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Mapped spectral methods and rational approximations

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ملخص

الهدف الرئيسي من هذه الأطروحة هو تقريب حلول بعض المسائل الرياضية في شكل معادلات تكامليه أو تفاضلية على المجالات غير المحدودة وتتمثل الاستراتيجية الشائعة والفعالة في التعامل مع المجالات غير المحدودة في استخدام تطبيق عكسي يحول المجال غير المحدود الي مجال محدود وفي هذه الأطروحة نقدم نظاما متعامدا جديدا يقوم على تركيب التطبيق العكسي بكثيرات حدود جاكوبي الكلاسيكية ومنه نتحصل على دوال جاكوبي المعينة والتي نوظفها لحل معادلات تفاضلية من الدرجة الثانية ومعادلات تكاملية غير خطية في مجال غير محدود.

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كلمات المفتاحية :

التقريب الطيفي ، التقريب الجذري، دوال جاكوبي المعينة، الاستيقطاب.

Abstract

The main aim of this thesis is to approach the solutions of some mathematical problems in the form of integral equations or differential equations on unbounded domains. A common and effective strategy in dealing with unbounded domains is to use a suitable mapping that transforms an infinite domain. In this thesis, we introduce a new orthogonal system of mapped Jacobi functions which is the images of classical Jacobi polynomials under the inverse mapping. The modified Jacobi spectral methods are proposed for second-order differential and nonlinear integral equations on the semi-infinite domain.

Key words:

Spectral approximation, rational approximation, mapped Jacobi polynomials, interpolation.

Résumé

L'objectif principal de cette thèse est d'approximer les solutions de certains problèmes mathématiques sous la forme d'équations intégrales ou d'équations différentielles sur des domaines non bornés. Une stratégie courante et efficace pour traiter les domaines illimités consiste à utiliser un mappage approprié qui transforme un domaine infini. Dans cette thèse, nous introduisons un nouveau système orthogonal de fonctions de Jacobi mappées qui sont les images des polynômes de Jacobi classiques sous l'application inverse. Les méthodes spectrales de Jacobi modifiées sont proposées pour les équations différentielles du second ordre et intégrales non linéaires sur le domaine semi-infini.

Mots clés:

Approximation spectrale, approximation rationnelle, polynômes de Jacobi cartographiés, interpolation.

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List of Symbols

Λ:	$[0,+\infty)$ or $(-\infty,+\infty)$.
I:	$[-1,1)$ or $(-1,1)$.
$\mathbb{R}$:	$(-\infty,+\infty)$.
$\mathcal{L}_n^\beta$:	Laguerre polynomial of degree n with $\beta>0$ defined in (1.40).
$\widehat{\mathcal{L}}_n^\beta$:	Laguerre function of degree n defined in (1.60).
H_n:	Hermite polynomial of degree n with $\alpha>0$ defined in (1.76).
H_n^α:	Hermite function of degree n defined in (1.95).
$J_n^{\alpha,\beta}$:	Jacobi polynomial of degree n with $\alpha,\beta>-1$ defined in (1.111).
G_n^α:	Gegenbauer polynomial of degree n with $\alpha>-1$ defined in (1.124).
L_n:	Legendre polynomial of degree n defined in (1.126).
T_n:	Chebyshev polynomial of degree n defined in (1.129).
$j_{s,n}^{\alpha,\beta}$:	Modified Jacobi function of degree n with $s>0$ defined in (1.148).
$E_{s,n}^{\alpha,\beta}$:	Modified Jacobi exponential function of degree n with $s>0$ defined in (1.191) and (1.199).
$R_{s,n}^{\alpha,\beta}$:	Modified Jacobi rational function of degree n with $s>0$ defined in (1.193) and (1.201).
$L_{s,n}$:	Modified Legendre function of degree n.
$E_{s,n}$:	Modified Legendre exponential function of degree n defined in (3.71) .
$R_{s,n}$:	Modified Legendre rational function of degree n defined in (3.71).
w_β:	Laguerre weight function: $w_\beta(y)=e^{-\beta y}$.
w:	Hermite weight function: $w(y)=e^{-y^2}$.
$w^{\alpha,\beta}$:	Jacobi weight function: $w^{\alpha,\beta}(y)=(1-y)^\alpha(1+y)^\beta$.
w^α:	Gegenbauer weight function: $w^\alpha(y)=(1-y^2)^{\alpha-\frac{1}{2}}$.
$w_s^{\alpha,\beta}$:	Modified Jacobi weight function:$\varpi_s^{\alpha,\beta}(x)=w^{\alpha,\beta}(\theta(x;s))\dfrac{dy}{dx}$ where $y=\theta(x;s)$.
$\mathcal{X}$:	Banach space.
$C(\Lambda)$:	Set of continues functions on Λ.
$C(I)$:	Set of continues functions on I.
$L^2(\Lambda)$:	Usual Hilbert space on Λ.
$L^2_{w^\beta}(\Lambda)$:	Weighted Hilbert space associated with Laguerre weight function .
$L^2_w(\Lambda)$:	Weighted Hilbert space associated with Hermite weight function.
$L^2_{w^{\alpha,\beta}}(I)$:	Weighted Hilbert on I associated with Jacobi weight function .
$H^m_\beta(\Lambda)$:	Weighted Sobolev space defined in (1.49).
$-H^m_\beta(\Lambda)$:	Weighted Sobolev space defined in (1.56).

$\widehat{H}_{\beta}^{m}(\Lambda)$: Weighted Sobolev space defined in (1.66).
$-\widehat{H}_{\beta}^{m}(\Lambda)$: Weighted Sobolev space defined in (1.72).
$H^{m}(\Lambda)$: Weighted Sobolev space defined in (1.88).
$H_{\alpha}^{m}(\Lambda)$: Weighted Sobolev space defined in (1.101).
$H_{\alpha,\beta}^{m}(I)$: Weighted Sobolev space defined in (1.133).
$\widetilde{H}_{\alpha,\beta}^{m,s}(\Lambda)$: Weighted Sobolev space defined in (1.166).
$\widetilde{H}^{m,s}(\Lambda)$: Weighted Sobolev space $\widetilde{H}_{\alpha,\beta}^{m,s}(\Lambda)$ with $\alpha = \beta = 0$.
$H_{\alpha,\beta}^{m,s}(\Lambda)$: Weighted Sobolev space defined in (2.12).
$\widehat{H}_{\alpha,\beta}^{m,s}(\Lambda)$: Weighted Sobolev space defined in (2.57).
$H_{w}^{1}(\Lambda)$: Weighted Sobolev space with $w(x) = x$.

$(\cdot,\cdot)_w$: Inner product of $L_w^2(\Lambda)$ or $L_w^2(I)$.
$(\cdot,\cdot)$: Inner product of $L^2(\Lambda)$.
$\|\cdot\|_{L_w^2(\Lambda)}$: Norm of $L_w^2(\Lambda)$.
$\|\cdot\|_{L^2(\Lambda)}$: Norm of $L^2(\Lambda)$.
$\|\cdot\|$: Norm of $\mathcal{X}$.
$\|\cdot\|_H$: Norm of $H(\Lambda)$ or $H(I)$.
$|\cdot|_H$: Semi-norm of $H(\Lambda)$ or $H(I)$.
$(\cdot,\cdot)_{w,N}$: Discrete inner product associated with a Gauss-type quadrature.
$(\cdot,\cdot)_N$: $(.,.)_N = (.,.)_{w,N}$ with $w \equiv 1$.
$\|\cdot\|_{w,N}$: Discrete norm associated with $(.,.)_{w,N}$.
$\|\cdot\|_N$: Discrete norm associated with $(.,.)_N$.

$\mathcal{X}_N$: Set of all real polynomials of degree $\leq N$.
$\mathcal{Q}_N^{\beta}$: Set of all scaled Laguerre polynomials of degree $\leq N$.
$\widehat{\mathcal{Q}}_N^{\beta}$: Set of scaled Laguerre functions $\widehat{\mathcal{Q}}_N^{\beta} := \{e^{-\frac{1}{2}\beta y}\psi(\beta y) \mid \psi \in \mathcal{Q}_N^{\beta}\}$.
$\mathcal{Q}_N$: Set of all Hermite polynomials of degree $\leq N$.
$\mathcal{Q}_N^{\alpha}$: Set of scaled Hermite functions $\mathcal{Q}_N^{\alpha} := \{e^{-\frac{1}{2}\alpha^2 y^2}\psi(\alpha y) \mid \psi \in \mathcal{Q}_N\}$.
$\mathbb{X}_N^{\alpha,\beta}$: Set of all Jacobi polynomials of degree $\leq N$.
$\mathbb{X}_{s,N}^{\alpha,\beta}$: Set of modified Jacobi functions $\mathbb{X}_{s,N}^{\alpha,\beta} := \{v \mid v(x) = \mu_s(\theta(x;s))\phi(\theta(x;s)), \forall \phi \in \mathbb{X}_N^{\alpha,\beta}\}$.

P : Projection operator.
P_N: Orthogonal projection operator.
P_N^{β}: The approximation operator of the Laguerre polynomial defined in (1.48).
I_N^{β}: The interpolation operator of the Laguerre polynomial defined in (1.51).
$\widehat{P}_N^{\beta}$: The approximation operator of the Laguerre function defined in (1.63).
$\widehat{I}_N^{\beta}$: The interpolation operator of the Laguerre function defined in (1.71).
$\mathcal{P}_N$: The approximation operator of the Hermite polynomial defined in (1.87).
$\mathcal{I}_N$: The interpolation operator of the Hermite polynomial defined in (1.90).
$\mathcal{P}_N^{\alpha}$: The approximation operator of the Hermite function defined in (1.100).
$\mathcal{I}_N^{\alpha}$: The interpolation operator of the Hermite function defined in (1.106).
$\pi_N^{\alpha,\beta}$: The approximation operator of the Jacobi polynomial defined in (1.132).
$I_N^{\alpha,\beta}$: The interpolation operator of the Jacobi polynomial defined in (1.135).
$\pi_{s,N}^{\alpha,\beta}$: The approximation operator of the modified Jacobi function defined in (1.162).

$\mathcal{I}_{s,N}^{\alpha,\beta}$: The interpolation operators of the modified Jacobi function defined in (1.175).

$\pi_{s,N}$: The approximation operator of the modified Legendre function.

$I_{s,N}$: The interpolation operator of the modified Legendre function.

Introduction

A series of problems in different fields such as physics and chemistry are modeled and represented mathematically by sets of integral and differential equations in unbounded domains. A small number of these integral/differential equations can be solved analytically and hence numerical methods are routinely used to solve such problems. Therefore, numerous methods have been developed and applied accordingly such as the spectral methods. The foundations of spectral methods are not recent, which were originally proposed in [1]. For many years, before the appearance of contemporary computers, the spectral methods stay in the theoretical studies, the first implementation was in [2], the authors have solved the barotropic vorticity equation using some special functions. Later, spectral methods have been implemented successfully for many challenging problems in bounded domains [3–6]. Basically, this is due to the abundance of convenient orthogonal basis sets in the finite computational domains such as Jacobi polynomials.

In the context of numerical schemes for integral and differential equations in unbounded domains, different spectral methods have been proposed for solving such problems, that can be classified as follows:

- Reduction of unbounded domains to bounded domains: This method based to replace the infinite limits in the unbounded domain by sufficiently large numbers called the truncation parameters, one solves the equation on the bounded domains, using piecewise polynomial basis functions and then letting the truncation parameters tend to the infinite limits. For partial differential equations, see [7–12]. The authors of [13–17] have been used to solve linear integral equations. In [18–21] the authors have been extended to nonlinear integral equations. The main disadvantage of this approach is that it can be used only for problems with rapidly decaying solutions or when accurate non-reflecting or exact boundary conditions are available at the artificial boundaries for differential equations.

- Approximation by classical orthogonal systems on unbounded domains: These approaches based on Laguerre or Hermite polynomials/functions, which have been used widely, for instance, the authors [22–29] have solved partial differential equations. The authors [30–33] have solved linear integral equations. Very recently the authors [34] have been extended the Laguerre polynomials as classical basis functions to solve nonlinear integral equation. Indeed, spectral methods based on Laguerre or Hermite polynomials are not advisable for problems with rapidly (exponentially) decaying solutions due to their wild behaviors at infinity. Instead, Laguerre or Hermite functions are strongly suggested in this context. However, both Laguerre or Hermite polynomials and functions are not effective when applied to problems with slow (algebraically) decaying solutions.

- Map unbounded domains to bounded domains: The main idea of this approach is to map the original problem into a singular problem on the finite domain through a suitable family of suitable mappings and then using the spectral methods associate with the classical orthogonal polynomials. For partial differential equations [35–41]. Very recently the authors [42, 43] have extended to solved linear integro differential and integral equations. The main advantage of this approach is that standard spectral approximation results can be used for the analysis, but its main disadvantage is that the mapped problem is usually very complicated and its analysis is often cumbersome.

- Approximation by mapped orthogonal functions (non-classical orthogonal systems): In this approach the domain of the solution is not transformed into a finite domain, but the solution is approximated using a new mapped orthogonal functions, which are obtained by applying a suitable mapping to classical orthogonal polynomials, see [44–48] for differential equations. Very recently the authors [49] have extended to solved linear integral equations. This approximation is suitable for capturing the localized rapid variations in the solution of the given problem. The analysis of this approach requires approximation results on the rate of convergence on the mapped orthogonal function series. The main advantage of this approach is that once these approximation results are established, it can be directly applied to a large class of problems in unbounded domains without using a transform.

Despite the concept of mapping is not new, the classical spectral methods based on the Laguerre or Hermite functions for integral and differential equations with underlying decaying solutions at infinity in unbounded domains have generated much interest in recent years, while the mapped Jacobi-spectral approximations have received only limited attention. Actually, the main purpose of this thesis are:

- To present a framework, for the analysis of mapped Jacobi and rational approximation, which leads to more concise results and optimal approximation results in most situations.

- To apply these methods for solving some kinds of integral and differential equations on the half line.

- To make a detailed comparison on the convergence rates of different methods for several typical solutions (which are algebraically or exponentially decay at infinity).

Our thesis is divided into three chapters, in the first Chapter we present some spectral methods i.e. the collocation and Galerkin methods with their convergence results. We carry out a review of the main classical techniques available for unbounded domains, namely Hermite and Laguerre approximations/interpolations based on polynomials/functions, we also implement Laguerre and Hermite functions to solve linear integral equations with decaying solutions and give some comments on the use of such functions on the half and real lines.

At the end of the first chapter, we introduce a new mapped Jacobi function, namely the Modified Jacobi function, which is the core content of this thesis. In more detail, the set of Modified Jacobi functions is mutually orthogonal with the uniform weight $w(x) \equiv 1$ on L^2 space in unbounded domains. The main advantage of this approach is the ability to achieve the superlinear convergence rate for some problems with rapidly and slowly decaying solutions at infinity.

The second Chapter is devoted to apply the modified Jacobi functions to solve the second-order differential equations on the half line, we starts in the first section with the modified Jacobi rational functions for solving differential equation slowly decaying solutions. The same way to solve differential equations with rapidly decaying solutions using the modified Jacobi exponential functions in the second section.

In the third Chapter, we apply the modified Legendre functions to solve Hammerstein integral equations with convolution and non-convolution kernels on the half-line. We will discuss the collocation methods and their superconvergence results. Finally, illustrate the feasibility of the present methods by instructive examples and compared them to the results obtained with some famous methods.

Finally, we summarise the work that has been carried out in this thesis and consider some possible further work.

Chapter 1

Spectral methods on unbounded domains

The spectral methods possess high accuracy, and so play an important role in numerical solutions of differential and integral equations, the main idea is to write the spectral solution of the problem as a finite sum of various orthogonal systems of infinitely differentiable global functions and then to choose the coefficients in the sum in order to satisfy such problem as well as possible. Different functions lead to different spectral approximations. For instance in bounded domains, trigonometric polynomials for periodic problems and Jacobi polynomials for non-periodic problems. While, Laguerre, Hermite and Mapped Jacobi functions for problems in unbounded domains.

This chapter is organized as follows. The first section is devoted to presenting the properties of projection operator, Collocation, and Galerkin methods for integral equations with their convergence analyses. In the second and third sections, we recall the classical orthogonal polynomials/functions on unbounded domains and three numerical tests for the linear integral equations, which are Laguerre or Hermite polynomials/functions. The last section is generalized to the first section in our paper, which presents to introduce the modified Jacobi function with its properties and gives some numerical approximation of functions in unbounded domains.

1.1 Projection methods

The projection methods have attracted much attention in solving (linear/nonlinear) differential and integral equations. In this section, we describe the Galerkin and Collocation methods to solve approximately the integral equation of the second kind

$$u - \mathcal{K}(u) = f, \text{ for all } f \in \mathcal{X}, \tag{1.1}$$

where $\mathcal{K}$ is integral operator and $\mathcal{X}$ is a Banach space. Before that, we recall some useful concepts of the projection operator in the following subsection.

Projection operator

A projection operator is a general strategy of approximating a function by a finite number of approximating functions. That is, the solution is approximated by a finite combination of simple, known functions.

Definition 1.1. Let U be a nontrivial subspace of $\mathcal{X}$. A bounded linear operator $P : \mathcal{X} \to U$ with the property $Pu = u$ for all $u \in U$ is called a projection operator from $\mathcal{X}$ into U.

Next, we give some useful properties of the projection operator.

Theorem 1.1. *If P is a projection operator from $\mathcal{X}$ into U, we have*

$$P^m u = u, \text{ for all } u \in U, \tag{1.2}$$

and

$$\|P\| \geq 1. \tag{1.3}$$

Proof. For any $u \in U$, we have

$$Pu = u. \tag{1.4}$$

Next, assume that for integer $m \geq 0$,

$$P^m u = u. \tag{1.5}$$

Then, a direct calculation indicates that

$$P^{m+1} u = P P^m u = Pu = u. \tag{1.6}$$

From (1.4), the projection operator satisfies that

$$\|u\| = \|Pu\| \leq \|P\| \|u\|, \quad \forall u \in U. \tag{1.7}$$

This implies

$$\|P\| \geq 1. \tag{1.8}$$

□

Orthogonal projection operators

An important case of projection operators is given by the so-called orthogonal projection, i.e., by the best approximation if $\mathcal{X}$ is Hilbert space. Let $\mathcal{X} = L^2(\Lambda)$ is Hilbert space on Λ associate with a scalar product $(\cdot, \cdot)$.

Definition 1.2 (Orthogonal projection operators)**.** A projection operator P is orthogonal if and only if

$$(Pu, (I - P)v) = 0, \quad \text{for all } v, u \in L^2(\Lambda). \tag{1.9}$$

We equate the following theorem from Theorem 13.3 of [50], which gives more details about the orthogonal projection operator.

Theorem 1.2. *Let U be a nontrivial subspace of $L^2(\Lambda)$. Then the operator P mapping each element $u \in L^2(\Lambda)$ into its unique best approximation with respect to U is a projection operator such that*

$$\|u - Pu\|_{L^2(\Lambda)} = \inf_{v \in U} \|u - v\|_{L^2(\Lambda)}. \tag{1.10}$$

Moreover, the orthogonal projection operator P satisfies $\|P\| = 1$.

In the following , we consider a finite dimension subsequence $\mathcal{X}_N \subset L^2(\Lambda)$, let $\{p_0, p_2, \dots, p_N\}$ an orthogonal basis of $\mathcal{X}_N$ and the $L^2(\Lambda)-$orthogonal projection operator $P_N: \ L^2(\Lambda) \to \mathcal{X}_N$ as

$$P_N u(x) = \sum_{n=0}^{N} c_n p_n(x), \ \ N \in \mathbb{N}^*, \tag{1.11}$$

the coefficients c_n are given by

$$c_n = \frac{1}{\|p_n\|^2} \int_\Lambda u(x) p_n(x) dx, \quad \|p_n\|^2 = \int_\Lambda p_n^2(x) dx, \tag{1.12}$$

with

$$\lim_{N \to \infty} \|P_N u - u\|_{L^2(\Lambda)} = 0. \tag{1.13}$$

Galerkin's method

In order to use the Galerkin's method for solving equation (1.1), we expand the approximate solution as

$$u_N(x) = \sum_{j=0}^{N} c_j p_j(x), \ \ N \in \mathbb{N}^*. \tag{1.14}$$

Let us introduce the residual function of the integral equation (1.1)

$$r_N(x) = u_N(x) - \mathcal{K}(u_N)(x) - f(x), \text{ for all } x \in \Lambda. \tag{1.15}$$

The spectral Galerkin method is to find $u_N \in \mathcal{X}_N$, so that the unknown coefficients c_n are computed by requiring the residual r_N to satisfy

$$(r_N, p_i) = (u_N - \mathcal{K}(u_N) - f, p_i) = 0, \quad i = 0, \dots, N. \tag{1.16}$$

If $\mathcal{K}$ is linear operator this yields the linear system

$$\sum_{j=0}^{N} c_j \{(p_j, p_i) - (\mathcal{K}(p_j), p_i)\} = (f, p_i), \quad i = 0, \dots, N, \tag{1.17}$$

else $\mathcal{K}$ is nonlinear operator this yields the nonlinear system

$$\sum_{j=0}^{N} c_j (p_j, p_i) - (\mathcal{K}(\sum_{j=0}^{N} c_j p_j), p_i) = (f, p_i), \quad i = 0, \dots, N, \tag{1.18}$$

where

$$(p_j, p_i) = \delta_{i,j}, \quad i, j = 0, \dots, N.$$

Collocation method

The idea of the collocation method is to choose a number of points $\{x_0, x_1, \ldots, x_N\}$ in the domain (called the set of collocation points), which satisfies

$$r_N(x_i) = 0, \quad i = 0, 1, \ldots, N. \tag{1.19}$$

From the residual in Eq.(1.19) and integral equation (1.1), we have

$$u_N(x_i) - \mathcal{K}(u_N)(x_i) = f(x_j), \; i = 0, 1, \ldots, N. \tag{1.20}$$

We can express the equation (1.20) as

$$\sum_{j=0}^{N} c_j p_j(x_i) - \mathcal{K}(\sum_{j=0}^{N} c_j p_j)(x_i) = f(x_i), \quad i = 0, 1, \ldots, N. \tag{1.21}$$

If $\mathcal{K}$ is a linear operator, we have

$$\sum_{j=0}^{N} c_j \{p_j(x_i) - \mathcal{K}(p_j)(x_i)\} = f(x_i), \quad i = 0, 1, \ldots, N. \tag{1.22}$$

This linear system has a unique solution if

$$\det[p_j(x_i) - \mathcal{K}(p_j)(x_i)] \neq 0.$$

Convergence analysis for a linear integral equation

In this subsection, we discuss the convergence analysis of the spectral solutions of integral equations with L^2 kernel:

. $\left(\int_\Lambda \int_\Lambda |k(x,t)|^2 dt dx\right)^{1/2} < \infty$.

Before starting the main results, we need to introduce some clarifications. From the residual function (1.15) the approximate solution satisfies that

$$u_N = P_N \mathcal{K} u_N + P_N f. \tag{1.23}$$

To this end, we define the iterated solution as

$$\widetilde{u}_N = \mathcal{K} u_N + f. \tag{1.24}$$

Applying P_N on both sides of Eq. (1.24), we obtain

$$P_N \widetilde{u}_N = P_N \mathcal{K} u_N + P_N f. \tag{1.25}$$

From Eqs. (1.23) and (1.25), it follows that $P_N \widetilde{u}_N = u_N$. Using this, we see that the iterated solution $\widetilde{u}_N$ satisfies the following equation

$$\widetilde{u}_N = \mathcal{K} P_N \widetilde{u}_N + f. \tag{1.26}$$

Now , we mention the following theorems which help us to prove the operator $\mathcal{K}$ is a compact, the inverse operator $(I - \mathcal{K})^{-1}$ exists and bounded in $L^2(\Lambda)$ space.

Theorem 1.3. *(Theorem 8 of [51, p.52]) Let $k(x,t)$ be an L^2 kernel on $L^2(\Lambda)$, then the operator*

$$\mathcal{K}u = \int_\Lambda k(x,t)u(t)dt,$$

is compact.

Theorem 1.4. *(Theorem 3.4 of [52]) Let $\mathcal{K}$ be a compact operator on $L^2(\Lambda)$ into itself. Then $I - \mathcal{K}$ is injective if and only if it is surjective. If $I - \mathcal{K}$ is injective (and therefore also bijective). Then the inverse operator $(I - \mathcal{K})^{-1} : L^2(\Lambda) \to L^2(\Lambda)$ exists and bounded.*

To discuss the convergence results, we quote the following Lemma from [53].

Lemma 1. *Assume $\mathcal{K}$ is a compact operator on $L^2(\Lambda)$ and $(I - \mathcal{K}) : L^2(\Lambda) \to L^2(\Lambda)$ be one-to-one. Further assume $\|\mathcal{K} - \mathcal{K}_n\| \to 0$ as $n \to \infty$. Then for all sufficiently large n, the operator $(I - \mathcal{K}_n)^{-1}$ exists and is uniformly bounded on $L^2(\Lambda)$.*

Next we prove the following lemma, which useful to our convergence analysis.

Lemma 2. *Let $\mathcal{K}$ be the linear integral operator with L^2 kernel from $L^2(\Lambda)$ into $L^2(\Lambda)$, if $I - \mathcal{K} : L^2(\Lambda) \to L^2(\Lambda)$ is one to one, then we have*

$$\|\mathcal{K} - \mathcal{K}P_N\| \to 0, \ \ as\ N \to \infty. \tag{1.27}$$

Furthermore, for all sufficiently large N, the operator $(I - \mathcal{K}P_N)^{-1}$ exists as a bounded operator from $L^2(\Lambda)$ to $L^2(\Lambda)$ and bounded

$$\sup_{N\in\mathbb{N}^*} \|(I - \mathcal{K}P_N)^{-1}\| < \infty. \tag{1.28}$$

Proof. For all $u \in L^2(\Lambda)$, we have

$$\begin{aligned} |\mathcal{K}u(x) - \mathcal{K}P_N u(x)|^2 &= \left|\int_\Lambda k(x,t)(u(t) - P_N u(t))dt\right|^2 \\ &\leq \left(\int_\Lambda |k(x,t)||u(t) - P_N u(t)|dt\right)^2. \end{aligned} \tag{1.29}$$

Applying Cauchy-Schwarz inequality, we get

$$|\mathcal{K}u(x) - \mathcal{K}P_N u(x)|^2 = \int_\Lambda |k(x,t)|^2 dt \|u - P_N u\|^2_{L^2(\Lambda)}. \tag{1.30}$$

Hence

$$\|\mathcal{K}u - \mathcal{K}P_N u\|^2_{L^2(\Lambda)} \leq \int_\Lambda\int_\Lambda |k(x,t)|^2 dtdx \|u - P_N u\|^2_{L^2(\Lambda)}. \tag{1.31}$$

Then $\|\mathcal{K} - \mathcal{K}P_N\| \to 0,$ as $N \to \infty$. Next, we have $\|\mathcal{K} - \mathcal{K}P_N\| \to 0$, where $\mathcal{K}$ is a compact operator by applying Lemma 1, we obtain (1.28). □

Therefore, we use the above theorems and lemmas for the convergence analysis of u_N and $\widetilde{u}_N$ to the exact solution u .

Theorem 1.5. *Let u the exact solution of the integral equation $(I-\mathcal{K})u = f$ and let $\tilde{u}_N$ and u_N be the approximate solutions of u defined as in (1.26) and (1.23). Then, we get*

$$\|u-\tilde{u}_N\|_{L^2(\Lambda)} \leq \|(I-\mathcal{K}P_N)^{-1}\|_{L^2(\Lambda)} \left(\int_\Lambda\int_\Lambda |k(x,t)|^2 dtdx\right)^{1/2} \|u-P_N u\|_{L^2(\Lambda)}, \tag{1.32}$$

and

$$\|u-u_N\|_{L^2(\Lambda)} \leq \left(1+\|(I-\mathcal{K}P_N)^{-1}\|_{L^2(\Lambda)} \left(\int_\Lambda\int_\Lambda |k(x,t)|^2 dtdx\right)^{1/2}\right) \|u-P_N u\|_{L^2(\Lambda)}. \tag{1.33}$$

Proof. Since

$$\begin{aligned} u-\tilde{u}_N &= \mathcal{K}u-\mathcal{K}P_N\tilde{u}_N \\ &= \mathcal{K}u-\mathcal{K}P_N u+\mathcal{K}P_N u-\mathcal{K}P_N\tilde{u}_N, \end{aligned} \tag{1.34}$$

we have

$$u-\tilde{u}_N = (I-\mathcal{K}P_N)^{-1}\mathcal{K}(I-P_N)u. \tag{1.35}$$

This implies

$$\begin{aligned} \|u-\tilde{u}_N\|_{L^2(\Lambda)} &\leq \|(I-\mathcal{K}P_N)^{-1}\|_{L^2(\Lambda)}\|\mathcal{K}(I-P_N)u\|_{L^2(\Lambda)} \\ &\leq \|(I-\mathcal{K}P_N)^{-1}\|_{L^2(\Lambda)} \left(\int_\Lambda\int_\Lambda |k(x,t)|^2 dtdx\right)^{1/2} \|u-P_N u\|_{L^2(\Lambda)}. \end{aligned} \tag{1.36}$$

Now, we estimate the error between the approximate and exact solutions where $u_N = P_N\tilde{u}_N$

$$\begin{aligned} u-u_N &= u-P_N\tilde{u}_N \\ &= u-P_N u+P_N u-P_N\tilde{u}_N. \end{aligned} \tag{1.37}$$

This implies

$$\|u-u_N\|_{L^2(\Lambda)} \leqslant \|u_0-P_N u\|_{L^2(\Lambda)} + \|P_N\|_{L^2(\Lambda)}\|u-\tilde{u}_N\|_{L^2(\Lambda)}. \tag{1.38}$$

From (1.36) and (1.38), we get

$$\begin{aligned} &\|u-u_N\|_{L^2(\Lambda)} \\ &\leq \left(1+\|P_N\|_{L^2(\Lambda)}\|(I-\mathcal{K}P_N)^{-1}\|_{L^2(\Lambda)} \left(\int_\Lambda\int_\Lambda |k(x,t)|^2 dtdx\right)^{1/2}\right) \|u-P_N u\|_{L^2(\Lambda)} \\ &\leq \left(1+\|(I-\mathcal{K}P_N)^{-1}\|_{L^2(\Lambda)} \left(\int_\Lambda\int_\Lambda |k(x,t)|^2 dtdx\right)^{1/2}\right) \|u-P_N u\|_{L^2(\Lambda)}. \end{aligned} \tag{1.39}$$

□

1.2 Laguerre polynomials and functions

Orthogonal polynomials and related functions of Laguerre have been developed by many authors. For instance, the authors in [54] introduced a new orthogonal family of generalized Laguerre approximations with the more general weight function $w_{\alpha,\beta}(y) = y^\alpha e^{-\beta y}, \alpha > -1, \beta > 0,\ y \in \Lambda$ where $\Lambda = [0,\infty)$. However, in practice, this family is also not suitable for the semi-infinite problems with decaying solutions at infinity. It is this fact that motivates the authors of [55] introduced the generalized Laguerre functions as basis functions.

In this section, we give some basic properties of generalized Laguerre polynomials and functions with $\alpha = 0$, the so-called scaled Laguerre polynomials and functions.

Scaled Laguerre polynomials

For an arbitrary positive real number $\beta > 0$; scaled Laguerre polynomials of degree n can be defined by

$$\mathcal{L}_n^{\beta}(y) = \frac{1}{n!} e^{\beta y} \partial_y^n (y^n e^{-\beta y}), \quad n \in \mathbb{N}, \tag{1.40}$$

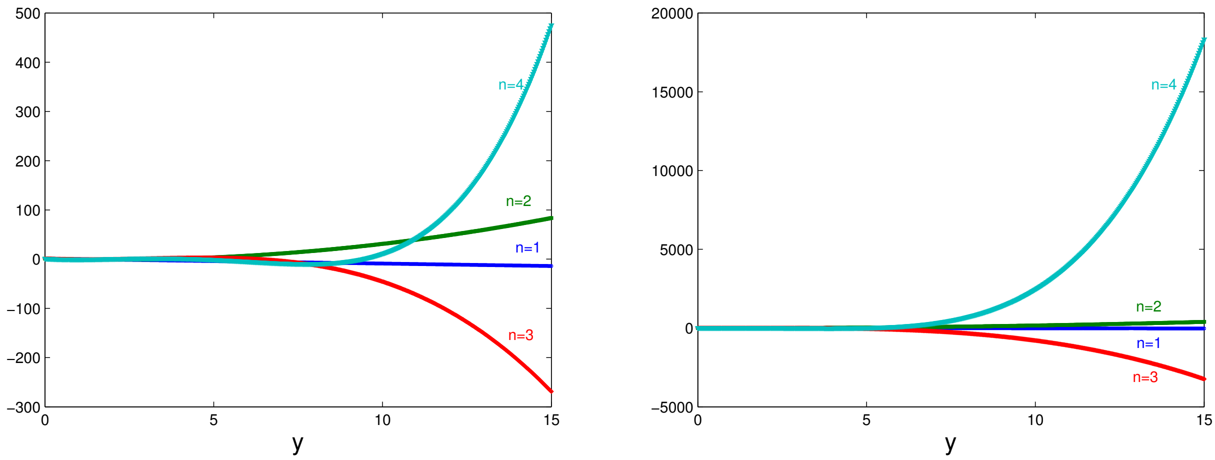
with the following properties

$$\begin{aligned} &\mathcal{L}_0^{\beta}(y) = 1, \quad \mathcal{L}_1^{\beta}(y) = (1 - \beta y); \\ &(n+1)\mathcal{L}_{n+1}^{\beta}(y) = (2n + 1 - \beta y)\mathcal{L}_n^{\beta}(y) - n\mathcal{L}_{n-1}^{\beta}(y), \ n \geq 1, \end{aligned} \tag{1.41}$$

the leading coefficient of $\mathcal{L}_n^{\beta}(y)$ is $(-\beta^n)/n!$ and $\mathcal{L}_n^{\beta}(0) = 1$. In the special case $\beta = 1$, it is common to drop the superscripts since we obtain the classical Laguerre polynomials $\mathcal{L}_n(y)$. Moreover it is important to show that

$$\mathcal{L}_n^{\beta}(y) = \mathcal{L}(x), \text{ where } x = \beta y. \tag{1.42}$$

Let consider the function weight $w_{\beta}(y) = e^{-\beta y}$. We have the following orthogonality



(a) $\beta = 1$. (b) $\beta = 2$.

Figure 1.1: Graphs of the scaled Laguerre polynomials $\mathcal{L}_n^{\beta}(y)$ versus $n = 1, 2, 3, 4$ with $\beta = 1, 2$.

relationship

$$\int_0^{+\infty} \mathcal{L}_n^{\beta}(y)\mathcal{L}_m^{\beta}(y)e^{-\beta y}dy = \frac{1}{\beta}\delta_{n,m}, \tag{1.43}$$

where $\delta_{n,m}$ is the Kronecker function. Moreover, the set of Laguerre polynomials forms a complete $L^2_{w_{\beta}}(\Lambda)-$orthogonal system. Where the $L^2_{w_{\beta}}(\Lambda)$ weighted Hilbert space defined as

$$L^2_{w_{\beta}}(\Lambda) = \{u \mid u \text{ is mesurable on } \Lambda \text{ and } \left(\int_0^{+\infty} |u(y)|^2 w_{\beta}(y)dy\right)^{\frac{1}{2}} < \infty\}, \tag{1.44}$$

equipped with the following inner product and norm:

$$(u, v)_{w_{\beta}} = \int_0^{+\infty} u(y)v(y)w_{\beta}(y)dy, \quad \|u\|_{L^2_{w_{\beta}}(\Lambda)} = (u, u)_{w_{\beta}}^{\frac{1}{2}}. \tag{1.45}$$

For any function $u \in L^2_{w_\beta}(\Lambda)$ can be expanded in series of scaled Laguerre polynomials

$$u(y) = \sum_{n=0}^{\infty} u_n^\beta \mathcal{L}_n^\beta(y), \quad u_n^\beta = \beta \int_0^{+\infty} u(y)\mathcal{L}_n^\beta(y) w_\beta(y) dy. \tag{1.46}$$

Scaled Laguerre polynomial approximations

Now, we define the weighted orthogonal projection operator $P_N^\beta : L^2_{w_\beta}(\Lambda) \to \mathcal{Q}_N^\beta$ as

$$(P_N^\beta u - u, \phi)_{w_\beta} = 0, \quad \forall \phi \in \mathcal{Q}_N^\beta, \tag{1.47}$$

with

$$P_N^\beta u(y) = \sum_{n=0}^{N} u_n^\beta \mathcal{L}_n^\beta(y), \quad u_n^\beta = \beta \int_0^{+\infty} u(y)\mathcal{L}_n^\beta(y) w_\beta(y) dy, \tag{1.48}$$

where $\mathcal{Q}_N^\beta$ denotes the set of all scaled Laguerre polynomials on Λ who have a degree at most N.

In order to describe the approximation results, we introduce the weighted space $H^m_\beta(\Lambda)$. For any integer $m \geq 0$

$$H^m_\beta(\Lambda) = \{u \mid y^{\frac{k}{2}} \partial_y^k u \in L^2_{w_\beta}(\Lambda),\ 0 \leq k \leq m\}, \tag{1.49}$$

equipped with the following semi-norm and norm:

$$|u|_{H^m_\beta(\Lambda)} = \|y^{\frac{m}{2}} \partial_y^m u\|_{L^2_{w_\beta}(\Lambda)}, \quad \|u\|_{H^m_\beta(\Lambda)} = \left(\sum_{k=0}^{m} |u|^2_{H^k_\beta(\Lambda)}\right)^{1/2}.$$

The following Lemma gives the error estimation between the approximate and exact solutions (see Theorem 2.1 in [54]).

Lemma 3. *For any $u \in H^m_\beta(\Lambda)$ and integer $m \geq 1$,*

$$\|P_N^\beta u - u\|_{L^2_{w_\beta}(\Lambda)} \leq c(\beta N)^{-\frac{m}{2}} |u|_{H^m_\beta(\Lambda)}, \tag{1.50}$$

where c is a positive constant independent of N and u.

Scaled Laguerre polynomial interpolation approximations

Now, we introduce the so-called, scaled Laguerre-Gauss interplante defined by

$$I_N^\beta u(y) = \sum_{n=0}^{N} u_n^\beta \mathcal{L}_n^\beta(y), \tag{1.51}$$

such that $I_N^\beta u(x_j^\beta) = u(x_j^\beta)$ where $x_j^\beta,\ 0 \leq j \leq N$, are the scaled Laguerre-Gauss points which are the distinct zeros of the scaled Laguerre polynomial $\mathcal{L}_N^\beta(y)$. The map $I_N^\beta : L^2_{w_\beta}(\Lambda) \to \mathcal{Q}_N^\beta$ is the orthogonal projection, with respect to the discrete inner product and norm,

$$(u, v)_{w_\beta, N} = \sum_{j=0}^{N} u(x_j^\beta) v(x_j^\beta) w_j^\beta, \quad \|u\|_{w_\beta, N} = (u, u)^{1/2}_{w_\beta, N}, \tag{1.52}$$

where $\{w_j^\beta\}_{j=0}^N$ are the discrete weights, such that:

$$\omega_j^\beta = \frac{1}{x_j^\beta [\partial_x \mathcal{L}_{N+1}^\beta (x_j^\beta)]^2}, \tag{1.53}$$

and for all $\psi(y) \in \mathcal{Q}_{2N+1}^\beta$, we have

$$\int_0^{+\infty} \psi(y) w_\beta(y) dy = \sum_{j=0}^N \psi(x_j^\beta) w_j^\beta. \tag{1.54}$$

From (1.52) and (1.54), we have:

$$(u, v)_{w_\beta} = (u, v)_{w_\beta, N} \text{ and } \|u\|_{L^2_{w_\beta}(\Lambda)} = \|u\|_{w_\beta, N}, \forall u, v \in \mathcal{Q}_N^\beta. \tag{1.55}$$

For any integer $m \geq 0$, we define the space

$$-H_\beta^m(\Lambda) = \{u \mid y^{\frac{k-1}{2}} \partial_y^k u \in L^2_{w_\beta}(\Lambda),\ 0 \leq k \leq m\}, \tag{1.56}$$

equipped with the following semi-norm and norm:

$$|u|_{-H_\beta^m(\Lambda)} = \|y^{\frac{m-1}{2}} \partial_y^m u\|_{L^2_{w_\beta}(\Lambda)}, \quad \|u\|_{-H_\beta^m(\Lambda)} = \left(\sum_{k=0}^m |u|^2_{-H_\beta^k(\Lambda)} \right)^{1/2}.$$

We have the following Lemma from Lemma 3.3 of [56].

Lemma 4. *For any $u \in H_\beta^m(\Lambda) \cap -H_\beta^m(\Lambda)$ and integer $m \geq 1$,*

$$\|I_N^\beta u - u\|_{L^2_{w_\beta}(\Lambda)} \leq c\beta^{-\frac{1}{2}} (\beta N)^{-\frac{m}{2}} \left(|u|_{H_\beta^m(\Lambda)} + |u|_{-H_\beta^m(\Lambda)} \right), \tag{1.57}$$

where c is a positive constant independent of N and u.

Scaled Laguerre functions

Let us define the $L^2(\Lambda)$ Hilbert space as

$$L^2(\Lambda) = \{u \mid u \text{ is mesurable on } \Lambda \text{ and } \left(\int_0^{+\infty} |u(y)|^2 dy \right)^{\frac{1}{2}} < \infty\}, \tag{1.58}$$

equipped with the following inner product and norm:

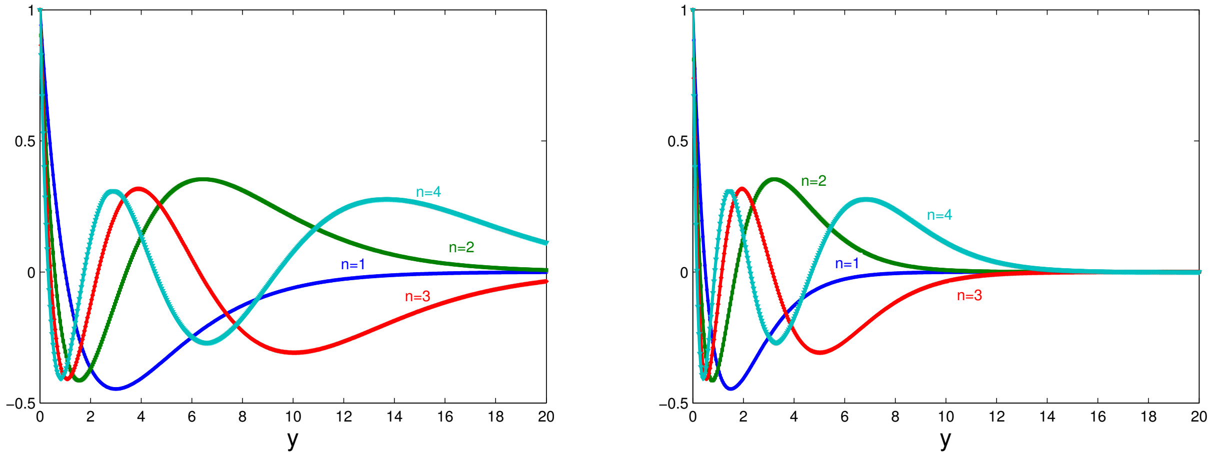
$$(u, v) = \int_0^{+\infty} u(x) v(x) dx, \quad \|u\|_{L^2(\Lambda)} = (u, u)^{\frac{1}{2}}. \tag{1.59}$$

For all $n \in \mathbb{N}$. It is well known that the scaled Laguerre functions can be defined by:

$$\widehat{\mathcal{L}}_n^\beta(y) = e^{-\frac{1}{2}\beta y} \mathcal{L}_n^\beta(y), \quad \beta > 0. \tag{1.60}$$

They can be determined also with the help of the following recurrence relation as

$$\begin{aligned} &\widehat{\mathcal{L}}_0^\beta(y) = e^{-\frac{1}{2}\beta y}, \quad \widehat{\mathcal{L}}_1^\beta(y) = e^{-\frac{1}{2}\beta y}(1 - \beta y); \\ &(n+1)\widehat{\mathcal{L}}_{n+1}^\beta(y) = (2n + 1 - \beta y)\widehat{\mathcal{L}}_n^\beta(y) - n\widehat{\mathcal{L}}_{n-1}^\beta(y). \end{aligned} \tag{1.61}$$



(a) $\beta = 1$ (b) $\beta = 2$

Figure 1.2: Graphs of the scaled Laguerre functions $\widehat{\mathcal{L}}_n^\beta(y)$ versus $n = 1, 2, 3, 4$ with $\beta = 1, 2$.

We have

$$\int_0^{+\infty} \widehat{\mathcal{L}}_n^\beta(y)\widehat{\mathcal{L}}_m^\beta(y)dy = \frac{1}{\beta}\delta_{n,m}, \tag{1.62}$$

where $\delta_{n,m}$ is the Kronecker function. In contrast to the scaled Laguerre polynomials, the scaled Laguerre functions are well-behaved with the decay property (see Figure 1.2)

$$\widehat{\mathcal{L}}_n^\beta(y) \to 0 \text{ as } y \to +\infty.$$

Therefore, the scaled Laguerre functions are suitable for approximation of functions which decay at infinity.

Scaled Laguerre function approximations

Let $N \in \mathbb{N}$, we denote by $\widehat{\mathcal{Q}}_N^\beta$, the finite dimensional approximation subspace spanned by $\{\widehat{\mathcal{L}}_n^\beta(y)\}_{n=0}^N$, which can be written as

$$\widehat{\mathcal{Q}}_N^\beta := \{e^{-\frac{1}{2}\beta y}\psi(\beta y) \mid \psi \in \mathcal{Q}_N^\beta\}.$$

We first consider the orthogonal projection: $\widehat{P}_N^\beta : L^2(\Lambda) \to \widehat{\mathcal{Q}}_N^\beta$. It is defined by

$$(\widehat{P}_N^\beta v - v, \phi) = 0, \quad \forall \phi \in \widehat{\mathcal{Q}}_N^\beta. \tag{1.63}$$

Now, we define the differential operator:

$$\partial_{y,\beta} = \partial_y + \frac{\beta}{2}, \tag{1.64}$$

$$\vdots$$

$$\partial_{y,\beta}^k = \sum_{l=0}^k \binom{k}{l} \left(\frac{\beta}{2}\right)^{k-l} \partial_y^l, \tag{1.65}$$

and we introduce the following Sobolev space

$$\widehat{H}_\beta^m(\Lambda) = \{u \mid y^{\frac{k}{2}}\partial_{y,\beta}^k u \in L^2(\Lambda),\ 0 \le k \le m\}, \tag{1.66}$$

equipped with the semi-norm and norm

$$|u|_{\widehat{H}^m_\beta(\Lambda)} = \|y^{\frac{m}{2}}\partial^m_{y,\beta}u\|_{L^2(\Lambda)}, \quad \|u\|_{\widehat{H}^m_\beta(\Lambda)} = \left(\sum_{k=0}^{m}|u|^2_{\widehat{H}^m_\beta(\Lambda)}\right)^{1/2}.$$

Next, we have the following result (see Lemma 3.1 in [57]).

Lemma 5. *For any $u \in \widehat{H}^m_\beta(\Lambda)$ and integer $m \geq 1$,*

$$\|\widehat{P}^\beta_N u - u\|_{L^2(\Lambda)} \leq c(\beta N)^{-\frac{m}{2}}|u|_{\widehat{H}^m_\beta(\Lambda)}, \tag{1.67}$$

where c is a positive constant independent of N and u.

Scaled Laguerre function interpolation approximations

For a given positive integer N, we introduce the set of scaled Laguerre function-Gauss quadrature set $\{\widehat{x}^\beta_j, \widehat{w}^\beta_j\}^N_{j=0}$ as

$$\widehat{x}^\beta_j = x^\beta_j, \quad \widehat{w}^\beta_j = e^{\beta\widehat{x}^\beta_j}w^\beta_j, \quad 0 \leqslant j \leqslant N, \tag{1.68}$$

where $\{x^\beta_j, w^\beta_j\}^N_{j=0}$ is the set of scaled Laguerre polynomial-Gauss quadrature.

Now, we proceed the discrete inner product and norm associated with the set of scaled Laguerre function-Gauss quadrature as

$$(u, v)_N = \sum_{j=0}^{N} u(\widehat{x}^\beta_j)v(\widehat{x}^\beta_j)\widehat{w}^\beta_j, \quad \|u\|_N = (u, u)^{1/2}_N, \tag{1.69}$$

and it is easy to satisfy that

$$(u, v)_N = (u, v), \quad \|u\|_N = \|u\|_{L^2(\Lambda)}, \text{ for all } u, v \in \mathcal{Q}^\beta_N. \tag{1.70}$$

Let $\widehat{I}^\beta_N$ is the interpolation operator-defined from $L^2(\Lambda)$ to $\mathcal{Q}^\beta_N$ as

$$\widehat{I}^\beta_N u(y) = \sum_{n=0}^{N} u^\beta_n \widehat{\mathcal{L}}^\beta_n(y), \quad u^\beta_n = \beta\sum_{j=0}^{N} u(\widehat{x}^\beta_j)\widehat{\mathcal{L}}^\beta_n(\widehat{x}^\beta_j)\widehat{w}^\beta_j. \tag{1.71}$$

In order to estimate the error between the interpolate and exact solution, we define the following Sobolev space

$$-\widehat{H}^m_\beta(\Lambda) = \{u \mid y^{\frac{k-1}{2}}\partial^k_{y,\beta}u \in L^2(\Lambda),\ 0 \leq k \leq m\}, \tag{1.72}$$

equipped with the semi-norm and norm

$$|u|_{-\widehat{H}^m_\beta(\Lambda)} = \|y^{\frac{m-1}{2}}\partial^m_{y,\beta}u\|_{L^2(\Lambda)}, \quad \|u\|_{-\widehat{H}^m_\beta(\Lambda)} = \left(\sum_{k=0}^{m}|u|^2_{-\widehat{H}^m_\beta(\Lambda)}\right)^{1/2}.$$

Next, we quote the following Lemma from Theorem 3.6 of [57].

Lemma 6. *For any $u \in \widehat{H}^m_\beta(\Lambda) \cap -\widehat{H}^m_\beta(\Lambda)$ and integer $m \geq 1$,*

$$\|\widehat{I}^\beta_N u - u\|_{L^2(\Lambda)} \leq c(\beta N)^{\frac{1-m}{2}}\left(\beta^{-1}|u|_{\widehat{H}^m_\beta(\Lambda)} + (1+\beta^{-\frac{1}{2}})\sqrt{\ln N}\|u\|_{-\widehat{H}^m_\beta(\Lambda)}\right), \tag{1.73}$$

where c is a positive constant independent of N and u.

Numerical examples

In order to examine the scaled Laguerre function collocation method for solving Eq.(1.1) on the half line with rapidly and slowly decaying solutions at infinity, we present some numerical examples and give the L^2-errors for the approximate solution of collocation method and their iterated versions.

Example 1.1. Consider the following integral equation with an exponentially decaying solution

$$u(x) - \int_0^{+\infty} e^{-x^2-t^2} u(t)dt = e^{-x} - \sqrt{\pi} e^{-x^2+1/4}(erf(\frac{1}{2}) - 1)/2, \tag{1.74}$$

where the exact solution $u(x) = e^{-x}$. If we apply scaled Laguerre functions collocation method to solve Example 1.1, we have the following results in Table 1.1.

Table 1.1: The L^2-error for $u(x) = e^{-x}$ with different factors β.

	$\beta = 1$		$\beta = 2$	
N	$\|u - u_N\|_{L^2(\Lambda)}$	$\|u - \tilde{u}_N\|_{L^2(\Lambda)}$	$\|u - u_N\|_{L^2(\Lambda)}$	$\|u - \tilde{u}_N\|_{L^2(\Lambda)}$
2	1.06e-01	1.01e-01	2.47e-02	2.46e-02
8	1.44e-02	1.44e-02	4.20e-04	4.20e-04
16	3.55e-04	3.55e-04	2.21e-07	2.21e-07
32	2.19e-06	2.19e-06	3.10e-10	3.10e-10
64	2.51e-10	2.51e-10	1.95e-13	1.93e-13

Example 1.2. (Sloan and Spence (1986) [58]) Consider Sloan Fredholm integral equation on the half-line:

$$u(x) - \int_0^{+\infty} \frac{1}{1 + x^2 + t^2} u(t)dt = \frac{1}{1 + x^2} - \frac{\pi}{2x^2}(1 - \frac{1}{\sqrt{1 + x^2}}), \tag{1.75}$$

the exact solution is $u(x) = \dfrac{1}{1 + x^2}$, which is a smooth function and decays algebraically at infinity. Table 1.2 shows the comparison errors of iterate and approximate solutions with different various of N at $\beta = 1, 2$.

Remark 1.1. We observed that the scaled Laguerre functions collocation method achieves a convergence at exponential rates for problems with rapidly decaying solutions at infinity whilst a convergence at algebraic rates for problems with slowly decaying solutions at infinity.

1.3 Hermite polynomials and functions

In this section, we shall derive some results on the Hermite orthogonal polynomials/functions approximations and interpolations, defined on the whole line $\Lambda = (-\infty, +\infty)$.

Table 1.2: The L^2-errors for $u(x) = \dfrac{1}{1+x^2}$ with different factors β.

	$\beta = 1$		$\beta = 2$	
N	$\|u - u_N\|_{L^2(\Lambda)}$	$\|u - \widetilde{u}_N\|_{L^2(\Lambda)}$	$\|u - u_N\|_{L^2(\Lambda)}$	$\|u - \widetilde{u}_N\|_{L^2(\Lambda)}$
4	6.84e-01	1.58e-01	2.34e-01	1.04e-01
8	2.97e-01	1.60e-01	3.20e-02	7.90e-03
16	3.23e-02	8.54e-03	2.23e-03	1.60e-03
32	3.04e-03	6.37e-06	3.44e-04	2.15e-04
64	1.12e-04	3.30e-06	4.32e-05	2.70e-05
128	3.40e-06	4.18e-07	6.29e-06	3.34e-06

Hermite polynomials

Let $H_n(x)$ be the usual Hermite polynomials,

$$H_n(x) = (-1)^n e^{x^2} \partial_x^n (e^{-x^2}), \quad n \in \mathbb{N}, \tag{1.76}$$

which are mutually orthogonal, with the weight function $w(x) = e^{-x^2}$, namely:

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) w(x) dx = \gamma_n \delta_{n,m}, \;\; \gamma_n = \sqrt{\pi} 2^n n!. \tag{1.77}$$

The Hermite polynomials satisfy the following recurrence relation:

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x), \;\; n \geq 1, \tag{1.78}$$

and the first few members are:

$$H_0(x) = 1, \tag{1.79}$$
$$H_1(x) = 2x, \tag{1.80}$$
$$H_2(x) = 4x^2 - 2. \tag{1.81}$$

One verifies by induction that leading coefficient of $H_n(x)$ is 2^n. The Hermite polynomials have a close connection with the generalized Laguerre polynomials:

$$H_{2n}(x) = (-1)^n 2^{2n} n! \mathcal{L}_n^{-1/2}(x^2), \tag{1.82}$$
$$H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x \mathcal{L}_n^{1/2}(x^2). \tag{1.83}$$

Let us also introduce the space of functions

$$L_w^2(\Lambda) = \{u \mid u \text{ is mesurable on } \Lambda \text{ and } \left(\int_{-\infty}^{+\infty} |u(x)|^2 w(x) dx \right)^{\frac{1}{2}} < \infty\}, \tag{1.84}$$

where the weighted L^2-norm and inner product are defined by:

$$(u, v)_w = \int_{-\infty}^{+\infty} u(x) v(x) w(x) dx, \quad \|u\|_{L_w^2(\Lambda)} = (u, u)_w^{\frac{1}{2}}. \tag{1.85}$$

Moreover, the set of Hermite polynomials forms a complete $L_w^2(\Lambda)-$orthogonal system.

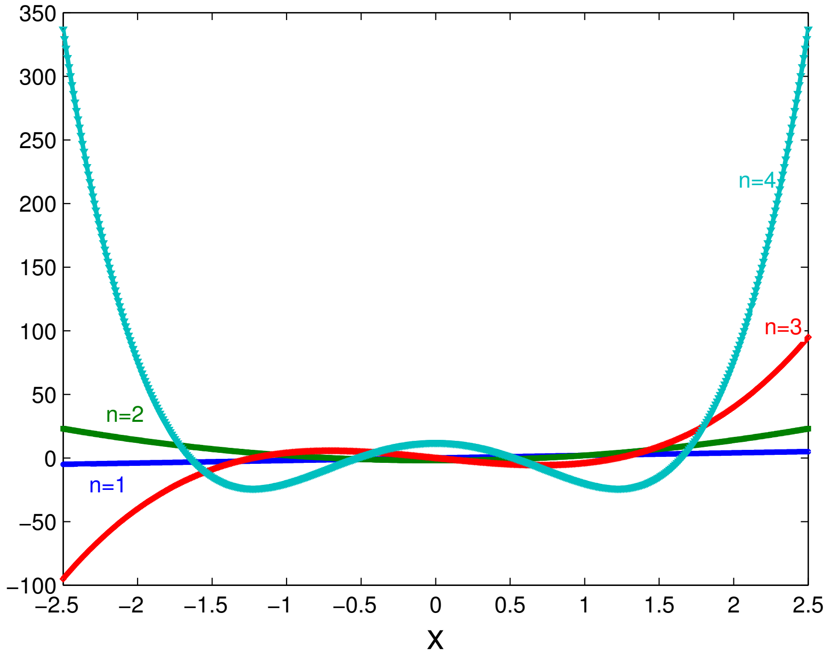


Figure 1.3: Graphs of the Hermite polynomials $H_n(x)$ versus $n = 1, 2, 3, 4$.

Hermite polynomial approximation

Let N be a positive integer and $\mathcal{Q}_N$ denotes the set of all Hermite polynomials on Λ who have a degree at most N. The L^2_w-orthogonal projection $\mathcal{P}_N : L^2_w(\Lambda) \to \mathcal{Q}_N$ as

$$(\mathcal{P}_N u - u, \phi)_w = 0, \quad \forall \phi \in \mathcal{Q}_N, \tag{1.86}$$

or equivalently,

$$\mathcal{P}_N u(x) = \sum_{n=0}^{N} u_n H_n(x) \text{ where } u_n = \frac{1}{\sqrt{\pi}2^n n!} \int_{-\infty}^{+\infty} u(x) H_n(x) w(x) dx. \tag{1.87}$$

Next for any integer $m \geq 0$, we define the weighted normed space as follows:

$$H^m(\Lambda) = \{u \mid \partial_x^k u \in L^2_{w(\Lambda)},\ 0 \leq k \leq m\}, \tag{1.88}$$

with the following semi-norm and norm, respectively,

$$|u|_{H^m(\Lambda)} = \|\partial_x^m u\|_{L^2_w(\Lambda)}, \quad \|u\|_{H^m(\Lambda)} = \left(\sum_{k=0}^{m} |u|^2_{H^k(\Lambda)}\right)^{1/2}.$$

Below we give the following Lemma, which proves the convergence of the approximation solution (see, Theorem 7.13 of [59]).

Lemma 7. *For any $u \in H^m(\Lambda)$ and integer $m \geq 1$,*

$$\|\mathcal{P}_N u - u\|_{L^2_w(\Lambda)} \leq 2^{-\frac{m}{2}} \sqrt{\frac{(N-m+1)!}{(N+1)!}} |u|_{H^m(\Lambda)}. \tag{1.89}$$

Hermite polynomial approximation interpolation

Now, we turn to the interpolation error estimates associated with the Hermite-Gauss quadrature set $\{x_j, w_j\}_{j=0}^N$. Let $\mathcal{I}_N : L^2(\Lambda) \to \mathcal{Q}_N$ be the interpolation operator defined as

$$\mathcal{I}_N u(x) = \sum_{n=0}^{N} u_n H_n(x), \quad u_n = \frac{1}{\sqrt{\pi}2^n n!} \sum_{j=0}^{N} u(x_j) H_n(x_j) w_j, \tag{1.90}$$

where

$$\omega_j = \frac{\Gamma(j+1)x_j}{(j+1)(j+1)![H_{N+1}(x_j)]^2}. \tag{1.91}$$

Accordingly, we introduce the following discrete inner product and norm,

$$(u,v)_{w,N} = \sum_{j=0}^{N} u(x_j)v(x_j)w_j, \quad \|u\|_{w,N} = (u,u)_{w,N}^{1/2}. \tag{1.92}$$

Moreover,

$$(u,v)_{w,N} = (u,v)_w, \quad \|u\|_{w,N} = \|u\|_{L^2_w(\Lambda)}, \text{ for all } u \in P_N, v \in P_{N+1}. \tag{1.93}$$

Next, we quote the following Lemma from Theorem 4.6 of [60]

Lemma 8. *For any $u \in H^m(\Lambda)$ and integer $m \geq 1$,*

$$\|\mathcal{I}_N u - u\|_{L^2_w(\Lambda)} \leq N^{\frac{1}{6}-\frac{m}{2}} |u|_{H^m(\Lambda)}. \tag{1.94}$$

Hermite functions

As in the Laguerre case, the Hermite polynomials are no very useful in practice due to its wild behavior at infinity. Therefore, the authors of [61, 62] introduced a new basis functions named scaled Hermite functions defined by:

$$H_n^\alpha(y) = \frac{1}{\sqrt{2^n n!}} e^{-\frac{1}{2}x} H_n(x), \text{ where } x = \alpha y \text{ and } \alpha > 0. \tag{1.95}$$

The scaled Hermite functions are orthogonal with respect to the weight uniform function, namely:

$$\int_{-\infty}^{+\infty} H_n^\alpha(y) H_m^\alpha(y) dy = \frac{\sqrt{\pi}}{\alpha} \delta_{n,m}. \tag{1.96}$$

For any $u \in L^2(\Lambda)$, we have

$$u(y) = \sum_{n=0}^{\infty} u_n^\alpha H_n^\alpha(y), \tag{1.97}$$

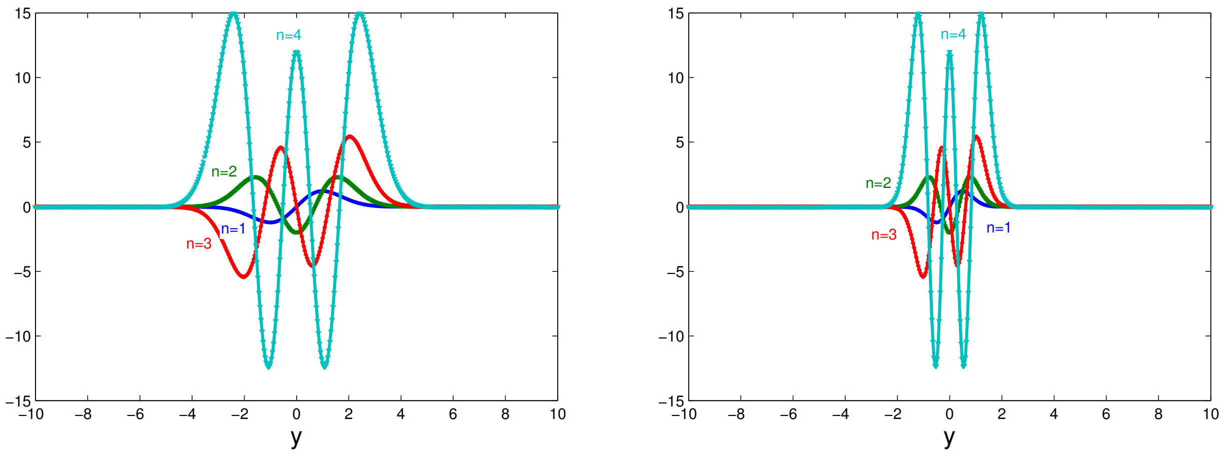
with

$$u_n^\alpha = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{+\infty} u(y) H_n^\alpha(y) dy. \tag{1.98}$$

In contrast to the Hermite polynomials, the scaled Hermite functions are well-behaved with the decay property (see Figure 1.4)

$$H_n^\alpha(y) \to 0 \text{ as } |y| \to \infty.$$

Therefore, the scaled Hermite functions are suitable for approximation of functions which decay at infinity.



(a) $\alpha = 1$. (b) $\alpha = 2$.

Figure 1.4: Graphs of the scaled Hermite functions $H_n^\alpha(y)$ versus $n = 1, 2, 3, 4$ with $\alpha = 1, 2$.

Scaled Hermite functions approximation

Let us define $\mathcal{Q}_N^\alpha := \{e^{-\frac{1}{2}\alpha^2 y^2}\psi(\alpha y) \mid \psi \in \mathcal{Q}_N\}$. Since, we define the orthogonal projection operator $\mathcal{P}_N^\alpha : L^2(\Lambda) \to \mathcal{Q}_N^\alpha$ as

$$(\mathcal{P}_N^\alpha u - u, \phi) = 0, \quad \forall \phi \in \mathcal{Q}_N, \tag{1.99}$$

with

$$\mathcal{P}_N^\alpha u(y) = \sum_{n=0}^{N} u_n^\alpha H_n^\alpha(y), \quad u_n^\alpha = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{+\infty} u(y) H_n^\alpha(y) dy. \tag{1.100}$$

For any integer $m \geq 0$, we define the space

$$H_\alpha^m(\Lambda) = \{u \mid (\alpha^4 y^2 + \alpha^2)^{\frac{m-k}{2}} \partial_y^k u \in L^2\Lambda),\ 0 \leq k \leq m\}, \tag{1.101}$$

equipped with the norm

$$|u|_{H_\alpha^k(\Lambda)} = \|(\alpha^4 y^2 + \alpha^2)^{\frac{m-k}{2}} \partial_y^k u\|_{L^2(\Lambda)}, \quad \|u\|_{H_\alpha^m(\Lambda)} = \left(\sum_{k=0}^{m} \|(\alpha^4 y^2 + \alpha^2)^{\frac{m-k}{2}} \partial_y^k u\|_{L^2(\Lambda)}^2\right)^{1/2}.$$

Lemma 9. *(2.1 of [62]). For any $u \in H_\alpha^m(\Lambda)$ and integer $m \geq 1$,*

$$\|\mathcal{P}_N^\alpha u - u\|_{L^2(\Lambda)} \leq c(\alpha^2 N)^{-\frac{m}{2}} |u|_{H_\alpha^m(\Lambda)}. \tag{1.102}$$

Scaled Hermite functions approximation interpolation

We now turn to the related Hermite–Gauss interpolation. Let x_j and w_j be the nodes and the weights of the standard Hermite–Gauss interpolation, $0 \leq j \leq N$ (see, [26]). We take the nodes and the weights of the scaled Hermite–Gauss interpolation as follows,

$$x_j^\alpha = \frac{x_j}{\alpha}, \quad w_j^\beta = \frac{1}{\alpha} e^{x_j^2} w_j, \quad 0 \leqslant j \leqslant N. \tag{1.103}$$

The corresponding discrete inner product and norm are as follows,

$$(u, v)_N = \sum_{j=0}^{N} u(x_j^\alpha) v(x_j^\alpha) w_j^\alpha, \quad \|u\|_N = (u, u)_N^{1/2}. \tag{1.104}$$

For any $u \in \mathcal{Q}_N^\alpha$ and $v \in \mathcal{Q}_{N+1}^\alpha$, we have

$$(u, v)_N = (u, v), \quad \|u\|_N = \|u\|_{L^2(\Lambda)}. \tag{1.105}$$

The scaled Hermite–Gauss interpolation operator $\mathcal{I}_N^\alpha$ defined from $L^2(\Lambda)$ to $\mathcal{Q}_N^\alpha$ as

$$\mathcal{I}_N^\alpha u(y) = \sum_{n=0}^{N} u_n^\alpha H_n^\alpha(y), \quad u_n^\alpha = \frac{\alpha}{\sqrt{\pi}} \sum_{j=0}^{N} u(x_j^\alpha) H_n^\alpha(x_j^\alpha) w_j^\alpha. \tag{1.106}$$

Lemma 10. *(Theorem 2.1 of [63]). If $u \in H_\alpha^m(\Lambda)$ and integer $m \geq 1$, we have*

$$\|\mathcal{I}_N^\alpha u - u\|_{L^2(\Lambda)} \leq c2(\alpha^2 N)^{\frac{1}{3}-\frac{m}{2}} |u|_{H_\alpha^m(\Lambda)}. \tag{1.107}$$

Numerical examples

This subsection presents some numerical examples to examine the scaled Hermite functions collocation method. Since the solution $u(x) \to 0$ as $|x| \to \infty$.

Example 1.3. [64] For the first example, consider the following of linear Fredholm equation with exponentially decaying solution

$$u(x) - \int_{-\infty}^{+\infty} \frac{u(t)}{(1+x^2)(1+t^2)} dt = f(x), \tag{1.108}$$

where $f(x)$ is chosen so that the exact solution is $u(x) = e^{-x^2}$, which is a smooth function and decays exponentially at infinity, if applying a technique described in the subsection 1.1 using the scaled Hermite functions for different degree n, we get the following results in Table 3.1.

Table 1.3: A comparison of the L^2-errors for different factor α.

	$\alpha = 1$		$\alpha = 2$	
N	$\|u - u_N\|_{L^2(\Lambda)}$	$\|u - \tilde{u}_N\|_{L^2(\Lambda)}$	$\|u - u_N\|_{L^2(\Lambda)}$	$\|u - \tilde{u}_N\|_{L^2(\Lambda)}$
4	2.63e-01	2.20e-01	3.39e-01	3.35e-01
8	3.25e-02	2.86e-02	2.54e-02	2.16e-02
16	1.30e-03	1.29e-03	2.42e-04	1.79e-04
32	1.40e-05	1.40e-05	3.31e-08	2.47e-08
64	2.08e-08	2.08e-08	2.43e-14	2.59e-14

Example 1.4. Consider integral equation with algebraically decaying solution

$$u(x) - \int_{-\infty}^{+\infty} \frac{u(t)}{(1+x^2)(1+t^2)} dt = f(x), \tag{1.109}$$

where $f(x)$ s chosen so that the exact solution is

$$u(x) = \frac{1}{(1+x^2)^2}. \tag{1.110}$$

Table 1.4: Example 1.4:A comparison of the L^2-errors for different factor α.

	$\alpha = 1$		$\alpha = 2$	
N	$\|u - u_N\|_{L^2(\Lambda)}$	$\|u - \widetilde{u}_N\|_{L^2(\Lambda)}$	$\|u - u_N\|_{L^2(\Lambda)}$	$\|u - \widetilde{u}_N\|_{L^2(\Lambda)}$
4	8.58e-01	7.06e-01	5.12e-01	4.78e-01
8	3.28e-01	1.39e-01	1.74e-01	1.16e-01
16	8.51e-02	9.99e-03	5.92e-02	2.41e-02
32	1.15e-02	5.41e-05	1.92e-02	4.54e-03
64	5.79e-04	2.60e-05	5.95e-03	8.12e-04
128	8.39e-06	4.55e-06	1.63e-03	1.42e-04

Remark 1.2. The numerical results in Table 1.3 and 1.4 show again the approximate and iterate solutions converge at exponential rates for rapidly decaying solutions while they converge at algebraic rates for slowly decaying solutions.

1.4 Modified Jacobi functions and Their Properties

As shown in the previous sections all these aforementioned spectral methods do not appear to be suitable for general efficient use. To fill in some of these gaps, first we derive the so-called modified Jacobi functions that can be obtained by combining the classical Jacobi polynomials with an appropriate invertible mapping. Note that a carefully choice of the mapping and its parameters is required to provide a very accurate approximations to the underlying solution. To this end, we will implement two alternative mappings: One is exponential and the other is rational (algebraic) and hence we generate two complementary basis functions that are mutually orthogonal on $L^2(\Lambda)-$space with respect to the uniform weight function. We also estimate an upper bound for function approximation based on modified Jacobi functions on unbounded domains and discuss the convergence of some cases of smooth functions with different decay properties.

Jacobi polynomial

In this subsection, we recall some basic results and give working tools of the classical Jacobi polynomials $J_n^{\alpha,\beta}(y)$ $(n \geq 0)$ are defined by (see[65])

$$(1-y)^{\alpha}(1+y)^{\beta} J_n^{\alpha,\beta}(y) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dy^n}\{(1-y)^{n+\alpha}(1+y)^{n+\beta}\}, \quad y \in I.$$

They are the eigenfunctions of the following Strum-Liouville problem

$$\partial_y\left((1-y)^{\alpha+1}(1+y)^{\beta+1}\partial_y\left(u(y)\right)\right) + \lambda_n^{\alpha,\beta}(1-y)^{\alpha}(1+y)^{\beta}u(y) = 0,$$

where $\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1)$. The Jacobi polynomials satisfy the following recurrence relations (see [66]) with $J_0^{\alpha,\beta}(y) = 1$, $J_1^{\alpha,\beta}(y) = \frac{1}{2}(\alpha+\beta+2)y + \frac{1}{2}(\alpha-\beta)$ and

$$J_{n+1}^{\alpha,\beta}(y) = (a_n y - b_n)J_n^{\alpha,\beta}(y) - c_n J_{n-1}^{\alpha,\beta}(y), \tag{1.111}$$

where

$$a_n = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \tag{1.112}$$

$$b_n = \frac{(\alpha^2-\beta^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \tag{1.113}$$

$$c_n = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \tag{1.114}$$

Let $w^{\alpha,\beta}(y) = (1-y)^\alpha(1+y)^\beta$ be the Jacobi weight function, for $\alpha, \beta > -1$. The Jacobi polynomials are mutually orthogonal in $L^2_{w^{\alpha,\beta}}(I)$, i.e.,

$$\int_I J_n^{\alpha,\beta}(y) J_m^{\alpha,\beta}(y) w^{\alpha,\beta}(y) dy = \gamma_n^{\alpha,\beta}\delta_{n,m}, \tag{1.115}$$

where

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \tag{1.116}$$

The derivatives of Jacobi polynomials

$$\partial_y J_n^{\alpha,\beta}(y) = C_n^{\alpha,\beta} J_{n-1}^{\alpha+1,\beta+1}(y), \quad n \geq 1, \tag{1.117}$$

where $C_n^{\alpha,\beta} = \dfrac{1}{2}(n+\alpha+\beta+1)$ and $C_0^{\alpha,\beta} = 0$, let $G_n^{\alpha,\beta}(y) = (J_0^{\alpha,\beta}(y), \ldots, J_n^{\alpha,\beta}(y))^T$ and $G_{n-1}^{\alpha+1,\beta+1}(y) = (0, J_0^{\alpha+1,\beta+1}(y), J_1^{\alpha+1,\beta+1}(y), \ldots, J_{n-1}^{\alpha+1,\beta+1}(y))^T$.

The relation between the row vector of $G_n^{\alpha,\beta}(y)$ and its derivative is given as

$$\partial_y G_n^{\alpha,\beta}(y) = D_n^{\alpha,\beta} G_{n-1}^{\alpha+1,\beta+1}(y), \quad n \geq 1, \tag{1.118}$$

where

$$D_n^{\alpha,\beta} = \begin{pmatrix} C_0^{\alpha,\beta} & 0 & \cdots & 0 \\ 0 & C_1^{\alpha,\beta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_n^{\alpha,\beta} \end{pmatrix}, \tag{1.119}$$

and

$$(1-y^2)\partial_y J_n^{\alpha,\beta}(y) = A_n J_{n-1}^{\alpha,\beta}(y) + B_n J_n^{\alpha,\beta}(y) + C_n J_{n+1}^{\alpha,\beta}(y), \tag{1.120}$$

where

$$A_n = \frac{2(n+\alpha)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \tag{1.121}$$

$$B_n = \frac{2n(\alpha-\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \tag{1.122}$$

$$C_n = \frac{2n(n+1)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}. \tag{1.123}$$

We shall now derive some important special cases of Jacobi polynomials:

Case 1: $J_n^{\alpha,\alpha}(y)$ for $\alpha > 1/2$ is Gegenbauer polynomials $G_n^\alpha(y)$ associated with $G_0^\alpha(y) = 1, G_1^\alpha(y) = 2\alpha y$ and

$$(n+1)G_{n+1}^\alpha(y) = 2y(\alpha+n)G_n^\alpha(y) - (2\alpha+n-1)G_{n-1}^\alpha(y), \tag{1.124}$$

for $n \geq 1$ and $y \in I$. The set of Gegenbauer polynomials is orthogonal with respect to the weight function $w^\alpha(y) = (1-y^2)^{\alpha-\frac{1}{2}}$. i.e.,

$$\int_I G_n^\alpha(y)G_m^\alpha(y)w^\alpha(y)dy = \gamma_n^\alpha\delta_{n,m}, \quad \gamma_n^\alpha = \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{(n+\alpha)n!\Gamma^2(\alpha)}. \tag{1.125}$$

Case 2: $J_n^{0,0}(y)$ is the Legendre polynomials, which satisfy the following recurrence relation, see [65]

$$(n+1)L_{1+n}(y) = (2n+1)yL_n(y) - nL_{n-1}(y), \quad n \geqslant 1, y \in I. \tag{1.126}$$

Besides

$$L_0(y) = 1,\ L_1(y) = y,\ L_n(1) = 1,\ L_n(-1) = (-1)^n. \tag{1.127}$$

The set of Legendre polynomials in I are mutually orthogonal with respect to the uniform weight function, namely,

$$\int_I L_n(y)L_m(y)dy = \frac{2}{2n+1}\delta_{n,m}, \quad n,m \in \mathbb{N}. \tag{1.128}$$

Case 3: $J_n^{-1/2,-1/2}(y)$ is the Chebyshev polynomials, which satisfy the three-term recurrence relation reads:

$$T_{1+n}(y) = 2yT_n(y) - T_{n-1}(y), \quad n \geqslant 1, y \in I, \tag{1.129}$$

with $T_0(y) = 1$ and $T_1(y) = y$. The set of Chebyshev polynomials are orthogonal with respect to the weight function $w(y) = (1-y^2)^{\frac{1}{2}}$.

Approximation using Jacobi polynomials

Since $J_n^{\alpha,\beta}(y)$ forms a complete orthogonal system in $L^2_{w^{\alpha,\beta}}(I)$, we define the finite dimensional approximation space

$$\mathbb{X}_N^{\alpha,\beta} := Span\{J_0^{\alpha,\beta}, J_1^{\alpha,\beta}, \dots, J_N^{\alpha,\beta}\}. \tag{1.130}$$

Let $\pi_N^{\alpha,\beta} : L^2_{w^{\alpha,\beta}}(I) \longrightarrow \mathbb{X}_N^{\alpha,\beta}$ be the $L^2_{w^{\alpha,\beta}}(I)$-orthogonal projection such that

$$(\pi_N^{\alpha,\beta}u - u, v)_{w^{\alpha,\beta}} = 0, \quad \forall v \in \mathbb{X}_N^{\alpha.\beta}, \tag{1.131}$$

or equivalently,

$$\pi_N^{\alpha,\beta}u(y) = \sum_{n=0}^N u_n^{\alpha,\beta}J_n^{\alpha,\beta}(y), \quad u_n^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}}\int_I u(y)J_n^{\alpha,\beta}(y)w^{\alpha,\beta}(y)dy. \tag{1.132}$$

To provide an error estimate of the truncation error in the $L^2_{w^{\alpha,\beta}}$-norm, let us define the following weighted Sobolev space

$$H^m_{\alpha,\beta}(I) = \{u : u \text{ is measurable in } I \text{ and } \|u\|_{H^m_{\alpha,\beta}} < \infty\}, \tag{1.133}$$

we have the following results from [67].

Lemma 11. *For any $u \in H^m_{\alpha,\beta}(I)$ with $\alpha, \beta > -1$ and $m \geq 1$, we have*

$$\|\pi_N^{\alpha,\beta}u - u\|_{L^2_{w^{\alpha,\beta}}(I)} \leq cN^{-m}\|\partial_y^m u\|_{L^2_{w^{\alpha+m,\beta+m}}}, \tag{1.134}$$

where c is a positive constant independent of N and u.

Interpolation approximation using Jacobi polynomials

Let $I_N^{\alpha,\beta}$ be the interpolation operator, $I_N^{\alpha,\beta}: C(I) \to \mathbb{X}_N^{\alpha,\beta}$ defined as

$$I_N^{\alpha,\beta}u(y) = \sum_{n=0}^{N} u_n^{\alpha,\beta} J_n^{\alpha,\beta}(y), \quad u_n^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \sum_{j=0}^{N} u(x_j^{\alpha,\beta}) J_n^{\alpha,\beta}(x_j^{\alpha,\beta}) w_j^{\alpha,\beta}, \tag{1.135}$$

where $\{x_j^{\alpha,\beta}, w_j^{\alpha,\beta}\}_{j=0}^{N}$ the Jacobi polynomial-Gauss quadrature set. The discrete inner product associate with Jacobi Gauss points defined as

$$(u,v)_{w^{\alpha,\beta},N} = \sum_{j=0}^{N} u(x_j^{\alpha,\beta}) v(x_j^{\alpha,\beta}) w_j^{\alpha,\beta}, \quad \|u\|_{w^{\alpha,\beta},N} = (u,u)_{w^{\alpha,\beta},N}^{1/2}, \text{ for all } u,v \in \mathbb{X}_N^{\alpha,\beta}. \tag{1.136}$$

The following interpolation approximation result can be found in [68]

Lemma 12. *For any $u \in H^m_{\alpha,\beta}(I)$ with $\alpha, \beta > -1$ and $m \geq 1$, we have*

$$\|I_N^{\alpha,\beta}u - u\|_{L^2_{w^{\alpha,\beta}}(I)} \leq cN^{-m} \|\partial_y^m u\|_{L^2_{w^{\alpha+m,\beta+m}}}, \tag{1.137}$$

where c is a positive constant independent of N and u.

Mappings

While the families of orthogonal polynomials as Jacobi, Chebyshev and Legendre are generally defined on I, the physical domains can be on an unbounded domain that can be half or real lines. In this subsection, we recall some one-to-one mappings from bounded to unbounded domains, which has two ways to use the first is the transformation of the unbounded domains into bounded while the second is to generate a great variety of new basis sets for the infinite interval that are the images under the change-of-coordinate of the classical orthogonal systems. The one to one invertible mapping between $y \in I$ and $x \in \Lambda$, of the form:

$$y = \theta(x;s), \quad x = \psi(y;s), \quad s > 0, \tag{1.138}$$

where s is a positive scaling factor. Without loss of generality, we further assume that the mapping is explicitly invertible, and denote its inverse mapping by

$$y = \theta(x;s) = \psi^{-1}(x;s), \quad x \in \Lambda \text{ and } \frac{dx}{dy} = \psi'(y;s) > 0, \quad y \in I, \tag{1.139}$$

that have been proposed and used from the authors of [60].

Mappings on the half-line

In the following, we consider a family of mappings where $x \in \Lambda = [0, +\infty)$ and $y \in I = [-1, 1)$:

$$\theta(0;s) = -1, \quad \theta(+\infty;s) = 1 \text{ and } \psi(-1;s) = 0, \quad \psi(1;s) = +\infty. \tag{1.140}$$

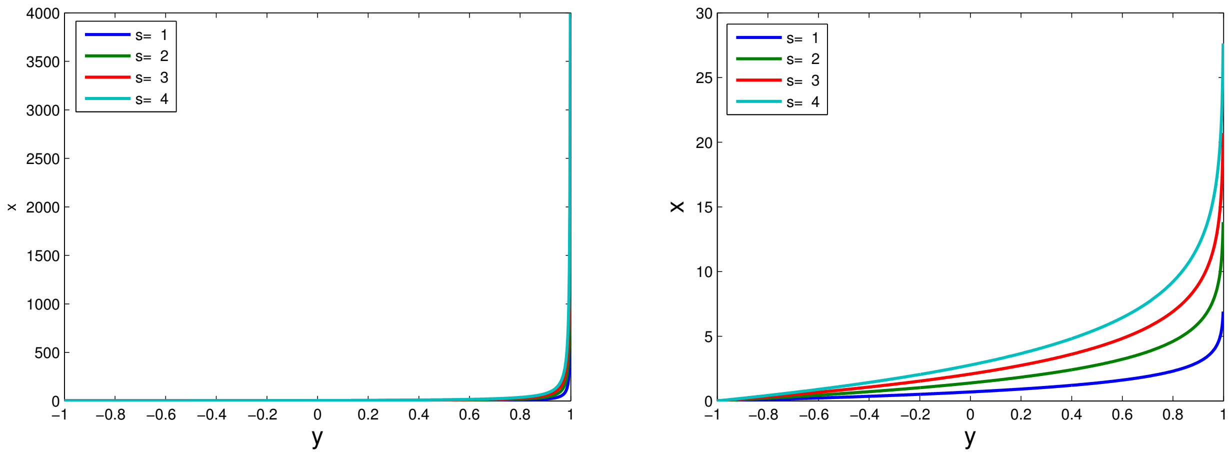
There are two general categories of the half-line stretching, given by the following formulas:

- Exponential mapping:

$$x = -s\ln\left(\frac{1-y}{2}\right), \quad y = 1 - 2e^{-x/s}. \tag{1.141}$$

- Algebraic mapping:

$$x = s\frac{1+y}{1-y}, \quad y = \frac{x-s}{x+s}. \tag{1.142}$$



(a) $x = s\frac{1+y}{1-y}$. (b) $x = 1 - 2e^{-x/s}$.

Figure 1.5: Graphs of mappings on the half line at scaling factors $s = 1, 2, 3, 4$.

Mappings on the real-line

Similar considerations apply to expansions on the real line as on the half line where $x \in \Lambda = (-\infty, +\infty)$ and $y \in I = (-1, 1)$:

$$\theta(-\infty; s) = -1, \quad \theta(+\infty; s) = 1 \text{ and } \psi(-1; s) = -\infty, \quad \psi(1; s) = +\infty. \tag{1.143}$$

- Exponential mapping:

$$x = s \text{ arctanh}(y) = s \ln(\frac{1+y}{1-y}), \quad y = \tanh(s^{-1}x) = \frac{e^{s^{-1}x} - 1}{e^{s^{-1}x} + 1}. \tag{1.144}$$

- Algebraic mapping:

$$x = \frac{sy}{\sqrt{1-y^2}}, \quad y = \frac{x}{\sqrt{s^2 + x^2}}. \tag{1.145}$$

Modified Jacobi functions

In this subsection, we introduce the properties of modified Jacobi functions. First, we define an auxiliary function $\mu_s(y)$ so that

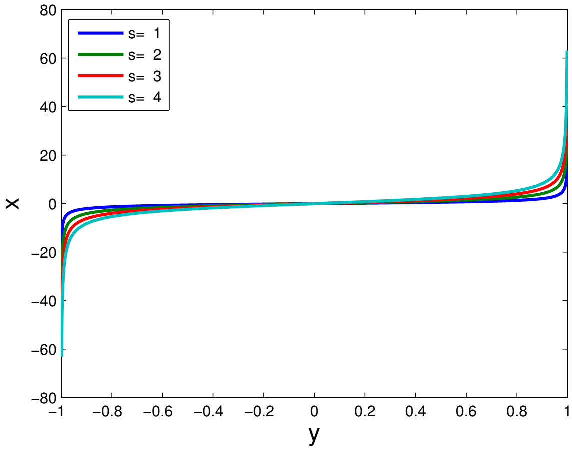
$$\mu_s^2(y)\frac{dx}{dy} = w^{\alpha,\beta}(y), \tag{1.146}$$

or equivalently,

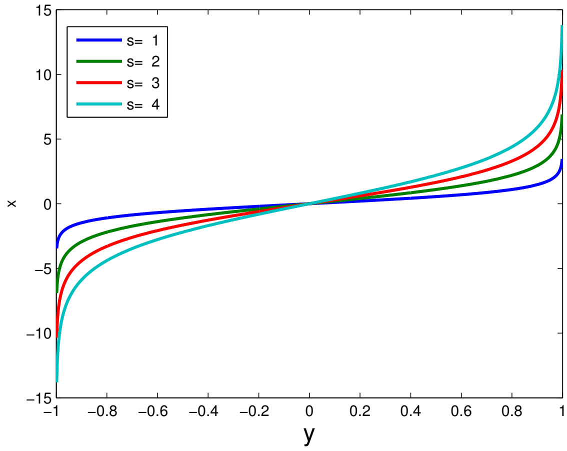
$$\mu_s(y) = \sqrt{\frac{w^{\alpha,\beta}(y)}{\psi'(y; s)}}. \tag{1.147}$$

Let $J_n^{\alpha,\beta}(y)$ be the $n-$th degree Jacobi polynomials defined in (1.111). We define the modified Jacobi functions as

$$j_{s,n}^{\alpha,\beta}(x) = \mu_s(\theta(x; s))J_n^{\alpha,\beta}(\theta(x; s)), \quad s > 0, \quad x \in \Lambda. \tag{1.148}$$



(a) $x = \frac{sy}{\sqrt{1-y^2}}$.



(b) $x = s\ \mathrm{arctanh}(y)$.

Figure 1.6: Graphs of mappings on the real line at scaling factors $s = 1, 2, 3, 4$.

$j^{\alpha,\beta}_{s,n}(x)$ are the eigenfunctions of the following Strum-Liouville problem:

$$\frac{\mu_s(\theta(x;s))}{\theta'(x;s)}\partial_x\left(\frac{(1-\theta(x;s))^{\alpha+1}(1+\theta(x;s))^{\beta+1}}{\theta'(x;s)}\partial_x\left(\frac{u(x)}{\mu_s(\theta(x;s))}\right)\right)$$
$$+\lambda^{\alpha,\beta}_n(1-\theta(x;s))^{\alpha}(1+\theta(x;s))^{\beta}u(x) = 0.$$

Moreover, by (1.111)-(1.114) and (1.138)-(1.147) , we derive that

$$j^{\alpha,\beta}_{s,n+1}(x) = (a_n\theta(x;s) - b_n)j^{\alpha,\beta}_{s,n}(x) - c_n j^{\alpha,\beta}_{s,n-1}(x), \tag{1.149}$$

$$j^{\alpha,\beta}_{s,0}(y) = \mu_s(\theta(x;s)), \quad j^{\alpha,\beta}_{s,1}(x) = \mu_s(\theta(x;s))\left(\frac{1}{2}(\alpha+\beta+2)\theta(x;s) + \frac{1}{2}(\alpha-\beta)\right). \tag{1.150}$$

In particular, if $\alpha = \beta$, the so-called Modified Gegenbauer functions $G^{\alpha}_{s,n}(x)$ associated with

$$G^{\alpha}_{s,0}(x) = \mu_s(\theta(x;s)), G^{\alpha}_{s,1}(x) = 2\alpha\theta(x;s)\mu_s(\theta(x;s)), \tag{1.151}$$

$$(n+1)G^{\alpha}_{s,n+1}(x) = 2\theta(x;s)(\alpha+n)G^{\alpha}_{s,n}(x) - (2\alpha+n-1)G^{\alpha}_{s,n-1}(x). \tag{1.152}$$

Also, by (1.120)-(1.123) and (1.148), we get

$$\frac{\mu_s(\theta(x;s))}{\theta'(x;s)}(1-\theta(x;s)^2)\partial_x\frac{j^{\alpha,\beta}_{s,n}(x)}{\mu_s(\theta(x;s))} = A_n j^{\alpha,\beta}_{s,n-1}(x) + B_n j^{\alpha,\beta}_{s,n}(x) + C_n J^{\alpha,\beta}_{s,n+1}(x). \tag{1.153}$$

The modified Jacobi functions $\{j^{\alpha,\beta}_{s,n}\}$ are orthogonal in $L^2(\Lambda)$ space, namely,

$$\int_\Lambda j^{\alpha,\beta}_{s,n}(x)j^{\alpha,\beta}_{s,m}(x)dx = \int_I J^{\alpha,\beta}_n(y)J^{\alpha,\beta}_m(y)w^{\alpha,\beta}(y)dy = \gamma^{\alpha,\beta}_n\delta_{n,m}, \quad n,m\in\mathbb{N}. \tag{1.154}$$

In order to the sequel, let the pairs of functions associated with invertible mapping introduced as follow:

$$u(x) = u(\psi(y;s)) := U_s(y), \tag{1.155}$$

respectively,

$$\widehat{u}_s(x) = \frac{u(x)}{\mu_s(\theta(x;s))} = \frac{U_s(y)}{\mu_s(y)} = \widehat{U}_s(y) := \widehat{U}_s(\theta(x;s)). \tag{1.156}$$

Proposition 1. *The set of modified Jacobi functions* $\{j_{s,n}^{\alpha,\beta}\}_{n=0}^{\infty}$ *forms a complete* $L^2(\Lambda)$*-orthogonal system.*

Proof. For any function $u \in L^2(\Lambda)$, we have $\widehat{U}_s \in L^2_{w^{\alpha,\beta}}(I)$. Indeed, using (1.147), (1.155) and (1.156)

$$\int_\Lambda |u(x)|^2 dx = \int_I \left(\frac{U_s(y)}{\mu_s(y)}\right)^2 w^{\alpha,\beta}(y)dy = \int_I \widehat{U}_s(y)^2 w^{\alpha,\beta}(y)dy. \tag{1.157}$$

Then, we can expand $\widehat{U}_s(y)$ in the infinite series of Jacobi polynomials as

$$\widehat{U}_s(y) = \sum_{n=0}^{\infty} \widehat{U}_{s,n}^{\alpha,\beta} J_n^{\alpha,\beta}(y),$$

with

$$\widehat{U}_{s,n}^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \int_I \frac{U_s(y)}{\mu_s(y)} J_n^{\alpha,\beta}(y) w^{\alpha,\beta}(y)dy = \frac{1}{\gamma_n^{\alpha,\beta}} \int_\Lambda u(x) j_{s,n}^{\alpha,\beta}(x)dx := u_{s,n}^{\alpha,\beta}. \tag{1.158}$$

Any function $u \in L^2(\Lambda)$ can be expanded in infinite series of modified Jacobi functions as

$$u(x) = \mu_s(\theta(x;s))\widehat{U}_s(\theta(x;s)) = \sum_{n=0}^{\infty} u_n^{\alpha,\beta} \mu_s(\theta(x;s)) J_n^{\alpha,\beta}(\theta(x;s)) = \sum_{n=0}^{\infty} u_{s,n}^{\alpha,\beta} j_{s,n}^{\alpha,\beta}(x). \tag{1.159}$$

This shows that $\{j_{s,n}^{\alpha,\beta}\}_{n=0}^{\infty}$ forms a complete $L^2(\Lambda)-$ orthogonal system. □

Approximation using modified Jacobi functions

For any given positive integer N, we define $\mathbb{X}_{s,N}^{\alpha,\beta}$ the finite dimensional approximation subspace spanned by the set of modified Jacobi functions as

$$\mathbb{X}_{s,N}^{\alpha,\beta} := \{v \mid v(x) = \mu_s(\theta(x;s))\phi(\theta(x;s)), \forall \phi \in \mathbb{X}_N^{\alpha,\beta}\}, \tag{1.160}$$

where $\mathbb{X}_N^{\alpha,\beta}$ defined in (1.130). Let $\pi_{s,N}^{\alpha,\beta} : L^2(\Lambda) \longrightarrow \mathbb{X}_{s,N}^{\alpha,\beta}$ be the $L^2(\Lambda)$-orthogonal projection such that

$$(\pi_{s,N}^{\alpha,\beta} u - u, v) = 0, \quad \forall v \in \mathbb{X}_{s,N}^{\alpha,\beta}, \tag{1.161}$$

where

$$\pi_{s,N}^{\alpha,\beta} u(x) = \sum_{n=0}^{N} u_n^{\alpha,\beta} j_{s,n}^{\alpha,\beta}(x), \quad u_{s,n}^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \int_\Lambda u(x) j_{s,n}^{\alpha,\beta}(x)dx. \tag{1.162}$$

To provide an error estimate of the truncation error in the L^2-norm, we need to introduce the differential operator

$$D_x u = V_s(x)\frac{d\hat{u}_s}{dx}, \qquad V_s(x) := \frac{dx}{dy}, \tag{1.163}$$

and an induction argument leads to

$$D_x^k u = V_s(x)\frac{d}{dx}\left(V_s(x)\frac{d}{dx}\left(\cdots\left(V_s(x)\frac{d\hat{u}_s}{dx}\right)\cdots\right)\right) = \partial_y^k \widehat{U}_s, \quad k = 0, 1, \ldots, \tag{1.164}$$

and then, let us define the following weighted Sobolev spaces

$$\widetilde{H}^{m,s}_{\alpha,\beta}(\Lambda) = \{u : u \text{ is measurable in } \Lambda \text{ and } \|u\|_{\widetilde{H}^{m,s}_{\alpha,\beta}} < \infty\}, \tag{1.165}$$

equipped with the norm and semi-norm

$$\|u\|_{\widetilde{H}^{m,s}_{\alpha,\beta}} = \left(\sum_{k=0}^{m} \|D^k_x u\|^2_{L^2_{\varpi_s^{\alpha+k,\beta+k}}(\Lambda)}\right)^{1/2}, |u|_{\widetilde{H}^{k,s}_{\alpha,\beta}} = \|D^k_x u\|_{L^2_{\varpi_s^{\alpha+k,\beta+k}}(\Lambda)}, \tag{1.166}$$

where the weight function $\varpi^{\alpha,\beta}_s(x) = w^{\alpha,\beta}(\theta(x;s))\dfrac{dy}{dx}$.

In the following, we prove the below Lemma, which estimates the error between the approximate and exact solutions.

Lemma 13. *For any $u \in \widetilde{H}^{m,s}_{\alpha,\beta}(\Lambda)$ with $m \geqslant 1$, we have*

$$\|\pi^{\alpha,\beta}_{s,N}u - u\|_{L^2(\Lambda)} \le cN^{-m}|u|_{\widetilde{H}^{m,s}_{\alpha,\beta}}, \tag{1.167}$$

where c is positive constant independent of N and u.

Proof. Let $\pi^{\alpha,\beta}_N$ be the $L^2_{w^{\alpha,\beta}}-$orthogonal projection operator associated with the Jacobi polynomials. By (1.158) and according to Lemma 11, we can write

$$\begin{aligned}
\|\pi^{\alpha,\beta}_{s,N}u - u\|^2_{L^2(\Lambda)} &= \sum_{n=N+1}^{\infty} \gamma^{\alpha,\beta}_n (u^{\alpha,\beta}_{s,n})^2 = \sum_{n=N+1}^{\infty} \gamma^{\alpha,\beta}_n (\widehat{U}^{\alpha,\beta}_{s,n})^2 \\
&= \|\pi^{\alpha,\beta}_N \widehat{U} - \widehat{U}\|^2_{L^2_{w^{\alpha,\beta}}(I)} \le c^2 N^{-2m} \|\partial^m_y \widehat{U}_s\|^2_{L^2_{w^{\alpha+m,\beta+m}}(I)} \\
&\le c^2 N^{-2m} \|D^m_x u\|^2_{L^2_{\varpi_s^{\alpha+m,\beta+m}}(\Lambda)} \\
&\le c^2 N^{-2m} |u|^2_{\widetilde{H}^m_{\alpha,\beta}}.
\end{aligned} \tag{1.168}$$

□

Modified Jacobi functions interpolation approximations

Now, we denote the set of modified Jacobi-Gauss for a given positive integer N by $\{\zeta^{\alpha,\beta}_{s,j}, \rho^{\alpha,\beta}_{s,j}\}^N_{j=0}$, which is defined as

$$\zeta^{\alpha,\beta}_{s,j} = \psi(x^{\alpha,\beta}_j; s), \quad \rho^{\alpha,\beta}_{s,j} = \frac{\omega^{\alpha,\beta}_j}{\mu^2_s(x^{\alpha,\beta}_j)}, \quad 0 \leqslant j \leqslant N, \tag{1.169}$$

where $x^{\alpha,\beta}_j$ are the roots of $J^{\alpha,\beta}_{N+1}(y)$ and $\omega^{\alpha,\beta}_j$ are the weights of the Jacobi-Gauss quadrature.

We now introduce the discrete inner product and the discrete norm associated with $\{\zeta^{\alpha,\beta}_{s,N,j}\}^N_{j=0}$.

$$(u,v)_N = \sum_{j=0}^{N} u(\zeta^{\alpha,\beta}_{s,j}) v(\zeta^{\alpha,\beta}_{s,j}) \rho^{\alpha,\beta}_{s,j}, \quad \|u\|_N = (u,u)^{1/2}_N. \tag{1.170}$$

Indeed, for any $u \in \mathbb{X}^{\alpha,\beta}_{s,N}$ and $v \in \mathbb{X}^{\alpha,\beta}_{s,N+1}$, we can write

$$u(x) = \mu_s(\theta(x;s))\phi_N(\theta(x;s)), \quad v(x) = \mu(\theta(x;s))\phi_{N+1}(\theta(x;s)), \tag{1.171}$$

with $\phi_{N+1} \in \mathbb{X}_{N+1}^{\alpha,\beta}$ and $\phi_N \in \mathbb{X}_N^{\alpha,\beta}$. By using (1.147) and the Jacobi-Gauss quadrature formula, we have

$$\begin{aligned}
(u,v) &= \int_\Lambda \phi_N(\theta(x;s))\phi_{N+1}(\theta(x;s))\mu_s^2(\theta(x;s))dx \\
&= \int_I \phi_N(y)\phi_{N+1}(y)w^{\alpha,\beta}(y)dy = \sum_{j=0}^{N} \phi_N(x_j^{\alpha,\beta})\phi_{N+1}(x_j^{\alpha,\beta})\omega_j^{\alpha,\beta} \\
&= \sum_{j=0}^{N} u(\zeta_{s,j}^{\alpha,\beta})v(\zeta_{s,j}^{\alpha,\beta})\rho_{s,j}^{\alpha,\beta} = (u,v)_N, \quad \forall u \in \mathbb{X}_{s,N}^{\alpha,\beta}, \forall v \in \mathbb{X}_{s,N+1}^{\alpha,\beta}.
\end{aligned} \tag{1.172}$$

In particular, we have

$$\|u\|_N = \|u\|_{L^2(\Lambda)}, \quad \forall u \in \mathbb{X}_{s,N}^{\alpha,\beta}. \tag{1.173}$$

The modified Jacobi-Gauss interpolation operator $\mathcal{I}_{s,N}^{\alpha,\beta} : C(\Lambda) \longrightarrow \mathbb{X}_{s,N}^{\alpha,\beta}$ is given by

$$\mathcal{I}_{s,N}^{\alpha,\beta}u \in \mathbb{X}_{s,N}^{\alpha,\beta} \text{ such that } (\mathcal{I}_{s,N}^{\alpha,\beta}u)(\zeta_{s,j}^{\alpha,\beta}) = u(\zeta_{s,j}^{\alpha,\beta}), \quad 0 \le j \le N, \tag{1.174}$$

which can be expanded as

$$\mathcal{I}_{s,N}^{\alpha,\beta}u(x) = \sum_{n=0}^{N} u_{s,n}^{\alpha,\beta} j_{s,n}^{\alpha,\beta}(x), \tag{1.175}$$

such that

$$u_{s,n}^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \sum_{j=0}^{N} u(\zeta_{s,j}^{\alpha,\beta}) j_{s,n}^{\alpha,\beta}(\zeta_{s,j}^{\alpha,\beta})\rho_{s,j}^{\alpha,\beta}. \tag{1.176}$$

Lemma 14. *For any $u \in \widetilde{H}_{\alpha,\beta}^m(\Lambda)$ with $m \ge 1$, we have*

$$\|\mathcal{I}_{s,N}^{\alpha,\beta}u - u\|_{L^2(\Lambda)} \le cN^{-m}|u|_{\widetilde{H}_{\alpha,\beta}^{m,s}}, \tag{1.177}$$

where c is positive constant independent of N and u.

Proof. Recall the Jacobi-Gauss interpolation operator $I_N^{\alpha,\beta} : C(I) \longrightarrow \mathbb{X}_N^{\alpha,\beta}$, of the form

$$I_N^{\alpha,\beta}\widehat{U}_s(y) = \sum_{n=0}^{N} \widehat{U}_{s,n}^{\alpha,\beta} J_n^{\alpha,\beta}(y), \tag{1.178}$$

such that

$$\widehat{U}_{s,n}^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \sum_{j=0}^{N} \widehat{U}_s(x_j^{\alpha,\beta}) J_n^{\alpha,\beta}(x_j^{\alpha,\beta})\omega_j^{\alpha,\beta}. \tag{1.179}$$

Firstly, by using (1.148), (1.169) and (1.176), we get

$$\begin{aligned}
u_{s,n}^{\alpha,\beta} &= \frac{1}{\gamma_n^{\alpha,\beta}} \sum_{j=0}^{N} u(\zeta_{s,j}^{\alpha,\beta}) j_{s,n}^{\alpha,\beta}(\zeta_{s,j}^{\alpha,\beta})\rho_{s,j}^{\alpha,\beta} \\
&= \frac{1}{\gamma_n^{\alpha,\beta}} \sum_{j=0}^{N} U(x_j^{\alpha,\beta})\mu_s(x_j^{\alpha,\beta}) J_n^{\alpha,\beta}(x_j^{\alpha,\beta}) \frac{\omega_j^{\alpha,\beta}}{\mu_s^2(\sigma_j^{\alpha,\beta})} \\
&= \frac{1}{\gamma_n^{\alpha,\beta}} \sum_{j=0}^{N} \frac{U(x_j^{\alpha,\beta})}{\mu_s(x_j^{\alpha,\beta})} J_n^{\alpha,\beta}(x_j^{\alpha,\beta})\omega_j^{\alpha,\beta} \\
&= \frac{1}{\gamma_n^{\alpha,\beta}} \sum_{j=0}^{N} \widehat{U}_s(x_j^{\alpha,\beta}) J_n^{\alpha,\beta}(x_j^{\alpha,\beta})\omega_j^{\alpha,\beta} = \widehat{U}_{s,n}^{\alpha,\beta},
\end{aligned} \tag{1.180}$$

and therefore

$$\mathcal{I}_{s,N}^{\alpha,\beta}u(x) = \sum_{n=0}^{N} u_{s,n}^{\alpha,\beta} j_{s,n}^{\alpha,\beta}(x) = \sum_{n=0}^{N} \widehat{U}_{s,n}^{\alpha,\beta} \mu_s(\theta(x;s)) J_n^{\alpha,\beta}(\theta(x;s)) = \mu_s(y) I_N^{\alpha,\beta} \widehat{U}_s(y). \tag{1.181}$$

Now, we can write

$$\begin{aligned} \int_{\Lambda} \mid \mathcal{I}_{s,N}^{\alpha,\beta}u(x) - u(x) \mid^2 dx &= \int_I \mid \mu_s(y) \left(I_N^{\alpha,\beta}\widehat{U}_s(y) - \frac{U(y)}{\mu_s(y)} \right) \mid^2 \frac{w^{\alpha,\beta}(y)}{\mu_s(y)^2} dy \\ &= \int_I \mid I_N^{\alpha,\beta}\widehat{U}_s(y) - \widehat{U}_s(y) \mid^2 w^{\alpha,\beta}(y) dy. \end{aligned} \tag{1.182}$$

Hence,

$$\|\mathcal{I}_{s,N}^{\alpha,\beta}u - u\|_{L^2(\Lambda)} = \|I_N^{\alpha,\beta}\widehat{U} - \widehat{U}\|_{L^2_{w^{\alpha,\beta}}(I)}. \tag{1.183}$$

Finally, according to Lemma 12, it is mentioned that for any $\widehat{U}_s \in L^2_{w^{\alpha,\beta}}(I)$ and $m \geq 1$,

$$\|I_N^{\alpha,\beta}\widehat{U} - \widehat{U}\|_{L^2_{w^{\alpha,\beta}}(I)} \leq cN^{-m} \|\partial_y^m \widehat{U}_s\|_{L^2_{w^{\alpha+m,\beta+m}}(I)}, \tag{1.184}$$

which implies

$$\|\mathcal{I}_{s,N}^{\alpha,\beta}u - u\|_{L^2(\Lambda)} \leq cN^{-m} |u|_{\widetilde{H}^{m,s}_{\alpha,\beta}}. \tag{1.185}$$

□

Modified Jacobi Rational and Exponential functions

We now introduce the Modified Jacobi Exponential and Rational functions that all of them are defined on the half and real lines.

First, we consider the family of auxiliary functions associated with mappings (1.142) and (1.141) on the half line by (1.147) as

- Exponential auxiliary function:

$$\mu_s^2(y) = \frac{(1-y)^{\alpha+1}(1+y)^{\beta}}{s}, \quad \mu_s^2(\theta(x;s)) = \frac{(2e^{-x/s})^{\alpha+1}(2-2e^{-x/s})^{\beta}}{s}, \tag{1.186}$$

with

$$\psi'(y;s) = \frac{s}{(1-y)}, \quad \theta'(x;s) = \frac{2e^{-x/s}}{s}. \tag{1.187}$$

- Algebraic auxiliary function:

$$\mu_s^2(y) = \frac{(1-y)^{\alpha+2}(1+y)^{\beta}}{2s}, \quad \mu_s^2(\theta(x;s)) = \frac{2^{\alpha+\beta+1}s^{\alpha+1}x^{\beta}}{(x+s)^{\alpha+\beta+2}}, \tag{1.188}$$

with

$$\psi'(y;s) = \frac{2s}{(1-y)^2}, \quad \theta'(x;s) = \frac{2s}{(s+x)^2}. \tag{1.189}$$

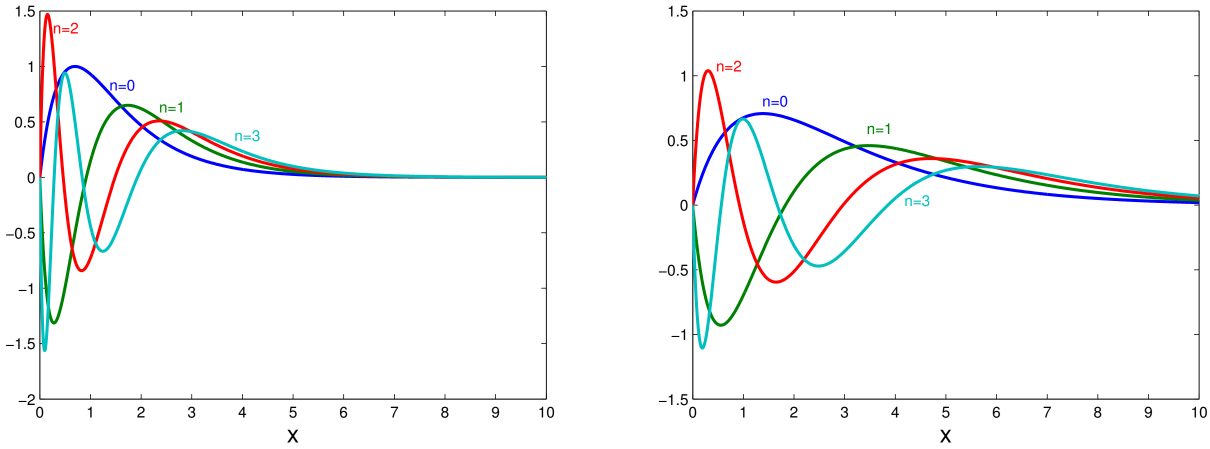
The Modified Jacobi Exponential functions-MJEFs is denoted by $E_{s,n}^{\alpha,\beta}(x)$ and defined as

$$E_{s,n}^{\alpha,\beta}(x) = \sqrt{\frac{(2e^{-x/s})^{\alpha+1}(2-2e^{-x/s})^{\beta}}{s}} J_n^{\alpha,\beta}\left(1-2e^{-x/s}\right), \tag{1.190}$$

with

$$\begin{cases} E_{s,0}^{\alpha,\beta}(x) = \sqrt{\dfrac{(2e^{-x/s})^{\alpha+1}(2-2e^{-x/s})^{\beta}}{s}}, \\ E_{s,1}^{\alpha,\beta}(x) = \sqrt{\dfrac{(2e^{-x/s})^{\alpha+1}(2-2e^{-x/s})^{\beta}}{s}} \left(\frac{1}{2}(\alpha+\beta+2)(1-2e^{-x/s}) + \frac{1}{2}(\alpha-\beta)\right), \\ \vdots \\ E_{s,n+1}^{\alpha,\beta}(x) = (a_n(1-2e^{-x/s}) - b_n)E_{s,n}^{\alpha,\beta}(x) - c_n E_{s,n-1}^{\alpha,\beta}(x). \end{cases} \tag{1.191}$$

The behavior of these four functions for $s = 1$ and $s = 2$ are plotted in Figure.1.7 Exclusive



(a) MJEFs with $s = 1$.

(b) MJEFs with $s = 2$.

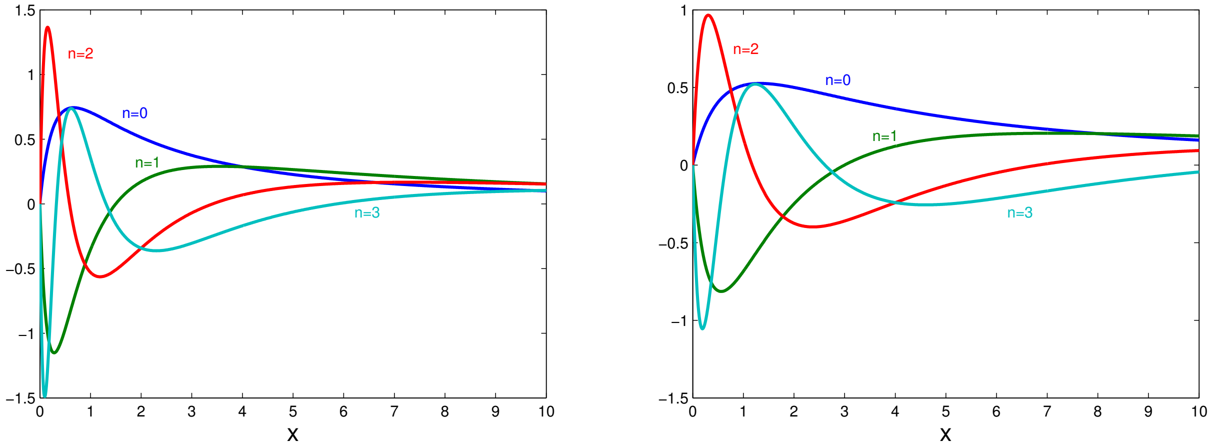
Figure 1.7: Graphs of the first four Modified Jacobi Exponential functions $E_{s,n}^{1,2}(x)$ versus with $n = 0,\ldots,3$ and scaling factors $(s = 1, 2)$.

of modified exponential functions we can use rational transformation to have new functions which are also defined on the real line. The Modified Jacobi Rational functions-MJRFs is denoted by $R_{s,n}^{\alpha,\beta}(x)$ and defined as

$$R_{s,n}^{\alpha,\beta}(x) = \sqrt{\frac{2^{\alpha+1}s^{\alpha+1}x^{\beta}}{(x+s)^{\alpha+\beta+2}}} J_n^{\alpha,\beta}\left(\frac{x-s}{x+s}\right), \tag{1.192}$$

with

$$\begin{cases} R_{s,0}^{\alpha,\beta}(x) = \sqrt{\dfrac{2^{\alpha+1}s^{\alpha+1}x^{\beta}}{(x+s)^{\alpha+\beta+2}}}, \\ R_{s,1}^{\alpha,\beta}(x) = \sqrt{\dfrac{2^{\alpha+1}s^{\alpha+1}x^{\beta}}{(x+s)^{\alpha+\beta+2}}} \left(\frac{1}{2}(\alpha+\beta+2)\dfrac{x-s}{x+s} + \frac{1}{2}(\alpha-\beta)\right), \\ \vdots \\ R_{s,n+1}^{\alpha,\beta}(x) = (a_n \dfrac{x-s}{x+s} - b_n)R_{s,n}^{\alpha,\beta}(x) - c_n R_{s,n-1}^{\alpha,\beta}(x). \end{cases} \tag{1.193}$$



(a) MJRFs with $s = 1$. (b) MJRFs with $s = 2$.

Figure 1.8: Graphs of the first four Modified Jacobi Rational functions $R^{1,2}_{s,n}(x)$ versus with $n = 0, \ldots, 3$ and scaling factors $(s = 1, 2)$.

The behavior of these four functions for $s = 1$ and $s = 2$ are plotted in Figure.1.8

As in the half-line. To define the modified Jacobi functions on the real line, we consider the family of auxiliary functions associated with mappings (1.144) and (1.145) on the real line by (1.147) as

- Exponential auxiliary function:

$$\mu_s^2(y) = \frac{(1-y)^{\alpha+1}(1+y)^{\beta+1}}{s}, \quad \mu_s^2(\theta(x;s)) = \frac{2^{\alpha+\beta+2}e^{\frac{\beta+1}{s}x}}{s(e^{s^{-1}x}+1)^{\alpha+\beta+2}}, \tag{1.194}$$

with

$$\psi'(y;s) = \frac{s}{1-y^2}, \quad \theta'(x;s) = \frac{1-\tanh(s^{-1}x)^2}{s}. \tag{1.195}$$

- Algebraic auxiliary function:

$$\mu_s^2(y) = \frac{(1-y)^{\alpha+3/2}(1+y)^{\beta+3/2}}{s}, \quad \mu_s^2(\theta(x;s)) = \frac{(1-\frac{x}{\sqrt{s^2+x^2}})^{\alpha+3/2}(1+\frac{x}{\sqrt{s^2+x^2}})^{\beta+3/2}}{s}, \tag{1.196}$$

with

$$\psi'(y;s) = \frac{s}{(1-y^2)^{3/2}}, \quad \theta'(x;s) = \frac{s^2}{(s^2+x^2)^{3/2}}. \tag{1.197}$$

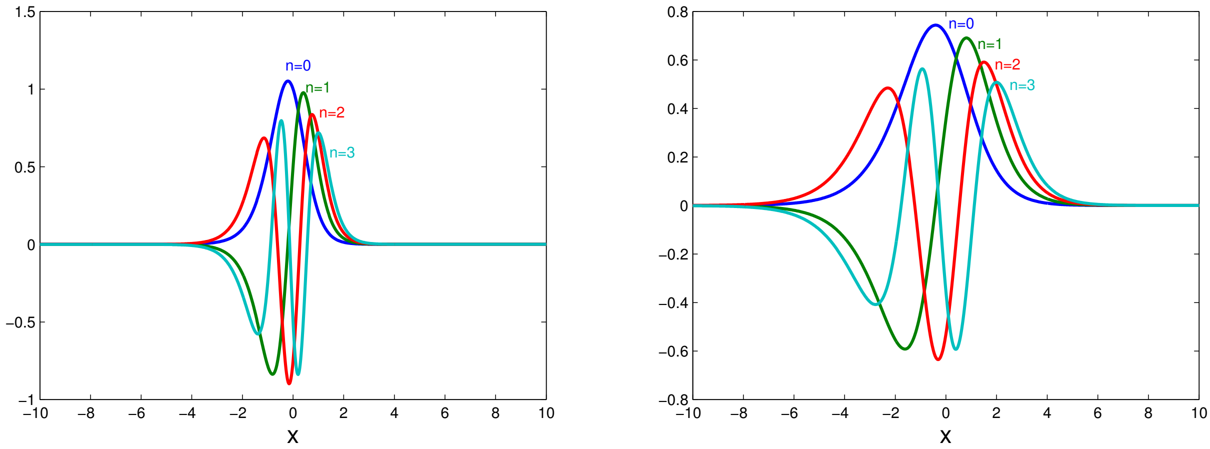
The Modified Jacobi Exponential functions on the half line is also denoted by the same for the real line $E^{\alpha,\beta}_{s,n}(x)$ and defined as

$$E^{\alpha,\beta}_{s,n}(x) = \frac{2^{\frac{\alpha+\beta+2}{2}}e^{\frac{\beta+1}{2s}x}}{\sqrt{s(e^{s^{-1}x}+1)^{\alpha+\beta+2}}} J^{\alpha,\beta}_n\left(\frac{e^{s^{-1}x}-1}{e^{s^{-1}x}+1}\right), \tag{1.198}$$

By using (1.149) and (1.150) recursive formula the Modified Exponential Jacobi functions are obtained as below:

$$\begin{cases} E^{\alpha,\beta}_{s,0}(x) = \dfrac{2^{\frac{\alpha+\beta+2}{2}} e^{\frac{\beta+1}{2s}x}}{\sqrt{s(e^{s^{-1}x}+1)^{\alpha+\beta+2}}}, \\ E^{\alpha,\beta}_{s,1}(x) = \dfrac{2^{\frac{\alpha+\beta+2}{2}} e^{\frac{\beta+1}{2s}x}}{\sqrt{s(e^{s^{-1}x}+1)^{\alpha+\beta+2}}} \left(\frac{1}{2}(\alpha+\beta+2)\dfrac{e^{s^{-1}x}-1}{e^{s^{-1}x}+1} + \frac{1}{2}(\alpha-\beta)\right), \\ \vdots \\ E^{\alpha,\beta}_{s,n+1}(x) = (a_n \dfrac{e^{s^{-1}x}-1}{e^{s^{-1}x}+1} - b_n) E^{\alpha,\beta}_{s,n}(x) - c_n E^{\alpha,\beta}_{s,n-1}(x). \end{cases} \tag{1.199}$$

The behavior of these four functions for $s = 1$ and $s = 2$ are plotted in Figure.1.9.



(a) MJEFs with $s = 1$.

(b) MJEFs with $s = 2$.

Figure 1.9: Graphs of the first four Modified Jacobi Exponential functions is denoted by $E^{2,1}_{s,n}(x)$ versus with $n = 0, \ldots, 3$ and scaling factors ($s = 1, 2$).

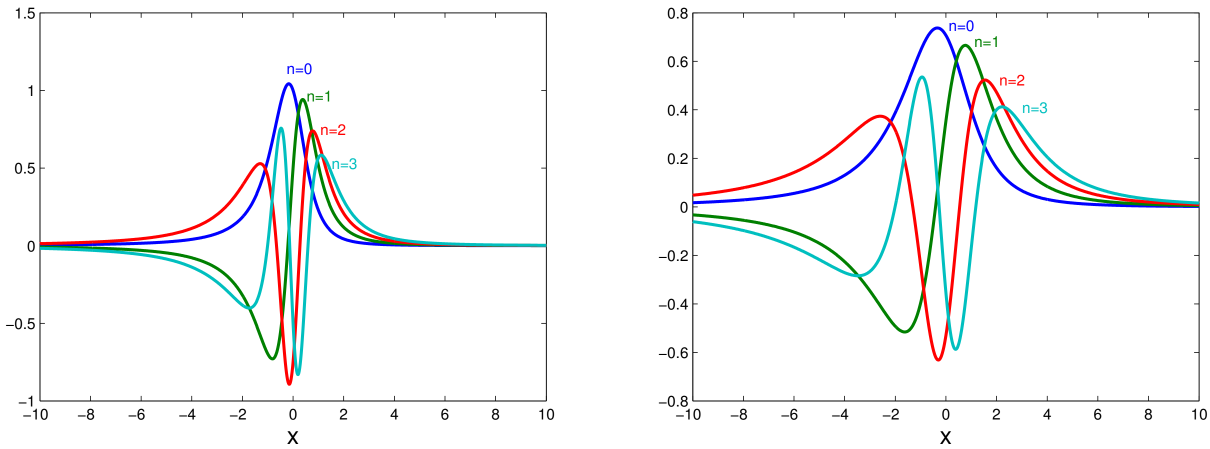
The Modified Jacobi Rational functions on the real line is also denoted by the same for the half-line $R^{\alpha,\beta}_{s,n}(x)$ and defined as

$$R^{\alpha,\beta}_{s,n}(x) = \frac{(1-\frac{x}{\sqrt{s^2+x^2}})^{\frac{\alpha+3/2}{2}}(1+\frac{x}{\sqrt{s^2+x^2}})^{\frac{\beta+3/2}{2}}}{s^{\frac{1}{2}}} J^{\alpha,\beta}_n\left(\frac{x}{\sqrt{s^2+x^2}}\right), \tag{1.200}$$

with

$$\begin{cases} R^{\alpha,\beta}_{s,0}(x) = \dfrac{(1-\frac{x}{\sqrt{s^2+x^2}})^{\frac{\alpha+3/2}{2}}(1+\frac{x}{\sqrt{s^2+x^2}})^{\frac{\beta+3/2}{2}}}{s^{\frac{1}{2}}}, \\ R^{\alpha,\beta}_{s,1}(x) = \dfrac{(1-\frac{x}{\sqrt{s^2+x^2}})^{\frac{\alpha+3/2}{2}}(1+\frac{x}{\sqrt{s^2+x^2}})^{\frac{\beta+3/2}{2}}}{s^{\frac{1}{2}}} \left(\frac{1}{2}(\alpha+\beta+2)\frac{x}{\sqrt{s^2+x^2}} + \frac{1}{2}(\alpha-\beta)\right), \\ \vdots \\ R^{\alpha,\beta}_{s,n+1}(x) = (a_n \frac{x}{\sqrt{s^2+x^2}} - b_n) R^{\alpha,\beta}_{s,n}(x) - c_n R^{\alpha,\beta}_{s,n-1}(x). \end{cases} \tag{1.201}$$

For various values n and $(\alpha, \beta) = (2, 1)$ Figure.1.10 plotters $R^{\alpha,\beta}_{s,n}$ for $s = 1$ and $s = 2$.



(a) MJRFs with $s = 1$.

(b) MJRFs with $s = 2$.

Figure 1.10: Graphs of the first four Modified Jacobi Rational functions $R^{2,1}_{s,n}(x)$ versus with $n = 0, \ldots, 3$ and scaling factors ($s = 1, 2$).

The graphs in the figures 1.7 , 1.8, 1.9 and 1.10 mean that the modified Jacobi Exponential and Rational functions are suitable for approximation of functions which decay at infinity. To examine the accuracy of the Modified Jacobi function approximations, we take $\alpha = \beta$ with different positive scaling parameters s and test the typical exact solutions as

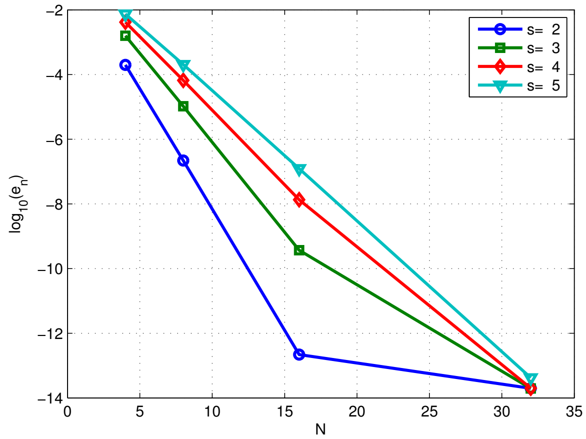
. $u(x) = \dfrac{1}{1+x}$, which decays algebraically at infinity on the half line. In Figure 1.11, we plot the L^2-errors for $\alpha = 0$ with $s = 2, 3, 4, 5$: (1.11a) using Jacobi rational functions, which indicate exponential convergence rates; (1.11c) using Jacobi exponential functions, which indicate algebraic convergence rates, as predicted by Lemma 13.

. $u(x) = e^{-x}$, that decays exponentially at infinity on the half line. In Figure 1.11, we plot the L^2-errors for $\alpha = 0$ with $s = 2, 3, 4, 5$: (1.11b) using Jacobi rational functions , which indicate exponential convergence rates; (1.11d) using Jacobi exponential functions, which indicate exponential convergence rates, as predicted by Lemma 13.

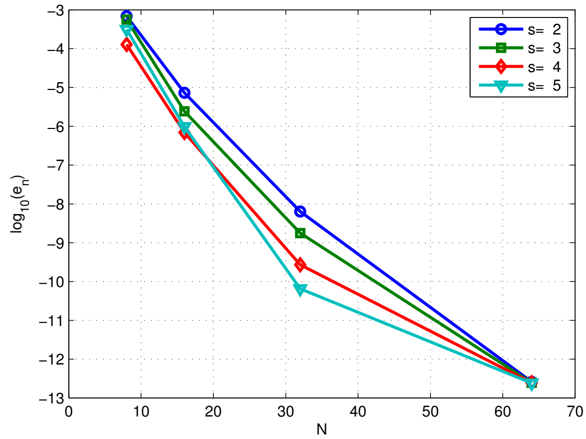
. $u(x) = \dfrac{1}{1+x^2}$, which decays algebraically on the real line. In Figure 1.12, we plot the L^2-errors for $\alpha = 1/2$ with $s =, 2, 3, 4, 5$: (1.12a) using Jacobi rational functions , which indicate exponential convergence rates; (1.12c) for using Jacobi exponential functions, which indicate algebraic convergence rates, as predicted by Lemma 13.

. $u(x) = e^{-x^2}$, which decays exponentially at infinity on the real line. In Figure 1.12, we plot the L^2-errors for $\alpha = 1/2$ with $s =, 2, 3, 4, 5$: (1.12b) using Jacobi rational functions , which indicate exponential convergence rates; (1.12d) for using Jacobi exponential functions, which indicate exponential convergence rates, as predicted by Lemma 13.

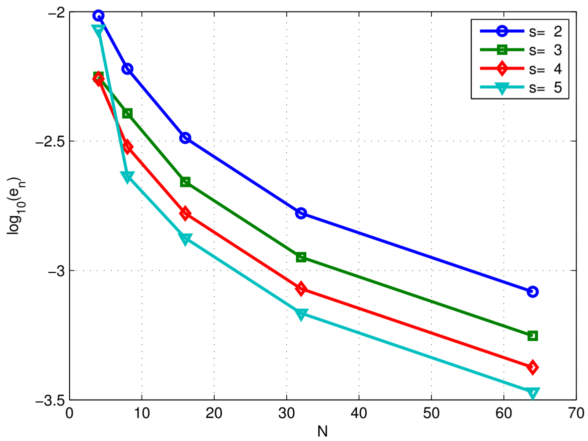
Remark 1.3. As we show in Figures 2.3 and 2.1, the choice of the mapping depends on the asymptotic behavior of the solution at infinity. While the parameters β,α and s played an essential role in designing the Modified Jacobi functions to improve the convergence rates. Note that the suitable choice of β to approximate a function with $u(0) \neq 0$ by using Modified Jacobi functions on the half line is $\beta = 0$ because $R^{\alpha,\beta}_{s,n}(0) = E^{\alpha,\beta}_{s,n}(0) = 0$ for all $n \in \mathbb{N}$ and $\beta > 0$.



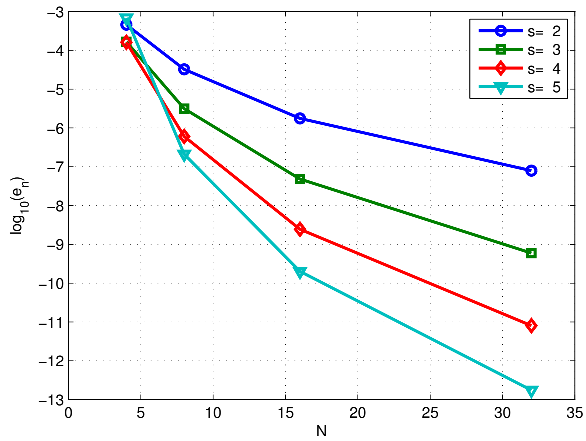
(a) $e_N = \|\dfrac{1}{1+x} - \sum_{n=0}^{N} u_n R_{s,n}(x)\|_{L^2(\Lambda)}$.



(b) $e_N = \|e^{-x} - \sum_{n=0}^{N} u_n R_{s,n}(x)\|_{L^2(\Lambda)}$.

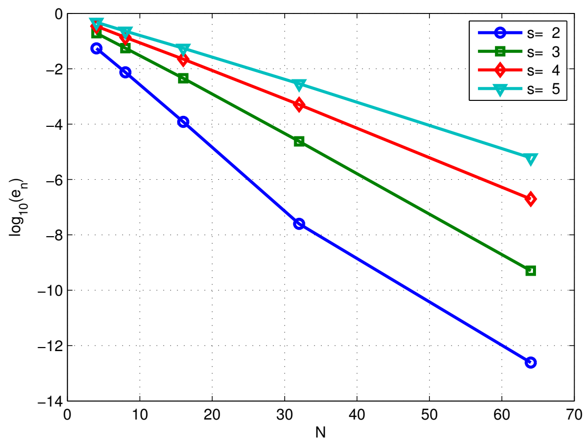


(c) $e_N = \|\dfrac{1}{1+x} - \sum_{n=0}^{N} u_n E_{s,n}(x)\|_{L^2(\Lambda)}$.

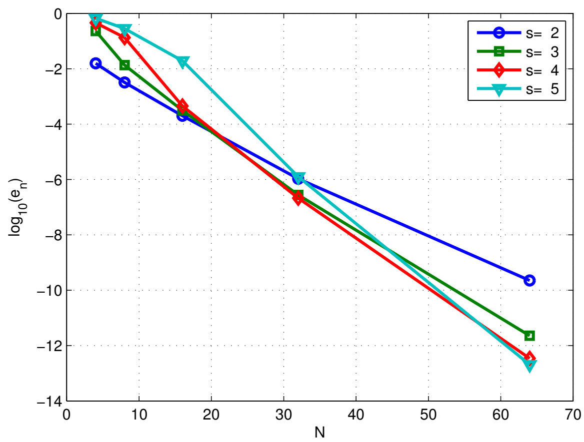


(d) $e_N = \|e^{-x} - \sum_{n=0}^{N} u_n E_{s,n}(x)\|_{L^2(\Lambda)}$.

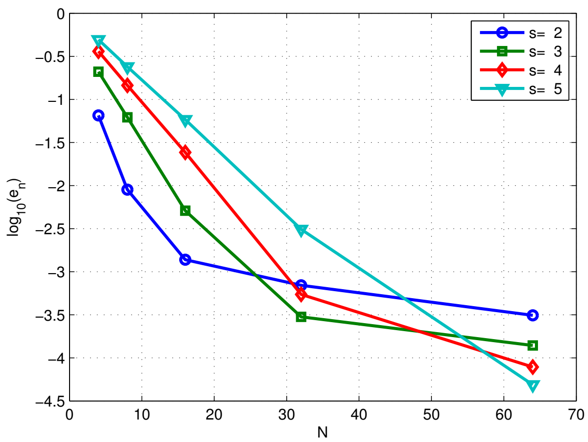
Figure 1.11: L^2-errors of modified Jacobi orthogonal approximations on the half-line:(1.11a) $u(x) = \dfrac{1}{1+x}$ using MJRFs; (1.11b) $u(x) = e^{-x}$; (1.11c) $u(x) = \dfrac{1}{1+x}$ using MJEFs; (1.11d) $u(x) = e^{-x}$ using MJEFs.



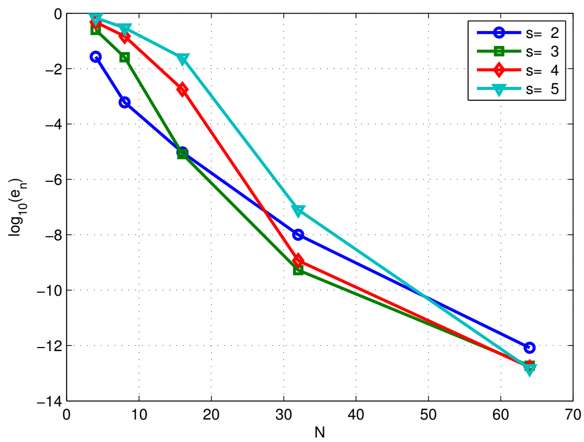
(a) $e_N = \|\dfrac{1}{1+x^2} - \sum_{n=0}^{N} u_n^{1,1} R_{s,n}^{1,1}(x)\|_{L^2(\Lambda)}$.



(b) $e_N = \|e^{-x^2} - \sum_{n=0}^{N} u_n^{1,1} R_{s,n}^{1,1}(x)\|_{L^2(\Lambda)}$.



(c) $e_N = \|\dfrac{1}{1+x^2} - \sum_{n=0}^{N} u_n^{1,1} E_{s,n}^{1,1}(x)\|_{L^2(\Lambda)}$.



(d) $e_N = \|e^{-x^2} - \sum_{n=0}^{N} u_n^{1,1} E_{s,n}^{1,1}(x)\|_{L^2(\Lambda)}$.

Figure 1.12: L^2-errors of modified Jacobi orthogonal approximations on the real-line:(1.12a) $u(x) = \dfrac{1}{1+x^2}$ using MJRFs; (1.12b) $u(x) = e^{-x^2}$; (1.12c) $u(x) = \dfrac{1}{1+x^2}$ using MJEFs; (1.12d) $u(x) = e^{-x^2}$ using MJEFs.

Chapter 2

Application of Modified Jacobi functions to differential equations on the half line

Many mathematical physics problems are given as differential equations on unbounded domains. Here, we shall consider the approximate solution to differential equations on the half-line. While there have been many existing numerous spectral algorithms that suggested significantly for solving differential equations on the half-line by many authors (see, [44, 69–71] and the references therein). On the basis of the theories of the last section of the previous chapter, we present and analyze in this chapter a numerical schema for the approximate solution of second order differential equations, using in the first section modified Jacobi rational functions for slowly decaying solutions and in the second section modified Jacobi exponential functions for rapidly decaying solutions. We also discuss spectral methods and obtain the convergence results in the weighted H^1_w and L^2 norms. To illustrate the efficiency and accuracy of our proposed methods, several selected numerical examples are presented with their approximate solutions.

2.1 Modified Jacobi rational functions

This section devoted to consider the approximation solution of differential equations on the half line using the Modified Jacobi rational functions and give its rate of convergence. To do this, we define the Strum-Liouville problem from (1.4) with eigenfunction $R^{\alpha,\beta}_{s,n}(x)$ as

$$\Big(x(x+s)^2u'(x)\Big)' + \left(-\frac{\beta^2}{4}x^{-1}(x+s)^2 + (\frac{\beta^2}{4} - \alpha^2 + 1)x\right)u(x) + \left(\lambda^{\alpha,\beta}_n + \frac{\beta^2}{2}(\beta+\alpha) + (\beta+\alpha+2)\right)u(x) = 0. \tag{2.1}$$

The Sturm-Liouville problem is very useful for the error estimation of spectral methods in the case of differential equations, so many authors have been used it in their convergence studies for instance [44, 69]. In this following, we give some basic results which are necessary for studying the convergence analyses.

Let $A_s^{\alpha,\beta}$ is linear differential operator defined as

$$A_s^{\alpha,\beta}v(x) = -\Big(x(x+s)^2u'(x)\Big)' + \left(\frac{\beta^2}{4}x^{-1}(x+s)^2 + (\alpha^2 - \frac{\beta^2}{4} - 1)x\right)u(x), \quad x \in \Lambda. \tag{2.2}$$

We can verify from (2.1) that

$$A_s^{\alpha,\beta}R_{s,n}^{\alpha,\beta}(x) = \widetilde{\lambda}_n^{\alpha,\beta}R_{s,n}^{\alpha,\beta}(x), \quad x \in \Lambda, \tag{2.3}$$

where

$$\widetilde{\lambda}_n^{\alpha,\beta} = \lambda_n^{\alpha,\beta} + \frac{\beta^2}{2}(\beta+\alpha) + (\beta+\alpha+2). \tag{2.4}$$

Moreover, from (2.3) for any $u \in L^2(\Lambda)$, we have

$$A_s^{\alpha,\beta}u(x) = \sum_{n=0}^{\infty}\widetilde{\lambda}_n^{\alpha,\beta}u_{s,n}^{\alpha,\beta}R_{s,n}^{\alpha,\beta}(x), \quad x \in \Lambda. \tag{2.5}$$

Let $a_s(u,v) := (A_s^{\alpha,\beta}u, v)$ is the bilinear operator, from (1.154) and (2.3) , we have

$$a_s(R_{s,n}^{\alpha,\beta}, R_{s,m}^{\alpha,\beta}) = (A_s^{\alpha,\beta}R_{s,n}^{\alpha,\beta}, R_{s,m}^{\alpha,\beta}) = \widetilde{\lambda}_n^{\alpha,\beta}(R_{s,n}^{\alpha,\beta}, R_{s,m}^{\alpha,\beta}) = \widetilde{\lambda}_n^{\alpha,\beta}\gamma_n^{\alpha,\beta}\delta_{n,m}, \quad \forall n,m \in \mathbb{N}. \tag{2.6}$$

By virtue of (2.5) and (2.6) for any $u, v \in L^2(\Lambda)$, we get

$$\begin{aligned}(A_s^{\alpha,\beta}u, v) = \sum_{n,j=0}^{\infty}\widetilde{\lambda}_n^{\alpha,\beta}u_{s,n}^{\alpha,\beta}v_{s,j}^{\alpha,\beta}(R_{s,n}^{\alpha,\beta}, R_{s,j}^{\alpha,\beta}) &= \sum_{n=0}^{\infty}\widetilde{\lambda}_n^{\alpha,\beta}u_{s,n}^{\alpha,\beta}v_{s,n}^{\alpha,\beta}(R_{s,n}^{\alpha,\beta}, R_{s,n}^{\alpha,\beta}) \\ &= (u, A_s^{\alpha,\beta}v).\end{aligned} \tag{2.7}$$

For any u and v in the domain of $A_s^{\alpha,\beta}$, applying integration by parts and using (2.7) leads to

$$\begin{aligned}a_s(u,v) =& (x^{1/2}(x+s)u', x^{1/2}(x+s)v') \\ &+ \frac{\beta^2}{2}(x^{-1/2}(x+s)u, x^{-1/2}(x+s)v) + (\alpha^2 - \frac{\beta^2}{4} - 1)(x^{1/2}u, x^{1/2}v).\end{aligned} \tag{2.8}$$

In particular, for $u \neq 0$ and $\alpha^2 > \frac{\beta^2}{4} + 1$, we have

$$a_s(u,u) = \|(x+s)u'\|^2_{L^2_w(\Lambda)} + \frac{\beta^2}{2}\|\frac{(x+s)}{x}u\|^2_{L^2_w(\Lambda)} + (\alpha^2 - \frac{\beta^2}{4} - 1)\|u\|^2_{L^2_w(\Lambda)} > 0. \tag{2.9}$$

Also, for all $u \in L^2(\Lambda)$, we have

$$a_s(u,u) = \sum_{n=0}^{\infty}\widetilde{\lambda}_n^{\alpha,\beta}(u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,\beta})^2. \tag{2.10}$$

From (2.7) and (2.9), the differential operator $A_s^{\alpha,\beta}$ is a self-adjoint and positive-definite operator for $\alpha^2 > \frac{\beta^2}{4} + 1$, then for every $r > 0$ the fractional power $(A_s^{\alpha,\beta})^r$ is well defined, and the associated norm for $r = 1/2$ can be characterized by [72]

$$\|(A_s^{\alpha,\beta})^{1/2}u\|^2_{L^2(\Lambda)} = a_s(u,u),$$

$$\|(A_s^{\alpha,\beta})^{1/2}u\|_{L^2(\Lambda)} = \left(\sum_{n=0}^{\infty}\widetilde{\lambda}_n^{\alpha,\beta}(u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,\beta})^2\right)^{1/2},$$

and for any non-negative integer m,

$$\|(A_s^{\alpha,\beta})^{m+1/2}u\|^2_{L^2(\Lambda)} = a_s((A_s^{\alpha,\beta})^m u, (A_s^{\alpha,\beta})^m u),$$
$$\|(A_s^{\alpha,\beta})^{m/2}u\|_{L^2(\Lambda)} = \left(\sum_{n=0}^{\infty}\left(\widetilde{\lambda}_n^{\alpha,\beta}\right)^m (u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,\beta})^2\right)^{1/2}. \tag{2.11}$$

Now, we define the following normed space for $m \in \mathbb{N}$ and $\alpha^2 > \frac{\beta^2}{4} + 1$ as

$$H^{m,s}_{\alpha,\beta}(\Lambda) = \{u \mid \|u\|_{H^{m,s}_{\alpha,\beta}(\Lambda)} < \infty\}, \quad \|u\|_{H^{m,s}_{\alpha,\beta}(\Lambda)} = \|(A_s^{\alpha,\beta})^{m/2}u\|^2_{L^2(\Lambda)}. \tag{2.12}$$

In the following Lemma, we derive the relationship of the norms $\|.\|_{L^2(\Lambda)}$ and $\|.\|_{H^{m,s}_{\alpha,\beta}(\Lambda)}$.

Lemma 15. *For all $u \in H^{m,s}_{\alpha,\beta}(\Lambda)$ and $\alpha+\beta \geq 0$, we have*

$$\|u\|_{L^2(\Lambda)} \leq 2^{-m/2}\|u\|_{H^{m,s}_{\alpha,\beta}(\Lambda)}, \tag{2.13}$$

and

$$\|\pi^{\alpha,\beta}_{s,N}u - u\|_{L^2(\Lambda)} \leq (N+1)^{-m}\|u\|_{H^{m,s}_{\alpha,\beta}(\Lambda)}. \tag{2.14}$$

Proof. We first estimate $1/\widetilde{\lambda}_n^{\alpha,\beta}$. Clearly, for all $N, n \in \mathbb{N}$ and $\alpha+\beta \geq 0$ we have that

$$\widetilde{\lambda}_n^{\alpha,\beta} = \lambda_n^{\alpha,\beta} + \frac{\beta^2}{2}(\beta+\alpha) + (\beta+\alpha+2) \geq \frac{\beta^2}{2}(\beta+\alpha) + (\beta+\alpha+2) \geq 2, \tag{2.15}$$

if $n > N$,

$$\widetilde{\lambda}_n^{\alpha,\beta} > \widetilde{\lambda}_N^{\alpha,\beta} \geq N^2 + \frac{\beta^2}{2}(\beta+\alpha) + (\beta+\alpha+2) \geq N^2. \tag{2.16}$$

This means

$$\left(\widetilde{\lambda}_n^{\alpha,\beta}\right)^{-1} \leq 2^{-1}, \text{ for all } n \in \mathbb{N}, \tag{2.17}$$

and

$$\left(\widetilde{\lambda}_n^{\alpha,\beta}\right)^{-1} \leq N^{-2}, \text{ for all } n \geq N > 0. \tag{2.18}$$

Now, using the inequality (2.17) for all $u \in L^2(\Lambda)$, we get

$$\begin{aligned}
\|u\|_{L^2(\Lambda)} = \left(\sum_{n=0}^{\infty}(u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,\beta})^2\right)^{1/2} &= \left(\sum_{n=0}^{\infty}\frac{\left(\widetilde{\lambda}_n^{\alpha,\beta}\right)^m}{\left(\widetilde{\lambda}_n^{\alpha,\beta}\right)^m}(u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,\beta})^2\right)^{1/2}\\
&\leq \max_{n\in\mathbb{N}}\{\left(\widetilde{\lambda}_n^{\alpha,\beta}\right)^{-m/2}\}\left(\sum_{n=0}^{\infty}\left(\widetilde{\lambda}_n^{\alpha,\beta}\right)^m (u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,\beta})^2\right)^{1/2}\\
&\leq 2^{-m/2}\|u\|_{H^{m,s}_{\alpha,\beta}(\Lambda)}.
\end{aligned} \tag{2.19}$$

Next using (2.18) and orthogonality of $\pi^{\alpha,\beta}_{s,N}$, for any $u \in L^2(\Lambda)$, we have

$$\begin{aligned}
\|\pi^{\alpha,\beta}_{s,N}u - u\|_{L^2(\Lambda)} &= \left(\sum_{n=N+1}^{\infty}(u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,\beta})^2\right)^{1/2}\\
&= \left(\sum_{n=N+1}^{\infty}\frac{\left(\widetilde{\lambda}_n^{\alpha,\beta}\right)^m}{\left(\widetilde{\lambda}_n^{\alpha,\beta}\right)^m}(u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,\beta})^2\right)^{1/2}\\
&\leq \max_{n\geq N+1>0}\{\left(\widetilde{\lambda}_n^{\alpha,\beta}\right)^{-m/2}\}\left(\sum_{n=N+1}^{\infty}\left(\widetilde{\lambda}_n^{\alpha,\beta}\right)^m (u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,\beta})^2\right)^{1/2}\\
&\leq (N+1)^{-m}\|u\|_{H^{m,s}_{\alpha,\beta}(\Lambda)}.
\end{aligned} \tag{2.20}$$

□

Hence from the inequality (2.13) of Lemma 15 for all $\alpha+\beta \geq 0$, the space $H^{m,s}_{\alpha,\beta}(\Lambda)$ is subspace of $L^2(\Lambda)$. This means that the $L^2(\Lambda)-$orthogonal projection $\pi^{\alpha,\beta}_{s,N}$ is also the $H^{m,s}_{\alpha,\beta}(\Lambda)-$orthogonal projection $\pi^{\alpha,\beta}_{s,N} : H^{m,s}_{\alpha,\beta}(\Lambda) \longrightarrow \mathbb{X}^{\alpha,\beta}_{s,N}$ such that

$$((A^{\alpha,\beta}_s)^{m/2}(\pi^{\alpha,\beta}_{s,N}u-u),(A^{\alpha,\beta}_s)^{m/2}v) = (\pi^{\alpha,\beta}_{s,N}u-u,(A^{\alpha,\beta}_s)^{m}v) = 0, \quad \forall v \in \mathbb{X}^{\alpha,\beta}_{s,N}. \tag{2.21}$$

Lemma 16. *For all $u \in H^{m,s}_{\alpha,\beta}(\Lambda)$, $m \geq k > 0$ and $\alpha+\beta \geq 0$, we have*

$$\|\pi^{\alpha,\beta}_{s,N}u-u\|_{H^{k,s}_{\alpha,\beta}(\Lambda)} \leq (N+1)^{k-m}\|u\|_{H^{m,s}_{\alpha,\beta}(\Lambda)}. \tag{2.22}$$

Proof. From (1.159) and (1.162), we have

$$\pi^{\alpha,\beta}_{s,N}u-u = \sum_{n=N+1}^{\infty} u^{\alpha,\beta}_{s,n}R^{\alpha,\beta}_{s,n}(x). \tag{2.23}$$

Therefore, using the inequality (2.18), we get

$$\begin{aligned}
\|\pi^{\alpha,\beta}_{s,N}u-u\|_{H^{k,s}_{\alpha,\beta}(\Lambda)} &= \left(\sum_{n=N+1}^{\infty}\left(\widetilde{\lambda}^{\alpha,\beta}_n\right)^k (u^{\alpha,\beta}_{s,n})^2(\gamma^{\alpha,\beta}_n)^2\right)^{1/2} \\
&\leq \max_{n\geq N+1>0}\{\left(\widetilde{\lambda}^{\alpha,\beta}_n\right)^{(k-m)/2}\}\left(\sum_{n=N+1}^{\infty}\left(\widetilde{\lambda}^{\alpha,\beta}_n\right)^m (u^{\alpha,\beta}_{s,n})^2(\gamma^{\alpha,\beta}_n)^2\right)^{1/2} \\
&\leq (N+1)^{k-m}\|u\|_{H^{m,s}_{\alpha,\beta}(\Lambda)}.
\end{aligned} \tag{2.24}$$

□

Application

In this subsection, we discuss the existence and convergence of the approximate solution of tow kinds of second-order differential equations.

Example 2.1. [69]

$$\begin{cases} -\partial_x^2 u(x) - \dfrac{1}{x}\partial_x u(x) + u(x) = f(x), \quad x \in \Lambda, \\ u(x) = \mathcal{O}(1), \text{ as } x \to 0, \quad \lim_{x\to\infty} xu(x) = 0. \end{cases} \tag{2.25}$$

Let $H^1_w(\Lambda)$ is weighted Sobolev space with $w(x) = x$ equipped with the norm

$$\|v\|_{H^1_w(\Lambda)} = \left(\|v\|^2_{L^2_w(\Lambda)} + \|\partial_x v\|^2_{L^2_w(\Lambda)}\right)^{1/2}. \tag{2.26}$$

For $v \in H^1_w(\Lambda)$, we have

$$\lim_{x\to 0} x\partial_x v(x) = \lim_{x\to\infty} x\partial_x v(x) = \lim_{x\to\infty} xv(x) = 0. \tag{2.27}$$

Hence, $xv(x)\partial_x u(x) \to 0$, as $x \to 0, \infty$. Then, we get the weak formulation of (2.25) as

$$B(u,v) := (\partial_x u, \partial_x v)_w + (u,v)_w = (f,v)_w, \quad v \in H^1_w(\Lambda). \tag{2.28}$$

Clearly, applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}|B(u,v)| &\le \|\partial_x u\|_{L^2_w(\Lambda)}\|\partial_x v\|_{L^2_w(\Lambda)} + \|u\|_{L^2_w(\Lambda)}\|v\|_{L^2_w(\Lambda)} \\ &\le 2\|u\|_{H^1_w(\Lambda)}\|v\|_{H^1_w(\Lambda)}.\end{aligned} \tag{2.29}$$

Moreover

$$|B(u,u)| = \|u\|^2_{H^1_w(\Lambda)}. \tag{2.30}$$

Therefore, by the Lax–Milgram lemma, for any $f \in (H^1_w(\Lambda))'$, (2.25) admits a unique solution. The spectral scheme for solving problem (2.25) is to seek $u^{\alpha,0}_{s,N} \in \mathbb{X}^{\alpha,0}_{s,N}$ such

$$B(u^{\alpha,0}_{s,N}, \phi) = (f,\phi)_w, \quad \phi \in \mathbb{X}^{\alpha,0}_{s,N}, \tag{2.31}$$

where $\mathbb{X}^{\alpha,0}_{s,N}$ associated with the set of Modified Jacobi rational functions.

We next estimate the numerical error between $u^{\alpha,0}_{s,N}$ and u of example 2.1.

Theorem 2.1. *If $u \in H^1_w(\Lambda) \cap H^{m,s}_{\alpha,0}(\Lambda)$, $s \ge 1$ and $\alpha^2 \ge 2$, we have*

$$\|u - u^{\alpha,0}_{s,N}\|_{H^1_w(\Lambda)} \le 2(N+1)^{1-m}\|u\|_{H^{m,s}_{\alpha,0}(\Lambda)}. \tag{2.32}$$

Proof. Let $U^{\alpha,0}_{s,N} = \pi^{\alpha,0}_{s,N}u$ and u the solution of (2.28), then for all $\phi \in \mathbb{X}^{\alpha,0}_{s,N}$, we have

$$(\partial_x U^{\alpha,0}_{s,N}, \partial_x\phi)_w + (U^{\alpha,0}_{s,N},\phi)_w + (\partial_x(u - U^{\alpha,0}_{s,N}), \partial_x\phi)_w + (u - U^{\alpha,0}_{s,N},\phi)_w = (f,\phi)_w. \tag{2.33}$$

Set $\eta^{\alpha,0}_{s,N} = u^{\alpha,0}_{s,N} - U^{\alpha,0}_{s,N}$. Clearly, by subtracting (2.33) from (2.31), we get

$$(\partial_x \eta^{\alpha,0}_{s,N}, \partial_x\phi)_w + (\eta^{\alpha,0}_{s,N},\phi)_w = (\partial_x(u - U^{\alpha,0}_{s,N}), \partial_x\phi)_w + (u - U^{\alpha,0}_{s,N},\phi)_w. \tag{2.34}$$

Taking $\phi = \eta^{\alpha,0}_{s,N}$ in (2.34), we obtain

$$\begin{aligned}&\|\eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} + \|\partial_x\eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} \\ &= (\partial_x(u - U^{\alpha,0}_{s,N}), \partial_x\eta^{\alpha,0}_{s,N})_w + (u - U^{\alpha,0}_{s,N}, \eta^{\alpha,0}_{s,N})_w \\ &\le \frac{1}{2}\|\eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} + \frac{1}{2}\|\partial_x\eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} + \frac{1}{2}\|(u - U^{\alpha,0}_{s,N})\|^2_{L^2_w(\Lambda)} + \frac{1}{2}\|\partial_x(u - U^{\alpha,0}_{s,N})\|^2_{L^2_w(\Lambda)}.\end{aligned}$$

Hence

$$\begin{aligned}\|\eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} + \|\partial_x\eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} &\le \|u - U^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} + \|\partial_x(u - U^{\alpha,0}_{s,N})\|^2_{L^2_w(\Lambda)} \\ &\le (\alpha^2 - 1)\|u - U^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} + \|(x+s)\partial_x(u - U^{\alpha,0}_{s,N})\|^2_{L^2_w(\Lambda)} \\ &\le \|u - U^{\alpha,0}_{s,N}\|^2_{H^{1,s}_{\alpha,0}(\Lambda)}.\end{aligned} \tag{2.35}$$

Therefore, by Lemma 16 we deduce

$$\|\eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} + \|\partial_x\eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} \le (N+1)^{2(1-m)}\|u\|^2_{H^{m,s}_{\alpha,0}(\Lambda)}. \tag{2.36}$$

Next we estimate $\|u - u^{\alpha,0}_{s,N}\|_{H^1_w(\Lambda)}$ as

$$\|u - u^{\alpha,0}_{s,N}\|_{H^1_w(\Lambda)} \le \|\eta^{\alpha,0}_{s,N}\|_{H^1_w(\Lambda)} + \|u - U^{\alpha,0}_{s,N}\|_{H^1_w(\Lambda)}. \tag{2.37}$$

Employing (2.22), (2.36) and (2.37), we have

$$\|u - u^{\alpha,0}_{s,N}\|_{H^1_w(\Lambda)} \le 2(N+1)^{1-m}\|u\|_{H^{m,s}_{\alpha,0}(\Lambda)}. \tag{2.38}$$

□

Example 2.2.

$$\begin{cases} -\partial_x \left(x\partial_x u(x)\right) + xu(x) = xf(x), & x \in \Lambda, \\ u(0) = 0, \quad \lim_{x\to\infty} xu(x) = 0. \end{cases} \tag{2.39}$$

In order to consider problem (2.39), we need to define the following space

$$H^1_{0,w}(\Lambda) = \{v \in H^1_w(\Lambda) \mid v(0) = 0\}. \tag{2.40}$$

A weak formulation of (2.39) is to find $u \in H^1_{0,w}(\Lambda)$ such that

$$B(u,v) := (\partial_x u, \partial_x v)_w + (u,v)_w = (f,v)_w, \quad v \in H^1_{0,w}(\Lambda). \tag{2.41}$$

Hence, by the Lax–Milgram lemma, the variational problem (2.41) has a unique solution for any $f \in (H^1_{0,w}(\Lambda))'$. Here, the spectral scheme for solving problem (2.25) is to seek $u^{\alpha,\beta}_{s,N} \in \mathbb{X}^{\alpha,\beta}_{s,N}$ such

$$B(u^{\alpha,\beta}_{s,N}, \phi) = (f,\phi)_w, \quad \phi \in \mathbb{X}^{\alpha,\beta}_{s,N} \text{ and } \beta > 0, \tag{2.42}$$

where $\mathbb{X}^{\alpha,\beta}_{s,N}$ associated with the set of Modified Jacobi rational functions.

The following theorem gives the convergence analyses of example 2.2.

Theorem 2.2. *If $u \in H^1_{0,w}(\Lambda) \cap H^{m,s}_{\alpha,\beta}(\Lambda)$, $s \geq 1$, $\beta > 0$ and $\alpha^2 - \frac{\beta^2}{4} \geq 2$, we have*

$$\|u - u^{\alpha,\beta}_{s,N}\|_{H^1_w(\Lambda)} \leq 2(N+1)^{1-m}\|u\|_{H^{m,s}_{\alpha,\beta}(\Lambda)}. \tag{2.43}$$

The proof of Theorem 2.2 is similar to the proof of Theorem 2.1.

Numerical Results

This section examine the efficiency and accuracy of the modified Jacobi rational spectral method for solving the differential equations with different underlining decaying solutions on the half line. Before starting the numerical tests we describe the scheme for examples 2.1 and 2.2 where

$$u^{\alpha,\beta}_{s,N}(x) = \sum_{n=0}^{N} u^{\alpha,\beta}_{s,n} R^{\alpha,\beta}_{s,n}(x).$$

Take $\phi(x) = R^{\alpha,\beta}_{s,j}(x),\ 0 \leq j \leq N$. Then we obtain

$$\sum_{n=0}^{N} u^{\alpha,\beta}_{s,n}(\partial_x R^{\alpha,\beta}_{s,n}(x), \partial_x R^{\alpha,\beta}_{s,j}(x))_w + (R^{\alpha,\beta}_{s,n}(x), R^{\alpha,\beta}_{s,j}(x))_w = (f, R^{\alpha,\beta}_{s,j}(x))_w. \tag{2.44}$$

Results of Example 2.1

We take $\beta = 0$ to consider Example 2.1 with the following two cases of the smooth solutions with algebraic decay properties:

- $u(x) = \dfrac{1}{(1+x)^h}$, which decays slowly at infinity. In Tables 2.1 and 2.2, we display the L^2- and H^1_w-errors of algorithm (2.44) for $h = 2,3$ with $\alpha = 2,4$ and $s = 2,4$. To better understand the solution behaviors, we present in Figure 2.1 the log_{10} of the L^2- and H^1_w-errors vs $log_{10}N$ with $h = 2,3$ at $s = 3$ and $\alpha = 2,3,4,5$.

Clearly, the approximate solutions converge at exponential rates for $h = 2$ with $\alpha = 2$ and for $h = 3$ with $\alpha = 2, 4$ while at algebraic rates for $h = 2$ with $\alpha = 3, 4, 5$ and for $h = 3$ with $\alpha = 3, 5$, as predicted by Theorem 2.1.

Table 2.1: A comparison of the errors with $\alpha = 2$ and different factors s for $u(x) = \dfrac{1}{(1+x)^2}$.

	$s=2$		$s=4$	
N	$\|u - u_{s,N}^{\alpha,0}\|_{L^2(\Lambda)}$	$\|u - u_{s,N}^{\alpha,0}\|_{H^1_w(\Lambda)}$	$\|u - u_{s,N}^{\alpha,0}\|_{L^2(\Lambda)}$	$\|u - u_{s,N}^{\alpha,0}\|_{H^1_w(\Lambda)}$
4	1.61e-03	7.29e-03	2.91e-02	1.24e-01
8	2.89e-06	2.98e-05	5.80e-04	2.30e-03
16	4.76e-12	7.39e-11	1.56e-07	1.47e-06
32	3.89e-15	9.53e-15	1.29e-14	1.62e-13
64	1.74e-15	8.80e-15	2.56e-15	1.45e-14

Table 2.2: A comparison of the errors with $\alpha = 4$ and different factors s for $u(x) = \dfrac{1}{(1+x)^3}$.

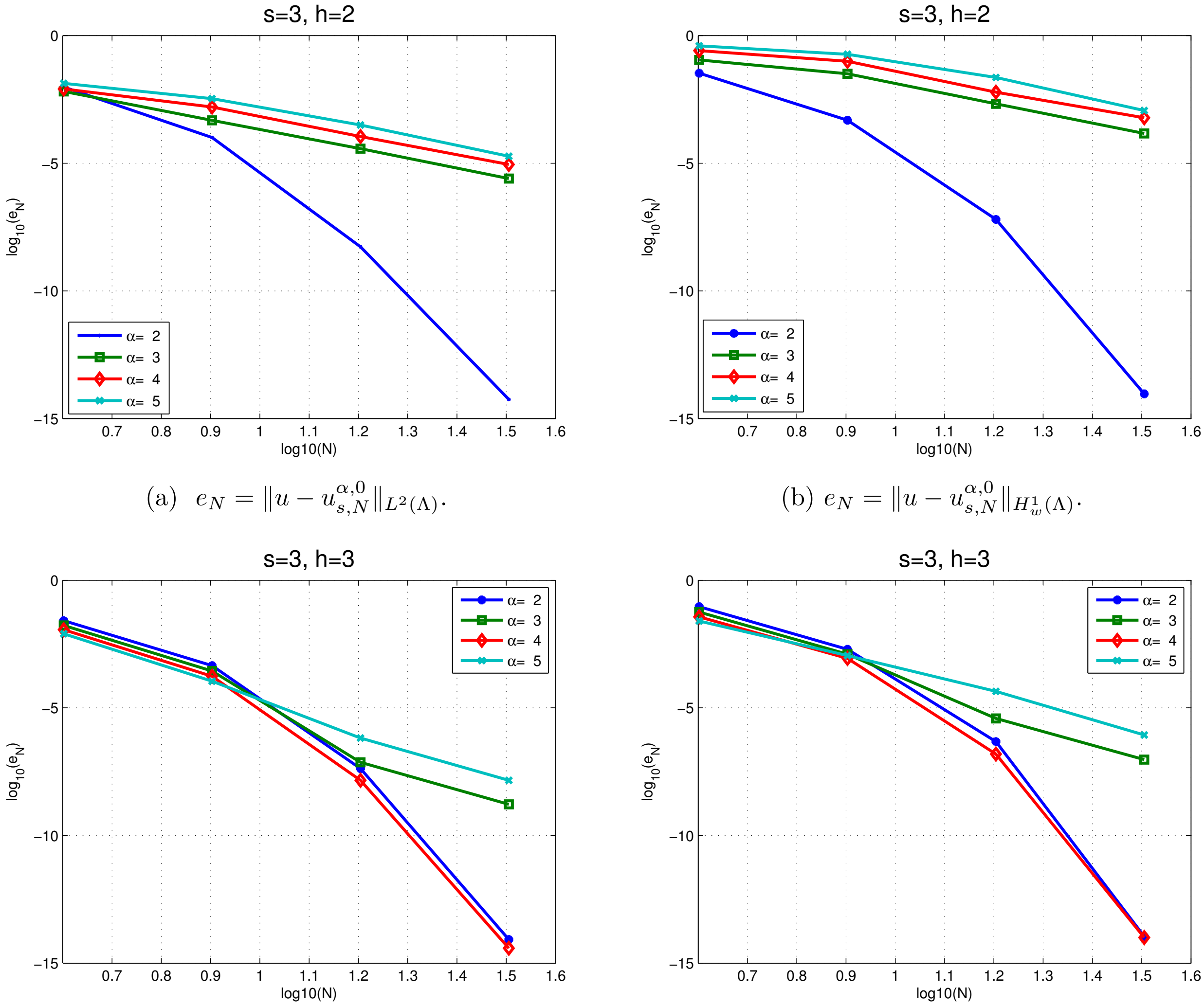
	$s=2$		$s=4$	
N	$\|u - u_{s,N}^{\alpha,0}\|_{L^2(\Lambda)}$	$\|u - u_{s,N}^{\alpha,0}\|_{H^1_w(\Lambda)}$	$\|u - u_{s,N}^{\alpha,0}\|_{L^2(\Lambda)}$	$\|u - u_{s,N}^{\alpha,0}\|_{H^1_w(\Lambda)}$
4	2.04e-03	7.69e-03	3.11e-02	1.06e-01
8	4.82e-06	4.47e-05	9.71e-04	3.92e-03
16	1.22e-11	2.32e-10	3.88e-07	3.73e-06
32	2.37e-15	5.78e-15	3.64e-14	6.74e-13
64	6.74e-15	1.84e-14	9.68e-15	3.08e-14

Results of Example 2.2

Here, we choose $\beta > 0$ to consider Example 2.2 with the following two cases of the smooth solutions with algebraic decay properties:

. $u(x) = \dfrac{x^2}{(1+x)^{4.5}}$, which decays algebraically at infinity with $u(0) = 0$. In Table 2.3, we present the values the L^2- and $H^1_{0,w}-$errors of algorithm (2.44) with $\beta = 2$, $\alpha = 3$ and $s = 2, 3$. The graphs of the log_{10} of the L^2- and $H^1_{0,w}-$errors vs $log_{10}N$ for $(s, \beta) = (4, 2)$ at $\alpha = 2, 3, 4, 5$, and $(s, \alpha) = (4, 3)$ at $\beta = 1, 2, 3, 4$ are displayed in Figure 2.2 to make it easier to show the essential role of the parameters α, and β for the convergence analysis.

The results in Table 2.2 and Figure 2.3 show that, the approximate solutions converge at exponential rates for $(\alpha, \beta) = (3, 2)$ and at algebraic rates for other values, as predicted by Theorem 2.2.



(a) $e_N = \|u - u_{s,N}^{\alpha,0}\|_{L^2(\Lambda)}$.

(b) $e_N = \|u - u_{s,N}^{\alpha,0}\|_{H_w^1(\Lambda)}$.

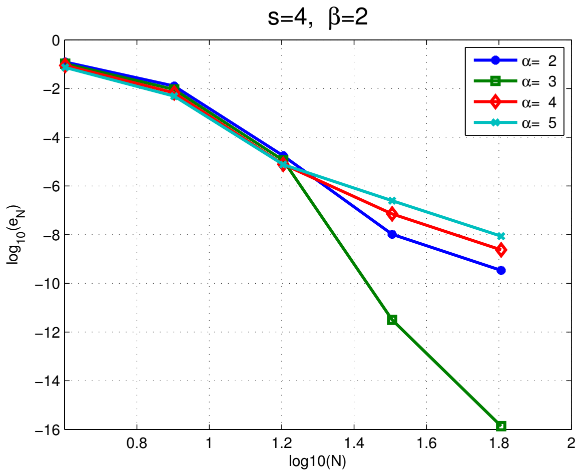
(c) $e_N = \|u - u_{s,N}^{\alpha,0}\|_{L^2(\Lambda)}$.

(d) $e_N = \|u - u_{s,N}^{\alpha,0}\|_{H_w^1(\Lambda)}$.

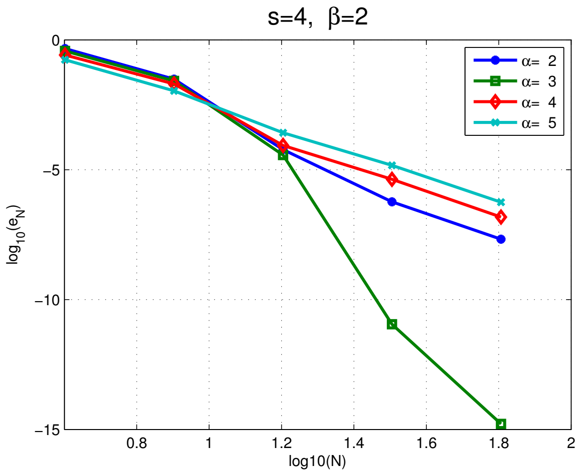
Figure 2.1: Graphs of the errors for $u(x) = \dfrac{1}{(1+x)^h}$ with $h = 2, 3$.

Table 2.3: A comparison of the errors with $(\beta, \alpha) = (2, 3)$ and different factors s for $u(x) = \dfrac{x^2}{(1+x)^{4.5}}$.

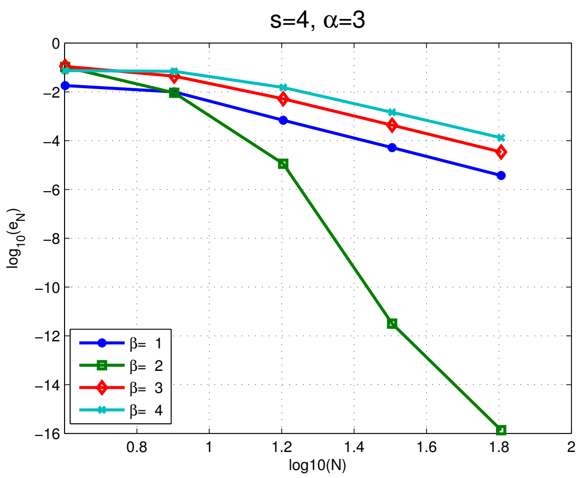
	$s=2$		$s=3$	
N	$\|u - u_{s,N}^{\alpha,\beta}\|_{L^2(\Lambda)}$	$\|u - u_{s,N}^{\alpha,\beta}\|_{H_{0,w}^1(\Lambda)}$	$\|u - u_{s,N}^{\alpha,\beta}\|_{L^2(\Lambda)}$	$\|u - u_{s,N}^{\alpha,\beta}\|_{H_{0,w}^1(\Lambda)}$
2	1.07e-01	2.21e-01	1.54e-01	4.55e-01
4	1.42e-02	4.31e-02	5.93e-02	1.91e-01
8	8.04e-05	3.72e-04	2.02e-03	7.01e-03
16	5.20e-10	2.38e-09	4.29e-07	1.66e-06
32	1.35e-15	1.83e-15	5.47e-15	1.73e-14



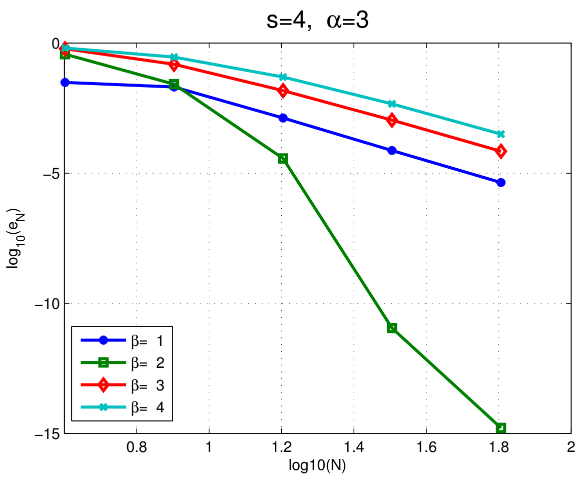
(a) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{L^2(\Lambda)}$.



(b) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{H^1_{0,w}(\Lambda)}$.



(c) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{L^2(\Lambda)}$.



(d) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{H^1_{0,w}(\Lambda)}$.

Figure 2.2: Graphs of the errors for $u(x) = \dfrac{x^2}{(1+x)^{4.5}}$.

2.2 Modified Jacobi exponential functions

Similar as the previous section to solve differential equations on the half-line and give the convergence analyses, we need to define the Sturm-Liouville problem from (1.4) associated with modified Jacobi exponential functions as

$$s^2\left((e^{x/s}-1)u'(x)\right)' + \frac{1}{4}\left((1-\alpha^2)e^{x/s} - \frac{\beta^2}{e^{x/s}-1}\right)u(x) + \left(\lambda_n^{\alpha,\beta} + \frac{1}{4}(2(\beta+1)(\alpha+1)+\alpha^2-1)\right)u(x) = 0. \tag{2.45}$$

Throughout this section, we take $\alpha, \beta \geq 0$. Therefore, the differential operator $B_s^{\alpha,\beta}$ defined as

$$B_s^{\alpha,\beta}u(x) = -s^2\left((e^{x/s}-1)u'(x)\right)' + \frac{(\alpha^2-1)}{4}e^{x/s}u(x) + \frac{\beta^2}{e^{x/s}-1}u(x), \quad x \in \Lambda. \tag{2.46}$$

From (2.45), it is readily seen that the differential operator $B_s^{\alpha,\beta}$ satisfies

$$B_s^{\alpha,\beta}E_{s,n}^{\alpha,\beta}(x) = \widehat{\lambda}_n^{\alpha,\beta}E_{s,n}^{\alpha,,\beta}(x), \quad x \in \Lambda, \tag{2.47}$$

where

$$\widehat{\lambda}_n^{\alpha,\beta} = \lambda_n^{\alpha,\beta} + \frac{\alpha+1}{4}(2\beta+\alpha+1). \tag{2.48}$$

Due to Proposition 1 and (2.47), for any u in the domain of $B_s^{\alpha,\beta}$ intersection with $L^2(\Lambda)$ we have

$$B_s^{\alpha,\beta}u(x) = \sum_{n=0}^{\infty}\widehat{\lambda}_n^{\alpha,\beta}u_{s,n}^{\alpha,\beta}E_{s,n}^{\alpha,\beta}(x), \quad x \in \Lambda. \tag{2.49}$$

We now introduce a new bilinear operator as

$$b_s(u,v) := (B_s^{\alpha,\beta}u, v). \tag{2.50}$$

By virtue of (2.47) and (1.154) , we have

$$b_s(E_{s,n}^{\alpha,\beta}, E_{s,m}^{\alpha,\beta}) = (B_s^{\alpha,\beta}E_{s,n}^{\alpha,\beta}, E_{s,m}^{\alpha,\beta}) = \widehat{\lambda}_n^{\alpha,\beta}\gamma_n^{\alpha,\beta}\delta_{n,m}, \quad n,m \in \mathbb{N}. \tag{2.51}$$

Indeed, using Proposition 1 and (2.51), for any u and v in the domain of $B_s^{\alpha,\beta}$ intersection with $L^2(\Lambda)$, we have

$$(B_s^{\alpha,\beta}u, v)_=(u, B_s^{\alpha,\beta}v), \tag{2.52}$$

applying integration by parts leads to

$$b_s(u,v) = (s^2(e^{x/s}-1)u', v') + \frac{\alpha^2-1}{4}(e^{x/s}u, v) + \beta^2((e^{x/s}-1)^{-1}u, v). \tag{2.53}$$

In particular, for $u \neq 0$ and $\alpha^2 > 1$, we have

$$b_s(u,u) = \|s(e^{x/s}-1)^{1/2}u'\|^2 + \frac{\alpha^2-1}{4}\|e^{x/2s}u\|^2 + \beta^2\|(e^{x/s}-1)^{-1/2}u\|^2 > 0. \tag{2.54}$$

Also, for all $u \in L^2(\Lambda)$, we have

$$b_s(u,u) = \sum_{n=0}^{\infty}\widehat{\lambda}_n^{\alpha,\beta}(u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,\beta})^2. \tag{2.55}$$

Since $B_s^{\alpha,\beta}$ is a self-adjoint and positive-definite operator for $\alpha^2 > 1$, then the fractional power $(B_s^{\alpha,\beta})^{1/2}$ is well defined, and the associated norm can be characterized by [72]

$$\|(B_s^{\alpha,\beta})^{1/2}u\|^2_{L^2(\Lambda)} = a_s(u,u),$$
$$\|(B_s^{\alpha,\beta})^{1/2}u\|_{L^2(\Lambda)} = \left(\sum_{n=0}^{\infty} \widehat{\lambda}_n^{\alpha,\beta}(u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,\beta})^2\right)^{1/2},$$

and for any non-negative integer m.

$$\|(B_s^{\alpha,\beta})^{m+1/2}u\|^2_{L^2(\Lambda)} = a_s((B_s^{\alpha})^m u, (B_s^{\alpha,\beta})^m u),$$
$$\|(B_s^{\alpha,\beta})^{m/2}u\|_{L^2(\Lambda)} = \left(\sum_{n=0}^{\infty} \left(\widehat{\lambda}_n^{\alpha,\beta}\right)^m (u_{s,n}^{\alpha,\beta})^2(\gamma_n^{\alpha,,\beta})^2\right)^{1/2}. \tag{2.56}$$

For any $m \in \mathbb{N}$ and $\alpha^2 > 1$, we define the normed space as follows:

$$\widehat{H}^{m,s}_{\alpha,\beta}(\Lambda) = \{u \mid \|u\|_{\widehat{H}^{m,s}_{\alpha,\beta}(\Lambda)} < \infty\}, \quad \|u\|_{\widehat{H}^{m,s}_{\alpha,\beta}(\Lambda)} = \|(B_s^{\alpha,\beta})^{m/2}u\|^2_{L^2(\Lambda)}. \tag{2.57}$$

We first prove the following preliminary result which is needed later on.

Lemma 17. *For all $u \in L^2(\Lambda)$ and $m \geq 1$, we have*

$$\|u\|_{L^2(\Lambda)} \leq 2^m \|u\|_{\widehat{H}^{m,s}_{\alpha,\beta}(\Lambda)}, \tag{2.58}$$

and

$$\|\pi_{s,N}^{\alpha,\beta}u - u\|_{L^2(\Lambda)} \leq (N+1)^{-m}\|u\|_{\widehat{H}^{m,s}_{\alpha,\beta}(\Lambda)}. \tag{2.59}$$

Proof. Similarly to the proof of Lemma 15, we estimate $1/\widehat{\lambda}_n^{\alpha,\beta}$, for all $N, n \in \mathbb{N}$ as

$$\widehat{\lambda}_n^{\alpha,\beta} = \lambda_n^{\alpha,\beta} + \frac{\alpha+1}{4}(2\beta+\alpha+1) \geq \frac{\alpha+1}{4}(2\beta+\alpha+1) \geq \frac{1}{4}, \tag{2.60}$$

if $n > N$,

$$\widehat{\lambda}_n^{\alpha,\beta} > \widehat{\lambda}_N^{\alpha,\beta} \geq N^2 + \frac{\alpha+1}{4}(2\beta+\alpha+1) \geq N^2. \tag{2.61}$$

This means

$$\left(\widehat{\lambda}_n^{\alpha,\beta}\right)^{-1} \leq 4, \text{ for all } n \in \mathbb{N}, \tag{2.62}$$

and

$$\left(\widehat{\lambda}_n^{\alpha,\beta}\right)^{-1} \leq N^{-2}, \text{ for all } n \geq N > 0. \tag{2.63}$$

And then, applying the same argument to prove (2.13) and (2.14), we get (2.58) and (2.59) respectively. □

From inequality (2.58) of Lemma 17 for all $\alpha \geq 0$, we observe that the space $\widehat{H}^{m,s}_{\alpha,\beta}(\Lambda)$ is subspace of $L^2(\Lambda)$. Then the $L^2(\Lambda)-$orthogonal projection $\pi_{s,N}^{\alpha,0}$ is also the $\widehat{H}^{m,s}_{\alpha,\beta}(\Lambda)-$orthogonal projection $\pi_{s,N}^{\alpha,\beta} : \widehat{H}^{m,s}_{\alpha,\beta}(\Lambda) \longrightarrow \mathbb{X}_{s,N}^{\alpha,\beta}$ such that

$$((B_s^{\alpha,\beta})^{m/2}(\pi_{s,N}^{\alpha,\beta}u - u), (B_s^{\alpha,\beta})^{m/2}v)_\Lambda = (\pi_{s,N}^{\alpha,\beta}u - u, (B_s^{\alpha,\beta})^m v)_\Lambda = 0, \quad \forall v \in \mathbb{X}_{s,N}^{\alpha,\beta}. \tag{2.64}$$

We are now in position to estimate the approximation error in the normed space $\widehat{H}^{m,s}_{\alpha,\beta}(\Lambda)$.

Lemma 18. *For all $u \in \widehat{H}^{m,s}_{\alpha,\beta}(\Lambda)$ and $m \geq k \geq 1$, we have*

$$\|\pi_{s,N}^{\alpha,\beta}u - u\|_{\widehat{H}^{k,s}_{\alpha,\beta}(\Lambda)} \leq (N+1)^{k-m}\|u\|_{\widehat{H}^{m,s}_{\alpha,\beta}(\Lambda)}. \tag{2.65}$$

We skip the proof as it is similar to the proof of Lemma 16.

Application

Based on the above analysis in the previous section for existence solutions of Examples 2.1 and 2.2 in the weighted Sobolev spaces $H^1_w(\Lambda)$ and $H^1_{0,w}(\Lambda)$ respectively. Here, we analyze the convergence of the spectral method based on the modified Jacobi exponential functions, where the differential equations have rapidly decaying solutions.

The spectral scheme for solving problem (2.25) is to seek $u^{\alpha,0}_{s,N} \in \mathbb{X}^{\alpha,0}_{s,N}$ such

$$B(u^{\alpha,0}_{s,N}, \phi) = (f, \phi)_w, \quad \phi \in \mathbb{X}^{\alpha,0}_{s,N}, \tag{2.66}$$

where $\mathbb{X}^{\alpha,0}_{s,N}$ associated with the set of Modified Jacobi exponential functions.

The next theorem estimates the numerical error between $u^{\alpha,0}_{s,N}$ and u of example 2.1.

Theorem 2.3. *If $u \in H^1_w(\Lambda) \cap \widehat{H}^{m,s}_{\alpha,0}(\Lambda)$, $s \geq 1$ and $\alpha^2 \geq 5$, we have*

$$\|u - u^{\alpha,0}_{s,N}\|_{H^1_w(\Lambda)} \leq 2(N+1)^{1-m}\|u\|_{\widehat{H}^{m,s}_{\alpha,0}(\Lambda)}. \tag{2.67}$$

Proof. In this proof, we use the same argument before (2.35) in the proof of Lemma 2.1 and then estimate (2.35) for $\alpha^2 \geq 5$ and $s \geq 1$ as

$$\begin{aligned}\|\eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} + \|\partial_x \eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} &\leq \|u - U^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} + \|\partial_x(u - U^{\alpha,0}_{s,N})\|^2_{L^2_w(\Lambda)} \\ . &\leq \frac{\alpha^2-1}{4}\|e^{x/s}(u - U^{\alpha,0}_{s,N})\|^2_{L^2(\Lambda)} + \|s(e^{x/s}-1)^{1/2}\partial_x(u - U^{\alpha,0}_{s,N})\|^2_{L^2(\Lambda)} \\ &\leq \|(u - U^{\alpha,0}_{s,N}\|_{\widehat{H}^{1,s}_{\alpha,0}(\Lambda)}.\end{aligned} \tag{2.68}$$

Then using the approximation inequality (2.65) in Lemma 18 with $\beta = 0$ and $s \geq 1$, we obtain

$$\|\eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} + \|\partial_x \eta^{\alpha,0}_{s,N}\|^2_{L^2_w(\Lambda)} \leq (N+1)^{2(1-m)}\|u\|^2_{\widehat{H}^{m,s}_{\alpha,0}(\Lambda)}. \tag{2.69}$$

Using the triangle inequality, we get

$$\|u - u^{\alpha,0}_{s,N}\|_{H^1_w(\Lambda)} \leq \|\eta^{\alpha,0}_{s,N}\|_{H^1_w(\Lambda)} + \|u - U^{\alpha,0}_{s,N}\|_{H^1_w(\Lambda)}. \tag{2.70}$$

Employing (2.65), (2.69) and (2.70), we have

$$\|u - u^{\alpha,0}_{s,N}\|_{H^1_w(\Lambda)} \leq 2(N+1)^{1-m}\|u\|_{\widehat{H}^{m,s}_{\alpha,0}(\Lambda)}. \tag{2.71}$$

□

The spectral scheme for solving problem (2.39) is to seek $u^{\alpha,\beta}_{s,N} \in \mathbb{X}^{\alpha,\beta}_{s,N}$ such

$$B(u^{\alpha,\beta}_{s,N}, \phi) = (f, \phi)_w, \quad \phi \in \mathbb{X}^{\alpha,\beta}_{s,N}, \tag{2.72}$$

where $\mathbb{X}^{\alpha,\beta}_{s,N}$ associated with the set of Modified Jacobi exponential functions.

Theorem 2.4. *If $u \in H^1_{0,w}(\Lambda) \cap \widehat{H}^{m,s}_{\alpha,\beta}(\Lambda)$, $\beta \geq 0$ and $\alpha^2 \geq 5$, we have*

$$\|u - u^{\alpha,\beta}_{s,N}\|_{H^1_w(\Lambda)} \leq 2(N+1)^{1-m}\|u\|_{\widehat{H}^{m,s}_{\alpha,\beta}(\Lambda)}. \tag{2.73}$$

We use the same technique in the proof of the previous theorem.

Numerical Results

In order to test the efficiency and accuracy of our spectral method, we present below some numerical results and describe the numerical implementations based on Modified Jacobi exponential function for examples 2.1 and 2.2 where

$$u_{s,N}^{\alpha,\beta}(x) = \sum_{n=0}^{N} u_{s,n}^{\alpha,\beta} E_{s,n}^{\alpha,\beta}(x).$$

Take $\phi_j(x) = E_{s,j}^{\alpha,\beta}(x),\ 0 \le j \le N$. Then we obtain

$$\sum_{n=0}^{N} u_{s,n}^{\alpha,\beta}(\partial_x E_{s,n}^{\alpha,\beta}(x), \partial_x E_{s,j}^{\alpha,\beta}(x))_w + (E_{s,n}^{\alpha,\beta}(x), E_{s,j}^{\alpha,\beta}(x))_w = (f, E_{s,j}^{\alpha,\beta}(x))_w. \tag{2.74}$$

Results of Example 2.1

In the following, we choose $\beta = 0$ to consider the Example 2.1 with the following rapidly decaying solutions at infinity and nonzero value at $x = 0$:

$$u(x) = e^{-xk}, \text{ where } k \ge 1, \quad x \in \Lambda,$$

which decays exponentially (rapidly) at infinity .

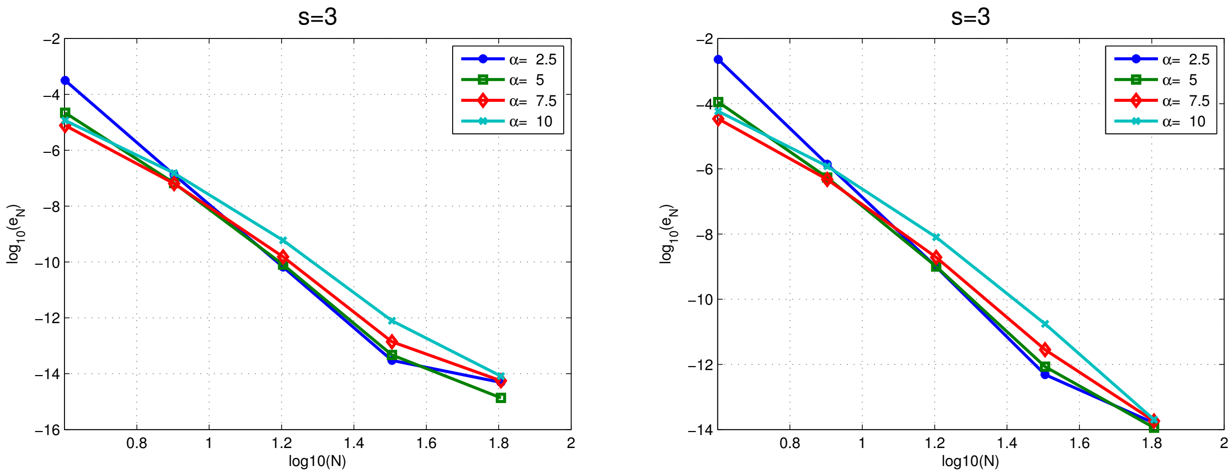
. In Tables 2.4 , we introduce of the values the L^2- and H_w^1-errors of algorithm (2.74) for $k = 1$ at $\alpha = 3$ and $s = 2, 3$. While, Figure 2.3 plots the errors at different values of N with $s = 3$ and $\alpha = 2.5, 5, 7.5, 10$: (2.3a) the L^2- and H_w^1-errors vs. $\log_{10} N$; (2.3b) the H_w^1-errors vs. $\log_{10} N$.

. In Tables 2.5, we give the values of the L^2- and H_w^1-errors of algorithm (2.74) for $k = 2$ at $\alpha = 2.5$ and $s = 3, 5$. While, Figure 2.4 plots the errors at different values of N with $s = 4$ and $\alpha = 3.4, 5, 6$: (2.4a) the L^2- and H_w^1-errors vs. $\log_{10} N$; (2.4b) the H_w^1-errors vs. $\log_{10} N$.

Clearly, The observed convergence rates are shown in Figure 2.3 and 2.4 agree with the theoretical result for the H_w^1-error, and the near straight lines indicate again an exponential convergence rate.

Table 2.4: A comparison of the errors with $\alpha = 3$ and different factors s for $u(x) = e^{-x}$.

	$s = 4$		$s = 6$	
N	$\lVert u - u_{s,N}^{\alpha,0}\rVert_{L^2(\Lambda)}$	$\lVert u - u_{s,N}^{\alpha,0}\rVert_{H_w^1(\Lambda)}$	$\lVert u - u_{s,N}^{\alpha,0}\rVert_{L^2(\Lambda)}$	$\lVert u - u_{s,N}^{\alpha,0}\rVert_{H_w^1(\Lambda)}$
4	7.12e-05	5.00e-04	5.90e-05	4.69e-04
8	1.14e-06	1.25e-05	7.08e-08	9.38e-07
16	9.48e-09	1.57e-07	5.00e-11	1.04e-09
32	5.15e-11	1.20e-09	2.13e-14	6.47e-13
64	2.16e-13	6.66e-12	1.51e-14	4.19e-14

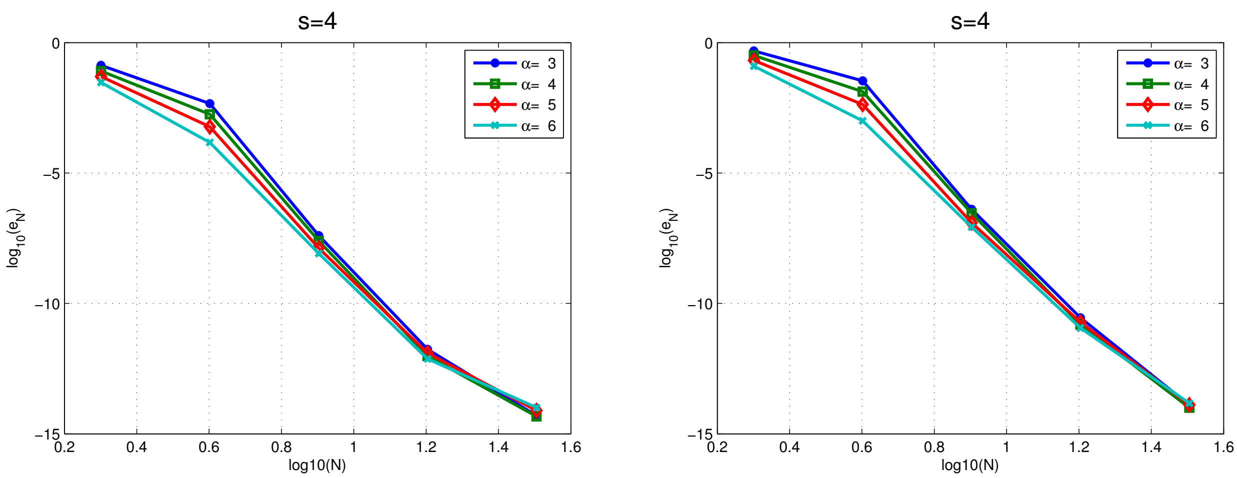


(a) $e_N = \|u - u_{s,N}^{\alpha,0}\|_{L^2(\Lambda)}$.　　(b) $e_N = \|u - u_{s,N}^{\alpha,0}\|_{H^1_w(\Lambda)}$.

Figure 2.3: Graphs of the errors for $u(x) = e^{-x}$.

Table 2.5: A comparison of the errors with $\alpha = 2.5$ and different factors s for $u(x) = e^{-2x}$.

	$s=3$		$s=5$	
N	$\|u - u_{s,N}^{\alpha,0}\|_{L^2(\Lambda)}$	$\|u - u_{s,N}^{\alpha,0}\|_{H^1_w(\Lambda)}$	$\|u - u_{s,N}^{\alpha,0}\|_{L^2(\Lambda)}$	$\|u - u_{s,N}^{\alpha,0}\|_{H^1_w(\Lambda)}$
4	3.15e-04	2.28e-03	2.51e-02	1.69e-01
8	1.33e-07	1.39e-06	8.52e-07	1.20e-05
16	6.62e-11	9.81e-10	1.99e-13	4.11e-12
32	3.02e-14	4.89e-13	2.55e-14	2.69e-14
64	4.92e-15	1.59e-14	7.46e-15	2.91e-14



(a) $e_N = \|u - u_{s,N}^{\alpha,0}\|_{L^2(\Lambda)}$.　　(b) $e_N = \|u - u_{s,N}^{\alpha,0}\|_{H^1_w(\Lambda)}$.

Figure 2.4: Graphs of the errors for $u(x) = e^{-2x}$.

Results of Example 2.2

Take $\beta > 0$ in (2.39) and consider the following case of smooth solutions with exponential asymptotic behaviors.

$$u(x) = x^2 e^{-xk}, \text{ where } k \geq 1, \quad x \in \Lambda,$$

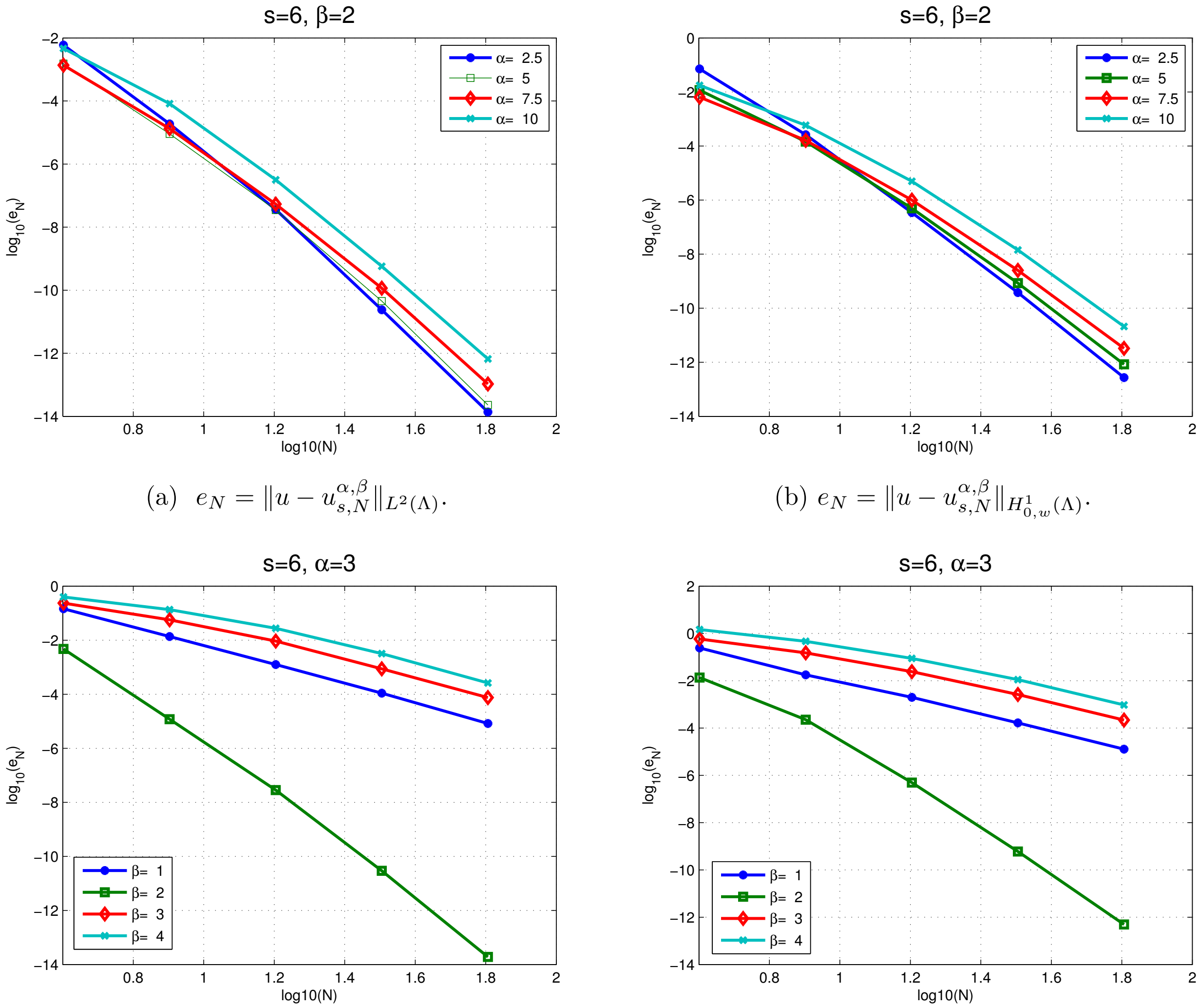
which decays exponential at infinity with $u(0) = 0$.

- In Tables 2.6, we list the values of the L^2- and $H^1_{0,w}-$errors of algorithm (2.74) for $k = 1$ at $(\beta, \alpha) = (2, 3)$ with $s = 8, 10$. Also, sub-figures (2.5a) and (2.5b) plot the errors at different values of N with $(s, \beta) = (6, 2)$ and $\alpha = 2.5, 5, 7.5, 10$. While, sub-figures (2.5c) and (2.5d) plot the errors at different values of N with $(s, \alpha) = (6, 3)$ and $\beta = 1, 2, 3, 4$: (2.5a) and (2.5c) the L^2-errors vs. $\log_{10} N$; (2.5b) and (2.5d) the $H^1_{0,w}-$errors vs. $\log_{10} N$.
- In Tables 2.7, we list the values of the L^2- and $H^1_{0,w}-$errors of algorithm (2.74) for $k = 2$ at $(\beta, \alpha) = (2, 2.5)$ with $s = 6, 8$. Also, sub-figures (2.6a) and (2.6b) plot the errors at different values of N with $(s, \beta) = (7, 2)$ and $\alpha = 3.4, 5, 6$. While, sub-figures (2.6c) and (2.6d) plot the errors at different values of N with $(s, \alpha) = (7, 2.5)$ and $\beta = 1, 2, 3, 4$: (2.6a) and (2.6c) the L^2-errors vs. $\log_{10} N$; (2.6b) and (2.6d) the $H^1_{0,w}-$errors vs. $\log_{10} N$.

From the sub-figures (2.5c), (2.5d), (2.6c) and (2.6d), we see that the numerical errors, which are obtained by using the modified Jacobi exponential functions with $\beta = 2$, are much better than other $\beta = 1, 3, 4$. Obviously, the spectral method with $\beta = 2$ gives an exponential convergence and for $\beta = 1, 3, 4$ gives an algebraic convergence.

Table 2.6: A comparison of the errors with $(\beta, \alpha) = (2, 3)$ and different factors s for $u(x) = x^2 e^{-x}$.

	$s = 8$		$s = 10$	
N	$\lVert u - u^{\alpha,\beta}_{s,N} \rVert_{L^2(\Lambda)}$	$\lVert u - u^{\alpha,\beta}_{s,N} \rVert_{H^1_{0,w}(\Lambda)}$	$\lVert u - u^{\alpha,\beta}_{s,N} \rVert_{L^2(\Lambda)}$	$\lVert u - u^{\alpha,\beta}_{s,N} \rVert_{H^1_{0,w}(\Lambda)}$
4	1.06e-01	6.32e-01	3.86e-01	9.92e-01
8	6.08e-06	7.38e-05	2.61e-05	2.62e-04
16	6.51e-10	1.80e-08	5.19e-11	1.85e-09
32	7.39e-14	1.58e-12	1.43e-14	2.87e-14



(a) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{L^2(\Lambda)}$.

(b) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{H^1_{0,w}(\Lambda)}$.

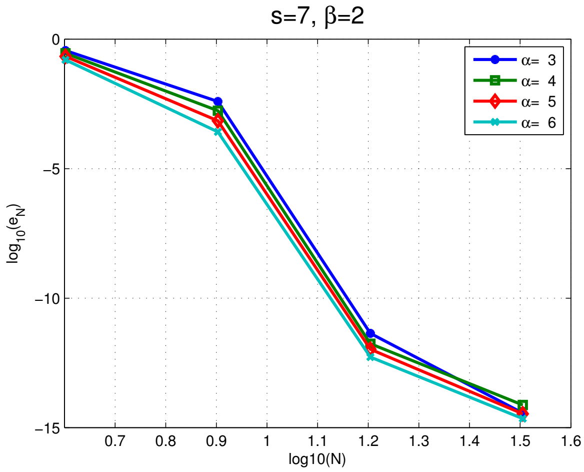
(c) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{L^2(\Lambda)}$.

(d) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{H^1_{0,w}(\Lambda)}$.

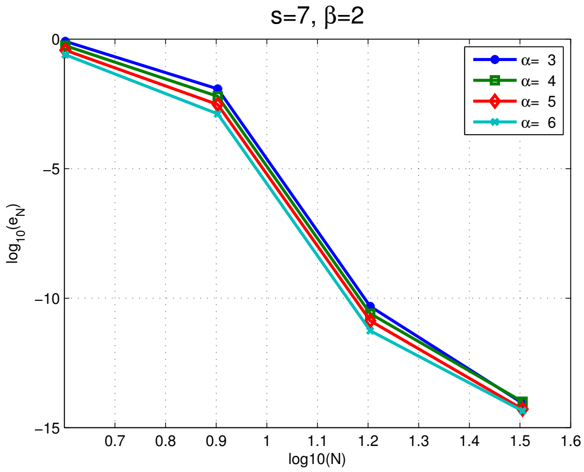
Figure 2.5: Graphs of the errors for $u(x) = x^2e^{-x}$.

Table 2.7: A comparison of the errors with $(\beta, \alpha) = (2, 2.5)$ and different factors s for $u(x) = x^2e^{-2x}$.

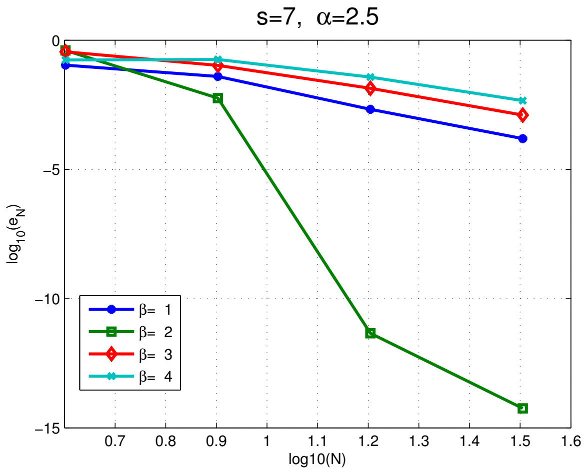
	$s = 6$		$s = 8$	
N	$\|u - u_{s,N}^{\alpha,\beta}\|_{L^2(\Lambda)}$	$\|u - u_{s,N}^{\alpha,\beta}\|_{H^1_{0,w}(\Lambda)}$	$\|u - u_{s,N}^{\alpha,\beta}\|_{L^2(\Lambda)}$	$\|u - u_{s,N}^{\alpha,\beta}\|_{H^1_{0,w}(\Lambda)}$
4	4.39e-02	7.75e-02	2.85e-01	5.72e-01
8	2.07e-06	2.72e-05	7.78e-04	3.96e-03
16	2.59e-10	1.65e-09	4.84e-12	1.10e-10
32	2.50e-14	2.46e-13	6.07e-15	7.30e-15



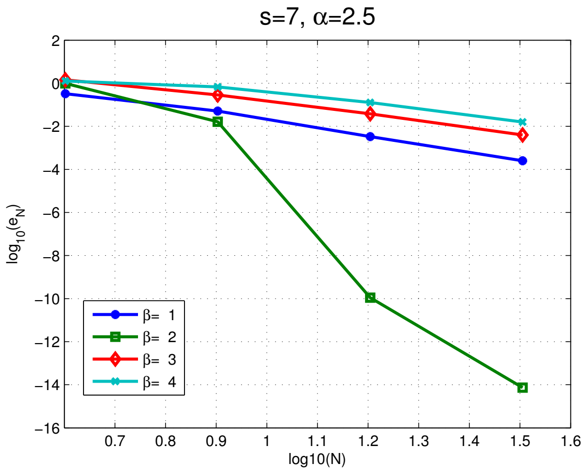
(a) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{L^2(\Lambda)}$.



(b) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{H^1_{0,w}(\Lambda)}$.



(c) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{L^2(\Lambda)}$.



(d) $e_N = \|u - u_{s,N}^{\alpha,\beta}\|_{H^1_{0,w}(\Lambda)}$.

Figure 2.6: Graphs of the errors for $u(x) = x^2 e^{-2x}$.

Chapter 3

Application of Modified Legendre functions to Hammerstein integral equations on the half line

This chapter is the subject of our research which was published in [73] where we discusses two efficient collocation methods for solving the Hammerstein integral equations on the semi-infinite domain, where the underlying solutions decay to zero at infinity. These methods are based upon modified Legendre rational and exponential functions, and reduces the Hammerstein integral equation to a nonlinear algebraic system. The error between the approximate and exact solutions in the usual L^2-norm is estimated. Finally, some numerical experiments are presented to examine and demonstrate the effectiveness and accuracy of the proposed methods in comparison to other approaches.

3.1 Introdution

Nonlinear integral equations arise in many scientific fields and engineering such as p-adic mathematical physics, theory of radiation transport, feedback control, kinetic theory of gases and chemical reactor theory (see, e.g., [74–81]). Although unarguably important, nonlinear integral equations are usually difficult to solve analytically and, as a result, one has to resort to numerical approximation of the solution. Even though many methods and algorithms have been developed and improved in the literature (see, e.g., [82–89] and the references therein), significant efforts are still needed to make further progress especially for those naturally set in unbounded domains.

In this paper, we shall consider the Hammerstein type of nonlinear integral equations on the semi-infinite interval $\Lambda = [0, +\infty)$ of the following type

$$u(x) - \int_0^{+\infty} k(x,t) f(t, u(t)) dt = g(x), \quad x \in \Lambda, \tag{3.1}$$

where f, k and g are given sufficiently smooth functions, with $f(t, v)$ is nonlinear in v, and u is the unknown function to be determined. The existence and approximation of solutions of such equations has been studied by some authors. For instance, the authors of [90, 91] have investigated the existence and uniqueness of solutions for nonlinear functional integral equations of convolution type. For that purpose, they have used Darbo's fixed point theorem associated with the Hausdorff measure of noncompactness on the space $L^p(\mathbb{R}_+)(1 \leq p < \infty)$, but no numerical method was presented in this work. Ganesh and Joshi [92] discussed the solvability of Eq. (3.1) with convolution and non-convolution kernels by using the Nyström method. Anselone and Lee [19] have treated the existence, uniqueness and finite-section approximation of solutions of nonlinear integral equations defined on the half-line. They obtained convergence results for the proposed approach under some hypotheses on the kernel. Notice that such method has been previously applied to solve linear integral equations on the half-line by some authors such as [13, 14, 93–95]. Recently, the authors [96–99] have proposed collocation and Galerkin spectral methods based on Laguerre polynomials/functions to solve linear integral equations on the half-line. These methods and their iterated versions are then discussed and extended to solve the nonlinear Hammerstein type integral equation on the half-line for both convolution and non-convolution kernels [20, 21]. Also Nahid and Nelakanti [34] have used Laguerre polynomials as classical basis functions to solve the same type of equations and obtained the convergence analysis of (multi-Galerkin/Galerkin) methods in both weighted L^2 and infinity norms. Very recently, the authors of [100] have used the Sinc-Nystrom method based on Single-Exponential and Double-Exponential transformations to solve Eq. (3.1) and obtained the convergence analysis in the infinity norm.

The main aim of this chapter is to extend the Modified Legendre spectral methods to an important class of nonlinear integral equations, namely Hammerstein equations with convolution and non-convolution kernels on the half-line. As stated in [101], these methods have the main advantage of being effective for solutions without oscillation at infinity.

The rest of the chapter is organized as follows. In section 1.4, we give some naturale assumptions on the kernel as well as the nonlinear function. In addition, discuss the existence of the unique solution of Eq. (3.1). In section 3.3, we describe the Newton method using modified Legendre functions approximation to solve Eq. (3.1). Section 3.4 discusses the convergence of the approximate solution to the exact solution in the $L^2(\mathbb{R}_+)-$space. In section 3.5 numerical examples are carried out in order to demonstrate the effectiveness and accuracy of the proposed methods.

3.2 The basics of Hammerstein integral equations and assumptions

In the following we suppose some conditions on the functions k and f to consider the nonlinear Hammerstein integral equation defined in (3.1):
For a non-convolution kernel, we suppose that

C1. $\left(\int_0^{+\infty}\int_0^{+\infty}|k(x,t)|^2dtdx\right)^{\frac{1}{2}} < \infty$.

C2. $\sup_{t\in\Lambda}\int_0^{+\infty}|k(x,t)|^2dx \leqslant M_1 < \infty$.

For a convolution kernel, we suppose that

C3. $\int_{-\infty}^{+\infty}|k(x)|dx < \infty$.

C4. $\sup_{x\in\mathbb{R}}|k(x)| \leqslant M_2 < \infty$.

From now onwards, we make the following assumptions on the nonlinear function $f(.,u(.))$:

C5. $f(t,u(t))$ is continuous on $\Lambda \times \mathbb{R}$.

C6. The partial derivative $f_u(x,u(x)) := \frac{df}{du}(x,u(x))$ of $f(.,u(.))$ exists and is continuous on $\Lambda \times \mathbb{R}$.

C7. The functions $f(x,u(x))$ and $f_u(x,u(x))$, are Lipschitz continuous in u i.e., for any $u_1, u_2 \in L^2(\Lambda)$, there exist constants M_3, M_4 such that

$$|f(x,u_1(x)) - f(x,u_2(x))| \leqslant M_3|u_1(x) - u_2(x)|,$$

and

$$|f_u(x,u_1(x)) - f_u(x,u_2(x))| \leqslant M_4|u_1(x) - u_2(x)|.$$

Next, we define the operator $\mathcal{T}$ on $L^2(\Lambda)$ by

$$\mathcal{T}(u) := \mathcal{K}(u) + g,$$

where $\mathcal{K}(u)(x) = \int_0^{+\infty} k(x,t)f(t,u(t))dt$, then can be written as

$$\mathcal{T}(u) = u. \tag{3.2}$$

The following theorem gives the conditions for the existence and uniqueness of the solution for Eq. (3.1) with convolution and non-convolution kernels in $L^2(\Lambda)$

Theorem 3.1. *If the following conditions are hold*

- $M_3 \left(\int_0^{+\infty} \int_0^{+\infty} |k(x,t)|^2 dtdx\right)^{\frac{1}{2}} < 1$, *for a non-convolution kernel.*
- $M_3 \int_{-\infty}^{+\infty} |k(x)|dx < 1$, *for a convolution kernel.*

Then the operator equation $\mathcal{T}(u) = u$ has a unique solution $u_0 \in L^2(\Lambda)$, i.e., we have $\mathcal{T}(u_0) = u_0$.

Proof. For all $u_1, u_2 \in L^2(\Lambda)$ with non-convolution kernel. From Lipschitz's continuity of f, we can write

$$\begin{aligned} |\mathcal{T}(u_1)(x) - \mathcal{T}(u_2)(x)| &= \left|\int_0^{+\infty} k(x,t)(f(t,u_1(t)) - f(t,u_2(t)))dt\right| \qquad (3.3) \\ &\leqslant M_3 \int_0^{+\infty} |k(x,t)||u_1(t) - u_2(t)|dt. \end{aligned}$$

By applying Cauchy-Schwarz inequality, we get

$$\|\mathcal{T}(u_1) - \mathcal{T}(u_2)\|_{L^2(\Lambda)} \leqslant M_3 \left(\int_0^{+\infty} \int_0^{+\infty} |k(x,t)|^2 dtdx\right)^{\frac{1}{2}} \|u_1 - u_2\|_{L^2(\Lambda)}. \tag{3.4}$$

For all $u_1, u_2 \in L^2(\Lambda)$ with convolution kernel, we have

$$\begin{aligned} |\mathcal{T}(u_1)(x) - \mathcal{T}(u_2)(x)| &= \left|\int_0^{+\infty} k(x-t)(f(t,u_1(t)) - f(t,u_2(t)))dt\right| \\ &\leqslant M_3 \int_0^{+\infty} |k(x-t)||u_1(t) - u_2(t)|dt. \qquad (3.5) \end{aligned}$$

Let us define

$$\tilde{k}(x-t) = \begin{cases} |k(x-t)|, & x \geqslant 0, t \geqslant 0, \\ 0 & \text{otherwise}, \end{cases} \tag{3.6}$$

and

$$\tilde{u}(t) = \begin{cases} |u_1(t) - u_2(t)|, & t \geqslant 0, \\ 0, & t < 0, \end{cases} \tag{3.7}$$

so that we can write

$$(\tilde{k} \star \tilde{u})(x) := \int_{-\infty}^{+\infty} \tilde{k}(x-t)\tilde{u}(t)dt = \int_0^{+\infty} |k(x-t)||u_1(t) - u_2(t)|dt.$$

Obviously, we have $\|\tilde{k}\|_{L^1(\mathbb{R})} = \|k\|_{L^1(\mathbb{R})}$ and $\|\tilde{u}\|_{L^2(\mathbb{R})} = \|u_1 - u_2\|_{L^2(\Lambda)}$. Hence, by using Young's theorem 4.15 [102, p.104], we obtain

$$\|\tilde{k} \star \tilde{u}\|_{L^2(\mathbb{R})} \leqslant \|\tilde{k}\|_{L^1(\mathbb{R})}\|\tilde{u}\|_{L^2(\mathbb{R})}. \tag{3.8}$$

This implies

$$\|\mathcal{T}(u_1) - \mathcal{T}(u_2)\|_{L^2(\Lambda)} \leqslant M_3\|k\|_{L^1(\mathbb{R})}\|u_1 - u_2\|_{L^2(\Lambda)}. \tag{3.9}$$

Finally, from the obtained results in equations (3.9) and (3.4), the operator $\mathcal{T}$ is a contraction for (convolution/non-convolution) kernel from $L^2(\Lambda)$ into $L^2(\Lambda)$. Then $\mathcal{T}$ has a unique solution $\mathcal{T}(u_0) = u_0$, where $u_0 \in L^2(\Lambda)$. □

3.3 Modified Legendre functions collocation method

In this section, we describe the Newton method used modified Legendre functions for solving the nonlinear integral defined Eq. (3.1), where the solution is assumed to belong to $L^2(\Lambda)$. According to Lemma 13, the function u can be expanded by a finite series of modified Legendre functions as follows:

$$u_{s,N}(x) = \sum_{n=0}^{N} u_{s,n}L_{s,n}(x) = \mathbf{r}(x)^{\mathrm{T}}\mathbf{u}, \tag{3.10}$$

where $\mathbf{u}$ and $\mathbf{L}$ are two vectors given by:

$$\mathbf{u} = (u_{s,0}, \ldots, u_{s,N})^{\mathrm{T}}, \quad \mathbf{L}(x) = (L_{s,0}(x), \ldots, L_{s,N}(x))^{\mathrm{T}}. \tag{3.11}$$

Let us introduce the residual function

$$\begin{aligned} \mathrm{R}(x) &= u_{s,N}(x) - \int_0^{+\infty} k(x,t)f(t, u_{s,N}(t))dt - g(x) \qquad (3.12) \\ &= \sum_{n=0}^{N} u_{s,n}L_{s,n}(x) - \int_0^{+\infty} k(x,t)f(t, \sum_{n=0}^{N} u_{s,n}L_{s,n}(t))dt - g(x), \quad x \in \Lambda, \end{aligned}$$

where the residual function satisfies that $\mathrm{R}(\zeta_{s,j}) = 0$ for all $0 \leq j \leq N$, then to find the unknown coefficients $u_{s,n}$ of the approximate solution $u_{s,N}$ satisfies that $u_{s,N} = \pi_{s,N}\mathcal{K}u_{s,N} + \pi_{s,N}g$, it is equivalently to solve

$$\sum_{n=0}^{N} u_{s,n}L_{s,n}(\zeta_{s,j}) + \int_0^{+\infty} k(\zeta_{s,j}, t)f(t, \sum_{n=0}^{N} u_{s,n}L_{s,n}(t))dt = g(\zeta_{s,j}), \quad 0 \leq j \leq N, \tag{3.13}$$

or equivalently,

$$\mathbf{L}(\zeta_{s,j})^{\mathrm{T}}\mathbf{u} + \int_0^{+\infty} k(\zeta_{s,j}, t) f(t, \mathbf{L}(t)^{\mathrm{T}}\mathbf{u}) dt = g(\zeta_{s,j}), \quad 0 \le j \le N. \tag{3.14}$$

The integral term in the above equation can be evaluated so approximately using modified Legendre-Gauss quadrature rule relative to the weights $\{\rho_{s,i}\}_{i=0}^{N}$ as follows:

$$\int_0^{+\infty} k(\zeta_{s,j}, t) f(t, \mathbf{L}(t)^{\mathrm{T}}\mathbf{u}) dt \approx \sum_{i=0}^{N} k(\zeta_{s,j}, \zeta_{s,i}) f(\zeta_{s,i}, \mathbf{r}(\zeta_{s,i})^{\mathrm{T}}\mathbf{u}) \rho_{s,i}. \tag{3.15}$$

Let us denote

$$\begin{aligned} &\mathrm{F}(\mathbf{u}) = \mathrm{diag}(f(\zeta_{s,i}, \mathbf{L}(\zeta_{s,i})^{\mathrm{T}}\mathbf{u})), \mathrm{G} = (g(\zeta_{s,0}), \ldots, g(\zeta_{s,N}))^{\mathrm{T}}, \mathrm{D} = \mathrm{diag}(\rho_{s,i}), \\ &P_{s,j} = L_{s,n}(\zeta_{s,j}), \mathrm{P} = (P_{s,j}), K^s_{j,i} = k(\zeta_{s,j}, \zeta_{s,i}), \mathrm{M} = (K^s_{j,i}). \end{aligned} \tag{3.16}$$

Then, (3.14) leads to the following algebraic system of nonlinear equations

$$\mathrm{P}\mathbf{u} - \mathrm{MDF}(\mathbf{u}) = \mathrm{G}. \tag{3.17}$$

Next to solve the nonlinear system (3.17), we define a new function Q as

$$\mathrm{Q}(\mathbf{u}) = \mathrm{P}\mathbf{u} - \mathrm{MDF}(\mathbf{u}) - \mathrm{G} = 0, \tag{3.18}$$

and then applying the Newton's method for solving the nonlinear function (3.18) to obtain the value of $\mathbf{u}$ as

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} - [J_{\mathrm{Q}}(\mathbf{u}^{(k)})]^{-1}\mathrm{Q}(\mathbf{u}^{(k)}), \tag{3.19}$$

where $J_{\mathrm{Q}}(\mathbf{u})$ is the Jacobian matrix of $\mathrm{Q}(\mathbf{u})$, defined by

$$[J_{\mathrm{Q}}(\mathbf{u})]_{ij} = \frac{\partial Q_i(\mathbf{u})}{\partial u^s_{N,j}}, \tag{3.20}$$

with the initial value (initial guess) $\mathbf{u}^{(0)} = g((\zeta_{\mathbf{s},\mathbf{N}}))$ where $(\zeta_{\mathbf{s},\mathbf{N}}) = (\zeta_{s,0}, \ldots, \zeta_{s,N})$.

3.4 Convergence theorems

In this section, we discuss the convergence analysis of the collocation solution of the nonlinear integral equation of the type (3.1) for (convolution/non-convolution) kernel, for to do that, we need to define the Frechet derivative, which is a linear integral operator from $L^2(\Lambda)$ into itself defined by

$$(\mathcal{T}'(u_0)v)(x) = (\mathcal{K}'(u_0)v)(x) = \int_0^{+\infty} k(x,t) f_u(t, u_0(t)) v(t) dt, \quad v \in L^2(\Lambda). \tag{3.21}$$

In the following Lemma, we prove $(I - \mathcal{K}'(u_0))$ is an invertible operator and its inverse is uniformly bounded, which is convenient for the convergence analysis.

Lemma 19. *Let u_0 is the solution of Eq. (3.2), if the following condition is hold*

$$f_u(t, 0) = 0, \quad \textit{for all } t \in \Lambda, \tag{3.22}$$

then $(I - \mathcal{K}'(u_0))^{-1} : L^2(\Lambda) \to L^2(\Lambda)$ exists and uniformly bounded.

Proof. We first prove that $f_u(t,u_0(t)) \in L^2(\Lambda)$. By using the condition (3.22) and Lipschitz's continuity of f_u, we get

$$\|f_u\|_{L^2(\Lambda)} \leqslant M_4 \|u_0\|_{L^2(\Lambda)} < \infty. \tag{3.23}$$

For non-convolution kernel, we shall prove that $k(x,t)f_u(t,u_0(t)) \in L^2(\Lambda\times\Lambda)$, for to do that let us denote

$$K'(x,t) = k(x,t)f_u(t,u_0(t)), \tag{3.24}$$

where $k(x,t)f_u(t,u_0(t))$ is the Frechet derivative kernel, for $t\in\Lambda$ we have

$$\begin{aligned}\int_0^{+\infty} |K'(x,t)|^2 dx &= \int_0^{+\infty} |k(x,t)f_u(t,u_0(t))|^2 dx\\ &= |f_u(t,u_0(t))|^2 \int_0^{+\infty} |k(x,t)|^2 dx\\ &\leqslant |f_u(t,u_0(t))|^2 \sup_{t\in\Lambda} \int_0^{+\infty} |k(x,t)|^2 dx, \end{aligned}\tag{3.25}$$

and, moreover,

$$\int_0^{+\infty}\int_0^{+\infty} |K'(x,t)|^2 dxdt \leqslant \|f_u\|^2_{L^2(\Lambda)} \sup_{t\in\Lambda} \int_0^{+\infty} |k(x,t)|^2 dx < \infty. \tag{3.26}$$

Applying Fubini's theorem to the left part of the inequality (3.26), we get

$$\left(\int_0^{+\infty}\int_0^{+\infty} |K'(x,t)|^2 dtdx\right)^{\frac{1}{2}} = \left(\int_0^{+\infty}\int_0^{+\infty} |K'(x,t)|^2 dxdt\right)^{\frac{1}{2}} < \infty. \tag{3.27}$$

The Frechet derivative kernel with convolution kernel

$$\int_0^{+\infty} |k(x-t)f_u(t,u_0(t))|^2 dt \leqslant M_2 \int_0^{+\infty} |k(x-t)||f_u(t,u_0(t))|^2 dt. \tag{3.28}$$

We denote

$$\tilde{k}\star\tilde{f}_u(x) = \int_{-\infty}^{+\infty} \tilde{k}(x-t)\tilde{f}_u(t,u_0(t))dt, \tag{3.29}$$

where $\tilde{k}$ defined in (3.6) and

$$\tilde{f}_u(t,u_0) = \begin{cases} |f_u(t,u_0(t))|^2 & t \geqslant 0,\\ 0 & t<0.\end{cases} \tag{3.30}$$

Obviously, we can verify easily that $\tilde{k},\tilde{f}_u \in L^1(\mathbb{R})$. Thus by applying Young's Theorem 4.15 of [102, p.104], we obtain

$$\|\tilde{k}\star\tilde{f}_u\|_{L^1(\mathbb{R})} \leqslant \|\tilde{k}\|_{L^1(\mathbb{R})}\|\tilde{f}_u\|_{L^2(\mathbb{R})}, \tag{3.31}$$

then, from Eqs.(3.28) and (3.31), we have

$$\begin{aligned}\left(\int_0^{+\infty}\int_0^{+\infty} |k(x-t)f_u(t,u_0(t))|^2 dtdx\right)^{\frac{1}{2}} &\leqslant M_2^{\frac{1}{2}} \|\tilde{k}\star\tilde{f}_u\|^{\frac{1}{2}}_{L^1(\mathbb{R})}\\ &\leqslant M_2^{\frac{1}{2}}\|k\|^{\frac{1}{2}}_{L^1(\mathbb{R})}\|f_u\|_{L^2(\Lambda)} < \infty. \end{aligned}\tag{3.32}$$

Finally, from the obtained results in equations (3.27) and (3.32), by using Theorem 1.3 the Frechet derivative $\mathcal{T}'(u_0)$ is a compact operator for (convolution/non-convolution) kernels from $L^2(\Lambda)$ into itself, and then by using Theorem 1.4, the operator $(I-\mathcal{K}'(u_0))^{-1}$ exists and uniformly bounded. □

Now we prove the following Lemma which useful to our convergence results.

Lemma 20. *For any $u, v \in L^2(\Lambda)$, the following hold:*
For a non-convolution kernel,

$$\|(\mathcal{K}'(u_0) - \mathcal{K}'(u))v\|_{L^2(\Lambda)} \leqslant M_4 M_1^{\frac{1}{2}} \|u_0 - u\|_{L^2(\Lambda)} \|v\|_{L^2(\Lambda)}. \tag{3.33}$$

For a convolution kernel,

$$\|(\mathcal{K}'(u_0) - \mathcal{K}'(u))v\|_{L^2(\Lambda)} \leqslant M_2^{\frac{1}{2}} M_4 \|k\|_{L^1(\mathbb{R})} \|u - u_0\|_{L^2(\Lambda)} \|v\|_{L^2(\Lambda)}. \tag{3.34}$$

Proof. Using Lipschitz's continuity of f_u and Cauchy–Schwarz inequality, we have for a non-convolution kernel

$$\begin{aligned}
\|(\mathcal{K}'(u_0) - \mathcal{K}'(u))v\|^2_{L^2(\Lambda)} &= \int_0^{+\infty} |(\mathcal{K}'(u_0) - \mathcal{K}'(u))v(x)|^2 dx \\
&\leqslant \int_0^{+\infty} \left(\int_0^{+\infty} |k(x,t)(f_u(t, u_0(t)) - f_u(t, u(t)))||v(t)| dt \right)^2 dx \\
&\leqslant M_4^2 \int_0^{+\infty} \int_0^{+\infty} (|k(x,t)||v(t)|)^2 dt dx \|u_0 - u\|^2_{L^2(\Lambda)}.
\end{aligned} \tag{3.35}$$

Now, by applying the same way to prove Lemma 19 for a non-convolution kernel, we get

$$\begin{aligned}
\int_0^{+\infty} \int_0^{+\infty} (|k(x,t)||v(t)|)^2 dt dx &= \int_0^{+\infty} \int_0^{+\infty} (|k(x,t)||v(t)|)^2 dx dt \\
&\leqslant \sup_{t \in \Lambda} \int_0^{+\infty} |k(x,t)|^2 dx \|v\|^2_{L^2(\Lambda)}.
\end{aligned} \tag{3.36}$$

This implies

$$\|(\mathcal{K}'(u_0) - \mathcal{K}'(u))v\|^2_{L^2(\Lambda)} \leqslant M_4^2 \sup_{t \in \Lambda} \int_0^{+\infty} |k(x,t)|^2 dx \|u_0 - u\|^2_{L^2(\Lambda)} \|v\|^2_{L^2(\Lambda)}. \tag{3.37}$$

For convolution kernel, we have

$$\begin{aligned}
\|(\mathcal{K}'(u_0) - \mathcal{K}'(u))v\|^2_{L^2(\Lambda)} &= \int_0^{+\infty} |(\mathcal{K}'(u_0) - \mathcal{K}'(u))v(x)|^2 dx \\
&\leqslant \int_0^{+\infty} \left(\int_0^{+\infty} |k(x-t)(f_u(t, u_0(t)) - f_u(t, u(t)))||v(t)| dt \right)^2 dx \\
&\leqslant M_2 M_4^2 \int_0^{+\infty} \int_0^{+\infty} |k(x-t)||v(t)|^2 dt dx \|u_0 - u\|^2_{L^2(\Lambda)}.
\end{aligned} \tag{3.38}$$

For estimate $\int_0^{+\infty} \int_0^{+\infty} |k(x-t)||v(t)|^2 dt dx$. If applying the same argument to prove Lemma 19 for a convolution kernel, we get

$$\int_0^{+\infty} \int_0^{+\infty} |k(x-t)||v(t)|^2 dt dx \quad \leq \quad \|k\|_{L^1(\mathbb{R})} \|v\|^2_{L^2(\Lambda)}. \tag{3.39}$$

This implies

$$\|(\mathcal{K}'(u_0) - \mathcal{K}'(u))v\|^2_{L^2(\Lambda)} \leqslant M_2 M_4^2 \|k\|_{L^1(\mathbb{R})} \|u - u_0\|^2_{L^2(\Lambda)} \|v\|^2_{L^2(\Lambda)}. \tag{3.40}$$

□

We now quote the following theorem from Vainikko of [103] which gives us the conditions under which the solvability of one equation leads to the solvability of other equations.

Theorem 3.2. *Let $\widetilde{T}$ and $\widehat{T}$ be two continuous operators over an open set Ω in a Hilbert space $\mathbb{X}$. Let the equation $u = \widehat{T}u$ possesses an isolated solution $\widehat{u}_0 \in \Omega$ and let the following conditions hold.*

- *The operator $\widetilde{T}$ is Frechet differentiable in some neighborhood of the point $\widehat{u}_0$, whereas the linear operator $(I - \widetilde{T}'(\widehat{u}_0))$ is continuously invertible.*
- *For some δ and $0 < q < 1$ the following inequalities are valid (the number δ is assumed to be so small that the sphere $\|u - \widehat{u}_0\| \leq \delta$ is contained within Ω)*

$$\sup_{\|u-\widehat{u}_0\|\leqslant\delta} \|(I - \widetilde{T}'(\widehat{u}_0)^{-1})(\widetilde{T}'(\widehat{u}) - \widetilde{T}'(\widehat{u}_0))\| \leqslant q, \tag{3.41}$$

$$\rho = \|(I - \widetilde{T}'(\widehat{u}_0)^{-1})(\widetilde{T}(\widehat{u}_0) - \widehat{T}(\widehat{u}_0))\| \leqslant \delta(1-q). \tag{3.42}$$

Then the equation $u = \widetilde{T}u$ has a unique solution $\widetilde{u}_0$ in the sphere $\|u - \widehat{u}_0\| \leqslant \delta$ and

$$\frac{\rho}{(1+q)} \leqslant \|\widetilde{u}_0 - \widehat{u}_0\| \leqslant \frac{\rho}{(1-q)}. \tag{3.43}$$

Next we discuss the existence and convergence rates of the approximate solution u_N^s to u_0, where $\pi_{s,N}\mathrm{R}(x) = 0$ or equivalently

$$u_{s,N} = \pi_{s,N}\mathcal{K}u_N^s + \pi_{s,N}g. \tag{3.44}$$

In order to facilitate the study of the existence and convergence of $u_{s,N}$, we define the iterated solution as

$$\widetilde{u}_{s,N} = \mathcal{K}u_{s,N} + g. \tag{3.45}$$

Applying $\pi_{s,N}$ on both sides of Eq. (3.45), we have

$$\pi_{s,N}\widetilde{u}_{s,N} = \pi_{s,N}\mathcal{K}u_{s,N} + \pi_{s,N}g. \tag{3.46}$$

From Eqs. (3.46) and (3.44), it follows that $\pi_{s,N}\widetilde{u}_{s,N} = u_{s,N}$, then Eq. (3.45) becomes

$$\widetilde{u}_{s,N} = \mathcal{K}\pi_{s,N}\widetilde{u}_{s,N} + g. \tag{3.47}$$

Next, define the operators $\mathcal{T}_N^s$, $\widetilde{\mathcal{T}}_N^s : L^2(\Lambda) \to L^2(\Lambda)$ as follows:

$$\mathcal{T}_N^s u := \mathcal{K}u + \pi_{s,N}g, \tag{3.48}$$

$$\widetilde{\mathcal{T}}_{s,N}u := \mathcal{K}\pi_{s,N}u + g, \tag{3.49}$$

the above equations satisfies that $\mathcal{T}_{s,N}u_{s,N} = u_{s,N}$, $\widetilde{\mathcal{T}}_{s,N}^s\widetilde{u}_{s,N} = \widetilde{u}_{s,N}$ and for all $v \in L^2(\Lambda)$ the Frechet derivatives of the operator $\widetilde{\mathcal{T}}_N^s$ at u_0 defined as

$$\widetilde{\mathcal{T}}_N^{s\prime}(u_0)v = \mathcal{K}'(\pi_{s,N}u_0)v. \tag{3.50}$$

In following theorem, we prove that the existence and convergence of iterated solution $\widetilde{u}_{s,N}$ to u_0 in $L^2(\Lambda)$ space. We will use the Theorem 2 in [103].

Theorem 3.3. *Let $u_0 \in L^2(\Lambda)$ be an isolated solution of Eq. (3.2) and $\mathcal{T}'(u_0)$ is the Frechet derivative of $\mathcal{T}$ at u_0. Let the orthogonal projection $\pi_{s,N} : L^2(\Lambda) \longrightarrow \mathbb{X}_{s,N}$ be given by (1.175). Then Eq. (3.49) has a unique solution $\widetilde{u}_{s,N} \in \mathcal{B}(u_0, \delta) := \{u \mid \|u - u_0\|_{L^2(\Lambda)} \leq \delta\}$ for some $\delta > 0$ and for sufficiently large N. Moreover, there exists a constant $0 < q < 1$, independent of N such that*

$$\frac{\alpha_N^s}{(1+q)} \leqslant \|\widetilde{u}_{s,N} - u_0\|_{L^2(\Lambda)} \leqslant \frac{\alpha_N^s}{(1-q)}, \tag{3.51}$$

where $\alpha_N^s = \|(I - \widetilde{\mathcal{T}}_N^{s\prime}(u_0))^{-1}(\widetilde{\mathcal{T}}_N^s(u_0) - \mathcal{T}(u_0))\|_{L^2(\Lambda)}$.

Proof. For all $v \in L^2(\Lambda)$ and u_0 is the solution of Eq. (3.2), we have

$$\|(\mathcal{T}'(u_0) - \widetilde{\mathcal{T}}_N^{s\prime}(u_0))v\|_{L^2(\Lambda)} = \|(\mathcal{K}'(u_0) - \mathcal{K}'(\pi_{s,N}u_0))v\|_{L^2(\Lambda)}. \tag{3.52}$$

From Lemma 20, we get for a non-convolution kernel,

$$\|(\mathcal{T}'(u_0) - \widetilde{\mathcal{T}}_N^{s\prime}(u_0))v\|_{L^2(\Lambda)} \leqslant M_4 M_1^{\frac{1}{2}} \|u_0 - \pi_{s,N}u_0\|_{L^2(\Lambda)} \|v\|_{L^2(\Lambda)}. \tag{3.53}$$

For a convolution kernel,

$$\|(\mathcal{T}'(u_0) - \widetilde{\mathcal{T}}_N^{s\prime}(u_0))v\|_{L^2(\Lambda)} \leqslant M_2^{\frac{1}{2}} M_4 \|k\|_{L^1(\mathbb{R})} \|u - \pi_N^s u_0\|_{L^2(\Lambda)} \|v\|_{L^2(\Lambda)}. \tag{3.54}$$

This implies $\|\mathcal{T}'(u_0) - \widetilde{\mathcal{T}}_N^{s\prime}(u_0)\| \longrightarrow 0$, as $N \to \infty$, $\widetilde{\mathcal{T}}_N^{s\prime}(u_0)$ is norm convergent to $\mathcal{T}'(u_0)$. Hence by Lemma 1, we have $(I - \widetilde{\mathcal{T}}_N^{s\prime}(u_0))^{-1}$ exists and uniformly bounded on $L^2(\Lambda)$, for some sufficiently large N, i.e., there exists some $M_5 > 0$ such that $\|(I - \widetilde{\mathcal{T}}_N^{s\prime}(u_0))^{-1}\| \leqslant M_5 < \infty$. Next, we are going to estimate $\|\widetilde{\mathcal{T}}_N^{s\prime}(u_0) - \widetilde{\mathcal{T}}_N^{s\prime}(u)\|$ for any $u \in \mathcal{B}(u_0, \delta)$ and $v \in L^2(\Lambda)$.

$$\|(\widetilde{\mathcal{T}}_N^{s\prime}(u_0) - \widetilde{\mathcal{T}}_N^{s\prime}(u))v\|_{L^2(\Lambda)} = \|(\mathcal{K}'(\pi_{s,N}u_0) - \mathcal{K}'((\pi_{s,N}u))v\|_{L^2(\Lambda)}. \tag{3.55}$$

Now from the estimate (3.33) for non-convolution kernel, we have

$$\|(\widetilde{\mathcal{T}}_N^{s\prime}(u_0) - \widetilde{\mathcal{T}}_N^{s\prime}(u))v\|_{L^2(\Lambda)} \leqslant M_4 M_1^{\frac{1}{2}} \delta \|\pi_{s,N}\| \|v\|^2_{L^2(\Lambda)}, \tag{3.56}$$

and from the estimate (3.34) for convolution kernel, we have

$$\|\widetilde{\mathcal{T}}_N^{s\prime}(u_0) - \widetilde{\mathcal{T}}_N^{s\prime}(u))v\|_{L^2(\Lambda)} \leqslant M_4 M_2^{\frac{1}{2}} \delta \|k\|_{L^1(\mathbb{R})} \|\pi_{s,N}\| \|v\|^2_{L^2(\Lambda)}. \tag{3.57}$$

This implies

$$\|\widetilde{\mathcal{T}}_N^{s\prime}(u_0) - \widetilde{\mathcal{T}}_N^{s\prime}(u)\|_{L^2(\Lambda)} \leqslant M_4 \delta p \|\pi_{s,N}\|, \tag{3.58}$$

where $p = \max\{M_2^{\frac{1}{2}} \|k\|_{L^1(\mathbb{R})}, M_1^{\frac{1}{2}}\}$.

Hence, we have

$$\sup_{\|u-u_0\|_{L^2(\Lambda)} \leqslant \delta} \|(I - \widetilde{\mathcal{T}}_N^{s\prime}(u_0))^{-1}(\widetilde{\mathcal{T}}_N^{s\prime}(u_0) - \widetilde{\mathcal{T}}_N^{s\prime}(u))\|_{L^2(\Lambda)} \leqslant M_4 M_5 \delta p \|\pi_{s,N}\|, \tag{3.59}$$

let $q = M_4 M_5 \delta p \|\pi_{s,N}\|$, we now pick δ so small that $0 < q < 1$, which proves Eq. (3.42) of Theorem 3.2.

Taking use of (1.134), (3.4) and (3.9) we have

$$\begin{aligned} \alpha_N^s &= \|(I-\widetilde{\mathcal{T}}_N^{s\prime}(u_0))^{-1}(\widetilde{\mathcal{T}}_N^s(u_0)-\mathcal{T}(u_0))\|_{L^2(\Lambda)} \\ &\leqslant M_5\|\widetilde{\mathcal{T}}_N^s(u_0)-\mathcal{T}(u_0)\|_{L^2(\Lambda)} \\ &= M_5\|\mathcal{T}(\pi_{s,N}u_0)-\mathcal{T}(u_0)\|_{L^2(\Lambda)} \\ &\leqslant M_5M_6\|\pi_{s,N}u_0-u_0\|_{L^2(\Lambda)} \\ &\leqslant M_5M_6cN^{-m}|u_0|_{\widetilde{H}^{m,s}}, \end{aligned} \tag{3.60}$$

where $M_6=\max\{M_2\left(\int_0^\infty\int_0^\infty k(x,t)^2dtdx\right)^{\frac{1}{2}}, M_3\|k\|_{L(\mathbb{R})}\}$.

By choosing N large enough such that $\alpha_N^s\leq\delta(1-q)$, Eq. (3.43) of Theorem 3.2 is satisfied. Hence by applying Theorem 3.2, we obtain

$$\frac{\alpha_N^s}{(1+q)}\leqslant\|\widetilde{u}_{s,N}-u_0\|_{L^2(\Lambda)}\leqslant\frac{\alpha_N^s}{(1-q)}. \tag{3.61}$$

Hence the theorem is proved. □

Now, we are ready to estimate the error between $u_{s,N}$ and u in the following theorem

Theorem 3.4. *If u_0 the solution of the operator equation (3.2), $u_{s,N}$ is the approximated solution of Eq. (3.48) and $\widetilde{u}_{s,N}$ is the iterated solution of (3.49), then we have the error between $u_{s,N}$ and u_0 is bound as below:*

$$\|u_0-u_{s,N}\|_{L^2(\Lambda)}\leqslant cN^{-m}(\|\pi_{s,N}\|_{L^2(\Lambda)}\frac{M_5M_6}{(1-q)}+1)|u_0|_{\widetilde{H}^{m,s}}. \tag{3.62}$$

Proof. We have $u_N^s=\pi_{s,N}\widetilde{u}_{s,N}$

$$u_0-u_{s,N}=u_0-\pi_{s,N}\widetilde{u}_{s,N}=u_0-\pi_{s,N}u_0+\pi_{s,N}u_0-\pi_{s,N}\widetilde{u}_{s,N}. \tag{3.63}$$

This is implies

$$\|u-u_{s,N}\|_{L^2(\Lambda)}\leqslant\|u_0-\pi_{s,N}u_0\|_{L^2(\Lambda)}+\|\pi_{s,N}\|\|u_0-\widetilde{u}_{s,N}\|_{L^2(\Lambda)}. \tag{3.64}$$

Employing (1.134), (3.51) and (3.60), we get

$$\|u_0-u_{s,N}\|_{L^2(\Lambda)}\leqslant cN^{-m}(\|\pi_{s,N}\|\frac{M_5M_6}{(1-q)}+1)|u_0|_{\widetilde{H}^{m,s}}. \tag{3.65}$$

□

3.5 Illustrative examples

In this section, we present some numerical examples to illustrate the convergence behavior of the proposed methods. Throughout this section, the following abbreviations are used: MLEFs-modified Legendre exponential functions; MLRFs-modified Legendre rational functions; CPU(s) running time in seconds corresponding to different N. The calculations performed in the examples are calculated by Matlab software, and a Core(TM) i3-5010U CPU@2.10GHz 2.10GHz and 4 GB RAM are used to run the programs.

Examples with exponentially decaying solutions

In the following, we solve the nonlinear integral equation of the type (3.1) for non-convolution and convolution kernels by using MLEFs-scheme. Due to (1.126), (1.127), (1.148) and (1.141), we can easily define the MLEFs, which satisfy the following recurrence relation

$$E_{s,0}(x) = \sqrt{\frac{2}{s}}e^{-x/2s}, \quad E_{s,1}(x) = \sqrt{\frac{2}{s}}e^{-x/2s}(1-2e^{-x/s}), \tag{3.66}$$

$$(n+1)E_{s,1+n}(x) = (2n+1)(1-2e^{-x/s})E_{s,n}(x) + nE_{s,n-1}(x), \tag{3.67}$$

where $x \in \Lambda$ and $n \geqslant 1$.

Example 3.1. [92] Consider the following nonlinear Hammerstein integral equation with non-convolution kernel:

$$u(x) + \int_0^{+\infty} e^{-(x+t)}u(t)^2dt = 6e^{-x}, \quad x \in \Lambda. \tag{3.68}$$

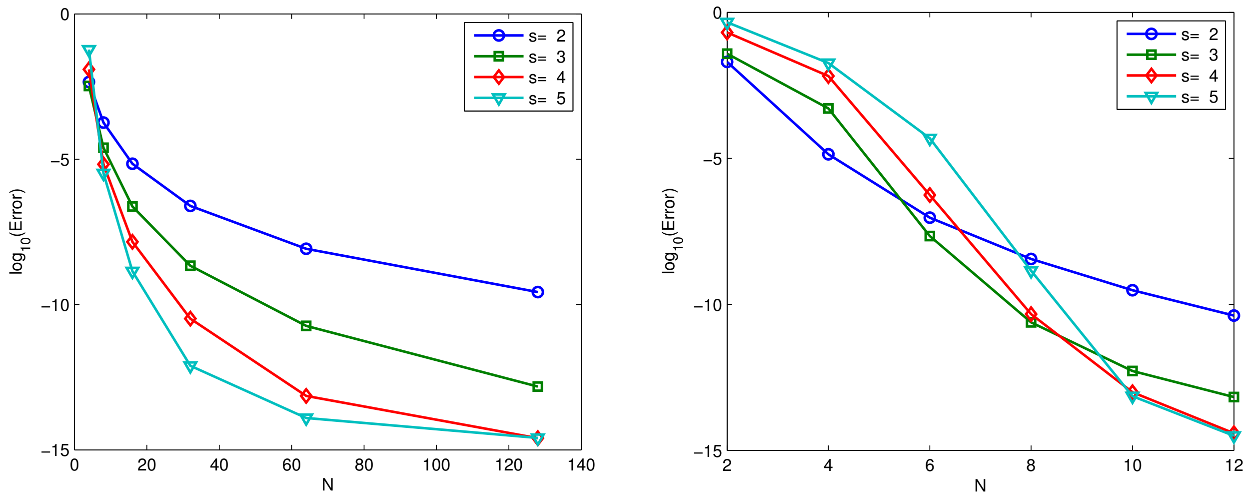
The exact solution is $u(x) = 3e^{-x}$, which is a smooth function and decays exponentially at infinity. The numerical errors obtained by applying the MLEFs-scheme described in section 3.3 with $s = 6$ are displayed in Table 3.1. These results are compared with those obtained by using multi-Galerkin and iterated multi-Galerkin methods based on piecewise polynomials [34]. A comparison with two other methods is also given in Table 3.2. The results from this comparison show that the proposed method is both accurate and efficient. In addition, the significant effect of the scale parameter on the convergence of the approximate and iterated solutions is shown in Figure 3.1.

Table 3.1: Comparison of the L^2-errors for Example 3.1.

	$s=6$				Method in [34]	
N	$\|u-u_{s,N}\|_{L^2(\Lambda)}$	CPU(s)	$\|u-\tilde{u}_{s,N}\|_{L^2(\Lambda)}$	CPU(s)	$\|u-u_N^M\|_{L^2}$	$\|u-\tilde{u}_N^M\|_{L^2}$
4	1.26e-01	0.17	2.87e-02	0.21	-	-
8	3.10e-06	0.19	6.61e-08	0.25	-	-
16	2.06e-10	0.25	4.02e-15	0.49	6.38e-04	3.26e-05
32	2.72e-14	0.81	3.92e-15	0.89	7.41e-05	2.03e-06
64	1.47e-14	0.88	1.13e-14	1.55	9.91e-06	1.26e-07
128	3.43e-15	1.50	4.39e-15	2.04	1.24e-06	7.89e-09

Table 3.2: Comparison of the $L^\infty-$errors for Example 3.1.

	$s=6$					
N	$\|u-u_{s,N}\|_\infty$	CPU(s)	$\|u-\tilde{u}_{s,N}\|_\infty$	CPU(s)	Method in [20]	Method in [21]
4	1.13e-01	0.16	7.54e-02	0.29	-	-
8	3.79e-06	0.25	5.43e-08	0.35	-	-
16	2.08e-10	0.57	3.11e-15	0.49	2.11e-05	4.74e-04
32	1.95e-14	0.67	4.44e-16	0.59	1.39e-06	2.94e-05
64	1.91e-14	1.10	2.13e-14	1.39	8.83e-08	1.83e-06
128	1.42e-14	2.36	4.44e-15	2.73	5.53e-09	1.14e-07



(a) $e_N = \|u - u_{s,N}\|_{L^2(\Lambda)}$.

(b) $e_N = \|u - \widetilde{u}_{s,N}\|_{L^2(\Lambda)}$.

Figure 3.1: Example 3.1: Convergence rates of the approximate and iterated solutions using MLEFs-scheme with various $s-$parameter.

Example 3.2. [92] Consider the following nonlinear Hammerstein integral equation with convolution kernel:

$$u(x) + \int_0^{+\infty} \frac{2}{e^{x-t} + e^{t-x}} u(t)^{\frac{3}{2}} dt = e^{-\frac{2}{3}x} + e^{-x} \ln(1 + e^{2x})/2, \quad x \in \Lambda. \tag{3.69}$$

The exact solution is $u(x) = e^{-\frac{2}{3}x}$. In Table 3.3 the L^2-errors of the MLEFs-scheme with $s = 5$ are compared with those obtained from [34]. A comparison of the infinity errors between the proposed method and those obtained from [21] and [20] is given in Table 3.4. It is obvious from the tables that the proposed method is better in term of accuracy when compared with the methods in [20, 21, 34]. Furthermore, from Figures 3.2 (a) and (b) we observe that the scaling parameter s can greatly enhance the convergence rates of the proposed method.

Table 3.3: Comparison of the L^2-errors for Example 3.2.

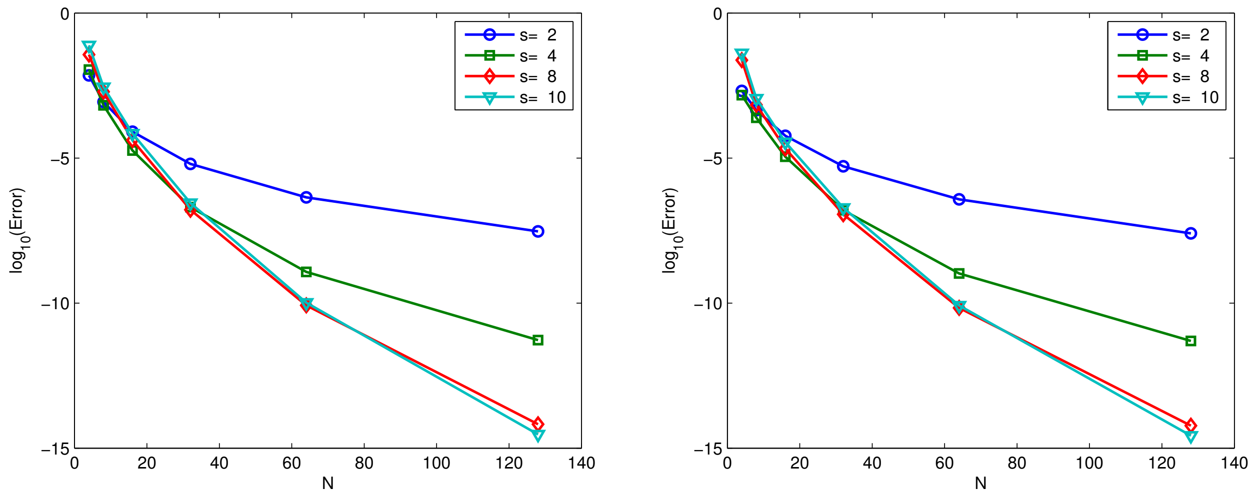
	$s = 5$				Method in [34]	
N	$\|u - u_{s,N}\|_{L^2(\Lambda)}$	CPU(s)	$\|u - \widetilde{u}_{s,N}\|_{L^2(\Lambda)}$	CPU(s)	$\|u - u_N^M\|_{L^2}$	$\|u - \widetilde{u}_N^M\|_{L^2}$
4	1.29e-02	0.29	3.57e-03	0.22	-	-
8	8.57e-04	0.32	2.93e-04	0.28	-	-
16	1.95e-05	0.43	1.11e-05	0.57	8.52e-04	5.44e-05
32	1.30e-07	0.69	9.76e-08	1.10	1.06e-04	3.38e-06
64	2.99e-10	1.71	2.57e-10	1.64	1.32e-05	2.11e-07
128	3.88e-13	3.10	3.59e-13	3.08	1.65e-06	1.32e-08
256	8.91e-16	6.66	7.81e-16	7.48	2.06e-07	8.14e-10

Example 3.3. Consider the nonlinear singular integral equation with non-convolution kernel:

$$u(x) + \int_0^{+\infty} \frac{e^{-x-t}}{x^{\alpha/2}} u(t)^2 dt = g(x), \quad 0 < \alpha < 1, \tag{3.70}$$

Table 3.4: Comparison of the $L^\infty-$errors for Example 3.2.

	$s=5$					
N	$\|u-u_{s,N}\|_\infty$	CPU(s)	$\|u-\widetilde{u}_{s,N}\|_\infty$	CPU(s)	Method in [20]	Method in [21]
4	1.33e-02	0.17	1.94e-02	0.29	-	-
8	5.80e-04	0.20	1.33e-03	0.39	-	-
16	1.09e-05	0.38	2.16e-05	0.56	1.72e-04	9.31e-05
32	6.38e-08	0.59	1.01e-07	0.95	1.06e-06	5.78e-06
64	1.36e-10	0.76	1.87e-10	1.63	6.65e-07	3.61e-07
128	1.71e-13	1.57	2.33e-13	2.59	4.15e-08	2.25e-08
256	1.33e-15	4.68	5.55e-16	7.44	2.56e-09	1.41e-09



(a) $e_N = \|u-u_{s,N}\|_{L^2(\Lambda)}$.　(b) $e_N = \|u-\widetilde{u}_{s,N}\|_{L^2(\Lambda)}$.

Figure 3.2: Example 3.2: Convergence rates of the approximate and iterated solutions using MLEFs-scheme with various $s-$parameter.

where $g(x)$ is chosen so that the exact solution is $u(x) = \sin(10x)e^{-x}$, which decays exponentially at infinity with oscillation. The L^2-errors for $u_{s,N}$ and $\widetilde{u}_{s,N}$ obtained by using MLEFs-schemes with $s=8$ for the case $\alpha = 0.5$ are displayed in Tables 3.5.

Table 3.5: The L^2-errors for Example 3.3 using MLEFs-scheme.

N	$\|u-u_{s,N}\|_{L^2(\Lambda)}$	CPU(s)	$\|u-\widetilde{u}_{s,N}\|_{L^2(\Lambda)}$	CPU(s)
4	2.80e-01	0.17	1.09e-01	0.22
8	3.10e-01	0.19	7.16e-02	0.25
16	4.49e-01	0.25	7.13e-03	0.38
32	2.36e-01	0.38	3.92e-03	0.62
64	1.84e-02	0.66	4.03e-06	1.14
128	3.12e-05	1.34	3.75e-12	2.23
256	2.06e-09	3.59	1.61e-16	5.33

Examples with algebraically decaying solutions

Now we solve the nonlinear integral equation of the type (3.1) for convolution and non-convolution kernels with algebraically decaying solutions at infinity by using MLRFs-scheme. Due to (1.126), (1.127), (1.148) and (1.142), we can easily define the MLRFs, which satisfy the following recurrence relation

$$
\begin{aligned}
&R_{s,0}(x) = \frac{\sqrt{2s}}{(x+s)}, \quad R_{s,1}(x) = \sqrt{2s}\frac{x-s}{(x+s)^2},\\
&(n+1)R_{s,1+n}(x) = (2n+1)\frac{x-s}{x+s}R_{s,n}(x) + nR_{s,n-1}(x),
\end{aligned} \tag{3.71}
$$

where $x \in \Lambda$ and $n \geqslant 1$.

Example 3.4. Consider the following nonlinear Hammerstein integral equation with non-convolution kernel:

$$
u(x) + \int_0^{+\infty} e^{-x^2-t^2}u(t)^3 dt = g(x), \quad x \in \Lambda, \tag{3.72}
$$

where $g(x)$ is chosen so that the exact solution is $u(x) = \frac{1}{\sqrt{x^2+1}}$. Table 3.6 shows the L^2-errors obtained by using the MLRFs-scheme described in section 3.3 with $s = 1$. It is observed that the desired exponential rate of convergence is obtained for a smooth solution with very slow decay.

Table 3.6: The L^2-errors for Example 3.4 using MLRFs-scheme.

N	$\|u - u_{s,N}\|_{L^2(\Lambda)}$	CPU(s)	$\|u - \widetilde{u}_{s,N}\|_{L^2(\Lambda)}$	CPU(s)
4	4.14e-02	0.16	2.85e-03	0.21
8	9.59e-04	0.19	3.43e-04	0.28
16	8.04e-07	0.26	2.30e-11	0.41
32	5.89e-12	0.40	1.95e-12	0.67
64	1.78e-15	0.68	1.78e-16	1.30

Example 3.5. Consider the following nonlinear Hammerstein integral equation with convolution kernel:

$$
u(x) + \int_0^{+\infty} \frac{e^{-4^2}}{(t-x)^2+1}u(t)^2 dt = g(x), \quad x \in \Lambda, \tag{3.73}
$$

where $g(x)$ is chosen so that the exact solution is $u(x) = \frac{1}{(x+1)^2}$, which is smooth function and decays algebraically at infinity. Tables 3.7-3.8 show the L^2-errors obtained by using the MLRFs-scheme with $s = 2$ and $s = 4$, respectively.

Example 3.6. Consider the following nonlinear integral equation

$$
u(x) - \frac{1}{2}\int_0^{+\infty} \frac{e^{-t}}{t^2+x^2+1}u(t)^2 dt = \frac{\arctan(x+1)}{x^2+1}, \quad x \in \Lambda, \tag{3.74}
$$

where the exact solution is unknown. In Tables 3.9-3.10, we display the numerical errors by computing the L^2-norm of the difference between $u_{s,128}$ and $u_{s,N}$, $\widetilde{u}_{s,128}$ and $\widetilde{u}_{s,N}$ by using the MLRFs-scheme with $s = 1$ and $s = 2$, respectively. Also, we represent the absolute values of the MLRFs coefficients and the numerical solution in Figure 3.3.

Table 3.7: The L^2-errors for Example 3.5 using MLRFs-scheme, $s = 2$.

N	$\|u - u_{s,N}\|_{L^2(\Lambda)}$	CPU(s)	$\|u - \widetilde{u}_{s,N}\|_{L^2(\Lambda)}$	CPU(s)
4	6.22e-03	0.17	8.84e-10	0.24
8	1.05e-05	0.20	2.73e-10	0.28
16	7.79e-11	0.28	6.65e-11	0.44
32	1.55e-11	0.42	1.44e-11	0.75
64	3.06e-12	0.75	2.96e-12	1.32
128	6.04e-13	1.39	5.93e-13	2.56

Table 3.8: The L^2-errors for Example 3.5 using MLRFs-scheme, $s = 4$.

N	$\|u - u_{s,N}\|_{L^2(\Lambda)}$	CPU(s)	$\|u - \widetilde{u}_{s,N}\|_{L^2(\Lambda)}$	CPU(s)
4	4.00e-02	0.20	1.65e-09	0.22
8	9.45e-04	0.24	2.35e-10	0.30
16	2.82e-07	0.27	4.72e-11	0.43
32	9.40e-12	0.42	8.70e-12	0.75
64	1.66e-12	0.71	1.60e-12	1.42
128	3.03e-13	1.61	2.98e-13	3.19

Table 3.9: The L^2-errors for Example 3.6 using MLRFs-scheme, $s = 1$.

N	$\|u_{s,128} - u_{s,N}\|_{L^2(\Lambda)}$	CPU(s)	$\|\widetilde{u}_{s,128} - \widetilde{u}_{s,N}\|_{L^2(\Lambda)}$	CPU(s)
4	2.68e-01	0.18	6.32e-02	0.25
8	8.93e-03	0.23	2.01e-04	0.32
16	8.99e-06	0.30	4.63e-10	0.49
32	7.97e-12	0.46	4.48e-15	0.82
64	5.73e-15	0.86	5.09e-15	1.53

Table 3.10: The L^2-errors for Example 3.6 using MLRFs-scheme, $s = 2$.

N	$\|u_{s,128} - u_{s,N}\|_{L^2(\Lambda)}$	CPU(s)	$\|\widetilde{u}_{s,128} - \widetilde{u}_{s,N}\|_{L^2(\Lambda)}$	CPU(s)
4	2.67e-01	0.22	1.15e-01	0.28
8	1.43e-02	0.25	4.58e-04	0.34
16	1.98e-05	0.34	2.04e-09	0.50
32	5.56e-11	0.52	1.62e-15	0.87
64	9.13e-15	0.92	9.34e-15	1.54

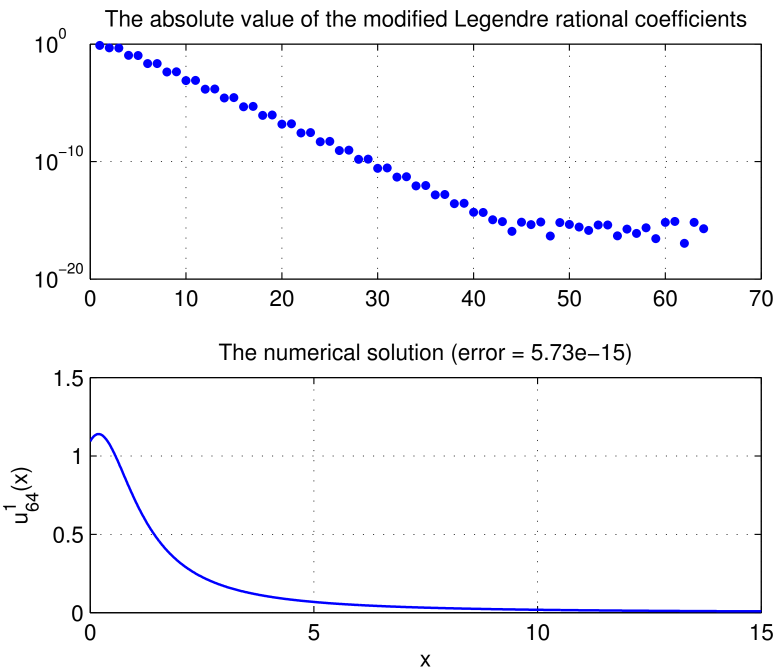


Figure 3.3: Numerical results of MLRFs-scheme for Example 3.6.

Appendix

Here we give some standard definitions and theorems used in this thesis. They do not into the aims covered by this thesis but are necessary fundamentals for the existence and convergence analysis. We list theorems, without proof, to remind the reader.

Standard Definitions

The Cauchy–Schwarz inequality

The Cauchy–Schwarz inequality also called Cauchy–Bunyakovsky–Schwarz inequality is considered one of the most important and widely used inequalities in mathematics, which is used to prove the existence of nonlinear integral equations with convolution kernels also the convergence of numerical methods proposed above to approximate integral and differential equations.

The inequality for integrals originally was published by the authors of [104] while the modern proofgiven by the authors of [105].

Definition 3.1 (Cauchy–Schwarz inequality)**.** Let H is Hilbert space with inner product $(,)_H$. Then for all $u, v \in H$, we have

$$\sqrt{|(u,v)_H|} \leq \|u\|_H \|v\|_H,$$

where $\|u\|_H = \sqrt{(u,u)_H}$.

Positive-difinite and Self-adjoint operator

We now turn our mention to some essential definitions of the linear operator, positive and Self-adjoint, which are used to provide new norms associated with the fractional power of the operators defined in (2.46) and (2.2) in chapter 2.

Definition 3.2 (Positive-difinite operator)**.** A linear operator T acting on an inner product space is called positive-definite if, for every $v \in \mathrm{Dom}(T)\backslash\{0\}$

$$(Tv, v) \in \mathbb{R} \text{ and } (Tv, v) > 0, \tag{3.75}$$

where $\mathrm{Dom}(T)$ is the domain of T.

Definition 3.3 (Self-adjoint operator)**.** Let $T : H \to H$ is linear operator between Hilbert space, we say that T is self-adjoint if

$$(Tv, u)_H = (v, Tu)_H, \quad \text{for all } v, u \in H, \tag{3.76}$$

where $(\cdot,\cdot)_H$ is the inner product in the Hilbert space H.

The Fréchet derivative

In mathematics, the Fréchet derivative is a derivative define on Banach spaces. This derivative is a class of directional derivatives, which has applications to nonlinear problems throughout the mathematical analysis and physical sciences, particularly in the calculus of variations and much of nonlinear analysis and nonlinear functional analysis. For the well-known properties of the Fréchet derivative of a nonlinear operator we refer to [106].

Definition 3.4. Let X, Y are Banach spaces, the directional derivative of $\mathcal{K} : X \to Y$ at $u \in U \subseteq X$ in the direction $v \in X$, denoted by the symbol $\mathcal{K}'(u; v)$, is defined by the equation

$$\mathcal{K}'(u)v = \lim_{t \longrightarrow 0} \frac{\mathcal{K}(u + rv) - \mathcal{K}(u)}{t},$$

whenever the limit on the right exists. We say $\mathcal{K}$ is Fréchet differentiable at u if there is bounded and linear operator $\mathcal{K}'(u) : X \to Y$ such that

$$\mathcal{K}'(u)v = \lim_{r \longrightarrow 0} \frac{\mathcal{K}(u + rv) - \mathcal{K}(u)}{r},$$

is uniform for every $v \in \mathrm{Dom}(\mathcal{K}'(u))$. The operator $\mathcal{K}'(u)$ is called the Fréchet derivative of $\mathcal{K}$ at u.

In chapter 3, we have considered the Hammerstein integral equations on the half-line where the nonlinear integral operator defined as

$$\left(\mathcal{K}'(u)v\right)(x) = \int_0^{+\infty} k(x,t) f(t, u(t)) dt, \tag{3.77}$$

while $f(., u) = u^p$, for $p = 3/2, 2$ and 3. The Fréchet derivative of the integral operator (3.77)

$$\begin{aligned} \left(\mathcal{K}'(u)\right) v(x) &= \lim_{r \longrightarrow 0} \frac{\int_0^{+\infty} k(x,t)(u(t) + rv(t))^p dt - \int_0^{+\infty} k(x,t)(u(t))^p dt}{r} \\ &= \lim_{r \longrightarrow 0} \frac{\int_0^{+\infty} k(x,t)\left((u(t) + rv(t))^p - (u(t))^p\right) dt}{r}. \end{aligned}$$

Using, the binomial Newton formula, we get

$$\left(\mathcal{K}'(u)\right) v(x) = p \int_0^{+\infty} k(x,t)(u(t))^{p-1} v(t) dt. \tag{3.78}$$

Newton's method

In numerical analysis, Newton's method is an iterative method that computes an approximate solution to the system of nonlinear equations. This method is to find successively better approximations to the zeroes of a real-valued function. In the following, we describe Newton's method to find a simple root of a system of nonlinear functions $f(x) = 0$ where

$$f(x) = \begin{cases} f_1(x_1, x_2, \ldots, x_N) & = 0, \\ f_2(x_1, x_2, \ldots, x_N) & = 0, \\ \vdots & \vdots \\ f_N(x_1, x_2, \ldots, x_N) & = 0. \end{cases} \tag{3.79}$$

In order to use the Newton method for solving system (3.79), we define the Jacobian matrix of f as.

$$\mathbf{J}(f) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{pmatrix}. \tag{3.80}$$

The process is repeated of newton method as

$$x^{k+1} = x^k - \triangle x^k, \tag{3.81}$$

where

$$\triangle\, x^k = \mathbf{J}(f)(x^k)^{-1} Q(x^k), \tag{3.82}$$

with initial guess x^0 chosen close to the exact solution and the Jacobian matrix $\mathbf{J}(Q)(x^0)^{-1}$ is exist.

Standard Theorems

In this sub-section, we list some Theorems which are necessary for our studies in the thesis. This theorem is quoted in [107]. It is used in Section 2.1 of Chapter 2 to prove the existence of the exact solutions of the second-order differential equations.

Theorem 3.5 (Lax-Milgram Theorem). *Let $b : H \times H \longrightarrow \mathbb{R}$ be a bounded bilinear form. If b is coercive, i.e., there exists $c > 0$ such that $b(u, u) \geq c\|u\|_H^2$ for every $u \in H$, then for any $f \in H'$ (H' is conjugate space of H) there exists a unique $u \in H$ such that*

$$b(u, v) = (f, v)_H \text{ for every } v \in H.$$

The below theorem is used in Section 3.4 of Chapter 3 to obtain the convergence analyses of the approximate and iterate solution.

Theorem 3.6 (Fubini's theorem). *Assume that $f \in L^1(\Lambda \times \Lambda)$. Then for a.e. $x \in \Lambda$, $f(x, y) \in L^1(\Lambda)$ and $\int_\Lambda |f(x, y)|dy \in L^1(\Lambda)$. Similarly, for a.e. $y \in \Lambda$, $f(x, y) \in L^1(\Lambda)$ and $\int_\Lambda |f(x, y)|dx \in L^1(\Lambda)$.*

Moreover, one has

$$\int_\Lambda \int_\Lambda |f(x, y)|dydx = \int_\Lambda \int_\Lambda |f(x, y)|dxdy.$$

A proof is given in [102]. The following Theorem gives the Young inequality, which is used in Sections 3.3 and 3.4 of Chapter 3 for the Hammerstein integral equation with convolution smooth kernel.

Theorem 3.7 (Young). *(Theorem 4.15 [102, p.104]) Let $f \in L^1(\mathbb{R})$ and let $g \in L^p(\mathbb{R})$ with $1 \leq p \leq \infty$. Then for a.e. $x \in \mathbb{R}$ the function $y \longmapsto f(x - y)g(y)$ is integrable on $\mathbb{R}$ and we define*

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x - y)g(y)dy.$$

In addition $f(x - y)g(y) \in L^p(\mathbb{R})$

$$\|f * g\|_{L^p(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}\|g\|_{L^p(\mathbb{R})}.$$

Concluding remarks

In the existing spectral methods in unbounded domains, one usually used the classical orthogonal approximations, Laguerre and Hermite functions. This thesis firstly has introduced and analyzed a new set of orthogonal functions in unbounded domains named modified Jacobi rational and exponential functions which are mutually orthogonal in the usual L^2 space. The error between the approximate and interpolate with exact solutions are estimated, we also made a detailed comparison of the convergence rates of this spectral method for solutions with typical decay behaviors. The following general observations can be made related to the convergence rates of the approximate solutions:

- For smooth functions which decay exponentially fast at infinity, the spectral methods based on Modified Jacobi functions converge exponentially, but the Modified Jacobi exponential functions better than the Modified Jacobi rational functions.
- For smooth functions which decay algebraically slow at infinity, the Modified Jacobi exponential functions converge algebraically while the Modified Jacobi rational functions converge exponentially.

The Modified Jacobi exponential and rational functions spectral methods have several fascinating merits:

- The advantages of these methods are that they can be implemented and analyzed using standard procedures and approximation results, they do not require domain truncation or reduction of the equation to a finite domain.
- The mapping in this approach played an essential role in designing the Modified Jacobi spectral schemes, that depends on the asymptotic behaviour of the exact solution. It is worth noticing that the accuracy can be greatly improved by using the method with appropriate scaling parameters β, α and s.
- The numerical error of the modified Jacobi functions approach decays faster than that of the Laguerre and Hermite functions spectral methods.

Secondly, we implemented the modified Jacobi functions spectral methods to solve second-order differential equations, where the solutions of such differential equations decay smoothly towards zero at infinity. Also, we construct two collocation schemes and provide an estimate of the error and convergence rate in the weighted Hilbert space H^1_w.

Finally, we applied the Modified Jacobi functions with $\alpha = \beta = 0$ which is the Modified Legendre functions for solving the nonlinear Hammerstein integral equation on the half-line, including the special case of convolution kernel, i.e., $k(x,t) = k(x-t)$. Also, the error estimation between the approximate and exact solutions has investigated in L^2-norm with the order $\mathcal{O}(N^{-m})$, where m is the smoothness degree of the equation's solution. The obtained results from the numerical examples have shown that the present methods are reliable and efficient.

Besides the already discussed current lines of research, they can be further applied to solve a large class of problems on the real-line as Hammerstein generalized integral equations with Green's kernel

$$\begin{cases} \partial_x^2 u(x) - a u(x) = f(t, u(x), \partial_x u(x), \ldots, \partial_x^m u(x)), \quad a > 0, \quad m \geq 1, \\ u(\mp\infty) = u'(\mp\infty) = 0. \end{cases} \tag{3.83}$$

The converted integral equation may therefore be expressed as follows:

$$u(x) = -\frac{1}{2\sqrt{a}} \int_{-\infty}^{+\infty} e^{-\sqrt{a}|x-t|} f(t, u(t), \partial_t u(t), \ldots, \partial_t^m u(t)) dt, \quad x \in \mathbb{R}, \tag{3.84}$$

that came up as possible future work.

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ملخص :

الهدف الرئيسي من هذه الأطروحة هو تقريب حلول بعض المسائل الرياضية في شكل معادلات تكامليه أو تفاضلية على المجالات غير المحدودة وتتمثل الاستراتيجية الشائعة والفعالة في التعامل مع المجالات غير المحدودة في استخدام تطبيق عكسي يحول المجال غير المحدود الي مجال محدود وفي هذه الأطروحة نقدم نظاما متعامدا جديدا يقوم على تركيب التطبيق العكسي بكثيرات حدود جاكوبي الكلاسيكية ومنه نتحصل على دوال جاكوبي المعينة والتي نوظفها لحل معادلات تفاضلية من الدرجة الثانية ومعادلات تكاملية غير خطية في مجال غير محدود.

.

كلمات المفتاحية :

التقريب الطيفي ، التقريب الجذري، دوال جاكوبي المعينة، الاستيقطاب.

Abstract

The main aim of this thesis is to approach the solutions of some mathematical problems in the form of integral equations or differential equations on unbounded domains. A common and effective strategy in dealing with unbounded domains is to use a suitable mapping that transforms an infinite domain. In this thesis, we introduce a new orthogonal system of mapped Jacobi functions which is the images of classical Jacobi polynomials under the inverse mapping. The modified Jacobi spectral methods are proposed for second-order differential and nonlinear integral equations on the semi-infinite domain.

Key words :

Spectral approximation, rational approximation, mapped Jacobi functions, interpolation.

Résumé

L'objectif principal de cette thèse est d'approximer les solutions de certains problèmes mathématiques sous la forme d'équations intégrales ou d'équations différentielles sur des domaines non bornés. Une stratégie courante et efficace pour traiter les domaines illimités consiste à utiliser un mappage approprié qui transforme un domaine infini. Dans cette thèse, nous introduisons un nouveau système orthogonal de fonctions de Jacobi mappées qui sont les images des polynômes de Jacobi classiques sous l'application inverse. Les méthodes spectrales de Jacobi modifiées sont proposées pour les équations différentielles du second ordre et intégrales non linéaires sur le domaine semi-infini.

Mots clés :

Approximation spectrale, approximation rationnelle, fonctions jacobi mappées, interpolation.