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Sinc-Nyström method for solving a class of nonlinear integral equations

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Dedication

♡ I dedicate this work to : Those who enlightened the path of my life. Those who taught me the perseverance, the trust in God and the determination in life. Those from whom I got my honor and my courage. Those who were the pillars striving for my success and happiness; their prayers were the beacon of my success ship. To my eternal beloved father and to the one I feel so blessed to have in my life, my mother.
To all my family, my brothers Dr. Mosaab and my dear Younes. To my darling sisters Aya, Kaouthar, and little Mayar.
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Introduction

The integral equations appear naturally in many fields, for example in mathematics, biology, physics, science, and technology, such that electricity, heat and mass transfer, population dynamics and disease spread...etc, where the majority of problems can be formulated as an integral equations.
The nonlinear integral equation is defined by this general formula:

$$u(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x, s, u(s))ds,$$

where, a and b are the limits of integration, $u(x)$ is the unknown function to be determined, $k(x, s, u)$ is the kernel of the equation, with the kernel and the function $f(x)$ in the integral equation are given functions, and λ is a constant parameter.
The objective of this work is to offer a brief theoretical study on nonlinear integral equations, and since the analytical resolution of these type of integral equations is still remains not possible and very difficult, we'll present an efficient and recent numerical method to solve nonlinear integral equations, namely "the Sinc-Nyström method", such that this method is a combination of Sinc-approximation method with Nyström method. Firstly, we started to explain the principles of the Sinc method, when the mathematics of Sinc methods is substantially different from the mathematics of classical numerical analysis.
This method is based on the Cardinal function $S(k, h)(x)$, which we will define later, where, if we will give a function f bounded, the representation of this

function by the Cardinal function is defined by:

$$C(f,h)(x) = \sum_{k=-\infty}^{+\infty} f(kh)S(k,h)(x),$$

this function occupies an important place in the theory of analytical functions, and it is particularly adept at solving one dimensional problems: interpolation, indefinite and definite integration and convolution...etc.
Secondly, we will talk about the Nyström method such that, where we will present the principle of this method for approximating a class of nonlinear integral equations of the second kind, in which the integral terms is approximated by an ordinary quadrature rule.
Finally, we will give the relationship and the combination between these two methods then we solve the nonlinear integral equations by the so-called Sinc-Nyström method.
The layout of our work is as follows:
The first chapter will present few basic concept from general theoretical framework, such as compactness, compact operators, and integral operators in Banach spaces, we will discuss the existence and uniqueness of solutions for nonlinear integral equations and mention some fundamental theorems and certain various results which will be used in the next chapters.
In the second chapter we will explain the Sinc-approximation method, where we will give some definitions of the Sinc function and then we talk about the Sinc quadrature and interpolation formulas in the Wiener space, infinite and finite Sinc approximation on $\mathbb{R}$ and Sinc approximation methods on arcs Γ, with some solved examples. Then we will talk about Nyström method and its principle.
The last chapter is an application of the Sinc-Nyström method for solving nonlinear integral equations, where we present the resolution of class of nonlinear Fredholm integral equations on bounded and unbounded intervals, and a class of nonlinear Volterra integral equation by Sinc-Nyström method, then, we will illustrate the efficiency of the present method by a several instructive examples.

Chapter 1

Preliminary concepts and basics about nonlinear integral equations

In this chapter, we recall some definitions and notions of compactness, compact operators, and integral operators in Banach spaces, we discuss the existence and uniqueness of solutions for nonlinear integral equations and we mention some fundamental theorems and certain various results which will be used in the next chapters.

1.1 Functional spaces

A normed vector space

Definition 1.1. A normed vector space consists of an underlying vector space E over a field of scalars (the real or complex numbers), together with a norm $\|.\|$:$E \to \mathbb{R}^+$ verify:
$\forall x, y \in E$, $\forall \alpha \in \mathbb{R}$:

- $\|x\| = 0 \Leftrightarrow x = 0$,
- $\|\alpha x\| = |\alpha| \, \|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$.

Definition 1.2. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the normed vector space $(E, \|.\|)$, we say that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if:

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n > N_\varepsilon, \forall m > N_\varepsilon, \|x_n - x_m\| \leq \varepsilon.$$

Complet space

A normed vector space $(E, \|.\|)$ is called complete (or a Cauchy space) if every Cauchy sequence of points in E has a limit that is also in E.

Banach space

A complete normed vector space is called Banach space.

Example 1.1. $(\mathbb{K}, |.|)$ is Banach spaces.

of continuous functions

The space of continuous functions consists of all continuous maps of the closed interval (Ω) into $\mathbb{R}$ denoted by $C(\Omega, \mathbb{R})$, the norm is the usual supremum norm, given by:

$$\|f\| = \sup_{x \in \Omega} |f(x)| \,.$$

L^2 space

Let $\Omega = [a, b]$ we define the space of all square integrable functions by:

$$L^2(\Omega) = \{f : \Omega \to \mathbb{R} \text{ } \textit{mesurable such that} \int_\Omega |f(t)|^2 \,\mathrm{d}t < \infty\}.$$

with the norm

$$\|f\|_2 = [\int_\Omega (|f(t)|^2 \,\mathrm{d}t]^{\frac{1}{2}},$$

is a complete space.

Compactness

Definition 1.3. A subset Ω of a normed spaces E, is called compact if every open cover H of Ω contains a finite subcover of Ω. In other words for every familly $H = \{V_j\}_{j \in I}$, of an open sets with the property

$$\Omega \subset \bigcup_{j \in I} V_j,$$

there exists a finite subfamily$\{V_{j(n)}\}$, $j(n) \in J$, $\quad n = 1, 2, ...N$
such that:

$$\Omega \subset \bigcup_{n=1}^{N} V_{j(n)}.$$

When V_j is a cover.

Definition 1.4. (Relatively compact set) A subset G of normed space E is relatively compact, if it's closer compact in E.

Theorem 1.1. *(See [3]) Any bounded and finite dimensional set of a normed space is relatively compact.*

Definition 1.5. we say that f is equicontinuous at point $x_0 \in \Omega$, if and only if:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall f \in C(\Omega, \mathbb{R}), \forall x \in \Omega, \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| \leq \varepsilon.$$

Theorem 1.2. (Arzela-Ascoli's theorem) *(See [4]) A subset G of $C(\Omega, \mathbb{R})$ is relatively compact if and only if the following conditions are satisfied:*
i) G is bounded,
ii) G is equicontinuous.

1.2 Compact operators

Definition 1.6. Let X and Y be two normed linear spaces and $A : X \to Y$ a linear operator between X and Y. A is called compact operator if and only if for all bounded sets $G \subseteq X$, $A(G)$ is relatively compact in Y.

1.3 Integral operators

Definition 1.7. Let $\Omega \in \mathbb{R}$ a compact subset, K a continuous function from $\Omega \times \Omega$ into $\mathbb{R}$, then the linear operator defined from $C(\Omega, \mathbb{R})$ into itself by:

$$T(u(x)) = \int_{\Omega} k(x, s)u(s)\mathrm{d}s, \quad x \in \Omega,$$

is called integral operator, and $k(x, s)$ is the kernel of the integral operator.

Some examples

Let $\Omega = [a, b]$ be a bounded and closed interval of $\mathbb{R}$.

- The Fredholm integral operator such that the region of integration is finite:

$$T(u(x)) = \lambda \int_a^b k(x, s, u(s))\mathrm{d}s.$$

- The Volterra integral operator:

$$T(u(x)) = \lambda \int_a^x k(x, s, u(s))\mathrm{d}s.$$

- The Abel integral operator:

$$T(u(x)) = \lambda \int_a^x \frac{u(s)}{(x-s)^\alpha}\mathrm{d}s, \quad 0 < \alpha < 1.$$

1.4 Classification of integral equations

In this section, we shall give definitions and classifications of some major types of linear and nonlinear integral equations.

1.4.1 Fredholm integrale equation

Definition 1.8. All integral equations in the form:

$$\alpha(x)u(x) = f(x) + \lambda \int_G k(x, s)u(s)\mathrm{d}s, \tag{1.1}$$

with $G = [a, b]$, is called Fredholm integral equation and it is given by the form:

$$\alpha(x)u(x) = f(x) + \lambda \int_a^b k(x, s)u(s)\mathrm{d}s. \tag{1.2}$$

1.4.2 Volterra integral equation

Definition 1.9. The integral equation with the upper limit b in the equation (1.1) is replaced by x and a is fixed, it is called Volterra integral equation and it is given by the general form:

$$\alpha(x)u(x) = f(x) + \lambda \int_a^x k(x,s)u(s)\mathrm{d}s. \tag{1.3}$$

It should be noted that the Volterra integral equation is a special case of Fredholm integral equation, it is enough to take the kernel

$$k(x,s,u(s)) = 0$$

if $a \leq x < s \leq b$.

If the unknown function u appears under the sign of integral we say that this integral equation is nonlinear, we define in this regard the nonlinear integral equation of Fredholm type by the general form:

$$\alpha(x)u(x) = f(x) + \lambda \int_a^b k(x,s,u(s))\mathrm{d}s, \tag{1.4}$$

but if

$$k(x,s,u(s)) = k(x,s)f(s,u(s)).$$

then, (1.4) is known as Hammerstein integral equation.

In the same manner we define the nonlinear Volterra integral equation by the form:

$$\alpha(x)u(x) = f(x) + \lambda \int_a^x k(x,s,u(s))\mathrm{d}s. \tag{1.5}$$

- If $\alpha(x) = 0$, the equation (1.1) written in the form:

$$f(x) + \lambda \int_G k(x,s)u(s)\mathrm{d}s = 0,$$

  and it is called integral equation of first kind.

- If $\alpha(x) = 1$, the equation (1.1) written in the form:

$$f(x) + \lambda \int_G k(x,s)u(s)\mathrm{d}s = u(x),$$

  and it is called integral equation of second kind.

- If $f(x) = 0$ then
$$u(x) = \lambda \int_G k(x,s)u(s)\mathrm{d}s,$$
the equation (1.1) is called homogeneous equation.

- If $f(x) \neq 0$ the equation (1.1) is called inhomogeneous equation.

Example 1.2.

- The equation
$$x + 1 + \lambda \int_{-1}^{1} (x^2 - s)u(s)\mathrm{d}s = 0,$$
is a Fredholm integral of first kind.
- The equation
$$u(x) = x^2 + cosx - 1 + \int_{-1}^{x} (x - s)u(s)\mathrm{d}s,$$
is a Volterra integral equation of second kind.
- The equation
$$u(x) = \lambda \int_{-1}^{1} (x^2 - s)u(s)\mathrm{d}s,$$
is homogeneous integral equation.

1.5 Existence and uniqueness of solution for nonlinear integral equations via Banach's fixed point theorem

1.5.1 Banach's fixed point theorem

This section is devoted to establishing certain results of existence and uniqueness for solving an integral equation in the form:
$$u = f + Tu,$$
where T is an operator defined on Banach space X.

These results are based on Banach's fixed-point theorem.

Definition 1.10. A bounded operator T on a Banach space X is a contraction, if there exists a constant L with $0 < L < 1$, such that:

$$\|Tu_1 - Tu_2\| \leq L\|u_1 - u_2\|, \qquad \forall u_1, u_2 \in X.$$

Theorem 1.3. (Banach 1922) *(See [3]) Let T be a contraction on a Banach space X.*
Then the equation

$$Tu = u, \tag{1.6}$$

has a unique solution $u_0 \in X$, this solution is the fixed point of operator T.

1.5.2 Application of Banach's fixed point theorem for nonlinear Volterra integral equation

Theorem 1.4. *(See [9]) Assume that $K(x, s, u)$ is defined and continuous on the square $a \leq x, s \leq b$ and that it satisfies a Lipschitz condition of the form:*

$$\|K(x, s, u_1) - K(x, s, u_2)\| \leq L\|u_1 - u_2\|.$$

Assume further that $f \in C[a, b]$. Then the nonlinear Volterra integral equation:

$$u(x) = f(x) + \lambda \int_a^x K(x, s, u(s))ds, \tag{1.7}$$

has a unique solution on the interval $[a, b]$ for every value of λ, where $a \leq x \leq b$.

Proof. We consider the Banach space $X = C(\Omega, \|.\|)$ of continuous functions defined on Ω into $\mathbb{R}$ equipped with the norm of uniform convergence:

$$\|u\| = \max_{x\in\Omega} |u(x)|.$$

We define in the space X the operator $T : X \to X$ by:

$$T(u(x)) = f(x) + \lambda \int_a^x K(x, s, u(s))\mathrm{d}s, \quad x \in \Omega, \tag{1.8}$$

it is clear that Tu is continuous, then the integral equation (1.7) is equivalent to the problem of fixed point:

$$Tu = u,$$

where T is defined by (1.8), if it can be proved that an adequate power of the continuous operator T is a contraction for every valued of λ, then it will be obvious as an application of theorem (1.3) that T has a unique fixed point, implying that the equation (1.7) has a unique solution.
Let $u_1, u_2 \in C[a, b]$ and $x \in [a, b]$, we will show that for any $n \geq 1$

$$\|T^n u_1 - T^n u_2\| \leq \frac{\lambda^n L^n (b-a)^n}{n!} \|u_1 - u_2\|. \tag{1.9}$$

For $n = 1$, we have:

$$\begin{aligned}
|T(u_1(x)) - T(u_2(x))| &= \left|\lambda \int_a^x [K(x, s, u_1(s)) - K(x, s, u_2(s))]\right| \mathrm{d}s, \\
&\leq |\lambda| \int_a^x |u_1(s) - u_2(s)|\,\mathrm{d}s, \\
&\leq |\lambda|\, L \|u_1 - u_2\| (x - a), \\
&\leq |\lambda|\, L(b - a) \|u_1 - u_2\|.
\end{aligned}$$

this implies that:

$$\|Tu_1 - Tu_2\| \leq |\lambda|\, L(b - a) \|u_1 - u_2\|.$$

Assume that the property is verified for $n = m$, and we will show that (1.9) is verified for $n = m + 1$.

$$\begin{aligned}
\left|T^{m+1}(u_1(x)) - T^{m+1}(u_2(x))\right| &= \left|T(T^m(u_1(x))) - T(T^m(u_2(x)))\right|, \\
&= \left|\lambda \int_a^x [K(x,s,T^m(u_1(x)) - K(x,s,T^m(u_2(x))]\right| \mathrm{d}s, \\
&\leq |\lambda| \int_a^x L \frac{|\lambda|^m L^m (b-a)^m}{m!} \|u_1 - u_2\| \mathrm{d}s, \\
&\leq \frac{|\lambda|^{m+1} L^{m+1} (b-a)^{m+1}}{(m+1)!} \|u_1 - u_2\|,
\end{aligned}$$

hence

$$\left\|T^{m+1}(u_1(x)) - T^{m+1}(u_2(x))\right\| \leq \frac{|\lambda|^{m+1} L^{m+1}(b-a)^{m+1}}{(m+1)!} \|u_1 - u_2\|.$$

then the property is valid for all $n > 0$.
Since the sequence $\frac{|\lambda|^m L^m (b-a)^m}{m!}$ is convergent to 0, there exists n_0 such that:

$$\frac{|\lambda|^{n_0} L^{n_0} (b-a)^{n_0}}{n_0!} < 1,$$

this prove that T^{n_0} is a contraction. □

1.5.3 Application of Banach's fixed point theorem for nonlinear Fredholm integral equation

Theorem 1.5. *(See [9]) Assume that $K(x,s,u)$ is defined and continuous on the square $a \leq x, s \leq b$ and that it satisfies a Lipschitz condition of the form:*

$$|K(x,s,u_1) - K(x,s,u_2)| < L|u_1 - u_2|.$$

assume further that $f \in C[a,b]$. Then the nonlinear Fredholm integral equation:

$$u(x) = f(x) + \lambda \int_a^b K(x,s,u(s))ds, \tag{1.10}$$

has a unique solution on the interval $[a,b]$ whenever $\lambda < 1/(L(b-a))$.

Proof. We consider the Banach space $X = C[a,b]$ of continuous functions defined on Ω into $\mathbb{R}$ equipped with the norm of uniform convergence:

$$\|u\| = \max_{x\in\Omega} |u(x)| .$$

We define in the space X the operator $T : X \to X$ by:

$$T(u(x)) = f(x) + \lambda \int_a^b K(x, s, u(s))\mathrm{d}s, \quad x \in \Omega, \tag{1.11}$$

it is clear that Tu is continuous, then the integral equation (1.10) is equivalent to the problem of fixed point:

$$Tu = u,$$

where T is defined by (1.11), if it can be proved that an adequate power of the continuous operator T is a contraction for every valued of λ, then it will be evident as an application of theorem (1.3) that T has a unique fixed point, implying that the equation (1.10) has a unique solution. If it can be proved that the operator T is a contraction for the constrained values of λ specified in the statements of the theorem, then it will be obvious as an application of Banach's fixed point theorem (theorem 1.3) that T has an unique fixed point, implying that the integral equation (1.10) has a unique solution.
Let $u_1, u_2 \in C[a, b]$ and $x \in [a, b]$, we have:

$$\begin{aligned}|Tu_1(x) - Tu_2(x)| &= |\lambda \int_a^b (K(x, s, u_1(s)) - K(x, s, u_2(s))ds|,\\ &\leq |\lambda|L(b - a)\|u_1 - u_2\|,\end{aligned}$$

this implies that

$$|\lambda|L(b - a) < 1.$$

(we can select $\lambda < \frac{1}{L(b-a)}$), then T is a contraction operator. $\square$

Chapter 2

Sinc-approximation method and Nyström method

In this chapter we will explain the Sinc-approximation method, where we give some definition of the Sinc function and then we will talk about the Sinc quadrature and interpolation formulas in the Wiener space, infinite and finite Sinc approximation on $\mathbb{R}$ and Sinc approximation methods on arcs Γ, with some solved examples. Then we will talk about the Nyström method and its principle.

2.1 Sinc-approximation method

2.1.1 The Sinc function

In mathematics, physics and engineering, the Sinc function is handy, and it denoted by sinc(x), this function has two forms: normalized and unnormalized functions.

Unnormalized sinc function

Definition 2.1. We call unnormalized Sinc function, the function defined by:

$$sinc(x) = \begin{cases} \frac{sin(x)}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{2.1}$$

Alternatively, the unnormalized Sinc function is often called sampling function, indicated as $\delta_a(x)$.

Normalized Sinc function

Definition 2.2. We named the normalized Sinc function, denoted sinc(x), the function defined by:

$$sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{2.2}$$

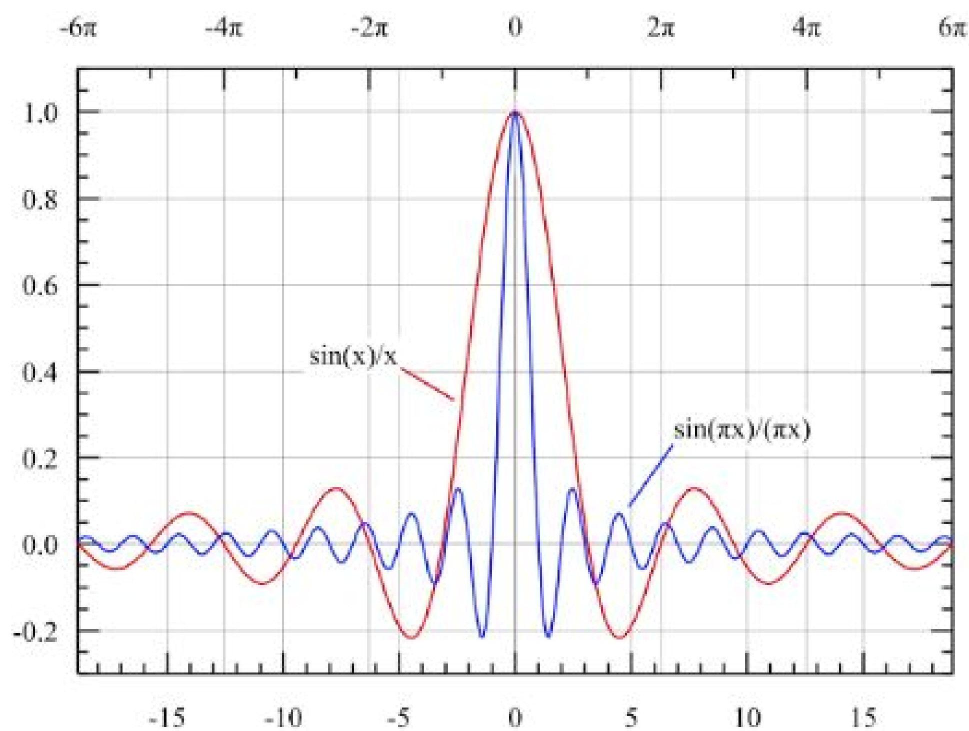

Figure 2.1: The Normalised and unnormalized Sinc function

Other notation of Sinc function:

We defined the B-Spline function by:

$$B_N(x) = \prod_{k=1}^{N} \left(1 - \frac{x^2}{k^2}\right). \tag{2.3}$$

May defined the Sinc function as the limit of the B-Spline's function by:

$$sinc(x) = \lim_{N\to+\infty} B_N(x) = \lim_{N\to+\infty} \prod_{k=1}^{N} \left(1 - \frac{x^2}{k^2}\right). \tag{2.4}$$

The sinc function $S(k, h)$ should be written as follows:

$$S(k,h)(x) = sinc\left(\frac{x}{h} - k\right). \tag{2.5}$$

We will also write the sinc function as:

$$S(k,h)(x) = \frac{sin[\pi(x-kh)/h]}{\pi(x-kh)/h},$$

when k an element of $\mathbb{Z}$ (the set of integers), and h is a positive number.

Sinc function does well as approximating functions, let be defined for all x on the real line, and from the sum:

$$F_h(x) = \sum_{k=-\infty}^{+\infty} f(kh)S(k,h)(x).$$

The series is known as the Whittaker cardinal function, if it is convergent. It is replete with many identities, and it is enable highly exacte approximation of smooth functions defined on $\mathbb{R}$ such as:

$$f(x) = \frac{1}{(2+(x+5)^2)},$$

or

$$f(x) = \frac{1}{cosh(x)},$$

and

$$f(x) = \exp(-x^4).$$

If the maximum difference for all x on $\mathbb{R}$ between $f(x)$ and $F_h(x)$ is ε $i.e$:

$$\forall \varepsilon > 0, |f(x) - F_h(x)| \leq \varepsilon,$$

then the maximum difference between $f(x)$ and $F_{\frac{h}{2}}(x)$ is less than ε^2 $i.e$:

$$\forall \varepsilon > 0, \left| f(x) - F_{\frac{h}{2}}(x) \right| \leq \varepsilon^2.$$

By replacing h with $\frac{h}{2}$, every second Sinc interpolation point kh remains intact allowing automatic verification of approximation correctness.

2.1.2 Exact Sinc-interpolation in the Wiener space $\mathbf{W}(\frac{\pi}{h})$

Let $\mathbb{R}$ denote the real line $(-\infty, +\infty)$, and let $\mathbb{C}$ denote the complex plane:

$$\{(x, y) : x + iy / x \in \mathbb{R}, y \in \mathbb{R}\},$$

with $i = \sqrt{-1}$.
Well known that any function $G \in L^2(-\pi/h, \pi/h)$ may be represented on $(-\pi/h, \pi/h)$, by its Fourier series:

$$G(x) = \sum_{k=-\infty}^{+\infty} C_k e^{ikhx}, \tag{2.6}$$

with:

$$C_k = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} G(x) e^{-ikhx} dx. \tag{2.7}$$

Where the convergence is in the L^2 norm $\|.\|$, it turns out that the Fourier series of the complex exponential it is a Sinc expansion over $\mathbb{R}$.

Theorem 2.1. *(See[6])*

Let $\xi \in \mathbb{C}$, and let $\mathbb{Z}$ denote the set of all integers then:

$$e^{i\xi x} = \sum_{n=-\infty}^{+\infty} S(n,h)(\xi) e^{inhx}, \quad -\pi/h < x < \pi/h. \tag{2.8}$$

Moreover, for all any $m \in \mathbb{Z}$:

$$\sum_{n=-\infty}^{+\infty} S(n,h)(\xi) e^{inhx} = \begin{cases} e^{i\xi(x-2m\pi}/h), & if \quad \frac{(2m-1)\pi}{h} < x < \frac{(2m+1)\pi}{h}, \\ cos(\pi\xi/h), & if \quad x = \frac{(2m\pm1)\pi}{h}. \end{cases} \tag{2.9}$$

Proof. When examined as a function of x, the function $G(x) = e^{i\xi x}$ clearly belong to $L^2(-\pi/h, \pi/h)$, and so has a Fourier series expansion of the form (2.6), on the interval $(-\pi/h, \pi/h)$ with C_k corresponding to (2.7). Thus:

$$C_k = \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{ix(\xi-kh)} dx, \tag{2.10}$$

when $C_k = S(k,h)(\xi)$.

However, the series sum of (2.9) indicated a periodic function of x just on real line $\mathbb{R}$, with period $2\pi/h$.
As a results, the top line on the right hand side of (2.10) is deduced.
It's important to note that the function $G(x) = e^{i\xi x}$ is Fourier series, wich is identical to this function on $(-\pi/h, \pi/h)$, defines a new function $H(x)$ is a periodic extension of G to all of $\mathbb{R}$.
Actually $H(x) = G(x)$ if $(-\pi/h < x < \pi/h)$, while if $|x| > \pi/h$ and $x = \xi + 2m\pi/h$ with $-\pi/h < \xi < \pi/h$, then H is defined by:

$$H(x) = H(\xi + 2m\pi/h) = H(\xi). \tag{2.11}$$

□

Fourier transform

Definition 2.3. Let a given function $f \in L^2(\mathbb{R})$, the Fourier transform $\tilde{f}$ of f is defined by:

$$\tilde{f}(x) = \int_{-\infty}^{+\infty} f(t) e^{ixt} dt, \tag{2.12}$$

the function $\tilde{f}$ also belongs to $L^2(\mathbb{R})$.

The Wiener space W($\frac{\pi}{h}$)

Definition 2.4. Given $\tilde{f}$, we can recover f via the inverse transform of $\tilde{f}$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixt} \tilde{f}(x) dx. \tag{2.13}$$

Let's start with a function $F \in L^2(-\pi/h, \pi/h)$, allow us to define a new function $\tilde{f}$ on $\mathbb{R}$:

$$\tilde{f}(x) = \begin{cases} F(x), & if \quad x \in (\frac{-\pi}{h}, \frac{\pi}{h}), \\ 0, & if \quad x \in \mathbb{R}, x \notin (\frac{-\pi}{h}, \frac{\pi}{h}). \end{cases} \tag{2.14}$$

The inverse Fourier transform of this function $\tilde{f}$, is given by:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixt} \tilde{f}(x) dx, \tag{2.15}$$

$$= \frac{1}{2\pi} \int_{-\pi/h}^{+\pi/h} e^{-ixt} F(x) dx.$$

Every function f defined in this way is said to belong to Wiener space $W(\frac{\pi}{h})$, equivalently, is said to be band limited.
In particular, it follows from (2.10) and (2.11) that:

$$f(nh) = \frac{1}{2\pi} \int_{-\pi/h}^{+\pi/h} e^{-inhx} \tilde{f}(x) dx, \tag{2.16}$$

we then get the cardinal function representation of f, given by theorem (2.1), as:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(x) \sum_{k=-\infty}^{+\infty} S(k,h)(t) e^{-ikhx} dx, \tag{2.17}$$

$$= \sum_{k=-\infty}^{+\infty} f(kh)S(k,h)(t).$$

The cardinal function representation of f, is denoted by $C(f,h)$, and given for $t \in \mathbb{R}$ (and generally for any complexe) by:

$$C(f,h)(t) = \sum_{k=-\infty}^{+\infty} f(kh)S(k,h)(t). \tag{2.18}$$

It turn out that the set of all function f, satisfy:

$$f = C(f,h),$$

is precisely the set of band limited function $W(\frac{\pi}{h})$.

2.1.3 Infinite Sinc approximation on $\mathbb{R}$

When an accurate representation for all functions, is no longer possible, $C(f,h)$ provides a highly approximation on $\mathbb{R}$, for all function whose Fourier transforms decay quickly.
In this section, we'll look at how to approximate functions that are not band limited.

Theorem 2.2. *(See [7]) Let f be defined on $\mathbb{R}$, and d is a positive constant, when $h \to 0$:*

$$\left\| f - \sum_{k=-\infty}^{+\infty} S(k,h)f(kh) \right\| = \mathcal{O}(e^{-\pi d/h}), \tag{2.19}$$

$$\mid \int_{-\infty}^{\infty} f(t)dt - h\sum_{k=-\infty}^{+\infty} f(kh) \mid = \mathcal{O}(e^{-2\pi d/h}).$$

Proof. By theorem (2.1) we have:

$$f(t) - \sum_{n=-\infty}^{+\infty} f(nh)S(n,h)(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widetilde{f}(x)\left[e^{-ixt} - \sum_{n=-\infty}^{+\infty} e^{-inhx}S(n,h)(t)\right]dx, \tag{2.20}$$

$$= \frac{1}{2\pi}\int_{|x|>\frac{\pi}{h}} \widetilde{f}(x)\left[e^{-int} - \sum_{n=-\infty}^{+\infty} e^{-inhx}S(n,h)(t)\right]dx.$$

But it follows from (2.14) , that if $x \in \mathbb{R}$ and $|x| > \frac{\pi}{h}$ then:

$$\sum_{n=-\infty}^{+\infty} e^{inhx}S(n,h)(t) = e^{i\xi t}. \tag{2.21}$$

For some $\xi \in \mathbb{R}$, so that this sum is bounded by 1 on $\mathbb{R}$, the first term e^{-ixt} inside the square brackets of the last equation (2.20) is also bounded by 1 on $\mathbb{R}$. That is the term in square brackets in the last equation in (2.21) is bounded by 2. Furthermore, under out assumption on $\widetilde{f}$ there exists a constant C such that:

$$|\widetilde{f}(x)| \leq Ce^{-d|x|}, \quad \forall x \in \mathbb{R}. \tag{2.22}$$

Therefore, we have:

$$\sup_{t\in\mathbb{R}} |f(t) - C(f,h)(t)| \leq \frac{4C}{2\pi} \int_{\pi/h}^{+\infty} e^{-dx} dx, \tag{2.23}$$

$$= \frac{4C}{2\pi d} e^{-\pi d/h}.$$

□

2.1.4 Finite Sinc approximation on $\mathbb{R}$

The Sinc approximation with infinite number of terms, may be very accurate as indicated by theorem (2.2), there could still be a difficulty in terms of numerical computation, this is especially true when the number of terms in the summations, we need achieve this accuracy is huge.

Definition 2.5. Let α and d are positive numbers, and let $L_\alpha(D_d)$ denote the family of all function f, with $\widetilde{f}$ means transforms Fourier, when:

$$f(t) = \mathcal{O}(\ exp(-\alpha\,|t|)), \quad t \to \pm\infty, \tag{2.24}$$
$$\widetilde{f}(x) = \mathcal{O}(\ exp(-\ d\,|x|)), \quad x \to \pm\infty.$$

This will be appropriate to replace the second equation of (2.24) the assumption that f is analytical in the domain:

$$D_d = \ \{z \in \mathbb{C} : |\Im(z)| < d\}.$$

Theorem 2.3. *(see [6]) Let $f \in L_\alpha(D_d)$, and corresponding to a positive integer $\mathbb{N}$,*

$$h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}, \tag{2.25}$$

$$h^* = \left(\frac{2\pi d}{\alpha N}\right)^{\frac{1}{2}},$$

$$\varepsilon_N = N^{1/2} exp\Big\{(-\pi d\alpha N)^{1/2}\Big\},$$

$$\varepsilon_N^* = exp\Big\{ - (2\pi d\alpha N)^{1/2}\Big\}.$$

Then, as $N \longrightarrow +\infty$

$$\left\| f - \sum_{k=-N}^{+N} f(kh)S(k,h) \right\| = \mathcal{O}(\varepsilon_N), \tag{2.26}$$

$$\left| \int_{-\infty}^{+\infty} f(t)dt - h^* \sum_{k=-N}^{+N} f(kh^*) \right| = \mathcal{O}(\varepsilon_N^*). \tag{2.27}$$

2.1.5 Sinc approximation methods on arcs Γ

In this section we'll expand the results for approximation on $\mathbb{R}$ to approximate functions over infinite, semi infinite, and finite intervals, in fact, over arcs Γ.
For this purpose a conformal maps ϕ that transform Γ to $\mathbb{R}$ are desirable.

Definition 2.6. Let D be a domain in $\mathbb{C}$ with boundary points $a \neq b$. Let ϕ denote a conformal map of $\mathcal{D}$ into D_d such that, $\phi(a) = -\infty$ and $\phi(b) = +\infty$.
Denote by $w = \psi(z)$ the inverse of mapping ϕ and let

$$\Gamma = \{w \in \mathbb{C} : w = \psi(x), x \in \mathbb{R}\} = \psi(\mathbb{R}).$$

and let $\mathcal{L}_\alpha(D)$ denote the set of all functions f analytic in $\mathcal{D}$, such that for some constant $C > 0$, and all $z \in \mathcal{D}$, we have

$$|f(z)| \leq C \frac{|e^{\phi(z)}|^\alpha}{(1 + |e^{\phi(z)}|)^{2\alpha}}.$$

Theorem 2.4. *(see [6]) Assume that $f\psi' \in \mathcal{L}_\alpha(\psi(D_d))$ for d with $0 < d < \frac{\pi}{2}$, let N be a positive integer, and*

$$h = \left(\frac{\pi d}{\alpha N}\right)^{1/2}, \qquad h^* = \left(\frac{2\pi d}{\alpha N}\right)^{1/2},$$

then there exist constants C and C^*, independent of N, such that:

a)

$$\left\| f - \sum_{k=-N}^{N} f(z_k) S(k,h) \circ \phi \right\| \leq C\sqrt{N} e^{-\sqrt{d\pi\alpha N}}. \tag{2.28}$$

b)

$$\left| \int_\Gamma f(t)dt - h^* \sum_{k=-N}^{N} f(\psi(kh)\psi'(kh) \right| \leq C^* e^{-\sqrt{2\pi\alpha d N}}. \tag{2.29}$$

Sinc-quadrature formula for $\int_a^x f(s)ds$:

The numerical indefinite integral formula introduced by employing the Sinc function as follows:

$$\int_a^x f(s)ds \approx \sum_{k=-N}^{+N} f(kh) \int_a^x S(k,h)(s)ds, \tag{2.30}$$

$$= \sum_{k=-N}^{+N} f(kh) j(k,h)(x), \quad x \in \mathbb{R},$$

where the basis function $j(k,h)$ is expressed as:

$$j(k,h)(x) = h\Big\{\frac{1}{2} + \frac{1}{\pi}Si[\pi(\frac{x}{h} - k)]\Big\},$$

with:

$$Si(x) = \int_0^x \frac{sin\mu}{\mu} d\mu.$$

Theorem 2.5. *(See[5]) Let $0 < d < \pi$ and let h be selected by the formula:*

$$h = \sqrt{\frac{\pi d}{\alpha N}}, \qquad where \quad N \in \mathbb{Z},$$

then:

$$\left| \int_a^x f(t)dt - h \sum_{k=-N}^{+N} f(\psi(kh))\psi'(kh)j(k,h)(x) \right| \le Ce^{-\sqrt{\pi d \alpha N}}. \tag{2.31}$$

Some examples

Example 2.1. The case of $\Gamma = [a,b]$, where $-\infty < a < b < +\infty$.
In this case we take:

$$w = \phi(z) = \ln \frac{z-a}{b-z},$$

hence

$$\psi(w) = \frac{a + be^w}{1 + e^w}.$$

The conformal map ϕ transforms the complex domain:

$$\mathcal{D} = \left\{ z = x + iy : \left| \arg \frac{z-a}{b-z} \right| < d < \frac{\pi}{2} \right\},$$

into the infinite strip

$$D_d = \left\{ w = \alpha + i\beta : |\beta| < d < \frac{\pi}{2} \right\}.$$

the basic functions in the interval (a,b) defined by:

$$\begin{aligned} S(k,h) \circ \phi(x) &= \begin{cases} \frac{\sin[\pi(\phi(x)-kh)/h]}{\pi(\phi(x)-kh)/h}, & \phi(x) \neq kh, \\ 1, & \phi(x) = kh, \end{cases} \\ &= \text{Sinc } [(\phi(x) - kh)/h]. \end{aligned}$$

Since

$$S(k,h)(jh) = \delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

Then, the nodes of the function $S(k,h) \circ \phi(x)$ are:

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}},$$

and the finite approximation of interpolation and quadrature, for a function $f(x)$ on $[a, b]$ are defined by:

$$f(x) \approx \sum_{k=-N}^{+N} f(x_k) S(k, h) \circ \ln \frac{x-a}{b-x},$$

$$\int_a^b f(x)dx \approx h(b-a) \sum_{k=-N}^{+N} \frac{f(x_k)e^{kh}}{(1+e^{kh})^2}.$$

Numerical test:

We wish to approximate the function g on the interval $(0, 1)$ where

$$g(x) = \frac{1}{\sqrt{5x^2+1}},$$

the numerical results of the approximation of quadrature and interpolation for the function g on $(0, 1)$, are represented in Table 2.1 and Figures 2.2-2.5.

Table 2.1: Absolute error between the exact and the Sinc Approximate integration of g

N	h	error
10	9.9350e-01	3.8804e-05
20	7.0250e-01	7.6189e-07
30	5.7360e-01	3.5330e-08
40	4.9670e-01	2.5130e-09
50	4.4430e-01	2.5130e-10
60	4.0560e-01	3.0752e-11
70	3.7550e-01	4.3343e-12
80	3.5120e-01	6.2528e-13
90	3.3120e-01	1.1369e-13

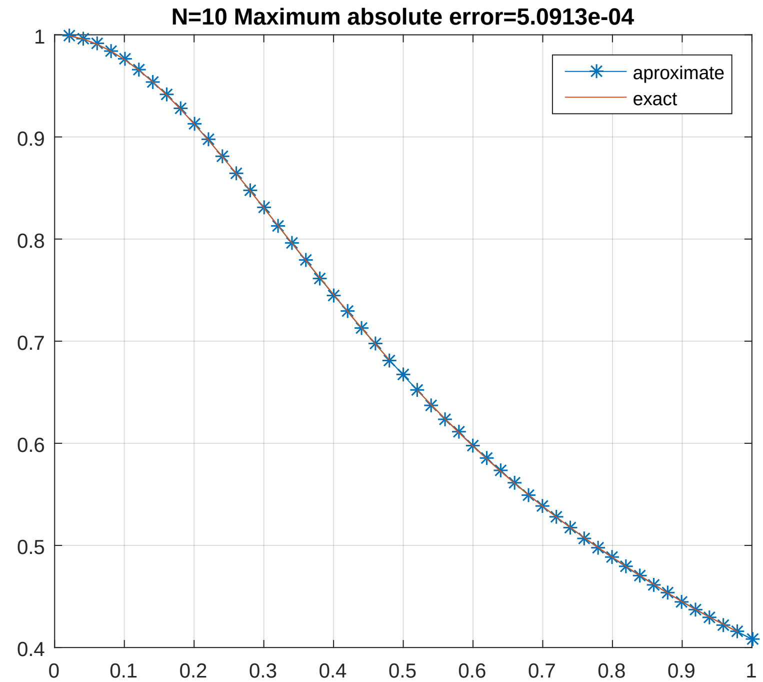

Figure 2.2: Sinc approximate interpolation of g for $N = 10$

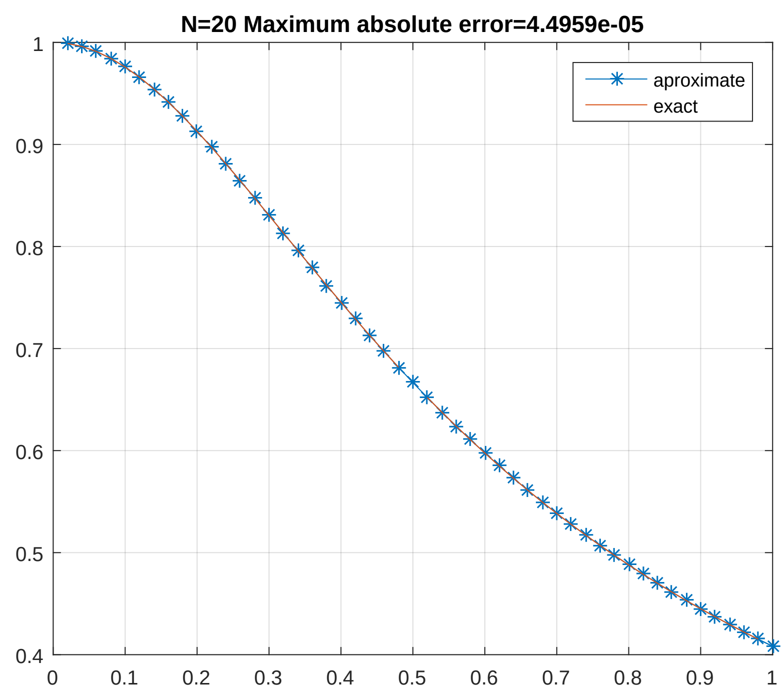

Figure 2.3: Sinc approximate interpolation of g for $N = 20$

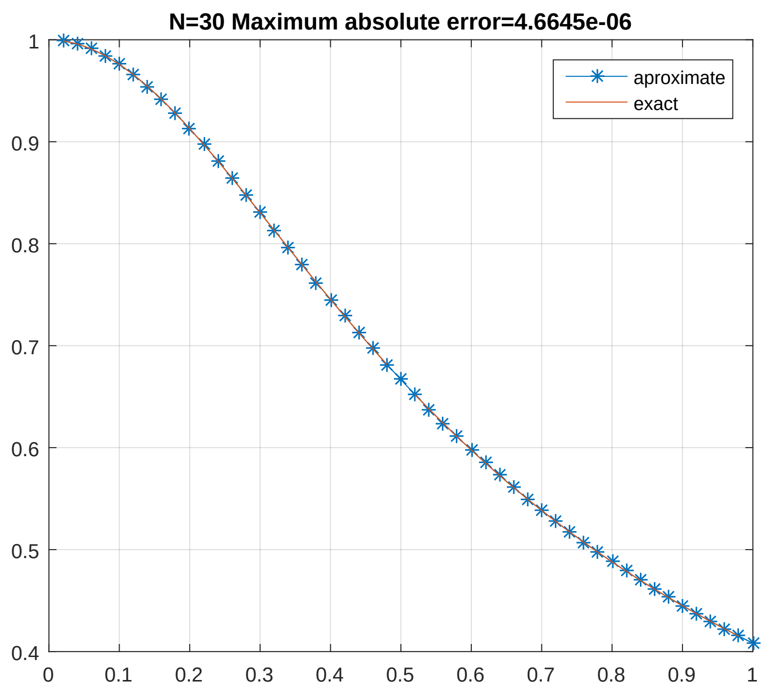

Figure 2.4: Sinc approximate interpolation of g for $N = 30$

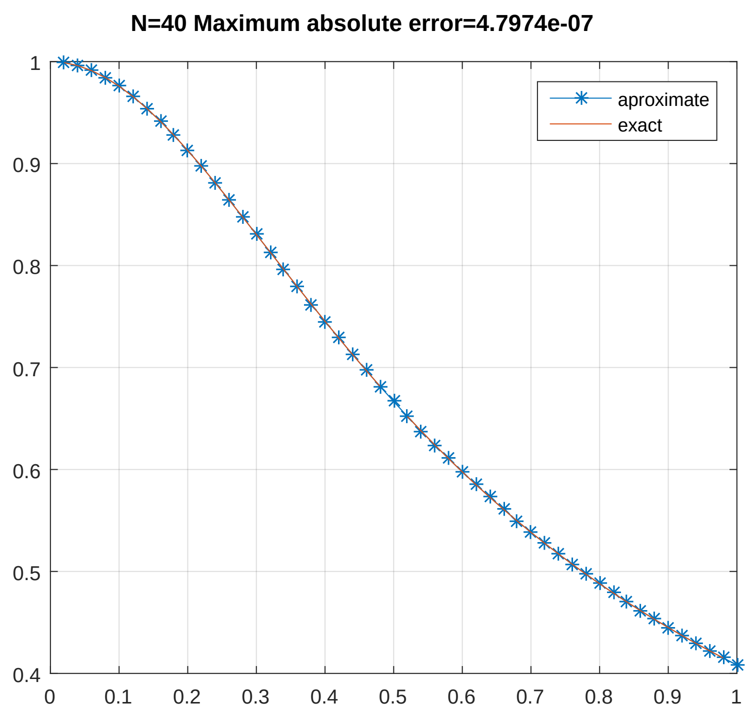

Figure 2.5: Sinc approximate interpolation of g for $N = 40$

Example 2.2. The case $\Gamma = [0, \infty)$
We take

$$w = \phi(z) = \log(z),$$

hence

$$\psi(w) = e^w.$$

The conformal map ϕ transforms the complex domain

$$\mathcal{D} = \{z \in \mathbb{C} : |\arg(z)| < d\},$$

into the infinite strip

$$D_d = \left\{w = \alpha + i\beta : |\beta| < d < \frac{\pi}{2}\right\}.$$

The nodes of the function $S(k,h) \circ \phi(x)$ are:

$$x_k = \phi^{-1}(kh) = e^{kh},$$

and the finite approximation of interpolation and quadrature for a function $f(x)$ on $[0, \infty)$ are defined by:

$$f(x) \approx \sum_{k=-N}^{N} f(x_k) S(k,h) \circ \log(x),$$

$$\int_0^\infty f(x)dx \approx h \sum_{k=-N}^{N} f(x_k) e^{kh}.$$

Numerical test:

We wish to approximate the function f on the interval $[0, +\infty)$ where:

$$f(x) = \frac{1}{2x^5 + 1},$$

the numerical results of the approximation of quadrature and interpolation, for the function f on $(0, 1)$, are represented in Table 2.2 and Figures 2.6-2.9.

Table 2.2: Absolute error between the exact and the Sinc approximate integration of f

N	h	error
10	9.935e-01	1.0859e-02
20	7.025e-01	4.5993e-03
30	5.736e-01	1.7618e-03
40	4.967e-01	7.1481e-04
50	4.443e-01	3.1096e-04
60	4.056e-01	1.4629e-04
70	3.755e-01	7.5842e-05
80	3.512e-01	4.4479e-05
90	3.312e-01	3.0061e-05

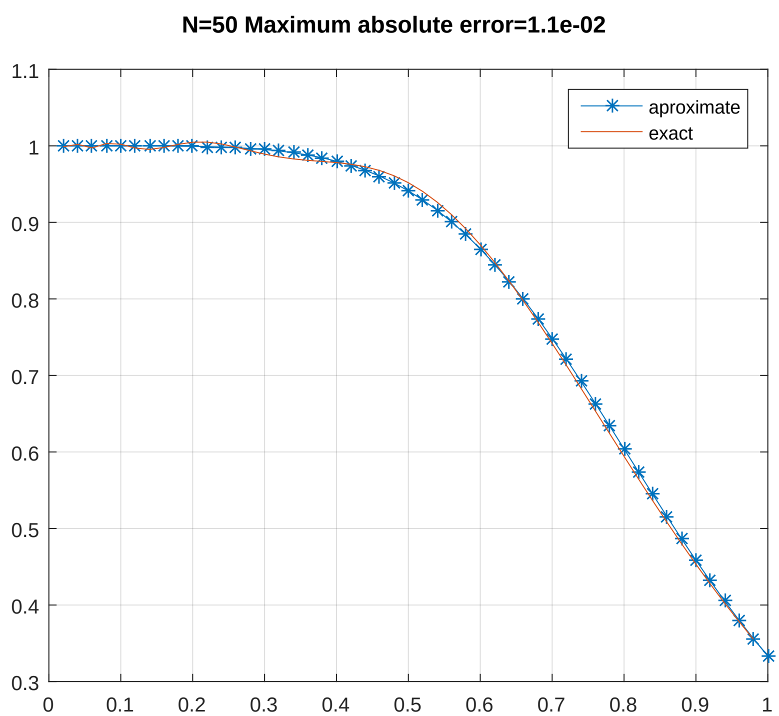

Figure 2.6: Sinc approximate interpolation of f for $N = 50$

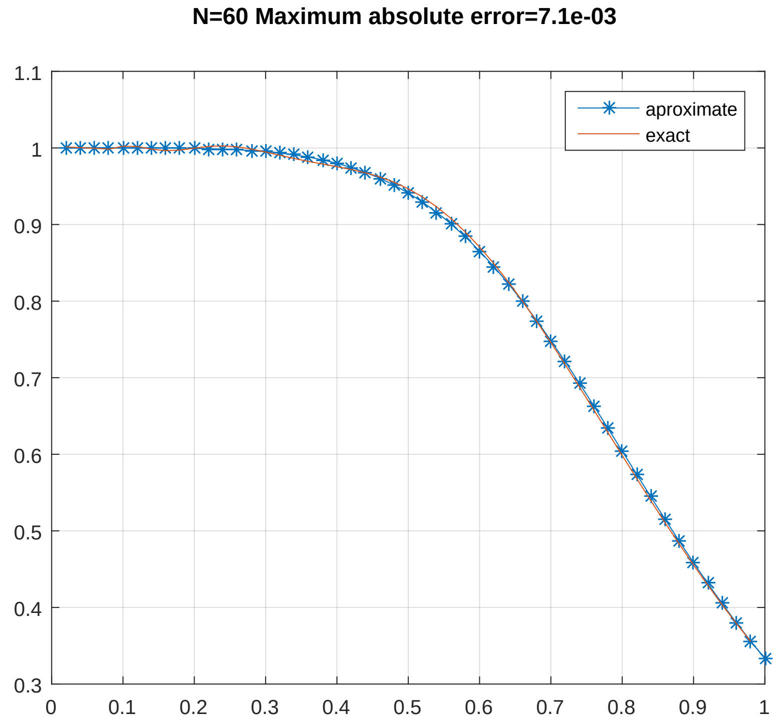

Figure 2.7: Sinc approximate interpolation of f for $N = 60$

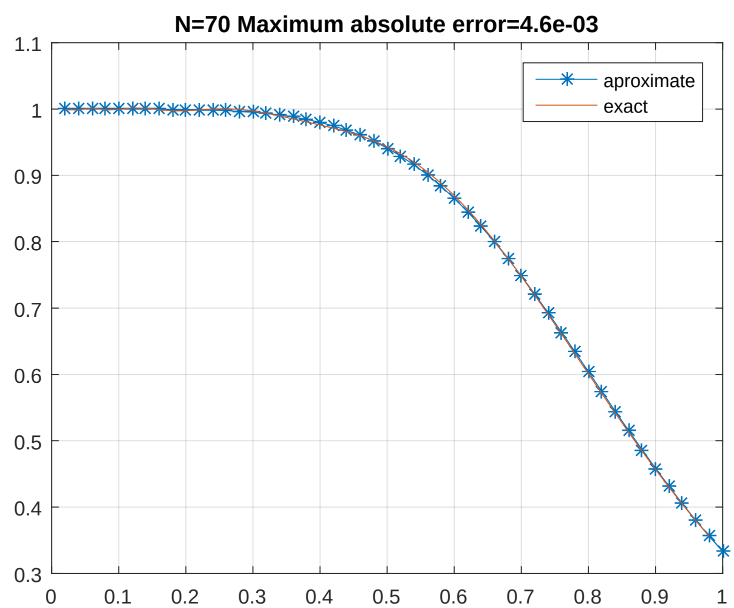

Figure 2.8: Sinc approximate interpolation of f for $N = 70$

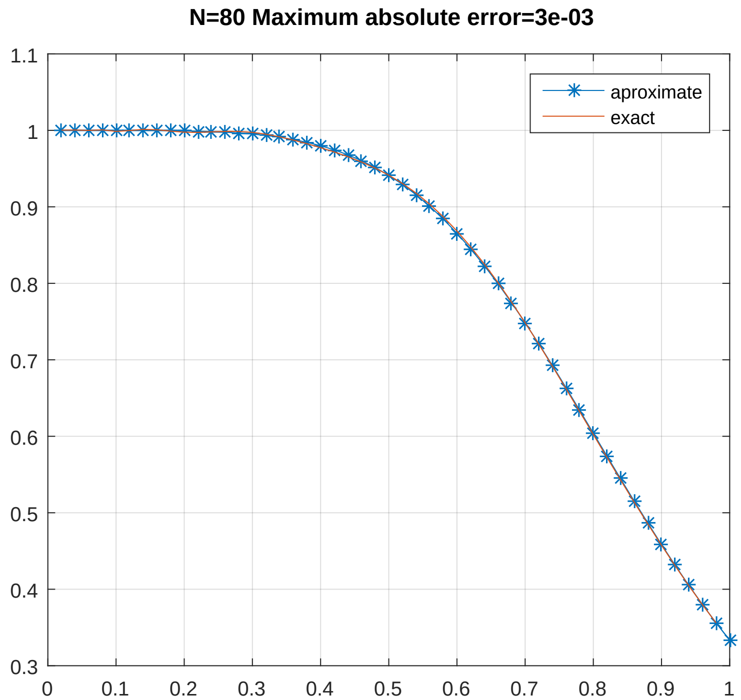

Figure 2.9: Sinc approximate interpolation of f for $N = 80$

2.2 Nyström method

In this section, we will approximate a class of nonlinear integral equations of the second kind by the Nyström method, in which the integral terms is approximated by an ordinary quadrature rule.
Let $Q : X = C[a, b] \to \mathbb{R}$ be an integral operator defined by:

$$Q(g) = \int_a^b g(s)ds,$$

and let $Q_n : X \to \mathbb{R}$ be a discrete operator defined by the quadrature rule:

$$Q_n(g) = \sum_{j=1}^{n} \omega_j^{(n)} g(x_j^{(n)}), \tag{2.32}$$

the values $\{x_j^{(n)}\}_{j=1}^n$ are called the quadrature nodes and $\{\omega_j^{(n)}\}_{j=1}^n$ are called weights. A sequence of quadrature rules $Q_n(g)$ is called convergent if

$$Q_n(g) \to Q(g), \quad \text{as} \quad n \to \infty, \text{ for all } g \in X.$$

i.e., if the sequence of linear functionals $Q_n(g)$ converges pointwise to the integral $Q(g)$.

2.2.1 Principle of Nyström method

Consider the nonlinear integral equation:

$$u(x) = f(x) + \int_a^b K(x,s)F(u(s))ds, \qquad x \in \Omega = [a,b], \tag{2.33}$$

where $f \in X = C(\Omega)$ and $K(x,s)F(u(s))$ given with an appropriate smoothness assumption, the right-hand side of (2.33) defines a completely continuous operator from some open domain $D \subset X$ into X, explicitly

$$\mathcal{K}(u)(x) = \int_a^b K(x,s)F(u(s))ds, \qquad x \in \Omega.$$

Thus, solving (2.33) is equivalent to solve the operator equation

$$u = f + \mathcal{K}(u), \tag{2.34}$$

using the quadrature formula to approximate the integral in (2.34) and the Nyström method to (2.33) is find $u_n(s)$ such that:

$$u_n(x) = f(x) + \sum_{j=1}^{n} \omega_j^{(n)} K(x, s_j^{(n)}) F(u_n(s_j^{(n)})), \qquad x \in \Omega, \tag{2.35}$$

where $u_n(x)$ is an approximation to $u(x)$, A solution to the equation (2.35) may be obtained by determining $\{u_n(x_i^{(n)})\}$, thus (2.35) is reduced to the finite nonlinear system

$$z_i = f(x_i^{(n)}) + \sum_{j=1}^{n} \omega_j^{(n)} K(x_i^{(n)}, s_j^{(n)}) F(z_j^{(n)}), \qquad i = 1, \cdots, n, \tag{2.36}$$

where the function:

$$z(x) = f(x) + \sum_{j=1}^{n} \omega_j^{(n)} K(x, s_j^{(n)}) F(z_j^{(n)}), \qquad x \in \Omega,$$

satisfies (2.35)

Chapter 3

Sinc-Nyström method for solving nonlinear integral equations

In this chapter we will talk about the Sinc-Nyström methods for solving nonlinear equations, when we present the resolusion of nonlinear Fredholm and Volterra integral equation by Sinc-Nyström methods, then we will illustrate the efficiency of the present method by a several instructive examples.

3.1 Resolusion of class of nonlinear Fredholm integral equations by Sinc-Nyström methods

In this section, we consider the Sinc-Nyström method for the numerical solution of the non-linear integral equation of Fredholm type:

$$u(x) - \int_I K(x,s)F(u(s))ds = g(x), \quad x \in I. \tag{3.1}$$

where $u(x)$ is an unknown function to be determined, and $k(x,s)$, $F(x)$ and $g(x)$ are given functions.
This equation (3.1) can be expressed in operator form as:

$$(I - \mathcal{K})u = g, \tag{3.2}$$

Where

$$(\mathcal{K}u)(x) = \int_I K(x,s)F(u(s))ds.$$

3.1.1 Sinc-schema

In our work we approximate the integral operator in (3.1), by the quadrature formula (2.29). Let $k(x,.)F(u(.))\psi'(.) \in \mathcal{L}_\alpha(D)$ for all $x \in I$. Then the integral in (3.1) be approximated by *theorem(2.4) and the following discrete operator can be defined by:*

$$\mathcal{K}_N(u)(x) = h \sum_{k=-N}^{+N} k(x, s_k) F(u(s_k)) \psi'(kh). \tag{3.3}$$

The Nyström method applied to (3.1) is to find u_N such that:

$$u_N(x) - h \sum_{k=-N}^{+N} k(x, s_k) F(u(s_k)) \psi'(kh) = g(x), \tag{3.4}$$

where the point s_k are defined by the formula:

$$s_k = \psi(kh), \quad j = -N, ..., N.$$

Solving (3.4) reduces to solving a finite dimensional nonlinear system.
For any solution of (3.4) the values $u_N(x_i)$ at the quadrature points satisfy the nonlinear system:

$$u_N(x_i) - h \sum_{k=-N}^{+N} k(x_i, s_k) F(u(s_k)) \psi'(kh) = g(x_i), \tag{3.5}$$

$$i = -N, ..., N.$$

Conversely, given a solution $u_N(x_i)$, $i = -N, ..., N$, of the system (3.3), then the function u_N defined by:

$$u_N(x) = h \sum_{k=-N}^{+N} k(x, s_k) F(u(s_k)) \psi'(kh) + g(x), \tag{3.6}$$

is readily seen to satisfy (3.6).
We rewrite equation (3.6) in operator notation as:

$$(I - \mathcal{K}_N) u_N = g. \tag{3.7}$$

3.1.2 Numerical examples

In the following, the theoretical results of the previous section are used for some numerical examples.
The numerical experiments are implemented in Matlab. The programs are executed on a PC Intel CORE-i5 dual processor with 2 GB RAM.
In order to analyse the error of the method the following notations are introduced:

$$\|E_N(h)\| = \max_{-N \le i \le N} |u(x_i) - u_N(x_i)|. \tag{3.8}$$

Example 3.1. We consider the nonlinear integral equation given by the formula:

$$u(x) + \int_0^1 cos(\pi(x+s)) u^2(s) ds = cos(\pi x) - \frac{2}{3\pi} sin(\pi x) \quad x \in [0,1], \tag{3.9}$$

where the exact solution is:

$$u(x) = cos(\pi x).$$

We try to reduce our nonlinear integral equation to a nonlinear system of algebraic equations with Sinc-Nyström method.
For this we use the quadrature formula (2.29) and we obtain:

$$\int_0^1 cos(\pi(x+s))u^2(s)ds \approx \sum_{k=-N}^{+N} cos(\pi(x+s_k))u_N^2(s_k)\psi'(kh), \tag{3.10}$$

wherever:

$$\psi = \phi^{-1}, \qquad \text{and} \quad \phi(s) = ln\frac{s}{1-s},$$

hence

$$\psi(s) = \frac{e^s}{1-e^s}, \qquad \text{and} \quad \psi'(s_k) = \frac{e^{s_k}}{(1+e^{s_k})^2}.$$

Table 3.1: The absolute errors for example 3.1

N	h	$\|E_N(h)\|_\infty$
10	9.9320e-01	5.2942e-04
20	7.0230e-01	3.4705e-06
30	5.7340e-01	5.0827e-08
40	4.9660e-01	1.2263e-09
50	4.4420e-01	4.1885e-11
60	4.0550e-01	1.8562e-12
70	3.7540e-01	1.0131e-13
80	3.5120e-01	6.5503e-15
90	3.3110e-01	4.9960e-16

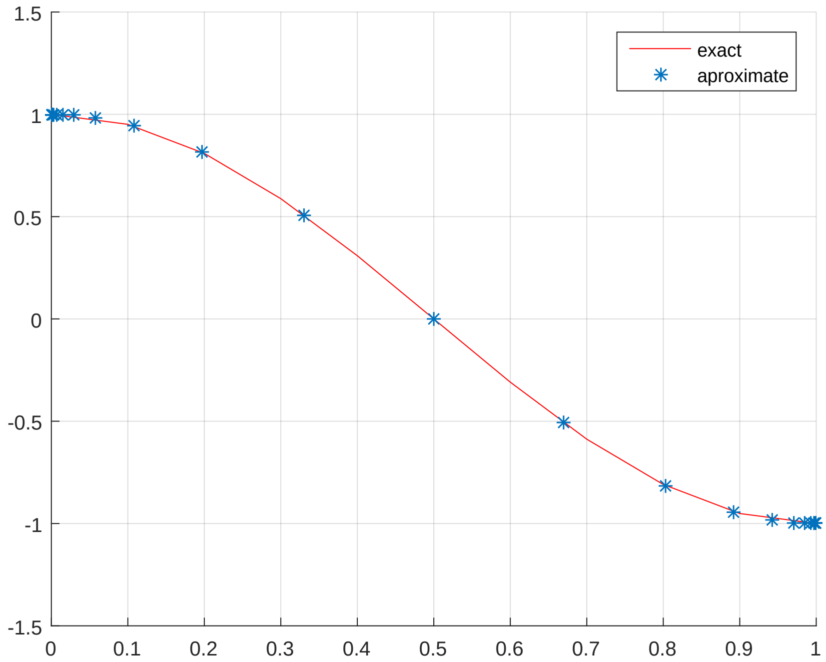

Figure 3.1: The exact and the approximate solution for example 3.1

Example 3.2. In this example, we consider solving a nonlinear Fredholm integral equation which is defined in the half-line as follows:

$$u(x) + \int_0^{+\infty} e^{-(x+s)} sin(u(s))ds = e^{-x}(1 + 2sin^2(1/2)), \tag{3.11}$$

where the exact solution is:

$$u(x) = e^{-x}.$$

We are attempt to simplify our nonlinear integral equation, to a nonlinear system of algebraic equation with Sinc-Nyström method.
We utilize the quadrature formula (2.29), we obtain:

$$\int_0^{+\infty} e^{-(x+s)} sin(u(s))ds \approx h \sum_{k=-N}^{+N} e^{-(x+s_k)} sin(u_N(s_k))\psi'(kh), \tag{3.12}$$

where

$$\psi = \phi^{-1}, \qquad \text{and} \quad \phi(s) = ln(s),$$

hence

$$\psi(s) = e^s, \qquad \text{and} \quad \psi'(s_k) = e^{s_k}$$

Table 3.2: The absolute errors for example 3.2

N	h	$\|E_N(h)\|_\infty$
10	9.9320e-01	3.5920e-05
20	7.0230e-01	1.5893e-06
30	5.7340e-01	1.4080e-07
40	4.9660e-01	6.8297e-09
50	4.4420e-01	3.3899e-10
60	4.0550e-01	7.9627e-11
70	3.7540e-01	1.9126e-11
80	3.5120e-01	4.6915e-12
90	3.3110e-01	6.3705e-13

3.2 Resolution of class of nonlinear Volterra integral equations by Sinc-Nyström methods

In this section, we consider the Sinc-Nyström method for the numerical solution of the nonlinear Volterra integral equation defined by the following formula:

$$u(x) - \int_a^x K(x,s)F(u(s))ds = g(x), \quad x \in [a,b]. \tag{3.13}$$

where $u(x)$ *is the unknown function to be determined, and* $K(x,s)$*,* $F(x)$ *and* $g(x)$ *are given functions.*
This equation can be expressed in operator form as:

$$(I - \mathcal{K})u = g, \tag{3.14}$$

where

$$(\mathcal{K}u)(x) = \int_a^x K(x,s)F(u(s))ds.$$

3.2.1 Sinc schema

In our work we approximate the integral operator in (3.13) by the quadrature formula(2.31)
Let $K(x,.)F(u(.))\psi'(.) \in \mathcal{L}_\alpha(D)$ *for all* $x \in [a,b]$*.*
Then the integral in (3.13)can be approximated by theorem (2.5) and the following discrete operator can be defined by:

$$\mathcal{K}_N(u)(x) = h\sum_{k=-N}^{+N} k(x,s_k)F(u(s_k))\psi'(kh)j(h,k)(x). \tag{3.15}$$

The Nyström method applied to (3.13) is to find u_N *such that:*

$$u_N(x) - h\sum_{k=-N}^{+N} k(x,s_k)F(u(s_k))\psi'(kh)j(h,k)(x) = g(x), \tag{3.16}$$

where the point s_k are defined by the formula:

$$s_k = \psi(kh), \quad k = -N, ..., N.$$

Solving (3.13) reduces to solving a finite dimensional nonlinear system.
For any solution of (3.13) the values $u_N(x_i)$ at the quadrature points satisfy the nonlinear system

$$u_N(x_i) - h \sum_{k=-N}^{+N} k(x_i, s_k)u(s_k))\psi'(kh)j(k,h)(x_i) = g(x_i), \tag{3.17}$$

$$i = -N, ..., N.$$

Conversely, given a solution $u_N(x_i)$, $\quad i = -N, ..., N$, of the system (3.15), then the function u_N defined by:

$$u_N(x) = h \sum_{k=-N}^{+N} k(x, s_k)F(u(s_k))\psi'(kh)j(h,k)(x) + g(x). \tag{3.18}$$

is readily seen to satisfy(3.18).
We rewrite equation (3.18) in operator notation as:

$$(I - \mathcal{K}_N)u_N = g, \tag{3.19}$$

3.2.2 Numerical example

Example 3.3. In this case, we'll solve the equation:

$$u(x) + \int_0^x e^{x-s}u^2(s)ds = e^{2x}, \quad x \in [0, 1] \tag{3.20}$$

where the exact solution is defined by:

$$u(x) = e^x.$$

We try to simplify our nonlinear integral equation, to a nonlinear system of algebraic equations, with Sinc-Nyström method.
We use the quadrature formula (2.31) for this:

$$\int_0^x e^{x-s}u^2(s)ds \approx h \sum_{k=-N}^{+N} e^{x-s_k}u_N^2(s_k)\psi'(kh)j(k,h)(x), \tag{3.21}$$

where

$$\psi = \phi^{-1}, \qquad \text{and} \quad \phi(s) = ln\frac{s}{1-s},$$

hence

$$\psi(s) = \frac{e^s}{1-e^s}, \qquad \text{and} \quad \psi'(s_k) = \frac{e^{s_k}}{(1+e^{s_k})^2}.$$

Table 3.3: The absolute errors for example 3.3

N	h	$\|E_N(h)\|_\infty$
10	7.0250e-01	4.6000e-03
20	4.9670e-01	2.8017e-04
30	4.0560e-01	3.1488e-05
40	4.5120e-01	4.9276e-07
50	3.1420e-01	9.5612e-07
60	2.8680e-01	2.1643e-07
70	2.6550e-01	5.5100e-08
80	2.4840e-01	1.5400e-08
90	2.3420e-01	4.6464e-09

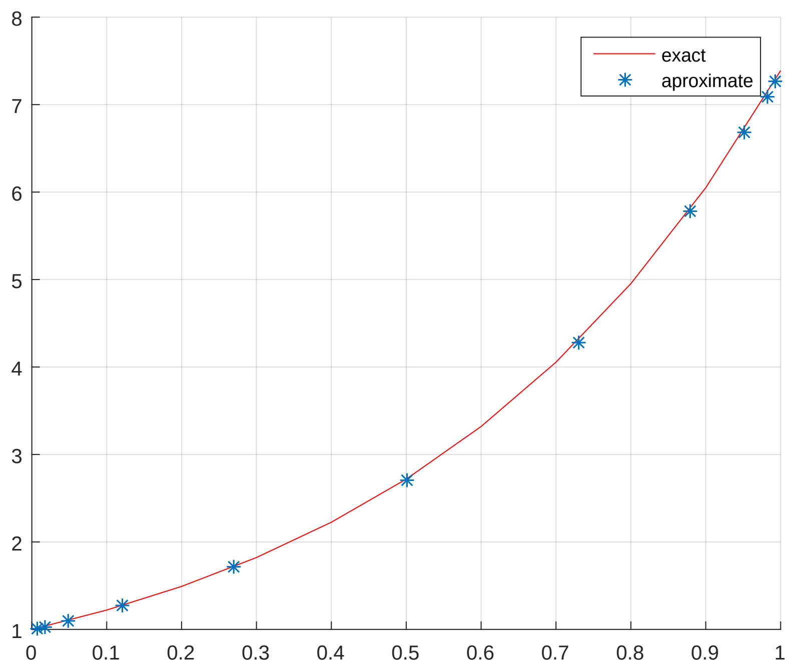

Figure 3.2: The exact and the approximate solution for example 3.3

Conclusion and prospects

In this work we've presented an efficient quadrature method, namely "the Sinc-Nyström method", such numerical method based on the so-called "Sinc-function ", where we have treated some techniques of approximation of functions and integrals and even the solving of mathematical problems, including the nonlinear Volterra and Fredholm integral equations, through our approaching of this type of integral equations, we deduced that this method is very effective and productive for solving nonlinear integral equations, and the numerical results have confirmed the theoretical prediction of the exponential rate of convergence.
This work could be extended to subject other classes of nonlinear integral equations, we can also combinate the sinc approximation with other classical methods such as: Collocation method.

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ملخص

الهدف من هذا العمل هو دراسة المعادلات التكاملية غير الخطية في فضاءات بناخ، وحل فئة من هذا النوع من المعادلات بطريقة عددية حديثة تسمى طريقة سينك-نيستروم، تم تطوير هذه الطريقة بواسطة طريقة نيستروم، حيث ان تقريب سينك هو الأساس الأمثل للتقريب في فضاءات الدوال التحليلية.

الكلمات المفتاحية: معادلات تكاملية غير خطية. تقريب السينك. طريقة نيستروم.

Abstract

The aim of this work is to study nonlinear integral equations in Banach spaces, and to solve a class of this type of equations by a recent numerical method called Sinc-Nyström method, this method is developed by means of the Nyström method, with the Sinc-approximation which is an optimal basis for approximation in spaces of functions that are analytic.

Keywords: Nonlinear integral equations. Sinc approximation. Nyström method.

Résumé

Le but de ce travail est d'étudier les équations intégrales non linéaires dans les espaces de Banach, et à résoudre une classe de ce type des équations par une méthode numérique récente appelée la méthode de Sinc- Nyström, cette méthode est développée au moyen de la méthode de Nyström, avec l'approximation Sinc qui est une base optimal pour l'approximation dans les espaces des fonctions analytiques.

Mots-clés: Equations intégrales non linéaires. L'approximation Sinc. Méthode de Nyström.