an hyperbolic unstable node at* p_2 *with its corresponding eigenvalues* $\lambda_1 = \sqrt{1-k^2}$ *and* $\lambda_2 = (a-1)\sqrt{1-k^2}$, *the third singularity at* p_3 *has the eigenvalues* $\lambda_{1,2} = \mp iB_1$. *These eigenvalues are purely imaginary, then this singular point is either a focus or a center, and due to the fact that system* (2.27) *is symetric with respect to x–axes, we know that* p_3 *is a center, and the fourth singularity* p_4 *with eigenvalues* $\lambda_{1,2} = \mp iB_2$. *These eigenvalues are purely imaginary, and due to the fact that systems* (2.27) *are symetric with respect to the x–axes, we know that* p_4 *is a center. Then the statement (ii) holds.*

If $a \in (c_1, c_2)$, *then* p_1 *is an hyperbolic stable node with eigenvalues* $\lambda_1 = -(a-1)\sqrt{1-k^2}$ *and* $\lambda_2 = -\sqrt{1-k^2}$, *and* p_2 *is an unstable node with eigenvalues* $\lambda_1 = \sqrt{1-k^2}$ *and* $\lambda_2 = (a-1)\sqrt{1-k^2}$. *Then the statement (iii) holds.*

If $a = c_1$ *where* $c_1 = 2 - \sqrt{1-k^2}$, *systems* (2.27) *become*

$$\begin{aligned} \dot{x} &= \frac{1}{2}\left(1-\sqrt{1-k^2}\right)x^2 + \frac{1}{2}\left(-\sqrt{1-k^2}-1\right)y^2 + ky + \frac{1}{2}\left(\sqrt{1-k^2}-1\right), \\ \dot{y} &= x(y-k). \end{aligned} \tag{2.28}$$

These systems have three singularities p_1, p_2 *and* p_3, *with* $p_3 = \left(0, \dfrac{1-\sqrt{1-k^2}}{k}\right)$.

At p_1 *we have the eigenvalues* $\lambda_1 = \sqrt{1-k^2}\left(\sqrt{1-k^2}-1\right)$ *and* $\lambda_2 - \sqrt{1-k^2}$. *So it's an hyperbolic stable node. The singularity* p_2 *is an hyperbolic unstable node with eigenvalues* $\lambda_1 = \sqrt{1-k^2}$ *and* $\lambda_2 = -\sqrt{1-k^2}\left(\sqrt{1-k^2}-1\right)$. *The third singularity* p_3 *is a nilpotent singular point with eigenvalues* $\lambda_1 = 0$ *and* $\lambda_2 = 0$. *By applying Theorem 2.15 of [42] we know that* p_3 *has two hyperbolic sectors. Then the statement (iv) holds.*

If $a = c_2$ where $c_2 = 2 + \sqrt{1-k^2}$ systems (2.27) become

$$\begin{aligned} \dot{x} &= \frac{1}{2}\left(1+\sqrt{1-k^2}\right)x^2 + \frac{1}{2}\left(\sqrt{1-k^2}-1\right)y^2 + ky + \frac{1}{2}\left(-\sqrt{1-k^2}-1\right), \\ \dot{y} &= x(y-k). \end{aligned} \tag{2.29}$$

These systems have three singularities p_1, p_2 and p_4, with $p_4 = \left(0, \dfrac{1+\sqrt{1-k^2}}{k}\right)$.

The singularity p_1 is an hyperbolic stable node with eigenvalues $\lambda_1 = -\sqrt{1-k^2}$ and $\lambda_2 = -\sqrt{1-k^2}\left(\sqrt{1-k^2}+1\right)$. The singularity p_2 is an unstable node with eigenvalues $\lambda_1 = \sqrt{1-k^2}\left(\sqrt{1-k^2}+1\right)$ and $\lambda_2 = \sqrt{1-k^2}$. The third singularity p_4 is a nilpotent singular point with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 0$. By applying Theorem 3.5 of [42] we know that p_4 has two hyperbolic sectors. Then the statement (v) holds.

If $a = 1$ systems (2.27) become

$$\dot{x} = -y(y-k), \qquad \dot{y} = x(y-k). \tag{2.30}$$

These systems have a line of singularities $y = k$. We take the change of variables $ds = (y-k)dt$, we get the following system $\dot{x} = -y$, $\dot{y} = x$, which has a center at the origin with its corresponding eigenvalues i and $(-i)$. Then the statement (vi) holds.

If $a = 3$ systems (2.27) become

$$\dot{x} = x^2 + ky - 1, \qquad \dot{y} = x(y-k). \tag{2.31}$$

These systems have three singularities p_1, p_2 and p_4 where $p_4 = \left(0, \dfrac{1}{k}\right)$. The singularities p_1 and p_2 have the eigenvalues $\lambda_1 = -2\sqrt{1-k^2}$ and $\lambda_2 = -\sqrt{1-k^2}$, and $\lambda_1 = \sqrt{1-k^2}$ and $\lambda_2 = 2\sqrt{1-k^2}$, respectively. So p_1 is an hyperbolic stable node and p_2 is an hyperbolic unstable node. The third singularity p_4 is a saddle with eigenvalues $\lambda_1 = -\sqrt{1-k^2}$ and $\lambda_2 = \sqrt{1-k^2}$. Then the statement (vii) holds.

Proof of statement (II) of Proposition 2.3.

If $a \in (-\infty, 1) \cup (1, 3) \cup (3, \infty)$ systems (2.27)) have two singularities, p_3 with eigenvalues $\lambda_{1,2} = \mp i B_1$, then $\lambda_1.\lambda_2 = -\dfrac{S\left((a-2)k + S\right)}{a-3}$ which means that p_3 is an hyperbolic saddle if $a \in (-\infty, 1) \cup (3, \infty)$ and a center if $a \in (1, 3)$, and the second singular point p_4 has the *eigenvalues* $\lambda_{1,2} = \mp i B_2$, *then* $\lambda_1.\lambda_2 = \dfrac{S\,((a-2)k-S)}{a-3}$. *We prouve easily that this singularity is a center. Then two statements* (i) *and* (ii) *hold.*

If $a = 1$ *systems* (2.27) *become*

$$\dot{x} = -y(y-k), \qquad \dot{y} = x(y-k). \tag{2.32}$$

These systems have a line of singularities $y = k$. *By performing the change of variables* $ds = (y-k)dt$, *we get the following system* $\dot{x} = -y$, $\dot{y} = x$, *which has a center at the origin. Then the statement* (iii) *holds.*

If $a = 3$ *systems* (2.27) *become*

$$\dot{x} = x^2 + ky - 1, \qquad \dot{y} = x(y-k). \tag{2.33}$$

These systems have one singularity at p_4 *which is a center with eigenvalues* $\lambda_1 = (-i)\sqrt{1-k^2}$ *and* $\lambda_2 = i\sqrt{1-k^2}$. *Then the statement* (iv) *holds.*

Proof of statement (III) of Proposition 2.3

If $a = 1$ *systems* (2.27) *become*

$$\dot{x} = -y(y-1), \qquad \dot{y} = x(y-1). \tag{2.34}$$

This system has a line of singularities $y = 1$. *We take the change of variables* $ds = (y-1)dt$, *we get the following system* $\dot{x} = -y$, $\dot{y} = x$, *which has a center at the origin. Then the statement* (i) *holds.*

If $a = 2$ *systems* (2.27) *become*

$$\dot{x} = \frac{1}{2}(x^2 - y^2 - 1) + y, \qquad \dot{y} = x(y-1). \tag{2.35}$$

This system has a linearly zero singular point at $p_3 = (0,1)$. *In order to know the nature of this singularity. First, we put this point at the origin of coordinates by performing the translation* $x = x_1$, $y = y_1 + 1$, *and we get* $\dot{x_1} = \frac{1}{2}(x_1 - y_1)(x_1 + y_1)$, $\dot{y_1} = x_1 y_1$. *Second, we need to do a blow-up* $y_1 = z x_1$ *for describing its local phase portrait. After eliminating the common factor* x_1 *of* $\dot{x}_1$ *and* $\dot{z}$, *by doing the rescaling of the independent variable* $ds = x_1 dt$, *we obtain the* *system* $\dot{x_1} = \frac{1}{2}(1-z)(1+z)$, $\dot{z} = \frac{1}{2}z(1+z^2)$*. This system has no singularity for* $x_1 = 0$*. Going back through the two changes of variables* $y_1 = zx_1$ *and* $x_1 dt = ds$ *and by taking into account the direction of the flow of the system on the axes of coordinates, we conclud that the local phase portrait of the origin consists of two elliptic sectors. Then the statement* (ii) *holds.*

If $a = 3$ *systems* (2.27) *become*

$$\dot{x} = x^2 + y - 1, \qquad \dot{y} = x(y-1). \tag{2.36}$$

This system has one nilpotent singularity at p_3*. By using Theorem 3.5 of [42] we obtain that its local phase portrait consists of two parabolic and two hyperbolic sectors. Then the statement* (iii) *holds.*

If $a \in \mathbb{R} \setminus \{1, 2, 3\}$ *systems* (2.27) *have two singularities, a nilpotent one at* p_3 *where its local phase portrait of* p_3 *consists of two parabolic, two elliptic and two hyperbolic sectors if* $a \in (3, \infty)$ *and two hyperbolic sectors if* $a \in (-\infty, 1) \cup (1, 2)$*, and an hyperbolic singularity at* p_4 *with eigenvalues* $\mp\sqrt{2}\sqrt{\frac{(a-2)^2}{a-3}}$*, then it is a saddle if* $a \in (3, \infty)$ *and a center if* $a \in (-\infty, 1) \cup (1, 2) \cup (2, 3)$*. Then the statement* (iv) *holds.* ∎

2.2.3 Infinite singular points

The main goal of this part is to give the local phase portraits of systems (2.27) at its infinite singular points. To give a full study about the infinite singular points in the Poincaré disc we present the analysis of the vector field at infinity.

Proposition 2.4 *The local phase portraits at the infinite singular points of systems* (2.27) *in the local chart* U_1 *consists of*

(i) *One singular point at* $q_1 = (0, 0)$ *which is an hyperbolic stable node if* $a \in (-\infty, 1)$ *and* $k \in (0, \infty)$*, an hyperbolic unstable node if* $a \in (3, \infty)$ *and* $k \in (0, \infty)$*, an hyperbolic saddle if* $a \in (1, 3)$ *and* $k \in (0, \infty)$*, and a semi-hyperbolic saddle-node if* $a = 1$ *and* $k \in (0, \infty)$*;*

(ii) *a line of singularities if* $a = 3$ *and* $k \in (0, \infty)$*.*

The local phase portrait at the origin of the local chart U_2 is not a singularity in all the cases.

Proof. *The expression of systems (2.27) in the local chart U_1 is given by*

$$\begin{aligned} \dot{u} &= \frac{1}{2}\left(-(a-3)u^3 + u\left(a(v^2-1) - v^2 + 3\right) - 2ku^2v - 2kv\right) + a(1+v), \\ \dot{v} &= -\frac{1}{2}v\left(-a(u^2+1) + (a-1)v^2 - 2kuv + 3u^2 + 1\right). \end{aligned} \tag{2.37}$$

Any arbitrary infinite singular point of differential systems (2.37) takes the form $(u_0, 0)$.

If $a \in (-\infty, 3) \cup (3, \infty)$ and $k \in (0, \infty)$ systems (2.37) have one singular point at $q_1 = (0,0)$ with eigenvalues $\frac{1-a}{2}$ and $\frac{3-a}{2}$. So q_1 is an hyperbolic unstable node if $a \in (-\infty, 1)$, an hyperbolic stable node if $a \in (3, \infty)$, an hyperbolic saddle if $a \in (1,3)$, and a semi-hyperbolic saddle-node if $a = 1$. Then the statement (i) holds.

If $a = 3$ and $k \in (0, \infty)$ systems (2.37) become

$$\begin{aligned} \dot{x} &= -v\left(k(u^2+1) - uv\right), \\ \dot{y} &= v\Big(v(v-ku) - 1\Big). \end{aligned} \tag{2.38}$$

These systems have infinity as a line of singularities. We take the change of variables $ds = v dt$, we get the following systems $\dot{x} = -ku^2 - k + uv$, $\dot{y} = -kuv + v^2 - 1$. These systems have no singularity. Then the statement (ii) holds.

The expression of systems (2.27) in the local chart U_2 is given by

$$\begin{aligned} \dot{u} &= \frac{1}{2}\left(a(u^2 - v^2 + 1) + u^2(2kv - 3) + 2kv + v^2 - 3\right), \\ \dot{v} &= uv(kv - 1). \end{aligned} \tag{2.39}$$

It is clear that the origin is not a singularity for these systems for all $a \in \mathbb{R}$ and $k \in (0, \infty)$. ■

2.2.4 Local and global phase portraits

If $k \in (0, 1)$, $a \neq 1$ and from Propositions 2.3 and 2.4 we obtain the local phase portraits of the finite and infinite singular points of systems (2.27). Due to the fact that the two singular points q_1 and q_2 belongs to the reducible invariant curve of the systems, we obtain some orbits on this invariant curve connecting those singular points, these connections vary if either $a \in (-\infty, 1) \cup (1, c_1) \cup (c_1, \infty)$, or $a \in (-\infty, 1)$, or $a \in (c_1, c_2)$, or $a = c_1$, or $a = c_2$, (see the local phase portrait 1, or 2, or 3, or 4, or 5 or 7 of Figure 2.7, respectively).

Since $\dot{x}_{|x=0} = \frac{1}{2}(a-3)y^2 + ky + \frac{1-a}{2}$, and $\dot{y}_{|y=0} = -kx < 0$ the separatrices for which we do not know their α- or ω-limit can be easily determined from the mentioned figures, obtaining the global phase portrait 1, or 2, or 3, or 4, or 5 or 7 of Figure 2.9, respectively).

If $k \in (0, \infty)$ and $a = 1$ the systems have $y = k$ as a line of singularities, and by doing the change of variables $(y-k)dt = ds$ we know that they have a center at the origin. Since $\dot{x}_{|x=0} = y(k-y)$, and $\dot{y}_{|y=0} = 0$, (see also the local phase portrait 6 of Figure 2.7) we get the global phase portrait 6 of Figure 2.9.

If $k \in (1, \infty)$ and $a \neq 1$ we get the local phase portrait 8, or 9, or 10 of Figure 2.8 for the fnite and infnite singular points of the systems. Since $\dot{x}_{|x=0} = \frac{1}{2}(a-3)y^2 + ky + \frac{1-a}{2}$, and $\dot{y}_{|y=0} = -kx < 0$ we get that the global phase portrait in this case is 8, or 9, or 10 of Figure 2.10, respectively.

If $k = 1$ from Propositions 2.3 and 2.4 we obtain the local phase portraits of the finite and infinite singular points of systems (2.27). Due to the fact that the singular point p_3 belongs to the reducible invariant curve of the system, we obtain some orbits on this invariant curve connecting this singular point, these connections vary in the interval $a \in (-\infty, \infty)$, (see the local phase portrait 11, or 12, or 13 or 14 of Figure 2.8, respectively).

Taking into account the following directions of the vector field of the systems in the axes $\dot{x}_{|x=0} = \frac{1}{2}(a-3)y^2 + y + \frac{1-a}{2}$, and $\dot{y}_{|y=0} = -x < 0$ we get to the global phase portrait 11, or 12, or 13 or 14 of Figure 2.10, respectively).

2.2.5 Statement of the main results

Now we provide the global phase portraits in the Poincaré disc of systems (2.27). More precisely our main result is the following

Theorem 2.4 *The global phase portraits in the Poincaré disc of systems (2.27) of Theorem 2.3 are:*

1 *for* $k \in (0,1)$ *and* $a \in (1, c_1) \cup (c_2, 3) \cup (3, \infty)$, *where* $c_1 = 2 - \sqrt{1-k^2}$ *and* $c_2 = 2 + \sqrt{1-k^2}$;

2 *for* $k \in (0,1)$ *and* $a \in (-\infty, 1)$;

3 *for* $k \in (0,1)$ *and* $a \in (c_1, c_2)$;

4 *for* $k \in (0,1)$ *and* $a = c_1$;

5 *for* $k \in (0,1)$ *and* $a = c_2$;

6 *for* $k \in (0,\infty)$ *and* $a = 1$;

7 *for* $k \in (0,1)$ *and* $a = 3$;

8 *for* $k \in (1,\infty)$ *and* $a \in (-\infty, 1) \cup (3, \infty)$;

9 *for* $k \in (1,\infty)$ *and* $a \in (1,3)$;

10 *for* $k \in (1,\infty)$ *and* $a = 3$;

11 *for* $k = 1$ *and* $a = 2$;

12 *for* $k = 1$ *and* $a = 3$;

13 *for* $k = 1$ *and* $a \in (-\infty, 1) \cup (1,2) \cup (2,3)$;

14 *for* $k = 1$ *and* $a \in (3, \infty)$.

2.2.6 Figure of local and global phase portraits

In what follows we give the local and the global phase portraits of systems mentioned in Theorem 2.4, respectively.

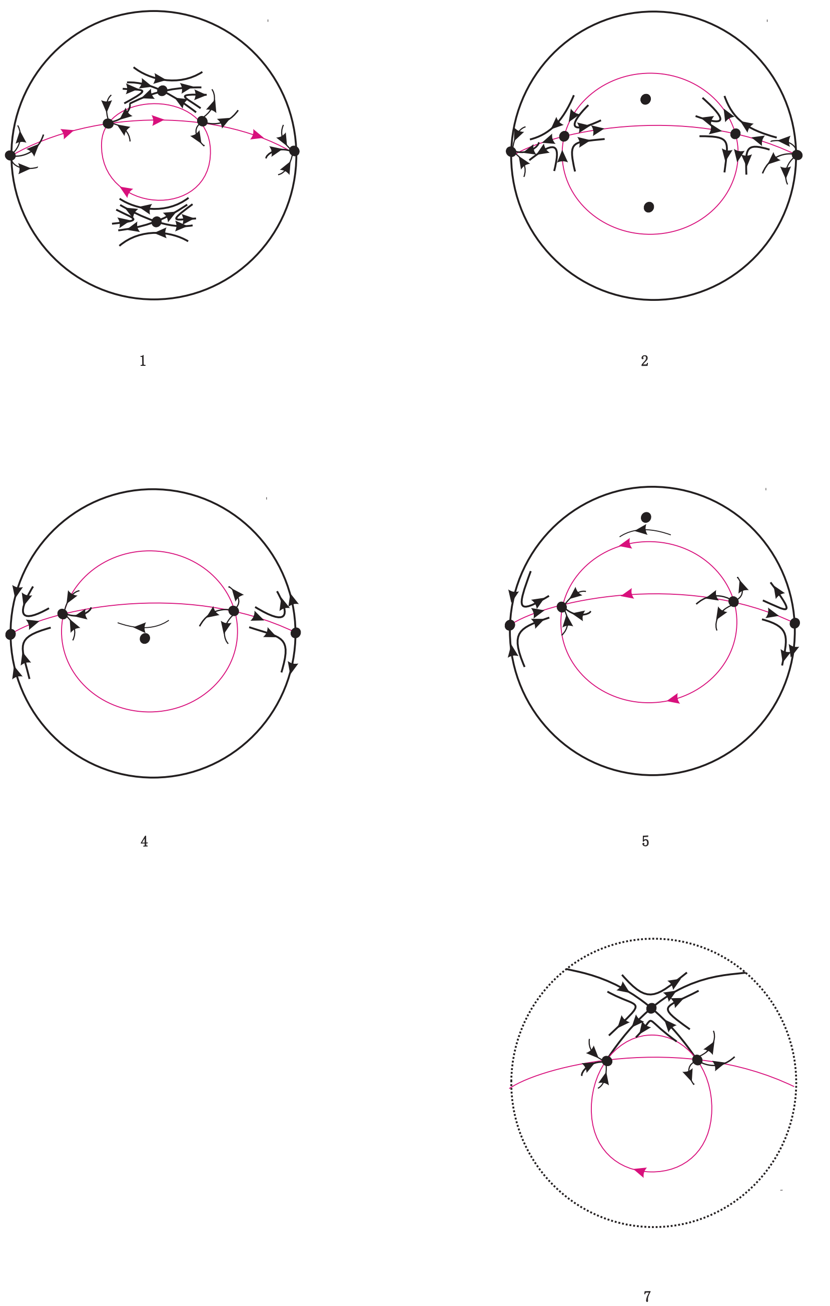


Figure 2.7: Local phase portraits in the Poincaré disc of systems (2.27).

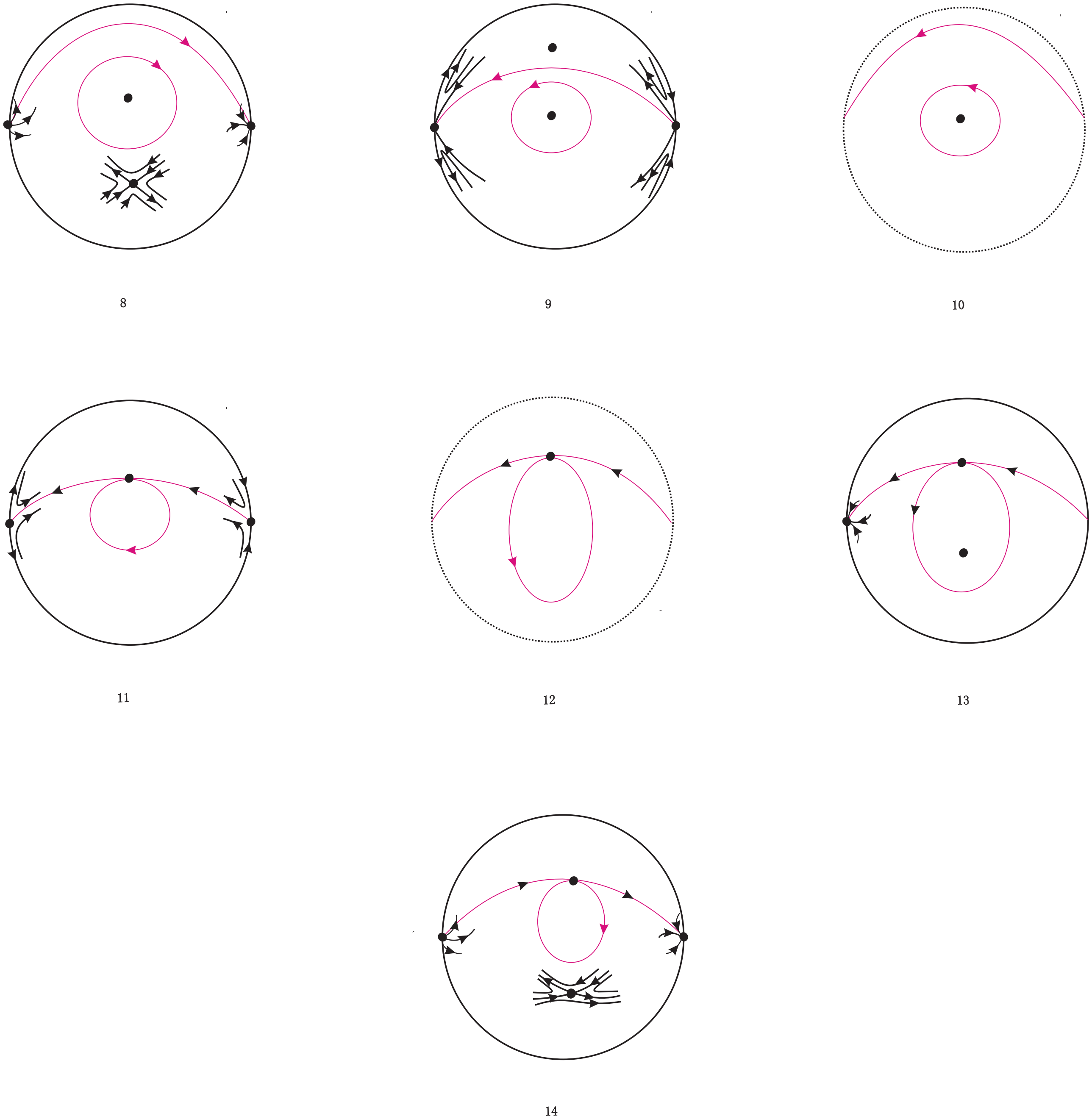


Figure 2.8: Continuation of Figure 2.7.

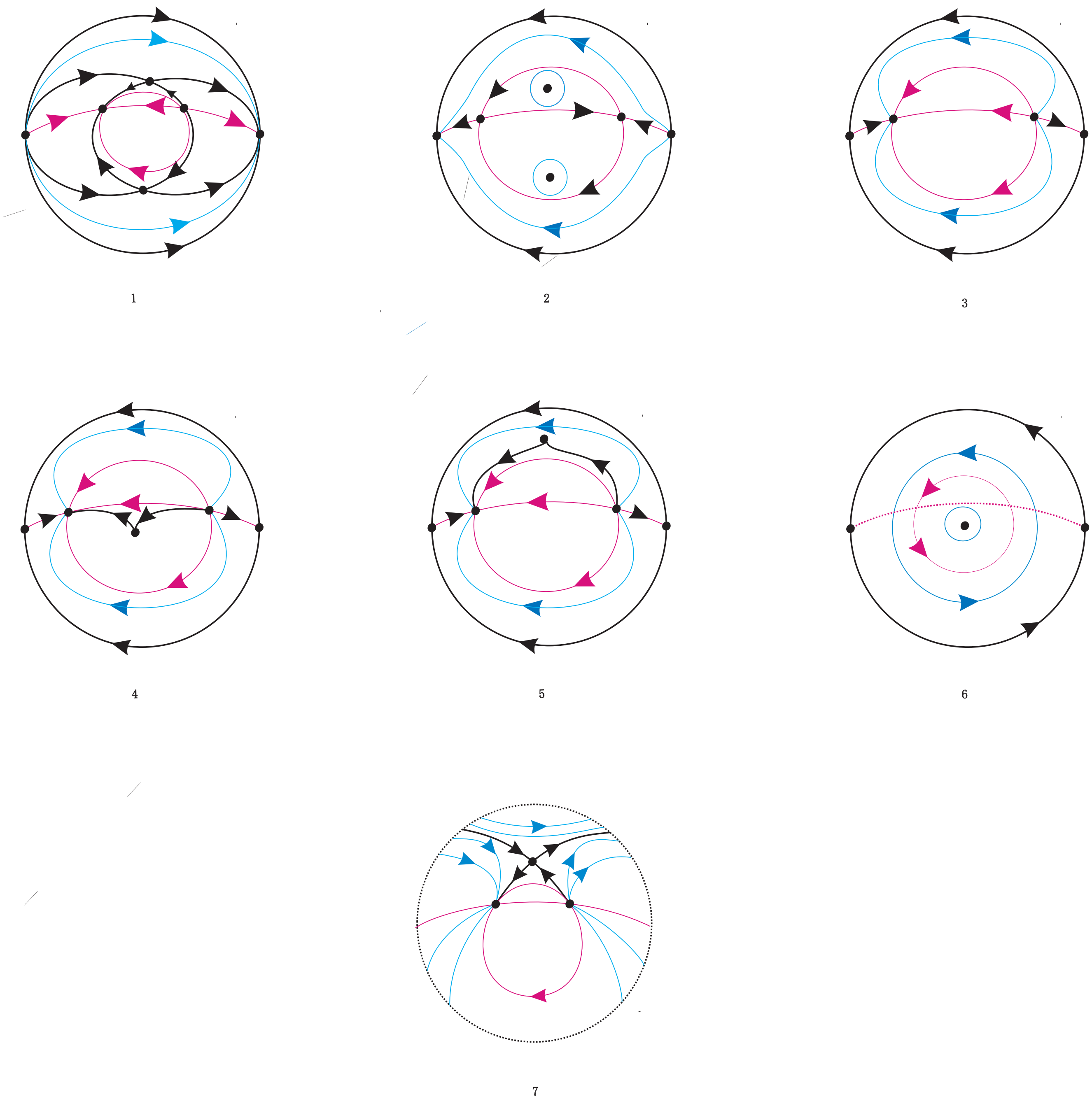


Figure 2.9: Global phase portraits in the Poincaré disc of systems (2.27).

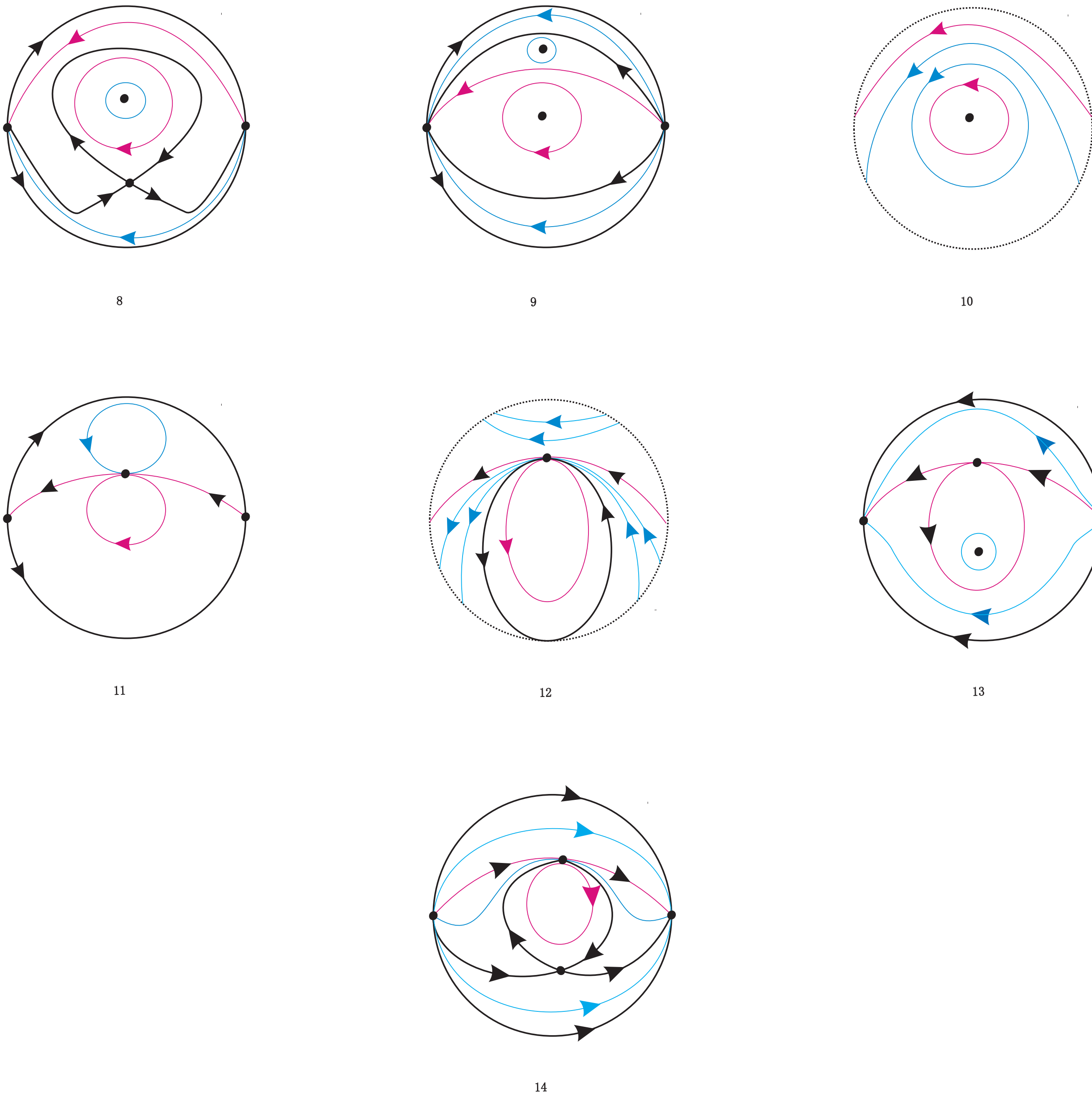


Figure 2.10: Continuation of Figure 2.9.

Chapter 3

Centers and Limit Cycles of Generalized Kukles Polynomial Differential Systems: Phase Portraits and Limit Cycles

We consider the so-called *Kukles homogeneous differential system*. Giné [49]

$$\dot{x} = -y, \quad \dot{y} = x + Q_n(x, y), \tag{3.1}$$

who has a center at the origin, where $Q_n(x, y)$ denotes a homogeneous real polynomial of degree n with the variables x and y over $\mathbb{R}$.

In 1999 Volokitin and Ivanov [95] prove that systems (3.1) have a center at the origin definitely if they are symmetric with respect to one of the coordinate axes. For $n = 2$ and $n = 3$, the authors of the conjecture knew that it holds. Giné [49] in 2002 proved the conjecture for $n = 4$ and $n = 5$. Giné et al. [50, 51] proved the conjecture for all n under an additional assumption, that the authors believe that it is redundant.

The phase portraits for quadratic systems with center written in the form (3.1), are known, see Vulpe [96]. The phase portraits of cubic differential systems symmetry with respect to a straight line are also known and in particular those of system (4.1) with $n = 3$, see Buzzi et al. [35], see also Malkin [77]; Vulpe Sibirskii [96] and Żołądek [99].

The phase portraits of systems (4.1) with $n = 4$ follows from Benterki and Llibre [25]. In [68, 69] Llibre and Silva classified the phase portraits of systems (3.1) for $n = 5, 6$. While the phase portraits of systems (3.1) with $n = 7$ follows from Benterki and Llibre [27].

In this chapter we studied global phase portraits of Kukles differential systems (3.1) for $n = 8$, and by using the averaging theory we studied the number of limit cycles which can bifurcated from the origin of coordinates.

Section 3.1 First Main Results

Our first main objective is to classify all the global phase portraits in the Poincaré disc of the of Kukles differential system written as

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + ax^8 + bx^4y^4 + cy^8. \end{aligned} \tag{3.2}$$

Where $x, y \in \mathbb{R}$ and $a, b, c \in \mathbb{R}$ with $a^2 + b^2 + c^2 \neq 0$.

Remark 7

(*a*) *System* (3.2) *is invariant under the change* $(t, x, y) \to (-t, x, -y)$*. Hence, the phase portrait of system* (3.2) *is symmetric with respect to the x-axis.*

(*b*) *System* (3.2) *is also invariant under the change* $(x, y, t, a, b, c) \to (-x, y, -t, -a, -b, -c)$ *then we only need to study the phase portrait of systems* (3.2) *when* ($a = 0$, $b \geqslant 0$ *and* $c \geqslant 0$), ($a = 0$, $b > 0$ *and* $c < 0$),($a > 0$, $b \geqslant 0$ *and* $c \geq 0$), ($a < 0$, $b > 0$ *and* $c = 0$), ($a > 0$, $b \geqslant 0$ *and* $c < 0$), ($a > 0$, $b^2 - 4ac = 0$, *and* $b < 0$) *and* ($a > 0$, $b^2 - 4ac > 0$, $b < 0$ *and* $c > 0$).

3.1.1 Finite and infinite singularities

To study the phase portrait of system (3.2) we identify all the finite singular points and their local phase portrait. We go through the same steps to study the local phase portrait for the infinite ones.

Finite singular points. We identify the finite singular points of the the generalized kukels polynomial differential system (3.2) in the following Proposition.

Proposition 3.1 *The differential system* (3.2) *has*

(i) Two finite singular points, a center at $(0,0)$ *and a hyperbolic saddle at* $(-\sqrt[7]{1/a},0)$*, if* $a \neq 0$*;*

(ii) one singular point at $(0,0)$ *wich is a center, if* $a = 0$.

Proof. *Clearly when* $a \neq 0$ *the system has two equilibria the origin, with eigenvalues* $\pm i$*, then we take into acount the symmetry of system* (3.2) *with respect to x-axis, we conclude that the origin is a center. The second equilibria is* $(-\sqrt[7]{1/a},0)$ *with eigenvalues* $\pm\sqrt{7}$*. So it is an hyperbolic saddle.* ∎

Infinite singular points. By using the preliminaries given in chapter 1 we study the infinite singular points and their nature in the Poincaré disc.

Proposition 3.2 *In the chart* U_1 *system* (3.2) *has*

(a) The origin as a linearly zero infinite critical point, and its local phase portrait consists of four hyperbolic sectors, if $a = 0$*,* $b \geqslant 0$ *and* $c \geqslant 0$*;*

(b) three infinite singular points, the origin mentioned in the previous case and two saddle-nodes at $(\pm\sqrt[4]{-b/c},0)$*;*

(c) no singularity, if $a > 0$*,* $b \geqslant 0$ *and* $c \geqslant 0$*;*

(d) two infinite semi-hyperbolic saddle-nodes, $(\pm\sqrt[4]{-a/b},0)$*, if* $c = 0$*,* $b > 0$ *and* $a < 0$ *;*

(e) two infinite semi-hyperbolic saddle-nodes at $\left(\pm\sqrt[4]{\dfrac{-b-\sqrt{b^2-4ac}}{2c}},0\right)$*, if* $a > 0$*,* $b \geqslant 0$ *and* $c < 0$*;*

(f) two infinite linearly zero singular points $(\pm\sqrt[4]{-2a/b},0)$ *with two hyperbolic and two parabolic sectors, if* $a > 0$*,* $c > 0$*,* $b^2 = 4ac$ *and* $b < 0$*;*

(g) four infinite semi-hyperbolic saddle-nodes, $\left(\pm\sqrt[4]{\dfrac{-b+\sqrt{b^2-4ac}}{2c}},0\right)$*, and* $\left(\pm\sqrt{\dfrac{b+\sqrt{b^2-4ac}}{-2c}},0\right)$*, if* $a>0$*,* $b^2-4ac>0$*,* $b<0$ *and* $c>0$*.*

The origin of the chart U_2 *is*

(h) an hyperbolic node, which is stable if $c>0$ *and unstable if* $c<0$*;*

(i) a linearly zero singular point that its local phase portrait consists of four parabolic sectors, if $c=0$*.*

Proof. *The differential system (3.2) in the chart* U_1 *is given by*

$$\begin{aligned}\dot{u} &= a+bu^4+cu^8+v^7+u^2v^7,\\ \dot{v} &= uv^8.\end{aligned} \tag{3.3}$$

If $b>0$*,* $a=0$ *and* $c\geqslant 0$ *system (3.3) is written as follows*

$$\begin{aligned}\dot{u} &= bu^4+cu^8+v^7+u^2v^7,\\ \dot{v} &= uv^8.\end{aligned} \tag{3.4}$$

System (3.4) has one linealrly zero singular point at the origin. Then to study its local phase portrait we have to do blow-up's. We take the directional blow-up $(u,v)\to(u,w)$ *with* $w=v/u$ *and have*

$$\begin{aligned}\dot{u} &= bu^4+cu^8+u^7w^7+u^9w^7,\\ \dot{w} &= -bu^3w-cu^7w-u^6w^8.\end{aligned} \tag{3.5}$$

We do the rescaling of the time $u^3dt=ds$*, and we get*

$$\begin{aligned}\dot{u} &= bu+cu^5+u^4w^7+u^6w^7,\\ \dot{w} &= -bw-cu^4w-u^3w^8,\end{aligned} \tag{3.6}$$

this system has one hyperbolic saddle at $(0,0)$*, with eigenvalues* $\pm b$*. Returning through the change of variables to system (3.3), we conclude that the local phase portrait at the origin trained by four hyperbolic sectors.*

If $a=0$*,* $b=0$ *and* $c>0$*, and after taking a rescaling of the time* $u^6dt=ds$ *we get the* *following system*

$$\begin{aligned} \dot{u} &= cu^2 + uw^7 + u^3w^7, \\ \dot{w} &= -cuw - w^8. \end{aligned} \tag{3.7}$$

System (3.7) *has a linearly zero singular point at the origin. Doing blow-up's by performing the directional* $(u,w) \to (u,z)$ *with* $z = w/u$ *and we get*

$$\begin{aligned} \dot{u} &= cu^2 + u^8z^7 + u^{10}z^7, \\ \dot{z} &= -2cuz - 2u^7z^8 - u^9z^8. \end{aligned} \tag{3.8}$$

Doing a rescaling of the time $u\,dt = ds$*, we get the following system*

$$\begin{aligned} \dot{u} &= cu + u^7z^7 + u^9z^7, \\ \dot{z} &= -2cz - 2u^6z^8 - u^8z^8. \end{aligned} \tag{3.9}$$

System (3.9) *has one hyperbolic saddle at the origin with eigenvalues* c *and* $-2c$*. Returning through the change of variables to system* (3.3)*, we conclude that the local phase portrait at the origin formed by four hyperbolic sectors.*

If $b > 0$*,* $a = 0$ *and* $c = 0$ *we have the follwing system*

$$\begin{aligned} \dot{u} &= bu^4 + u^7w^7 + u^9w^7, \\ \dot{w} &= -bu^3w - u^6w^8. \end{aligned} \tag{3.10}$$

Doing a change of variable $u^3dt = ds$*, we get the following system*

$$\begin{aligned} \dot{u} &= bu + u^4w^7 + u^6w^7, \\ \dot{w} &= -bw - u^3w^8. \end{aligned} \tag{3.11}$$

System (3.11) *has one hyperbolic saddle at the origin with eigenvalues* b *and* $-b$*. Returning through the change of variables, we know that the local phase portrait at the origin of system* (3.3)*, when* $a = 0$*,* $b > 0$ *and* $c = 0$*, consists of four hyperbolic sectors. Then the statement (a) holds.*

If $b > 0$*,* $a = 0$ *and* $c < 0$ *system* (3.4) *has in addition to the origin (the same case in satement (a)) two infinite semi-hyperbolic singular points, namely* $(\pm\sqrt[4]{-b/c}, 0)$*, with eigenvalues* $\pm 4b\sqrt[4]{-b^3/c^3}$ *and* 0*. Applying Theorem 2.19 of [42] we know that these points are* *saddle-nodes. Then statement (b) holds.*

If $a > 0$*,* $b \geqslant 0$ *and* $c \geqslant 0$ *system* (3.3) *has no singular point.*

If $c = 0$*,* $b > 0$ *and* $a < 0$ *system* (3.3) *becomes*

$$\begin{aligned} \dot{u} &= a + bu^4 + v^7 + u^2v^7, \\ \dot{v} &= uv^8, \end{aligned} \tag{3.12}$$

this system has two semi-hyperbolic singular points, $(\pm\sqrt[4]{-a/b}, 0)$ *with eigenvalues* $\lambda_1 = \pm 4b(-a^3/b^3)^{(1/4)}$ *and* $\lambda_2 = 0$*. We perform the translation* $u = z \pm (-a/b)^{(1/4)}$ *to system* (3.12)*. Applying Theorem 2.19 of [42] we know that the points are saddle-nodes. Then (d) is proved.*

If $a > 0$*,* $c > 0$*,* $b^2 = 4ac$*, and* $b < 0$ *system* (3.3) *becomes*

$$\begin{aligned} \dot{u} &= (2a + bu^4)^2/(4a) + (1 + u^2)v^7, \\ \dot{v} &= uv^8. \end{aligned} \tag{3.13}$$

This system has two singular points $(\pm\sqrt[4]{(-2a)/b}, 0)$ *which are linearly zero. We study at first the point* $(\sqrt[4]{(-2a)/b}, 0)$ *after performing the translation* $u = z + \sqrt[4]{(-2a)/b}$*. Doing blow-up's by taking the directional* $(z, v) \longrightarrow (z, w)$ *with* $w = v/z$*, and eleminating the common factor* z *between* $\dot{z}$ *and* $\dot{w}$*, we get the following differential system*

$$\begin{aligned} \dot{z} = & -8bz\sqrt{\frac{-2a}{b}} - 24\left(\frac{-2a}{b}\right)^{\frac{1}{4}} bz^2 - 34bz^3 + \frac{1}{a}14(-2a/b)^{\frac{3}{4}} b^2z^4 \\ & + \frac{7}{a}\sqrt{\frac{-2a}{b}} b^2z^5 + \frac{2^{\frac{5}{4}}}{a}\left(\frac{-a}{b}\right)^{\frac{1}{4}} b^2z^6 + w^7z^6 + \sqrt{\frac{-2a}{b}} w^7z^6 \\ & + w^7z^8 + \frac{b^2z^7}{4a} + 2^{\frac{5}{4}}\left(\frac{-a}{b}\right)^{\frac{1}{4}} w^7z^7, \\ \dot{w} = & \; 8\sqrt{\frac{-2a}{b}} bw + 24\left(\frac{-2a}{b}\right)^{\frac{1}{4}} bwz + 34bwz^2 - \frac{14}{a}\left(\frac{-2a}{b}\right)^{\frac{3}{4}} b^2wz^3 \\ & - \frac{7}{a}\sqrt{\frac{-2a}{b}} b^2wz^4 - \frac{2}{a}(-2a/b)^{\frac{1}{4}} b^2wz^5 - w^8z^5 - \sqrt{\frac{-2a}{b}} w^8z^5 \\ & - \frac{b^2wz^6}{4a} - \left(\frac{-2a}{b}\right)^{\frac{1}{4}} w^8z^6. \end{aligned} \tag{3.14}$$

For $z = 0$*, system* (3.14) *has one hyperbolic saddle at the origin with eigenvalues* $-8\sqrt{2}b\sqrt{(-a/b)}$

and $8\sqrt{2}b\sqrt{(-a/b)}$. *Going back through the change of variables to system* (3.13), *we conclude that the local phase portrait at the singular point* $(\sqrt[4]{(-2a)/b}, 0)$ *formed by two hyperbolic and two parabolic sectors. We get the same local phase portrait for the singular point* $(-\sqrt[4]{(-2a)/b}, 0)$ *as the critical point* $(\sqrt[4]{(-2a)/b}, 0)$.

If $a > 0$, $b^2 - 4ac > 0$, $b < 0$ *and* $c > 0$ *system* (3.3) *has four semi-hyperbolic singular points,* $\left(\pm\sqrt[4]{(-b+\sqrt{b^2-4ac})/2c}, 0\right)$, *with eigenvalues* $\lambda_1 = 2^{\frac{5}{4}}(b^2-4ac)^{\frac{1}{4}}\left(\dfrac{-b-\sqrt{b^2-4ac}}{c}\right)^{\dfrac{3}{4}}$ *and* $\lambda_2 = 0$, *and the points* $\left(\pm\sqrt[4]{(-b-\sqrt{b^2-4ac})/2c}, 0\right)$, *with eigenvalues* $\lambda_1 = -2^{\frac{5}{4}}(b^2 - 4ac)^{(1/4)}\left(\dfrac{-b-\sqrt{b^2-4ac}}{c}\right)^{\dfrac{3}{4}}$ *and* $\lambda_2 = 0$. *Hence the four singular points are semi-hyperbolic.*

In the chart U_2 *the differential system* (3.2) *becomes*

$$\begin{aligned} \dot{u} &= -cu - bu^3 - v^3 - au^5 - u^2v^3, \\ \dot{v} &= -cv - buv - au^4v - bu^2v - uv^4. \end{aligned} \tag{3.15}$$

If $c \neq 0$ *the origin is an hyperbolic node of system* (3.15), *with eigenvalues* $-c$ *and* $-c$, *then it is stable if* $c > 0$ *and unstable if* $c < 0$. *So statement* (*e*) *holds.*

If $c = 0$, *system* (3.15) *becomes*

$$\begin{aligned} \dot{u} &= -bu^3 - v^3 - au^5 - u^2v^3, \\ \dot{v} &= -buv - au^4v - bu^2v - uv^4. \end{aligned} \tag{3.16}$$

The origin is a linearly zero singular point of the differential system (3.16). *We have to do blow-up's to know the localphase portrait at this point. We take the directional blow-up* $w = v/u$, *then we get*

$$\begin{aligned} \dot{u} &= -bu^3 - au^5 - u^3w^3 - u^5w^3, \\ \dot{w} &= -buw + u^2w^4. \end{aligned} \tag{3.17}$$

We do the rescaling $udt = ds$ *and we get the system*

$$\begin{aligned} \dot{u} &= -bu^2 - au^4 - u^2w^3 - u^4w^3, \\ \dot{w} &= -bw + uw^4. \end{aligned} \tag{3.18}$$

When $u = 0$; *the origin is the only singular point of system* (3.18), *with eigenvalues* 0 *and*

$-b$. Then, it is a semi-hyperbolic singular point. By using Theorem 2.19 of [42] we conclude that the origin is a saddle-node. Going back through the change of variables to system (3.16), we know that its local phase portrait formed by four parabolic sectors. ■

3.1.2 Global phase portraits on the Poincaré disc

Taking into account the results on the finite and infinite singular points given in Sections 3.1.1, we shall obtain the different phase portraits of the system (3.2) that we describe in what follows.

Case 1. When $a = 0$, $b \geqslant 0$ and $c \geqslant 0$ system (3.2) has one finite singular point, a center at $(0,0)$. And from statement (a) of Proposition 3.2 we know that in the chart U_1 the system has one singular point at $(0,0)$ is the only wich is linearly zero and with local phase portrait consists of four hyperbolic sectors. From statement (h) of Proposition 3.2, the origin of U_2 is a hyperbolic stable node. So, the phase portrait of system (3.2) is given by Figure 3.1(1), and its immediate that $S = 10$ and $R = 2$.

Case 2. When $a = 0$, $b > 0$ and $c < 0$ system (3.2) has one finite singular point at the origin of coodrinates, wich is a center. From statement (b) of Proposition 3.2 and in the local chart U_1 the system has three infinite singular points, a center at the origin and two semi-hyperbolic saddle-nodes. In U_2 and from statement (h) of Proposition 3.2, the origin is a hyperbolic unstable node. Then, the phase portrait of in this case is topologically equivalent to Figure 3.1(2), and its immediate that $S = 20$ and $R = 4$.

Case 3. When $a > 0$, $b \geqslant 0$ and $c \geqslant 0$ system (3.2) has two finite singular points, a center at $(0,0)$ and a hyperbolic saddle at $(-\sqrt[7]{1/a}, 0)$, and we get the same finite singular points for the following cases. From statement (c) of Proposition 3.2 the system has no singular points in the local chart U_1. In U_2 and from statement (h) of Proposition 3.2, the origin is a hyperbolic stable node if $c > 0$, and from statement (i) of the same proposition the origin is a linearly zero singular point and its local phase portrait formed by four parabolic sectors if $c = 0$. So, in this case the phase portrait is topologically equivalent to Figure 3.1(3), and its immediate that $S = 9$ and $R = 3$.

Case 4. When $a < 0$, $b > 0$ and $c = 0$ and from statement (d) of Proposition 3.2 the system has two infinite semi-hyperbolic saddle-nodes in the local chart U_1. In U_2 and from statement (i) of Proposition 3.2, the origin is a linearly zero singular point with local phase portrait formed by four parabolic sectors. Therefore in this case, the phase portrait of system (4.2) is topologically equivalent to the Figure 3.1(4), and its immediate that $S = 22$ and $R = 7$.

Case 5. When $a > 0$, $b \geqslant 0$ and $c < 0$ system (3.2) and from statement (e) of Proposition 3.2 we obtain that the system has two infinite semi-hyperbolic saddle-nodes in the local chart U_1. The origin of the chart U_2 is a hyperbolic unstable node. So, the phase portrait of system (3.2) is topologically equivalent to Figure 3.1(5), and its immediate that $S = 19$ and $R = 5$.

Case 6. When $a > 0$, $b^2 = 4ac$ and $b < 0$ and from statement (f) of Proposition 3.2 the system has two infinite semi-hyperbolic saddle-nodes in the local chart U_1. In U_2 and from statement (h) of Proposition 3.2, the origin is a hyperbolic unstable node. Then, we conclude that the phase portrait of system (3.2) is topologically equivalent to Figure 3.1(6), and its immediate that $S = 18$ and $R = 3$.

Case 7. When $a > 0$, $b^2 - 4ac > 0$, $b < 0$ and $c > 0$ system (3.2) has two finite singular points, a center at the origin and a hyperbolic saddle at $(-\sqrt[7]{1/a}, 0)$. From statement (g) of Proposition 3.2 we obtain that the system has four infinite semi-hyperbolic saddle-nodes in the local chart U_1. In U_2 and from statement (h) of Proposition 3.2, the origin is a hyperbolic unstable node. Therefore in this case, the phase portrait of system (3.2) is topologically equivalent to the Figure 3.1(7), and its immediate that $S = 29$ and $R = 6$.

Theorem 3.1 *The set of all global phase portraits in the Poincaré disc of the differential system* (3.2) *with* $a^2 + b^2 + c^2 \neq 0$ *are topologically equivalent to the phase portraits given in Figure* 3.1.

3.1.3 Figure of global phase portraits

In what follows we give the global phase portraits of system (3.2) mentioned in Theorem 3.1

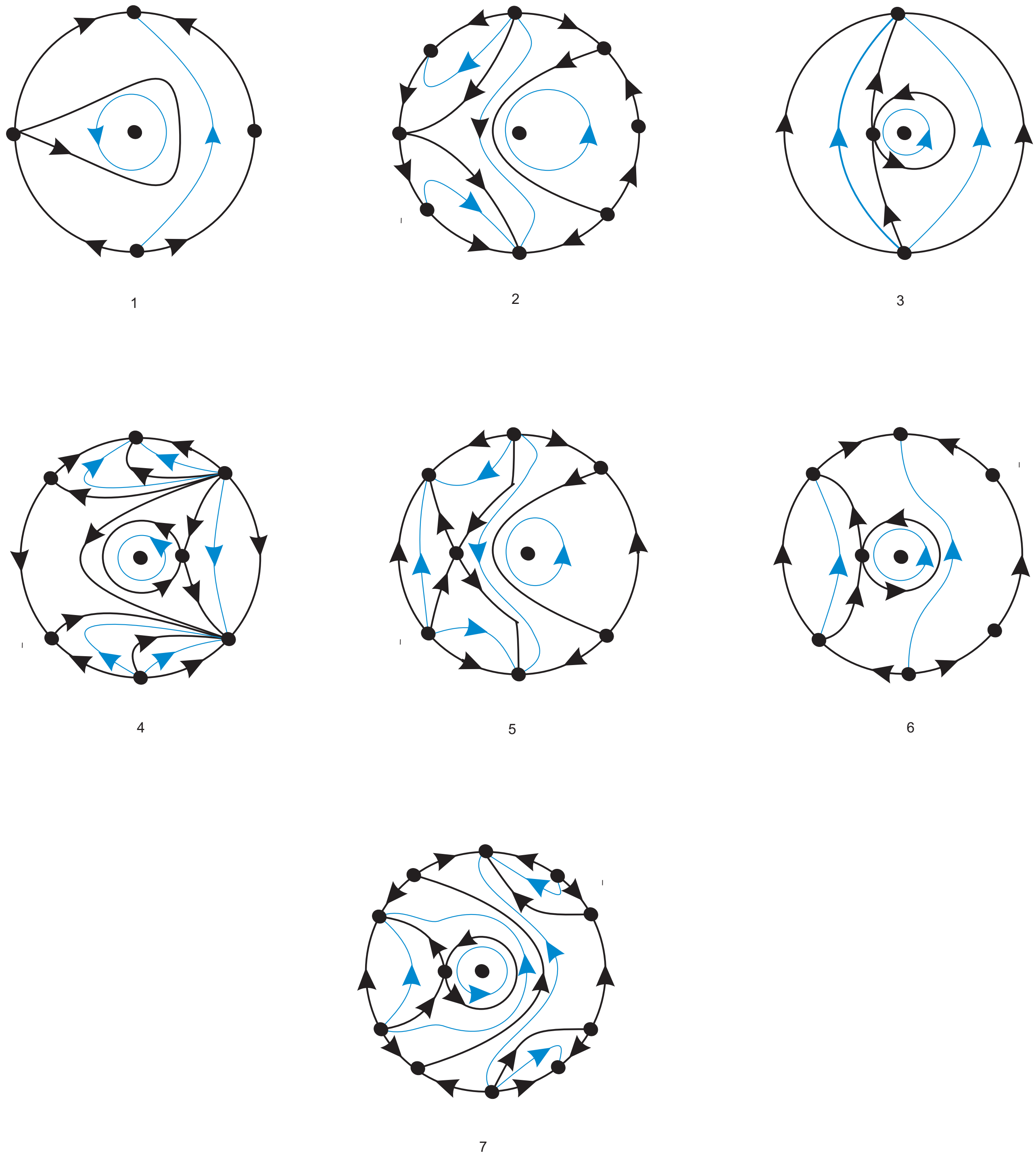


Figure 3.1: Global phase portraits of differential system (3.2).

3.1.4 Limit cycles of Kukles differential systems via averaging theory

We now consider the application of the developed averaging methods. As the first application we perturbed the polynomial differential system (3.2) with polynomials of degree eight, we get

$$\begin{aligned} \dot{x} &= -y + \sum_{s=1}^{6} \epsilon^s \sum_{0\leqslant i+j\leqslant 8} \alpha_{ij}^{(s)} x^i y^j, \\ \dot{y} &= x + ax^8 + bx^4y^4 + cy^8 + \sum_{s=1}^{6} \epsilon^s \sum_{0\leqslant i+j\leqslant 8} \beta_{ij}^{(s)} x^i y^j, \end{aligned} \tag{3.19}$$

where $i, j \in \mathbb{N}$. For more information about the averaging theory of higher order see chapter 1.

Section 3.2 Second Main Results

Our second main result is given in the following Theorem.

Theorem 3.2 *The number of limit cycles of the differential system (3.19) with $\varepsilon \neq 0$ is*

(a) No limit cycles if we use the averaging theory of order 1 and 2,

(b) one limit cycle if we use the averaging theory of order 3 and 4,

(c) two limit cycles if we use the averaging theory of order 5 and 6.

Proof. *Consider system (3.2), we shall study which periodic solutions of the center become limit cycles when we perturb the center inside the class of polynomial differential systems of degree 8. This study will be done by applying the averaging theory, we work as follows.*

Before doing the scaling $x = \epsilon X$, $y = \epsilon Y$, with ϵ is a small parameter we get a new differential system $(\dot{X}, \dot{Y})$. After we perform the polar change of coordinates $X = r\cos\theta$, $Y = r\sin\theta$, then we get a differential system $(\dot{r}, \dot{\theta})$. We take the independent variable the angle θ we get the differential equation $dr/d\theta$, and by doing a Taylor expansion up to 6-th order in ε

we obtain the differential equation

$$r' = \frac{dr}{d\theta} = \sum_{i=1}^{6} \epsilon^i F_i(\theta, r) + O(\epsilon^7). \tag{3.20}$$

The functions $F_i(\theta, r)$ $i = 1, \ldots, 6$ of the differential system (3.20) are analytic, and since the independent variable θ appears through the sinus and cosinus of θ, they are 2π–periodic. Hence the assumptions for applying the averaging theory given in [63] are satisfied. To know how the averaging theory for differential equation works we advice the lecture to see [63]. We give only the expression of functions $F_1(r, \theta)$ and $F_2(r, \theta)$.

The explicit expression of $F_i(r, \theta)$ with $i = 3, \ldots, 6$ is quite large so we omit.

The functions $F_i(\theta, r)$ $i = 1, \ldots, 6$ and $R(t, x, \varepsilon)$ of system (3.20) are analytic, and since the variable appears through sinus and cosinus of θ, they are 2π periodic. Hence the assumptions of Theorem 1.2 are satisfied.

The expressions of $F_1(r, \theta)$ and $F_2(r, \theta)$ are

$$\begin{aligned}
F_1(r,\theta) = {} & +\beta_{00}^{(2)} \sin\theta + \tfrac{r}{2}(\alpha_{10}^{(1)} + \beta_{01}^{(1)} + (\alpha_{10}^{(1)} - \beta_{01}^{(1)}) \cos 2\theta + (\alpha_{01}^{(1)} + \beta_{10}^{(1)}) \sin 2\theta). \\
F_2(r,\theta) = {} & \frac{1}{r}(\alpha_{00}^{(2)} \cos\theta + \alpha_{10}^{(1)} r \cos^2\theta + \beta_{00}^{(2)} \sin\theta + \beta_{10}^{(1)} r \cos\theta \sin\theta + \alpha_{01}^{(1)} r \cos\theta \sin\theta + \beta_{01}^{(1)} r \\
& \sin^2\theta(-\beta_{00}^{(2)} \cos\theta - \beta_{10}^{(1)} r \cos^2\theta + \alpha_{10}^{(1)} r \cos\theta \sin\theta + \alpha_{01}^{(1)} r \sin^2\theta) + (\alpha_{00}^{(3)} \cos\theta + \alpha_{10}^{(2)} \\
& r \cos^2\theta + \alpha_{20}^{(1)} r^2 \cos^3\theta + \alpha_{00}^{(2)} \sin\theta - \beta_{01}^{(1)} r \cos\theta \sin\theta + \beta_{00}^{(3)} \sin\theta + \beta_{10}^{(2)} r \cos\theta \sin\theta \\
& + \alpha_{01}^{(2)} r \cos\theta \sin\theta + \beta_{20}^{(1)} r^2 \cos^2\theta \sin\theta + \alpha_{11}^{(1)} r^2 \cos^2\theta \sin\theta + \beta_{01}^{(2)} r \sin^2\theta + \beta_{11}^{(1)} r^2 \\
& \cos\theta \sin^2\theta + \alpha_{02}^{(1)} r^2 \cos\theta \sin^2\theta + \beta_{02}^{(1)} r^2 \sin^3\theta).
\end{aligned}$$

Using the formulas given in section 4.1 of [63] the averaged function of first order is

$$f_1(r) = (\alpha_{10}^{(1)} + \beta_{01}^{(1)}) r.$$

Clearly equation $f_1(r) = 0$ has no positive zeros. Thus the averaging method of first order does not provide limit cycles.

We put $\alpha_{10}^{(1)} = -\beta_{01}^{(1)}$ we obtain $f_1(r) = 0$. We apply the averaging theory of second order, we *get the averaging function of second order*

$$f_2(r) = (\alpha_{10}^{(2)} + \beta_{01}^{(2)})r.$$

We see that the equation $f_2(r) = 0$ has no positive zeros, it follows that there is no limit cycle by applying the averaging method of second order.

To apply the averaging method of third order we must put $\alpha_{10}^{(2)} = -\beta_{01}^{(2)}$ we get $f_2(r) = 0$. The third averaging function is

$$\begin{aligned} f_3(r) = & -(\beta_{11}^{(1)}\beta_{00}^{(2)} - \beta_{01}^{(3)} + 2\beta_{00}^{(2)}\alpha_{20}^{(1)} - 2\beta_{02}^{(1)}\alpha_{00}^{(2)} - \alpha_{11}^{(1)}\alpha_{00}^{(2)} - \alpha_{10}^{(3)})r + (1/4)(3\beta_{03}^{(1)} + \beta_{21}^{(1)} + 3\alpha_{30}^{(1)} \\ & +\alpha_{12}^{(1)})r^3 \end{aligned}$$

So, $f_3(r)$ can have at most one positive real root. Then we have the proof of the theorem for $k = 3$.

To apply the averaging method of fourth order, we need to have $f_3(r) = 0$, then we set

$$\begin{aligned} \alpha_{10}^{(3)} = & \ \beta_{11}^{(1)}\beta_{00}^{(2)} - \beta_{01}^{(3)} + 2\beta_{00}^{(2)}\alpha_{20}^{(1)} - 2\beta_{02}^{(1)}\alpha_{00}^{(2)} - \alpha_{11}^{(1)}\alpha_{00}^{(2)} \quad \textit{and} \\ \alpha_{12}^{(1)} = & -(3\beta_{03}^{(1)} + \beta_{21}^{(1)} + 3\alpha_{30}^{(1)}). \end{aligned}$$

The averaging function of fourth order is $f_4(r) = r(A_1 + A_2r^2)$, where

$$\begin{aligned} A_1 = & \Big(\beta_{10}^{(1)}\beta_{11}^{(1)}\beta_{00}^{(2)} + 2\beta_{01}^{(1)}\beta_{02}^{(1)}\beta_{00}^{(2)} - \beta_{00}^{(2)}\beta_{11}^{(2)} - \beta_{11}^{(1)}\beta_{00}^{(3)} + 2\beta_{10}^{(1)}\beta_{00}^{(2)}\alpha_{20}^{(1)} - 2\beta_{00}^{(3)}\alpha_{20}^{(1)} + \beta_{01}^{(1)}\beta_{00}^{(2)} \\ & \alpha_{11}^{(1)} - \beta_{01}^{(1)}\beta_{11}^{(1)}\alpha_{00}^{(2)} + 2\beta_{02}^{(2)}\alpha_{00}^{(2)} + 2\beta_{02}^{(1)}\alpha_{01}^{(1)}\alpha_{00}^{(2)} - 2\beta_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{00}^{(2)} + \alpha_{01}^{(1)}\alpha_{11}^{(1)}\alpha_{00}^{(2)} - 2\beta_{00}^{(2)} \\ & \alpha_{20}^{(2)} + \alpha_{00}^{(2)}\alpha_{11}^{(2)} + 2\beta_{02}^{(1)}\alpha_{00}^{(3)} + \alpha_{11}^{(1)}\alpha_{00}^{(3)} + \alpha_{10}^{(4)} + \beta_{01}^{(4)}\Big). \\ A_2 = & \frac{-1}{4}\Big(\beta_{20}^{(1)}\beta_{11}^{(1)} + \beta_{11}^{(1)}\beta_{02}^{(1)} + \beta_{10}^{(1)}\beta_{21}^{(1)} + 2\beta_{01}^{(1)}\beta_{12}^{(1)} - 3\beta_{03}^{(2)} - \beta_{21}^{(2)} + \beta_{21}^{(1)}\alpha_{01}^{(1)} - \alpha_{20}^{(1)}\alpha_{11}^{(1)} - 2\beta_{02}^{(1)} \\ & \alpha_{02}^{(1)} - \alpha_{11}^{(1)}\alpha_{02}^{(1)} + 2\beta_{01}^{(1)}\alpha_{21}^{(1)} + 3\beta_{10}^{(1)}\alpha_{30}^{(1)} + 3\alpha_{01}^{(1)}\alpha_{30}^{(1)} - 3\alpha_{30}^{(2)} + 2\beta_{20}^{(1)}\alpha_{20}^{(1)} - \alpha_{12}^{(2)}\Big). \end{aligned}$$

According to the expression of the function f_4 we conclude that can get at most one limit cycle.

Solving $A_1 = 0$ and $A_2 = 0$ we obtain $f_4(r) = 0$, so we can apply the averaging theory of order 5, and its corresponding averaging function is $f_5(r) = r(B_1 + B_2r^2 + B_3r^4)$.

The explicit expression of B_i, with $i = 1, 2\,3$ are given in the appendix of chapter 3.

The rank of the largest square matrix of the Jacobian matrix $\mathcal{B} = (B_1, B_2, B_3)$ is 3. Then the coefficients B_1, B_2 and B_3 are linearly independent in their variables. By the Descartes Theorem (or by the roots of a quadratic polynomial in the variable r^2) it follows that we can get at mosst two positive real roots of $f_5(r)$. So statement (c) holds. Solving $B_1 = 0$, $B_2 = 0$ and $B_3 = 0$ we obtain $f_5(r) = 0$.

Now if we apply the averaging method of order six we get

$$f_6(r) = \left(K_1 + K_2 r^2 + K_3 r^4\right) r.$$

The rank of the Jacobian matrix $\mathcal{K} = (K_1, K_2, K_3)$ with respect to its variables is 3. We have three of the coefficients K_i, $i = 1, 2, 3$ which are linearly independent in their variables, where these valeurs are given in the appendix of chapter 3. Therefore by Theorem 1.3, it follows that $f_6(r) = 0$ has 2 positive real roots. Consequently, the differential system (3.2) has at least 2 limit cycles. This ends the proof of the theorem. ■

Section 3.3 Appendix of Chapter 3

Here we provide the values B_1, B_2 and B_3 of the avreraging fonction f_5.

$$\begin{aligned}
B_1 = -\ \Big(& (\beta_{10}^{(1)})^2 \beta_{11}^{(1)} \beta_{00}^{(2)} + (\beta_{01}^{(1)})^2 \beta_{11}^{(1)} \beta_{00}^{(2)} + 2\beta_{10}^{(1)} \beta_{01}^{(1)} \beta_{02}^{(1)} \beta_{00}^{(2)} + 2\beta_{00}^{(2)} \alpha_{20}^{(3)} + \beta_{00}^{(2)} \beta_{11}^{(3)} - \beta_{10}^{(1)} \beta_{00}^{(2)} \beta_{11}^{(2)} \\
& -2\beta_{01}^{(1)} \beta_{00}^{(2)} \beta_{02}^{(2)} - \beta_{10}^{(1)} \beta_{11}^{(1)} \beta_{00}^{(3)} - 2\beta_{01}^{(1)} \beta_{02}^{(1)} \beta_{00}^{(3)} + \beta_{11}^{(2)} \beta_{00}^{(3)} - \alpha_{10}^{(5)} - \beta_{10}^{(1)} \beta_{01}^{(1)} \beta_{11}^{(1)} \alpha_{00}^{(2)} - 2\beta_{02}^{(1)} \\
& \beta_{00}^{(2)} \beta_{60}^{(2)} - 2\beta_{10}^{(1)} \beta_{00}^{(3)} \alpha_{20}^{(1)} - 2\beta_{02}^{(1)} \alpha_{00}^{(4)} - \alpha_{11}^{(1)} \alpha_{00}^{(4)} - \beta_{01}^{(5)} - 3(\beta_{00}^{(2)})^2 \alpha_{30}^{(1)} - \alpha_{01}^{(1)} \alpha_{11}^{(1)} \alpha_{00}^{(3)} - 2 \\
& \beta_{02}^{(1)} (\alpha_{01}^{(1)})^2 \alpha_{00}^{(2)} - \alpha_{11}^{(1)} \alpha_{00}^{(2)} \alpha_{01}^{(2)} - \alpha_{00}^{(2)} \alpha_{11}^{(3)} - 2\beta_{01}^{(1)} \beta_{02}^{(1)} \beta_{00}^{(2)} \alpha_{01}^{(1)} + 2(\beta_{10}^{(1)})^2 \beta_{00}^{(2)} \alpha_{20}^{(1)} + 2(\beta_{01}^{(1)})^2 \\
& \beta_{00}^{(2)} \alpha_{20}^{(1)} - 2\beta_{00}^{(2)} \beta_{10}^{(2)} \alpha_{20}^{(1)} + 2\beta_{00}^{(4)} \alpha_{20}^{(1)} + \beta_{10}^{(1)} \beta_{01}^{(1)} \beta_{00}^{(2)} \alpha_{11}^{(1)} - \beta_{00}^{(2)} \beta_{01}^{(2)} \alpha_{11}^{(1)} - \beta_{01}^{(1)} \beta_{00}^{(3)} \alpha_{11}^{(1)} - \beta_{01}^{(1)} \\
& \beta_{00}^{(2)} \alpha_{01}^{(1)} \alpha_{11}^{(1)} + 3\alpha_{30}^{(1)} (\alpha_{00}^{(2)})^2 - 2(\beta_{01}^{(1)})^2 \beta_{02}^{(1)} \alpha_{00}^{(2)} + 2\beta_{12}^{(1)} \beta_{00}^{(2)} \alpha_{00}^{(2)} + 2\beta_{00}^{(2)} \alpha_{21}^{(1)} \alpha_{00}^{(2)} + \beta_{21}^{(1)} (\alpha_{00}^{(2)})^2 \\
& +\beta_{11}^{(1)} \beta_{01}^{(2)} \alpha_{00}^{(2)} + \beta_{01}^{(1)} \beta_{11}^{(2)} \alpha_{00}^{(2)} - 2\beta_{02}^{(3)} \alpha_{00}^{(2)} + \beta_{01}^{(1)} \beta_{11}^{(1)} \alpha_{01}^{(1)} \alpha_{00}^{(2)} - 2\beta_{02}^{(2)} \alpha_{01}^{(1)} \alpha_{00}^{(2)} - 2\beta_{10}^{(1)} \beta_{01}^{(1)} \alpha_{20}^{(1)} \\
& \alpha_{00}^{(2)} + 2\beta_{01}^{(2)} \alpha_{20}^{(1)} \alpha_{00}^{(2)} + 2\beta_{01}^{(1)} \alpha_{01}^{(1)} \alpha_{20}^{(1)} \alpha_{00}^{(2)} - (\beta_{01}^{(1)})^2 \alpha_{11}^{(1)} \alpha_{00}^{(2)} - 2\beta_{02}^{(1)} \alpha_{00}^{(2)} \alpha_{01}^{(2)} - 2\beta_{10}^{(1)} \beta_{00}^{(2)} \alpha_{20}^{(2)} \\
& +2\beta_{00}^{(3)} \alpha_{20}^{(2)} + 2\beta_{01}^{(1)} \alpha_{00}^{(2)} \alpha_{20}^{(2)} - \beta_{01}^{(1)} \beta_{00}^{(2)} \alpha_{11}^{(2)} - \alpha_{01}^{(1)} \alpha_{00}^{(2)} \alpha_{11}^{(2)} + \beta_{01}^{(1)} \beta_{11}^{(1)} \alpha_{00}^{(3)} - 2\beta_{02}^{(2)} \alpha_{00}^{(3)} - 2\beta_{02}^{(1)}
\end{aligned}$$

$$\alpha_{01}^{(1)}\alpha_{00}^{(3)}+2\beta_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{00}^{(3)}-\beta_{21}^{(1)}(\beta_{00}^{(2)})^2-\beta_{11}^{(1)}\beta_{00}^{(2)}\beta_{10}^{(2)}-\alpha_{11}^{(2)}\alpha_{00}^{(3)}-(\alpha_{01}^{(1)})^2\alpha_{11}^{(1)}\alpha_{00}^{(2)}+\beta_{11}^{(1)}\beta_{00}^{(4)}\Big).$$

$$\begin{aligned}B_2 = {} & \frac{1}{4}\Big(2\beta_{10}^{(1)}\beta_{20}^{(1)}\beta_{11}^{(1)}+\beta_{01}^{(1)}(\beta_{11}^{(1)})^2+2\beta_{01}^{(1)}\beta_{20}^{(1)}\beta_{02}^{(1)}+\beta_{10}^{(1)}\beta_{11}^{(1)}\beta_{02}^{(1)}+2\beta_{01}^{(1)}(\beta_{02}^{(1)})^2+(\beta_{10}^{(1)})^2\beta_{21}^{(1)}+2\\
& \beta_{10}^{(1)}\beta_{01}^{(1)}\beta_{12}^{(1)}-3\beta_{31}^{(1)}\beta_{00}^{(2)}-3\beta_{13}^{(1)}\beta_{00}^{(2)}-\beta_{21}^{(1)}\beta_{10}^{(2)}-2\beta_{12}^{(1)}\beta_{01}^{(2)}+\alpha_{12}^{(3)}-\beta_{20}^{(1)}\beta_{11}^{(2)}-\beta_{02}^{(1)}\beta_{11}^{(2)}-\beta_{11}^{(1)}\\
& \beta_{02}^{(2)}-\beta_{10}^{(1)}\beta_{21}^{(2)}-2\beta_{01}^{(1)}\beta_{12}^{(2)}+3\beta_{03}^{(3)}+\beta_{21}^{(3)}+3\alpha_{30}^{(3)}-2\beta_{00}^{(2)}\alpha_{22}^{(1)}+\beta_{20}^{(1)}\beta_{11}^{(1)}\alpha_{01}^{(1)}+\beta_{10}^{(1)}\beta_{21}^{(1)}\alpha_{01}^{(1)}-\\
& \beta_{21}^{(2)}\alpha_{01}^{(1)}+4\beta_{10}^{(1)}\beta_{20}^{(1)}\alpha_{20}^{(1)}+2\beta_{20}^{(1)}\alpha_{01}^{(1)}\alpha_{20}^{(1)}-2\beta_{01}^{(1)}(\alpha_{20}^{(1)})^2+\beta_{01}^{(1)}\beta_{20}^{(1)}\alpha_{11}^{(1)}-\beta_{01}^{(1)}\beta_{02}^{(1)}\alpha_{11}^{(1)}-\beta_{10}^{(1)}\\
& \alpha_{20}^{(1)}\alpha_{11}^{(1)}+2\beta_{02}^{(2)}\alpha_{02}^{(1)}+2\beta_{02}^{(1)}\alpha_{01}^{(1)}\alpha_{02}^{(1)}-2\beta_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{02}^{(1)}+\alpha_{01}^{(1)}\alpha_{11}^{(1)}\alpha_{02}^{(1)}+2\beta_{10}^{(1)}\beta_{01}^{(1)}\alpha_{21}^{(1)}+3(\beta_{10}^{(1)})^2\\
& \alpha_{30}^{(1)}-3\beta_{10}^{(2)}\alpha_{30}^{(1)}+12\beta_{04}^{(1)}\alpha_{00}^{(2)}+2\beta_{22}^{(1)}\alpha_{00}^{(2)}+3\alpha_{31}^{(1)}\alpha_{00}^{(2)}-2\beta_{01}^{(1)}\alpha_{21}^{(2)}+3\alpha_{13}^{(1)}\alpha_{00}^{(2)}-\beta_{21}^{(1)}\alpha_{01}^{(2)}-\\
& 2\beta_{20}^{(1)}\alpha_{20}^{(2)}+\alpha_{11}^{(1)}\alpha_{20}^{(2)}+\alpha_{20}^{(1)}\alpha_{11}^{(2)}+\alpha_{02}^{(1)}\alpha_{11}^{(2)}+\alpha_{11}^{(1)}\alpha_{02}^{(2)}-\beta_{01}^{(1)}\beta_{11}^{(1)}\alpha_{02}^{(1)}+\beta_{01}^{(1)}\beta_{11}^{(1)}\alpha_{20}^{(1)}-2\beta_{20}^{(2)}\\
& \alpha_{20}^{(1)}-\beta_{11}^{(1)}\beta_{20}^{(2)}-\beta_{01}^{(1)}(\alpha_{11}^{(1)})^2-2\beta_{01}^{(2)}\alpha_{21}^{(1)}+2\beta_{02}^{(1)}\alpha_{02}^{(2)}-3\beta_{10}^{(1)}\alpha_{30}^{(2)}-3\alpha_{01}^{(1)}\alpha_{30}^{(2)}-12\beta_{00}^{(2)}\alpha_{40}^{(1)}\\
& +3\beta_{10}^{(1)}\alpha_{01}^{(1)}\alpha_{30}^{(1)}-3\alpha_{30}^{(1)}\alpha_{01}^{(2)}\Big).\end{aligned}$$

$$B_3 = \frac{1}{8}(\beta_{23}^{(1)}+\beta_{41}^{(1)}+5\beta_{05}^{(1)}+5\alpha_{50}^{(1)}+\alpha_{14}^{(1)}+\alpha_{32}^{(1)}).$$

Here we provide the values K_1, K_2 and K_3 of the avreraging fonction f_6.

$$\begin{aligned}K_1 = {} & (\beta_{10}^{(1)})^3\beta_{11}^{(1)}\beta_{00}^{(2)}+2\beta_{10}^{(1)}(\beta_{01}^{(1)})^2\beta_{11}^{(1)}\beta_{00}^{(2)}+2(\beta_{10}^{(1)})^2\beta_{01}^{(1)}\beta_{02}^{(1)}\beta_{00}^{(2)}+2(\beta_{01}^{(1)})^3\beta_{02}^{(1)}\beta_{00}^{(2)}-\beta_{20}^{(1)}\beta_{11}^{(1)}\\
& (\beta_{00}^{(2)})^2-2\beta_{11}^{(1)}\beta_{02}^{(1)}(\beta_{00}^{(2)})^2-2\beta_{10}^{(1)}\beta_{21}^{(1)}(\beta_{00}^{(2)})^2-2\beta_{01}^{(1)}\beta_{12}^{(1)}(\beta_{00}^{(2)})^2-2\beta_{10}^{(1)}\beta_{11}^{(1)}\beta_{00}^{(2)}\beta_{10}^{(2)}-2\beta_{01}^{(1)}\\
& \beta_{02}^{(1)}\beta_{00}^{(2)}\beta_{10}^{(2)}-2\beta_{01}^{(1)}\beta_{11}^{(1)}\beta_{00}^{(2)}\beta_{01}^{(2)}-2\beta_{10}^{(1)}\beta_{02}^{(1)}\beta_{00}^{(2)}\beta_{01}^{(2)}-(\beta_{10}^{(1)})^2\beta_{00}^{(2)}\beta_{11}^{(2)}-(\beta_{01}^{(1)})^2\beta_{00}^{(2)}\beta_{11}^{(2)}+\\
& \beta_{00}^{(2)}\beta_{10}^{(2)}\beta_{11}^{(2)}-2\beta_{10}^{(1)}\beta_{01}^{(1)}\beta_{00}^{(2)}\beta_{02}^{(2)}+2\beta_{00}^{(2)}\beta_{01}^{(2)}\beta_{02}^{(2)}+(\beta_{00}^{(2)})^2\beta_{21}^{(2)}-(\beta_{10}^{(1)})^2\beta_{11}^{(1)}\beta_{00}^{(3)}-(\beta_{01}^{(1)})^2\beta_{11}^{(1)}\\
& \beta_{00}^{(3)}-2\beta_{10}^{(1)}\beta_{01}^{(1)}\beta_{02}^{(1)}\beta_{00}^{(3)}+2\beta_{21}^{(1)}\beta_{00}^{(2)}\beta_{00}^{(3)}+\beta_{11}^{(1)}\beta_{10}^{(2)}\beta_{00}^{(3)}+2\beta_{02}^{(1)}\beta_{01}^{(2)}\beta_{00}^{(3)}+\beta_{10}^{(1)}\beta_{11}^{(2)}\beta_{00}^{(3)}+2\beta_{01}^{(1)}\\
& \beta_{02}^{(2)}\beta_{00}^{(3)}+\beta_{11}^{(1)}\beta_{00}^{(2)}\beta_{10}^{(3)}+2\beta_{02}^{(1)}\beta_{00}^{(2)}\beta_{01}^{(3)}+\beta_{10}^{(1)}\beta_{00}^{(2)}\beta_{11}^{(3)}+2\beta_{01}^{(1)}\beta_{00}^{(2)}\beta_{02}^{(3)}+\beta_{10}^{(1)}\beta_{11}^{(1)}\beta_{00}^{(4)}+2\beta_{01}^{(1)}\\
& \beta_{02}^{(1)}\beta_{00}^{(4)}-\beta_{11}^{(2)}\beta_{00}^{(4)}-\beta_{00}^{(2)}\beta_{11}^{(4)}-\beta_{11}^{(1)}\beta_{00}^{(5)}+\beta_{01}^{(6)}-(\beta_{01}^{(1)})^2\beta_{11}^{(1)}\beta_{00}^{(2)}\alpha_{01}^{(1)}-2\beta_{10}^{(1)}\beta_{01}^{(1)}\beta_{02}^{(1)}\beta_{00}^{(2)}\alpha_{01}^{(1)}\\
& +2\beta_{02}^{(1)}\beta_{00}^{(2)}\beta_{01}^{(2)}\alpha_{01}^{(1)}+2\beta_{01}^{(1)}\beta_{00}^{(2)}\beta_{02}^{(2)}\alpha_{01}^{(1)}+2\beta_{01}^{(1)}\beta_{02}^{(1)}\beta_{00}^{(3)}\alpha_{01}^{(1)}+2\beta_{01}^{(1)}\beta_{02}^{(1)}\beta_{00}^{(2)}(\alpha_{01}^{(1)})^2+2(\beta_{10}^{(1)})^3\\
& \beta_{00}^{(2)}\alpha_{20}^{(1)}+4\beta_{10}^{(1)}(\beta_{01}^{(1)})^2\beta_{00}^{(2)}\alpha_{20}^{(1)}-2\beta_{20}^{(1)}(\beta_{00}^{(2)})^2\alpha_{20}^{(1)}-2\beta_{02}^{(1)}(\beta_{00}^{(2)})^2\alpha_{20}^{(1)}-4\beta_{10}^{(1)}\beta_{00}^{(2)}\beta_{10}^{(2)}\alpha_{20}^{(1)}-4\end{aligned}$$

$$\beta_{01}^{(1)}\beta_{00}^{(2)}\beta_{01}^{(2)}\alpha_{20}^{(1)} - 2(\beta_{10}^{(1)})^2\beta_{00}^{(3)}\alpha_{20}^{(1)} - 2(\beta_{01}^{(1)})^2\beta_{00}^{(3)}\alpha_{20}^{(1)} + 2\beta_{10}^{(2)}\beta_{00}^{(3)}\alpha_{20}^{(1)} + 2\beta_{00}^{(2)}\beta_{10}^{(3)}\alpha_{20}^{(1)}\beta_{10}^{(1)}\beta_{00}^{(4)}$$

$$\alpha_{20}^{(1)} - 2\beta_{00}^{(5)}\alpha_{20}^{(1)} - 2(\beta_{01}^{(1)})^2\beta_{00}^{(2)}\alpha_{01}^{(1)}\alpha_{20}^{(1)} + (\beta_{10}^{(1)})^2\beta_{01}^{(1)}\beta_{00}^{(2)}\alpha_{11}^{(1)} + (\beta_{01}^{(1)})^3\beta_{00}^{(2)}\alpha_{11}^{(1)} - \beta_{11}^{(1)}(\beta_{00}^{(2)})^2$$

$$\alpha_{11}^{(1)} - \beta_{01}^{(1)}\beta_{00}^{(2)}\beta_{10}^{(2)}\alpha_{11}^{(1)} - \beta_{10}^{(1)}\beta_{00}^{(2)}\beta_{01}^{(2)}\alpha_{11}^{(1)} - \beta_{10}^{(1)}\beta_{01}^{(1)}\beta_{00}^{(3)}\alpha_{11}^{(1)} + \beta_{01}^{(2)}\beta_{00}^{(3)}\alpha_{11}^{(1)} + \beta_{00}^{(2)}\beta_{01}^{(3)} + \alpha_{11}^{(1)}$$

$$\beta_{01}^{(1)}\beta_{00}^{(4)}\alpha_{11}^{(1)} - \beta_{10}^{(1)}\beta_{01}^{(1)}\beta_{00}^{(2)}\alpha_{01}^{(1)}\alpha_{11}^{(1)} + \beta_{00}^{(2)}\beta_{01}^{(2)}\alpha_{01}^{(1)}\alpha_{11}^{(1)} + \beta_{01}^{(1)}\beta_{00}^{(3)}\alpha_{01}^{(1)}\alpha_{11}^{(1)} + \beta_{01}^{(1)}\beta_{00}^{(2)}(\alpha_{01}^{(1)})^2\alpha_{11}^{(1)}$$

$$-(\beta_{00}^{(2)})^2\alpha_{20}^{(1)}\alpha_{11}^{(1)} - 2\beta_{01}^{(1)}(\beta_{00}^{(2)})^2\alpha_{21}^{(1)} - 6\beta_{10}^{(1)}(\beta_{00}^{(2)})^2\alpha_{30}^{(1)} + 6\beta_{00}^{(2)}\beta_{00}^{(3)}\alpha_{30}^{(1)} - \beta_{00}^{(3)}\beta_{11}^{(3)} - (\beta_{10}^{(1)})^2\beta_{01}^{(1)}$$

$$\beta_{11}^{(1)}\alpha_{00}^{(2)} - (\beta_{01}^{(1)})^3\beta_{11}^{(1)}\alpha_{00}^{(2)} - 2\beta_{10}^{(1)}(\beta_{01}^{(1)})^2\beta_{02}^{(1)}\alpha_{00}^{(2)} + (\beta_{11}^{(1)})^2\beta_{00}^{(2)}\alpha_{00}^{(2)} + 4(\beta_{02}^{(1)})^2\beta_{00}^{(2)}\alpha_{00}^{(2)} + 2\beta_{10}^{(1)}$$

$$\beta_{12}^{(1)}\beta_{00}^{(2)}\alpha_{00}^{(2)} + \beta_{01}^{(1)}\beta_{11}^{(1)}\beta_{10}^{(2)}\alpha_{00}^{(2)} + \beta_{10}^{(1)}\beta_{11}^{(1)}\beta_{01}^{(2)}\alpha_{00}^{(2)} + 4\beta_{01}^{(1)}\beta_{02}^{(1)}\beta_{01}^{(2)}\alpha_{00}^{(2)} - \beta_{01}^{(2)}\beta_{11}^{(2)}\alpha_{00}^{(2)} + 2(\beta_{01}^{(1)})^2$$

$$\beta_{02}^{(2)}\alpha_{00}^{(2)} - \beta_{11}^{(1)}\beta_{01}^{(2)}\alpha_{01}^{(1)}\alpha_{00}^{(2)} - 2\beta_{00}^{(2)}\beta_{12}^{(2)}\alpha_{00}^{(2)} - 2\beta_{12}^{(1)}\beta_{00}^{(3)}\alpha_{00}^{(2)} - \beta_{11}^{(1)}\beta_{01}^{(3)}\alpha_{00}^{(2)} - \beta_{01}^{(1)}\beta_{11}^{(3)}\alpha_{00}^{(2)} + 2$$

$$\beta_{02}^{(4)}\alpha_{00}^{(2)} + \beta_{10}^{(1)}\beta_{01}^{(1)}\beta_{11}^{(1)}\alpha_{01}^{(1)}\alpha_{00}^{(2)} + 4(\beta_{01}^{(1)})^2\beta_{02}^{(1)}\alpha_{01}^{(1)}\alpha_{00}^{(2)} - 2\beta_{12}^{(1)}\beta_{00}^{(2)}\alpha_{01}^{(1)}\alpha_{00}^{(2)} + \beta_{10}^{(1)}\beta_{01}^{(1)}\beta_{11}^{(2)}\alpha_{00}^{(2)}$$

$$-\beta_{01}^{(1)}\beta_{11}^{(2)}\alpha_{01}^{(1)}\alpha_{00}^{(2)} + 2\beta_{02}^{(3)}\alpha_{01}^{(1)}\alpha_{00}^{(2)} - \beta_{01}^{(1)}\beta_{11}^{(1)}(\alpha_{01}^{(1)})^2\alpha_{00}^{(2)} + 2\beta_{02}^{(2)}(\alpha_{01}^{(1)})^2\alpha_{00}^{(2)} + 2\beta_{02}^{(1)}(\alpha_{01}^{(1)})^3\alpha_{00}^{(2)}$$

$$-2(\beta_{10}^{(1)})^2\beta_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{00}^{(2)} - 2(\beta_{01}^{(1)})^3\alpha_{20}^{(1)}\alpha_{00}^{(2)} + 2\beta_{11}^{(1)}\beta_{00}^{(2)}\alpha_{20}^{(1)}\alpha_{00}^{(2)} + 2\beta_{01}^{(1)}\beta_{10}^{(2)}\alpha_{20}^{(1)}\alpha_{00}^{(2)} + 2\beta_{10}^{(1)}\beta_{01}^{(2)}$$

$$\alpha_{20}^{(1)}\alpha_{00}^{(2)} - 2\beta_{01}^{(3)}\alpha_{20}^{(1)}\alpha_{00}^{(2)} + 2\beta_{10}^{(1)}\beta_{01}^{(1)}\alpha_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{00}^{(2)} - 2\beta_{01}^{(2)}\alpha_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{00}^{(2)} - 2\beta_{01}^{(1)}(\alpha_{01}^{(1)})^2\alpha_{20}^{(1)}\alpha_{00}^{(2)}$$

$$-\beta_{10}^{(1)}(\beta_{01}^{(1)})^2\alpha_{11}^{(1)}\alpha_{00}^{(2)} + 2\beta_{02}^{(1)}\beta_{00}^{(2)}\alpha_{11}^{(1)}\alpha_{00}^{(2)} + 2\beta_{01}^{(1)}\beta_{01}^{(2)}\alpha_{11}^{(1)}\alpha_{00}^{(2)} + 2(\beta_{01}^{(1)})^2\alpha_{01}^{(1)}\alpha_{11}^{(1)}\alpha_{00}^{(2)} + (\alpha_{01}^{(1)})^3$$

$$\alpha_{11}^{(1)}\alpha_{00}^{(2)} + 2\beta_{10}^{(1)}\beta_{00}^{(2)}\alpha_{21}^{(1)}\alpha_{00}^{(2)} - 2\beta_{00}^{(3)}\alpha_{21}^{(1)}\alpha_{00}^{(2)} - 2\beta_{00}^{(2)}\alpha_{01}^{(1)}\alpha_{21}^{(1)}\alpha_{00}^{(2)} + \beta_{10}^{(1)}\beta_{21}^{(1)}(\alpha_{00}^{(2)})^2 - \beta_{21}^{(2)}(\alpha_{00}^{(2)})^2$$

$$+2\beta_{02}^{(1)}\alpha_{00}^{(5)} - \beta_{21}^{(1)}\alpha_{01}^{(1)}(\alpha_{00}^{(2)})^2 + 2\beta_{20}^{(1)}\alpha_{20}^{(1)}(\alpha_{00}^{(2)})^2 - 2\beta_{02}^{(1)}\alpha_{20}^{(1)}(\alpha_{00}^{(2)})^2 - \alpha_{20}^{(1)}\alpha_{11}^{(1)}(\alpha_{00}^{(2)})^2 + 3\beta_{10}^{(1)}$$

$$\alpha_{30}^{(1)}(\alpha_{00}^{(2)})^2 - 3\alpha_{01}^{(1)}\alpha_{30}^{(1)}(\alpha_{00}^{(2)})^2 + \beta_{01}^{(1)}\beta_{00}^{(2)}\alpha_{11}^{(1)}\alpha_{01}^{(2)} + \alpha_{00}^{(2)}\alpha_{11}^{(4)} - 2\beta_{00}^{(4)}\alpha_{20}^{(2)} - \beta_{01}^{(1)}\beta_{11}^{(1)}\alpha_{00}^{(2)}\alpha_{01}^{(2)} +$$

$$2\beta_{02}^{(2)}\alpha_{00}^{(2)}\alpha_{01}^{(2)} + 4\beta_{02}^{(1)}\alpha_{01}^{(1)}\alpha_{00}^{(2)}\alpha_{01}^{(2)} - 2\beta_{00}^{(2)}\alpha_{20}^{(4)} - 2\beta_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{00}^{(2)}\alpha_{01}^{(2)} + 2\alpha_{01}^{(1)}\alpha_{11}^{(1)}\alpha_{00}^{(2)}\alpha_{01}^{(2)} - 2$$

$$(\beta_{10}^{(1)})^2\beta_{00}^{(2)}\alpha_{20}^{(2)} - 2 + 2\beta_{10}^{(1)}\beta_{01}^{(1)}\alpha_{00}^{(2)}\alpha_{20}^{(2)} - 2\beta_{01}^{(2)}\alpha_{00}^{(2)}\alpha_{20}^{(2)} - 2\beta_{01}^{(1)}\alpha_{01}^{(1)}\alpha_{00}^{(2)}\alpha_{20}^{(2)} + \alpha_{00}^{(3)}\alpha_{11}^{(3)} - \beta_{10}^{(1)}$$

$$\beta_{01}^{(1)}\beta_{00}^{(2)}\alpha_{11}^{(2)} + \beta_{00}^{(2)}\beta_{01}^{(2)}\alpha_{11}^{(2)} + \beta_{01}^{(1)}\beta_{00}^{(3)}\alpha_{11}^{(2)} + \beta_{01}^{(1)}\beta_{00}^{(2)}\alpha_{01}^{(1)}\alpha_{11}^{(2)} + (\beta_{01}^{(1)})^2\alpha_{00}^{(2)}\alpha_{11}^{(2)} + (\alpha_{01}^{(1)})^2\alpha_{00}^{(2)}$$

$$\alpha_{11}^{(2)} + \alpha_{00}^{(2)}\alpha_{01}^{(2)}\alpha_{11}^{(2)} - 2\beta_{00}^{(2)}\alpha_{00}^{(2)}\alpha_{21}^{(2)} + 3(\beta_{00}^{(2)})^2\alpha_{30}^{(2)} - 3(\alpha_{00}^{(2)})^2\alpha_{30}^{(2)}\beta_{10}^{(1)}\beta_{01}^{(1)}\beta_{11}^{(1)}\alpha_{00}^{(3)} + 2(\beta_{01}^{(1)})^2$$

$$\beta_{02}^{(1)}\alpha_{00}^{(3)} - 2\beta_{12}^{(1)}\beta_{00}^{(2)}\alpha_{00}^{(3)} - \beta_{11}^{(1)}\beta_{01}^{(2)}\alpha_{00}^{(3)} - \beta_{01}^{(1)}\beta_{11}^{(2)}\alpha_{00}^{(3)} + 2\beta_{02}^{(3)}\alpha_{00}^{(3)} - \beta_{01}^{(1)}\beta_{11}^{(1)}\alpha_{01}^{(1)}\alpha_{00}^{(3)} + 2\beta_{02}^{(2)}$$

$$\alpha_{01}^{(1)}\alpha_{00}^{(3)} + 2\beta_{02}^{(1)}(\alpha_{01}^{(1)})^2\alpha_{00}^{(3)} + 2\beta_{10}^{(1)}\beta_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{00}^{(3)} - 2\beta_{01}^{(2)}\alpha_{20}^{(1)}\alpha_{00}^{(3)} - 2\beta_{01}^{(1)}\alpha_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{00}^{(3)} + (\beta_{01}^{(1)})^2$$

$$\alpha^{(1)}_{11}\alpha^{(3)}_{00} - 2\beta^{(2)}_{00}\alpha^{(1)}_{21}\alpha^{(3)}_{00} - 2\beta^{(1)}_{21}\alpha^{(2)}_{00}\alpha^{(3)}_{00} - 6\alpha^{(1)}_{30}\alpha^{(2)}_{00}\alpha^{(3)}_{00} + 2\beta^{(1)}_{02}\alpha^{(2)}_{01}\alpha^{(3)}_{00} + \alpha^{(1)}_{11}\alpha^{(2)}_{01}\alpha^{(3)}_{00}$$

$$-2\beta^{(1)}_{01}\alpha^{(2)}_{20}\alpha^{(3)}_{00} + \alpha^{(1)}_{01}\alpha^{(2)}_{11}\alpha^{(3)}_{00} + 2\beta^{(1)}_{02}\alpha^{(2)}_{00}\alpha^{(3)}_{01} + \alpha^{(1)}_{11}\alpha^{(2)}_{00}\alpha^{(3)}_{01} + 2\beta^{(1)}_{10}\beta^{(2)}_{00}\alpha^{(3)}_{20} - 2\beta^{(3)}_{00}\alpha^{(3)}_{20}$$

$$+\alpha^{(1)}_{11}\alpha^{(5)}_{00} - 2\beta^{(1)}_{01}\alpha^{(2)}_{00}\alpha^{(3)}_{20} + \beta^{(1)}_{01}\beta^{(2)}_{00}\alpha^{(3)}_{11} + \alpha^{(1)}_{01}\alpha^{(2)}_{00}\alpha^{(3)}_{11} + 2\beta^{(1)}_{01}\beta^{(1)}_{02}\beta^{(2)}_{00}\alpha^{(2)}_{01} + \beta^{(1)}_{20}\beta^{(1)}_{11}(\alpha^{(2)}_{00})^2$$

$$-\beta^{(1)}_{01}\beta^{(1)}_{11}\alpha^{(4)}_{00} + 2\beta^{(2)}_{02}\alpha^{(4)}_{00} + 2\beta^{(1)}_{02}\alpha^{(1)}_{01}\alpha^{(4)}_{00} - 2\beta^{(1)}_{01}\alpha^{(1)}_{20}\alpha^{(4)}_{00} + \alpha^{(1)}_{01}\alpha^{(1)}_{11}\alpha^{(4)}_{00} + \alpha^{(2)}_{11}\alpha^{(4)}_{00}.$$

$$K_2 = \frac{-1}{4}(3(\beta^{(1)}_{10})^2\beta^{(1)}_{20}\beta^{(1)}_{11} + (\beta^{(1)}_{01})^2\beta^{(1)}_{20}\beta^{(1)}_{11} + 2\beta^{(1)}_{10}\beta^{(1)}_{01}(\beta^{(1)}_{11})^2 + 4\beta^{(1)}_{10}\beta^{(1)}_{01}\beta^{(1)}_{20}\beta^{(1)}_{02} + (\beta^{(1)}_{10})^2\beta^{(1)}_{11}$$

$$\beta^{(1)}_{02} + 2(\beta^{(1)}_{10})^2\beta^{(1)}_{01}\beta^{(1)}_{12} - 6\beta^{(1)}_{10}\beta^{(1)}_{31}\beta^{(2)}_{00} - 3\beta^{(1)}_{10}\beta^{(1)}_{13}\beta^{(2)}_{00} - 3\alpha^{(4)}_{30} - 6\beta^{(1)}_{22}\beta^{(1)}_{01}\beta^{(2)}_{00} - 6\beta^{(1)}_{02}\beta^{(1)}_{03}$$

$$\beta^{(2)}_{00} - 4\beta^{(1)}_{20}\beta^{(1)}_{21}\beta^{(2)}_{00} - 4\beta^{(1)}_{02}\beta^{(1)}_{21}\beta^{(2)}_{00} - 3\beta^{(1)}_{11}\beta^{(1)}_{30}\beta^{(2)}_{00} - 3\beta^{(1)}_{11}\beta^{(1)}_{12}\beta^{(2)}_{00} - 2\beta^{(1)}_{20}\beta^{(1)}_{11}\beta^{(2)}_{10} - \beta^{(1)}_{11}$$

$$\beta^{(1)}_{02}\beta^{(2)}_{10} - 2\beta^{(1)}_{10}\beta^{(1)}_{21}\beta^{(2)}_{10} - 2\beta^{(1)}_{01}\beta^{(1)}_{12}\beta^{(2)}_{10} + 3\beta^{(2)}_{00}\beta^{(2)}_{31} + 3\beta^{(2)}_{00}\beta^{(2)}_{13} - (\beta^{(1)}_{11})^2\beta^{(2)}_{01} - 2\beta^{(1)}_{20}\beta^{(1)}_{02}$$

$$\beta^{(2)}_{01} - 2\beta^{(1)}_{10}\beta^{(1)}_{12}\beta^{(2)}_{01} - 2\beta^{(1)}_{10}\beta^{(1)}_{11}\beta^{(2)}_{20} - 2(\beta^{(1)}_{02})^2\beta^{(2)}_{01} + \beta^{(1)}_{21}\alpha^{(3)}_{01} - 2\beta^{(1)}_{01}\beta^{(1)}_{02}\beta^{(2)}_{20} - 2\beta^{(1)}_{10}\beta^{(1)}_{20}$$

$$\beta^{(2)}_{11} - 2\beta^{(1)}_{02}\alpha^{(3)}_{02} - 4\beta^{(1)}_{01}\beta^{(1)}_{02}\beta^{(2)}_{02} + \beta^{(2)}_{11}\beta^{(2)}_{02} - (\beta^{(1)}_{10})^2\beta^{(2)}_{21} + \beta^{(2)}_{10}\beta^{(2)}_{21} - 2\beta^{(1)}_{10}\beta^{(1)}_{01}\beta^{(2)}_{12} + 2\beta^{(1)}_{20}$$

$$\alpha^{(3)}_{20} + 3(\beta^{(1)}_{01})^2\beta^{(1)}_{11}\beta^{(1)}_{02} + 2\beta^{(2)}_{01}\beta^{(2)}_{12} + 3\beta^{(1)}_{31}\beta^{(3)}_{00} + 3\beta^{(1)}_{13}\beta^{(3)}_{00} + \beta^{(1)}_{21}\beta^{(3)}_{10} + 2\beta^{(1)}_{12}\beta^{(3)}_{01} + \beta^{(1)}_{11}$$

$$\beta^{(3)}_{20} + \beta^{(1)}_{20}\beta^{(3)}_{11} + \beta^{(1)}_{02}\beta^{(3)}_{11} + \beta^{(1)}_{11}\beta^{(3)}_{02} + \beta^{(1)}_{10}\beta^{(3)}_{21} + 2\beta^{(1)}_{01}\beta^{(3)}_{12} - \alpha^{(1)}_{11}\alpha^{(3)}_{20} - 2\beta^{(1)}_{10}\beta^{(2)}_{00}\alpha^{(1)}_{22} + 2$$

$$\beta^{(3)}_{00}\alpha^{(1)}_{22} - 3\beta^{(1)}_{01}\beta^{(2)}_{00}\alpha^{(1)}_{13} + 2\beta^{(1)}_{10}\beta^{(1)}_{20}\beta^{(1)}_{11}\alpha^{(1)}_{01} - 2\beta^{(1)}_{01}(\beta^{(1)}_{02})^2\alpha^{(1)}_{01} - \alpha^{(1)}_{02}\alpha^{(3)}_{11} - \beta^{(1)}_{10}\beta^{(1)}_{02}\beta^{(2)}_{11}$$

$$-\alpha^{(1)}_{20}\alpha^{(3)}_{11} + 12\beta^{(3)}_{00}\alpha^{(1)}_{40} + 3\alpha^{(1)}_{30}\alpha^{(3)}_{01} - 12\beta^{(1)}_{04}\beta^{(1)}_{01}\beta^{(2)}_{00} - \beta^{(1)}_{10}\beta^{(1)}_{11}\beta^{(2)}_{02} - 2\beta^{(1)}_{01}\beta^{(1)}_{11}\beta^{(2)}_{11} + 2$$

$$\beta^{(1)}_{10}\beta^{(1)}_{01}(\beta^{(1)}_{02})^2 - \alpha^{(4)}_{12} + \beta^{(2)}_{20}\beta^{(2)}_{11} + 3\alpha^{(1)}_{01}\alpha^{(3)}_{30} + (\beta^{(1)}_{10})^2\beta^{(1)}_{21}\alpha^{(1)}_{01} - 3\beta^{(1)}_{31}\beta^{(2)}_{00}\alpha^{(1)}_{01} - \beta^{(1)}_{21}\beta^{(2)}_{10}$$

$$\alpha^{(1)}_{01} - \beta^{(1)}_{11}\beta^{(2)}_{20}\alpha^{(1)}_{01} - \beta^{(1)}_{20}\beta^{(2)}_{11}\alpha^{(1)}_{01} - \beta^{(1)}_{10}\beta^{(2)}_{21}\alpha^{(1)}_{01} + \beta^{(3)}_{21}\alpha^{(1)}_{01} - 12\beta^{(2)}_{00}\alpha^{(1)}_{40}\alpha^{(1)}_{01} + 6(\beta^{(1)}_{10})^2$$

$$\beta^{(1)}_{20}\alpha^{(1)}_{20} + 2(\beta^{(1)}_{01})^2\beta^{(1)}_{20}\alpha^{(1)}_{20} + \beta^{(1)}_{01}\beta^{(1)}_{02}\alpha^{(1)}_{01}\alpha^{(1)}_{11} - \beta^{(1)}_{01}\beta^{(1)}_{20}\alpha^{(2)}_{11} - 2\beta^{(1)}_{01}\beta^{(1)}_{20}\beta^{(2)}_{02} - (\alpha^{(1)}_{01})^2\alpha^{(1)}_{11}$$

$$\alpha^{(1)}_{02} - 3\alpha^{(1)}_{04}\alpha^{(1)}_{11} + 2\beta^{(1)}_{10}\beta^{(1)}_{01}\beta^{(1)}_{11}\alpha^{(1)}_{20} - 6\beta^{(1)}_{30}\beta^{(2)}_{00}\alpha^{(1)}_{20} - 4\beta^{(1)}_{20}\beta^{(2)}_{10}\alpha^{(1)}_{20} - \beta^{(1)}_{11}\beta^{(2)}_{01}\alpha^{(1)}_{20} - 4\beta^{(1)}_{10}$$

$$\beta^{(2)}_{20}\alpha^{(1)}_{20} - \beta^{(1)}_{01}\beta^{(2)}_{11}\alpha^{(1)}_{20} + 2\beta^{(3)}_{20}\alpha^{(1)}_{20} + 4\beta^{(1)}_{10}\beta^{(1)}_{20}\alpha^{(1)}_{01}\alpha^{(1)}_{20} - 2\beta^{(2)}_{20}\alpha^{(1)}_{01}\alpha^{(1)}_{20} - 4\beta^{(1)}_{10}\beta^{(1)}_{01}(\alpha^{(1)}_{20})^2$$

$$+2\beta^{(2)}_{01}(\alpha^{(1)}_{20})^2 + 2\beta^{(1)}_{10}\beta^{(1)}_{01}\beta^{(1)}_{20}\alpha^{(1)}_{11} - \beta^{(1)}_{10}\beta^{(1)}_{01}\beta^{(1)}_{02}\alpha^{(1)}_{11} - 3\beta^{(1)}_{03}\beta^{(2)}_{00}\alpha^{(1)}_{11} - 2\beta^{(1)}_{21}\beta^{(2)}_{00}\alpha^{(1)}_{11} - 24$$

$$\beta^{(1)}_{10}\beta^{(2)}_{00}\alpha^{(1)}_{40} - \beta^{(1)}_{20}\beta^{(2)}_{01}\alpha^{(1)}_{11} + \beta^{(1)}_{02}\beta^{(2)}_{01}\alpha^{(1)}_{11} - \beta^{(1)}_{01}\beta^{(2)}_{20}\alpha^{(1)}_{11} + 2\beta^{(1)}_{11}\beta^{(1)}_{21}\alpha^{(2)}_{00} - (\beta^{(1)}_{10})^2\alpha^{(1)}_{20}\alpha^{(1)}_{11}$$

$$-3(\beta^{(1)}_{01})^2\alpha^{(1)}_{20}\alpha^{(1)}_{11} + \beta^{(2)}_{10}\alpha^{(1)}_{20}\alpha^{(1)}_{11} - \beta^{(1)}_{10}\beta^{(1)}_{01}(\alpha^{(1)}_{11})^2 + \beta^{(2)}_{01}(\alpha^{(1)}_{11})^2 + \beta^{(1)}_{01}\alpha^{(1)}_{01}(\alpha^{(1)}_{11})^2 - \beta^{(1)}_{10}$$

$$\beta^{(1)}_{01}\beta^{(1)}_{11}\alpha^{(1)}_{02} - 2(\beta^{(1)}_{01})^2\beta^{(1)}_{02}\alpha^{(1)}_{02} + 2\beta^{(1)}_{12}\beta^{(2)}_{00}\alpha^{(1)}_{02} + \beta^{(1)}_{11}\beta^{(2)}_{01}\alpha^{(1)}_{02} + \beta^{(1)}_{01}\beta^{(2)}_{11}\alpha^{(1)}_{02} - 2\beta^{(3)}_{02}\alpha^{(1)}_{02} +$$

$$\beta_{01}^{(1)}\beta_{11}^{(1)}\alpha_{01}^{(1)}\alpha_{02}^{(1)} - 2\beta_{02}^{(2)}\alpha_{01}^{(1)}\alpha_{02}^{(1)} - 2\beta_{02}^{(1)}(\alpha_{01}^{(1)})^2\alpha_{02}^{(1)} - 2\beta_{10}^{(1)}\beta_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{02}^{(1)} + 2\beta_{01}^{(2)}\alpha_{20}^{(1)}\alpha_{02}^{(1)} +$$

$$2\beta_{01}^{(1)}\alpha_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{02}^{(1)} - 12\beta_{04}^{(2)}\alpha_{00}^{(2)} - 2\beta_{22}^{(2)}\alpha_{00}^{(2)} + 2(\beta_{10}^{(1)})^2\beta_{01}^{(1)}\alpha_{21}^{(1)} - 2\beta_{11}^{(1)}\beta_{00}^{(2)}\alpha_{21}^{(1)} - 2\beta_{01}^{(1)}\beta_{10}^{(2)}$$

$$\alpha_{21}^{(1)} - 2\beta_{10}^{(1)}\beta_{01}^{(2)}\alpha_{21}^{(1)} + 2\beta_{01}^{(3)}\alpha_{21}^{(1)} + 2\beta_{00}^{(2)}\alpha_{20}^{(1)}\alpha_{21}^{(1)} + 2\beta_{00}^{(2)}\alpha_{02}^{(1)}\alpha_{21}^{(1)} + 3(\beta_{10}^{(1)})^2\alpha_{01}^{(1)}\alpha_{30}^{(1)} - 3$$

$$\beta_{10}^{(2)}\alpha_{01}^{(1)}\alpha_{30}^{(1)} - 3\beta_{00}^{(2)}\alpha_{11}^{(1)}\alpha_{30}^{(1)} + 2\beta_{10}^{(1)}\beta_{22}^{(1)}\alpha_{00}^{(2)} + 3\beta_{31}^{(1)}\beta_{01}^{(1)}\alpha_{00}^{(2)} + 9\beta_{13}^{(1)}\beta_{01}^{(1)}\alpha_{00}^{(2)} + 3\beta_{11}^{(1)}\beta_{03}^{(1)}$$

$$\alpha_{00}^{(2)} + 2\beta_{01}^{(1)}\alpha_{02}^{(1)}\alpha_{20}^{(2)} + 2\beta_{20}^{(1)}\beta_{12}^{(1)}\alpha_{00}^{(2)} + 6\beta_{02}^{(1)}\beta_{12}^{(1)}\alpha_{00}^{(2)} + 12\beta_{01}^{(1)}\alpha_{40}^{(1)}\alpha_{00}^{(2)} + \beta_{10}^{(1)}\alpha_{11}^{(1)}\alpha_{20}^{(2)} -$$

$$(\beta_{01}^{(1)})^2\alpha_{11}^{(1)}\alpha_{02}^{(1)} + 3\beta_{10}^{(1)}\alpha_{31}^{(1)}\alpha_{00}^{(2)} + 6\beta_{01}^{(1)}\alpha_{22}^{(1)}\alpha_{00}^{(2)} - 12\beta_{04}^{(1)}\alpha_{01}^{(1)}\alpha_{00}^{(2)} - 3\alpha_{13}^{(1)}\alpha_{01}^{(1)}\alpha_{00}^{(2)} + 6\beta_{03}^{(1)}$$

$$\alpha_{20}^{(1)}\alpha_{00}^{(2)} + 4\beta_{21}^{(1)}\alpha_{20}^{(1)}\alpha_{00}^{(2)} + 4\beta_{21}^{(1)}\alpha_{02}^{(1)}\alpha_{00}^{(2)} - 6\beta_{02}^{(1)}\alpha_{03}^{(1)}\alpha_{00}^{(2)} - 3\alpha_{11}^{(1)}\alpha_{03}^{(1)}\alpha_{00}^{(2)} + 2\beta_{20}^{(1)}\alpha_{21}^{(1)}\alpha_{00}^{(2)}$$

$$+4\beta_{02}^{(1)}\alpha_{21}^{(1)}\alpha_{00}^{(2)} - \alpha_{11}^{(1)}\alpha_{21}^{(1)}\alpha_{00}^{(2)} + 3\beta_{11}^{(1)}\alpha_{30}^{(1)}\alpha_{00}^{(2)} + 6\alpha_{20}^{(1)}\alpha_{30}^{(1)}\alpha_{00}^{(2)} + 12\alpha_{02}^{(1)}\alpha_{30}^{(1)}\alpha_{00}^{(2)} + 12\beta_{00}^{(2)}$$

$$\alpha_{40}^{(2)} - 3\alpha_{00}^{(2)}\alpha_{31}^{(2)} + 2\beta_{00}^{(2)}\alpha_{22}^{(2)} - 3\alpha_{00}^{(2)}\alpha_{13}^{(2)} - \beta_{20}^{(1)}\beta_{11}^{(1)}\alpha_{01}^{(2)} + 4\beta_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{20}^{(2)} - 3\beta_{03}^{(4)} - \beta_{21}^{(4)} -$$

$$\beta_{10}^{(1)}\beta_{21}^{(1)}\alpha_{01}^{(2)} + \beta_{21}^{(2)}\alpha_{01}^{(2)} - 2\beta_{20}^{(1)}\alpha_{20}^{(1)}\alpha_{01}^{(2)} - 2\beta_{02}^{(1)}\alpha_{02}^{(1)}\alpha_{01}^{(2)} - \alpha_{11}^{(1)}\alpha_{02}^{(1)}\alpha_{01}^{(2)} - 9\beta_{01}^{(1)}\beta_{00}^{(2)}\alpha_{31}^{(1)} - 3$$

$$\beta_{10}^{(1)}\alpha_{30}^{(1)}\alpha_{01}^{(2)} - 4\beta_{10}^{(1)}\beta_{20}^{(1)}\alpha_{11}^{(2)} - \alpha_{01}^{(1)}\alpha_{20}^{(2)} - \beta_{01}^{(1)}\beta_{11}^{(1)}\alpha_{20}^{(2)} + 2\beta_{20}^{(2)}\alpha_{20}^{(2)}\alpha_{11}^{(1)}\alpha_{02}^{(3)} + \beta_{10}^{(1)}\alpha_{20}^{(1)}\alpha_{11}^{(2)}$$

$$+2\beta_{01}^{(1)}\alpha_{11}^{(1)}\alpha_{11}^{(2)} - \alpha_{01}^{(1)}\alpha_{02}^{(1)}\alpha_{11}^{(2)} - \alpha_{20}^{(2)}\alpha_{11}^{(2)} + \beta_{01}^{(1)}\beta_{11}^{(1)}\alpha_{02}^{(2)} - 2\beta_{20}^{(1)}\alpha_{01}^{(1)}\alpha_{20}^{(2)} - 2\beta_{02}^{(2)}\alpha_{02}^{(2)} - 2$$

$$\beta_{02}^{(1)}\alpha_{01}^{(1)}\alpha_{02}^{(2)} + 2\beta_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{02}^{(2)} - \alpha_{01}^{(1)}\alpha_{11}^{(1)}\alpha_{02}^{(2)} - \alpha_{11}^{(2)}\alpha_{02}^{(2)} + (\beta_{10}^{(1)})^3\beta_{21}^{(1)} - 2\beta_{10}^{(1)}\beta_{01}^{(1)}\alpha_{21}^{(2)} + 2$$

$$\beta_{01}^{(2)}\alpha_{21}^{(2)} - 3(\beta_{10}^{(1)})^2\alpha_{30}^{(2)} + 3\beta_{10}^{(2)}\alpha_{30}^{(2)} - 3\beta_{10}^{(1)}\alpha_{01}^{(1)}\alpha_{30}^{(2)} + 3\beta_{10}^{(1)}\alpha_{30}^{(3)} + 3\alpha_{01}^{(2)}\alpha_{30}^{(2)} - 12\beta_{04}^{(1)}\alpha_{00}^{(3)}$$

$$-2\beta_{22}^{(1)}\alpha_{00}^{(3)} - 3\alpha_{31}^{(1)}\alpha_{00}^{(3)} - 3\alpha_{13}^{(1)}\alpha_{00}^{(3)} + 2\beta_{01}^{(1)}\alpha_{21}^{(3)} + \beta_{01}^{(1)}\beta_{02}^{(1)}\alpha_{11}^{(2)}).$$

$$K_3 = \frac{-1}{24}(3\beta_{10}^{(1)}\beta_{41}^{(1)} + 12\beta_{14}^{(1)}\beta_{01}^{(1)} + 6\beta_{32}^{(1)}\beta_{01}^{(1)} - 15\beta_{10}^{(1)}\beta_{05}^{(1)} + 5\beta_{31}^{(1)}\beta_{20}^{(1)} + 3\beta_{13}^{(1)}\beta_{20}^{(1)} + 3\beta_{40}^{(1)}\beta_{11}^{(1)} -$$

$$5\beta_{04}^{(1)}\beta_{11}^{(1)} + \beta_{22}^{(1)}\beta_{11}^{(1)} + 7\beta_{31}^{(1)}\beta_{02}^{(1)} + 9\beta_{13}^{(1)}\beta_{02}^{(1)} + 3\beta_{21}^{(1)}\beta_{30}^{(1)} + 6\beta_{03}^{(1)}\beta_{12}^{(1)} + 3\beta_{21}^{(1)}\beta_{12}^{(1)} - 3\beta_{23}^{(2)} -$$

$$3\beta_{41}^{(2)} - 15\beta_{05}^{(2)} + 20\beta_{20}^{(1)}\alpha_{40}^{(1)} - 15\alpha_{50}^{(2)} + 22\beta_{02}^{(1)}\alpha_{40}^{(1)} - 6\beta_{02}^{(1)}\alpha_{04}^{(1)} + 2\beta_{20}^{(1)}\alpha_{22}^{(1)} + 4\beta_{02}^{(1)}\alpha_{22}^{(1)} -$$

$$2\beta_{11}^{(1)}\alpha_{13}^{(1)} - 3\alpha_{32}^{(2)} + 15\beta_{10}^{(1)}\alpha_{50}^{(1)} + 6\beta_{01}^{(1)}\alpha_{23}^{(1)} - 3\beta_{10}^{(1)}\alpha_{14}^{(1)} + 12\beta_{01}^{(1)}\alpha_{41}^{(1)} + 3\beta_{41}^{(1)}\alpha_{01}^{(1)} - 3\alpha_{14}^{(2)}$$

$$-15\beta_{05}^{(1)}\alpha_{01}^{(1)} + 15\alpha_{50}^{(1)}\alpha_{01}^{(1)} - 3\alpha_{14}^{(1)}\alpha_{01}^{(1)} + 6\beta_{40}^{(1)}\alpha_{20}^{(1)} - 22\beta_{04}^{(1)}\alpha_{20}^{(1)} - 4\beta_{22}^{(1)}\alpha_{20}^{(1)} - 9\alpha_{31}^{(1)}\alpha_{20}^{(1)}$$

$$-7\alpha_{13}^{(1)}\alpha_{20}^{(1)} + 2\beta_{31}^{(1)}\alpha_{11}^{(1)} + 5\alpha_{40}^{(1)}\alpha_{11}^{(1)} + 9\alpha_{03}^{(1)}\alpha_{30}^{(1)} - \alpha_{22}^{(1)}\alpha_{11}^{(1)} - 20\beta_{04}^{(1)}\alpha_{02}^{(1)} - 2\beta_{22}^{(1)}\alpha_{02}^{(1)}) - 3$$

$$\alpha_{31}^{(1)}\alpha_{02}^{(1)} - 5\alpha_{13}^{(1)}\alpha_{02}^{(1)} + 3\beta_{21}^{(1)}\alpha_{03}^{(1)} + 6\beta_{03}^{(1)}\alpha_{21}^{(1)} + 3\beta_{21}^{(1)}\alpha_{21}^{(1)} + 9\beta_{30}^{(1)}\alpha_{30}^{(1)} + 3\beta_{12}^{(1)}\alpha_{30}^{(1)} + 3\alpha_{21}^{(1)}).$$

Chapter 4

Limit Cycles of Planar Discontinuous Piecewise Linear Hamiltonian Systems Without Singular Points Separated by Irreducible Cubics

The second part of the 16–th Hilber problem asks for the maximum number of limit cycles and their possible configuration of planar polynomial vector field of degree n. We know that, planar polynomial differential systems have no limit cycles, while it is not the case for piecewise linear differential systems. These systems revent to Andronov, Vitt and Khaikin [1]. Owing to the simplicity of this kind of differential systems, researchers had given a big interest on studying them and they have extensively a large relevance in the domain of engineering sciences, for example, we can model key of component in even simple electronic circuit, also the Diodes and transistors as a piecewise linear differential systems, etc., see for instance the survey of Makarenkov and Lamb [76], and the books of Bernardo and Simpson [93]. Even now, the 16–th Hilbert problem still open for this type of systems. We recall that a *limit cycle* of a differential system is an isolated periodic orbit in the set of all periodic orbit of this system.

Recently, many authors have been interesting to solve the second part of Hilbert 16–th problem for these systems in $\mathbb{R}^2$ when we separate them by a curve Σ. In [62] authors proved that if Σ is a straight line then discontinuous piecewise linear differential centers have no crossing limit cycle. There are many other papers devoted to study the existence and the number of *limit cycles* of these systems when the curve of separation is a straight line, see for instance [4, 34, 43, 45, 46, 47, 64, 70].

In particular in the papers [30, 56, 72] the authors studied the maximum number of limit cycles of piecewise linear centers separated by algebraic curves of the form $y = x^n$, or by a conic, or by a reducible or irreducible cubic curve.

In [29] authors studied the number of crossing limit cycles of discontinuous piecewise linear differential Hamiltonian separated by a parabola, a hyperbola or an ellipse; they can have at most 2, 3 or 3 crossing limit cycles, respectively.

This chapter is devoted to study the limit cycles of planar discontinuous piecewise linear Hamiltonian systems without singular points separated by irreducible cubics. We study the limit cycles that intersect the cubic in two or four points. We provide upper bounds for the maximum number of limit cycles intersecting the cubic either in two points, or in four points, or in both classes simultaneously.

Section 4.1 Classification of the Ireducible Cubics Curves

An *algebraic cubic curve* or simple a *cubic* is the set of points $(x, y) \in \mathbb{R}^2$ satisfying $P(x, y) = 0$ for some polynomial $P(x, y)$ of degree three. This real cubic is *irreducible* (respectively *reducible*) if the polynomial $P(x, y)$ is irreducible (respectively reducible) in the ring of all real polynomials in the variables x and y.

A point (x_0, y_0) of a cubic $P(x, y) = 0$ is *singular* if $P_x(x_0, y_0) = P_y(x_0, y_0) = 0$. A cubic curve is *singular* if it has some singular point.

A *flex* of an algebraic curve C is a point p of C such that C is nonsingular at p and the tangent at p of the curve C intersects C at least three times at p.

Theorem 4.1 *The following statements classify all the irreducible cubic algebraic curves.*

(a) A cubic is nonsingular and irreducible if and only if it can be transformed with an

affine transformation into one of the following two curves

$$c_1 = c_1(x,y) = y^2 - x(x^2 + bx + 1) = 0 \quad \text{with } b \in (-2,2), \text{ or}$$
$$c_2 = c_2(x,y) = y^2 - x(x-1)(x-r) = 0 \quad \text{with } r > 1.$$

(b) A cubic is singular and irreducible if and only if it can be transformed with an affine transformation into one of the following three curves:

$$c_3 = c_3(x,y) = y^2 - x^3 = 0, \qquad \text{or}$$
$$c_4 = c_4(x,y) = y^2 - x^2(x-1) = 0, \quad \text{or}$$
$$c_5 = c_5(x,y) = y^2 - x^2(x+1) = 0.$$

Statement (a) of Theorem 4.1 is proved in Theorem 8.3 of the book [33] under the additional assumption that the cubic has a flex, but in section 12 of that book it is shown that every nonsingular irreducible cubic curve has a flex. While statement (b) of Theorem 4.1 follows directly from Theorem 8.4 of [33].

We consider planar discontinuous piecewise linear Hamiltonian systems without singular points separated by an irreducible cubic. In the following lemma we give the normal form of linear Hamiltonian system without singular points.

Lemma *An arbitrary linear differential Hamiltonian system in $\mathbb{R}^2$ without singular points can be written as*

$$\mathcal{X}_i(x,y) = (-\lambda_i b_i x + b_i y + \mu_i, -\lambda_i^2 b_i x + \lambda_i b_i y + \sigma_i),$$

where $\sigma_i \neq \lambda_i \mu_i$ and $b_i \neq 0$ for $i = 1 \ldots 4$, see for details [45]. The Hamiltonian function associated to the Hamiltonian vector field $\mathcal{X}_i$ is

$$H_i(x,y) = (-\lambda_i^2 b_i/2)x^2 + \lambda_i b_i xy - (b_i/2)y^2 + \sigma_i x - \mu_i y.$$

The goal in this section is to study the maximum number of crossing limit cycles of PLHS formed by linear differential Hamiltonian systems and separated by any cubic which only intersect the discontinuity curve in two points.

4.2.1 Statement of the main result

We denote by C_k the class of planar discontinuous piecewise linear Hamiltonian systems without singular points separated by the irreducible cubic $c_k = 0$ for $k = 1, \ldots, 5$.

Our first objective is to provide the maximum number of limit cycles with two points on the cubic for the discontinuous piecewise linear Hamiltonian systems separated by a cubic $c_k = 0$, with $k = 1, \ldots, 5$. We note that such limit cycles are contained only in two pieces of the discontinuous piecewise linear Hamiltonian system.

Theorem 4.2 *For $k = 1, \ldots, 5$ the maximum number of limit cycles of the discontinuous piecewise linear Hamiltonian systems intersecting the cubic $c_k = 0$ in two points is three.*

This maximum is reached in Figures C_1, C_3 of Figure 4.1, and C_4 of Figure 4.2 for the classes C_1, C_3 and C_4 respectively; and in Figures C_2^1 of Figure 4.2, and C_2^2 and C_2^3 of Figure 4.3 for the class C_2; and in Figure 4.4 for the class C_5.

4.2.2 Proof of the main result

We shall prove that the maximum number of limit cycles of the discontinuous piecewise linear Hamiltonian systems intersecting the cubic $c_3 = 0$ in two points is three. For the other four cubics the proof is similar.

We consider the discontinuous piecewise linear Hamiltonian system such that in the region $R_1 = \{(x, y) : y^2 - x^3 \geq 0\}$ is defined as

$$\dot{x} = -\lambda_1 b_1 x + b_1 y + \mu_1, \quad \dot{y} = -\lambda_1^2 b_1 x + \lambda_1 b_1 y + \sigma_1, \tag{4.1}$$

with $b_1 \neq 0$ and $\sigma_1 \neq \lambda_1 \mu_1$. This system has the first integral

$$H_1(x, y) = -(\lambda_1^2 b_1/2)x^2 + \lambda_1 b_1 xy - (b_1/2)y^2 + \sigma_1 x - \mu_1 y.$$

In the region $R_2 = \{(x, y) : y^2 - x^3 \leq 0\}$ we consider the linear Hamiltonian system

$$\dot{x} = -\lambda_2 b_2 x + b_2 y + \mu_2, \quad \dot{y} = -\lambda_2^2 b_2 x + \lambda_2 b_2 y + \sigma_2, \tag{4.2}$$

with $b_2 \neq 0$ and $\sigma_2 \neq \lambda_2 \mu_2$. Its corresponding Hamiltonian first integral is

$$H_2(x, y) = -(\lambda_2^2 b_2/2)x^2 + \lambda_2 b_2 xy - (b_2/2)y^2 + \sigma_2 x - \mu_2 y.$$

In order to have a limit cycle which intersects the cubic $y^2 - x^3 = 0$ in the points (x_i, y_i) and (x_k, y_k), these points must satisfy the system

$$\begin{aligned} &H_1(x_i, y_i) - H_1(x_k, y_k) = 0, \\ &H_2(x_i, y_i) - H_2(x_k, y_k) = 0, \\ &y_i^2 - x_i^3 = 0, \quad y_k^2 - x_k^3 = 0. \end{aligned} \tag{4.3}$$

Suppose that the piecewise differential system formed by the systems (4.1) and (4.2) has four limit cycles. Then system (4.3) must have four pairs of points of solutions of the form $p_i = (r_i^2, r_i^3)$ and $q_i = (s_i^2, s_i^3)$ for $i = 1, \ldots 4$. Due to the fact that these points must satisfy the first two equations of system (4.3), from these two equations and for $i = 1$ we get that

$$\sigma_1 = \frac{1}{2(r_1^2 - s_1^2)}\Big(b_1 r_1^6 - 2b_1\lambda_1 r_1^5 + b_1\lambda_1^2 r_1^4 - b_1 s_1^6 + 2b_1\lambda_1 s_1^5 - b_1\lambda_1^2 s_1^4 + 2\mu_1 r_1^3 - 2\mu_1 s_1^3\Big),$$

and σ_2 has the same expression as σ_1 changing (b_1, λ_1, μ_1) by (b_2, λ_2, μ_2).

Since the points $p_2 = (r_2^2, r_2^3)$ and $q_2 = (s_2^2, s_2^3)$ also satisfy system (4.3), we obtain that parameters μ_1 and μ_2 must be $\mu_1 = A/B$, where

$$\begin{aligned} A = & -(b_1(r_1^5(r_2+s_2)+r_1^4(r_2+s_2)(s_1-2\lambda_1)+r_1^3(r_2+s_2)(s_1-\lambda_1)^2+r_1^2 s_1(r_2+s_2)(s_1-\lambda_1)^2-r_1 \\ & (r_2^5+r_2^4(s_2-2\lambda_1)+r_2^3(s_2-\lambda_1)^2+r_2^2 s_2(s_2-\lambda_1)^2+r_2(-s_1^4+s_2^2(s_2-\lambda_1)^2+2s_1^3\lambda_1-s_1^2\lambda_1^2) \\ & +s_2(-s_1^4+s_2^2(s_2-\lambda_1)^2+2s_1^3\lambda_1-s_1^2\lambda_1^2))+s_1(-r_2^5-r_2^4(s_2-2\lambda_1)-r_2^3(s_2-\lambda_1)^2-r_2^2 s_2(s_2 \\ & -\lambda_1)^2+r_2(s_1^4-s_2^2(s_2-\lambda_1)^2-2s_1^3\lambda_1+s_1^2\lambda_1^2)+s_2(s_1^4-s_2^2(s_2-\lambda_1)^2-2s_1^3\lambda_1+s_1^2\lambda_1^2))), \\ B = & \ 2(r_1^2(r_2+s_2)+s_1(-r_2^2+r_2(s_1-s_2)+(s_1-s_2)s_2)-r_1(r_2^2+r_2(-s_1+s_2)+s_2(-s_1+s_2))). \end{aligned}$$

And μ_2 has the same expression as μ_1 changing (b_1, λ_1) by (b_2, λ_2).

Again the points $p_3 = (r_3^2, r_3^3)$ and $q_3 = (s_3^2, s_3^3)$ satisfy system (4.3), then we obtain two values of λ_1 we name them $\lambda^{(1)}$ and $\lambda^{(2)}$ and the same values of λ_2. The first value of λ_1 and λ_2 is given by $\lambda^{(1)} = (C-(1/2)\sqrt{D})/E$ and their second value is $\lambda^{(2)} = (C+(1/2)\sqrt{D})/E$, where the values of C, D and E are given in the appendix of chapter 4.

We replace μ_1, $\lambda^{(i)}$ and σ_1 in the expression of $H_1(x,y)$, and μ_2, $\lambda^{(i)}$ and σ_2 in the expression of $H_2(x,y)$ and we obtain $H_1(x,y) = H_2(x,y)$, for $i = 1,2$. Hence the discontinuous piecewise linear differential system becomes a linear differential system, and consequently the system has no limit cycles. So the maximum number of limit cycles in this case is two.

Now we consider the pairs either $\lambda^{(1)}$ and $\lambda^{(2)}$, or $\lambda^{(2)}$ and $\lambda^{(1)}$. By replacing the expressions of σ_1, μ_1 and $\lambda^{(1)}$ (resp. $\lambda^{(2)}$) in the expression of $H_1(x,y)$, and σ_2, μ_2 and $\lambda^{(2)}$ (resp. $\lambda^{(1)}$) in the expression of $H_2(x,y)$ we obtain that $H_1(x,y) \neq H_2(x,y)$. Since the points $p_4 = (r_4^2, r_4^3)$ and $q_4 = (s_4^2, s_4^3)$ satisfy system (4.3), then we obtain $b_1 = 0$ and $b_2 = 0$. This is a contradiction because by the assumptions they are not zero. In summary, we proved that the maximum number of limit cycles for planar discontinuous piecewise linear Hamiltonian systems without singular points separated by a irreducible cubic curve $c_3 = 0$ is at most three.

In order to complete the proof of the theorem we shall provide discontinuous piecewise linear Hamiltonian systems without singular points separated by the cubic $c_k = 0$ with three limit cycles for $k = 1, \ldots, 5$.

Example with three limit cycles when the cubic of separation is $c_1 = 0$. In the region $R_1 = \{(x,y) : y^2 - x(x^2+x+1) \geq 0\}$, we consider the linear Hamiltonian system

$$(\dot{x},\dot{y}) = \left(-\frac{9x}{5} + 3y + \frac{1}{5}, -\frac{27x}{25} + \frac{9y}{5} + 1\right). \tag{4.4}$$

It has the Hamiltonian function $H_1(x,y) = -27x^2/50 + 9xy/5 - 3y^2/2 - y/5$. Now we consider the second linear Hamiltonian system

$$(\dot{x},\dot{y}) \approx (5.54426\ldots x - 2y - 9.52503\ldots, 15.3694\ldots x - 5.54426\ldots y - 38.5097\ldots), \tag{4.5}$$

in the region $R_2 = \{(x,y) : y^2 - x(x^2+x+1) \leq 0\}$. This Hamiltonian system has the Hamiltonian function $H_2(x,y) \approx 7.68471\ldots x^2 - 5.54426\ldots xy - 38.5097\ldots x + y^2 + 9.52503\ldots y$.

The discontinuous piecewise differential system (4.4)–(4.5) has exactly three limit cycles, because the system of equations

$$\begin{aligned} &H_1(\alpha,\beta) - H_1(\gamma,\delta) = 0,\\ &H_2(\alpha,\beta) - H_2(\gamma,\delta) = 0,\\ &c_i(\alpha,\beta) = 0, \;\; c_i(\gamma,\delta) = 0, \end{aligned} \tag{4.6}$$

when $i = 1$, has only three real solutions

$$\begin{aligned} &(\alpha_1,\beta_1,\gamma_1,\delta_1) \approx (0.393342\ldots, -0.780303\ldots, 0.908209\ldots, 1.5755\ldots),\\ &(\alpha_2,\beta_2,\gamma_2,\delta_2) \approx (0.558506\ldots, -1.02208\ldots, 1.19256\ldots, 2.07625\ldots),\\ &(\alpha_3,\beta_3,\gamma_3,\delta_3) \approx (0.680997, -1.20854\ldots, 1.3862\ldots, 2.44365\ldots), \end{aligned}$$

see C_1 of Figure 4.1.

Example with three limit cycles when the cubic of separation is $c_3 = 0$. In the region $R_1 = \{(x,y) : x^3 - y^2 \leq 0\}$ we consider the Hamiltonian system

$$(\dot{x},\dot{y}) = \left(-\frac{x}{2} + 5y + \frac{1}{5}, -\frac{x}{20} + \frac{y}{2} + \frac{4}{5}\right). \tag{4.7}$$

It has the Hamiltonian function $H_1(x,y) = -x^2/40 + xy/2 + 4x/5 - 5y^2/2 - y/5$.

Now we consider the second Hamiltonian system

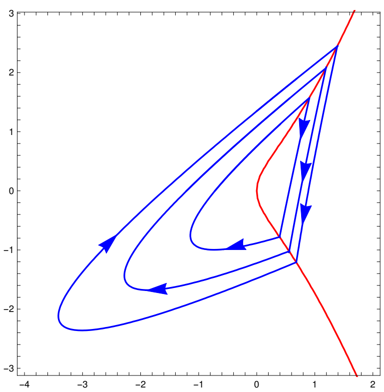
$$(\dot{x},\dot{y}) \approx (2.8254\ldots x + \frac{51y}{100} - 9.47986\ldots, -15.6528\ldots x - 2.8254\ldots y - 132.539\ldots), \tag{4.8}$$

in the region $R_2 = \{(x,y) : x^3 - y^2 \geq 0\}$. This Hamiltonian system has the Hamiltonian function $H_2(x,y) \approx -7.82638\ldots x^2 - 2.8254\ldots xy - 132.539\ldots x - \dfrac{51y^2}{200} + 9.47986\ldots y$.

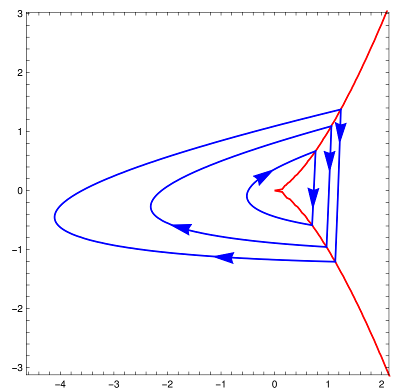
The discontinuous piecewise differential system (4.7)–(4.8) has exactly three limit cycles, because the system of equations (4.6) when $i = 3$ has only three real solutions

$$
\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1) &\approx (0.700707\ldots, -0.58655\ldots, 0.765078\ldots, 0.669204\ldots),\\
(\alpha_2, \beta_2, \gamma_2, \delta_2) &\approx (0.969795\ldots, -0.955036\ldots, 1.06038\ldots, 1.09192\ldots),\\
(\alpha_3, \beta_3, \gamma_3, \delta_3) &\approx (1.13263\ldots, -1.2054\ldots, 1.23647\ldots, 1.37492\ldots),
\end{aligned}
$$

see C_3 of Figure 4.1.



C_1



C_3

Figure 4.1: The three limit cycles of the discontinuous piecewise differential system C_1 for (4.4)–(4.5), and C_3 for (4.7)–(4.8).

Example with three limit cycles when the cubic of separation is $c_4 = 0$. We consider the Hamiltonian system

$$
(\dot{x}, \dot{y}) = \left(-\frac{4x}{5} + 4y + \frac{3}{10}, -\frac{4x}{25} + \frac{4y}{5} + \frac{3}{5}\right), \tag{4.9}
$$

in the region $R_1 = \{(x,y) : y^2 - x^2(x-1) \leq 0\}$. This Hamiltonian system has the Hamiltonian function $H_1(x,y) = -2x^2/25 + 4xy/5 + 3x/5 - 2y^2 - 3y/10$.

In the region $R_2 = \{(x,y) : y^2 - x^2(x-1) \geq 0\}$ we consider the Hamiltonian system

$$
(\dot{x}, \dot{y}) \approx (-4.28711\ldots x + 9y/100 - 13.3828\ldots, -204.215\ldots x + 4.28711\ldots y + 151.777\ldots), \tag{4.10}
$$

which has the Hamiltonian function

$$H_2(x,y) \approx -102.107\ldots x^2 + 4.28711\ldots xy + 151.777\ldots x - 9y^2/200 + 13.3828\ldots y.$$

The discontinuous piecewise differential system (4.9)–(4.10) has exactly three limit cycles, because the system of equations (4.6) when $i = 4$ has only three real solutions

$$\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1) &\approx (1.05134\ldots, -0.238211\ldots, 1.24153\ldots, 0.610155\ldots),\\
(\alpha_2, \beta_2, \gamma_2, \delta_2) &\approx (1.21453\ldots, -0.562538\ldots, 1.46092\ldots, 0.991826\ldots),\\
(\alpha_3, \beta_3, \gamma_3, \delta_3) &\approx (1.33077\ldots, -0.765367\ldots, 1.59962\ldots, 1.23867\ldots),
\end{aligned}$$

see C_4 of Figure 4.2.

Three examples with three limit cycles when the cubic of separation is $c_2 = 0$. We define the following regions associated to the curve $c_2 = 0$

$$\begin{aligned}
R_1 &= \{(x,y) : y^2 - x(x-1)(x-3) \geq 0\},\\
R_2 &= \{(x,y) : y^2 - x(x-1)(x-3) \leq 0, x \geq 3\},\\
R_3 &= \{(x,y) : y^2 - x(x-1)(x-3) \leq 0, 0 \leq x \leq 1\}.
\end{aligned} \tag{4.11}$$

For the first configuration of limit cycles separated by the curve $c_2 = 0$ we consider the Hamiltonian system

$$(\dot{x}, \dot{y}) = \left(-\frac{13x}{20} + 5y + \frac{7}{10}, -\frac{169x}{2000} + \frac{13y}{20} + \frac{19}{10}\right), \tag{4.12}$$

in the region R_1. It has the Hamiltonian function $H_1(x,y) = -169x^2/4000 + 13xy/20 + 19x/10 - 5y^2/2 - 7y/10$. In the region R_2 we consider the Hamiltonian system

$$(\dot{x}, \dot{y}) \approx (44.7429\ldots x - 3y/10 + 289.458\ldots, 6673.09\ldots x - 44.7429\ldots y - 15544.3\ldots), \tag{4.13}$$

which has the Hamiltonian function

$$H_2(x,y) \approx 3336.54\ldots x^2 - 44.7429\ldots xy - 15544.3\ldots x + 3y^2/20 - 289.458\ldots y.$$

The discontinuous piecewise differential system (4.12)–(4.13) has exactly three limit cycles, because the system of equations (4.6) when $i = 2$, has only the three real solutions

$$(\alpha_1, \beta_1, \gamma_1, \delta_1) \approx (3.00602\ldots, -0.190529\ldots, 3.09131\ldots, 0.768297\ldots),$$
$$(\alpha_2, \beta_2, \gamma_2, \delta_2) \approx (3.04571\ldots, -0.533662\ldots, 3.18176\ldots, 1.12329\ldots),$$
$$(\alpha_3, \beta_3, \gamma_3, \delta_3) \approx (3.09077\ldots, -0.765857\ldots, 3.25463\ldots, 1.36691\ldots),$$

see C_2^1 of Figure 4.2.

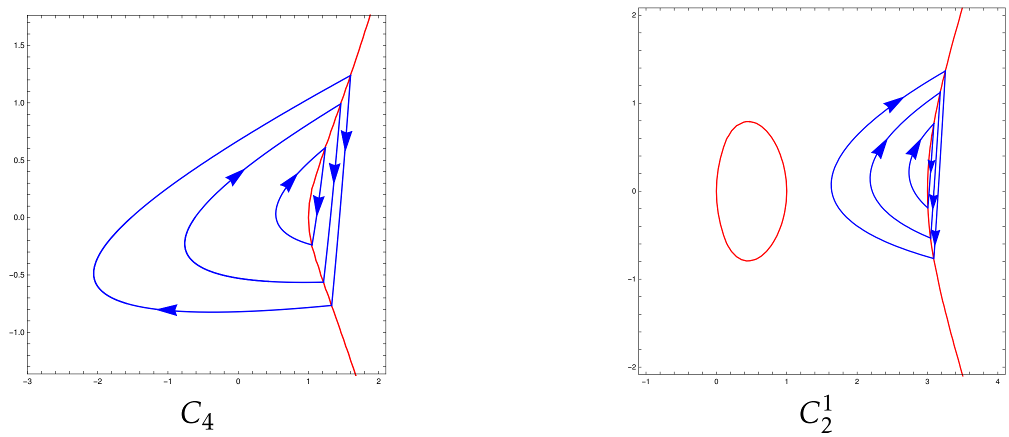


Figure 4.2: The three limit cycles of the discontinuous piecewise differential system C_4 for (4.9)–(4.10), and C_2^1 for (4.12)–(4.13).

For the second configuration we consider the Hamiltonian system

$$(\dot{x}, \dot{y}) = \left(-\frac{14x}{5} - 7y + \frac{163}{100}, \frac{28x}{25} + \frac{14y}{5} + \frac{2}{5}\right), \tag{4.14}$$

in the region R_1. It has the Hamiltonian function $H_1(x,y) = 14x^2/25 + 14xy/5 + 2x/5 + 7y^2/2 - 163y/100$.

Now we consider the second Hamiltonian system

$$(\dot{x}, \dot{y}) \approx (-0.860462\ldots x + 3y/10 + 0.4357\ldots, -2.46798\ldots x + 0.860462\ldots y + 0.115128\ldots), \tag{4.15}$$

in the region R_3. This Hamiltonian system has the Hamiltonian function

$$H_2(x,y) \approx -1.23399\ldots x^2 + 0.860462\ldots xy + 0.115128\ldots x - 3y^2/20 - 0.4357\ldots y.$$

The discontinuous piecewise differential sysem (4.14)–(4.15) has exactly three limit cycles, because the system of equations (4.6) when $i = 2$, has only the three real solutions

$$
\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1) &\approx (0.605489\ldots, -0.756295\ldots, 0.748414\ldots, 0.651116\ldots),\\
(\alpha_2, \beta_2, \gamma_2, \delta_2) &\approx (0.747078\ldots, -0.652453\ldots, 0.911161\ldots, 0.4112\ldots),\\
(\alpha_3, \beta_3, \gamma_3, \delta_3) &\approx (0.87431\ldots, -0.483319\ldots, 0.988069\ldots, 0.154006\ldots),
\end{aligned}
$$

see C_2^2 of Figure 4.3.

To obtain the third configuration we consider in the region R_1 the Hamiltonian system

$$
(\dot{x}, \dot{y}) = \left(\frac{23x}{100} + \frac{23y}{10} - \frac{1}{5}, -\frac{23x}{1000} - \frac{23y}{100} + 1\right), \tag{4.16}
$$

which has the Hamiltonian function $H_1(x,y) = -23x^2/2000 - 23xy/100 + x - 23y^2/20 + y/5$. In the region R_2 we consider the Hamiltonian system

$$
(\dot{x}, \dot{y}) \approx (0.0984281\ldots x + 0.000233032\ldots y + 2, -41.574\ldots x - 0.0984281\ldots y + 96.8899\ldots), \tag{4.17}
$$

This differential system has the Hamiltonian function $H_2(x,y) \approx -20.787\ldots x^2 - 0.0984281 \ldots xy + 96.8899\ldots x - 0.000116516\ldots y^2 - 2y$.

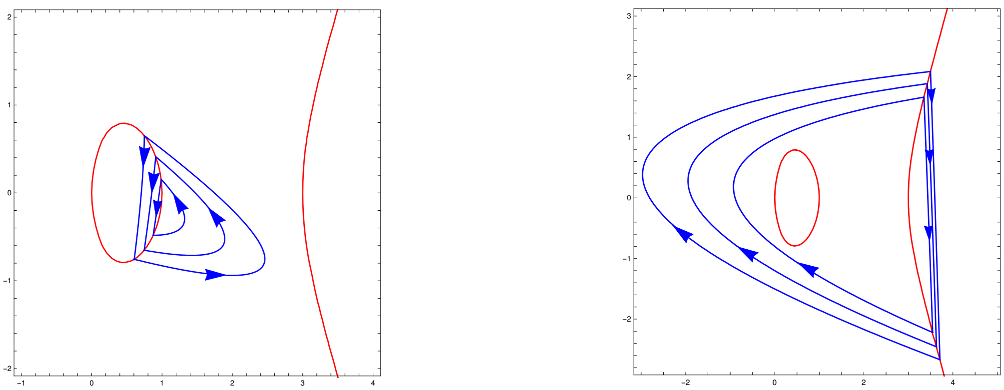
When $i = 2$ in the system of equations (4.6) the discontinuous piecewise differential system (4.16)–(4.17) has exactly three limit cycles intersecting the cubic curve $c_2 = 0$ in the points

$$
\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1) &\approx (3.54594\ldots, -2.22005\ldots, 3.35042\ldots, 1.66118\ldots),\\
(\alpha_2, \beta_2, \gamma_2, \delta_2) &\approx (3.63153\ldots, -2.45666\ldots, 3.42694\ldots, 1.88438\ldots),\\
(\alpha_3, \beta_3, \gamma_3, \delta_3) &\approx (3.70911\ldots, -2.66935\ldots, 3.49745\ldots, 2.08449\ldots),
\end{aligned}
$$

see C_2^3 of Figure 4.3.

Example with three limit cycles when the cubic of separation is $c_5 = 0$. We define the following three regions associated to the curve $c_5 = 0$

$$
\begin{aligned}
R_1 &= \{(x,y) : y^2 - x^2(x+1) \le 0, x \ge 0\},\\
R_2 &= \{(x,y) : y^2 - x^2(x+1) \ge 0\},\\
R_3 &= \{(x,y) : y^2 - x^2(x+1) \le 0, -1 \le x \le 0\}.
\end{aligned} \tag{4.18}
$$



C_2^2 C_2^3

Figure 4.3: The three limit cycles of the discontinuous piecewise differential system C_2^2 for (4.14)–(4.15), and C_2^3 for (4.16)–(4.17).

For the class C_5 and in the region R_1 we consider the Hamiltonian system

$$(\dot{x},\dot{y}) = \left(-\frac{9x}{10} + 3y + \frac{1}{5}, -\frac{27x}{100} + \frac{9y}{10} + 1\right), \tag{4.19}$$

which has the Hamiltonian function $H_1(x,y) = -27x^2/200 + 9xy/10 + x - 3y^2/2 - y/5$. Now we consider the Hamiltonian system

$$(\dot{x},\dot{y}) \approx (-0.192471\ldots x + y/1000 - 6.61081\ldots, -37.0449\ldots x + 0.192471\ldots y - 26.7945\ldots), \tag{4.20}$$

in the region R_2. This differential system has the Hamiltonian function $H_2(x,y) \approx -18.5225\ldots x^2 + 0.192471\ldots xy - 26.7945\cdots - y^2/2000 + 6.61081\ldots y$.

The discontinuous piecewise differential system (4.19)–(4.20) has exactly three limit cycles, because the system of equations (4.6) when $i = 5$, has the three real solutions

$$(\alpha_1,\beta_1,\gamma_1,\delta_1) \approx (0.863262\ldots, -1.17836\ldots, 1.16993\ldots, 1.72339\ldots),$$
$$(\alpha_2,\beta_2,\gamma_2,\delta_2) \approx (0.986378\ldots, -1.39019\ldots, 1.32103\ldots, 2.01258\ldots),$$
$$(\alpha_3,\beta_3,\gamma_3,\delta_3) \approx (1.0876\ldots, -1.57142\ldots, 1.44408\ldots, 2.25761\ldots),$$

see Figure 4.4.

This completes the proof of Theorem 4.2.

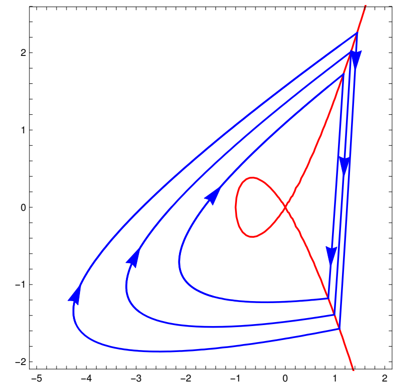


Figure 4.4: The three limit cycles of the discontinuous piecewise differential system (4.19)–(4.20).

Section 4.3 Limit Cycles with Four Discontinuity or Four and Two Discontinuity Intersection Points

In this section we study the maximum number of simultaneous limit cycles with two or four points on the cubic for the discontinuous piecewise linear Hamiltonian systems which intersect the cubics $c_2 = 0$ or $c_5 = 0$. We note that such limit cycles are contained in three pieces of the discontinuous piecewise linear Hamiltonian system.

4.3.1 Statement of the main result

Theorem 4.3 *The following statements hold.*

(a) The maximum number of limit cycles of the discontinuous piecewise linear Hamiltonian systems intersecting in four points the cubics c_2 or c_5 is three. See C_2^1 and C_5^1 of Figure 4.5 for the classes C_2 and C_5, respectively.

(b) The maximum number of limit cycles of the discontinuous piecewise linear Hamiltonian systems intersecting simultaneously in four points and two points the cubics c_2 or c_5 is three.

This maximum is reached in C_2^2 and C_2^3 of Figure 4.6 for the class C_2, and C_5^2 of Figure 4.7 for the class C_5 where there are examples of systems exhibiting simultaneously one limit cycle with four intersection points and two limit cycles *with two intersection points with the cubic.*

The maximum is also reached in C_2^4 of Figure 4.7 and C_2^5 of Figure 4.8 for the class C_2, and C_5^3 of Figure 4.8 for the class C_5 where there are examples of systems exhibiting simultaneously two limit cycles with four intersection points and one limit cycles with two intersection points with the cubic.

4.3.2 Proof of the main result

We give the proof for the maximum number of limit cycles for the statements (a) and (b) of the class C_5, and the proof for the class C_2 is similar.

Proof of statement (a) of Theorem 4.3.

We consider the Hamiltonian systems

$$\begin{aligned}
&\dot{x} = -\lambda_1 b_1 x + b_1 y + \mu_1, \quad \dot{y} = -\lambda_1^2 b_1 x + \lambda_1 b_1 y + \sigma_1, \text{ in the region } R_1,\\
&\dot{x} = -\lambda_2 b_2 x + b_2 y + \mu_2, \quad \dot{y} = -\lambda_2^2 b_2 x + \lambda_2 b_2 y + \sigma_2, \text{ in the region } R_2,\\
&\dot{x} = -\lambda_3 b_3 x + b_3 y + \mu_3, \quad \dot{y} = -\lambda_3^2 b_3 x + \lambda_3 b_3 y + \sigma_3, \text{ in the region } R_3,
\end{aligned} \tag{4.21}$$

with $b_i \neq 0$ and $\sigma_i \neq \lambda_i \mu_i$, when $i = 1, 2, 3$. The regions R_i for $i = 1, 2, 3$ are defined in (4.18). Their corresponding Hamiltonian first integrals are

$$\begin{aligned}
H_1(x,y) &= -(\lambda_1^2 b_1/2)x^2 + \lambda_1 b_1 xy - (b_1/2)y^2 + \sigma_1 x - \mu_1 y,\\
H_2(x,y) &= -(\lambda_2^2 b_2/2)x^2 + \lambda_2 b_2 xy - (b_2/2)y^2 + \sigma_2 x - \mu_2 y,\\
H_3(x,y) &= -(\lambda_3^2 b_3/2)x^2 + \lambda_3 b_3 xy - (b_3/2)y^2 + \sigma_3 x - \mu_3 y.
\end{aligned} \tag{4.22}$$

In order that the discontinuous piecewise differential system (4.21) has limit cycles which intersect the cubic $c_5 = 0$ in the points $p_1^{(i)} = (r_i^2-1, r_i(r_i^2-1))$, $p_2^{(i)} = (s_i^2-1, s_i(s_i^2-1))$, $p_3^{(i)} = (u_i^2-1, u_i(u_i^2-1))$ and $p_4^{(i)} = (w_i^2-1, w_i(w_i^2-1))$ they must satisfy the following system

$$\begin{aligned}
&H_1(r_i^2-1, r_i(r_i^2-1)) - H_1(s_i^2-1, s_i(s_i^2-1)) = 0,\\
&H_2(s_i^2-1, s_i(s_i^2-1)) - H_2(w_i^2-1, w_i(w_i^2-1)) = 0,\\
&H_2(r_i^2-1, r_i(r_i^2-1)) - H_2(u_i^2-1, u_i(u_i^2-1)) = 0,\\
&H_3(u_i^2-1, u_i(u_i^2-1)) - H_3(w_i^2-1, w_i(w_i^2-1)) = 0.
\end{aligned} \tag{4.23}$$

Now we consider the first and the last equations of system (4.23), by solving the first equation for $i = 1, 2, 3$ we get the expressions of λ_1, μ_1 and σ_1, and we get λ_3, μ_3 and σ_3 by solving the last equation. If we suppose that these two equations have a fourth solution, then from the first we get $b_1 = 0$ and from the last one we get $b_3 = 0$. This is a contradiction because by the assumptions they are not zero. Then we proved that the maximum number of limit cycles intersecting the cubic $c_5 = 0$ in four points is at most three.

Example with three limit cycles intersecting the curve $c_2 = 0$ in four points.

We consider the Hamiltonian systems

$$
\begin{aligned}
\dot{x} = \; & -9x/25 - 18y/5 + 1/10, \dot{y} = 9x/250 + 9y/25 - 19/10, \text{ in } R_1, \\
\dot{x} \approx \; & -409.623\ldots x - 3y/25 - 91053.6\ldots, \dot{y} \approx 1.39826 \times 10^6 x + 409.623\ldots y \\
& -3.35682 \times 10^6, \text{ in } R_2, \\
\dot{x} \approx \; & -23.0405\ldots x - y - 49.2425\ldots, \dot{y} \approx 530.865\ldots x + 23.0405\ldots \\
& y - 2056.81\ldots, \text{ in } R_3,
\end{aligned}
\tag{4.24}
$$

where the regions R_i for $i = 1, 2, 3$ are defined in (4.11). The Hamiltonian first integrals of the Hamiltonian systems (4.24) are

$$
\begin{aligned}
& H_1(x,y) = 9x^2/500 + 9xy/25 - 19x/10 + 9y^2/5 - y/10, \\
& H_2(x,y) \approx 699130\ldots x^2 + 409.623\ldots xy - 3.35682 \times 10^6 x + 3y^2/50 + 91053.6\ldots y, \\
& H_3(x,y) \approx 265.432\ldots x^2 + 23.0405\ldots xy - 2056.81\ldots x + y^2/2 + 49.2425\ldots y,
\end{aligned}
$$

In order that discontinuous piecewise differential system (4.24) has limit cycles which intersect the cubic $c_2 = 0$ in the points $p_1^{(i)} = (\alpha_i, \beta_i)$, $p_2^{(i)} = (\gamma_i, \delta_i)$, $p_3^{(i)} = (f_i, g_i)$ and $p_4^{(i)} = (h_i, k_i)$ these points must satisfy the system

$$
\begin{aligned}
& H_1(\alpha_i, \beta_i) - H_1(\gamma_i, \delta_i) = 0, \\
& H_2(\alpha_i, \beta_i)) - H_2(f_i, g_i) = 0, \\
& H_2(\gamma_i, \delta_i) - H_2(h_i, k_i) = 0, \\
& H_3(f_i, g_i) - H_3(h_i, k_i) = 0, \\
& c_2(f_i, g_i) = c_2(h_i, k_i) = 0, \\
& c_2(\alpha_i, \beta_i) = c_2(\gamma_i, \delta_i) = 0.
\end{aligned}
\tag{4.25}
$$

For the discontinuos piecewise differential system (4.24) all the real solutions of the system of equations (4.25) are

$$
\begin{aligned}
(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) \approx \;& (3.22996\ldots,1.28698\ldots,3.45846\ldots,-1.97435\ldots,0.960385\\
&\ldots,0.278564\ldots,0.931044\ldots,-0.364457\ldots),\\
(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) \approx \;& (3.28571\ldots,1.46483\ldots,3.52436\ldots,-2.15987\ldots,0.763003\\
&\ldots,0.636015\ldots,0.710476\ldots,-0.686261\ldots),\\
(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) \approx \;& (3.33822\ldots,1.62479\ldots,3.58465\ldots,-2.3274\ldots,0.400766\\
&\ldots,0.790071\ldots,0.351882\ldots,-0.777131\ldots),
\end{aligned}
$$

Then the discontinuous piecewise linear differential system (4.24) has exactly three limit cycles, see C_2^1 of Figure 4.5.

Example with three limit cycles intersecting the curve $c_5 = 0$ in four points.

We consider the following Hamiltonian systems

$$
\begin{aligned}
\dot{x} \approx \;& -82.0596\ldots x - y - 176.297\ldots, \dot{y} \approx 6733.77x + 82.0596y\\
& +333.127, \text{ in } R_1,\\
\dot{x} = \;& -9x/25 - 18y/5 + 1/10, \dot{y} = 9x/250 + 9y/25 - 19/10, \text{ in } R_2,\\
\dot{x} \approx \;& -138.156\ldots x + 2y + 550.05\ldots, \dot{y} \approx -9543.6\ldots x + 138.156\ldots y\\
& -9983.81\ldots, \text{ in } R_3.
\end{aligned}
\tag{4.26}
$$

The regions R_i for $i = 1,2,3$ are defined in (4.18). The Hamiltonian first integrals of the Hamiltonian systems (4.26) are

$$
\begin{aligned}
&H_1(x,y) \approx 3366.89\ldots x^2 + 82.0596\ldots xy + 333.127\ldots x + \; y^2/2 + 176.297\ldots y,\\
&H_2(x,y) = 9x^2/500 + 9xy/25 - 19x/10 + 9y^2/5 - y/10,\\
&H_3(x,y) \approx -4771.8\ldots x^2 + 138.156\ldots xy - 9983.81\ldots x - y^2 - 550.05\ldots y,
\end{aligned}
$$

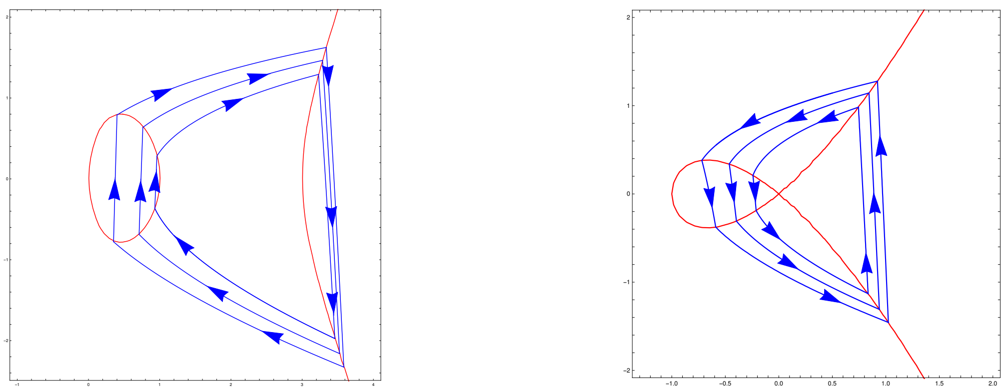
respectively.

For the discontinuos piecewise linear differential system (4.26) the real solutions of the system of equations (4.23) are

$$
\begin{aligned}
(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) \approx \;& (0.744242\ldots,0.982918\ldots,0.834499\ldots,-1.13028\ldots,-0.24\\
&1407\ldots,0.210259\ldots,-0.211795\ldots,-0.188034\ldots),
\end{aligned}
$$

$$(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) \approx (0.842745\ldots,1.14401\ldots,0.939388\ldots,-1.30821\ldots,-0.462869\ldots,0.339233\ldots,-0.395867\ldots,-0.307691\ldots),$$
$$(\alpha_3,\beta_3,\gamma_3,\delta_3,f_3,g_3,h_3,k_3) \approx (0.92187\ldots,1.278\ldots,1.02368\ldots,-1.45624\ldots,-0.718929\ldots,0.381148\ldots,-0.589074\ldots,-0.377617\ldots),$$

where $\alpha_i = r_i^2-1$, $\beta_i = r_i(r_i^2-1)$, $\gamma_i = s_i^2-1$, $\delta_i = s_i(s_i^2-1)$, $f_i = u_i^2-1$, $g_i = u_i(u_i^2-1)$, $h_i = w_i^2-1$, $k_i = w_i(w_i^2-1)$. Then the discontinuous piecewise linear differential system (4.26) has exactly three limit cycles, see C_5^1 of Figure 4.5.



C_2^1 C_5^1

Figure 4.5: The three limit cycles of the discontinuous piecewise differential systems C_2^1 for (4.24), and C_5^1 for (4.26).

This completes the proof of statement (a) of Theorem 4.3.

Proof of statement (b) of Theorem 4.3.

We consider the discontinuous piecewise differential system (4.21) with their corresponding first integrals (4.22). Assume that there are piecewise differential systems (4.21) having two limit cycles intersecting the cubic $c_2=0$ in four points and two limit cycles intersecingt $c_2=0$ in two points. Then such systems must have two solutions in system (4.25), and two solutions in system (4.6) with $i=2$ where eventually H_1 can be permuted with H_2.

From the fourth first equations of these two systems related with the first integral H_1 and the fourth mentioned solutions, we obtain the expressions of the parameters λ_1, μ_1, σ_1 and b_1, and it results that $b_1=0$, which is a contradiction.

Now assume that the discontinuous piecewise differential system (4.21) has one (resp. three) limit cycle intersecting the cubic $c_2 = 0$ in four points and three (resp. one) limit cycles intersecingt $c_2 = 0$ in two points, we get in the region R_3, defined in (4.11), four equations on H_3 from which we obtain the expressions of the parameters λ_3, μ_3, σ_3 and a zero value for b_3, which is again a contradiction.

In summary, we conclude that the maximum number of simultaneous limit cycles intersecting the cubic $c_2 = 0$ in four points and two points is three.

Examples with one limit cycle with four points on $c_2 = 0$ and two limit cycles with two points on $c_2 = 0$. As usual we consider the regions defined in (4.11). For the first possible configuration we consider the following Hamiltonian systems separated by the cubic $c_2 = 0$

$$\begin{aligned} \dot{x} \approx \quad & 308.837x - y/10 - 67017.8\ldots, \dot{y} \approx 953806.x - 308.837\ldots y \\ & -2.2674 \times 10^6, \text{ in } R_1, \\ \dot{x} = \quad & -7x/10 - 7y + 1/10, \dot{y} = 7x/100 + 7y/10 - 3, \text{ in } R_2, \\ \dot{x} = \quad & -4x + 2y + 3.55037, \dot{y} = -8x + 4y + 3, \text{ in } R_3. \end{aligned} \tag{4.27}$$

The Hamiltonian systems in (4.27) have the Hamiltonian first integrals

$$\begin{aligned} & H_1(x,y) \approx 476903\ldots x^2 - 308.837\ldots xy - 2.2674 \times 10^6 x + y^2/20 + 67017.8\ldots y, \\ & H_2(x,y) = 7x^2/200 + 7xy/10 - 3x + 7y^2/2 - y/10, \\ & H_3(x,y) \approx -4x^2 + 4xy + 3x - y^2 - 3.55037\ldots y. \end{aligned}$$

The discontinuos piecewise differential system (4.27) has one limit cycle intersecting the cubic $c_2 = 0$ in four points satisfying system (4.25) and two limit cycles intersecting the cubic $c_2 = 0$ in two points satisfying system (4.6) with $i = 2$, because all the real solutions of these two systems are

$$\begin{aligned} (\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) \approx \quad & (3.12267\ldots, 0.901721\ldots, 3.3261\ldots, -1.58839\ldots, 0.949883 \\ & \ldots, 0.312404\ldots, 0.907114\ldots, -0.419931\ldots), \\ (\alpha_2, \beta_2, \gamma_2, \delta_2) \approx \quad & (3.15388\ldots, 1.0224\ldots, 3.36809\ldots, -1.71344\ldots), \\ (\alpha_3, \beta_3, \gamma_3, \delta_3) \approx \quad & (3.18433\ldots, 1.1323\ldots, 3.40728\ldots, -1.82772\ldots). \end{aligned}$$

Then the discontinuous piecewise differential system (4.27) has exactly three limit cycles, see C_2^2 Figure 4.6.

For the second possible configuration we consider the following Hamiltonian systems separated by the cubic $c_2 = 0$

$$\begin{aligned}
\dot{x} \approx&\ -1.02031\ldots x + 12y - 3/10, \dot{y} \approx -0.0867522\ldots x + 1.02031\ldots y - 2, \text{ in } R_1,\\
\dot{x} =&\ -13x/5 + 26y + 7/10, \dot{y} = -13x/50 + 13y/5 - 11/5, \text{ in } R_2,\\
\dot{x} \approx&\ -2.71966\ldots x - y/5 + 1.64749\ldots, \dot{y} \approx 36.9826\ldots x + 2.71966\ldots y\\
&\ -23.1126\ldots, \text{ in } R_3.
\end{aligned} \tag{4.28}$$

The Hamiltonian first integrals of the Hamltonian systemsn (4.28) are

$$\begin{aligned}
H_1(x,y) &\approx -0.0433761\ldots x^2 + 1.02031\ldots xy - 2x - 6y^2 + 3y/10,\\
H_2(x,y) &= -13x^2/100 + 13xy/5 - 11x/5 - 13y^2 - 7y/10,\\
H_3(x,y) &\approx 18.4913\ldots x^2 + 2.71966\ldots xy - 23.1126\ldots x + y^2/10 - 1.64749\ldots y,
\end{aligned}$$

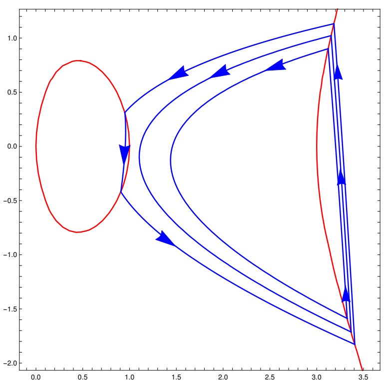
respectively.

The discontinuos piecewise differential system (4.28) has one limit cycle intersecting the cubic $c_2 = 0$ in four points satisfying system (4.25) and two limit cycles intersecting the cubic $c_2 = 0$ in two points satisfying system (4.6) with $i = 2$ and H_3 instead of H_1, because all the real solutions of these two systems are

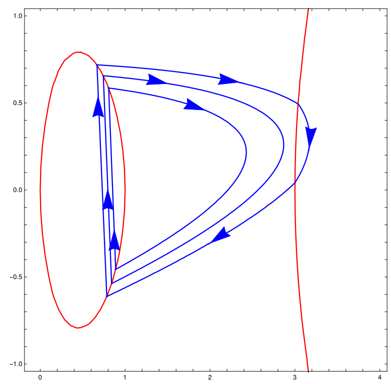
$$\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) \approx&\ (3.03932\ldots, 0.493657\ldots, 3.00028\ldots, 0.04126\ldots, 0.806003\\
&\ \ldots, 0.585712\ldots, 0.889093\ldots, -0.456234\ldots),\\
(\alpha_2, \beta_2, \gamma_2, \delta_2) \approx&\ (0.743027\ldots, 0.656462., 0.840689\ldots, -0.537772\ldots.),\\
(\alpha_3, \beta_3, \gamma_3, \delta_3) \approx&\ (0.668452\ldots, 0.718837\ldots, 0.78419\ldots, -0.612369\ldots).
\end{aligned}$$

Then the discontinuous piecewise differential system (4.28) has exactly three limit cycles, see C_2^3 Figure 4.6.

Example with one limit cycle with four points on $c_5 = 0$ and two limit cycles with two points on $c_5 = 0$. As usual we consider the regions defined in (4.18).



C_2^2



C_2^3

Figure 4.6: The three limit cycles of the discontinuous piecewise differential systems C_2^2 for (4.27), and C_2^3 for (4.28).

We consider the following Hamiltonian systems

$$
\begin{aligned}
\dot{x} \approx & \ -12x - 4y - 3.26985\ldots, \dot{y} = 36x + 12y + 2, \text{ in } R_1, \\
\dot{x} = & \ -16x - 40y - 41/5, \dot{y} = 32x/5 + 16y + 699/100, \text{ in } R_2, \\
\dot{x} \approx & \ -17.5227\ldots x + 20y - 2.73401\ldots, \dot{y} \approx -15.3522\ldots x \\
& +17.5227\ldots y - 2.04797\ldots, \text{ in } R_3.
\end{aligned} \tag{4.29}
$$

The Hamiltonian first integrals of these Hamiltonian systems are

$$
\begin{aligned}
&H_1(x,y) \approx 18x^2 + 12xy + 2x + 2y^2 + 3.26985\ldots y, \\
&H_2(x,y) = 16x^2/5 + 16xy + \ 699x/100 + 20y^2 + 41y/5, \\
&H_3(x,y) \approx -7.67611\ldots x^2 + 17.5227\ldots xy - 2.04797\ldots x - 10y^2 + 2.73401\ldots y,
\end{aligned}
$$

respectively. The discontinuos piecewise differential system (4.29) has one limit cycle intersecting the cubic $c_5 = 0$ in four points satisfying system (4.23) and two limit cycles intersecting the cubic $c_5 = 0$ in two points satisfying system (4.6) with $i = 5$ and H_3 instead of H_1, because all the real solutions of these two systems are

$$
\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) \approx & \ (0.0127029\ldots, 0.0127834\ldots, 0.280458\ldots, -0.317359\ldots, -0 \\
& .14376\ldots, 0.133026\ldots, -0.571076\ldots, -0.374011\ldots), \\
(\alpha_2, \beta_2, \gamma_2, \delta_2) \approx & \ (-0.028627\ldots, -0.028214\ldots, -0.480257\ldots, -0.346233\ldots), \\
(\alpha_3, \beta_3, \gamma_3, \delta_3) \approx & \ (-0.085601\ldots, -0.081855\ldots, -0.386674\ldots, -0.302824\ldots),
\end{aligned}
$$

where $\alpha_1 = r_1^2 - 1$, $\beta_1 = r_1(r_1^2 - 1)$, $\gamma_1 = s_1^2 - 1$, $\delta_1 = s_1(s_1^2 - 1)$, $f_1 = u_1^2 - 1$, $g_1 = u_1(u_1^2 - 1)$, $h_1 = w_1^2 - 1$, $k_1 = w_1(w_1^2 - 1)$. Then the discontinuous piecewise differential system (4.29) has exactly three limit cycles, see C_5^2 Figure 4.8.

Examples with two limit cycles with four points on $c_2 = 0$ and one limit cycle with two points on $c_2 = 0$.

We consider the regions defined in (4.11). For the first configuration of the class C_2 we consider the following Hamiltonian systems

$$
\begin{aligned}
\dot{x} \approx &\ 1937.84\ldots x - 7y/10 - 377775\ldots, \dot{y} \approx 5.36462 \times 10^6 x - 1937.84\ldots y \\
&-1.27506 \times 10^7, \text{ in } R_1, \\
\dot{x} = &\ -7x/10 - 7y + 1/10, \dot{y} = 7x/100 + 7y/10 - 3, \text{ in } R_2, \\
\dot{x} \approx &\ -6x + 3y - 11.0527\ldots, \dot{y} \approx -12x + 6y - 275.389\ldots, \text{ in } R_3,
\end{aligned}
\tag{4.30}
$$

with the Hamiltonian first integrals

$$
\begin{aligned}
&H_1(x,y) \approx 2.68231 \times 10^6 x^2 - 1937.84\ldots xy - 1.27506 \times 10^7 x + 7y^2/20 + 377775\ldots y, \\
&H_2(x,y) = 7x^2/200 + 7xy/10 - 3x + 7y^2/2 - y/10, \\
&H_3(x,y) \approx -6x^2 + 6xy - 275.389\ldots x - 3y^2/2 + 11.0527\ldots y,
\end{aligned}
$$

respectively. The discontinuos piecewise differential system (4.30) has two limit cycles intersecting the cubic $c_2 = 0$ in four points satisfying system (4.25) and one limit cycle intersecting the cubic $c_2 = 0$ in two points satisfying system (4.6) with $i = 2$ and H_3 instead of H_1, because all the real solutions of these two systems are

$$
\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) \approx &\ (3.14459\ldots, 0.987464\ldots, 3.35582\ldots, -1.67719\ldots, 0.975858 \\
&\ldots, 0.218376\ldots, 0.943351\ldots, -0.331522\ldots), \\
(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) \approx &\ (3.17528\ldots, 1.1003\ldots, 3.39578\ldots, -1.79441\ldots, 0.88154747 \\
&\ldots, 0.470332\ldots, 0.822573\ldots, -0.563727\ldots), \\
(\alpha_3, \beta_3, \gamma_3, \delta_3) \approx &\ (3.20514\ldots, 1.20413\ldots, 3.4333\ldots, -1.9026\ldots).
\end{aligned}
$$

Then the discontinuous piecewise differential system (4.30) has exactly three limit cycles, see C_2^4 Figure 4.7.

For the second possible configuration of the class C_2 we consider the following

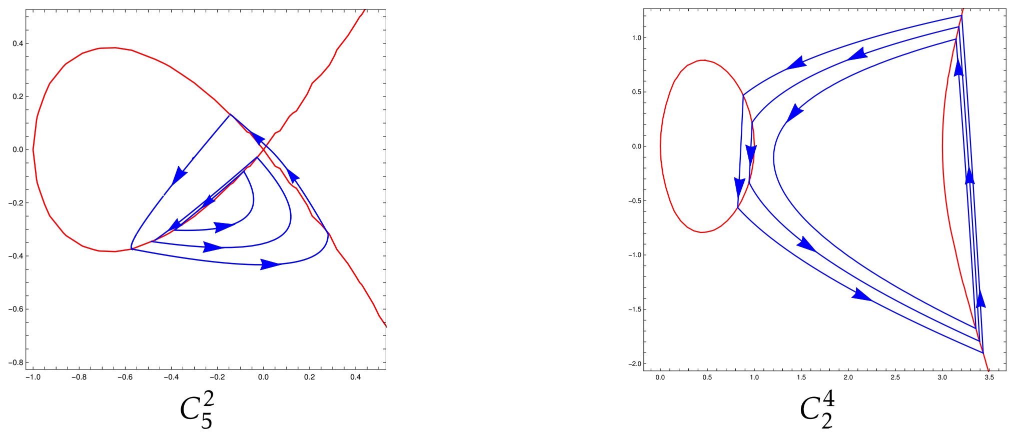


Figure 4.7: The three limit cycles of the discontinuous piecewise differential systems C_5^2 for (4.29), and C_2^4 for (4.30).

Hamiltonian systems

$$
\begin{aligned}
\dot{x} \approx\ & 10y - 1.05447\ldots x, \dot{y} \approx -0.11119\ldots x + 1.05447\ldots y - 0.426696\ldots, \text{ in } R_1 \\
\dot{x} \approx\ & -29x/10 + 29y - 3/5, \dot{y} = -29x/100 + 29y/10 - 5/2, \text{ in } R_2 \\
\dot{x} \approx\ & -1.11707\ldots x - y/10 + 0.370838\ldots, \dot{y} \approx 12.4785\ldots x + 1.11707\ldots y \\
& -6.64471\ldots, \text{ in } R_3,
\end{aligned}
\tag{4.31}
$$

which have the Hamiltonian first integrals

$$
\begin{aligned}
&H_1(x,y) \approx -0.0555949\ldots x^2 + 1.05447\ldots xy - 0.426696\ldots x - 5y^2, \\
&H_2(x,y) = -29x^2/200 + 29xy/10 - 5x/2 - 29y^2/2 + 3y/5, \\
&H_3(x,y) \approx 6.23927\ldots x^2 + 1.11707\ldots xy - 6.64471\ldots x + y^2/20 - 0.370838\ldots y,
\end{aligned}
$$

respectively. The discontinuos piecewise differential system (4.31) has two limit cycles intersecting the cubic $c_2 = 0$ in four points satisfying system (4.25) and one limit cycle intersecting the cubic $c_2 = 0$ in two points satisfying system (4.6) with $i = 2$ and H_3 instead of H_1, because all the real solutions of these two systems are

$$
\begin{aligned}
(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1) \approx\ & (3.03545\ldots, 0.468004\ldots, 3.00426\ldots, 0.160123\ldots, 0.72123 \\
& \ldots, 0.676877\ldots, 0.892948\ldots, -0.448796\ldots), \\
(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2) \approx\ & (3.06074\ldots, 0.618961\ldots, 3.00002\ldots, 0.0115046\ldots, 3.00002 \\
& \ldots, 0.0115046\ldots, 0.854489\ldots, -0.516495\ldots), \\
(\alpha_3,\beta_3,\gamma_3,\delta_3) \approx\ & (0.508698\ldots, 0.789074\ldots, 0.810902\ldots, -0.579376\ldots).
\end{aligned}
$$

Then the discontinuous piecewise differential system (4.31) has exactly three limit cycles, see C_2^5 Figure 4.8.

Example with two limit cycles with four points on $c_5 = 0$ **and one limit cycle with two points on** $c_5 = 0$. Here we consider the regions defined in (4.18).

We consider the Hamiltonian systems

$$\begin{aligned}
\dot{x} \approx\ & -3x/10 - y/10 - 0.0118645\ldots, \dot{y} \approx 9x/10 + 3y/10 - 0.049529\ldots, \text{ in } R_1,\\
\dot{x} =\ & -203x/25 - 29y - 5, \dot{y} = 1421x/625 + 203y/25 + 5/2, \text{ in } R_2,\\
\dot{x} \approx\ & -13.0887x - 2y - 23.1224\ldots, \dot{y} \approx 85.6566\ldots x + 13.0887\ldots y\\
& +1.62877\ldots, \text{ in } R_3.
\end{aligned} \tag{4.32}$$

The Hamiltonian systems in (4.24) have the Hamiltonian first integrals

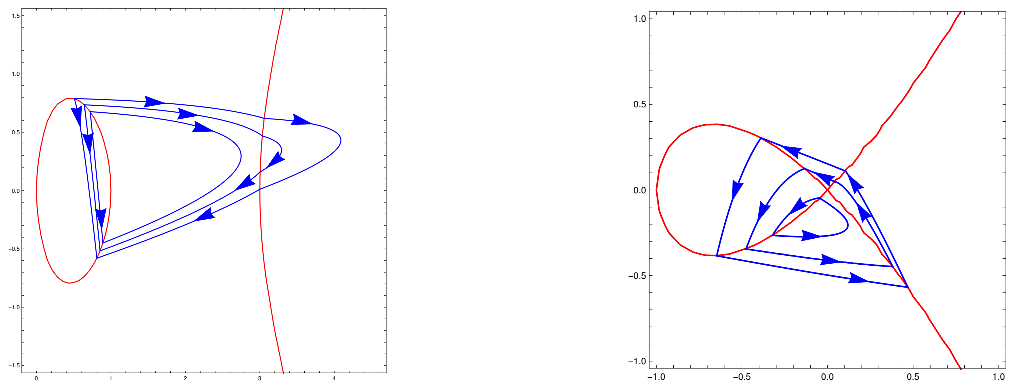
$$\begin{aligned}
&H_1(x,y) \approx 9x^2/20 + 3xy/10 - 0.049529\ldots x + y^2/20 + 0.0118645\ldots y,\\
&H_2(x,y) = 1421x^2/1250 + 203xy/25 + 5x/2 + 29y^2/2 + 5y,\\
&H_3(x,y) \approx 42.8283\ldots x^2 + 13.0887\ldots xy + 1.62877\ldots x + y^2 + 23.1224\ldots y.
\end{aligned}$$

The discontinuos piecewise differential system (4.32) has two limit cycles intersecting the cubic $c_5 = 0$ in four points satisfying system (4.23) and one limit cycle intersecting the cubic $c_5 = 0$ in two points satisfying system (4.6) with $i = 5$ and H_3 instead of H_1, because all the real solutions of these two systems are

$$\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) \approx\ & (0.0457703\ldots, 0.0468061\ldots, 0.380929\ldots, -0.44764\ldots, -0\\
& .0477081\ldots, -0.0465562\ldots, -0.322071\ldots, -0.265182\ldots),\\
(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) \approx\ & (0.105172\ldots, 0.110564\ldots, 0.468911\ldots, -0.568314\ldots, -0\\
& .134399\ldots, 0.125042\ldots, -0.476399\ldots, -0.344724\ldots),\\
(\alpha_3, \beta_3, \gamma_3, \delta_3) \approx\ & (-0.388977\ldots, 0.304056\ldots, -0.647556\ldots, -0.384435\ldots),
\end{aligned}$$

where $\alpha_i = r_i^2 - 1$, $\beta_i = r_i(r_i^2 - 1)$, $\gamma_i = s_i^2 - 1$, $\delta_i = s_i(s_i^2 - 1)$, $f_i = u_i^2 - 1$, $g_i = u_i(u_i^2 - 1)$, $h_i = w_i^2 - 1$, $k_i = w_i(w_i^2 - 1)$ for $i = 1, 2$. Then the discontinuous piecewise differential system (4.32) has exactly three limit cycles, see C_5^3 Figure 4.8.

This completes the proof of statement (b) of Theorem 4.3.



C_2^5 C_5^3

Figure 4.8: The three limit cycles of the discontinuous piecewise differential systems C_2^5 for (4.31), and C_5^3 for (4.32).

Section 4.4 Appendix of Chapter 4

Here we provide the values of C, D and E that appear in the proof of Theorem 4.2.

$$
\begin{aligned}
C = \; & r_1^4r_2^2r_3 - r_1^2r_2^4r_3 - r_1^4r_2r_3^2 + r_1r_2^4r_3^2 + r_1^2r_2r_3^4 - r_1r_2^2r_3^4 + r_1^3r_2^2r_3s_1 - r_1r_2^4r_3s_1 - r_1^3r_2r_3^2s_1 + r_2^4 \\
& r_3^2s_1 + r_1r_2r_3^4s_1 - r_2^2r_3^4s_1 + r_1^2r_2^2r_3s_1^2 - r_1^2r_2r_3^2s_1^2 + r_2r_3^4s_1^2 + r_1r_2^2r_3s_1^3 - r_1r_2r_3^2s_1^3 + r_2^2r_3s_1^4 - r_2 \\
& r_3^2s_1^4 - r_1^4r_3^2s_2 + r_1r_2^3r_3^2s_2 + r_1^2r_3^4s_2 - r_1r_2r_3^4s_2 + r_1^3r_2r_3s_1s_2 - r_1r_2^3r_3s_1s_2 + r_2^3r_3^2s_1s_2 + r_1r_3^4s_1 \\
& s_2 - r_2r_3^4s_1s_2 + r_1^2r_2r_3s_1^2s_2 - r_2^3r_3s_1^2s_2 - r_1^2r_3^2s_1^2s_2 + r_1r_2r_3s_1^3s_2 - r_1r_3^2s_1^3s_2 + r_2r_3s_1^4s_2 - r_3^2s_1^4 \\
& s_2 + r_1^4r_3s_2^2 - r_1^2r_2^2r_3s_2^2 - r_1r_3^4s_2^2 + r_1^3r_3s_1s_2^2 - r_1r_2^2r_3s_1s_2^2 + r_2^2r_3^2s_1s_2^2 - r_3^4s_1s_2^2 + r_1^2r_3s_1^2s_2^2 + r_1 \\
& r_3s_1^3s_2^2 + r_3s_1^4s_2^2 - r_1^2r_2r_3s_2^3 + r_1r_2r_3^2s_2^3 - r_1r_2r_3s_1s_2^3 + r_2r_3^2s_1s_2^3 - r_1^2r_3s_2^4 + r_1r_3^2s_2^4 - r_1r_3s_1s_2^4 \\
& + r_3^2s_1s_2^4 - r_3s_1^2s_2^4 + r_1^4r_2^2s_3 - r_1^2r_2^4s_3 + r_1r_2^4r_3s_3 + r_1^2r_2r_3^3s_3 - r_1r_2^2r_3^3s_3 + r_1^3r_2^2s_1s_3 - r_1r_2^4s_1s_3 \\
& - r_1^3r_2r_3s_1s_3 + r_1r_2r_3^3s_1s_3 - r_2^2r_3^3s_1s_3 + r_1^2r_2^2s_1^2s_3 - r_2^4s_1^2s_3 - r_1^2r_2r_3s_1^2s_3 + r_2r_3^3s_1^2s_3 - r_1r_2r_3 \\
& s_1^3s_3 + r_2^2s_1^4s_3 - r_2r_3s_1^4s_3 + r_1^4r_2s_2s_3 - r_1^2r_2^3s_2s_3 - r_1^4r_3s_2s_3 + r_1r_3^3s_1s_2s_3 - r_2r_3^3s_1s_2s_3 + r_1^2r_2 \\
& s_1^2s_2s_3 - r_2^3s_1^2s_2s_3 - r_1^2r_3s_1^2s_2s_3 + r_1r_2s_1^3s_2s_3 - r_1r_3s_1^3s_2s_3 + r_2s_1^4s_2s_3 - r_3s_1^4s_2s_3 + r_1^4s_2^2s_3 \\
& - r_1^2r_2^2s_2^2s_3 - r_1r_3^3s_2^2s_3 + r_1^3s_1s_2^2s_3 - r_1r_2^2s_1s_2^2s_3 + r_2^2r_3s_1s_2^2s_3 - r_3^3s_1s_2^2s_3 + r_1^2s_1^2s_2^2s_3 + r_1s_1^3s_2^2 \\
& s_3 + s_1^4s_2^2s_3 - r_1^2r_2s_2^3s_3 + r_1r_2r_3s_2^3s_3 - r_1r_2s_1s_2^3s_3 + r_2r_3s_1s_2^3s_3 - r_1^2s_2^4s_3 + r_1r_3s_2^4s_3 - r_1s_1s_2^4 \\
& s_3 + r_3s_1s_2^4s_3 - s_1^2s_2^4s_3 - r_1^4r_2s_3^2 + r_1r_2^4s_3^2 - r_1r_2^2r_3^2s_3^2 - r_1^3r_2s_1s_3^2 + r_2^4s_1s_3^2 + r_1r_2r_3^2s_1s_3^2 - r_2^2 \\
& r_3^2s_1s_3^2 - r_1^2r_2s_1^2s_3^2 - r_1r_2s_1^3s_3^2 - r_2s_1^4s_3^2 - r_1^4s_2s_3^2 + r_1r_2^3s_2s_3^2 + r_1^2r_3^2s_2s_3^2 - r_1r_2r_3^2s_2s_3^2 + r_2^3s_1s_2 \\
& s_3^2 + r_1r_3^2s_1s_2s_3^2 - r_2r_3^2s_1s_2s_3^2 - r_1^2s_1^2s_2s_3^2 + r_3^2s_1^2s_2s_3^2 - r_1s_1^3s_2s_3^2 + r_1r_2^2s_2^2s_3^2 - r_1r_3^2s_2^2s_3^2 + r_2^2s_1 \\
& s_2^2s_3^2 - r_3^2s_1s_2^2s_3^2 + r_1r_2s_2^3s_3^2 + r_2s_1s_2^3s_3^2 + s_1s_2^4s_3^2 + r_1^2r_2r_3s_3^3 - r_1r_2^2r_3s_3^3 + r_1r_2r_3s_1s_3^3 - r_2^2r_3s_1 \\
& s_3^3 + r_2r_3s_1^2s_3^3 - r_1r_2r_3s_2s_3^3 + r_1r_3s_1s_2s_3^3 - r_2r_3s_1s_2s_3^3 + r_3s_1^2s_2s_3^3 - r_1r_3s_2^2s_3^3 - r_3s_1s_2^2s_3^3 - r_1
\end{aligned}
$$

$$r_2^2s_3^4+r_1r_2s_1s_3^4-r_2^2s_1s_3^4+r_2s_1^2s_3^4+r_1^2s_2s_3^4-r_1r_2s_2s_3^4+r_1s_1s_2s_3^4+s_1^2s_2s_3^4-r_1s_2^2s_3^4-s_1s_2^2s_3^4$$
$$-r_1^2r_2^3r_3s_2-r_2s_1s_2s_3^4+r_1^2r_2s_3^4+r_1^2r_3s_2s_3^3+r_1s_2^4s_3^2-s_1^4s_2s_3^2+r_2r_3^2s_1^2s_3^2-r_2s_1^2s_2^3s_3+r_1^2r_2r_3^2$$
$$s_3^2+r_1r_2^2r_3s_2^2s_3-r_2^2r_3s_1^2s_2^2-r_2^4r_3s_1^2+r_1^4r_2r_3s_2-r_1^3r_3^2s_1s_2+r_3^4s_1^2s_2+r_1r_2^2r_3^2s_2^2-r_2r_3s_1^2s_2^3$$
$$-r_1^4r_2r_3s_3+r_2^4r_3s_1s_3+r_1r_2^3r_3s_2s_3-r_1^3s_1s_2s_3^2-r_2^2s_1^2s_2^2s_3+r_2^3r_3s_1s_2s_3+r_1r_2^2s_1^3s_3+r_3^3s_1^2s_2s_3,$$

The parameter D is equal to

$$4(r_1^4(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+s_3)))$$
$$+r_1^3s_1(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+s_3)))$$
$$+r_1^2(-r_2^4(r_3+s_3)-r_2^3s_2(r_3+s_3)+r_2^2(s_1^2-s_2^2)(r_3+s_3)+r_2(r_3^4+r_3^3s_3+r_3^2(-s_1^2+s_3^2)+r_3(-s_2^3$$
$$+s_1^2(s_2-s_3)+s_3^3)+s_3(-s_2^3+s_1^2(s_2-s_3)+s_3^3))+s_2(r_3^4+r_3^3s_3+r_3^2(-s_1^2+s_3^2)+r_3(-s_2^3+s_1^2(s_2$$
$$-s_3)+s_3^3)+s_3(-s_2^3+s_1^2(s_2-s_3)+s_3^3)))+r_1(r_2^4(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))+r_2^3s_2(r_3^2$$
$$+r_3(-s_1+s_3)+s_3(-s_1+s_3))-r_2^2(r_3^4+r_3^3s_3+r_3^2(-s_2^2+s_3^2)+r_3(-s_1^3+s_1s_2^2-s_2^2s_3+s_3^3)+s_3$$
$$(-s_1^3+s_1s_2^2-s_2^2s_3+s_3^3))+r_2(r_3^4(s_1-s_2)+r_3^3(s_1-s_2)s_3+r_3^2(-s_1^3+s_2^3+s_1s_3^2-s_2s_3^2)+r_3(s_1^3$$
$$(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))+s_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3)))+s_2$$
$$(r_3^4(s_1-s_2)+r_3^3(s_1-s_2)s_3+r_3^2(-s_1^3+s_2^3+s_1s_3^2-s_2s_3^2)+r_3(s_1^3(s_2-s_3)(s_2^2-s_3^2)+s_1(-s_2^3$$
$$+s_3^3))+s_2s_3+s_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))))+s_1(r_2^4(r_3^2+r_3(-s_1+s_3)+s_3$$
$$(-s_1+s_3))+r_2^3s_2(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))-r_2^2(r_3^4+r_3^3s_3+r_3^2(-s_2^2+s_3^2)+r_3(-s_1^3$$
$$+s_1s_2^2-s_2^2s_3+s_3^3)+s_3(-s_1^3+s_1s_2^2-s_2^2s_3+s_3^3))+r_2(r_3^4(s_1-s_2)+r_3^3(s_1-s_2)s_3+r_3^2(-s_1^3$$
$$+s_2^3+s_1s_3^2-s_2s_3^2)+r_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))+s_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2$$
$$-s_3^2)+s_1(-s_2^3+s_3^3)))+s_2(r_3^4(s_1-s_2)+r_3^3(s_1-s_2)s_3+r_3^2(-s_1^3+s_2^3+s_1s_3^2-s_2s_3^2)+r_3(s_1^3(s_2$$
$$-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))+s_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))))))^2-4$$
$$(r_1^2(-r_2^3(r_3+s_3)+r_2^2(s_1-s_2)(r_3+s_3)+r_2(r_3^3+r_3(s_1-s_2-s_3)(s_2-s_3)+(s_1-s_2-s_3)(s_2$$
$$(-s_1+s_3))-s_3)s_3+r_3^2+s_2(r_3^3+r_3(s_1-s_2-s_3)(s_2-s_3)+(s_1-s_2-s_3)(s_2-s_3)s_3+r_3^2(-s_1$$
$$+s_3)))+r_1^3(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2$$
$$+s_3)))+r_1(r_2(s_1-s_2)(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)+(s_1$$
$$-s_2)s_2(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)+r_2^3(r_3^2+r_3(-s_1+$$
$$s_3)+s_3(-s_1+s_3))-r_2^2(r_3^3+r_3^2(-s_2+s_3)-r_3(s_1-s_3)(s_1-s_2+s_3)-(s_1-s_3)s_3(s_1-s_2+$$
$$s_3)))+s_1(r_2(s_1-s_2)(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)+(s_1$$
$$-s_2)s_2(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)+r_2^3(r_3^2+r_3(-s_1+$$
$$s_3)+s_3(-s_1+s_3))-r_2^2(r_3^3+r_3^2(-s_2+s_3)-r_3(s_1-s_3)(s_1-s_2+s_3)-(s_1-s_3)s_3(s_1-s_2+$$

$$s_3))))(r_1^5(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+$$
$$s_3)))+r_1^4s_1(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2$$
$$+s_3)))+r_1^3s_1^2(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2$$
$$+s_3)))+r_1^2(-r_2^5(r_3+s_3)-r_2^4s_2(r_3+s_3)-r_2^3s_2^2(r_3+s_3)+r_2^2(s_1^3-s_2^3)(r_3+s_3)+r_2(r_3^5+r_3^4s_3$$
$$+r_3^3s_3^2+r_3^2(-s_1^3+s_3^3)+r_3(-s_2^4+s_1^3(s_2-s_3)+s_3^4)+s_3(-s_2^4+s_1^3(s_2-s_3)+s_3^4))+s_2(r_3^5+r_3^4s_3$$
$$+r_3^3s_3^2+r_3^2(-s_1^3+s_3^3)+r_3(-s_2^4+s_1^3(s_2-s_3)+s_3^4)+s_3(-s_2^4+s_1^3(s_2-s_3)+s_3^4)))+r_1(r_2^5(r_3^2+$$
$$r_3(-s_1+s_3)+s_3(-s_1+s_3))+r_2^4s_2(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))+r_2^3s_2^2(r_3^2+r_3(-s_1+s_3)$$
$$+s_3(-s_1+s_3))-r_2^2(r_3^5+r_3^4s_3+r_3^3s_3^2+r_3^2(-s_2^3+s_3^3)+r_3(-s_1^4+s_1s_2^3-s_2^3s_3+s_3^4)+s_3(-s_1^4+s_1$$
$$s_2^3-s_2^3s_3+s_3^4))+r_2(r_3^5(s_1-s_2)+r_3^4(s_1-s_2)s_3+r_3^3(s_1-s_2)s_3^2+r_3^2(-s_1^4+s_2^4+s_1s_3^3-s_2s_3^3)+$$
$$r_3(s_1^4(s_2-s_3)+s_2s_3(s_2^3-s_3^3)+s_1(-s_2^4+s_3^4))+s_3(s_1^4(s_2-s_3)+s_2s_3(s_2^3-s_3^3)+s_1(-s_2^4+s_3^4)))$$
$$+s_2(r_3^5(s_1-s_2)+r_3^4(s_1-s_2)s_3+r_3^3(s_1-s_2)s_3^2+r_3^2(-s_1^4+s_2^4+s_1s_3^3-s_2s_3^3)+r_3(s_1^4(s_2-s_3)+$$
$$s_2s_3(s_2^3-s_3^3)+s_1(-s_2^4+s_3^4))+s_3(s_1^4(s_2-s_3)+s_2s_3(s_2^3-s_3^3)+s_1(-s_2^4+s_3^4))))+s_1(r_2^5(r_3^2+r_3$$
$$(-s_1+s_3)+s_3(-s_1+s_3))+r_2^4s_2(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))+r_2^3s_2^2(r_3^2+r_3(-s_1+s_3)+$$
$$s_3(-s_1+s_3))-r_2^2(r_3^5+r_3^4s_3+r_3^3s_3^2+r_3^2(-s_2^3+s_3^3)+r_3(-s_1^4+s_1s_2^3-s_2^3s_3+s_3^4)+s_3(-s_1^4+s_1$$
$$s_2^3-s_2^3s_3+s_3^4))+r_2(r_3^5(s_1-s_2)+r_3^4(s_1-s_2)s_3+r_3^3(s_1-s_2)s_3^2+r_3^2(-s_1^4+s_2^4+s_1s_3^3-s_2s_3^3)+$$
$$r_3(s_1^4(s_2-s_3)+s_2s_3(s_2^3-s_3^3)+s_1(-s_2^4+s_3^4))+s_3(s_1^4(s_2-s_3)+s_2s_3(s_2^3-s_3^3)+s_1(-s_2^4+s_3^4)))$$
$$+s_2(r_3^5(s_1-s_2)+r_3^4(s_1-s_2)s_3+r_3^3(s_1-s_2)s_3^2+r_3^2(-s_1^4+s_2^4+s_1s_3^3-s_2s_3^3)+r_3(s_1^4(s_2-s_3)+$$
$$s_2s_3(s_2^3-s_3^3)+s_1(-s_2^4+s_3^4))+s_3(s_1^4(s_2-s_3)+s_2s_3(s_2^3-s_3^3)+s_1(-s_2^4+s_3^4))))),$$

Finally the parameter E is equal to

$$r_1^2(-r_2^3(r_3+s_3)+r_2^2(s_1-s_2)(r_3+s_3)+r_2(r_3^3+r_3(s_1-s_2-s_3)(s_2-s_3)+(s_1-s_2-s_3)(s_2-s_3)$$
$$s_3+r_3^2(-s_1+s_3))+s_2(r_3^3+r_3(s_1-s_2-s_3)(s_2-s_3)+(s_1-s_2-s_3)(s_2-s_3)s_3+r_3^2(-s_1+s_3)))$$
$$+r_1^3(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+s_3)))+r_1$$
$$(r_2(s_1-s_2)(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)+(s_1-s_2)s_2(r_3^3+r_3$$
$$(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)+r_2^3(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))$$
$$-r_2^2(r_3^3+r_3^2(-s_2+s_3)-r_3(s_1-s_3)(s_1-s_2+s_3)-(s_1-s_3)s_3(s_1-s_2+s_3)))+s_1(r_2(s_1-s_2)(r_3^3$$
$$+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)+(s_1-s_2)s_2(r_3^3+r_3(s_1-s_3)(s_2-$$
$$s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)+r_2^3(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))-r_2^2(r_3^3+r_3^2$$
$$(-s_2+s_3)-r_3(s_1-s_3)(s_1-s_2+s_3)-(s_1-s_3)s_3(s_1-s_2+s_3))).$$

Conclusion

In this work, firstly we classified the global phase portraits of five quadratic polynomial differential systems exhibiting five classical cubic invariant algebraic curves. We used the classical technique for analyzing four quadratic systems and the invariant theory to analyze the fifth one. We also studied the dynamics of a quadratic polynomial differential system exhibiting a reducible invariant curve of degree three, where we realized that this system produced 13 topologically different phase portraits in the Poincaré disc.

Secondly, we considered a Kukles differential systems of degree eight and we provided all their global phase portraits in the Poincaré disk, by using the classical method. Our second contribution for these systems consists in solving the second part of the 16th Hilbert problem by applying the averaging theory, and we succeeded in showing a certain number of limit cycles.

Finaly, we have solved the second part of the 16th Hilbert problem for a planar discontinuous piecewise differential Hamiltonian systems without equilibrium point separated by irreducible cubic algebraic curves.

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