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Théme

Crossing limit cycles for some classes of piecewise linear systems

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Introduction

The study of piecewise linear differential systems is relatively recent. Such that the dynamics of the piecewise linear differential systems started to be studied around 1930, mainly in the book of Andronov et al [1], which is russian version. The contribution of Andronov, Vitt and Khaikin [1] provided the basis for the development of the theory for this system. Many researchers from different fields interested this kind of differential systems, one of the reasons for this interest in the mathematical community is that these systems are widely used to model phenomena appearing in mechanics, electronics, economy, neuroscience,..., and it can be used to model applied problems, such as electronic circuits, biological systems, mechanical devices, etc, see for instance the book [6].

A limit cycle is a periodic orbit of a differential system in $\mathbb{R}^2$ isolated in the set of all periodic orbits of that system. The study of the limit cycles goes back essentially to Poincaré [20] at the end of the 19th century. One of the main problems in the dynamics of the differential systems in the plane is to control the existence and the number of their limit cycles. This problem restricted to polynomial differential systems is the famous 16th Hilbert's problem. see more in [12, 14, 13]. The existence of limit cycles became important in the applications to the real world, because many phenomena are related with their existence,see for instance the van der Pol oscillator [22].

Thus in recent years, the theory of piecewise linear differential systems has been increasingly developed and studied in order to understand the dynamics that such systems may have. In this sense one of the points of greatest interest is to obtain a lower bound for the maximum number of limit cycles that may arise around a single equilibrium point on the discontinuity set (i.e., on the region separating the linear differential systems).

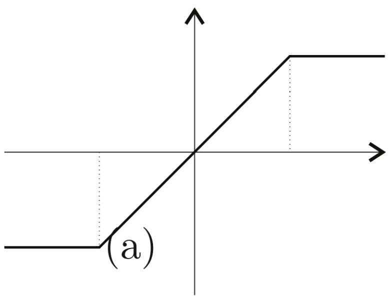
This investigation started with the simplest possible case: the continuous piecewise linear differential systems with two zones separated by a staight line. Lum and Chua [18, 19] in 1991 conjectured that such differential systems have at most one limit cycle. Later this conjecture was proved by Freire, Ponce, Rodrigo and Torres [9] in 1998.
while for the planar discontinuous piecewise linear differential systems in $\mathbb{R}^2$. Of course, the simplest piecewise linear differential systems in $\mathbb{R}^2$ are the ones having

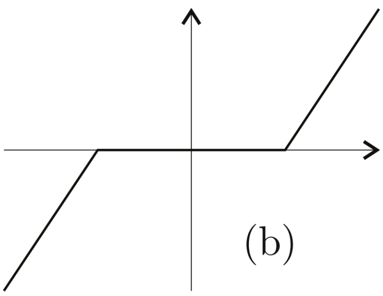
only two pieces separated by a curve, and when this curve is a straight line. Han and Zhang [11] obtained differential systems having two limit cycles and conjured that the maximum number of limit cycles of such class of differential systems is two.

But in this last years many authors have studied the limit cycles of discontinuous piecewise linear differential systems in $\mathbb{R}^2$. Thus the limit cycles of these last class of discontinuous piecewise linear differential systems has been intensively studied, see [2, 3, 7]. Up to know the results of all these papers only provide examples that the discontinuous piecewise linear differential systems in $\mathbb{R}^2$ separated by a straight line can have 3 crossing limit cycles.
we consider piecewise linear systems on the form

$$\dot{x} = A_i x + a_i \qquad for \quad x \in X_i \ .$$

Here X_i is a partition of the state space into operating regimes. The dynamics in each region is described by a system of linear differential equations. Piecewise linear systems have a wide applicability in a range of engineering sciences. Many publications on piecewise linear differential systems come from applications, as for instance control theory and electric circuits design. Non-linearities that appear in real dynamical systems are very often modeled by smooth functions. Hence, results and tools from smooth dynamics and local bifurcation theory can be fruitfully applied. But, in some cases, considering piecewise linear functions is an alternative that fits better, qualitatively and quantitatively, the experiment [1],[10]. Standard piecewise linear functions are: saturation, to model amplifiers and motors, see Figure (a); dead zone, to model valves and motors, see Figure (b); friction, to model the static friction of motors, see Figure(c); and sign, to model relays, see Figure (d).





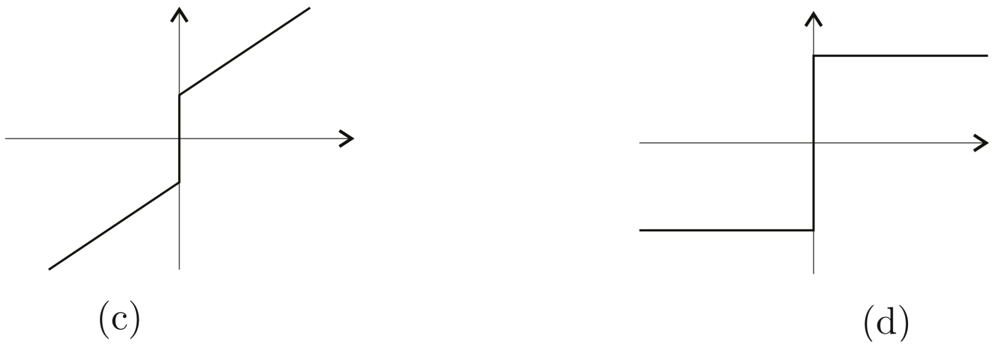


Figure 1: piecewise linear functions (a)saturaction ;(b)dead zone (c)friction;(d)sign.

In the following example, we show one of usual applications where PWLS arise in a natural way. In our opinion these applications justify the interest in these systems.
Wien bridge: [17]
In electronic circuits design also arises a large family of examples modeled by fundamental systems [1],[5]. In the following example we introduce a well-known circuit, the Wien bridge oscillator formed by two resistors, two capacitances and one operational amplifier (op-amp) with negative feedback see Figure 2.

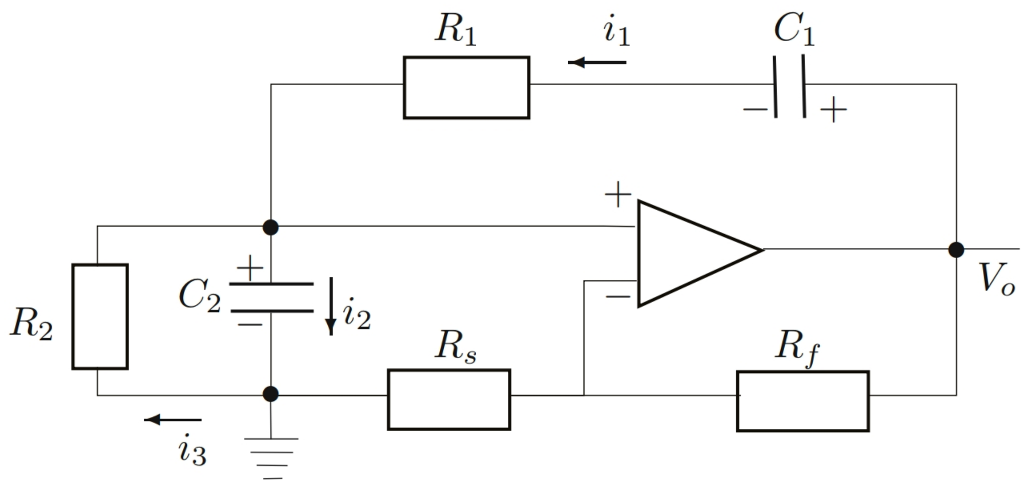


Figure 2: Wien bridge circuit .

The circuit is formed by two loops. The first one contains the resistor R_1 and the capacitors C_1 and C_2. The second loop is formed by the resistor R_2 and the capacitor C_2. For the sake of simplicity, we consider that the circuit is clockwise oriented in the first loop and anticlockwise oriented in the second one. Kirchhof laws can be used to describe the evolution of the voltages V_{C_1} and V_{C_2} across the capacitors C_1 and C_2, respectively, leading to the differential equations

$$\begin{cases} R_1C_1\dot{V_{C_1}} = -V_{C_1} - V_{C_2} - V_0 \ , \\ R_1C_2\dot{V_{C_2}} = -V_{C_1} - (1+\frac{R_1}{R_2})V_{C_2} - V_0 \ , \end{cases} \tag{1}$$

where V_0 is the output voltage of the op-amp. The characteristic function of an op-amp depends only on the difference between the voltage at the non-inverting terminal and the voltage at the inverting terminal (V_{C_2} and 0, respectively, in the Wien bridge). In an ideal framework, this function is considered to be linear and the slope of the function is called the open-loop gain of the amplifier. In practice, the op-amp has a limited response range (-E,E), beyond which the amplifier is saturated. Taking this into account, a more realistic characteristic function for the op-amp is given by

$$V_0 = \begin{cases} E \ \ sign(-\alpha V_{C_2} + E), & if \ |\alpha V_{C_2}| > E \ , \\ -\alpha V_{C_2}, & if \ |\alpha V_{C_2}| \leq E \ , \end{cases}$$

where $\alpha = 1 + \frac{R_F}{R_S}$ is the gain of the op-amp. Using the above expression of V_0 and making the change of variables

$$x_1 = \alpha\frac{V_{C_2}}{E} \ \ , \ \ x_2 = \alpha\frac{V_{C_1}}{E} \ ,$$

the system of differential equations (1) can be rewritten as the fundamental system

$$\dot{x} = \begin{cases} Ax + b, & if \ \ x_1 > 1, \\ Bx, & if \ \ |x_1| \leq 1, \\ Ax - b, & if \ \ x_1 < -1, \end{cases}$$

where

$$b = \begin{pmatrix} \dfrac{\alpha}{R_1C_2} \\ \dfrac{\alpha}{R_1C_1} \end{pmatrix}, A = \begin{pmatrix} -(\dfrac{1}{R_1C_2} + \dfrac{1}{R_2C_2}) & -\dfrac{1}{R_1C_2} \\ -\dfrac{1}{R_1C_1} & -\dfrac{1}{R_1C_1} \end{pmatrix}$$

and $B = A + b^T e_1$.

This thesis is structured as follows:

The first chapter is devoted to reminders of some preliminary notions on piecewise differential systems. We define the piecewise linear differential system, the solution of the continuous and the discontinuous piecewise linear differential system, the periodic orbit of this discontinuous piecewise linear differential system, also first integral and the bezout theorem.

In the second chapter, we will study the maximum number of crossing limit cycles that can have the planar Hamiltonian discontinuous planar piecewise differential systems formed by two or three linear differential systems separated by one or two straight line, such that both linear differential have no equilibria, neither real nor virtual.

In the third and last chapter, we will study the maximum number of crossing limit cycles that can have the planar Hamiltonian continuous and discontinuous planar piecewise differential systems formed by two or three linear differential systems separated by one or two straight line, such that one of this differential systems is linear center while the seconde and third differential systems have no equilibria, neither real nor virtual.

Chapter 1

Preliminaries

1.1 Introduction

Here some basic concepts, results, and tools necessary to the development of this work are presented. Most part of the results are given without proof, however references where they can be found, are included. In this work, we concern about planar discontinuous vector fields defined in two or more zones, for this reason, we present a generic definition for this type of vector fields.

1.2 Piecewise linear differential system

Definition 1.2.1 *A differential system defined on an open region $S \subseteq \mathbb{R}^n$ is said to be a piecewise linear differential system on S if there exists a set of 3-tuples $\{(A_i, b_i, S_i)\}$, $i \in I$ such that:*
A_i is a $n \times n$ real matrix; $b_i \in \mathbb{R}^n$;
$S_i \subset S$ is an open set in $\mathbb{R}^n$ satisfying that
$S_i \cap S_j = \emptyset$ if $i \neq j$ and $\bigcup_{i \in I} \overline{S_i} = S$;
and $A_i x + b_i$ is the vector field defined by the system when $x \in S_i$. As usual $\overline{S_i}$ denotes the closure of S_i.

Remark 1.2.1 *Thus the vector field defined by a piecewise linear differential system is a linear map on each of the disjoint regions S_i, but is not globally linear on the whole S.*

Definition 1.2.2 *let $\Gamma_{ij} = \partial\overline{S_i} \cap \partial\overline{S_j}$ be the common boundary of the regions $\overline{S_i}$ and $\overline{S_j}$.*

If $A_i p + b_i = A_j p + b_j$ for every $p \in \Gamma_{ij}$, is said to be continuous, otherwise the piecewise linear differential system is said to be discontinuous.

In discontinuous piecewise linear differential system, two different vectors $\dot{x}$, namely $f_i(x)$ and $f_j(x)$, can be associated to a point $x \in \Sigma_{ij}$. If the transversal components of $f_i(x)$ and $f_j(x)$ have the same sign, the orbit crosses the boundary and has, at that point, a discontinuity in its tangent vector.

On the contrary, if the transversal components of $f_i(x)$ and $f_j(x)$ are of opposite sign, i.e. if the two vector fields are "pushing" in opposite directions, the state of the system is forced to remain on the boundary and slide on it. Although, in principle, motions on the boundary could be defined in different ways, the most natural one is Filippov convex method.

1.3 Solution of Continuous piecewise linear differential system

Theorem 1.3.1 *Since the piecewise linear differential system is formed by linear differential systems in each region $\overline{S_i}$, then the solution of linear differential system*

$\dot{x}_i = A_i x + b_i$ *starting at* p_0 *isgiven by*

$$x(s, p_0) = e^{A_i s} p_0 + \int_0^s e^{A_i(s-r)} b_i dr. \tag{1.1}$$

Since, for a continuous piecewise linear differential system we have $A_i p + b_i = A_j p + b_j$ at any point of the boundary Γ_{ij} separating two adjacent regions S_i and S_j, then for these systems the vector $A_i p + b_i$ is uniquely defined at any point of the state space and the orbits in region S_i approaching transversely the boundary Γ_{ij}, cross it and enter into the adjacent region S_j.
In particular if the vector field

$$\dot{X} = \begin{cases} f_1(X) = A_1 X + b_1, & \text{if } H(x) > 0, \\ f_2(X) = A_2 X + b_2, & \text{if } H(x) < 0, \end{cases} \tag{1.2}$$

with the boundary

$$\Gamma = \left\{ x \in \mathbb{R}^2 : H(x) = 0 \right\},$$

and two regions

$$S_1 = \left\{ x \in \mathbb{R}^2 : H(x) > 0 \right\}, \quad S_2 = \left\{ x \in \mathbb{R}^2 : H(x) < 0 \right\},$$

is continuous

Let $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions of system (1.2) on S_1 and S_2 respectively. Then, the trajectory corresponding to the initial condition $X_0 = (x_{01}, y_{01})$ of the system (1.2) on S_1 is crossed the curve $H(x) = 0$, at the instance t^* in this case the initial condition of the second system (on S_2) is $(x_{02}, y_{02}) = (x_2(t^*), y_2(t^*))$.
Furthermore, for continuous piecewise linear differential system (1.2), we have if

$$X(s, p_0) = e^{A_1 s} p_0 + \int_0^s e^{A_1(s-r)} b_1 dr,$$

is a solution of linear differential system piecewise linear differential system starting at p_0 in S_1, then there exist a point $q = (x_1, y_1) \in \Gamma$ and the finite time t^* such that the orbit of linear differential system in S_1 starting at the point p is crossed the curve $H(x) = 0$, at the instance t^* at the point

$$q_0 = (x_1, y_1) = e^{A_1 t^*} p_0 + \int_0^{t^*} e^{A_1(t^*-r)} b_1 dr,$$

by the continuity of piecewise linear differential system, the solution of this system in S_2 is

$$X(s, q_0) = e^{A_2 s} q_0 + \int_0^s e^{A_2(s-r)} b_2 dr.$$

1.4 Solution of discontinuous piecewise linear differential system

We consider planar Filippov systems and assume, for simplicity, that there are only two regions S_i, i.e.

$$\dot{x} = \begin{cases} f_1(x), & x \in S_1, \\ f_2(x), & x \in S_2. \end{cases} \tag{1.3}$$

Moreover, the discontinuity boundary separating these two regions is described as

$$\Sigma = \left\{ x \in \mathbb{R}^2 : H(x) = 0 \right\},$$

where H is a smooth scalar function with nonvanishing gradient $\nabla H(x) = \left(\frac{\partial H}{\partial x_i}\right)^T$ on Σ, and

$$S_1 = \left\{ x \in \mathbb{R}^2 : H(x) > 0 \right\}, \quad S_2 = \left\{ x \in \mathbb{R}^2 : H(x) < 0 \right\}.$$

The boundary is either closed or goes to infinity in both directions and $f_1 \neq f_2$ on Σ.

1.4.1 Standard and sliding solutions

Definition 1.4.1 *The standard solutions of system* (1.3) *are the solution of this system in the regions S_i and is given by*

$$\varphi_i(s, p_0) = e^{A_i s} p_0 + \int_0^s e^{A_i(s-r)} b_i dr,$$

and sliding solutions of system (1.3) *are the solution of this system in the boundary $\Sigma_{i,j}$.*

Sliding solutions

The sliding solutions on Σ obtained with the well-known Filippov convex method. Let

$$\delta(x) = \langle \nabla H(x), f_1(x) \rangle \langle \nabla H(x), f_2(x) \rangle,$$

where $\langle ., . \rangle$ denotes the standard scalar product.

Definition 1.4.2 *We define the crossing set Σ_c as*

$$\Sigma_c = \{ x \in \Sigma : \delta(x) > 0 \} \subset \Sigma.$$

It is the set of all points $x \in \Sigma$, where the two vectors $f_i(x)$ have nontrivial normal components of the same sign. By definition, at these points the orbit of (1.3) *crosses Σ.*
We define the sliding set Σ_s as the complement to Σ_c in Σ, i.e.

$$\Sigma_s = \{ x \in \Sigma : \delta(x) \leq 0 \} \subset \Sigma.$$

Remark 1.4.1 *The crossing set is open, while the sliding set is the union of closed sliding segments and isolated sliding points.*

Definition 1.4.3 *Points $x^* \in \Sigma_s$, where*

$$\langle \nabla H(x^*), f_1(x^*) - f_2(x^*) \rangle = 0,$$

are called singular sliding points.

At such points, either both vectors $f_1(x)$ and $f_2(x)$ are tangent to Σ, or one of them vanishes while the other is tangent to Σ, or they both vanish.

Remark 1.4.2 *In general we define*
Escaping region (unstable sliding):

$$\Sigma_{es} = \left\{ x \in \Sigma : \langle \nabla H(x), f_1(x) \rangle > 0 \text{ and } \langle \nabla H(x), f_2(x) \rangle < 0 \right\}.$$

Attractive sliding region (stable sliding):

$$\Sigma_{as} = \left\{ x \in \Sigma : \langle \nabla H(x), f_1(x) \rangle < 0 \text{ and } \langle \nabla H(x), f_2(x) \rangle > 0 \right\}.$$

Then, the sliding set

$$\begin{aligned} \Sigma_s &= \{x \in \Sigma : \delta(x) \leq 0\} \\ &= \{x \in \Sigma : \langle \nabla H(x), f_1(x) \rangle > 0 \text{ and } \langle \nabla H(x), f_2(x) \rangle < 0\} \\ &\quad \cup \{x \in \Sigma : \langle \nabla H(x), f_1(x) \rangle < 0 \text{ and } \langle \nabla H(x), f_2(x) \rangle > 0\} \\ &\quad \cup \{x \in \Sigma : \langle \nabla H(x), f_1(x) \rangle \langle \nabla H(x), f_2(x) \rangle = 0\}. \end{aligned}$$

Remark 1.4.3 *At the points belonging to Σ_c, the standard solutions of the two systems can be joined to form a solution whose orbit crosses the discontinuity collector.*

Example 1.4.1 *Consider the planar system piecewise linear differential system with two zones separated by the straight line and*

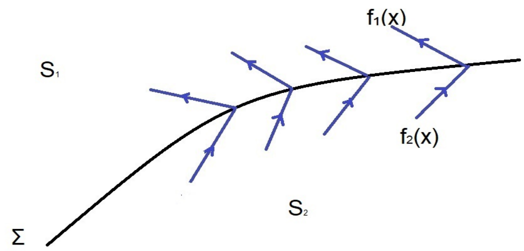
$$\Sigma = \left\{ x \in \mathbb{R}^2 : H(x) = 0 \right\} = \left\{ x \in \mathbb{R}^2 : y = 0 \right\},$$

and

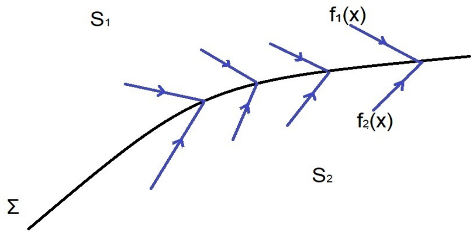
$$\begin{aligned} f_1(x) &= \begin{cases} \dot{x} = -6x - 6y - 30, \\ \dot{y} = 6x + 6y - 2, \end{cases} \quad \text{if } y > 0, \\ f_2(x) &= \begin{cases} \dot{x} = -x - y + 2, \\ \dot{y} = x + y + 1, \end{cases} \quad \text{if } y < 0, \end{aligned} \tag{1.4}$$

We have

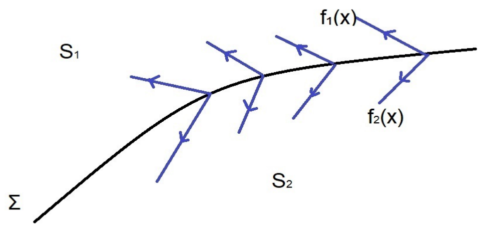
$$\nabla H(x) = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y} \right)^T = \begin{pmatrix} 0 & 1 \end{pmatrix}^T = k.$$



(a) Crossing region



(b) Attracting region (stable sliding)



(c) Escaping region (instable sliding)

The sliding region (stable sliding) is

$$\begin{aligned}\Sigma_{as} &= \left\{x \in \Sigma : k^T f_2(x) > 0 \text{ and } k^T f_1(x) < 0\right\}, \\ &= \left\{x \in \Sigma : -1 \leq x \leq \frac{1}{3}\right\}.\end{aligned}$$

The crossing set

$$\Sigma_c = \left\{x \in \Sigma : \left(k^T f_2(x)\right).\left(k^T f_1(x)\right) > 0\right\} =]-\infty, -1[\cup \left]\frac{1}{3}, +\infty\right[.$$

1.4.2 Filippov method

Within the sliding set, the Filippov method can be used to construct solutions, to be considered as extensions for solutions of (1.3). Such a method consists in defining a new vector field computed from an adequate convex combination $g(x)$ of the two original vector fields $f_i(x)$ to each nonsingular sliding point $x \in \Sigma_s$, namely

$$g(x) = \lambda f_1(x) + (1 - \lambda) f_2(x),$$

where for each $x \in \Sigma_s$ the value of λ is selected such that $\langle \nabla H(x), f_2(x) - f_1(x) \rangle \neq 0$. A simple computation shows that

$$\lambda = \lambda(x) = \frac{\langle \nabla H(x), f_2(x) \rangle}{\langle \nabla H(x), f_2(x) - f_1(x) \rangle},$$

provided the above denominator does not vanish and then, by using the definition of Σ_s, one concludes that

$$0 \leq \lambda(x) \leq 1.$$

Therefore, we have a explicit definition for the sliding vector field, namely

$$g(x) = \frac{\langle \nabla H(x), f_1(x) \rangle f_2(x) - \langle \nabla H(x), f_2(x) \rangle f_1(x)}{\langle \nabla H(x), f_2(x) - f_1(x) \rangle} \tag{1.5}$$

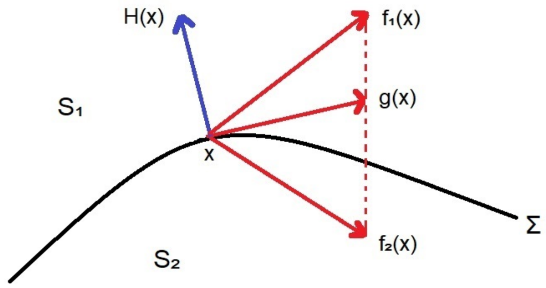


Figure 1.1: Construction de Filippov in the case unstable sliding region

Example 1.4.2 *For system* (1.4) *we have* $\Sigma_s = \left\{ x \in \Sigma : -1 \leq x \leq \frac{1}{3} \right\}$, *and*

$$g(0, x) = \frac{\langle \nabla H(x), f_1(x) \rangle f_2(x) - \langle \nabla H(x), f_2(x) \rangle f_1(x)}{\langle \nabla H(x), f_2(x) - f_1(x) \rangle} = \begin{pmatrix} -\frac{(50x+26)}{5x-3} \\ 0 \end{pmatrix}$$

Remark 1.4.4 *If* $\langle \nabla H(x), f_2(x) - f_1(x) \rangle = 0$ *for some* $x \in \Sigma_s$, *then we say that such* x *is a singular sliding point. This can happen in three cases: both vector fields are tangent; one is tangent and the other vanishes; both vector fields vanish at the point. In all these singular cases, if we exclude infinitely degenerate cases and the sliding point is non-isolated, it is possible to define the vector field* g *by continuity arguments. For isolated singular sliding points it will be taken* $g(x) = 0$. *Clearly, the boundary of the sliding set* Σ_s *(as a subset of* Σ*) is associated to each one of the two equalities* $\langle \nabla H(x), f_1(x) \rangle = 0$, $\langle \nabla H(x), f_2(x) \rangle = 0$; *that is,* $\lambda(x) = 0$ *or* $\lambda(x) = 1$ *with* $x \in \Sigma$.

1.4.3 The constant solutions

As usual in the analysis of dynamical systems, we must look for the simplest solutions organizing the dynamics, namely the constant solutions associated to rest points normally called equilibria. Of course, the Filippov system (1.3) inherit the equilibria of each vector field $f_i(x)$, but we must be cautious and distinguish between real or virtual equilibria. In particular,

Definition 1.4.4 *We call*
- *admissible or real equilibrium points to all the solutions of* $f_1(x) = 0$ *that belong to* S_1 *and the solutions of* $f_2(x) = 0$ *that belong to* S_2*, while*
- *virtual equilibrium points are the solutions of* $f_1(x) = 0$ *that belong to* S_2*, and the solutions of* $f_2(x) = 0$ *that belong to* S_1*.*

Remark 1.4.5 *Although virtual equilibria are not equilibria of* X*, they can still organize the dynamics in the corresponding region.*

Example 1.4.3 *For system*

$$\begin{array}{lll} \dot{x} = 2(-4+4x+5y), & \dot{y} = -8(1+x+y), & x > 1, \\ \dot{x} = 1+2y, & \dot{y} = 1-2x, & x < 1, \end{array} \tag{1.6}$$

we have

$$S_1 = \left\{(x,y) \in \mathbb{R}^2 : x > 1\right\}, \ S_2 = \left\{(x,y) \in \mathbb{R}^2 : x < 1\right\}.$$

The equilibria of the first system is $(-9,8) \notin S_1$*, then* $(-9,8)$ *is a virtual equilibrium points of* (1.6)*.*
The equilibria of the second system is $\left(\frac{1}{2},-\frac{1}{2}\right) \in S_2$*, then* $\left(\frac{1}{2},-\frac{1}{2}\right)$ *is a real equilibrium points of* (1.6)*.*

Regarding now the induced dynamical system $\dot{x} = g(x), x \in \Sigma_s$. As indicated in Fig 1.1, at nonisolated sliding points $x \in \Sigma_s$ we have

$$\langle \nabla H(x), g(x) \rangle = 0,$$

i.e. $g(x)$ is tangent to sliding segments of Σ_s. We set $g(x) = 0$ at isolated singular sliding points. Thus,

$$\dot{x} = g(x), \quad x \in \Sigma_s, \tag{1.7}$$

defines a scalar differential equation on Σ_s, which is smooth on one-dimensional sliding intervals of Σ_s. Solutions of this equation are called sliding solutions.

Remark 1.4.6 *We note that apart from isolated singular sliding points, the sliding vector field* g *can vanish in other points which behave like real equilibria for such dynamical system if we restrict our attention to the set* Σ_s*. They also are, in some sense, equilibria for system* (1.3) *and will be called pseudo-equilibrium points. For instance, when both vectors* f_i *are transversal to in a certain point of this surface and furthermore they are anti-collinear, that is, there exist* $\lambda_1, \lambda_2 > 0$*, such that*

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) = 0.$$

Definition 1.4.5 *The equilibria of* (1.7)*, where the vectors* $f_i(x)$ *are transversal to* Σ_s *and anti-collinear, are called pseudo-equilibria of* (1.3) *(or quasi-equilibria). This implies that a pseudo-equilibrium* P *is an internal point of a sliding segment.*

Example 1.4.4 *Consider the system*

$$\left\{\begin{array}{l}\dot{x}=3,\\ \dot{y}=1,\end{array}\right., \quad 2x-y<0\ , \quad \left\{\begin{array}{l}\dot{x}=-2,\\ \dot{y}=x-y,\end{array}\right., \quad 2x-y>0\ .$$

Using $H(x,y)=2x-y$ *we obtain* $\langle\nabla H(x),f_1(x)\rangle=5$ *and* $\langle\nabla H(x),f_2(x)\rangle\rangle=x-4$ *(on* $y=2x$*), and so* $y=2x$ *is an attracting sliding region for* $x<4$*. By evaluating* (1.5) *we find that on this region we have*

$$g\left(x,2x\right)=\left(\begin{array}{c}\frac{2-3x}{x-9}\\ \frac{4-6x}{x-9}\end{array}\right),$$

thus

$$\dot{x}=\frac{-3x+2}{-x+9},\quad \dot{y}=2\dot{x}.$$

Solving $\dot{x}=0$ *gives the pseudo-equilibrium* $(x,y)=\left(\frac{2}{3},\frac{4}{3}\right)$*. This equilibrium is stable because the sliding region is attracting and*

$$\left.\frac{d}{dx}\dot{x}\right|_{x=\frac{2}{3}}=-\frac{9}{25}<0.$$

Definition 1.4.6 *An equilibrium* $X\in\Sigma_s$ *of* (1.7)*, where one of the vectors* $f_i(X)$ *vanishes, is called a boundary equilibrium.*

Example 1.4.5 *For system*

$$\begin{aligned} f_1\left(x,y\right) &= \left\{\begin{array}{l}\dot{x}=-6x-6y-30,\\ \dot{y}=6x+6y-2,\end{array}\right. \quad \textit{if } y>0, \\ f_2\left(x,y\right) &= \left\{\begin{array}{l}\dot{x}=-x-y+2,\\ \dot{y}=x-2,\end{array}\right. \quad \textit{if } y<0,\end{aligned} \tag{1.8}$$

we have : $\Sigma_s=\left\{(x,0)\in\Sigma:\frac{1}{3}\leq x\leq 2\right\}$ *and*

$$g(x)=\frac{\langle\nabla H(x),f_1(x)\rangle\rangle f_2(x)-\langle\nabla H(x),f_2(x)\rangle f_1(x)}{\langle\nabla H(x),f_2(x)-f_1(x)\rangle}=\left(\begin{array}{c}\frac{32x-64}{-5x}\\ 0\end{array}\right),$$

we have $g(x)=0$ *if* $x=2$*, then* $(2,0)\in\Sigma_s$ *is an equilibrium of* $\dot{x}=g\left(x\right)$*. Moreover, notice that* $f_2\left(2,0\right)=0$*, then* $(2,0)$ *is a boundary equilibrium of* (1.8)*.*

Definition 1.4.7 *A sliding segment is delimited either by a boundary equilibrium* X*, or by a point* p *(called tangent point) where the vectors* $f_i(p)$ *are nonzero but one of them is tangent to* Σ*.*

Example 1.4.6 *For system*

$$\begin{aligned} f_1\left(x,y\right) &= \left\{\begin{array}{l}\dot{x}=-6x-6y-30,\\ \dot{y}=6x+6y-2,\end{array}\right. \quad \textit{if } y>0, \\ f_2\left(x,y\right) &= \left\{\begin{array}{l}\dot{x}=-x-y+2,\\ \dot{y}=x+y+1,\end{array}\right. \quad \textit{if } y<0,\end{aligned}$$

we have $\Sigma_s = \left\{(x,0) \in \Sigma : -1 \leq x \leq \frac{1}{3}\right\}$.
Notice that $f_2\left(-1,0\right) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ *is tangent to* Σ, *then* $(-1,0)$ *is a tangent point and* $f_1\left(\frac{1}{3},0\right) = \begin{pmatrix} -32 \\ 0 \end{pmatrix}$ *is tangent to* Σ, *then* $\left(\frac{1}{3},0\right)$ *is a tangent point.*

1.5 Periodic orbit γ of the discontinuous piecewise linear differential system

Definition 1.5.1 *A periodic orbit* γ *of the discontinuous piecewise linear differential system* (1.3) *is a smooth piecewise curve which is formed by pieces of orbits of each linear differential system,* $\gamma = \cup_{i\in I}\gamma_i$, *contained in the regions* S_i, *respectively,such that* $\varphi(p, s+T) = \varphi(p,s)$, *for some* $T > 0$, *where* T *is called the period of the orbit periodic* γ.
If $\gamma_i \cap \Sigma = \Sigma_c$ *for all* $i = 1, ..., n$, *then the periodic orbit* γ *is called the crossing periodic orbit, otherwise is called sliding periodic orbit.*

Definition 1.5.2 *If a periodic orbit* γ *is isolated in the set of all periodic orbits of discontinuous piecewise linear differential system, then it is called limit cycle of piecewise linear differential system* (1.3).

1.6 First integrals

The aim of this section is to introduce the terminology of the Darboux theory of integrability for real planar polynomial differential systems. For a detailed discussion of this theory see [4]. A real planar polynomial differential system or simply a polynomial system will be a differential system of the form

$$\frac{dx}{ds} = \dot{x} = P(x,y), \quad \frac{dy}{ds} = \dot{y} = Q(x,y), \tag{1.9}$$

where x and y are real variables, the independent one (the time) s is real, and P and Q are polynomials in the variables x and y with real coefficients. The degree of polynomial system (1.9) is defined as $m = \max\{\deg P, \deg Q\}$. The vector field X associated to system (1.9) is defined by

$$X = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q.$$

System (1.9) is integrable on an open subset U of $\mathbb{R}^2$ if there exists a non constant analytic function $H : U \to C$, called a first integral of (1.9) on U, which is constant on all orbits of system (1.9) contained on U, i. e.

$$\frac{dH}{dt} = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} \cong 0.$$

1.7 Bezout Theorem

In this section we consider the intersection of two algebraic curves. Suppose the curves are given by $F(x, y) = 0$ and $G(x, y) = 0$, then we shall be interested in their common solutions.

Consider two algebraic curves C, D given by the equations $F(x, y) = 0$ and $G(x, y) = 0$. We shall say that C, D have a common component if F, G have non-constant common divisor $H \in k[x, y]$. The common component is then the curve given by the equation $H(x, y) = 0$.

If F, G do not have a non-constant common factor we say that the curves do not have a common component. Any point x_0, y_0 satisfying $F(x_0, y_0) = G(x_0, y_0) = 0$ can be seen as an intersection point of C and D. So we see that two algebraic curves without common component intersect in finitely many points. We can say a bit more though.

Theorem 1.7.1 *(Bezout Theorem) Let C, D be two algebraic curves of degree m, n respectively. Suppose that the curves have no common component. Then the number of intersection points of C, D is at most mn.*

For more details, see [21].

Chapter 2

Limit cycles for a class of piecewise linear differential system whithout equilibrea

2.1 Introduction

The study to provide a sharp upper bound for the maximum number of crossing limit cycles for discontinuous piecewise linear differential system separated by a curve is a very difficult problem, even when this curve is a straight line. And there are two reasons that make difficult the analysis of this problem. First, even one can easily integrate the solution of every linear differential system X_i, it is difficult to determine explicitly the time that an orbit expends in each region governed by each linear differential system. And second, the number of parameters needed to analyze all possible cases is in general not small.

In this chapter we study the maximum number of crossing limit cycles that can have the planar Hamiltonian discontinuous planar piecewise differential systems formed by two or three linear differential systems separated by one or two straight line, such that both linear differential have no equilibria, neither real nor virtual.

In order to reduce the number of parameters on which the piecewise linear differential system depends we use the canonical forms in the Lemma .

2.2 Canonical forms

Other normal form which is independent of the change of coordinates it is provide in the following Lemmas.

Lemma 2.2.1 *A Hamiltonian linear differential systems has no equilibrium points can be written as*

$$\dot{x} = d(ax - ady + c) + b, \qquad \dot{y} = ax - ady + c,$$

where a, b, c and d real constants such that $ab \neq 0$; Moreover, this systems has the first integral

$$H_1(x, y) = ax^2 - 2adxy + ad^2y^2 + 2cx - 2(cd + b)y.$$

Proof. Consider a general linear differential system which has no equilibrium point

$$\dot{x} = d(ax + \beta y + c) + b, \qquad \dot{y} = ax + \beta y + c, \tag{2.1}$$

with $b \neq 0$ and it has the first integral

$$H_1(x, y) = \gamma x + \varepsilon y + kxy + mx^2 + ny^2,$$

then

$$\dot{x}\frac{\partial H_1}{\partial x} + \dot{y}\frac{\partial H_1}{\partial y} \equiv 0,$$

thus

$$\begin{gathered}(ak + 2adm)\,x^2 + (k\beta + 2an + 2dm\beta + adk)\,xy + (a\varepsilon + 2m\,(b + cd) + ck + ad\gamma)\,x \\ + (2n\beta + dk\beta)\,y^2 + (\beta\varepsilon + k\,(b + cd) + 2cn + d\beta\gamma)\,y + (\gamma\,(b + cd) + c\varepsilon) = 0,\end{gathered}$$

is equivalent to

$$
\begin{aligned}
&k + 2dm = 0,\\
&k\beta + 2an + 2dm\beta + adk = 0,\\
&a\varepsilon + ck + 2bm + ad\gamma + 2cdm = 0,\\
&2n + dk = 0,\\
&\beta\varepsilon + bk + 2cn + d\beta\gamma + cdk = 0,\\
&b\gamma + c\varepsilon + cd\gamma = 0.
\end{aligned}
$$

The solution of this system is

$$
\begin{aligned}
k &= \frac{ad\varepsilon}{b+cd},\\
m &= -a\frac{\varepsilon}{2b+2cd},\\
n &= -ad^2\frac{\varepsilon}{2b+2cd},\\
\beta &= -ad,\\
\gamma &= -c\frac{\varepsilon}{b+cd}.
\end{aligned}
$$

If we take $\varepsilon = -2(dc+b)$ then, the system (2.1) become

$$
\begin{cases}
\dot{x} = d(ax - ady + c) + b,\\
\dot{y} = ax - ady + c,
\end{cases}
$$

with $b \neq 0$ and

$$H_1(x, y) = 2cx - 2adxy + ax^2 + ad^2y^2 - 2(dc+b)y.$$

This will complete the proof of Lemma 2.2.1. ∎

2.3 Main result

Theorem 2.3.1 *A Hamiltonian discontinuous planar piecewise differential systems formed by two linear differential systems separated by a straight line, such that both linear differential have no equilibria, neither real nor virtual, have no limit cycles.*

Proof. Assume that we have a discontinuous piecewise linear differential system separated by one straight line and formed by two linear systems which has no equilibrium points. we can suppose that the straight line of discontinuity is $x = 0$, and that the linear system in the half-plane $x > 0$ is given by the system

$$\dot{x} = d(ax - ady + c) + b, \qquad \dot{y} = ax - ady + c,$$

while the linear system in the half-plane $x < 0$ is given by the system

$$\dot{x} = \gamma(\alpha x - \alpha\gamma y + \lambda) + \beta, \qquad \dot{y} = \alpha x - \alpha\gamma y + \lambda,$$

where α, β, λ and γ real constants such that $\alpha\beta \neq 0$; Moreover, this systems has the first integral

$$\begin{aligned} H_1(x,y) &= ax^2 - 2adxy + ad^2y^2 + 2cx - 2(cd+b)y, \\ H_2(x,y) &= \alpha x^2 - 2\alpha\gamma xy + \alpha\gamma^2y^2 + 2\lambda x - 2(\lambda\gamma+\beta)y, \end{aligned}$$

in half-plane $x > 0$ and $x < 0$ respectively.
Therefore if the piecewise linear differential system has a periodic orbit candidate to be a limit cycle it must intersect the line $x = 0$ in exactly two points, namely $(0, y_1)$ and$(0, y_2)$ with $y_1 < y_2$. Since H_1 and H_2 are two first integrals they must satisfy

$$\begin{aligned} H_1(0,y_1) - H_1(0,y_2) &= 0, \\ H_2(0,y_1) - H_2(0,y_2) &= 0, \end{aligned}$$

that is

$$\begin{cases} ad^2y_1^2 - 2(cd+b)y_1 - ad^2y_2^2 + 2(cd+b)y_2 = 0, \\ \alpha\gamma^2y_1^2 - 2(\lambda\gamma+\beta)y_1 - \alpha\gamma^2y_2^2 + 2(\lambda\gamma+\beta)y_2 = 0. \end{cases}$$

is equivalent to the system

$$\begin{cases} (y_1-y_2)(ad^2(y_1+y_2) + 2(cd+b)) = 0, \\ (y_1-y_2)(\alpha\gamma^2(y_1+y_2) + 2(\lambda\gamma+\beta)) = 0, \end{cases}$$

Since $y_1 < y_2$ the previous system is equivalent to the system

$$\begin{cases} ad^2(y_1+y_2) + 2(cd+b) = 0, \\ \alpha\gamma^2(y_1+y_2) + 2(\lambda\gamma+\beta) = 0. \end{cases} \tag{2.2}$$

Since $ad^2 \neq 0$ from the first equation of (2.2) we isolated y_1 then

$$y_1 = -y_2 - \frac{2(cd+b)}{ad^2},$$

and substitute it in the second equation of (2.2) , and we get

$$-\alpha\gamma^2\frac{2(cd+b)}{ad^2} + 2(\lambda\gamma+\beta) = 0.$$

Then if $-2\alpha\gamma^2(cd+b) + 2ad^2(\lambda\gamma+\beta) = 0$ we have

$$y_1 = -y_2 - \frac{2(cd+b)}{ad^2},$$

but if $-2\alpha\gamma^2(cd+b) + 2ad^2(\lambda\gamma+\beta) \neq 0$ there is no solution .

Finally, this system has either no solutions (y_1, y_2) satisfying the necessary condition $y_1 < y_2$, or it has a continuum of solutions. So the continuous piecewise linear differential system either does not have periodic solutions, or it has a continuum of periodic orbits, and consequently this differential system has no limit cycles. ■

Here we shall prove that Theorem 2.3.1 cannot be extended to discontinuous piecewise linear differential system separated by two parallel straight lines formed by three linear differentiel systems have no equilibria.
Thus our main result is:

Theorem 2.3.2 *A Hamiltonian discontinuous planar piecewise differential systems formed by three linear differential systems separated by two parallel straight lines, such that the three linear differential have no equilibria, neither real nor virtual, can have at most one crossing algebraic limit cycle. Moreover there are systems in this class having one limit cycle.*

Proof. Using the notations

$$\begin{aligned} \Sigma_+ &= \left\{(x,y)\in\mathbb{R}^2 : x>1\right\}, \\ \Sigma_0 &= \left\{(x,y)\in\mathbb{R}^2 : -1<x<1\right\}, \\ \Sigma_- &= \left\{(x,y)\in\mathbb{R}^2 : x<-1\right\}. \end{aligned}$$

Assume that we have a discontinuous piecewise linear differential system separated by two parallel straight lines formed by three linear system which has no equilibrium points .we can suppose without loss of generality that the two discontinuous parallel straight lines are $x=\pm1$, then the discontinuous piecewise linear differential system becomes

$$\begin{array}{lll} \dot{x}=s(mx-msy+h)+n, & \dot{y}=mx-msy+h & in\ \ \Sigma_+, \\ \dot{x}=d(ax-ady+c)+b, & \dot{y}=ax-ady+c & in\ \ \Sigma_0, \\ \dot{x}=\gamma(\alpha x-\alpha\gamma y+\lambda)+\beta, & \dot{y}=\alpha x-\alpha\gamma y+\lambda & in\ \ \Sigma_-, \end{array}$$

this systems has the first integrals

$$\begin{aligned} H_3(x,y) &= mx^2-2msxy+ms^2y^2+2hx-2(hs+n)y, \\ H_1(x,y) &= ax^2-2adxy+ad^2y^2+2cx-2(cd+b)y, \\ H_2(x,y) &= \alpha x^2-2\alpha\gamma xy+\alpha\gamma^2y^2+2\lambda x-2(\lambda\gamma+\beta)y, \end{aligned}$$

in Σ_+ , Σ_0 ,and Σ_- respectively.
A possible limit cycle of this discontinuous piecewise linear differential system must intersect each discontinuous straight line in two points, namely $(1,y_1),(1,y_2),(-1,y_3)$ and $(-1,y_4)$ with $y_1<y_2$ and $y_4<y_3$, and the first integrals H_1,H_2 and H_3 must satisfy

$$\begin{aligned} &H_3(1,y_1)-H_3(1,y_2)=0, \\ &H_1(1,y_2)-H_1(-1,y_3)=0, \\ &H_2(-1,y_3)-H_2(-1,y_4)=0, \\ &H_1(-1,y_4)-H_1(1,y_1)=0. \end{aligned} \tag{2.3}$$

Or equivalently

$$\begin{cases} (y_1 - y_2)\left(2n + 2hs + 2ms - ms^2 y_1 - ms^2 y_2\right) = 0, \\ 4c + \left(ad^2\left(y_2 - y_3\right) - 2ad\right)\left(y_2 + y_3\right) - 2\left(b + cd\right)\left(y_2 - y_3\right) = 0, \\ (y_4 - y_3)\left(2\alpha\gamma - 2\beta - 2\lambda\gamma + \alpha\gamma^2\left(y_3 + y_4\right)\right) = 0, \\ ad\left(2 - d\left(y_1 - y_4\right)\right)\left(y_1 + y_4\right) + 2\left(y_1 - y_4\right)\left(b + cd\right) - 4c = 0. \end{cases} \tag{2.4}$$

By hypothesis $y_1 < y_2$ and $y_3 > y_4$ and therefore this system is equivalently to the system

$$\begin{cases} \left(2n + 2hs + 2ms - ms^2\left(y_1 + y_2\right)\right) = 0, \\ 4c + \left(ad^2\left(y_2 - y_3\right) - 2ad\right)\left(y_2 + y_3\right) - 2\left(b + cd\right)\left(y_2 - y_3\right) = 0, \\ \left(2\alpha\gamma - 2\beta - 2\lambda\gamma + \alpha\gamma^2\left(y_3 + y_4\right)\right) = 0, \\ ad\left(-2 + d\left(y_1 - y_4\right)\right)\left(y_1 + y_4\right) - 2\left(y_1 - y_4\right)\left(b + cd\right) + 4c = 0. \end{cases}$$

This last system can be written as

$$\begin{cases} \gamma\left(y_1 + y_2\right) + \gamma_2 = 0, \\ 4c + l_1\left(y_1^2 - y_3^2\right) - l_2\left(y_1 - y_3\right) - l_3\left(y_1 + y_3\right) = 0, \\ \delta_1 - \delta_2\left(y_3 + y_4\right) = 0, \\ 4c + l_1\left(y_2^2 - y_4^2\right) - l_2\left(y_2 - y_4\right) - l_3\left(y_2 + y_4\right) = 0, \end{cases} \tag{2.5}$$

where $\gamma = -ms^2, \gamma_2 = 2n + 2hs + 2ms, \delta_1 = 2\alpha\gamma - 2\beta - 2\lambda\gamma, \delta_2 = \alpha\gamma^2, l_1 = ad^2, l_2 = 2\left(b + cd\right), l_3 = 2ad$. With $\gamma \neq 0$ and $\delta_2 \neq 0$.
Looking at system (2.5) we remark that if (y_1, y_2, y_3, y_4) is a solution, then (y_2, y_1, y_4, y_3) is also a solution, but due to the fact that $y_1 < y_2$ and $y_3 > y_4$, at most one of these two solutions will be satisfactory.
Since $\gamma \neq 0$ and $\delta_2 \neq 0$ from the first and third equations of (2.5) we can isolated y_1 and y_4, respectively. Then, we obtain

$$y_1 = -\frac{\left(\gamma_2 + \gamma y_2\right)}{\gamma}, \qquad y_4 = \frac{\delta_1 - \delta_2 y_3}{\delta_2}.$$

Now replacing these expressions of y_1 and y_4 in the second and fourth equations of (2.5), we have the system of two equations

$$\begin{aligned} l_1 y_2^2 + \left(l_2 + l_3 + \tfrac{2}{\gamma}\gamma_2 l_1\right) y_2 - l_1 y_3^2 + \left(l_2 - l_3\right) y_3 + \left(4c + \tfrac{1}{\gamma}\gamma_2 l_2 + \tfrac{1}{\gamma}\gamma_2 l_3 + \tfrac{1}{\gamma^2}\gamma_2^2 l_1\right) = 0, \\ l_1 y_2^2 - \left(l_2 + l_3\right) y_2 - l_1 y_3^2 + \left(l_3 - l_2 + 2\tfrac{\delta_1}{\delta_2} l_1\right) y_3 + \left(4c + \tfrac{\delta_1}{\delta_2} l_2 - \tfrac{\delta_1}{\delta_2} l_3 - \tfrac{\delta_1^2}{\delta_2^2} l_1\right) = 0. \end{aligned} \tag{2.6}$$

Summing up the first equation and the second equation of (2.6), we get

$$\begin{aligned} &\left(2l_2 + 2l_3 + \tfrac{2}{\gamma}\gamma_2 l_1\right) y_2 + \left(2l_2 - 2l_3 - 2\tfrac{\delta_1}{\delta_2} l_1\right) y_3 \\ &+ \left(\tfrac{1}{\gamma}\gamma_2 l_2 + \tfrac{1}{\gamma}\gamma_2 l_3 - \tfrac{\delta_1}{\delta_2} l_2 + \tfrac{\delta_1}{\delta_2} l_3 + \tfrac{1}{\gamma^2}\gamma_2^2 l_1 + \tfrac{\delta_1^2}{\delta_2^2} l_1\right) = 0, \end{aligned} \tag{2.7}$$

from (2.7) we isolated y_2 we obtain

$$y_2 = \gamma \frac{\delta_1 l_1 - \delta_2 l_2 + \delta_2 l_3}{\delta_2 (\gamma l_2 + \gamma l_3 + \gamma_2 l_1)} y_3 - \frac{\gamma^2 \delta_1^2 l_1 + \gamma_2^2 \delta_2^2 l_1 + \gamma \gamma_2 \delta_2^2 l_2 - \gamma^2 \delta_1 \delta_2 l_2 + \gamma \gamma_2 \delta_2^2 l_3 + \gamma^2 \delta_1 \delta_2 l_3}{2\gamma \delta_2^2 (\gamma l_2 + \gamma l_3 + \gamma_2 l_1)}. \tag{2.8}$$

Substituting (2.8) into the second equation of (2.6), we get

$$m_0 + m_1 y_3 + m_2 y_3^2 = 0,$$

where

$$\begin{aligned}
m_0 &= 4c + \frac{\delta_1}{\delta_2}(l_2 - l_3) - \frac{\delta_1^2}{\delta_2^2} l_1 \\
&\quad + \frac{(l_2 + l_3)(\gamma^2 \delta_1^2 l_1 + \gamma_2^2 \delta_2^2 l_1 + \gamma \gamma_2 \delta_2^2 l_2 - \gamma^2 \delta_1 \delta_2 l_2 + \gamma \gamma_2 \delta_2^2 l_3 + \gamma^2 \delta_1 \delta_2 l_3)}{2\gamma \delta_2^2 (\gamma l_2 + \gamma l_3 + \gamma_2 l_1)} \\
&\quad + \frac{l_1 (\gamma^2 \delta_1^2 l_1 + \gamma_2^2 \delta_2^2 l_1 + \gamma \gamma_2 \delta_2^2 l_2 - \gamma^2 \delta_1 \delta_2 l_2 + \gamma \gamma_2 \delta_2^2 l_3 + \gamma^2 \delta_1 \delta_2 l_3)^2}{4\gamma^2 \delta_2^4 (\gamma l_2 + \gamma l_3 + \gamma_2 l_1)^2}, \\
m_1 &= \frac{-\delta_1 l_1 (\gamma \delta_1 l_1 - 2\gamma \delta_2 l_2 - \gamma_2 \delta_2 l_1)(\gamma \delta_1 l_1 + 2\gamma \delta_2 l_3 + \gamma_2 \delta_2 l_1)}{\delta_2^3 (\gamma l_2 + \gamma l_3 + \gamma_2 l_1)^2}, \\
m_2 &= \frac{\gamma^2 l_1 (\delta_1 l_1 - \delta_2 l_2 + \delta_2 l_3)^2}{\delta_2^2 (\gamma l_2 + \gamma l_3 + \gamma_2 l_1)^2} - l_1.
\end{aligned} \tag{2.9}$$

Now, solving (2.5) reduces to solve

$$\begin{aligned}
m_0 + m_1 y_3 + m_2 y_3^2 &= 0, \\
k_0 + k_1 y_3 + k_2 y_2 &= 0,
\end{aligned} \tag{2.10}$$

where m_0, m_1, and m_2 are defined in (2.9) and

$$\begin{aligned}
k_0 &= \left(\gamma^2 \delta_1^2 l_1 + \gamma_2^2 \delta_2^2 l_1 + \gamma \gamma_2 \delta_2^2 l_2 - \gamma^2 \delta_1 \delta_2 l_2 + \gamma \gamma_2 \delta_2^2 l_3 + \gamma^2 \delta_1 \delta_2 l_3\right), \\
k_1 &= -2\gamma^2 \delta_2 (\delta_1 l_1 - \delta_2 l_2 + \delta_2 l_3), \\
k_2 &= 2\gamma \delta_2^2 (\gamma l_2 + \gamma l_3 + \gamma_2 l_1).
\end{aligned}$$

Eventually system (2.10) could have a continuum of solutions (y_2, y_3) if some coefficients of the polynomials that there appear are zero, but then the possible periodic solutions would not be limit cycles. So assume that system (2.10) has finitely many solutions. We recall that Bezout Theorem states that if a polynomial differential system of equations has finitely many solutions, then the number of its solutions is at most the product of the degrees of the polynomials which appear in the system. Then by Bezout Theorem system (2.10) has at most two solutions. Finally from these at most two solutions (y_2, y_3) of system (2.10) we get two solutions (y_2, y_1, y_4, y_3) of system (2.4), but from the previous remark at most one of these two solution would satisfy $y_1 < y_2$ and $y_3 > y_4$. In summary, we have proved that at most we can have one limit cycle. This will complete the proof of Theorem 2.3.2. ∎

Remark 2.3.1 *Concerning Theorem 2.3.2, we stress that since the three linear differentials systems have no equilibria then, the limit cycles, if they exists, are surround a sliding set or the origin.*

The next propositions shows that there are discontinuous piecewise linear differential systems separated by the set Σ with one, two or three (respectively) crossing algebraic limit cycles intersecting in a unique point with each of the three branches of Σ.

Proposition 2.3.1 *The discontinuous piecewise linear differential system defined by*

$$\begin{array}{lll} \dot{x} = -y - x + \dfrac{1}{2}, & \dot{y} = x + y + 1 & \text{in } \Sigma_+, \\ \dot{x} = -\left(lx + ky - k\right), & \dot{y} = 2x + ly - f & \text{in } \Sigma_0, \\ \dot{x} = -2\left(x + y + \sqrt{2} - \dfrac{\sqrt{3}}{2} + 1\right), & \dot{y} = 2x + 2y - 1 & \text{in } \Sigma_-, \end{array} \tag{2.11}$$

where $k = \left(168\sqrt{2} + 136\sqrt{3} + 96\sqrt{6} + 238\right)$, $f = \left(20\sqrt{12} + 32\sqrt{2} + 29\sqrt{3} + \frac{93}{2}\right)$ *and* $l = \left(4\sqrt{6} + 8\sqrt{2} + 6\sqrt{3} + 12\right)$ *has a unique crossing limit cycle surrounding the sliding segment:*

$$\Sigma_s = \left\{(1, y) : -\frac{1}{2} \leq y \leq 4\sqrt{2} - 2\sqrt{6} + 3\sqrt{3} - 5\right\}$$

$$\cup \left\{(-1, y) : \frac{1}{2}\sqrt{3} - \frac{\sqrt{18}}{3} \leq y \leq 2\sqrt{6} - 4\sqrt{2} - 3\sqrt{3} + 7\right\}.$$

Moreover, this limit cycle is algebraic of degree $(2, 2, 2)$ *and writes as*

$$\begin{aligned} \Gamma \;=\; & \{(x, y) \in \Sigma_+ : H_{11}(x, y) - 9 = 0\} \\ & \cup \{(x, y) \in \Sigma_0 : H_{21}(x, y) + 102.96 = 0\} \\ & \cup \{(x, y) \in \Sigma_0 : H_{21}(x, y) - 6776.8 = 0\} \\ & \cup \{(x, y) \in \Sigma_- : H_{31}(x, y) - 13.798 = 0\}, \end{aligned}$$

with

$$\begin{aligned} H_{11}(x, y) &= x^2 + 2xy + y^2 + 2x - y, \\ H_{21}(x, y) &= x^2 + 2\left(\sqrt{3} + 2\right)\left(2\sqrt{2} + 3\right)xy + 2\left(-10\sqrt{6} - 16\sqrt{2} - \frac{29}{2}\sqrt{3} - \frac{93}{4}\right)x \\ &\quad + \left(4\sqrt{3} + 7\right)\left(12\sqrt{2} + 17\right)\left(y^2 - 2y\right), \\ H_{31}(x, y) &= 2x^2 + 4xy + 2y^2 - 2x + 2\left(2\sqrt{2} - \sqrt{3} + 2\right)y. \end{aligned}$$

and it is traveled in counterclockwise sense, see it in the figure

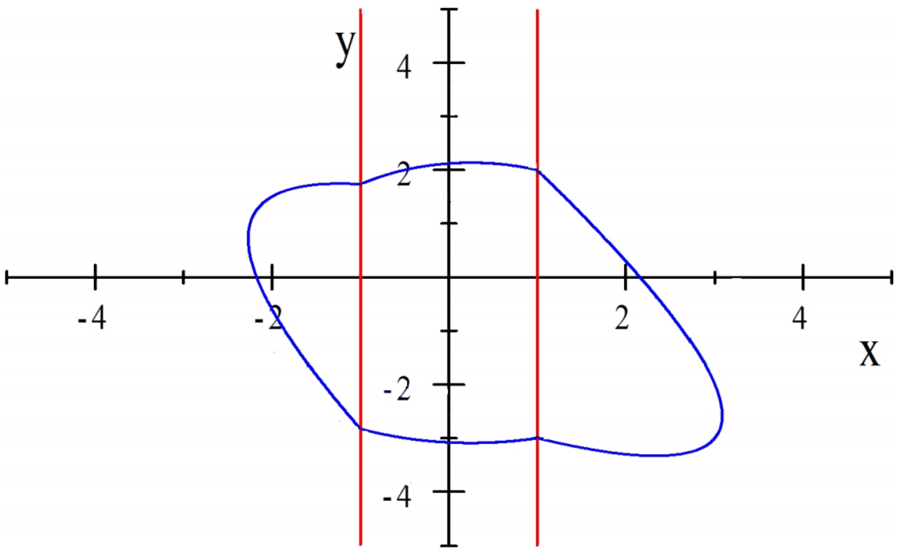


Figure 2.1: The limit cycle of the discontinuous piecewise linear differential system 2.11

Proof. The discontinuous piecewise linear differential system defined in (2.11) have no equilibrium point, neither real nor virtual.
The first integrals of the three linear differential systems (2.11) are

$$\begin{aligned} H_{11}(x,y) &= x^2+2xy+y^2+2x-y, \qquad (2.12)\\ H_{21}(x,y) &= x^2+2\left(\sqrt{3}+2\right)\left(2\sqrt{2}+3\right)xy+2\left(-10\sqrt{6}-16\sqrt{2}-\frac{29}{2}\sqrt{3}-\frac{93}{4}\right)x\\ &\quad+\left(4\sqrt{3}+7\right)\left(12\sqrt{2}+17\right)\left(y^2-2y\right),\\ H_{31}(x,y) &= 2x^2+4xy+2y^2-2x+2\left(2\sqrt{2}-\sqrt{3}+2\right)y. \end{aligned}$$

Then, the solutions of (2.11) can be obtained gluing pieces of parabolas and pieces of ellipses like curves defined by the level sets $H_{j1}=h_{j1}$ varying $h_j, j=1,2,3$. The piecewise algebraic curves are periodic orbits only if they are connex in the region Σ_+, Σ_0 and Σ_- where they are defined and they do not contain any real equilibrium point.
Now we shall use the notation and the expressions of the proof of theorem 2.3.2. System (2.3) can be written as the

$$\begin{cases} (y_1+y_2+1)(y_1-y_2)=0,\\ (y_1-y_3)(y_1+y_3)-1=0,\\ (y_2-y_4)(y_2+y_4)-1=0,\\ (y_3-y_4)\left(y_3+y_4+2\sqrt{2}-\sqrt{3}\right)=0. \end{cases}$$

Since $ad^2\neq 0$, $\alpha\beta^2\neq 0, y_1>y_2$ and $y_3>y_4$ the previous system is equivalent to the system

$$\begin{cases} (y_1 + y_2 + 1) = 0, \\ (y_1 - y_3)(y_1 + y_3) - 1 = 0, \\ (y_2 - y_4)(y_2 + y_4) - 1 = 0, \\ \left(y_3 + y_4 + 2\sqrt{2} - \sqrt{3}\right) = 0. \end{cases}$$

The unique solution (y_1, y_2, y_3, y_4) of this last system satisfying the necessary conditions $y_1 > y_2$ and $y_3 > y_4$ is $\left(y_1 = 2, y_2 = -3, y_3 = \sqrt{3}, y_4 = -2\sqrt{2}\right)$. Straightforward computations show that :
The implicit form of the solution of the first linear differential system of (2.11) passing through the crossing points $(1, y_1)$ and $(1, y_2)$ is

$$H_{11}(x, y) - 9 = 0.$$

The implicit form of the solution of the second linear differential system of (2.11) passing through the crossing points $(1, y_1)$ and $(-1, y_3)$ is

$$H_{21}(x, y) + 102.96 = 0.$$

The implicit form of the solution of the second linear differential system of (2.11) passing through the crossing points $(-1, y_4)$ and $(1, y_2)$ is

$$H_{21}(x, y) - 6776.8 = 0$$

and the implicit form of the solution of the third linear differential system of (2.11) passing through the crossing points $(-1, y_3)$ and $(-1, y_4)$ is

$$H_{31}(x, y) - \left(4\sqrt{6} + 4\right) = 0.$$

Then, the crossing periodic orbit Γ is algebraic of degree $(2, 2, 2)$ and writes as

$$\begin{aligned} \Gamma &= \{(x, y) \in \Sigma_+ : H_{11}(x, y) - 9 = 0\} \\ &\cup \{(x, y) \in \Sigma_0 : H_{21}(x, y) + 102.96 = 0\} \\ &\cup \{(x, y) \in \Sigma_0 : H_{21}(x, y) - 6776.8 = 0\} \\ &\cup \{(x, y) \in \Sigma_- : H_{31}(x, y) - 13.798 = 0\}. \end{aligned}$$

On the other hand, the orbit arc in Σ_+ starting from $(1, y_2)$ satisfies $\dot{x}_{|(1,y_2)} > 0$ and $\dot{y}_{|(1,y_2)} < 0$, so it runs in counterclockwise. The orbit arc in Σ_0 starting from $(1, y_1)$ satisfies $\dot{x}_{|(1,y_1)} < 0$ and $\dot{y}_{|(1,y_1)} < 0$, and so it runs in counterclockwise. The orbit arc in Σ_- starting from $(-1, y_3)$ satisfies $\dot{x}_{|(-1,y_3)} < 0$ and $\dot{y}_{|(-1,y_3)} > 0$, and so it runs in counterclockwise. The orbit arc in Σ_0 starting from $(-1, y_4)$ satisfies $\dot{x}_{|(-1,y_4)} > 0$ and $\dot{y}_{|(-1,y_4)} < 0$, and so it runs in counterclockwise. Furthermore, notice that system (2.11) has a sliding segment, namely
$\Sigma_s = \left\{(1, y) : -\frac{1}{2} \leq y \leq 4\sqrt{2} - 2\sqrt{6} + 3\sqrt{3} - 5\right\}$
$\cup \left\{(-1, y) : \frac{1}{2}\sqrt{3} - \frac{\sqrt{18}}{3} \leq y \leq 2\sqrt{6} - 4\sqrt{2} - 3\sqrt{3} + 7\right\}$,
that it is inside the periodic orbit. Drawing the orbit Γ we obtain the limit cycle of figure 2.1, which is traveled in counterclockwise sense.
This completes the proof of proposition 2.3.1. ■

2.4 Discussions and conclusions

In this chapter we studied the number of crossing limit cycles of the discontinuous planar piecewise differential systems formed by two or three linear differential systems separated by one or two parallel straight lines, such that the two or three linear differential have no equilibria, neither real nor virtual. We prove that if the discontinuous planar piecewise differential systems separated by one straight line they have no limit cycles. But when the piecewise differential systems are discontinuous separated by two parallel straight lines, we show that they can have at most one crossing algebraic limit cycle, and that there exist such systems with one limit cycle.

Chapter 3

Limit cycles of a class of piecewise linear differential system with only one equilibria which is a center

3.1 Introduction

In this chapter we study the maximum number of crossing limit cycles that can have the planar Hamiltonian continuous and discontinuous planar piecewise differential systems formed by two or three linear differential systems separated by one or two straight line, such that one of these differential systems is linear centre while other systems have no equilibria, neither real nor virtual.

3.2 Discontinuous piecewise linear differential system

Lemma 3.2.1 *A linear differential system having a center can be written as*

$$\dot{x} = -Bx - \frac{(4B^2 + w^2)}{4A}y + \delta, \qquad \dot{y} = Ax + By + D, \tag{3.1}$$

with $A > 0$ and $w > 0$. Moreover, this system has the first integral

$$H_2(x, y) = 4(Ax + By)^2 + 8A(Dx - \delta y) + w^2y^2 \tag{3.2}$$

Proof. Consider a general linear differential system in the $\mathbb{R}^2$

$$\dot{x} = ax + by + \delta, \qquad \dot{y} = Ax + By + D, \tag{3.3}$$

and assume that it has a center.
The eigenvalues of this system are

$$\frac{a + B \pm \sqrt{(a + B)^2 - 4(aB - Ab)}}{2}.$$

Then this system has a center if $a + B = 0$ and $(a + B)^2 - 4(aB - Ab) = -w^2$ for some $w > 0$ and $Ab < 0$, i.e. if $a = -B$, $b = -\dfrac{w^2 + 4B^2}{4A}$ and $A > 0$.
The system (3.3) has the first integral

$$H_2(x, y) = \alpha x + \eta y + Lxy + px^2 + qy^2, \tag{3.4}$$

then

$$\dot{x}\frac{\partial H_2}{\partial x} + \dot{y}\frac{\partial H_2}{\partial y} = 0,$$

thus

$$(-BL - \frac{4B^2 + w^2}{4A}\alpha + \delta L + B\eta + 2Dq)y + (-B\alpha + 2\delta p + A\eta + DL)x + (-2Bp + AL)x^2$$
$$+(2Aq - 2p\frac{4B^2 + w^2}{4A})xy + (-\frac{4B^2 + w^2}{4A}L + 2Bq)y^2 + \delta\alpha + D\eta = 0,$$

is equivalent to

$$\begin{cases} -BL - \dfrac{4B^2+w^2}{4A}\alpha + \delta L + B\eta + 2Dq = 0, \\ -B\alpha + 2\delta p + A\eta + DL = 0, \\ -2p\dfrac{4B^2+w^2}{4A} + 2Aq = 0, \\ -2Bp + AL = 0, \\ -\dfrac{4B^2+w^2}{4A}L + 2Bq = 0, \\ \delta\alpha + D\eta = 0. \end{cases}$$

As we solve this equations, we get

$$L = \frac{2B}{A}p, \quad q = \frac{4B^2+w^2}{4A^2}p, \quad \eta = \frac{-2\delta}{A}p, \quad \alpha = \frac{2D}{A}p,$$

if we take $p = 4A^2$ we get

$$\alpha = 8AD, \quad \eta = -8\delta A, \quad q = 4B^2 + w^2, \quad L = 8AB,$$

then (3.4) become

$$H_2(x,y) = 4(Ax+By)^2 + 8A(Dx-\delta y) + w^2y^2.$$

This completes the proof of the Lamma. ■

Theorem 3.2.1 . *A discontinuous piecewise linear differential system separated by one straight line with two linear systems, one of them is linear center the second have no equilibria, neither real nor virtual; has no algebraic limit cycles.*

Proof. Assume that we have a discontinuous piecewise linear differential system separated by one straight line with two linear systems, one of them is linear center the second have no equilibria, neither real nor virtual. Without loss of generality we can assume that the straight line is $x = 0$. We can assume that the linear center in the half-plane $x < 0$ can be given by the system

$$\dot{x} = -Bx - \frac{4B^2+w^2}{4A}y + \delta, \quad \dot{y} = Ax + By + D,$$

has the first integral

$$H_2(x,y) = 4(Ax+By)^2 + 8A(Dx-\delta y) + w^2y^2.$$

and finally from lemma 2.2.1 that the linear differential system which has no equilibrium points, and all its solutions are algebraic and given by parabolas in the half-plane $x > 0$ has a general expression as

$$\dot{x} = d(ax - ady + c) + b, \qquad \dot{y} = ax - ady + c,$$

this systems has the first integral

$$H_1(x, y) = ax^2 - 2adxy + ad^2y^2 + 2cx - 2(cd + b)y.$$

Therefore if the piecewise linear differential system has a periodic orbit candidate to be a limit cycle it must intersect the line $x = 0$ in exactly two points, namely $(0, y_1)$ and $(0, y_2)$ with $y_1 > y_2$. Since H_1 and H_2 are two first integrals they must satisfy the closing equations given by

$$\begin{aligned} H_1(0, y_1) - H_1(0, y_2) = 0, \\ H_2(0, y_1) - H_2(0, y_2) = 0, \end{aligned}$$

or equivalently

$$\begin{cases} (y_1 - y_2)\,(2b + 2cd - ad^2y_1 - ad^2y_2) = 0, \\ (y_1 - y_2)\,(-8A\delta + (4B^2 + w^2)\,(y_1 + y_2)) = 0. \end{cases}$$

since $y_1 > y_2$ the previous system is equivalent to this system

$$\begin{cases} (2b + 2cd - ad^2y_1 - ad^2y_2) = 0, \\ (-8A\delta + (4B^2 + w^2)\,(y_1 + y_2)) = 0, \end{cases}$$

then if $-8B^2b - 2bw^2 - 8B^2cd - 2cdw^2 + 8Aad^2\delta = 0$ we have

$$y_1 = \frac{1}{ad^2}\left(2b + 2cd - ad^2y_2\right),$$

but if $-8B^2b - 2bw^2 - 8B^2cd - 2cdw^2 + 8Aad^2\delta \neq 0$, this system has no solutions .

This system has either no solutions (y_1, y_2) satisfying the necessary condition $y_1 > y_2$, or it has a continuum of solutions. So the discontinuous piecewise linear differential system either does not have periodic solutions, or it has a continuum of periodic orbits, and consequently this differential system has no limit cycles. This completes the proof of the theorem 3.2.1. ■

Here we shall prove that Theorem 3.2.1 cannot be extended to discontinuous piecewise linear differential system separated by two parallel straight lines formed by one linear center.
Thus our main result is:

Theorem 3.2.2 *A discontinuous piecewise linear differential system separated by two parallel straight lines with three linear systems, two of them have no equilibria, neither real nor virtual and the third is a linear center, can have at most one crossing algebraic limit cycle. Moreover there are systems in this class having one limit cycle.*

Proof. Using the notations

$$\begin{aligned}\Sigma_+ &= \left\{(x,y)\in\mathbb{R}^2 : x>1\right\},\\ \Sigma_0 &= \left\{(x,y)\in\mathbb{R}^2 : -1<x<1\right\},\\ \Sigma_- &= \left\{(x,y)\in\mathbb{R}^2 : x<-1\right\}.\end{aligned}$$

Assume that we have a discontinuous piecewise linear differential system separated by two parallel straight lines with three linear systems, two of them have no equilibria, neither real nor virtual and the third is a linear center.
There are two possible cases as follows.

Case 1: the centre is located in the zone Σ_0.

Case 2: the centre is located in the zone Σ_- or Σ_+.

Case 1: We consider that the only center is defined in the region Σ_0. Doing a linear change of coordinates we can write the matrix of the linear center in Σ_0 in its real Jordan normal form, so the system in Σ_0 becomes

$$\dot{x} = -y-\omega, \quad \dot{y} = x-\xi,$$

and it has the first integral

$$H_3(x,y) = (y+\omega)^2 + (x-\xi)^2.$$

In the region Σ_+ we consider the arbitrary linear differential system

$$\dot{x} = d(ax-ady+c)+b, \qquad \dot{y} = ax-ady+c,$$

this system has the first integral

$$H_1(x,y) = ax^2 - 2adxy + ad^2y^2 + 2cx - 2(cd+b)y.$$

In the region Σ_- we consider the arbitrary linear differential system which have no equilibria

$$\dot{x} = \beta\left(\alpha x - \alpha\beta y + \lambda\right) + \gamma, \quad \dot{y} = \alpha x - \alpha\beta y + \lambda,$$

and having the first integral

$$H_4\left(x,y\right) = \alpha x^2 - 2\alpha\beta xy + \alpha\beta^2y^2 + 2\lambda x - 2\left(\lambda\beta+\gamma\right)y,$$

In order that the discontinuous piecewise linear differential system separated by two parallel straight lines formed by one linear center has a crossing limit cycle Γ, it must intersect each straight line $x=\pm1$ in exactly two points, namely $(1,y_1),(1,y_2),(-1,y_3)$ and $(-1,y_4)$ with $y_1>y_2$ and $y_3>y_4$. Hence the first integrals H_1, H_3 and H_4 must satisfy

$$\begin{aligned}&H_1(1,y_1) - H_1(1,y_2) = 0,\\ &H_3(1,y_1) - H_3(-1,y_3) = 0,\\ &H_3(1,y_2) - H_3(-1,y_4) = 0,\\ &H_4(-1,y_4) - H_4(-1,y_3) = 0,\end{aligned} \tag{3.5}$$

or equivalently

$$\begin{cases} (y_1 - y_2)(2b + 2ad + 2cd - ad^2y_1 - ad^2y_2) = 0, \\ (y_1 - y_3)(2\omega + y_1 + y_3) - 4\xi = 0, \\ (y_2 - y_4)(2\omega + y_2 + y_4) - 4\xi = 0, \\ (y_3 - y_4)(2\gamma - 2\alpha\beta + 2\lambda\beta - \alpha\beta^2y_3 - \alpha\beta^2y_4) = 0. \end{cases}$$

Since $y_1 > y_2$ and $y_3 > y_4$ the previous system is equivalent to the system

$$\begin{cases} F - L(y_1 + y_2) = 0, \\ (y_1 - y_3)(2\omega + y_1 + y_3) - 4\xi = 0, \\ (y_2 - y_4)(2\omega + y_2 + y_4) - 4\xi = 0, \\ G - N(y_3 + y_4) = 0, \end{cases} \tag{3.6}$$

where $F = 2b + 2ad + 2cd$, $L = ad^2$, $N = \alpha\beta^2$ and $G = 2\gamma - 2\alpha\beta + 2\lambda\beta$. Using maple the solutions (y_1, y_2, y_3, y_4) of this last system when

$$\frac{G^2L^2 - F^2N^2 + 16L^2N^2\xi - 4FLN^2\omega + 4GL^2N\omega}{(GL - FN)^3(GL + FN + 4LN\omega)} > 0$$

are $(y_1^*, y_2^*, y_3^*, y_4^*)$ and $(y_2^*, y_1^*, y_4^*, y_3^*)$ where

$$\begin{aligned} y_1^* &= \tfrac{F}{2L} + \tfrac{(G+2N\omega)(GL-FN)}{2N}\sqrt{\tfrac{G^2L^2-F^2N^2+16L^2N^2\xi-4FLN^2\omega+4GL^2N\omega}{(GL-FN)^3(GL+FN+4LN\omega)}}, \\ y_2^* &= \tfrac{F}{2L} - \tfrac{(G+2N\omega)(GL-FN)}{2N}\sqrt{\tfrac{G^2L^2-F^2N^2+16L^2N^2\xi-4FLN^2\omega+4GL^2N\omega}{(GL-FN)^3(GL+FN+4LN\omega)}}, \\ y_3^* &= \tfrac{G}{2N} + \tfrac{(F+2L\omega)(GL-FN)}{2L}\sqrt{\tfrac{G^2L^2-F^2N^2+16L^2N^2\xi-4FLN^2\omega+4GL^2N\omega}{(GL-FN)^3(GL+FN+4LN\omega)}}, \\ y_4^* &= \tfrac{G}{2N} - \tfrac{(F+2L\omega)(GL-FN)}{2L}\sqrt{\tfrac{G^2L^2-F^2N^2+16L^2N^2\xi-4FLN^2\omega+4GL^2N\omega}{(GL-FN)^3(GL+FN+4LN\omega)}}. \end{aligned}$$

But due to the fact that $y_1 > y_2$ and $y_3 > y_4$, at most one of these two solutions will be satisfactory. So, the discontinuous piecewise linear differential system separated by two parallel straight lines formed by one linear center. can have at most one limit cycle.

Case 2: We consider a planar discontinuous piecewise linear differential system separated by two parallel straight lines and formed by three linear systems , two of them have no equilibria, neither real nor virtual and the third is a linear center defined in Σ_+, we have that these linear systems can be as follows

$$\begin{array}{lll} \dot{x} = -Bx - \frac{4B^2+w^2}{4A}y + \delta, & \dot{y} = Ax + By + D, & \Sigma_+, \\ \dot{x} = \beta(\alpha x - \alpha\beta y + \lambda) + \gamma, & \dot{y} = \alpha x - \alpha\beta y + \lambda & \Sigma_0, \\ \dot{x} = d(ax - ady + c) + b, & \dot{y} = ax - ady + c, & \Sigma_-. \end{array} \tag{3.7}$$

These linear systems have the first integrals

$$\begin{aligned} H_1(x, y) &= 4(Ax + By)^2 + 8A(Dx - \delta y) + w^2y^2, \\ H_2(x, y) &= \alpha x^2 - 2\alpha\beta xy + \alpha\beta^2y^2 + 2\lambda x - 2(\lambda\beta + \gamma)y, \\ H_3(x, y) &= ax^2 - 2adxy + ad^2y^2 + 2cx - 2(cd + b)y. \end{aligned}$$

respectively.
We are going to analyze if the discontinuous piecewise linear differential system (3.7) has crossing periodic solutions. Since the orbits in each region Σ_0, and Σ_-, are pieces of one parabola and the orbit in region Σ_+ is pieces of one ellipse, we have that if there is a crossing limit cycle this must intersect each straight line $x = \pm 1$ in exactly two points namely $(1, y_1), (1, y_2), (-1, y_3)$ and $(-1, y_4)$ with $y_1 > y_2$ and $y_3 > y_4$. Therefore we must study the solutions of the closing equations

$$\begin{aligned} &H_1(1, y_1) - H_1(1, y_2) = 0, \\ &H_2(1, y_1) - H_2(-1, y_3) = 0, \\ &H_2(1, y_2) - H_2(-1, y_4) = 0, \\ &H_3(-1, y_4) - H_3(-1, y_3) = 0, \end{aligned} \tag{3.8}$$

or equivalently, we have the system

$$\begin{cases} (y_1 - y_2)\left(-8A\delta + 4B^2 y_1 + 4B^2 y_2 + w^2 y_1 + w^2 y_2 + 8AB\right) = 0, \\ 4\lambda + \alpha\beta^2\left(y_1^2 - y_3^2\right) - 2\left(\gamma + \beta\lambda\right)\left(y_1 - y_3\right) - 2\alpha\beta\left(y_1 + y_3\right) = 0, \\ 4\lambda + \alpha\beta^2\left(y_2^2 - y_4^2\right) - 2\left(\gamma + \beta\lambda\right)\left(y_2 - y_4\right) - 2\alpha\beta\left(y_2 + y_4\right) = 0, \\ (y_3 - y_4)\left(2b - 2ad + 2cd - ad^2 y_3 - ad^2 y_4\right) = 0, \end{cases} \tag{3.9}$$

By hypothesis $y_1 > y_2$ and $y_3 > y_4$ and therefore system (3.9) is equivalently to the system

$$\begin{cases} \left(4B^2 + w^2\right)(y_1 + y_2) + 8A\left(B - \delta\right) = 0, \\ 4\lambda + \alpha\beta^2\left(y_1^2 - y_3^2\right) - 2\left(\gamma + \beta\lambda\right)\left(y_1 - y_3\right) - 2\alpha\beta\left(y_1 + y_3\right) = 0, \\ 4\lambda + \alpha\beta^2\left(y_2^2 - y_4^2\right) - 2\left(\gamma + \beta\lambda\right)\left(y_2 - y_4\right) - 2\alpha\beta\left(y_2 + y_4\right) = 0, \\ ad^2\left(y_3 + y_4\right) + 2\left(-b + ad - cd\right) = 0, \end{cases}$$

thus

$$\begin{cases} \gamma\left(y_1 + y_2\right) + \gamma_2 = 0, \\ 4\lambda + l_1\left(y_1^2 - y_3^2\right) - l_2\left(y_1 - y_3\right) - l_3\left(y_1 + y_3\right) = 0, \\ 4\lambda + l_1\left(y_2^2 - y_4^2\right) - l_2\left(y_2 - y_4\right) - l_3\left(y_2 + y_4\right) = 0, \\ \delta_1 - \delta_2\left(y_3 + y_4\right) = 0, \end{cases} \tag{3.10}$$

where $\gamma = \left(4B^2 + w^2\right), \gamma_2 = 8A\left(B - \delta\right), \delta_1 = 2b - 2ad + 2cd, \delta_2 = ad^2, l_1 = \alpha\beta^2, l_2 = 2\left(\gamma + \beta\lambda\right), l_3 = 2\alpha\beta$. As $\gamma \neq 0$ and $\delta_2 \neq 0$,
Looking at system (3.10) we remark that if (y_1, y_2, y_3, y_4) is a solution, then (y_2, y_1, y_4, y_3) is also a solution, but due to the fact that $y_1 > y_2$ and $y_3 > y_4$, at most one of these two solutions will be satisfactory.
Since $\gamma \neq 0$ and $\delta_2 \neq 0$ from the first and fourth equations of (3.10) we can isolated y_1 and y_4, respectively. Then, we obtain

$$y_1 = -\frac{(\gamma_2 + \gamma y_2)}{\gamma}, \qquad y_4 = \frac{\delta_1 - \delta_2 y_3}{\delta_2}.$$

Now replacing these expressions of y_1 and y_4 in the second and third equations of (3.10), we have the system of two equations

$$\begin{aligned}&l_1 y_2^2+\left(l_2+l_3+\tfrac{2}{\gamma}\gamma_2 l_1\right)y_2-l_1 y_3^2+(l_2-l_3)\,y_3+\left(4\lambda+\tfrac{1}{\gamma}\gamma_2 l_2+\tfrac{1}{\gamma}\gamma_2 l_3+\tfrac{1}{\gamma^2}\gamma_2^2 l_1\right)=0,\\&l_1 y_2^2-(l_2+l_3)\,y_2-l_1 y_3^2+\left(l_3-l_2+2\tfrac{\delta_1}{\delta_2}l_1\right)y_3+\left(4\lambda+\tfrac{\delta_1}{\delta_2}l_2-\tfrac{\delta_1}{\delta_2}l_3-\tfrac{\delta_1^2}{\delta_2^2}l_1\right)=0.\end{aligned}\tag{3.11}$$

Summing up the first equation and the second equation of (3.11), we get

$$\begin{aligned}&\left(2l_2+2l_3+\tfrac{2}{\gamma}\gamma_2 l_1\right)y_2+\left(2l_2-2l_3-2\tfrac{\delta_1}{\delta_2}l_1\right)y_3\\&+\left(\tfrac{1}{\gamma}\gamma_2 l_2+\tfrac{1}{\gamma}\gamma_2 l_3-\tfrac{\delta_1}{\delta_2}l_2+\tfrac{\delta_1}{\delta_2}l_3+\tfrac{1}{\gamma^2}\gamma_2^2 l_1+\tfrac{\delta_1^2}{\delta_2^2}l_1\right)=0,\end{aligned}\tag{3.12}$$

from (3.12) we isolated y_2 we obtain

$$y_2=\gamma\frac{\delta_1 l_1-\delta_2 l_2+\delta_2 l_3}{\delta_2\left(\gamma l_2+\gamma l_3+\gamma_2 l_1\right)}y_3-\frac{\gamma^2\delta_1^2 l_1+\gamma_2^2\delta_2^2 l_1+\gamma\gamma_2\delta_2^2 l_2-\gamma^2\delta_1\delta_2 l_2+\gamma\gamma_2\delta_2^2 l_3+\gamma^2\delta_1\delta_2 l_3}{2\gamma\delta_2^2\left(\gamma l_2+\gamma l_3+\gamma_2 l_1\right)}.\tag{3.13}$$

Substituting (3.13) into the second equation of (3.11), we get

$$m_0+m_1 y_3+m_2 y_3^2=0,$$

where

$$\begin{aligned}m_0&=4\lambda+\frac{\delta_1}{\delta_2}(l_2-l_3)-\frac{\delta_1^2}{\delta_2^2}l_1\\&+\frac{(l_2+l_3)\left(\gamma^2\delta_1^2 l_1+\gamma_2^2\delta_2^2 l_1+\gamma\gamma_2\delta_2^2 l_2-\gamma^2\delta_1\delta_2 l_2+\gamma\gamma_2\delta_2^2 l_3+\gamma^2\delta_1\delta_2 l_3\right)}{2\gamma\delta_2^2\left(\gamma l_2+\gamma l_3+\gamma_2 l_1\right)}\\&+\frac{l_1\left(\gamma^2\delta_1^2 l_1+\gamma_2^2\delta_2^2 l_1+\gamma\gamma_2\delta_2^2 l_2-\gamma^2\delta_1\delta_2 l_2+\gamma\gamma_2\delta_2^2 l_3+\gamma^2\delta_1\delta_2 l_3\right)^2}{4\gamma^2\delta_2^4\left(\gamma l_2+\gamma l_3+\gamma_2 l_1\right)^2},\\m_1&=\frac{-\delta_1 l_1\left(\gamma\delta_1 l_1-2\gamma\delta_2 l_2-\gamma_2\delta_2 l_1\right)\left(\gamma\delta_1 l_1+2\gamma\delta_2 l_3+\gamma_2\delta_2 l_1\right)}{\delta_2^3\left(\gamma l_2+\gamma l_3+\gamma_2 l_1\right)^2},\\m_2&=\frac{\gamma^2 l_1\left(\delta_1 l_1-\delta_2 l_2+\delta_2 l_3\right)^2}{\delta_2^2\left(\gamma l_2+\gamma l_3+\gamma_2 l_1\right)^2}-l_1.\end{aligned}\tag{3.14}$$

Now, solving (3.11) reduces to solve

$$\begin{aligned}m_0+m_1 y_3+m_2 y_3^2&=0,\\k_0+k_1 y_3+k_2 y_2&=0,\end{aligned}\tag{3.15}$$

where m_0, m_1, and m_2 are defined in (3.14) and

$$\begin{aligned}k_0&=\left(\gamma^2\delta_1^2 l_1+\gamma_2^2\delta_2^2 l_1+\gamma\gamma_2\delta_2^2 l_2-\gamma^2\delta_1\delta_2 l_2+\gamma\gamma_2\delta_2^2 l_3+\gamma^2\delta_1\delta_2 l_3\right),\\k_1&=-2\gamma^2\delta_2\left(\delta_1 l_1-\delta_2 l_2+\delta_2 l_3\right),\\k_2&=2\gamma\delta_2^2\left(\gamma l_2+\gamma l_3+\gamma_2 l_1\right).\end{aligned}$$

Eventually system (3.15) could have a continuum of solutions (y_2, y_3) if some coefficients of the polynomials that there appear are zero, but then the possible periodic solutions would not be limit cycles. So assume that system (3.15) has finitely many solutions. We recall that Bezout Theorem states that if a polynomial differential system of equations has finitely many solutions, then the number of its solutions is at most the product of the degrees of the polynomials which appear in the system. Then by Bezout Theorem system (3.15) has at most two solutions. Finally from these at most two solutions (y_2, y_3) of system (3.15) we get two solutions (y_1, y_2, y_3, y_4) of system (3.9), but from the previous remark at most one of these two solution would satisfy $y_1 > y_2$ and $y_3 > y_4$. In summary, we have proved that at most we can have one limit cycle. This will complete the proof of Theorem 3.2.2. ■

The next proposition shows that there are discontinuous piecewise linear differential system separated by two parallel straight lines formed by one linear center with one crossing algebraic limit cycle:

Proposition 3.2.1 *(**case 1**) The discontinuous piecewise linear differential system defined by*

$$\begin{array}{lll} \dot{x} = -x - y + \dfrac{1}{2}, & \dot{y} = x + y + 1 & \textit{in } \Sigma_+, \\ \dot{x} = -y, & \dot{y} = x - \dfrac{1}{4} & \textit{in } \Sigma_0, \\ \dot{x} = 2\left(x - y - \sqrt{2} + \dfrac{\sqrt{3}}{2} + 1\right), & \dot{y} = 2x - 2y - 1 & \textit{in } \Sigma_-, \end{array} \tag{3.16}$$

where a, d, c, α, β and λ real constants, has a unique crossing limit cycle surrounding the $\Sigma_s = \left\{(1, y) : -\frac{1}{2} \leq y \leq 0\right\} \cup \left\{(-1, y) : -\sqrt{2} + \frac{1}{2}\sqrt{3} \leq y \leq 0\right\}$ and the center $\left(\frac{1}{4}, 0\right)$ when $d \neq 0, \beta \neq 0, \alpha\lambda < 0, ac > 0$. Moreover, this limit cycle is algebraic of degree $(2, 2, 2)$ and writes as

$$\begin{aligned} \Gamma_1 &= \{(x, y) \in \Sigma_+ : H_{11}(x, y) - 9 = 0\} \\ &\cup \left\{(x, y) \in \Sigma_0 : H_{21}(x, y) - \frac{73}{16} = 0\right\} \\ &\cup \left\{(x, y) \in \Sigma_0 : H_{21}(x, y) - \frac{153}{16} = 0\right\} \\ &\cup \{(x, y) \in \Sigma_- : H_{31}(x, y) - 13.798 = 0\}, \end{aligned}$$

with

$$\begin{aligned} H_{11}(x, y) &= x^2 + 2xy + y^2 + 2x - y, \\ H_{21}(x, y) &= y^2 + \left(x - \frac{1}{4}\right)^2, \\ H_{31}(x, y) &= 2x^2 - 4xy + 2y^2 - 2x + 2\left(\left(2\sqrt{2} - \sqrt{3}\right) - 2\right) y. \end{aligned}$$

and it is traveled in counterclockwise sense, see it in the figure

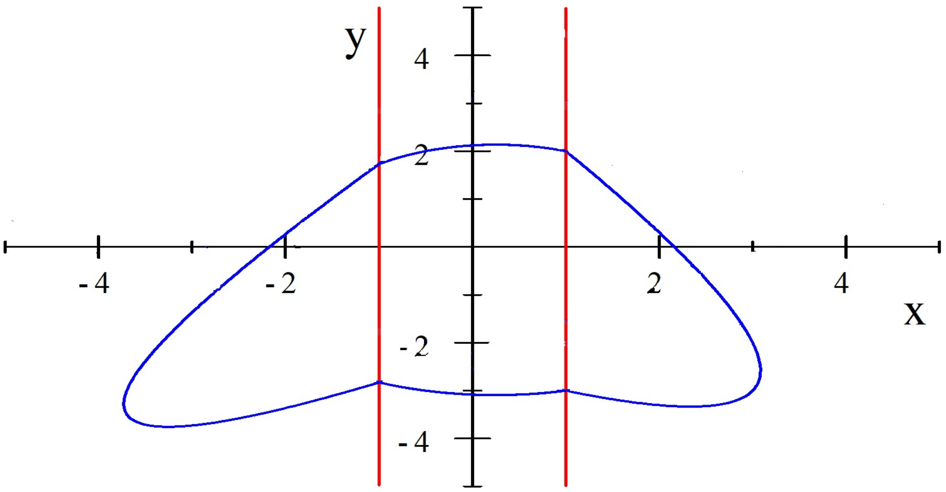


Figure 3.1: The limit cycle of the discontinuous piecewise linear differential system 3.16

Proof. The discontinuous piecewise linear differential system defined in (3.16) have one equilibrium point $\left(\frac{1}{4}, 0\right)$ Since $\pm i$ are the eigenvalues of the matrix of the second linear differential systems of (3.16), this system have its equilibria as center.
The first integrals of the three linear differential systems (3.16) are

$$\begin{aligned} H_{11}(x,y) &= x^2+2xy+y^2+2x-y, \\ H_{21}(x,y) &= y^2+\left(x-\frac{1}{4}\right)^2, \\ H_{31}(x,y) &= 2x^2-4xy+2y^2-2x+2\left(\left(2\sqrt{2}-\sqrt{3}\right)-2\right)y. \end{aligned}$$

Then, the solutions of (3.16) can be obtained gluing pieces of parabolas and pieces of ellipses like curves defined by the level sets $H_{j1}=h_{j1}$ varying $h_j, j=1,2,3$, . The piecewise algebraic curves are periodic orbits only if they are connex in the region Σ_+, Σ_0 and Σ_- where they are defined and they do not contain any real equilibrium point.
Now we shall use the notation and the expressions of the proof of theorem 3.2.2. System (3.5) can be written as the

$$\begin{cases} (y_1+y_2+1)(y_1-y_2)=0, \\ (y_1-y_3)(y_1+y_3)-1=0, \\ (y_2-y_4)(y_2+y_4)-1=0, \\ (y_3-y_4)\left(y_3+y_4+2\sqrt{2}-\sqrt{3}\right)=0. \end{cases}$$

Since $ad^2\neq 0$, $\alpha\beta^2\neq 0, y_1>y_2$ and $y_3>y_4$ the previous system is equivalent to the system

$$\begin{cases} (y_1 + y_2 + 1) = 0, \\ (y_1 - y_3)(y_1 + y_3) - 1 = 0, \\ (y_2 - y_4)(y_2 + y_4) - 1 = 0, \\ \left(y_3 + y_4 + 2\sqrt{2} - \sqrt{3}\right) = 0. \end{cases}$$

The unique solution (y_1, y_2, y_3, y_4) of this last system satisfying the necessary conditions $y_1 > y_2$ and $y_3 > y_4$ is $\left(y_1 = 2, y_2 = -3, y_3 = \sqrt{3}, y_4 = -2\sqrt{2}\right)$. Straightforward computations show that :
The implicit form of the solution of the first linear differential system of (3.16) passing through the crossing points $(1, y_1)$ and $(1, y_2)$ is

$$H_{11}(x, y) - 9 = 0.$$

The implicit form of the solution of the second linear differential system of (3.16) passing through the crossing points $(1, y_1)$ and $(-1, y_3)$ is

$$H_{21}(x, y) - \frac{73}{16} = 0.$$

The implicit form of the solution of the second linear differential system of (3.16) passing through the crossing points $(-1, y_4)$ and $(1, y_2)$ is

$$H_{21}(x, y) - \frac{153}{16} = 0$$

and the implicit form of the solution of the third linear differential system of (3.16) passing through the crossing points $(-1, y_3)$ and $(-1, y_4)$ is

$$H_{31}(x, y) - \left(4\sqrt{6} + 4\right) = 0.$$

Then, the crossing periodic orbit Γ is algebraic of degree $(2, 2, 2)$ and writes as

$$\begin{aligned} \Gamma_1 &= \{(x, y) \in \Sigma_+ : H_{11}(x, y) - 9 = 0\} \\ &\cup \left\{(x, y) \in \Sigma_0 : H_{21}(x, y) - \frac{73}{16} = 0\right\} \\ &\cup \left\{(x, y) \in \Sigma_0 : H_{21}(x, y) - \frac{153}{16} = 0\right\} \\ &\cup \{(x, y) \in \Sigma_- : H_{31}(x, y) - 13.798 = 0\}, \end{aligned}$$

On the other hand, the orbit arc in Σ_+ starting from $(1, y_2)$satisfies $\dot{x}_{|(1,y_2)} > 0$ and $\dot{y}_{|(1,y_2)} < 0$, so it runs in counterclockwise. The orbit arc in Σ_0 starting from $(1, y_1)$ satisfies $\dot{x}_{|(1,y_1)} < 0$ and $\dot{y}_{|(1,y_1)} > 0$, and so it runs in counterclockwise. The orbit arc in Σ_- starting from $(-1, y_3)$ satisfies $\dot{x}_{|(-1,y_3)} < 0$ and $\dot{y}_{|(-1,y_3)} < 0$, and so it runs in counterclockwise. The orbit arc in Σ_0 starting from $(-1, y_4)$ satisfies

$\dot{x}_{|(-1,y_4)} > 0$ and $\dot{y}_{|(-1,y_4)} < 0$, and so it runs in counterclockwise. Furthermore, notice that system (3.16) has a sliding segment, namely $\Sigma_s = \left\{(1,y) : -\frac{1}{2} \leq y \leq 0\right\} \cup \left\{(-1,y) : -\sqrt{2}+\frac{1}{2}\sqrt{3} \leq y \leq 0\right\}$, that it is inside the periodic orbit Drawing the orbit Γ we obtain the limit cycle of figure3.1, which is traveled in counterclockwise sense.

This completes the proof of proposition 3.2.1 ∎

Proposition 3.2.2 ***(case 2)****The discontinuous piecewise linear differential system defined by*

$$\begin{array}{lll} \dot{x} = -y - \frac{1}{2}, & \dot{y} = x - 1 & in \ \Sigma_+, \\ \dot{x} = -\left(lx + ky - k\right), & \dot{y} = 2x + ly + f & in \ \Sigma_0, \\ \dot{x} = -2\left(x + y + \sqrt{2} - \dfrac{\sqrt{3}}{2} + 1\right), & \dot{y} = 2x + 2y - 1 & in \ \Sigma_-, \end{array} \tag{3.17}$$

where $k = \left(168\sqrt{2} + 136\sqrt{3} + 96\sqrt{6} + 238\right), f = \left(20\sqrt{6} + 32\sqrt{2} + 29\sqrt{3} + \frac{93}{2}\right)$ *and* $l = \left(4\sqrt{6} + 8\sqrt{2} + 6\sqrt{3} + 12\right)$ *has a unique crossing limit cycle surrounding the sliding set*

$$\Sigma_s = \left\{(1,y) : -\frac{1}{2} \leq y \leq 4\sqrt{2} - 2\sqrt{2}\sqrt{3} + 3\sqrt{3} - 5\right\}$$

$$\cup \left\{(-1,y) : \frac{1}{2}\sqrt{3} - \frac{\sqrt{18}}{3} \leq y \leq 4\sqrt{2} - 2\sqrt{2}\sqrt{3} + 3\sqrt{3} - 5\right\},$$

and the center $\left(1, -\frac{1}{2}\right)$*. Moreover, this limit cycle is algebraic of degree* $(2,2,2)$ *and writes as*

$$\begin{aligned} \Gamma_1 \ &= \ \left\{(x,y) \in \Sigma_+ : H_{11}(x,y) - \frac{25}{4} = 0\right\} \\ &\quad \cup \left\{(x,y) \in \Sigma_0 : H_{21}(x,y) + 102.96 = 0\right\} \\ &\quad \cup \left\{(x,y) \in \Sigma_0 : H_{21}(x,y) - 6776.8 = 0\right\} \\ &\quad \cup \left\{(x,y) \in \Sigma_- : H_{31}(x,y) - \left(4\sqrt{6} + 4\right) = 0\right\}, \end{aligned}$$

with

$$\begin{aligned} H_{11}(x,y) \ &= \ x^2 - 2x + y^2 + y + \frac{5}{4}, \\ H_{21}(x,y) \ &= \ x^2 + \left(2\sqrt{3}+4\right)\left(2\sqrt{2}+3\right)xy + \left(-20\sqrt{6} - 32\sqrt{2} - 29\sqrt{3} - \frac{93}{2}\right)x \\ &\quad + \left(4\sqrt{3}+7\right)\left(12\sqrt{2}+17\right)y^2 - 2\left(4\sqrt{3}+7\right)\left(12\sqrt{2}+17\right)y, \\ H_{31}(x,y) \ &= \ 2x^2 + 4xy + 2y^2 - 2x + 2\left(2\sqrt{2} - \sqrt{3} + 2\right)y, \end{aligned}$$

and it is traveled in counterclockwise sense, see it in the figure

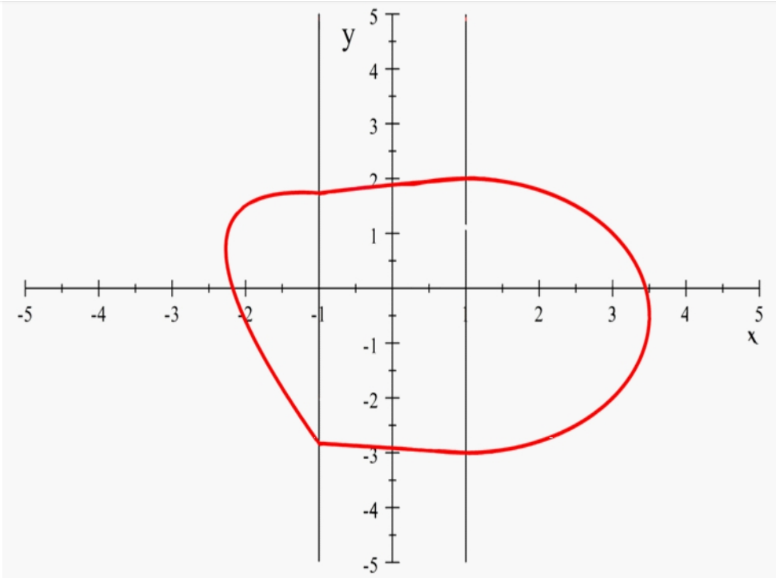


Figure 3.2: The limit cycle of the discontinuous piecewise linear differential system 3.17

Proof. The discontinuous piecewise linear differential system defined in (3.17) have one equilibrium point $\left(1, -\frac{1}{2}\right)$ Since $\pm i$ are the eigenvalues of the matrix of the first linear differential systems of (3.17), this system have its equilibria as center.
The first integrals of the three linear differential systems (3.17) are

$$
\begin{aligned}
H_{11}(x,y) &= x^2 - 2x + y^2 + y + \frac{5}{4}, \\
H_{21}(x,y) &= x^2 + \left(2\sqrt{3}+4\right)\left(2\sqrt{2}+3\right)xy + \left(-20\sqrt{6} - 32\sqrt{2} - 29\sqrt{3} - \frac{93}{2}\right)x \\
&\quad + \left(4\sqrt{3}+7\right)\left(12\sqrt{2}+17\right)y^2 - 2\left(4\sqrt{3}+7\right)\left(12\sqrt{2}+17\right)y, \\
H_{31}(x,y) &= 2x^2 + 4xy + 2y^2 - 2x + 2\left(2\sqrt{2} - \sqrt{3} + 2\right)y.
\end{aligned}
$$

The piecewise algebraic curves are periodic orbits only if they are connex in the region Σ_+, Σ_0 and Σ_- where they are defined and they do not contain any real equilibrium point.
Now we shall use the notation and the expressions of the proof of theorem 3.2.2. System (3.8) can be written as the

$$
\begin{cases}
(y_1 + y_2 + 1)(y_1 - y_2) = 0, \\
(y_1 - y_3)(y_1 + y_3) - 1 = 0, \\
(y_2 - y_4)(y_2 + y_4) - 1 = 0, \\
(y_3 - y_4)\left(y_4 + y_4 + 2\sqrt{2} - \sqrt{3}\right) = 0.
\end{cases}
$$

Since $y_1 > y_2$ and $y_3 > y_4$ the previous system is equivalent to the system

$$\begin{cases} (y_1 + y_2 + 1) = 0, \\ (y_1 - y_3)(y_1 + y_3) - 1 = 0, \\ (y_2 - y_4)(y_2 + y_4) - 1 = 0, \\ \left(y_3 + y_4 + 2\sqrt{2} - \sqrt{3}\right) = 0. \end{cases}$$

The unique solution (y_1, y_2, y_3, y_4) of this last system satisfying the necessary conditions $y_1 > y_2$ and $y_3 > y_4$ is $\left(y_1 = 2, y_2 = -3, y_3 = \sqrt{3}, y_4 = -2\sqrt{2}\right)$. Straightforward computations show that :
The implicit form of the solution of the first linear differential system of (3.17) passing through the crossing points $(1, y_1)$ and $(1, y_2)$ is

$$H_{11}(x, y) - \frac{25}{4} = 0.$$

The implicit form of the solution of the second linear differential system of (3.17) passing through the crossing points $(1, y_1)$ and $(-1, y_3)$ is

$$H_{21}(x, y) + 102.96 = 0.$$

The implicit form of the solution of the second linear differential system of (3.17) passing through the crossing points $(-1, y_4)$ and $(1, y_2)$ is

$$H_{21}(x, y) - 6776.8 = 0$$

and the implicit form of the solution of the third linear differential system of (3.17) passing through the crossing points $(-1, y_3)$ and $(-1, y_4)$ is

$$H_{31}(x, y) - \left(4\sqrt{6} + 4\right) = 0.$$

Then, the crossing periodic orbit Γ is algebraic of degree $(2, 2, 2)$ and writes as

$$\begin{aligned} \Gamma_1 \;=\; & \left\{(x, y) \in \Sigma_+ : H_{11}(x, y) - \frac{25}{4} = 0\right\} \\ & \cup \{(x, y) \in \Sigma_0 : H_{21}(x, y) + 102.96 = 0\} \\ & \cup \{(x, y) \in \Sigma_0 : H_{21}(x, y) - 6776.8 = 0\} \\ & \cup \left\{(x, y) \in \Sigma_- : H_{31}(x, y) - \left(4\sqrt{6} + 4\right) = 0\right\}. \end{aligned}$$

On the other hand, the orbit arc in Σ_+ starting from $(1, y_2)$ satisfies $\dot{x}_{|(1,y_2)} > 0$ and $\dot{y}_{|(1,y_2)} = 0$, so it runs in counterclockwise. The orbit arc in Σ_0 starting from $(1, y_1)$ satisfies $\dot{x}_{|(1,y_1)} < 0$ and $\dot{y}_{|(1,y_1)} < 0$, and so it runs in counterclockwise. The orbit arc in Σ_- starting from $(-1, y_3)$ satisfies $\dot{x}_{|(-1,y_3)} < 0$ and $\dot{y}_{|(-1,y_3)} > 0$, and so it runs in counterclockwise. The orbit arc in Σ_0 starting from $(-1, y_4)$ satisfies $\dot{x}_{|(-1,y_4)} > 0$ and $\dot{y}_{|(-1,y_4)} < 0$, and so it runs in counterclockwise. Furthermore,

notice that system (3.16) has a sliding segment, namely
$\Sigma_s = \left\{(1, y) : -\frac{1}{2} \le y \le 4\sqrt{2} - 2\sqrt{6} + 3\sqrt{3} - 5\right\}$
$\cup \left\{(-1, y) : \frac{1}{6} - \frac{\sqrt{18}}{3} \le y \le 4\sqrt{2} - 2\sqrt{6} + 3\sqrt{3} - 5\right\}$,
that it is inside the periodic orbit. Drawing the orbit Γ we obtain the limit cycle of figure 3.2, which is traveled in counterclockwise sense.
This completes the proof of proposition 3.2.2 ■

3.3 Continuous piecewise linear differential system

The easiest continuous piecewise linear differential systems are formed by two linear differential systems separated by a straight line. It is known that such systems have at most one limit cycle. But it is also known that if one linear differential systems are linear center then the continuous piecewise linear differential system has no limit cycles.

Lemma 3.3.1 *A continuous piecewise linear differential system separated by one straight line formed by one linear center can be written as*

$$\begin{aligned} \dot{x} &= -Bx - \frac{4B^2 + w^2}{4A}y + \delta, \qquad \dot{y} = \frac{4AB^2}{4B^2 + w^2}x + By + D, \qquad \textit{in } x < 0, \\ \dot{x} &= -Bx - \frac{4B^2 + w^2}{4A}y + \delta, \qquad \dot{y} = Ax + By + D, \qquad \textit{in } x > 0, \end{aligned}$$

with $-4B^2D - w^2D - 4AB\delta \neq 0, A > 0$ *and* $w > 0$*. Moreover, this systems has the first integral*

$$\begin{aligned} H_1(x, y) &= 4A\frac{B^2}{4B^2 + w^2}x2 + 2Bxy + 2Dx + \frac{1}{4A}(4B^2 + w^2)y^2 - 2\delta y, \quad (3.18) \\ H_2(x, y) &= 4(Ax + By)^2 + 8A(Dx - \delta y) + w^2y^2, \quad (3.19) \end{aligned}$$

in $x < 0$ *and* $x > 0$ *respectively.*

Proof. Assume that we have a continuous piecewise linear differential system separated by one straight line with two linear systems, one of them is linear center the second have no equilibria, neither real nor virtual. Without loss of generality, we can assume that the straight line of continuity is $x = 0$,and that the linear center in the half-plane $x > 0$ is given by the system

$$\dot{x} = -Bx - \frac{(4B^2 + w^2)}{4A}y + \delta, \qquad \dot{y} = Ax + By + D, \quad (3.20)$$

which has the first integral

$$H_2(x, y) = 4(Ax + By)^2 + 8A(Dx - \delta y) + w^2y^2, \quad (3.21)$$

while the linear system in the half-plane $x < 0$ is given by the system

$$\dot{x} = d(ax - ady + c) + b, \qquad \dot{y} = ax - ady + c, \tag{3.22}$$

this systems has the first integral

$$H_1(x, y) = ax^2 - 2adxy + ad^2y^2 + 2cx - 2(cd + b)y.$$

Since we must have a continuous piecewise linear differential system , both systems, the (3.20) and the (3.22) must coincide on $x = 0$, therefore

$$\begin{cases} -ad^2y + cd + b = -\dfrac{4B^2 + w^2}{4A}y + \delta, \\ -ady + c = By + D, \end{cases}$$

then

$$ad^2 = \frac{4B^2 + w^2}{4A}, \qquad cd + b = \delta, \qquad ad = -B, \quad c = D,$$

thus

$$\begin{array}{ll} d = -\dfrac{4B^2 + w^2}{4BA}, & a = 4A\dfrac{B^2}{4B^2 + w^2}, \\ b = \delta - D\dfrac{4B^2 + w^2}{-4BA}, & c = D, \end{array}$$

then the system (3.20) and (3.22) can be written as

$$\begin{array}{lll} \dot{x} = -Bx - \dfrac{4B^2 + w^2}{4A}y + \delta, & \dot{y} = \dfrac{4AB^2}{4B^2 + w^2}x + By + D & x < 0, \\ \dot{x} = -Bx - \dfrac{4B^2 + w^2}{4A}y + \delta, & \dot{y} = Ax + By + D, & x > 0, \end{array}$$

with $-4B^2D - w^2D - 4AB\delta \neq 0, A > 0$ and $w > 0$ and we get the first integral of this systems

$$\begin{array}{rcl} H_1(x, y) & = & \dfrac{4AB^2}{4B^2 + w^2}x2 + 2Bxy + 2Dx + \dfrac{1}{4A}(4B^2 + w^2)y^2 - 2\delta y, \\ H_2(x, y) & = & 4(Ax + By)^2 + 8A(Dx - \delta y) + w^2y^2, \end{array}$$

in $x < 0$ and $x > 0$ respectively. ■

Theorem 3.3.1 *A continuous piecewise linear differential system separated by one straight line with two linear systems, one of them is linear center the second have no equilibria, neither real nor virtual, has no limit cycles.*

Proof. Assume that we have a continuous piecewise linear differential system separated by one straight line with two linear systems, one of them is linear center the second have no equilibria, neither real nor virtual. Without loss of generality we can assume that the straight line of continuity is $x = 0$, and that the linear center in the half-plane $x > 0$ is given by

$$\dot{x} = -Bx - \frac{4B^2 + w^2}{4A}y + \delta, \dot{y} = Ax + By + D, \quad x > 0,$$

and the linear system in the half-plane $x < 0$ is given by

$$\dot{x} = -Bx - \frac{4B^2 + w^2}{4A}y + \delta, \qquad \dot{y} = \frac{4AB^2}{4B^2 + w^2}x + By + D \quad x < 0,$$

with the first integrals

$$\begin{aligned} H_1(x,y) &= \frac{4AB^2}{4B^2 + w^2}x2 + 2Bxy + 2Dx + \frac{1}{4A}(4B^2 + w^2)y^2 - 2\delta y \\ H_2(x,y) &= 4(Ax + By)^2 + 8A(Dx - \delta y) + w^2y^2, \end{aligned}$$

in $x < 0$ and $x > 0$ respectively.

Therefore if the piecewise linear differential system has a periodic orbit candidate to be a limit cycle it must intersect the line $x = 0$ in exactly two points, namely $(0, y_1)$ and $(0, y_2)$ with $y_1 < y_2$. Since H_1 and H_2 are two first integrals they must satisfy

$$\begin{aligned} H_1(0, y_1) - H_1(0, y_2) &= 0, \\ H_2(0, y_1) - H_2(0, y_2) &= 0, \end{aligned}$$

or equivalently

$$\begin{cases} (y_1 - y_2)((y_1 + y_2)(\frac{2}{4A}(4B^2 + w^2)) - 4\delta) = 0, \\ (y_1 - y_2)((y_1 + y_2)2(4B^2 + w^2) - 2(8A\delta)) = 0, \end{cases}$$

since $y_1 < y_2$ the previous system is equivalent to the system

$$\begin{cases} ((y_1 + y_2)(\frac{2}{4A}(4B^2 + w^2)) - 4\delta) = 0, \\ ((y_1 + y_2)2(4B^2 + w^2) - 2(8A\delta)) = 0. \end{cases}$$

The solutions (y_1, y_2) of this last system satisfying the necessary condition $y_1 < y_2$ are

$$y_1 = -\frac{-8A\delta + 4B^2y_2 + w^2y_2}{4B^2 + w^2},$$

with y_2 arbitrary. So the periodic orbits of the continuous piecewise linear differential system are in a continuum of periodic orbits, and consequently this differential system has no limit cycles. ■

Here we shall prove that Theorem 3.3.1 can be extended to continuous piecewise linear differential system separated by two parallel straight lines formed by one linear center.

Thus our main result is:

Theorem 3.3.2 *A continuous piecewise linear differential system separated by two parallel straight lines with three linear systems, two of them have no equilibria, neither real nor virtual and the third is a linear center, has no limit cycles.*

Proof. Suppose that we have a continuous piecewise linear differential system separated by two parallel straight lines with three linear systems, two of them have no equilibria, neither real nor virtual and the third is a linear center. we can assume without loss of generality that the two parallel straight lines of continuity are $x = \pm 1$. we can assume that the linear center in the strip $-1 < x < 1$ is given by the system

$$\dot{x} = -Bx - \frac{4B^2 + w^2}{4A} y + \delta, \qquad \dot{y} = Ax + By + D, \tag{3.23}$$

with the first integral

$$H_2(x, y) = 4(Ax + By)^2 + 8A(Dx - \delta y) + w^2 y^2,$$

while the linear system in the half-plane $x < -1$ is given by the system

$$\dot{x} = d(ax - ady + c) + b, \qquad \dot{y} = ax - ady + c, \tag{3.24}$$

with the first integral

$$H_1(x, y) = ax^2 - 2adxy + ad^2y^2 + 2cx - 2(cd + b)y,$$

and finally the linear system in the half-plane $x > 1$ is given by the system

$$\dot{x} = \gamma\left(\alpha x - \alpha\gamma y + \lambda\right) + \beta, \quad \dot{y} = \alpha x - \alpha\gamma y + \lambda, \tag{3.25}$$

with the first integral

$$H_4\left(x, y\right) = \alpha x^2 - 2\alpha\gamma xy + \alpha\gamma^2 y^2 + 2\lambda x - 2\left(\lambda\gamma + \beta\right) y,$$

Since we have a continuous piecewise linear differential system it follows that system (3.23) and (3.25) must coincide on the straight line $x = 1$, and system (3.24) and (3.23) must coincide on the straight line $x = -1$. so we obtain that

$$\begin{cases} -B - \dfrac{(4B^2 + w^2)}{4A} y + \delta = \gamma(\alpha - \alpha\gamma y + \lambda) + \beta, \\ A + By + D = \alpha - \alpha\gamma y + \lambda, \\ d(-a - ady + c) + b = B - \dfrac{(4B^2 + w^2)}{4A} y + \delta, \\ -a - ady + c = -A + By + D, \end{cases}$$

then

$$\frac{4B^2 + w^2}{4A} = \alpha\gamma^2 \ , \ -B + \delta = \gamma(\alpha + \lambda) + \beta \ \ , B = -\alpha\gamma,$$

$$A + D = \alpha + \lambda, \quad \frac{4B^2 + w^2}{4A} = ad^2, \quad B + \delta = d(c - a) + b,$$

$$B = -ad, \; c - a = D - A,$$

thus

$$\begin{aligned}
d &= \gamma = -\frac{(4B^2 + w^2)}{4AB}, \\
c &= D - A + \frac{4AB^2}{4B^2 + w^2}, \\
b &= B + \delta + \frac{1}{4AB}(4B^2 D - Aw^2 - 4AB^2 + w^2 D), \\
\beta &= \delta - B + \frac{1}{4AB}(4AB^2 + Aw^2 + 4B^2 D + w^2 D), \\
a &= \alpha = \frac{4AB^2}{4B^2 + w^2}, \\
\lambda &= A + D - \frac{4AB^2}{4B^2 + w^2}.
\end{aligned}$$

If this continuous piecewise linear differential system has a limit cycle this must intersect the three open regions Σ_-, Σ_0 and Σ_+. Since the orbits in each one of these three regions are ellipses or a piece of one ellipse, a possible limit cycle must intersect each straight line $x = \pm 1$ in exactly two points,namely $(1, y_1), (1, y_2), (-1, y_3)$ and $(-1, y_4)$ with $y_1 < y_2$ and $y_4 < y_3$. Hence the first integrals H_1, H_2 and H_4 must satisfy the four equations

$$\begin{array}{l}
H_4(1, y_1) - H_4(1, y_2) = 0, \\
H_2(1, y_2) - H_2(-1, y_3) = 0, \\
H_1(-1, y_3) - H_3(-1, y_4) = 0, \\
H_2(-1, y_4) - H_2(1, y_1) = 0,
\end{array}$$

which now for our continuous piecewise linear differential system are equal to

$$\begin{cases}
-2\left(\alpha\gamma + (\lambda\gamma + \beta)\right)(y_1 - y_2) + \alpha\gamma^2(y_1^2 - y_2^2) = 0, \\
(4B^2 + w^2)(y_2^2 - y_3^2) + 8AB(y_2 + y_3) - 8A\delta(y_2 - y_3) + 16AD = 0, \\
ad^2(y_3^2 - y_4^2) + 2(ad - (cd + b))(y_3 - y_4) = 0, \\
(4B^2 + w^2)\,(y_4^2 - y_1^2) - 8AB(y_4 + y_1) - 8A\delta(y_4 - y_1) - 16AD = 0
\end{cases}$$

this system is equal to

$$\begin{cases}
(y_1 - y_2)(8AB - 8A\delta + (4B^2 + w^2)(y_1 + y_2)) = 0, \\
(y_2 - y_3)((4B^2 + w^2)(y_2 + y_3) - 8A\delta) + 8AB(y_2 + y_3) + 16AD = 0, \\
(y_3 - y_4)(-8AB - 8A\delta + (4B^2 + w^2)(y_3 + y_4)) = 0, \\
(y_4 - y_1)((4B^2 + w^2)(y_4 + y_1) - 8A\delta) - 8AB(y_4 + y_1) - 16AD = 0.
\end{cases}$$

Since $y_1 < y_2$ and $y_4 < y_3$ the previous system is equivalent to the system

$$\begin{cases}
L - K + F(y_1 + y_2)) = 0, \\
(y_2 - y_3)(F(y_2 + y_3) - K) + L(y_2 + y_3) + \Gamma = 0, \\
-L - K + F(y_3 + y_4) = 0, \\
(y_4 - y_1)(F(y_4 + y_1) - K) - L(y_4 + y_1) - \Gamma = 0,
\end{cases} \tag{3.26}$$

where

$$F = 4B^2 + w^2, \quad \Gamma = 16AD, \quad L = 8AB, \quad K = 8A\delta.$$

Since $F \neq 0$ from the third equation of (3.26) we isolated y_3 then

$$y_3 = \frac{K + L - Fy_4}{F},$$

and substitute it in the second equation of (3.26), and we get

$$(y_2 - \frac{K + L - Fy_4}{F})(F(y_2 + \frac{K + L - Fy_4}{F}) - K) + L(y_2 + \frac{K + L - Fy_4}{F}) + \Gamma = 0,$$

is equivalent to

$$F(y_2^2 - y_4^2) + L(y_2 + y_4) - K(y_2 - y_4) + \Gamma = 0. \tag{3.27}$$

In a similar from the first equation of (3.26) we isolated y_1 then

$$y_1 = \frac{K - L - Fy_2}{F},$$

and substitute it in the fourth equation of (3.26), and we obtain

$$(y_4 - \frac{K - L - Fy_2}{F})(F(y_4 + \frac{K - L - Fy_2}{F}) - K) - L(y_4 + \frac{K - L - Fy_2}{F}) - \Gamma = 0,$$

is equivalent to

$$F(y_2^2 - y_4^2) + L(y_4 + y_2) - K(y_2 - y_4) + \Gamma = 0, \tag{3.28}$$

we combine equations (3.27)and (3.28) and we get

$$2F(y_2^2 - y_4^2) + 2L(y_4 + y_2) - 2K(y_2 - y_4) + 2\Gamma = 0, \tag{3.29}$$

from (3.29) we isolated y_2 we obtain

$$y_2^{(1)} = \frac{1}{F}\left(\frac{1}{2}K - \frac{1}{2}L - \frac{1}{2}\sqrt{K^2 - 2KL - 4FKy_4 - 4FLy_4 - 4F\Gamma + L^2 + 4F^2y_4^2}\right).$$

or

$$y_2^{(2)} = \frac{1}{F}\left(\frac{1}{2}K - \frac{1}{2}L + \frac{1}{2}\sqrt{K^2 - 2KL - 4FKy_4 - 4FLy_4 - 4F\Gamma + L^2 + 4F^2y_4^2}\right).$$

Finally, the solutions of the system (3.26) are $(y_1^{(1)}, y_2^{(1)}, y_3, y_4)$ and $(y_1^{(2)}, y_2^{(2)}, y_3, y_4)$ where

$$y_1^{(1)} = -\frac{1}{F}(L - K) - y_2^{(1)},$$

$$y_1^{(2)} = -\frac{1}{F}(L - K) - y_2^{(2)},$$

$$y_2^{(1)} = \frac{1}{F}\left(\frac{1}{2}K - \frac{1}{2}L + \frac{1}{2}\sqrt{K^2 - 2KL - 4FKy_4 - 4FLy_4 - 4F\Gamma + L^2 + 4F^2y_4^2}\right),$$

$$y_2^{(2)} = \frac{1}{F}\left(\frac{1}{2}K - \frac{1}{2}L - \frac{1}{2}\sqrt{K^2 - 2KL - 4FKy_4 - 4FLy_4 - 4F\Gamma + L^2 + 4F^2y_4^2}\right),$$

$$y_3 = -\frac{1}{F}\left(Fy_4 - L - K\right),$$

and y_4 arbitrary. But due to the fact that $y_1 > y_2$ and $y_3 > y_4$, at most one of these two solutions will be satisfactory. So, the periodic orbits of the continuous piecewise linear differential system are in a continuum of periodic orbits, and consequently this differential system has no limit cycles. ■

3.4 Discussions and conclusions

In this chapter we studied the number of crossing limit cycles of the continuous and discontinuous planar piecewise differential systems formed only by one linear center separated by one or two parallel straight lines. we prove that when these piecewise differential systems are continuous or discontinuous separated by a unique straight line they have no limit cycles. But when the piecewise differential systems are discontinuous separated by two parallel straight lines, we show that they can have at most one crossing algebraic limit cycle, and that there exist such systems with one limit cycle,and when the piecewise differential systems are continuous separated by two parallel straight lines, we show that they have no limit cycles .

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