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Spectral properties of posinormal operators

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Dedication

This work is dedicated to my Parents. They have genuinely encouraged and supported me,

and for that, I will be eternally grateful.

I also dedicate this work to my sister "Hadjer", and to my brother's "Bassem","Ziyed" and "Fourat El Rahmen".

Additionally, I dedicate this work to my mother, who always encouraged me to do my best and to follow my dream.

She is my inspiration and my role model.

Finally, I would like to express my sincere thanks and appreciation to "Dr. Dehimi Souheyb" for

helping and supervising us through this work.

A. Bourahla

Dedication

I dedicate this work to my family, friends and everyone helped me.

I am also dedicating this work to my beloved mother, my dear Father,

who always encouraged me.

Additionally, this work is dedicated to my husband,

My beloved son "Idris", my brothers

, my sisters, my classmates, and my dear professors.

Finally, I offer my greetings and thanks to "Dr. Dehimi Souheyb"

for helping us through this work.

N. Gasmi

Contents

List of symbols

$< . >$: Inner product.

H : Hilbert space.

$B(H)$: The set of all bounded linear operator defined on a Hilbert space H .

T : A Bounded linear operator defined on a Hilbert space$H$.

I : The identity operator where $I \in B(H)$.

$||T||$: The norm of T.

$\ker(T)$: The kernel of T.

$Im(T)$: Image of T.

T^* : Adjoint operator of T.

$\sigma(T)$: The spectrum of T.

$\rho(T)$: The resolvent set of T.

T^{-1} : The inverse of T.

$\sigma_p(T)$: Punctual spectrum of T.

$\sigma_c(T)$: Continuous spectrum of T.

$\sigma_r(T)$: Residual spectrum of T.

$\sigma_a p(T)$: Approximate spectrum of T.

$|T|$: The absolute value of T.

$P(H)$: The set of all posinormal operator.

$p(T)$: Posispectrum of T.

p-$P(H)$: The set of all p-posinormal operator.

Introduction

Spectral theory of linear operators on Hilbert spaces is a pillar in several developments in mathematics, physics and quantum mechanics. Its concepts like the spectrum of a linear operator, eigenvalues and vectors, spectral radius, spectral integrals among others have useful applications in quantum mechanics.

In this work we study the properties of a large subclass of bounded linear operators on a Hilbert space H, which is posinormal operators. This class was first introduced by Rhaly (see [13]). A bounded linear operator T on a Hilbert space H is said to be posinormal if there exist a positive operator P, such that

$$TT^* = T^*PT.$$

Also, We have discussed the relation between posinormal operators and other classes of bounded linear operators.

The first chapter contains an introduction to our work. We have presented some fundamental properties, for example: Hilbert spaces, bounded linear operators, spectrum of bounded linear operator and we have also defined some classes of bounded linear operators.

In the second chapter, we presented several proprieties of posinormal operators. Also, we showed the deferences between the deferent classes of bounded linear operators, and we finished the chapter by studying the spectrum of posinormal operators.

In the last chapter, we studied the powers of posinormal operators, more precisely, we have shown the relation between a posinormal operator and its powers.

Chapter 1

Essential background

In this chapter we present some fundamental properties of bounded linear operators. We also prove some results related to the spectrum of bounded linear operators and we present deferent classes of bounded linear operators.

1.1 Hilbert spaces

Let E be a linear space over $\mathbb{K} = \mathbb{C}$.

Definition 1.1.1. *An inner product on E is a function $< ., . >: E \times E \to \mathbb{C}$ with:*

(a) $< \lambda_1 x_1 + \lambda_2 x_2, y >= \lambda_1 < x_1, y > +\lambda_2 < x_2, y >$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$ (Linearity)

(b) $< x, y >= \overline{< y, x >}$ for all $x, y \in E$ (Hermitian property)

(c) $< x, x >\geq 0$ for all $x \in E$, and $< x, x >= 0$ implies $x = 0$ (positive definiteness).

Example 1.1.2. • *Let $E = \mathbb{C}^n$, and let $x, y \in \mathbb{C}$ where*

$$x = (x_1, ..., x_n) \quad and \quad y = (y_1, ..., y_n).$$

Then

$$< x, y >= \sum_{k=1}^{n} x_k \overline{y_k}$$

defines an inner product on the space $\mathbb{C}^n$.

Definition 1.1.3 (Hilbert space)**.** *A complete normed inner product space is said to be a Hilbert space.*

Proposition 1.1.4 (Cauchy-Schwarz Inequality)**.** *Let $(H, <.,.>)$ be a Hilbert space. For all $x, y \in H$ we have*

$$| < x, y > | \leq \|x\|\|y\|.$$

1.2 Bounded linear operators

In this section, we studied the proprieties of bounded linear operators.

Definition 1.2.1. *Let V, W be two vector espaces over the same field $\mathbb{K}$.*
A map $T : V \to W$ is a linear map if the following two condition are satisfied:
1- $T(x + y) = T(x) + T(y)$ for all $x, y \in V$.
2- $T(\lambda x) = \lambda T(x)$ for all $x \in V$ and $\lambda \in \mathbb{K}$.

Definition 1.2.2. *Let $T : H \longrightarrow H$ be a linear operator where H is a Hilbert space. We said that T is bounded linear operator if there exist $c > 0$ such that $\| Tx \| \leq c \| x \|$ for all $x \in H$.*

The set of all bounded linear operator on H is denoted by $B(H)$ for any $T \in B(H)$ one sets:

$$\begin{aligned} \| T \| &= \inf\{c > 0 : \|Tx\| \leqslant c\|x\| \quad \text{for all } x \in H\} \\ &= \sup\{\tfrac{\|Tx\|}{\|x\|} : x \neq 0 \quad \text{and } x \in H\} \\ &= \sup\{\|Tx\| : x \in v \quad \text{and } \| x \| = 1\}. \end{aligned}$$

Definition 1.2.3. *Let $T \in B(H)$. The Kernel of T is defined by*

$$\ker(T) = \{x \in H : T(x) = 0\}.$$

Example 1.2.4. *We consider the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:*

$$T(x, y) = (2x + y, x - y).$$

Then

$$\ker(T) = \{(x, y) \in \mathbb{R}^2 : (2x + y, x - y) = (0, 0)\}$$

which means

$$2x + y = 0 \quad and \quad x - y = 0,$$

therefore

$$x = y = 0.$$

so

$$\ker(T) = \{(0, 0)\}.$$

Definition 1.2.5. *Let $T \in B(H)$. The image of T is defined by*

$$Im(T) = \{T(x) : x \in H\}.$$

Example 1.2.6. *We consider the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:*

$$T(x, y) = (2x + y, x - y).$$

Then

$$Im(T) = \{(X, Y) \in \mathbb{R}^2 : T(x, y) = (X, Y)\}$$

which means

$$2x + y = X \quad and \quad x - y = Y,$$

thus

$$x = \frac{X + Y}{3} \quad and \quad y = \frac{X - 2Y}{3}.$$

So for any $(X, Y) \in \mathbb{R}^2$ we find an antecedent $(x, y) = \{\frac{X+Y}{3}, \frac{X-2Y}{3}\}$. Which therefore verifies $T(x, y) = (X, Y)$. So $Im(T) = \mathbb{R}^2$.

Theorem 1.2.7 ([7])**.** *If T is an operator on a Hilbert space H. Then*

$$\begin{aligned} < Tx, y >= \; & \frac{1}{4}(< T(x + y), x + y > - < T(x - y), x - y >) \\ & + \frac{1}{4}i(< T(x + y), x + iy > - < T(x - iy), x - iy >) \end{aligned}$$

holds for any $x, y \in H$.

The next Theorem was proved in [7].

Theorem 1.2.8. *If T is an operator on a Hilbert space H over the complex scalars $\mathbb{C}$, then the following (i), (ii) and (iii) are mutually equivalent:*
i) $T = 0$.
ii) $< Tx, x >= 0$ *for all* $x \in H$.
iii) $< Tx, y >= 0$ *for all* $x, y \in H$.

Now, let's define the adjoint of bounded linear operator.

Definition 1.2.9. *Let H be a Hilbert space and let $T \in B(H)$ then,*

$$T^* : H \longrightarrow H$$

is called the (Hilbert-) adjoint operator of T is defined by:

$$< Tx, y >=< x, T^*y >,$$

for all x,y $\in H$.

Theorem 1.2.10. *Let T be a bounded operator on a Hilbert space H, then T^* is also an operator on H, and the following properties holds:*
1) $\| T^* \|=\| T \|$.
2) $(T_1 + T_2)^* = (T_1)^* + (T_2)^*$.
3) $(\alpha T)^* = \overline{\alpha} T^*$ *for any* $\alpha \in \mathbb{C}$.
4) $(T^*)^* = T$.
5) $(ST)^* = T^*S^*$, *where S is the bounded operator*
6) $I^* = I$, *where I is the identity operator.*
7) $O^*_{B(H)} = O_{B(H)}$, *where $O_{B(H)}$ is the zero operator on H.*

Corollary 1.2.11. *Let T be an operator. Then*
i) $\| T^*T \|=\| TT^* \|=\| T \|^2$.
ii) $T^*T = 0$ *if and only if* $T = 0$.

Proof. i) Since $\parallel T^* \parallel = \parallel T \parallel$ by (1) in Theorem 1.2.10,

$$\parallel T^*T \parallel \leq \parallel T \parallel^2 .$$

Conversely we have

$$\parallel Tx \parallel^2 = < Tx, Tx > = < T^*Tx, x > \leq \parallel T^*Tx \parallel \parallel x \parallel \leq \parallel T^*T \parallel \parallel x \parallel^2$$

so

$$\parallel T \parallel^2 \leq \parallel T^*T \parallel .$$

Thus

$$\parallel T \parallel^2 = \parallel T^*T \parallel .$$

Replacing T by T^* to get $\parallel T \parallel^2 = \parallel T^* \parallel^2 = \parallel TT^* \parallel$, so the proof is complete.
(ii) obvious by (i). □

In the next theorem, we will show the relation between $\ker(T)$ and $Im(T)$.

Theorem 1.2.12. *If $T \in B(H)$, then*

$$\ker(T^*) = Im(T)^{\perp} \quad \textit{and} \quad \ker(T) = Im(T^*)^{\perp}.$$

Proof. Each of the following four statements is clearly equivalent to the one that follows and /or precedes it.

1) $T^*y = 0$.
2) $< x, T^*y > = 0$ for every $x \in H$.
3) $< Tx, y > = 0$ for every $x \in H$.
4) $y \in Im(T)^{\perp}$.

Thus, $\ker(T^*) = Im(T)^{\perp}$. Since $T^{**} = T$, the second assertion follows from the first if T is replaced by T^*.

□

1.3 Spectrum of bounded linear operators

In this section, we defined the spectrum of bounded linear operators, and we have given some examples related the spectrum.

Definition 1.3.1. *A bounded linear operator $T \in B(H)$, is said to be invertible operator if there exists an operator $S \in B(H)$, such that $ST = TS = I$.*
Where I is the identity operator.

Remark 1.3.2. *Let $T \in B(H)$*
- *T is injective $\Leftrightarrow \ker(T) = \{0\}$.*
- *T is surjective $\Leftrightarrow Im(T) = H$.*
- *T is bijective $\Leftrightarrow$ T is invertible $\Leftrightarrow$ T is injective and surjective.*

Theorem 1.3.3 ([7])**.** *If T is an operator and c is positive number such that $\| Tx \| \geq c \| x \|$ for every vector $x \in H$, then $Im(T)$ is closed.*

Proof. Assume $y_n = Tx_n$ for $n = 1, 2, ...$ and $y_n \to y_0$. Since

$$\| y_n - y_m \| = \| Tx_n - Tx_m \| = \| T(x_n - x_m) \| \geq c \| x_n - x_m \|$$

and $\{y_n\}$ is a Cauchy sequence, $\{x_n\}$ is also a Cauchy sequence, and there exists $x_0 \in H$ such that $x_n \to x_0$ because H is a Hilbert space. Now,

$$\| y_0 - Tx_0 \| \leq \| y_0 - Tx_n \| + \| Tx_n - Tx_0 \|$$

so

$$\| y_0 - Tx_0 \| \leq \| y_0 - y_n \| + \| T \| \| x_n - x_0 \| \to 0$$

as $n \to \infty$, and $y_0 = Tx_0 \in Im(T)$, that is $Im(T)$ is closed. □

Theorem 1.3.4 ([8])**.** *An operator T on a Hilbert space H is invertible if and only if the following (i) and (ii) hold:*
i) There exists a positive number c such that

$$\| Tx \| \geq c \| x \|$$

holds for any $x \in H$.
ii) $Im(T)$ the range of T is dense in H, that is $\overline{Im(T)} = H$.

1.3.1 Resolvent and spectrum of an operator

Definition 1.3.5. *Let $T \in B(H)$, we say that $\lambda \in \mathbb{C}$ belongs to the resolving set of T if $(T - \lambda I_H)$ is a bijection from H to H and that $(T - \lambda I_H)^{-1} \in B(H)$. The resolving set of T is noted by $\rho(T)$, and:*

$$\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I_H) \text{ is invertible } \}.$$

Definition 1.3.6. *Let $T \in B(H)$. The spectrum of T is the set of $\lambda \in \mathbb{C}$ such as $(T - \lambda I_H)$ is not invertible. The resolving set of T is noted by $\sigma(T)$, and:*

$$\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I_H) \text{ is not invertible } \}.$$

One can easily check that:

- $\sigma(T) \cup \rho(T) = \mathbb{C}$,
- $\sigma(T) \cap \rho(T) = \phi$.

Definition 1.3.7. *- The punctual spectrum of T is the set of eigenvalues of T and denoted by $\sigma_p(T)$ such that:*

$$\sigma_p(T) = \{\lambda \in \sigma(T), (T - \lambda I_H) \text{ is not injective } \},$$

- $\sigma_c(T)$ of T is defined as follows:

$$\sigma_c(T) = \{\lambda \in \sigma(T), (T - \lambda I_H) \text{ is injective and } Im(T - \lambda I_H) \neq \overline{Im(T - \lambda I_H)} = H\},$$

and $\sigma_c(T)$ is said to be the continuous spectrum of T.

- The residual spectrum $\sigma_r(T)$ of T is defined by:

$$\sigma_r(T) = \{\lambda \in \sigma(T), (T - \lambda I_H) \text{ is injective and } Im(T - \lambda I_H) \neq H\}.$$

- The set of approximate eigenvalues (which includes the point spectrum) is called the approximate point spectrum of T, denoted $\sigma_{ap}(T)$ and defined by:

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C}, (T - \lambda I_H) \text{ is not inferiorly bounded } \}$$

Remark 1.3.8. *The spectrum $\sigma(T)$ is the disjoint union of three sets:*

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

Example 1.3.9. *Let $T \in M_2(\mathbb{R})$ such as:*

$$T = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix},$$

we have:

$$\det(T - \lambda I_H) = \begin{vmatrix} -\lambda & 1 \\ 4 & \lambda \end{vmatrix},$$

and

$$\lambda^2 - 4 = (\lambda - 2)(\lambda + 2),$$

thus

$$\det(T - \lambda I_H) = 0 \Rightarrow \lambda = 2 \ or \ \lambda = -2,$$

so

$$\sigma(T) = \sigma_p(T) = \{2, -2\}.$$

Theorem 1.3.10 ([8])**.** *If T is a bounded operator, then $\sigma(T)$ is a compact subset of the complex plane. If $\lambda \in \sigma(T)$, then $|\lambda| \leq \| T \|$.*

1.4 Some classes of bounded linear operators

We start the section by giving definitions of some classes of bounded linear operators.

Definition 1.4.1. *Let $T \in B(H)$. Then,*

- *T is said to be self-adjoint if and only if $T^* = T$.*
- *T is said to be normal if and only if $T^*T = TT^*$.*
- *T is said to be quasinormal if and only if $T(T^*T) = (T^*T)T$.*
- *T is said to be projection if and only if $T^2 = T$ (idempotent) and $T^* = T$.*
- *T is said to be unitary if and only if $T^*T = TT^* = I$.*
- *T is said to be isometry operator if and only if $T^*T = I$.*

– *T is said to be positive (denoted by $T \geq 0$) if and only if $< Tx, x > \geq 0$ for all $x \in H$.*

– *T is said to be hyponormal if and only if $T^*T \geq TT^*$, where $A \geq B$ means $A - B \geq 0$ for self-adjoint operators A and B.*

Theorem 1.4.2 ([8])**.** *If T is an operator on a Hilbert space H over the complex scalars $\mathbb{C}$, then the following (i), (ii), (iii) and (iv) hold:*
(i) T is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in H$.
(ii) T is self-adjoint if and only if $< Tx, x >$ is real for all $x \in H$.
(iii) T is unitary if and only if $\|Tx\| = \|T^*x\| = \|x\|$ for all $x \in H$.
(iv) T is hyponormal if and only if $\|Tx\| \geq \|T^*x\|$ for all $x \in H$.

Proof. (i) Recall the following:

$$\|Tx\|^2 - \|T^*x\|^2 =< (T^*T - TT^*)x, x >,$$

for all $x \in H$.
If T is normal operator, then $\|Tx\| = \|T^*x\|$.
Conversely, assume $\|Tx\| = \|T^*x\|$ for all $x \in H$. Then $T^*T - TT^* = 0$, that is, T is a normal operator.
(ii) If T is self-adjoint operator, i.e., $T^* = T$; then the proof of the result follows by

$$< Tx, x >=< T^*x, x >=< x, Tx >= \overline{< Tx, x >}.$$

Conversely, assume $< Tx, x >$ is real for all $x \in H$. Then for all $x \in H$,

$$< Tx, x >= \overline{< Tx, x >} =< x, Tx >=< T^*x, x > .$$

Hence it follows that $(T - T^*) = 0$, that is, T is a self-adjoint operator.
(iii) If T is a unitary operator, i.e., $T^*T = TT^* = I$, then the proof of the result follows by

$$\|Tx\|^2 =< T^*Tx, x >=< TT^*x, x >= \|T^*x\|^2 = \|x\|^2.$$

Conversely, assume $\|Tx\| = \|T^*x\| = \|x\|$ for all $x \in H$. Then for all $x \in H$,

$$\|Tx\|^2 = \|T^*x\|^2 = \|x\|^2 \Longleftrightarrow < T^*Tx, x > = < TT^*x, x > = < x, x >$$

so

$$< (T^*T - I)x, x > = 0 \text{ and } < (TT^* - I)x, x > = 0.$$

Hence, $T^*T - I = 0$ and $TT^* - I = 0$ by Theorem 3, that is, T is unitary operator.
(iv) The proof easily follows by (1). □

Theorem 1.4.3 ([8]). *An operator $T \in B(H)$ is normal if and only if*

$$\|Tx\| = \|T^*x\|,$$

for every $x \in H$. Normal operators T have the following properties:
(a) $\ker(T) = \ker(T^*)$.
(b) $Im(T)$ is dense in H if and only if T is one-to-one.
(c) T is invertible if and only if there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for every $x \in H$.
(d) If $Tx = \alpha x$ for some $x \in H$, $\alpha \in \mathbb{C}$, then $T^*x = \overline{\alpha}x$.

Theorem 1.4.4 ([7]). *If T is a self-adjoint operator on a Hilbert space H, then all the eigenvalues of T are real numbers.*

Proof. If $Tx = \lambda x$ holds, then $\overline{\lambda} = \lambda$ as follows:

$$\lambda < x, x > = < \lambda x, x > = < Tx, x > = < x, Tx > = < x, \lambda x > = \overline{\lambda} < x, x >.$$

□

Theorem 1.4.5. *If T is a self-adjoint operator on a Hilbert space H, then $(T + iI)$ has a bounded inverse operator.*

Theorem 1.4.6. *If T is normal operator then $\sigma_{ap}(T) = \sigma(T)$.*

Theorem 1.4.4 can be generalized to the forthcoming theorem.

Theorem 1.4.7. *If an operator T is self-adjoint, then $\sigma(T)$ is a subset of the real line.*

Proof. If λ is not a real number, then for all non-zero vector x,

$$\begin{aligned} 0 < |\lambda - \overline{\lambda}| \|x\|^2 &= |< (T-\lambda)x, x > - < (T-\overline{\lambda})x, x > | \\ &= |< (T-\lambda)x, x > - < x, (T-\lambda)x > | \quad (\text{since } T^* = T) \\ &\leq 2\|Tx - \lambda x\| \|x\| \qquad (\text{by Schwarz inequality}). \end{aligned}$$

Therefor $\lambda \notin \sigma_{ap}$ and the proof is complete since $\sigma(T) = \sigma_{ap}(T)$ holds for a self-adjoint operator T by the previous theorem.

□

Theorem 1.4.8. *Let T be a normal operator, such that $Tx = \lambda x$ and $Ty = \mu y$, where $\lambda \neq \mu$. Then $< x, y >= 0$.*

Proof. Recall that $Ty = \mu y \Longleftrightarrow T^*y = \overline{\mu}y$ since $\|Ty - \mu y\| = \|T^*y - \overline{\mu}y\|$ by the normality of T. By an easy calculation,

$$\lambda < x, y >=< \lambda x, y >=< Tx, y >=< x, T^*y >=< x, \overline{\mu}y >= \mu < x, y >$$

so that $< x, y >= 0$ whenever $\lambda \neq \mu$. □

Chapter 2

Characterization of posinormal operators

In this chapter we study the properties of posinormal operators. We give a characterization of this class of bounded linear operators, also we present the spectral proprieties of posinormal operators.

2.1 Definitions and Examples

Definition 2.1.1. *If $T \in B(H)$, then T is posinormal if there exists a positive operator $P \in B(H)$ such that $TT^* = T^*PT$.*

Remark 2.1.2. - *$P(H)$ will denote the set of all posinormal operators on H. T is coposinormal if T^* is posinormal.*

- We note that if T is posinormal with interrupter P and V is an isometry (that is, $V^*V = I$), then, as one can easily check, VTV^* is posinormal with interrupter VPV^*, consequently, posinormality is a unitary invariant (that is, if T is posinormal and A is unitarily equivalent to T, then A is also posinormal).

If the posinormal operator T is non zero, the associated interrupter P must satisfy the condition $\|P\| \geq 1$.

Since

$$\|T\|^2 = \|TT^*\| = \|T^*PT\| \leq \|T^*\|\|P\|\|T\| = \|P\|\|T\|^2.$$

So

$$\|P\| \geq 1.$$

We will make repeated use $\sqrt{P}$ whose existence is guaranteed by the functional calculus for (positive) self-adjoint operators.

P need not be unique, as we will soon see; the following result gives a sufficient condition for the uniqueness of P.

The next Theorem was mentioned in [13].

Theorem 2.1.3. *If T is posinormal with interrupter P and T has dense range, then P is unique.*

Proof. Assume P_1 and P_2 both serve as interrupters for T, then

$$T^*P_1T = TT^* = T^*P_2T,$$

so

$$T^*(P_1 - P_2)T = 0.$$

Since T has dense range, T^* is one to one and, consequently, $(P_1 - P_2)T = 0$. We again apply the fact that T has dense range to conclude that $P_1 - P_2 = 0$. □

Corollary 2.1.4. *If T has dense range and S serves as an interrupter for T, then T is posinormal and the interrupter S is positive and unique.*

Example 2.1.5. *Let $H = \mathbb{R}^2$ and let T be defined on H as*

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Then $T^*T \neq TT^*$. Hence T is not normal. On the other hand, consider a positive operator

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Then $TT^* = T^*PT$. Hence T is posinormal.

From example 2.1.5 we conclude that posinormal operator on a finite dimensional Hilbert space are not necessary normal.

Example 2.1.6. *For an example of an operator that is not posinormal, we consider U^*, the adjoint of the unilateral shift $U : l^2 \longrightarrow l^2$; recall that U has matrix entries*

$$u_{jk} = \begin{cases} 0 & \text{if} \quad j \neq k+1 \\ 1 & \text{if} \quad j = k+1. \end{cases}$$

Here $T = U^*$ cannot be posinormal since $TT^* = I$ while $T^*PT \neq I$ for all P (the trouble comes in the northwest corner: $0 \neq 1$).

If $T \in B(H)$ is hyponormal and quasinilpotent (i.e: $\sigma(T) = \{0\}$), then T is a zero operator. But it is not true for the case of posinormal operator.

Example 2.1.7. *The example which motivated this study is the Cesàro matrix:*

$$C = \begin{pmatrix} 1 & 0 & 0 & .... \\ \frac{1}{2} & \frac{1}{2} & 0 & .... \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & .... \\ . & . & . & .... \\ . & . & . & .... \\ . & . & . & .... \end{pmatrix}$$

regarded as an operator on $H = l^2$. The standard orthonormal basis for l^2 will be denoted by $\{e_n : n = 0, 1, 2, ...\}$.
If D is the diagonal operator with diagonal $\{\frac{n+1}{n+2} : n = 0, 1, 2, ...\}$, then a routine computation verifies that

$$C^*DC = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & .... \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & .... \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & .... \\ . & . & . & .... \\ . & . & . & .... \\ . & . & . & .... \end{pmatrix} = CC^*.$$

So the Cesàro operator (on l^2) is posinormal with interrupter D. C is known to be hyponormal, even subnormal, C is shown to be hyponormal by looking at determinants of finite sections of $[C^*, C] = (C^*C - CC^*)$. We include here a brief and different proof-one that takes advantage of the availability of D.

Theorem 2.1.8. *C is hyponormal.*

Proof. Since $(I - D)$ is a positive operator, we have

$$< [C^*, C]x, x >=< (I - D)Cx, Cx >\geq 0, \qquad \text{for all } x.$$

□

We have, in the Cesàro operator, an example of a nonnormal posinormal operator. The next proposition provides us with a large supply of additional example, including the unilateral shift U.

Proposition 2.1.9. *Every unilateral weighted shift with non zero weights is posinormal.*

Proof. In matrix form, the weighted shift $T = [a_{jk}]$ with non zero weights w_k has entries.

$$a_{jk} = \begin{cases} 0 & \text{if} \quad j \neq k+1 \\ w_k & \text{if} \quad j = k+1. \end{cases}$$

Take P to be the diagonal matrix with diagonal entries $P_{00} \geq 0$, $P_{11} = 0$, and
$P_{kk} =\mid \dfrac{w_{k-2}}{w_{k-1}} \mid^2$ for $k \geq 2$.
It is easy to verify that $TT^* = T^*PT$, as required.
(Note: The freedom possible here for P_{00} illustrates the nonuniqueness of P when T does not have dense range.) □

It is easy to see that if T is the unilateral weighted shift with weights w_k, then $[T^*, T]$ is the diagonal matrix with diagonal entries $\{w_0^2, w_1^2 - w_0^2, w_2^2 - w_1^2, ...\}$.
If $\{w_k\}$ is increasing, then T is hyponormal. The special case when $w_0 = 2$ and $w_k = 1$ for all $k \geq 1$ provides an examples of a posinormal operator that is neither hyponormal nor cohyponormal.

2.2 Some properties of posinormal operators

We have just seen that posinormality does not imply hyponormality, but our experience with the Cesàro matrix and the unilateral shift suggests the plausibility of the reverse implication.

Theorem 2.2.1 (Douglas [5]). *For $T_1, T_2 \in B(H)$ the following statements are equivalent:*
1) $Im(T_1) \subseteq Im(T_2)$
2) $T_1T_1^* \leq \lambda^2 T_2T_2^*$ for some $\lambda \geq 0$; and
3) There exists a $T_3 \in B(H)$ such that $T_1 = T_2T_3$.
Moreover, if (1), (2), and (3) hold, then there is a unique operator T_3 such that
(a) $\|T_3\|^2 = inf\{u| \ T_1T_1^* \leq uT_2T_2^*\}$
(b) $\ker(T_1) = \ ker(T_3)$; and
(c) $Im(T_3) \subseteq \overline{Im(T_2^*)}$

The next result is an indication of the somewhat limited extent to which posinormal operators display behavior associated with hyponormal operators.
Recall that a hyponormal operator T must satisfy the inequality $\|T^*x\| \leq \|Tx\|$ for all x. Statement (a) of the following proposition give us an analogous result for posinormal operators.

Proposition 2.2.2. *If T is posinormal with (positive) interrupter P, then the following statements hold:*
(a) $\|T^*x\| = \|\sqrt{P}Tx\| \leq \|\sqrt{P}\|\|Tx\|$ for every x in H.
(b) $\|\sqrt{P}T\| = \|T\|$.

Proof. (a) Since T is posinormal and P is positive, then

$$\|T^*x\|^2 =< TT^*x, x >=< T^*PTx, x >= \|\sqrt{P}Tx\|^2 \leq \|\sqrt{P}\|^2\|Tx\|^2,$$

for all x in H.
(b) From (a) we see that

$$\|T^*\| = \|\sqrt{P}T\|$$

and $\|T\| = \|T^*\|$ is universal. □

We note that if T is posinormal, then condition (2) in theorem 2.2.1 is satisfied with

$$\lambda = \|\sqrt{P}\| \quad and \quad T = T^*.$$

If condition (3) in theorem 2.2.1 holds, then there is an operator $S \in B(H)$ such that $T = T^*S$, so $T^* = S^*T$, consequently; T is posinormal with interrupter TT^*.
Thus Douglas theorem has led almost immediately to the following result.

Theorem 2.2.3 ([13]). *For $T \in B(H)$ the following statements are equivalent:*
1) T is posinormal.
2) $Im(T) \subseteq Im(T^*)$
3) $TT^* \leq \lambda^2 T^*T$ for some $\lambda \geq 0$; and
4) there exists a $S \in B(H)$ such that $T = T^*S$.
Moreover, if (1),(2),(3) and (4) hold, then there is a unique operator S such that
(a) $\|S\|^2 = inf\{u \backslash \ TT^* \leq uT^*T\}$
(b) $\ker(T) = \ker(S)$; and
(c) $Im(S) \subseteq \overline{(ImT)}$

Corollary 2.2.4. *Every hyponormal operator is posinormal.*

Proof. If T is hyponormal, then condition (3) of theorem 2.2.3 is satisfied with $\lambda = 1$. □

Let $[T] = \{ST : S \in B(H)\}$, the left ideal in $B(H)$ generated by T. If T is posinormal, then, because of (4) of theorem 2.2.3, we have

$$T^* = S^*T$$

for some bounded operator S. So $T^* \in [T]$.
Conversely, if $T^* \in [T]$
then $T^* = KT$
for some $K \in B(H)$, so T is posinormal with interrupter $P = K^*K$ in summary.

Corollary 2.2.5. *T is posinormal if and only if $T^* \in [T]$.*

Corollary 2.2.6. *If T is posinormal, then $\ker(T) \subseteq \ker(T^*)$; in particular, $\ker(T)$ is a reducing subspace for the posinormal operator T.*

In example 2.1.6, if we did not already know that U^* fails to be posinormal, we would know now, for $\ker(U^*) \neq \{0\}$ while $\ker(U) = \{0\}$.

In fact, the adjoint of any unilateral weighted shift with non zero weights will fail to be posinormal.

It is trivial that the posinormal is invariant under unitary equivalence. But the similarity does not preserve the posinormal.

Example 2.2.7. *Let $H = \mathbb{R}^3$ and let A and T be defined on H as*

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad T = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix},$$

then A is posinormal, and if we take

$$X = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

then

$$T = XAX^{-1}.$$

Hence T is similar to A. Now

$$\ker(T) = \{x = (x_1, x_2, x_3) \in H : x_1 = x_2 + x_3\},$$

and

$$\ker(T^*) = \{x = (x_1, x_2, x_3) \in H : x_2 = -x_3\}.$$

Hence

$$\ker(T) \nsubseteq \ker(T^*).$$

Thus T is not posinormal.

Corollary 2.2.8. *In order for a cohyponormal operator T to be posinormal it is necessary that* $\ker(T) = \ker(T^*)$*.*

Proposition 2.2.9 ([9])**.** *If $T \in B(H)$ is posinormal, then* $\ker(T) = \ker(T^2)$*.*

Proof. It suffices to show that $\ker(T^2) \subset \ker(T)$.
If $x \in \ker(T^2)$, then

$$T^2x = 0.$$

Hence

$$Tx \in \ker(T).$$

Since

$$ker(T) \subset \ker(T^*),$$

so

$$Tx \in \ker(T^*),$$

hence

$$T^*Tx = 0.$$

Now

$$\|Tx\|^2 =< Tx, Tx >=< T^*Tx, x >\leqslant \|T^*Tx\|\|x\| = 0.$$

Thus $Tx = 0$, and so we have $x \in \ker(T)$. □

Theorem 2.2.10 ([13])**.** *Assume T is posinormal with interrupter P, then*
(a) T is hyponormal if and only if the restriction of $(I-P)$ to $Im(T)$ is a positive operator.
(b) If $\| P \|= 1$ then T is hyponormal.

Proof. (a) The assertion follows immediately from the fact that

$$< [T^*, T]x, x >=< (I - P)Tx, Tx >,$$

for all x in H.
(b) If $\|P\| = 1$, then

$$\|\sqrt{P}\| = 1$$

also, so

$$\|T^*x\| \leq \|\sqrt{P}\|\|Tx\| = \|Tx\|$$

one of the equivalent conditions for the hyponormality of T. □

Corollary 2.2.11. *If T hyponormal and has dense range, then the unique interrupter P associated with T must satisfy $\| P \|= 1$.*

Proof. Since T is hyponormal and the range of T is dense, we conclude from theorem 2.2.10 (a) that $(I - P)$ is a positive operator (P is unique by theorem 2.1.3).
It follows that

$$\|P\| \leq \|I\| = 1,$$

and since

$$\|P\| \geq 1,$$

is universal for nonzero T, the proof is complete. □

Theorem 2.2.12. *Assume T is posinormal with interrupter P. if $I \geqslant P$ (that is $(I - P)$ is a positive operator) and Q is a positive operator satisfying $I \geqslant Q \geqslant P$, then:*
(a) The operator $Z = \sqrt{Q}T\sqrt{Q}$ is hyponormal .
(b) $T^*QT - TQT^*$ is a positive operator.

Proof. (a) we have :

$$[Z^*, Z] = \sqrt{Q}T^*QT\sqrt{Q} - \sqrt{Q}T^*PT\sqrt{Q} + \sqrt{Q}T^*PT\sqrt{Q} - \sqrt{Q}TQT^*\sqrt{Q}.$$

Hence

$$[Z^*, Z] = \sqrt{Q}T^*(Q - P)T\sqrt{Q} + \sqrt{Q}T(I - Q)T^*\sqrt{Q}.$$

Therefore

$$< [Z^*, Z]x, x >=< (Q - P)T\sqrt{Q}x, T\sqrt{Q}x > + < (I - Q)T^*\sqrt{Q}x, T^*\sqrt{Q}x >\geq 0$$

for all x, as needed. The proof of (b) is similar. □

Corollary 2.2.13. *If T is posinormal with interrupter P and $I \geq P$, then T is hyponormal.*

2.3 Invertibility, translates, and posispectrum

We start this section by looking at the relationship between invertibility and posinormality. A posinormal operator need not be invertible (example: the unilateral shift), but the following theorem tells us that an invertible operator must be posinormal.

Theorem 2.3.1. *Every invertible operator is posinormal.*

Proof. If T is invertible, then

$$T^* = T^*(T^{-1}T) = (T^*T^{-1})T,$$

so $T^* \in [T]$. □

Corollary 2.3.2. *Every invertible operator is coposinormal.*

Corollary 2.3.3. *Assume $T \in B(H)$ and $\lambda \notin \sigma(T)$, the spectrum of T. Then $(T - \lambda I)$ is posinormal.*

Brown introduced a class of operators T satisfying the condition that T^*T commutes with T. These operators have since been referred to as quasinormal. Routine computation indicate that if T is quasinormal and $\lambda \neq 0$, then λT is quasinormal but the translate $(T + \lambda I)$ can be quasinormal only if T is normal.
As noted previously, the result is different for T hyponormal; in that case both λT and $(T + \lambda I)$ are also hyponormal.
The following theorem considers the same questions for posinormal operators.

Theorem 2.3.4 ([13])**.** *Assume T is posinormal which interrupter P and $\lambda \neq 0$.*
(a) Then λT is posinormal (which interrupter P).
(b) The translate $(T + \lambda I)$ need not be posinormal.

Proof. (a) Let T be a posinormal operator with interrupter P, then

$$(\lambda T)(\lambda T)^* = |\lambda|^2 TT^* = |\lambda|^2 T^* PT = (\lambda T)^* P(\lambda T).$$

(b) Consider the case where

$$T = U^* - 2I$$

and $\lambda = 2$ (recall that U^* is the adjoint of the unilateral shift). Since 2 is not in $\sigma(U^*)$, T is posinormal. But

$$(T + 2I) = U^*,$$

is not posinormal. □

Definition 2.3.5. *For $T \in B(H)$ the posispectrum of T, denoted $p(T)$, is the set*

$$p(T) = \{\lambda : (T - \lambda I) \text{ is not posinormal}\}.$$

Corollary 2.3.3 makes it clear that $p(T)$ is a subset of $\sigma(T)$.

Proposition 2.3.6. *If T is hyponormal, then $p(T) = \phi$.*

Proof. Since translates of a hyponormal operator are hyponormal, $(T - \lambda I)$ is hyponormal and hence posinormal for every λ. □

Definition 2.3.7. *We say that an operator T is totally posinormal if the translates $(T + \lambda I)$ are posinormal for all λ.*

Definition 2.3.8. *An operator $T \in B(H)$ is dominant if and only if*

$$Im(T - \lambda I) \subseteq Im(T - \lambda I)^*$$

for all $\lambda \in \sigma(T)$

As a consequence of Theorem 2.2.3, we have the following result:

Proposition 2.3.9. *T is totally posinormal if and only if T is dominant.*

As a consequence of theorem 2.3.1 and proposition 2.3.9 we have the following :

$$\{ \text{ dominant operators } \} \cup \{ \text{ invertible operators } \} \subseteq \{ \text{ posinormal operators } \}.$$

We have seen that posinormality is not preserved under the taking of adjoints. The next theorem is a modest result in the same direction.

Theorem 2.3.10. *If T is posinormal with an invertible interrupter P, then*

$$B = \sqrt{P}T^*\sqrt{P}$$

is posinormal which interrupter P^{-1}.

Theorem 2.3.11. *Assume T is invertible. If P serves as the interrupter for the posinormal operator T^*, then:*
(1) P is invertible and
(2) P^{-1} serves as the interrupter for the posinormal operator T^{-1}.

Since $(T - \lambda I)$ is posinormal for "many" values of λ, to what extent does the interrupter depend on λ? As we see here, if T is to be nonnormal, the dependence of P on λ is rather severe.

Theorem 2.3.12. *Assume $(T - \lambda I)$ is posinormal for four distinct complex values $\lambda = 0$, λ_1, λ_2 and λ_3 where*

$$\lambda_3 = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2},$$

and assume that the same positive operator P functions as an interrupter for $(T - \lambda I)$ in each of those four cases. Then T is normal.

Proof. Since

$$(T - \lambda)(T - \lambda)^* = (T - \lambda)^* P (T - \lambda),$$

for $\lambda = 0$, λ_1, λ_2 and λ_3, we find that, for $k = 1, 2,$ and 3

$$(T - \lambda_k)(T - \lambda_k)^* = (T - \lambda_k)^* P (T - \lambda_k),$$

reduces to the equation

$$\frac{1}{2}(I - P) = Re\big[\frac{1}{\lambda_k}(I - P)T\big].$$

Therefore

$$Re\big[\frac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2}(I - P)T\big] = Re\big[\frac{1}{\lambda_2}(I - P)T\big] - Re\big[\frac{1}{\lambda_1}(I - P)T\big] = 0,$$

from which it follows that

$$\frac{1}{2}(I-P) = Re\big[(\frac{1}{\lambda_3}(I-P)T\big] = 0,$$

and hence T is normal. □

Corollary 2.3.13. *Assume $(T-\lambda I)$ is posinormal for three distinct real values of λ and that the same positive operator P functions as an interrupter for $(T-\lambda I)$ for each of those three values. Then T is normal.*

Chapter 3

Powers of posinormal operators

In this chapter we introduce a new class of operators which is p-posinormal operators. We also prove that if T is p-posinormal then T^n is $\frac{p}{n}$-posinormal for all positive integer n.

3.1 p-Posinormal operators

Definition 3.1.1. *An operator T is said to be a p-hyponormal if*

$$(T^*T)^p \geq (TT^*)^p.$$

Where $p > 0$.

It's clear that p-hyponormal operators are hyponormal if $p > 1$.

Definition 3.1.2. *An operator $T \in B(H)$ is said to be p-posinormal operators if*

$$(TT^*)^p \leq \lambda^2 (T^*T)^p$$

for some positive numbers λ and p.

We denote the set off all p-posinormal operators by p-P(H). By Rhaly's characterization of posinormality, according to Theorem 2.2.3 we can see that 1-posinormal operators are posinormal.

Definition 3.1.3. *An operator $T \in B(H)$ is said to be M-paranormal if there exists $\lambda > 0$ such that*

$$\|Tx\|^2 \leq \lambda \|T^2 x\|,$$

for $x \in H$ with $\|x\| = 1$.

Let call P an interrupter of T with degree p if

$$|T^*|^{2p} = |T|P|T|^p.$$

Theorem 3.1.4 (Fujii, Nakamoto and Watanabe [6])**.** *Let $A \geq 0$ and $B \geq 0$. If T satisfies*

$$T^*T \leq A^2,$$

and

$$TT^* \leq B^2,$$

then inequality

$$\big| < T|T|^{p+q-1}x, y > \big| \leq \|A^p x\| \|B^q y\|$$

holds for all $x, y \in H$, $0 \leq p,\ q \leq 1$ with $p + q \geq 1$.

According to the previous Theorem, we have the following lemma:

Lemma 3.1.5. *If $T \in P(H)$, then there exists $\lambda > 0$ such that*

$$\big| < T|T|^{p+q-1}x, y > \big| \leq \big\| |T|^P x \big\| \big\| \lambda^q |T|^q y \big\|,$$

for all $x, y \in H$, $0 \leq p, q \leq 1$ with $p + q \geq 1$.

Proof. In Theorem 3.1.4, put $A = |T|$ and $B = \lambda |T|$.
Since

$$T^*T = |T|^2,$$

and

$$T^*T \leq \lambda^2|T|^2 = B^2,$$

by Theorem 2.2.3 we have:

$$\big| < T|T|^{p+q-1}x, y > \big| \leq \big\||T|^p x\big\|\big\|\lambda^q|T|^q y\big\|$$

for all $x, y \in H$, $0 \leq p, q \leq 1$ with $p + q \geq 1$. □

The following theorem is another characterization of posinormality, which is different from Rhaley's one.

Theorem 3.1.6 ([10])**.** *T is posinormal if and only if, there exists $\lambda > 0$ such that*

$$\big| < T|T|x, y > \big| \leq \lambda\big\||T|x\big\|\big\||T|y\big\|,$$

for all $x, y \in H$

Proof. Assume that T is posinormal. By Lemma 3.1.5 we have :

$$\big| < T|T|^{p+q-1}x, y > \big| \leq \lambda^p\big\||T|^p x\big\|\big\||T|^q y\big\|$$

for all $x, y \in H$, $0 \leq p, q \leq 1$ with $p + q \geq 1$.
Letting $p = q = 1$, we have

$$\big| < T|T|x, y > \big| \leq \lambda\big\||T|x\big\|\big\||T|y\big\|.$$

Conversely, assume that

$$\big| < T|T|x, y > \big| \leq \lambda\big\||T|x\big\|\big\||T|y\big\|,$$

holds. Let $T = U|T|$ be the polar decomposition of T.
For any $y \in H$, we put $x = U^*y$. Then

$$\big| < U|T|^2U^*y, y > \big| \leq \lambda\big\||T|U^*y\big\|\big\||T|y\big\|.$$

Since

$$< U|T|^2U^*y, y >= \big(T^*y, T^*y\big) = \big\|T^*y\big\|^2,$$

and

$$\left\||T|y\right\| = \left\|Ty\right\|,$$

we have

$$\left\|T^*y\right\|^2 \leq \lambda\left\|T^*y\right\|\left\|Ty\right\|.$$

Hence

$$\left\|T^*y\right\|^2 \leq \lambda^2\left\|Ty\right\|^2$$

that is

$$TT^* \leq \lambda^2 T^*T.$$

By theorem 2.2.3, we have $T \in P(H)$ □

Proposition 3.1.7. *If T is posinormal, then T is M-paranormal.*

Proof. By the hypothesis, there exists a positive number λ such that

$$\|T^*x\| \leq \lambda\|Tx\|,$$

for all $x \in H$ with $\|x\| = 1$. Hence

$$\|Tx\|^2 =< T^*Tx, x >\leq \|T^*Tx\|\|x\| = \|T^*Tx\| \leq \lambda\|T^2x\|$$

that is T is M-paranormal. □

By the definitions of p-hyponormal and p-posinormal operators, we can easily have the following results.

Proposition 3.1.8 ([10])**.** *1) If T is posinormal, then $T \in$ p-$P(H)$ for every p $(0 < p \leq 1)$*
2) If T is p-hyponormal, then $T \in$ p-$P(H)$
3) $T \in$ p-$P(H)$ if and only if there exists $\lambda > 0$ such that

$$\left\||T^*|^p x\right\| \leq \lambda\left\||T|^p x\right\|,$$

for all $x \in H$

Proposition 3.1.9. *For $T \in B(H)$, the following statements are equivalent:*

1) $T \in p$-$P(H)$.

2) $Im\big(|T^*|^P\big) \subset Im\big(|T|^p\big)$.

3) There exists $S \in B(H)$ such that $|T^*|^P = |T|^p S$.

4) There exists a positive operator such that $|T^*|^{2p} = |T|^p P |T|^p$.

Theorem 3.1.10. *[1] Let $T = U|T| \in B(H)$ be the polar decomposition of T. Then $T \in p$-$P(H)$ if and only if, there exists a positive number λ such that*

$$\big| < U|T|^{2p}x, y > \big| \leq \lambda \big\||T|^p x\big\| \big\||T|^p y\big\|,$$

for all $x, y \in H$

Proof. Suppose that

$$\big| < U|T|^{2p}x, y > \big| \leq \lambda \big\||T|^p x\big\| \big\||T|^p y\big\|,$$

holds. For any $y \in H$, put $x = U^*y$.Then

$$\big| < U|T|^{2p}U^*y, y > \big| \leq \lambda \big\||T|^p U^*y\big\| \big\||T|^p y\big\|,$$

since

$$\big\|T|^p U^*y\big\|^2 = < U|T|^{2p}U^*y, y > = < |T^*|^{2p}y, y > = \big\||T^*|^p y\big\|^2,$$

that is $T \in p$-$P(H)$.

Next, suppose $T \in p$-$P(H)$. Then, by (3) of proposition 3.1.8, we have

$$\big| < U|T|^{2p}x, y > \big| = \big| < |T|^{2p}x, U^*y > \big| \leq \big\||T|^p x\big\| \big\||T|^p U^*y\big\|$$

$$= \big\||T|^p x\big\| \big\||T^*|^p y\big\| \leq \lambda \big\||T|^p x\big\| \big\||T|^p y\big\|.$$

Hence (2) holds. This completes the proof. □

For the proof of Theorem 3.1.12, we need the following theorem.

Theorem 3.1.11. *Let $A \geq 0$. Then for all $x \in H$,*

1) $< Ax, x >^r \leq \|x\|^{2(r-1)} < A^r x, x >$ if $1 \leq r$; and

2) $< Ax, x >^r \geq \|x\|^{2(r-1)} < A^r x, x >$ if $0 \leq r \leq 1$.

Theorem 3.1.12 ([10])**.** *If $T \in p$-$P(H)$, then T is M-paranormal.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Since $T \in p$-$P(H)$, it is clear that

$$U|T|^{2p}U^* \leq \lambda^2|T|^{2p},$$

and

$$|T|^{2p} \leq \lambda^2 U^*|T|^{2p}U$$

Hence, by Theorem 3.1.11, for all $x \in H$ with $\|x\| = 1$, we have:

$$\begin{aligned}
\lambda^2\|T^2x\|^2 &= \lambda^2 < T^*TTx, Tx > \\
&= \lambda^2\langle(|T|^{2p})^{\frac{1}{r}}\frac{Tx}{\|Tx\|}, \frac{Tx}{\|Tx\|}\rangle \\
&\geq \lambda^2\frac{\langle|T|^{2p}Tx, Tx\rangle^{\frac{1}{p}}\|Tx\|^2}{\|Tx\|^{\frac{2}{p}}} \\
&= \lambda^2\frac{\langle U^*|T|^{2p}U|T|x, |T|x\rangle^{\frac{1}{p}}\|Tx\|^2}{\|Tx\|^{\frac{2}{p}}} \\
&\geq \frac{\langle|T|^{2p+2}x, x\rangle^{\frac{1}{p}}\|Tx\|^2}{\|Tx\|^{\frac{2}{p}}} \\
&= \frac{\langle(|T|^2)^{p+1}x, x\rangle^{\frac{1}{p}}\|Tx\|^2}{\|Tx\|^{\frac{2}{p}}} \\
&\geq \frac{\langle|T|^2x, x\rangle^{\frac{1}{p+1}}\|Tx\|^2}{\|Tx\|^{\frac{2}{p}}} \\
&= \|Tx\|^4.
\end{aligned}$$

Therefore $\|Tx\|^2 \leq \lambda\|T^2x\|$, that is T is M-paranormal.

□

Proposition 3.1.13. *Let $T = U|T|$ be p-posinormal operator for $0 < p < 1$. Then*
1) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is $\left(p + \frac{1}{2}\right)$-posinormal for $0 < p < \frac{1}{2}$.
2) $\tilde{T}$ is posinormal for $\frac{1}{2} \leq p < 1$

Theorem 3.1.14 ([11])**.** *Let T be a p-posinormal for some $0 < p < 1$, that is*

$$(TT^*)^p \leq \mu^2(T^*T)^p.$$

Then

1. $\left(T^{n*}T^{n}\right)^{\frac{p+1}{n}} \geq \mu^{\frac{-p+1}{p}(n-1)}\left(T^{*}T\right)^{p+1}$,

2. $\left(TT^{*}\right)^{p+1} \geq \mu^{\frac{-p+1}{p}(n-1)}\left(T^{n}T^{n*}\right)^{\frac{p+1}{n}}$,

hold for all positive integer n.

From Theorem 3.1.14, we have the next result.

Corollary 3.1.15. *If T is p-posinormal, then T^n is $\frac{p}{n}$-posinormal for all positive integer n.*

Proof. Let $\left(TT^{*}\right)^{p} \leq \mu^{2}\left(T^{*}T\right)^{p}$. Then, by Theorem 3.1.14,

$$\left(T^{n*}T^{n}\right)^{\frac{p}{n}} \geq \mu^{\frac{p}{p+1}}\left(T^{*}T\right)^{p} \geq \mu^{\frac{p}{p+1}}\mu^{2}\left(TT^{*}\right)^{p} \geq \mu^{-2n}\left(T^{n}T^{n*}\right)^{\frac{p}{n}}.$$

So, T^n is $\frac{p}{n}$-posinormal. □

Conclusion

In this work, we have studied the spectral proprieties of posinormal operators. We have given a several proprieties for this class of bounded linear operator, for example, we have proved that every bounded normal, hyponormal, or even invertible operators are posinormal. We have also provided this work with a several examples to show that the class of bounded posinormal operators is different from the other classes.

Moreover, we have shown the relation between a bounded posinormal operator and its powers, and we have showed that if T is p-posinormal operator then T^n is $\frac{p}{n}$-posinormal for all integer n.

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